The optimal gradient estimates for the perfect conductivity problem with $C^{1,\alpha}$ inclusions" by Yu Chen, Haigang Li and Longjuan Xu

Gradient estimates for the perfect conductivity problem and similar problems for linear systems of elasticity have received much attention. This paper has given nice extensions to some of the earlier works, and has also found some new phenomena.

Theorem 1.1 extends some earlier works in dimensions $n \geq 3$ for $C^{2,\alpha}$ inclusions ∂D_1 and ∂D_2 to $C^{1,\alpha}$ inclusions. The have also obtained in dimension $n = 2$ doe $C^{1,\alpha}$ inclusions an $\alpha-$dependence optimal gradient estimate Theorem 1.4 shows an interesting new phenomena that when two inclusions have flat pieces facing to each other, see Figure 1 on page 5, the gradient is actually bounded.

In this report, I will give some suggestions so that the authors can make a revision, and I will be glad to give another reading after the revision.

1. In the assumption of Theorem 1.4, it is assumed that ∂D_1 and ∂D_2 are $C^{2,\alpha}$. It is also assumed that (1.11) and (1.13) hold. These two assumptions are in contradition to each other, since (1.11) forces the Hessians of h_i to vanish on $\partial\Sigma'$. It is reasonable to believe that the conclusion of the theorem, i.e. the boundedness of the gradient ∇u, does not need the assumption (1.13). The authors should make a revision concerning this.

2. The essential part of Theorem 1.7 was already proved in [7], see (2.16) on page 212 there.

3. The statement in Remark 1.2 on page 4 should be stated as there are examples of domains Ω and D_i, and for some boundary data φ which has the lower bound. Note that when the inclusions are $C^{2,\alpha}$, lower bound examples were given in [7], see Theorem 1.2 and Remark 1.3 there. Here, the authors should also mention earlier works on upper bounds for $C^{2,\alpha}$ inclusions (Theorem 1.1 is for $C^{1,\alpha}$ inclusions.

4. In the abstract, the authors attribute an iteration technique to Bao-Li-Li (ARMA 2015). A similar statement is also made in the introduction (page 3, line

16-17). This technique was actually first used in [33], see Theorem 1.1 and its proof there. The statements should be revised.

5. On page 3, line 9-11, it was stated that only $m = 2 + \alpha$ was studied in earlier works, but it is not accurate, see (2.16) in [7] as mentioned in Item 2 above.

6. Line 5 in the abstract: “we derive the upper and lower bounds...” should be “we derive upper and lower bounds...”.

7. Line 9 in the abstract: “an pointwise upper bound” should be “a pointwise upper bound”.

8. The title could be changed to “Optimal gradient estimates for the perfect conductivity problem ...” from “The optimal gradient estimates for perfect conductivity problem...”.