UPPER AND LOWER BOUNDS FOR STRESS CONCENTRATION IN LINEAR ELASTICITY WHEN $C^{1,\alpha}$ INCLUSIONS ARE CLOSE TO BOUNDARY

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Abstract. In this paper, we establish boundary gradient estimates for both Lamé systems with partially infinite coefficients and perfect conductivity problem. As a supplement of boundary blow-up analysis in Li-Zhao (SIMA 2020) where m-convex domains ($m \geq 2$) are considered, we treat with the case $1 < m = 1 + \alpha < 2$ and give the specific examples of Dirichlet boundary data leading to the existence of nontrivial blow-up factors, which guarantee the gradients blow up, and obtain the lower bound gradient estimates in all dimensions indicating that the blow-up rates of the gradients with respect to the distance between the interfacial surfaces are optimal.

1. Introduction and main results

1.1. **Background and problem formulation.** There is an interesting research concerning with electrostatic, magnetic, and elastic fields in composites which consist of the fiber inclusions being packed in the background matrix whose material properties differ from that of inclusions. When the high-contrast inclusions are closely spaced in composite materials, the physical fields, such as the stress field or electric field always concentrates in narrow regions between inclusions. This enhancement inspires us to investigate the micro-structural effects, especially on the distance, say ε, between the close-to-touching interfacial surfaces. In this paper, we focus on the boundary gradient estimates of solutions for Lamé system with partially infinite coefficients and perfect conductivity problem to reveal that the concentration of physical fields (stress tensor and electric field) has the dependence on the distance ε between the inclusion and the boundary of the matrix domain as well as the interaction from Dirichlet boundary data.

This problem is stimulated by the research of the prediction for the damage initiation and growth in carbon-fiber epoxy composite materials in the well-known work of Babuška et al. [6]. They considered the system of linear elasticity

$$\mu\Delta \mathsf{u} + (\lambda + \mu)\nabla\nabla \cdot \mathsf{u}$$

in unidirectional composites, where u is a vector-valued function representing the displacement, and numerically observed that the stress, represented by $\nabla \mathsf{u}$, is still bounded when the inclusions touch or nearly touch. Bonnetier and Vogelius [13] considered the elliptic scalar equation in dimension $d = 2$,

$$\nabla \cdot \big((1 + (k-1)\chi_{D_1\cup D_2})\nabla u\big) = 0, \quad k \neq 1, \tag{1.1}$$

which models anti-plane shear problem or electrical conductivity problem, where u is a scalar function, representing the electric potential, and D_1, D_2 are two touching disks with comparable radii. By virtue of Möbius transformation and

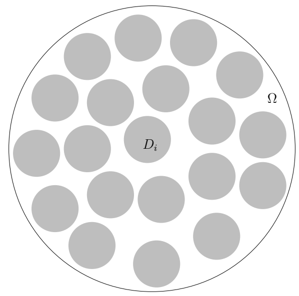

FIGURE 1. Illustration of the domain. $\Omega := D \setminus \cup_{i=1}^{L} \overline{D}_i$.

the maximum principle, they rigorously proved that the gradient of solution for (1.1), ∇u, representing the electric field, is indeed bounded. The general results were obtained by Li and Vogelius [38] considering a class of divergence form elliptic equations with piecewise Hölder continuous coefficients in a bounded domain D consisting of L disjoint inclusions D_i; see Figure 1 (for simplicity we let each of the inclusions be identical disks). The global Lipschitz estimates and piecewise $C^{1,\alpha}$ estimates were established in all dimensions. For the general elliptic systems including the system of linear elasticity, these results in [38] were extended by Li and Nirenberg [37], which, in particular, gives an affirmative theoretical answer to the numerical indication initially aroused in [6] for the boundedness of stress regardless of the distance ε. Here, we draw the attention of readers to the open problems on page 894 in [38]. Until now, with respect to the higher order derivative estimates, some progress has been obtained only for scalar equations in dimension $d = 2$, see [17, 18, 38].

We notice from [37, 38] that these estimates are dependent on the ellipticity of coefficients and are independent of the distance ε between these surfaces. To figure out the effect of ε, one can assume that the coefficients in inclusions degenerate to infinity, which makes the situation become quite different. For the scalar case, it is the well-known perfect conductivity problem. Keller [27] was the first to compute the effective electrical field and found the singularity of it for a composite media containing a dense array of perfect conducting spheres or circular cylinders. Recently, More rigorous approaches have been developed to obtain the blow-up rates as the distance ε tends to zero. There is a long list of literature in this direction of research, see the examples [2–5, 8, 9, 12, 16, 19, 21, 30, 31, 41–43] and their references therein. It is known that the blow-up rate of $|\nabla u|$ is $\varepsilon^{-1/2}$ in dimension $d = 2$, $|\varepsilon \ln \varepsilon|^{-1}$ in $d = 3$, and ε^{-1} in $d \geq 4$. Moreover, it is interesting and important for the practical application in engineer and the requirement of numerical algorithm design to characterize the singular behavior of ∇u, which are developed in [3, 22, 23, 26, 29, 32, 35, 40]. While, if the coefficients in inclusions degenerate to zero, it is the insulated conductivity problem. The blow-up gradient estimates and characterization are obtained in [2, 9, 39, 44].

For the system of linear elasticity, Lamé system, with partially infinite coefficients, Bao Li and Li [10, 11] applied an ingenious iteration technique for the energy to obtain the point-wise upper bound estimates of $|\nabla \mathsf{u}|$ in all dimensions. Hou and Li [20] used the polynomial function to reveal the relationship between the blow-up rates of $|\nabla \mathsf{u}|$ and the order of convexity. As for the optimal estimates,

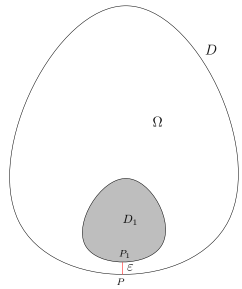

FIGURE 2. One inclusion

significant progress has been made by Kang and Yu [25]. They exquisitely constructed singular functions as elaborated linear combinations of nuclei of strain to characterize the behavior of $|\nabla \mathsf{u}|$, precisely, which shows that the optimal blow-up rate in dimension $d = 2$ is $\varepsilon^{-1/2}$. Subsequently, Li [28] established the lower bound estimates of $|\nabla \mathsf{u}|$ in dimensions $d = 2, 3$. Based on these results, we see that the blow-up rates of the gradients with respect to the distance ε between adjacent inclusions, $\varepsilon^{-1/2}$ in $d = 2$ and $|\varepsilon \ln \varepsilon|^{-1}$ in $d = 3$, are optimal. Moreover, due to the random character of the fibers' locations and the interaction from the boundary data, Bao, Ju and Li [7] considered the situation that stiff inclusions are located close to matrix boundary and established the point-wise upper bound and lower boundary estimates of $|\nabla \mathsf{u}|$ in all dimensions. Furthermore, Li and Zhao [36] investigated the dependence of blow-up rates on the inclusion's convexity and the boundary data's order of growth. They used a range of growth from the boundary data to guarantee the blow-up occurs and increase the blow-up rates.

We notice that all these blow-up rate estimates mentioned above are obtained under the $C^{2,\alpha}$, $0 < \alpha < 1$, or more smooth assumption for inclusions. Recently, Chen, Li and Xu [14] established the gradient estimates for perfect conductivity problem with adjacent $C^{1,\alpha}$ inclusions located far away from the matrix's boundary. They applied De Giorgi-Nash estimates and Campanato's approach to adapt the iteration technique, developed in [10, 33], and obtain the optimal blow-up rate in dimension $d = 2$. Subsequently, Chen and Li [15] extended these results to Lamé system. As a continuation of [14, 15], when the inclusions are located close to the boundary of background matrix, the interaction from boundary data brings that the solutions become more irregular. We shall establish the boundary gradient estimates for both Lamé system and perfect conductivity problem under $C^{1,\alpha}$ assumptions. Moreover, we give the specific examples of the boundary data which guarantee the blow-up occurs from the lower bound and do not change the blow-up rates. On the other hand, from the dependence of blow-up rates on the order of convexity, $C^{1,\alpha}$ inclusions imply that this order becomes $1 < m < 2$. Our results supplement the boundary gradient estimates from m-convex domains ($m \geq 2$) in [36], which classify the relationship between the singularity of $|\nabla u|$ and the boundary data to a large extent.

Before stating our main results precisely, we first fix our domains and notations. Let $D \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain and $D_1 \subset D$ be a sub-domain. We assume that there exist two points $P \in \partial D$ and $P_1 \in \partial D_1$ such that

$$\operatorname{dist}(P_1, P) := \operatorname{dist}(\partial D_1, \partial D) = \varepsilon.$$

Through translation and rotation, if necessary, we can further assume that

$$P_1 = (0', \varepsilon) \in \partial D_1, \quad P = (0', 0) \in \partial D.$$

See Figure 2. Here, we use superscripts prime to denote $d-1$ dimensional variables and domains, such as x' and B'. We assume that ∂D_1 and ∂D are both of $C^{1,\alpha}$, $0 < \alpha < 1$ and there exists a constant $0 < R_0 < 1$, independent of ε, such that ∂D_1 and ∂D near the origin can be represented by graphs

$$x_d = \varepsilon + h_1(x') \quad \text{and} \quad x_d = h(x'), \quad \text{for } |x'| \leq 2R_0, \tag{1.2}$$

where $h_1,\, h \in C^{1,\,\alpha}(B'_{2R_0}(0'))$ and satisfy

$$h(x') < \varepsilon + h_1(x'), \quad \text{for } |x'| < 2R_0, \tag{1.3}$$

$$h_1(0') = h(0') = 0, \quad \nabla_{x'} h_1(0') = \nabla_{x'} h(0') = 0, \tag{1.4}$$

$$\kappa_0 |x'|^\alpha \leq |\nabla_{x'} h_1(x')|, |\nabla_{x'} h(x')| \leq \kappa_1 |x'|^\alpha, \quad \text{for } |x'| < 2R_0, \tag{1.5}$$

and

$$\|h_1\|_{C^{1,\alpha}(B'_{2R_0})} + \|h\|_{C^{1,\alpha}(B'_{2R_0})} \leq \kappa_2, \tag{1.6}$$

where $\kappa_0,\, \kappa_1,\, \kappa_2 > 0$ are constants independent of ε. We define the narrow region

$$\Omega_r := \left\{(x', x_d) \in \mathbb{R}^d \;\middle|\; h(x') < x_d < \varepsilon + h_1(x'),\ |x'| < r\right\},$$

where $0 < r \leq 2R_0$. Throughout this paper, we use the notations u, v, w, g to denote vector-valued functions and u, v, w, g to denote scalar-valued ones. We divide two subsections to state our main results.

1.2. **Lamé system with partially infinite coefficients.** Let $\mathsf{u} = (u^{(1)}, u^{(2)}, \cdots, u^{(d)})^{\mathrm{T}} : D \to \mathbb{R}^d$ be a vector-valued function, representing the displacement, and verify the boundary value problem

$$\begin{cases} \mathcal{L}_{\lambda,\mu}\mathsf{u} := \nabla \cdot (\mathbb{C}^0 e(\mathsf{u})) = 0, & \text{in } \Omega := D \setminus \overline{D}_1, \\ \mathsf{u}|_+ = \mathsf{u}|_-, & \text{on } \partial D_1, \\ e(\mathsf{u}) = 0, & \text{in } D_1, \\ \displaystyle\int_{\partial D_1} \frac{\partial \mathsf{u}}{\partial \nu_0}\Big|_+ \cdot \psi_l = 0, & l = 1, 2, \cdots, \dfrac{d(d+1)}{2}, \\ \mathsf{u} = \mathsf{g}(x), & \text{on } \partial D. \end{cases} \tag{1.7}$$

where boundary data $\mathsf{g} = (g^{(1)}, g^{(2)}, \cdots, g^{(d)})^{\mathrm{T}} \in C^{1,\alpha}(\partial D; \mathbb{R}^d)$ and the elastic tensor $\mathbb{C}^0$ is

$$C^0_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad i, j, k, l = 1,\, 2, \cdots, d,$$

δ_{ij} is the kronecker symbol: $\delta_{ij} = 0$ for $i \neq j$, $\delta_{ij} = 1$ for $i = j$. The Lamé pair (λ, μ) satisfies the strong convexity condition $\mu > 0$ and $d\lambda + 2\mu > 0$. We further assume that for some $\delta_0 > 0$,

$$\delta_0 \leq \mu, d\lambda + 2\mu \leq \frac{1}{\delta_0}.$$

Moreover,

$$e(\mathsf{u}) := \frac{1}{2}(\nabla \mathsf{u} + (\nabla \mathsf{u})^{\mathrm{T}}),$$

is the strain tensor, and

$$\frac{\partial \mathsf{u}}{\partial \nu_0}\Big|_+ := (\mathbb{C}^0 e(\mathsf{u}))\mathsf{n} = \lambda(\nabla \cdot \mathsf{u})\mathsf{n} + \mu(\nabla \mathsf{u} + (\nabla \mathsf{u})^{\mathrm{T}})\mathsf{n}, \tag{1.8}$$

the vector n is the outward unit normal of its domain. Here and throughout this paper, the subscript $\pm$ indicates the limit from outside and inside the domain, respectively. We introduce the linear space of rigid displacement in $\mathbb{R}^d$:

$$\Psi := \{\psi \in C^1(\mathbb{R}^d;\mathbb{R}^d) \mid \nabla\psi + (\nabla\psi)^{\mathrm{T}} = 0\}.$$

We denote the standard basis of $\mathbb{R}^d$ by $e_1, \cdots, e_d$, then we see that

$$\left\{\ e_i,\ x_j e_k - x_k e_j\ \Big|\ 1 \le i \le d,\ 1 \le j < k \le d\ \right\} \tag{1.9}$$

are the basis of Ψ. For simplicity, we use $\big\{\psi_l \mid l = 1,\, 2,\, \cdots,\, d(d+1)/2\big\}$ to denote the basis of Ψ.

The existence, uniqueness and regularity of solutions for (1.7) are referred to the appendix in [8], with a minor modification. In particular, the H^1 weak solution of (1.7) is in $C^1(\overline{\Omega};\mathbb{R}^d) \cap C^1(\overline{D}_1;\mathbb{R}^d)$. Denote

$$\rho_{d,\,\alpha}(\varepsilon) = \begin{cases} \varepsilon^{\frac{\alpha}{1+\alpha}}, & d = 2, \\ 1, & d \ge 3. \end{cases} \tag{1.10}$$

We have the upper bound gradient estimates of solution for (1.7) as follows:

Theorem 1.1. *Let D_1 be a bounded $C^{1,\alpha}$ sub-domain of $D \subset \mathbb{R}^d$, $d \ge 2$, with ε apart. We assume that* (1.2)-(1.6) *hold. Let $\mathsf{u} \in H^1(D) \cap C^1(\Omega)$ be the solution of* (1.7) *with $\mathsf{g} \in C^{1,\alpha}(\partial D,\mathbb{R}^d)$. Then for sufficiently small $\varepsilon > 0$, we have*

$$|\nabla\mathsf{u}(x',x_d)| \le C\Big(\frac{\rho_{d,\,\alpha}(\varepsilon)}{\varepsilon + |x'|^{1+\alpha}} + \frac{|x'|}{\varepsilon + |x'|^{1+\alpha}} + 1\Big)\cdot\|\mathsf{g}\|_{C^{1,\,\alpha}(\partial D;\mathbb{R}^d)}, \quad x \in \Omega_{R_0}, \tag{1.11}$$

and

$$|\nabla\mathsf{u}(x)| \le C\|\mathsf{g}\|_{C^{1,\alpha}(\partial D;\mathbb{R}^d)}, \quad x \in \Omega \setminus \Omega_{R_0}, \tag{1.12}$$

where $C > 0$ is a constant independent of $\varepsilon > 0$.

The lower bound estimates of $|\nabla\mathsf{u}|$ on the segment $\overline{P_1P}$ can also be established with some specific boundary data g given. From these results, we can see that the blow-up rates of gradients with respect to ε, obtained in Theorem 1.1, are optimal. Denote $D_1^* := \{x \in \mathbb{R}^d \mid x + P_1 \in D_1\}$. Let u_0^* verify the boundary value problem:

$$\begin{cases} \mathcal{L}_{\lambda,\mu}\mathsf{u}_0^* = 0, & \text{in } \Omega^* := D \setminus \overline{D_1^*}, \\ \mathsf{u}_0^* = 0, & \text{on } \partial D_1^*, \\ \mathsf{u}_0^* = \mathsf{g}(x) - \mathsf{g}(P), & \text{on } \partial D. \end{cases} \tag{1.13}$$

We define the following functionals of g

$$b_l^* := b_l^*[\mathsf{g}] := \int_{\partial D_1^*} \frac{\partial\mathsf{u}_0^*}{\partial\nu_0}\Big|_+ \cdot \psi_l, \quad l = 1,\, 2, \cdots,\, \frac{d(d+1)}{2}. \tag{1.14}$$

Here, we give the specific boundary data that will be used to illustrate some of blow-up factors b_l^* are nontrivial, and guarantee $|\nabla\mathsf{u}|$ blows up from lower bounds.

For $d = 2$, we choose a scalar-valued function $\varphi_0 \in C^{1,\alpha}(\partial D)$ satisfying

$$\varphi_0(x_1, h(x_1)) = |x_1|^\tau, \quad \text{on } \partial D \cap B_{R_0}, \tag{1.15}$$

and

$$\max_{x \in \partial D \setminus B_{R_0}} \big|\varphi_0(x)\big| \le R_0^\tau, \tag{1.16}$$

where $\tau \ge 1 + \alpha$.

For $d \geq 3$, we assume that a scalar-valued function $\varphi_1 \in C^{1,\alpha}(\partial D)$ such that

$$\varphi_1(x', h(x')) = \begin{cases} \zeta_0 |x'|^{\tau}, & \text{on } \partial D \cap B_{R_0}, \\ \zeta_0^{|x'|^2 - (3R_0 + \frac{1}{2R_0})|x'| + \frac{3}{2} + 2R_0^2} |x'|^{\tau}, & \text{on } \partial D \cap \big(B_{2R_0} \setminus B_{R_0}\big), \end{cases} \tag{1.17}$$

and

$$\max_{x \in \partial D \setminus B_{2R_0}} \big|\varphi_1(x)\big| \leq \zeta_0^{1/2} (2R_0)^{\tau}, \tag{1.18}$$

where $\tau \geq 1 + \alpha$ and $\zeta_0 > 0$ is a constant independent of ε. Then, we have

Theorem 1.2. *Under the assumptions of Theorem 1.1, let* $\mathsf{u} \in H^1(D; \mathbb{R}^d) \cap C^1(\overline{\Omega}; \mathbb{R}^d)$ *be the solution of* (1.7) *with* $\mathsf{g} = (g^{(1)}, \cdots, g^{(d)}) \in C^{1,\alpha}(\partial D, \mathbb{R}^d)$.

(i) *For $d = 2$, if there exists $i_0 \in \{1, 2\}$ such that*

$$g^{(i_0)} = \varphi_0, \quad \text{and} \quad g^{(j)} = 0, \quad j \neq i_0, \tag{1.19}$$

where φ_0 is defined in (1.15) *and* (1.16). *Then, we have*

$$\big|b_{i_0}^*[\mathsf{g}]\big| > 0, \tag{1.20}$$

and for sufficiently small $\varepsilon > 0$, we have

$$|\nabla \mathsf{u}(x)| \geq \frac{1}{C\varepsilon^{1/(1+\alpha)}} \cdot \big|b_{i_0}^*[\mathsf{g}]\big|, \quad x \in \overline{P_1 P}. \tag{1.21}$$

(ii) *For $d \geq 3$, if there exists $i_0 \in \{1, 2, \cdots, d\}$ such that*

$$g^{(i_0)} = \varphi_1, \quad \text{and} \quad g^{(j)} = 0, \quad j \neq i_0, \tag{1.22}$$

where φ_1 is defined in (1.17) *and* (1.18). *Then, for ζ_0 large enough, we still have* (1.20) *and for sufficiently small $\varepsilon > 0$, we have*

$$|\nabla \mathsf{u}(x)| \geq \frac{1}{C\varepsilon} \cdot \big|b_{i_0}^*[\mathsf{g}]\big|, \quad x \in \overline{P_1 P}, \tag{1.23}$$

where $C > 0$ is a constant independent of $\varepsilon > 0$.

Remark 1.3. In view of (1.11), (1.21) and (1.23), we can see that if the blow-up happens, it occurs on the segment $\overline{P_1 P}$. These estimates also imply that the blow-up rates with respect to ε, $\varepsilon^{-1/(1+\alpha)}$ in dimension $d = 2$ and ε^{-1} in $d \geq 3$, are optimal. Comparing these results to those in [7], it is interesting to see that the blow-up rates under $C^{1,\alpha}$ assumption become larger in dimensions $d = 2$ and 3 than those under the assumption of $C^{2,\alpha}$ inclusions, which are $\varepsilon^{-1/2}$ in $d = 2$ and $|\varepsilon \ln \varepsilon|^{-1}$ in $d = 3$.

Remark 1.4. We notice from (1.13) that u_0^* is uniquely determined by the given boundary data $\mathsf{g}(\cdot) - \mathsf{g}(P)$. Theorem 1.2 indicates that some of the blow-up factors $b_l^*[\mathsf{g}]$, dependent on some specific boundary data g, are nontrivial, which guarantee that $|\nabla \mathsf{u}|$ blows up. Moreover, we also see that these nontrivial blow-up factors $b_{i_0}^*[\mathsf{g}]$ do not change the blow-up rates. These are the main novelties and differences with [7, 15, 36]. In [36], a range of growth from boundary data are used to obtain the nontrivial blow-up factors and increase the blow-up rates.

Remark 1.5. The assumption (1.5) on ∂D_1 and ∂D in Theorems 1.1 and 1.2 can be replaced by the following weaker relative convexity assumptions

$$\kappa_0 |x'|^{1+\alpha} \leq h_1(x') - h(x') \leq \kappa_1 |x'|^{1+\alpha}, \quad \text{and} \quad |\nabla_{x'} h_1|, |\nabla_{x'} h| \leq \kappa_2 |x'|^{\alpha}, \tag{1.24}$$

where the index $1+\alpha$ is usually considered to be the order of convexity. We can see from the proof that optimal blow-up rates depend not only on the distance ε, but also on these convexity orders. The results of this paper in some sense can be regarded as a supplement for $1<m=1+\alpha<2$ to those in [7, 20, 36] where the cases of orders $m\geq 2$ have been investigated. While, for $m=1$, the Lipschitz inclusions, the solution u will become singular near the corner and the gradient estimates will be an interesting and challenging problem. We refer to Kang and Yun [24] for the scalar case with bow-tie structure.

1.3. **Perfect conductivity problem.** In this subsection, we introduce the following perfect conductivity problem

$$\begin{cases}\Delta u=0, & \text{in } \Omega:=D\setminus\overline{D}_1,\\ u=C_1, & \text{on } \overline{D}_1,\\ \int_{\partial D_1}\frac{\partial u}{\partial \mathsf{n}}ds=0, & \\ u=\varphi, & \text{on } \partial D,\end{cases}\tag{1.25}$$

where $\varphi\in C^{1,\alpha}(\partial D)$ is a given scalar function. The constant C_1 will be determined later, and

$$\frac{\partial u}{\partial \mathsf{n}}:=\lim_{\tau\to 0^+}\frac{u(x+\mathsf{n}\tau)-u(x)}{\tau}.$$

Here n is the outward unit normal of its domain.

We have the upper bound gradient estimates of solution for (1.25) as follows:

Theorem 1.6. *Let D_1 be a bounded $C^{1,\alpha}$ sub-domain of $D\subset\mathbb{R}^d$, $d\geq 2$, with ε apart. We assume that (1.2)-(1.6) hold. Let $u\in H^1(D)\cap C^1(\Omega)$ be the solution of (1.25) with $\varphi\in C^{1,\alpha}(\partial D)$. Then, we have for small $\varepsilon>0$ and $x\in\Omega_{R_0}$,*

$$|\nabla u(x',x_d)|\leq C\Big(\frac{\rho_{d,\,\alpha}(\varepsilon)}{\varepsilon+|x'|^{1+\alpha}}+\frac{|x'|}{\varepsilon+|x'|^{1+\alpha}}+1\Big)\cdot\|\varphi\|_{C^{1,\,\alpha}(\partial D)},\tag{1.26}$$

where $C>0$ is a constant independent of $\varepsilon>0$.

We also have the lower bound estimates of $|\nabla u|$ with some specific φ given. Let v_0^* be the solution of

$$\begin{cases}\Delta v_0^*=0, & \text{in } \Omega^*,\\ v_0^*=0, & \text{on } \partial D_1^*,\\ v_0^*=\varphi-\varphi(P), & \text{on } \partial D.\end{cases}$$

Define

$$Q^*[\varphi]:=\int_{\partial D_1^*}\frac{\partial v_0^*}{\partial \mathsf{n}}ds.\tag{1.27}$$

Then, we have

Theorem 1.7. *Under the assumptions of Theorem 1.6, let $u\in H^1(D)\cap C^1(\Omega)$ be the solution of (1.25) with $\varphi\in C^{1,\alpha}(\partial D)$.*

(i) *For $d=2$, if we choose $\varphi=\varphi_0$ satisfying (1.15) and (1.16), then, we have*

$$|Q^*[\varphi]|>0,\tag{1.28}$$

and hence, for small $\varepsilon>0$,

$$|\nabla u(x)|\geq\frac{1}{C\varepsilon^{1/(1+\alpha)}}\cdot|Q^*[\varphi]|,\qquad x\in\overline{PP_1}.\tag{1.29}$$

(ii) *For $d \geq 3$, if we assume that $\varphi = \varphi_1$ defined in* (1.17) *and* (1.18), *then, for $\zeta_0 >$ large enough, we obtain that* (1.28) *and hence, for small $\varepsilon > 0$,*

$$|\nabla u(x)| \geq \frac{1}{C\varepsilon} \cdot |Q^*[\varphi]|, \qquad x \in \overline{PP_1}, \tag{1.30}$$

where $C > 0$ is a constant independent of $\varepsilon > 0$.

Remark 1.8. In view of (1.26), (1.29) and (1.30), we see that optimal blow-up rates of $|\nabla u|$, $\varepsilon^{-1/(1+\alpha)}$ in $d = 2$ and ε^{-1} in $d \geq 3$, are the same as those of $|\nabla \mathsf{u}|$ for (1.7). Comparing to the results in [34], these blow-up rates obtained from Theorems 1.6 and 1.7 also become larger in dimensions $d = 2$, 3 than those under the assumption of $C^{2,\alpha}$ inclusions. Moreover, we remark that the assumption (1.5) in Theorem 1.6 can also be replaced by (1.24).

Throughout this paper, unless otherwise stated, C denotes a constant, whose value may vary from line to line, depending only on d, δ_0, κ_0, κ_1, κ_2, but not on ε. Also, we call a constant having such dependence a *universal constant*.

The rest of this paper is organized as follows. The poof of Theorem 1.1 are arranged in Sections 2. In this section, we first decompose the solution of (1.7) by $\mathsf{u} = \sum C^l \mathsf{u}_l + \mathsf{u}_0$ in (2.1). Then the proof of Theorem 1.1 are divided into three subsections to estimate each terms in this decomposition. We establish the upper bound gradient estimates for u_0 and u_l, see Propositions 2.1 and 2.2; the proof of Proposition 2.1 is given in the Appendix for readers' convenience. Subsection 2.3 is devoted to the estimates for coefficients C^l. Section 3 focuses on the proof of the lower bound estimates in Theorem 1.2, which mainly investigate the convergence of the functionals $b_l[\mathsf{g}]$ to blow-up factors $b_l^*[\mathsf{g}]$, defined in (1.14), see Proposition 3.1, and $|b_{i_0}^*[\mathsf{g}]| \neq 0$, see Proposition 3.2. Finally, for the scalar case i.e. perfect conductivity problem, the proofs of Theorems 1.6 and 1.7 are given in Section 4.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We follow the idea, developed in [7], to decompose the solution u of (1.7). Then the proof are divided to estimate each term in this decomposition. For easy writing, we assume that $\mathsf{g}(P) = 0$. Otherwise, we replace u by $\mathsf{u} - \mathsf{g}(P)$. We decompose the solution of (1.7) as follows:

$$\mathsf{u} = \sum_{l=1}^{d(d+1)/2} C^l \psi_l, \quad \text{in } \overline{D}_1,$$

for some constants C^l, $l = 1, 2, \cdots, \frac{d(d+1)}{2}$, to be determined by the forth line of (1.7), and

$$\mathsf{u} = \sum_{l=1}^{d(d+1)/2} C^l \mathsf{u}_l + \mathsf{u}_0, \quad \text{in } \Omega, \tag{2.1}$$

where $\mathsf{u}_l \in C^1(\overline{\Omega}; \mathbb{R}^d) \cap C^{1,\alpha}(\Omega; \mathbb{R}^d)$, $l = 1, 2, \cdots, d(d+1)/2$, respectively, satisfy

$$\begin{cases} \mathcal{L}_{\lambda,\mu} \mathsf{u}_l = 0, & \text{in } \Omega, \\ \mathsf{u}_l = \psi_l, & \text{on } \partial D_1, \\ \mathsf{u}_l = 0, & \text{on } \partial D, \end{cases} \tag{2.2}$$

and $\mathsf{u}_0 \in C^1(\overline{\Omega};\mathbb{R}^d)\cap C^{1,\alpha}(\Omega;\mathbb{R}^d)$ satisfies

$$\begin{cases}\mathcal{L}_{\lambda,\mu}\mathsf{u}_0=0, & \text{in }\Omega,\\ \mathsf{u}_0=0, & \text{on }\partial D_1,\\ \mathsf{u}_0=\mathsf{g}, & \text{on }\partial D.\end{cases}\tag{2.3}$$

By the decomposition (2.1), we have

$$\nabla\mathsf{u}=\sum_{l=1}^{d}C^l\nabla\mathsf{u}_l+\sum_{l=d+1}^{d(d+1)/2}C^l\nabla\mathsf{u}_l+\nabla\mathsf{u}_0,\quad\text{in }\Omega.\tag{2.4}$$

Thus, in order to prove Theorem 1.1, it suffices to estimate each term in (2.4), respectively.

2.1. **The estimates of $|\nabla\mathsf{u}_0|$.** From (2.3), we notice that u_0 is uniquely determined by boundary data g. Due to the interaction from g, u_0 becomes more irregular near the boundary of D. The role of g will be embodied in the estimates of $|\nabla\mathsf{u}_0|$. This is the main difference with [15], in which the situation that $C^{1,\alpha}$ inclusions located far away from ∂D is investigated.

In order to estimate $|\nabla\mathsf{u}_0|$, we further decompose the solution u_0 of (2.3) as follows:

$$\mathsf{u}_0=\mathsf{u}_{01}+\mathsf{u}_{02}+\cdots+\mathsf{u}_{0d},$$

then, we notice that u_{0l}, $l=1,2,\cdots,d$, satisfy, respectively,

$$\begin{cases}\mathcal{L}_{\lambda,\mu}\mathsf{u}_{0l}=0, & \text{in }\Omega,\\ \mathsf{u}_{0l}=0, & \text{on }\partial D_1,\\ \mathsf{u}_{0l}=(0,\cdots,g^{(l)}(x),0,\cdots,0)^{\mathrm{T}}, & \text{on }\partial D.\end{cases}$$

To estimate $|\nabla\mathsf{u}_{0l}|$, we introduce an auxiliary function $\bar{u}\in C^{1,\alpha}(\mathbb{R}^d)$, such that $\bar{u}=1$ on ∂D_1, $\bar{u}=0$ on ∂D and

$$\bar{u}(x)=\frac{x_d-h(x')}{\varepsilon+h_1(x')-h(x')},\ x\in\Omega_{2R_0},\tag{2.5}$$

and

$$\|\bar{u}\|_{C^{1,\alpha}(\Omega\setminus\Omega_{R_0})}\le C.\tag{2.6}$$

Denoting $\partial_i:=\partial/\partial x_i$ and in view of (1.3)–(1.6), a direct calculation yields for $i=1,\cdots,d-1$, and for $x\in\Omega_{2R_0}$,

$$|\partial_i\bar{u}(x)|\le\frac{C|x'|^\alpha}{\varepsilon+|x'|^{1+\alpha}},\quad\text{and}\quad\partial_d\bar{u}(x)=\frac{1}{\delta(x')},\tag{2.7}$$

where

$$\delta(x'):=\varepsilon+h_1(x')-h(x').\tag{2.8}$$

Moreover, we extend $\mathsf{g}\in C^{1,\alpha}(\partial D;\mathbb{R}^d)$ to $\mathsf{g}\in C^{1,\alpha}(\overline{\Omega};\mathbb{R}^d)$ such that $\|g^{(l)}\|_{C^{1,\alpha}(\overline{\Omega\setminus\Omega_{R_0}})}\le C\|g^{(l)}\|_{C^{1,\alpha}(\partial D)}$, $l=1,2,\cdots,d$ and take a cut-off function $\rho\in C^2(\overline{\Omega})$ such that $0\le\rho\le1$, $|\nabla\rho|\le C$ on $\overline{\Omega}$ and

$$\rho=1\text{ on }\overline{\Omega}_{2R_0},\quad\text{and}\quad\rho=0\text{ on }\overline{\Omega}\setminus\Omega_{5R_0/2}.$$

Then, we define a vector-valued function $\tilde{\mathsf{u}}_{0l} = (\tilde{u}_{0l}^{(1)}, \tilde{u}_{0l}^{(2)}, \cdots, \tilde{u}_{0l}^{(d)})$ such that for $x \in \Omega$,

$$\begin{aligned} &\tilde{u}_{0l}^{(l)}(x) = \Big(\rho(x)g^{(l)}(x', h(x')) + \big(1-\rho(x)\big)g^{(l)}(x)\Big)(1-\bar{u})(x), \\ &\tilde{u}_{0l}^{(i)}(x) = 0, \quad i \neq l, \quad l = 1, 2, \cdots, d. \end{aligned} \tag{2.9}$$

In particular, we notice that

$$\tilde{u}_{0l}^{(l)}(x) = g^{(l)}(x', h(x'))(1-\bar{u})(x), \quad x \in \Omega_{2R_0}. \tag{2.10}$$

It follows from the Taylor expansion and $\mathsf{g}(0) = 0$ that

$$g^{(l)}(x', h(x')) = \nabla_{x'} g^{(l)}(0) \cdot x' + O(|x'|^{1+\alpha}). \tag{2.11}$$

Then, by virtue of (2.7), we have for $x \in \Omega_{2R_0}$,

$$|\nabla_{x'}\tilde{\mathsf{u}}_{0l}(x)| \leq \frac{C|\nabla_{x'}g^{(l)}(0)| \cdot |x'|^{1+\alpha}}{\varepsilon + |x'|^{1+\alpha}} + C\|g^{(l)}\|_{C^{1,\alpha}(\partial D)} \leq C\|g^{(l)}\|_{C^{1,\alpha}(\partial D)}, \tag{2.12}$$

and

$$\frac{|g^{(l)}(x', h(x'))|}{C(\varepsilon + |x'|^{1+\alpha})} \leq |\partial_d \tilde{\mathsf{u}}_{0l}(x)| \leq \frac{C|\nabla_{x'}g^{(l)}(0)| \cdot |x'|}{\varepsilon + |x'|^{1+\alpha}} + C\|g^{(l)}\|_{C^{1,\alpha}(\partial D)}. \tag{2.13}$$

We have the estimates of $|\nabla \mathsf{u}_{0l}|$ as follows:

Proposition 2.1. *Under the assumptions of Theorem 1.1, we have for small $\varepsilon > 0$,*

$$|\nabla(\mathsf{u}_{0l} - \tilde{\mathsf{u}}_{0l})|(x)| \leq C\|g^{(l)}\|_{C^{1,\alpha}(\partial D)}, \quad x \in \Omega_{R_0}, \tag{2.14}$$

for $l = 1, 2, \cdots, d$,. Consequently,

$$|\nabla_{x'}\mathsf{u}_{0l}(x)| \leq C\|g^{(l)}\|_{C^{1,\alpha}(\partial D)}, \quad x \in \Omega_{R_0}, \tag{2.15}$$

and for $x \in \Omega_{R_0}$,

$$\frac{|g^{(l)}(x', h(x'))|}{C(\varepsilon + |x'|^{1+\alpha})} \leq |\partial_d \mathsf{u}_{0l}(x)| \leq \frac{C|\nabla_{x'}g^{(l)}(0)| \cdot |x'|}{\varepsilon + |x'|^{1+\alpha}} + C\|g^{(l)}\|_{C^{1,\alpha}(\partial D)}. \tag{2.16}$$

To prove Proposition 2.1, we adapt Bao-Li-Li's iteration technique [10], whose embryonic form although first used in [33], to our setting that the right hand side is of divergence form. The proof is given in the Appendix.

2.2. **The estimates of $|\nabla \mathsf{u}_l|$.** In order to estimate $|\nabla \mathsf{u}_l|$, we define a vector-valued auxiliary function $\tilde{\mathsf{u}}_l = (\tilde{u}_l^{(1)}, \tilde{u}_l^{(2)}, \cdots, \tilde{u}_l^{(d)})$ as follows

$$\tilde{\mathsf{u}}_l(x) := \bar{u}(x)\psi_l, \quad x \in \Omega, \ l = 1, 2, \cdots, \frac{d(d+1)}{2}.$$

In particular, we notice from (1.9) that $\psi_l = e_l$, the standard basis of $\mathbb{R}^d$, $l = 1, 2, \cdots, d$. And hence, for $x \in \Omega_{2R_0}$, we see that

$$\tilde{u}_l^{(l)}(x) = \bar{u}(x), \quad \text{for } l = 1, 2, \cdots, d, \quad \text{and} \quad \tilde{u}_l^{(i)}(x) = 0, \ i \neq l. \tag{2.17}$$

In view of (2.7), we have for $x \in \Omega_{2R_0}$ and for $l = 1, 2, \cdots, d$,

$$|\nabla_{x'}\tilde{\mathsf{u}}_l(x)| \leq \frac{C|x'|^\alpha}{\varepsilon + |x'|^{1+\alpha}}, \quad \text{and} \quad \frac{1}{C(\varepsilon + |x'|^{1+\alpha})} \leq |\partial_d \tilde{\mathsf{u}}_l(x)| \leq \frac{C}{\varepsilon + |x'|^{1+\alpha}}, \tag{2.18}$$

and

$$|\nabla \tilde{\mathsf{u}}_l(x)| \leq \frac{C(\varepsilon + |x'|)}{\varepsilon + |x'|^{1+\alpha}}, \quad \text{for } l = d+1, \cdots, \frac{d(d+1)}{2}. \tag{2.19}$$

We have the estimates of $|\nabla \mathsf{u}_l|$ as follows:

Proposition 2.2. *Under the assumptions of Theorem 1.1, we have for small $\varepsilon > 0$*

$$|\nabla(\mathsf{u}_l - \tilde{\mathsf{u}}_l)(x)| \leq \frac{C}{(\varepsilon + |x'|^{1+\alpha})^{\frac{1}{1+\alpha}}}, \quad x \in \Omega_{R_0}, \quad l = 1,\, 2, \cdots, d, \tag{2.20}$$

and

$$|\nabla(\mathsf{u}_l - \tilde{\mathsf{u}}_l)(x)| \leq C, \quad x \in \Omega_{R_0}, \quad l = d+1, \cdots, \frac{d(d+1)}{2}.$$

Consequently, it follows from (2.18) *and* (2.19) *that*

$$|\nabla_{x'} \mathsf{u}_l(x)| \leq \frac{C}{(\varepsilon + |x'|^{1+\alpha})^{\frac{1}{1+\alpha}}}, \quad x \in \Omega_{R_0}, \quad l = 1,\, 2, \cdots, d,$$

$$\frac{1}{C(\varepsilon + |x'|^{1+\alpha})} \leq |\partial_d \mathsf{u}_l(x)| \leq \frac{C}{\varepsilon + |x'|^{1+\alpha}}, \quad x \in \Omega_{R_0}, \quad l = 1,\, 2, \cdots, d,$$

$$|\nabla \mathsf{u}_l(x)| \leq \frac{C(\varepsilon + |x'|)}{\varepsilon + |x'|^{1+\alpha}}, \quad x \in \Omega_{R_0}, \quad l = d+1, \cdots, \frac{d(d+1)}{2}, \tag{2.21}$$

and

$$|\nabla \mathsf{u}_l(x)| \leq C, \quad x \in \Omega \setminus \Omega_{R_0}, \quad l = 1,\, 2,\, \cdots, \frac{d(d+1)}{2}.$$

Since the proof of Proposition 2.2 is the same as that of proposition 2.1 in [15], we omit the details.

2.3. **The estimates of the coefficients C^l.** After estimating $|\nabla \mathsf{u}_l|$ and $|\nabla \mathsf{u}_0|$ in previous subsections, we recall from the decomposition (2.4) that it suffices to give the estimates of coefficients C^l. Denote for $k,\, l = 1,\, 2,\, \cdots, d(d+1)/2$,

$$a_{kl} := -\int_{\partial D_1} \frac{\partial \mathsf{u}_k}{\partial \nu_0}\Big|_+ \cdot \psi_l, \quad b_l := b_l[\mathsf{g}] := \int_{\partial D_1} \frac{\partial \mathsf{u}_0}{\partial \nu_0}\Big|_+ \cdot \psi_l,. \tag{2.22}$$

It follows from the decomposition (2.4) and the fourth line of (1.7) that

$$\sum_{k=1}^{d} C^k a_{kl} + \sum_{k=d+1}^{d(d+1)/2} C^k a_{kl} = b_l, \quad l = 1,\, 2, \cdots, \frac{d(d+1)}{2}. \tag{2.23}$$

Moreover, in view of (2.2) and (2.3), using integration by parts, we have for $k,\, l = 1,\, 2,\, \cdots, \frac{d(d+1)}{2}$,

$$a_{kl} = -\int_{\partial D_1} \frac{\partial \mathsf{u}_k}{\partial \nu_0}\Big|_+ \cdot \psi_l = \int_\Omega \Big(\mathbb{C}^0 e(\mathsf{u}_k), e(\mathsf{u}_l)\Big) dx,$$

and

$$b_l = -\int_\Omega \Big(\mathbb{C}^0 e(\mathsf{u}_0), e(\mathsf{u}_l)\Big) dx.$$

We rewrite (2.23) into the matrix equation as follows

$$AX = B, \tag{2.24}$$

where

$$A^{\mathrm{T}} = \big(a_{kl}\big)_{k,\, l=1}^{d(d+1)/2},$$

and

$$X = (C^1,\, C^2,\, \cdots,\, C^{\frac{d(d+1)}{2}})^{\mathrm{T}}, \quad B = (b_1[\mathsf{g}],\, b_2[\mathsf{g}],\, \cdots,\, b_{\frac{d(d+1)}{2}}[\mathsf{g}])^{\mathrm{T}}.$$

Before estimating C^l, we need the following Lemma:

Lemma 2.3. *Under the assumptions of Theorem 1.1, we have for small $\varepsilon > 0$,*

$$\frac{1}{C\rho_{d,\,\alpha}(\varepsilon)} \leq a_{kk} \leq \frac{C}{\rho_{d,\,\alpha}(\varepsilon)}, \quad k = 1,\, 2,\, \cdots,\, d, \tag{2.25}$$

$$\frac{1}{C} \leq a_{kk} \leq C, \quad k = d+1, \cdots, \frac{d(d+1)}{2},$$

$$|a_{kl}| \leq C, \quad k = 1, 2, \cdots, \frac{d(d+1)}{2},\ l = d+1, \cdots, \frac{d(d+1)}{2},\ k \neq l.$$

In particular, for $d = 2$,

$$|a_{12}| = |a_{21}| \leq C|\ln\varepsilon|,$$

and for $d \geq 3$,

$$|a_{kl}| = |a_{lk}| \leq C, \quad k, l = 1,\, 2,\, \cdots,\, d,\ k \neq l.$$

Consequently,

$$\frac{1}{C\big(\rho_{d,\,\alpha}(\varepsilon)\big)^d} \leq \det A \leq \frac{C}{\big(\rho_{d,\,\alpha}(\varepsilon)\big)^d},$$

where C is a universal constant independent of ε.

Since the proof of Lemma 2.3 is almost the same as lemma 3.1 in [15], we omit the proof.

Lemma 2.4. *Under the assumptions of Theorem 1.1, we have for small $\varepsilon > 0$,*

$$|b_l| \leq C, \quad l = 1,\, 2, \cdots, \frac{d(d+1)}{2}, \tag{2.26}$$

where C is a universal constant independent of ε.

Proof. For $l = 1,\, 2, \cdots, d$, we only prove the case of $l = 1$, since they are the same as other cases. We notice that

$$b_1 = \sum_{i=1}^{d} \int_{\partial D_1} \frac{\partial \mathsf{u}_{0i}}{\partial \nu_0}\Big|_+ \cdot \psi_1 =: \sum_{i=1}^{d} b_{1i}.$$

From the definition (1.8), we see that

$$b_{11} = \int_{\partial D_1} \frac{\partial \mathsf{u}_{01}}{\partial \nu_0}\Big|_+ \cdot \psi_1 = \int_{\partial D_1} \Big(\lambda \sum_{i=1}^{d} \partial_i u_{01}^{(i)} n_1 + \mu \sum_{j=1}^{d} (\partial_1 u_{01}^{(j)} + \partial_j u_{01}^{(1)}) n_j\Big) ds,$$

where

$$\mathsf{n} = (n_1, n_2, \cdots, n_d) = \frac{(\nabla_{x'} h_1(x'), -1)}{\sqrt{1 + |\nabla_{x'} h_1(x')|^2}}, \quad \text{on } \partial D_1 \cap B_{R_0}. \tag{2.27}$$

By using (2.15), (2.16) and (2.27), we have

$$\begin{aligned} \mathrm{I} := \Big| \int_{\partial D_1} \sum_{i=1}^{d} \partial_i u_{01}^{(i)} n_1 ds \Big| &\leq \int_{\partial D_1} \Big| \sum_{i=1}^{d-1} \partial_i u_{01}^{(i)} n_1 \Big| + \Big| \partial_d u_{01}^{(d)} n_1 \Big| ds \\ &\leq \int_{\partial D_1 \cap B_{R_0}} \Big(\frac{C|x'|^\alpha}{\sqrt{1 + |\nabla_{x'} h_1(x')|^2}} + \frac{C|\nabla_{x'} g^{(1)}(0)|}{\sqrt{1 + |\nabla_{x'} h_1(x')|^2}} \Big) ds + C \leq C. \end{aligned} \tag{2.28}$$

Moreover, we denote

$$\begin{aligned}\mathrm{II} &:= \int_{\partial D_1} \sum_{j=1}^{d} (\partial_1 u_{01}^{(j)} + \partial_j u_{01}^{(1)}) n_j ds \\ &= \int_{\partial D_1} \sum_{j=1}^{d-1} (\partial_1 u_{01}^{(j)} + \partial_j u_{01}^{(1)}) n_j ds + \int_{\partial D_1} \partial_1 u_{01}^{(d)} n_d ds + \int_{\partial D_1} \partial_d u_{01}^{(1)} n_d ds \\ &=: \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3.\end{aligned}$$

By virtue of (2.14), (2.15) and (2.27), one has

$$|\mathrm{II}_1| \leq \Big| \int_{\partial D_1} \sum_{j=1}^{d-1} (\partial_1 u_{01}^{(j)} + \partial_j u_{01}^{(1)}) n_j ds \Big| \leq \int_{\partial D_1 \cap B_{R_0}} \frac{C|x'|^{\alpha}}{\sqrt{1 + |\nabla_{x'} h_1(x')|^2}} ds + C \leq C,$$

and

$$\begin{aligned}|\mathrm{II}_2| &\leq \Big| \int_{\partial D_1} \partial_1 u_{01}^{(d)} n_d ds \Big| \\ &\leq \int_{\partial D_1 \cap B_{R_0}} |\partial_1 \tilde{u}_{01}^{(d)} n_d| ds + \int_{\partial D_1 \cap B_{R_0}} |\partial_1 (u_{01}^{(d)} - \tilde{u}_{01}^{(d)}) n_d| ds + C \\ &\leq \int_{\partial D_1 \cap B_{R_0}} \frac{C}{\sqrt{1 + |\nabla_{x'} h_1(x')|^2}} ds + C \leq C.\end{aligned}$$

For II_3, using (2.13), (2.14) and (2.27), we have

$$\begin{aligned}|\mathrm{II}_3| &\leq \Big| \int_{\partial D_1 \cap B_{R_0}} \partial_d \tilde{u}_{01}^{(1)} n_d ds \Big| + \Big| \int_{\partial D_1 \cap B_{R_0}} \partial_d (u_{01}^{(1)} - \tilde{u}_{01}^{(1)}) n_d ds \Big| + C \\ &\leq \int_{|x'| \leq R_0} \frac{C|x'|}{\varepsilon + |x'|^{1+\alpha}} dx' + C \leq C.\end{aligned}$$

Thus, we have

$$|\mathrm{II}| \leq |\mathrm{II}_1| + |\mathrm{II}_2| + |\mathrm{II}_3| \leq C.$$

This, together with (2.28), leads to

$$|b_{11}| \leq |\lambda \mathrm{I} + \mu \mathrm{II}| \leq C.$$

Similarly, we obtain

$$|b_{1i}| \leq C, \quad i = 2, \cdots, d.$$

Hence

$$|b_1| \leq \sum_{i=1}^{d} |b_{1i}| \leq C.$$

Next, to prove (2.26) for $l = d+1, \cdots, d(d+1)/2$, by using (2.16) and (2.21), we have

$$\begin{aligned}|b_l| &= \Big| \int_{\Omega} \Big(\mathbb{C}^0 e(\mathsf{u}_0), e(\mathsf{u}_l) \Big) dx \Big| \leq C \int_{\Omega} |\nabla \mathsf{u}_0| |\nabla \mathsf{u}_l| dx \\ &\leq \int_{\Omega_{R_0}} \frac{C |\nabla_{x'} \mathsf{g}(0)| \cdot |x'| \cdot (\varepsilon + |x'|)}{(\varepsilon + |x'|^{1+\alpha})^2} dx + C \leq C.\end{aligned}$$

Thus, (2.26) is proved. □

Then, we can give the estimates of coefficients C^l.

Proposition 2.5. *Under the assumptions of Theorem 1.1 and with the normalization* $\|\mathsf{g}\|_{C^{1,\alpha}(\partial D)}=1$. *Then*

$$|C^l|\leq C\rho_{d,\,\alpha}(\varepsilon),\quad l=1,\,2,\,\cdots,\,d,\tag{2.29}$$

and

$$|C^l|\leq C,\quad l=d+1,\,\cdots,\,\frac{d(d+1)}{2},\tag{2.30}$$

where $\rho_{d,\,\alpha}(\varepsilon)$ *is defined in* (1.10).

Proof. We first prove the case of $d=2$. In this case, we rewrite (2.24) precisely as follows

$$AX=\begin{pmatrix} a_{11} & a_{21} & a_{31}\\ a_{12} & a_{22} & a_{32}\\ a_{13} & a_{23} & a_{33}\end{pmatrix}\begin{pmatrix} C^1\\ C^2\\ C^3\end{pmatrix}=\begin{pmatrix} b_1[\mathsf{g}]\\ b_2[\mathsf{g}]\\ b_3[\mathsf{g}]\end{pmatrix}.$$

Then, it follows from the Cramer' rule that

$$C^1=\frac{1}{\det A}\Big(b_1[\mathsf{g}]\begin{vmatrix} a_{22} & a_{32}\\ a_{23} & a_{33}\end{vmatrix}-b_2[\mathsf{g}]\begin{vmatrix} a_{21} & a_{31}\\ a_{23} & a_{33}\end{vmatrix}+b_3[\mathsf{g}]\begin{vmatrix} a_{21} & a_{31}\\ a_{22} & a_{32}\end{vmatrix}\Big).$$

By virtue of Lemma 2.3, we have

$$C^1=\frac{a_{22}(b_1[\mathsf{g}]a_{33}-b_3[\mathsf{g}]a_{31})+O(|\ln\varepsilon|)}{a_{11}a_{22}a_{33}+O(\varepsilon^{-\frac{\alpha}{1+\alpha}})}=\frac{1}{a_{11}}\big(b_1[\mathsf{g}]-b_3[\mathsf{g}]\frac{a_{31}}{a_{33}}\big)+O(\varepsilon^{\frac{2\alpha}{1+\alpha}}|\ln\varepsilon|).\tag{2.31}$$

By the same reason, we have

$$C^2=\frac{1}{a_{22}}\big(b_2[\mathsf{g}]-b_3[\mathsf{g}]\frac{a_{32}}{a_{33}}\big)+O(\varepsilon^{\frac{2\alpha}{1+\alpha}}|\ln\varepsilon|),\quad C^3=\frac{b_3[\mathsf{g}]}{a_{33}}+O(\varepsilon^{\frac{\alpha}{1+\alpha}}).\tag{2.32}$$

Combining with the estimates in Lemma 2.3, we have

$$|C^l|\leq C\varepsilon^{\frac{\alpha}{1+\alpha}},\quad l=1,\,2,\quad\text{and}\quad |C^3|\leq C.$$

For $d\geq 3$, since A is invertible, from (2.24), we have

$$X=A^{-1}B=\frac{A^*}{\det A}B,\tag{2.33}$$

where $A^*=(a^*_{kl})$ is the adjoint matrix of A and a^*_{kl} is the cofactor of a_{kl}, $k,l=1,\,2,\,\cdots,d(d+1)/2$. Then, by virtue of Lemma 2.3, we see that $|a^*_{kl}|\leq C$, and in view of (2.26), we obtain

$$|X|\leq C.$$

Hence, (2.29) and (2.30) for $d\geq 3$ are proved. □

Now, we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. From the decomposition (2.4), by using the estimates of $|\nabla\mathsf{u}_0|$, $|\nabla\mathsf{u}_l|$ and C^l given in Propositions 2.1, 2.2 and 2.5, we have, for $x\in\Omega_{R_0}$,

$$\begin{aligned}|\nabla\mathsf{u}|&\leq\sum_{l=1}^{d}|C^l|\cdot|\nabla\mathsf{u}_l|+\sum_{l=d+1}^{d(d+1)/2}|C^l|\cdot|\nabla\mathsf{u}_l|+|\nabla\mathsf{u}_0|\\ &\leq C\left(\frac{\rho_{d,\,\alpha}(\varepsilon)}{\varepsilon+|x'|^{1+\alpha}}+\frac{|x'|}{\varepsilon+|x'|^{1+\alpha}}+1\right)\|\mathsf{g}\|_{C^{1,\,\alpha}(\partial D)}.\end{aligned}$$

Moreover, (1.12) follows from the standard interior and boundary estimates for elliptic systems (see [1]). The proof is completed. □

3. Proof of Theorem 1.2

This section is devoted to establishing the lower bound estimates of $|\nabla \mathsf{u}|$. From the decomposition (2.4), we see that

$$|\nabla \mathsf{u}(x)| \geq \Big|\sum_{l=1}^{d} C^l \nabla \mathsf{u}_l(x)\Big| - \Big|\sum_{l=d+1}^{d(d+1)/2} C^l \nabla \mathsf{u}_l(x)\Big| - |\nabla \mathsf{u}_0(x)|, \quad x \in \Omega_{R_0}. \tag{3.1}$$

In view of Propositions 2.1, 2.2 and 2.5, we notice that for $x = (0', x_d) \in \overline{P_1 P}$,

$$\Big|\sum_{l=d+1}^{d(d+1)/2} C^l \nabla \mathsf{u}_l(x)\Big| \leq \frac{C(\varepsilon + |x'|)}{\varepsilon + |x'|^{1+\alpha}} \leq C, \;\; |\nabla \mathsf{u}_0(x)| \leq \frac{C|\nabla_{x'} \mathsf{g}(0)| \cdot |x'|}{\varepsilon + |x'|^{1+\alpha}} + C \leq C. \tag{3.2}$$

Then, it suffices to estimate the lower bound of $|C^l|$, $l = 1, 2, \cdots, d$. To this end, we first prove the convergence of $b_l[\mathsf{g}]$ with respect to ε. The proof follows from the idea in [7], for readers' convenience, we only provide the proof with different part and omit the same part.

Let u_l^*, $l = 1, 2, \cdots, d(d+1)/2$, satisfy, respectively,

$$\begin{cases} \mathcal{L}_{\lambda,\mu} \mathsf{u}_l^* = 0, & \text{in } \Omega^*, \\ \mathsf{u}_l^* = \psi_l, & \text{on } \partial D_1^* \setminus \{0\}, \\ \mathsf{u}_l^* = 0, & \text{on } \partial D. \end{cases} \tag{3.3}$$

We have

Proposition 3.1. *Assume* $\mathsf{g} \in C^{1,\alpha}(\partial D, \mathbb{R}^d)$. *Let* $b_l[\mathsf{g}]$ *and* $b_l^*[\mathsf{g}]$ *be defined in* (2.22) *and* (1.14), *respectively. Then for small* $\varepsilon > 0$, *we have*

$$|b_l[\mathsf{g}] - b_l^*[\mathsf{g}]| \leq C\varepsilon^{\frac{d-1-\alpha}{(1+2\alpha)(d-\alpha)}}, \quad l = 1, 2, \cdots, \frac{d(d+1)}{2}. \tag{3.4}$$

where C *is a universal constant.*

Proof. Since the proof are similar to other cases, we only prove the case of $l = 1$. In view of (2.2) and (2.3), it follows from integration by parts that

$$b_1[\mathsf{g}] = \int_{\partial D_1} \frac{\partial \mathsf{u}_0}{\partial \nu_0}\Big|_+ \cdot \psi_1 = -\int_\Omega \big(\mathbb{C}^0 e(\mathsf{u}_1), e(\mathsf{u}_0)\big) dx = -\int_{\partial D} \frac{\partial \mathsf{u}_1}{\partial \nu_0}\Big|_+ \cdot \mathsf{g}.$$

By the same reason, we have

$$b_1^*[\mathsf{g}] = -\int_{\partial D} \frac{\partial \mathsf{u}_1^*}{\partial \nu_0}\Big|_+ \cdot \mathsf{g}.$$

Then

$$b_1 - b_1^* = -\int_{\partial D} \frac{\partial(\mathsf{u}_1 - \mathsf{u}_1^*)}{\partial \nu_0}\Big|_+ \cdot \mathsf{g}. \tag{3.5}$$

In order to prove (3.4), we need to estimate $|\mathsf{u}_1 - \mathsf{u}_1^*|$. To this end, we introduce an auxiliary function $\bar{u}^*$ satisfying $\bar{u}^* = 1$ on $\partial D_1^* \setminus \{0\}$, $\bar{u}^* = 0$ on ∂D, and

$$\bar{u}^* = \frac{x_d - h(x')}{h_1(x') - h(x')}, \text{ on } \Omega_{2R_0}^*, \quad \|\bar{u}^*\|_{C^{1,\alpha}(\overline{\Omega^* \setminus \Omega_{R_0}^*})} \leq C, \tag{3.6}$$

where $\Omega_R^* := \{x \in \Omega^* \mid |x'| \leq R\}$ for $R \leq R_0$. Through a direct calculation, we have

$$|\nabla_{x'} \bar{u}^*(x)| \leq \frac{C}{|x'|}, \quad \frac{1}{C|x'|^{1+\alpha}} \leq \partial_d \bar{u}^*(x) \leq \frac{C}{|x'|^{1+\alpha}}, \quad \text{for } x \in \Omega_{R_0}^*. \tag{3.7}$$

We define $\tilde{\mathsf{u}}_l^* := (\tilde{u}_l^{*(1)}, \tilde{u}_l^{*(2)}, \cdots, \tilde{u}_l^{*(d)})^{\mathrm{T}}$, $l = 1, 2, \cdots, d$, such that

$$\tilde{u}_l^{*(l)} = \bar{u}^*, \quad \tilde{u}_l^{*(k)} = 0, \ k \neq l. \tag{3.8}$$

Then, by virtue of (1.4) and (1.5), we have for $x \in \Omega^*_{R_0}$,

$$|\nabla_{x'}(\tilde{u}_1^{(1)} - \tilde{u}_1^{*(1)})(x)| \leq |\nabla_{x'}\tilde{u}_1^{(1)}(x)| + |\nabla_{x'}\tilde{u}_1^{*(1)}(x)| \leq \frac{C}{|x'|}, \tag{3.9}$$

and

$$|\partial_d(\tilde{u}_1^{(1)} - \tilde{u}_1^{*(1)})(x)| \leq \frac{C\varepsilon}{|x'|^{1+\alpha}(\varepsilon + |x'|^{1+\alpha})}. \tag{3.10}$$

Applying Proposition 2.2 for u_l^* in (3.3), $l = 1, 2, \cdots, d$, we obtain

$$|\nabla(\mathsf{u}_l^* - \tilde{\mathsf{u}}_l^*)(x)| \leq \frac{C}{|x'|}, \quad x \in \Omega^*_{R_0}, \tag{3.11}$$

and

$$|\nabla_{x'}\mathsf{u}_l^*(x)| \leq \frac{C}{|x'|}, \quad |\partial_d \mathsf{u}_l^*(x)| \leq \frac{C}{|x|^{1+\alpha}}, \quad x \in \Omega^*_{R_0},$$

and

$$|\nabla \mathsf{u}_l^*(x)| \leq C, \quad x \in \Omega^* \setminus \Omega^*_{R_0}. \tag{3.12}$$

Moreover, we notice that $\mathsf{u}_1 - \mathsf{u}_1^*$ satisfies

$$\begin{cases} \mathcal{L}_{\lambda,\mu}(\mathsf{u}_1 - \mathsf{u}_1^*) = 0, & \text{in } D \setminus (\overline{D_1 \cup D_1^*}), \\ \mathsf{u}_1 - \mathsf{u}_1^* = \psi_1 - \mathsf{u}_1^*, & \text{on } \partial D_1 \setminus D_1^*, \\ \mathsf{u}_1 - \mathsf{u}_1^* = \mathsf{u}_1 - \psi_1, & \text{on } \partial D_1^* \setminus (D_1 \cup \{0\}), \\ \mathsf{u}_1 - \mathsf{u}_1^* = 0, & \text{on } \partial D. \end{cases}$$

Then, by using the same proof of lemma 4.1 in [15], we obtain that for any $0 < \tilde{\theta} < \alpha/(1+2\alpha)$,

$$|\nabla(\mathsf{u}_1 - \mathsf{u}_1^*)(x)| \leq C\varepsilon^{\tilde{\theta}}, \quad x \in D \setminus (\overline{D_1 \cup D_1^* \cup \mathcal{C}_{\varepsilon^{1/(1+2\alpha)-\tilde{\theta}}}}),$$

where $\mathcal{C}_r$ is a cylinder defined by

$$\mathcal{C}_r := \Big\{x \in \mathbb{R}^d \;\Big|\; |x'| < r, \quad 0 \leq x_d \leq \varepsilon + 2\max_{|x'|=r} h_1(x')\Big\}. \tag{3.13}$$

Hence, we have

$$\Big|\int_{\partial D \setminus \mathcal{C}_{\varepsilon^{1/(1+2\alpha)-\tilde{\theta}}}} \frac{\partial(\mathsf{u}_1 - \mathsf{u}_1^*)}{\partial \nu_0}\Big|_+ \cdot \mathsf{g}\Big| \leq C\varepsilon^{\tilde{\theta}}. \tag{3.14}$$

On the other hand, in view of (3.5) and (3.14), it suffices to estimate

$$\begin{aligned} &\int_{\partial D \cap \mathcal{C}_{\varepsilon^{1/(1+2\alpha)-\tilde{\theta}}}} \frac{\partial(\mathsf{u}_1 - \mathsf{u}_1^*)}{\partial \nu_0}\Big|_+ \cdot \mathsf{g} \\ &= \int_{\partial D \cap \mathcal{C}_{\varepsilon^{1/(1+2\alpha)-\tilde{\theta}}}} \frac{\partial(\tilde{\mathsf{u}}_1 - \tilde{\mathsf{u}}_1^*)}{\partial \nu_0}\Big|_+ \cdot \mathsf{g} + \int_{\partial D \cap \mathcal{C}_{\varepsilon^{1/(1+2\alpha)-\tilde{\theta}}}} \frac{\partial(\mathsf{w}_1 - \mathsf{w}_1^*)}{\partial \nu_0}\Big|_+ \cdot \mathsf{g} \\ &=: \mathrm{I}_u + \mathrm{I}_w, \end{aligned}$$

where $\mathsf{w}_1 := \mathsf{u}_1 - \tilde{\mathsf{u}}_1$ and $\mathsf{w}_1^* := \mathsf{u}_1^* - \tilde{\mathsf{u}}_1^*$. We notice that

$$\mathsf{n} = (n_1, n_2, \cdots, n_d) = \frac{(\nabla_{x'}h(x'), -1)}{\sqrt{1 + |\nabla_{x'}h(x')|^2}}, \quad \text{on } \partial D \cap B_{R_0}. \tag{3.15}$$

It follows from the definition in (1.8) that

$$\mathrm{I}_u=\int_{\partial D\cap\mathcal{C}_{\varepsilon^{1/(1+2\alpha)-\tilde\theta}}}\Big(\lambda\sum_{i=1}^{d}\partial_1(\tilde u_1^{(1)}-\tilde u_1^{*(1)})n_ig^{(i)}$$
$$+\mu\sum_{j=1}^{d}\partial_j(\tilde u_1^{(1)}-\tilde u_1^{*(1)})\big(n_1g^{(j)}+n_jg^{(1)}\big)\Big)ds.$$

By using (3.9), (3.10), (3.15), (2.11) and $ds=\sqrt{1+|\nabla_{x'}h(x')|^2}dx'$, we have

$$\begin{aligned}|\mathrm{I}_{u,1}|&:=\Big|\int_{\partial D\cap\mathcal{C}_{\varepsilon^{1/(1+2\alpha)-\tilde\theta}}}\sum_{i=1}^{d-1}\partial_1(\tilde u_1^{(1)}-\tilde u_1^{*(1)})n_ig^{(i)}ds\Big|\\&\leq\int_{\partial D\cap\mathcal{C}_{\varepsilon^{1/(1+2\alpha)-\tilde\theta}}}\frac{C}{|x'|}\cdot\frac{|x'|^{1+\alpha}\cdot|\nabla_{x'}\mathsf{g}(0)|}{\sqrt{1+|\nabla_{x'}h(x')|^2}}ds\leq C|\nabla_{x'}\mathsf{g}(0)|\varepsilon^{(\frac{1}{1+2\alpha}-\tilde\theta)(d+\alpha-1)},\end{aligned}$$

and

$$\begin{aligned}|\mathrm{I}_{u,2}|&:=\Big|\int_{\partial D\cap\mathcal{C}_{\varepsilon^{1/(1+2\alpha)-\tilde\theta}}}\partial_1(\tilde u_1^{(1)}-\tilde u_1^{*(1)})n_dg^{(d)}ds\Big|\\&\leq\int_{\partial D\cap\mathcal{C}_{\varepsilon^{1/(1+2\alpha)-\tilde\theta}}}\frac{C}{|x'|}\cdot\frac{|\nabla_{x'}\mathsf{g}(0)|\cdot|x'|}{\sqrt{1+|\nabla_{x'}h(x')|^2}}ds\leq C|\nabla_{x'}\mathsf{g}(0)|\varepsilon^{(\frac{1}{1+2\alpha}-\tilde\theta)(d-1)}.\end{aligned}$$

By the same reason, a direct calculation yields

$$\begin{aligned}|\mathrm{I}_{u,3}|&:=\Big|\int_{\partial D\cap\mathcal{C}_{\varepsilon^{1/(1+2\alpha)-\tilde\theta}}}\sum_{j=1}^{d}\partial_j(\tilde u_1^{(1)}-\tilde u_1^{*(1)})\big(n_1g^{(j)}+n_jg^{(1)}\big)ds\Big|\\&\leq C|\nabla_{x'}\mathsf{g}(0)|\varepsilon^{(\frac{1}{1+2\alpha}-\tilde\theta)(d-1-\alpha)}.\end{aligned}$$

Then, we have

$$|\mathrm{I}_u|=|\lambda(\mathrm{I}_{u,1}+\mathrm{I}_{u,2})+\mu\mathrm{I}_{u,3}|\leq C|\nabla_{x'}\mathsf{g}(0)|\varepsilon^{(\frac{1}{1+2\alpha}-\tilde\theta)(d-1-\alpha)}.\tag{3.16}$$

For I_w, by virtue of (2.20), (3.11) and (2.11), we have

$$|\mathrm{I}_w|\leq C|\nabla_{x'}\mathsf{g}(0)|\varepsilon^{(\frac{1}{1+2\alpha}-\tilde\theta)(d-1)}.\tag{3.17}$$

Hence, it follows form (3.16) and (3.17) that

$$\Big|\int_{\partial D\cap\mathcal{C}_{\varepsilon^{1/(1+2\alpha)-\tilde\theta}}}\frac{\partial(\mathsf{u}_1-\mathsf{u}_1^*)}{\partial\nu_0}\Big|_+\cdot\mathsf{g}\Big|\leq|\mathrm{I}_u|+|\mathrm{I}_w|\leq C|\nabla_{x'}\mathsf{g}(0)|\varepsilon^{(\frac{1}{1+2\alpha}-\tilde\theta)(d-1-\alpha)}.$$

Combining with (3.14) and taking $\tilde\theta=(\frac{1}{1+2\alpha}-\tilde\theta)(d-1-\alpha)$, i.e. $\tilde\theta=\frac{d-1-\alpha}{(1+2\alpha)(d-\alpha)}$, we obtain (3.4) and hence the proof is completed. □

The following proposition is the main novelty and difference with [7, 15, 36]. We see that some of the blow-up factors with given specific boundary data g are nontrivial and do not change the blow-up rates. In [36], we see that the nontrivial blow-up factors are obtained from the boundary data's order of growth, which increase the blow-up rates.

Proposition 3.2. *Under the assumptions of Theorem 1.2, let $b_l^*[\mathsf{g}]$ be defined in (1.14) with $\mathsf{g}\in C^{1,\alpha}(\partial D;\mathbb{R}^d)$.*

(i) *For $d=2$, if there exists $i_0\in\{1,\,2\}$ such that* g *satisfies* (1.19), *then, we have*

$$\big|b^*_{i_0}[\mathsf{g}]\big|>0, \tag{3.18}$$

and

$$\Big|\sum_{l=1}^{3}b^*_l[\mathsf{g}]\Big|\geq\frac{1}{C}\big|b^*_{i_0}[\mathsf{g}]\big|>0. \tag{3.19}$$

(ii) *For $d\geq 3$, if there exists $i_0\in\{1,\,2,\cdots,d\}$ such that* g *satisfies* (1.22), *then for ζ_0 large enough, we also have* (3.18) *and*

$$\Big|\sum_{l=1}^{d(d+1)/2}b^*_l[\mathsf{g}]\Big|\geq\frac{1}{C}\big|b^*_{i_0}[\mathsf{g}]\big|>0. \tag{3.20}$$

where $C>0$ is a constant independent of ε.

Proof. First, we replace R_0 in (3.11), (3.12) and the assumptions (1.15)-(1.18) by γR_0, where $0<\gamma\leq 1$ is to de determined later. Then, we divide $b^*_l[\mathsf{g}]$ as follows

$$b^*_l[\mathsf{g}]=-\int_{\partial D\cap B_{\gamma R_0}}\frac{\partial\tilde{\mathsf{u}}^*_l}{\partial\nu_0}\Big|_+\cdot\mathsf{g}-\int_{\partial D\cap B_{\gamma R_0}}\frac{\partial(\mathsf{u}^*_l-\tilde{\mathsf{u}}^*_l)}{\partial\nu_0}\Big|_+\cdot\mathsf{g}-\int_{\partial D\setminus B_{\gamma R_0}}\frac{\partial\mathsf{u}^*_l}{\partial\nu_0}\Big|_+\cdot\mathsf{g}.$$

Next, we estimate these three integrations above. By using (3.11) and the definition of g, we have

$$\Big|\int_{\partial D\cap B_{\gamma R_0}}\frac{\partial(\mathsf{u}^*_l-\tilde{\mathsf{u}}^*_l)}{\partial\nu_0}\Big|_+\cdot\mathsf{g}\Big|\leq\begin{cases}C\displaystyle\int_{|x'|\leq\gamma R_0}|x'|^{\tau-1}dx'\leq C(\gamma R_0)^{\tau},\ d=2,\\ C\zeta_0\displaystyle\int_{|x'|\leq\gamma R_0}|x'|^{\tau-1}dx'\leq C\zeta_0(\gamma R_0)^{\tau+d-2},\ d\geq 3.\end{cases} \tag{3.21}$$

Meanwhile, for $d=2$, using (1.16), we have

$$\Big|\int_{\partial D\setminus B_{\gamma R_0}}\frac{\partial\mathsf{u}^*_l}{\partial\nu_0}\Big|_+\cdot\mathsf{g}\Big|\leq C(\gamma R_0)^{\tau}. \tag{3.22}$$

For $d\geq 3$, using (1.17) and (1.18), one has

$$\begin{aligned}\Big|\int_{\partial D\setminus B_{\gamma R_0}}\frac{\partial\mathsf{u}^*_l}{\partial\nu_0}\Big|_+\cdot\mathsf{g}\Big|&\leq C\zeta_0\int_{\gamma R_0\leq|x'|\leq 2\gamma R_0}|x'|^{\tau}dx'+\int_{\partial D\setminus B_{2\gamma R_0}}C\zeta_0^{1/2}(2\gamma R_0)^{\tau}ds\\&\leq C\big(\zeta_0(\gamma R_0)^{\tau+d-1}+\zeta_0^{1/2}(\gamma R_0)^{\tau}\big).\end{aligned} \tag{3.23}$$

Moreover, we notice from (1.8) and (3.8) that for $l=1,\,2,\cdots,d$,

$$\frac{\partial\tilde{\mathsf{u}}^*_l}{\partial\nu_0}\Big|_+\cdot\mathsf{g}=\lambda\partial_l\tilde{\mathsf{u}}^{*(l)}_l\sum_{i=1}^{d}n_ig^{(i)}+\mu\sum_{i=1}^{d}\partial_i\tilde{\mathsf{u}}^{*(l)}_l n_ig^{(l)}+\mu\sum_{j=1}^{d}\partial_j\tilde{\mathsf{u}}^{*(l)}_l n_lg^{(j)}. \tag{3.24}$$

Then, for $k=1,\,2,\cdots,d$ and $k\neq i_0$, by virtue of (3.7), (3.8) and the definition of g, we have

$$\begin{aligned}\Big|\int_{\partial D\cap B_{\gamma R_0}}\frac{\partial\tilde{\mathsf{u}}^*_k}{\partial\nu_0}\Big|_+\cdot\mathsf{g}\Big|&=\Big|\int_{\partial D\cap B_{\gamma R_0}}\lambda\partial_k\tilde{u}^{*(k)}_k n_{i_0}g^{(i_0)}+\mu\partial_{i_0}\tilde{u}^{*(k)}_k n_kg^{(i_0)}ds\Big|\\&\leq\begin{cases}C(\gamma R_0)^{\tau}, & d=2,\\ C\zeta_0(\gamma R_0)^{\tau+d-2}, & d\geq 3.\end{cases}\end{aligned}$$

Thus, combining with (3.21) - (3.23), we have for $k \neq i_0$ and $k = 1, 2, \cdots, d$,

$$|b_k^*[\mathsf{g}]| \leq \begin{cases} C(\gamma R_0)^\tau, & d = 2, \\ C\big(\zeta_0(\gamma R_0)^{\tau+d-2} + \zeta_0^{1/2}(\gamma R_0)^\tau\big), & d \geq 3. \end{cases} \tag{3.25}$$

By the same way, we can obtain that for $l = d+1, \cdots, d(d+1)/2$,

$$|b_l^*[\mathsf{g}]| \leq \begin{cases} C(\gamma R_0)^\tau, & d = 2, \\ C\big(\zeta_0(\gamma R_0)^{\tau+d-1-\alpha} + \zeta_0^{1/2}(\gamma R_0)^\tau\big), & d \geq 3. \end{cases} \tag{3.26}$$

While, using (3.7), (3.8) and (3.24), we have

$$\begin{aligned} &\Big| \int_{\partial D \cap B_{\gamma R_0}} \frac{\partial \tilde{\mathsf{u}}_{i_0}^*}{\partial \nu_0}\Big|_+ \cdot \mathsf{g} \Big| \\ &= \Big| \int_{\partial D \cap B_{\gamma R_0}} \Big(\lambda \partial_{i_0} \tilde{\mathsf{u}}_{i_0}^{*(i_0)} n_{i_0} g^{(i_0)} + \mu \sum_{i=1}^d \partial_i \tilde{\mathsf{u}}_{i_0}^{*(i_0)} n_i g^{(i_0)} + \mu \partial_{i_0} \tilde{\mathsf{u}}_{i_0}^{*(i_0)} n_{i_0} g^{(i_0)} \Big) ds \Big| \\ &\geq \begin{cases} (\gamma R_0)^{\tau-\alpha} \big(\dfrac{1}{C_0} - C(\gamma R_0)^{2\alpha} \big) & d = 2, \\ \zeta_0 (\gamma R_0)^{\tau+d-2-\alpha} \big(\dfrac{1}{\widetilde{C}_0} - C(\gamma R_0)^{2\alpha} \big) & d \geq 3. \end{cases} \end{aligned} \tag{3.27}$$

These, together with (3.21) and (3.22), yield that for $d = 2$,

$$\begin{aligned} |b_{i_0}^*[\mathsf{g}]| &\geq (\gamma R_0)^{\tau-\alpha} \Big(\frac{1}{C_0} - C(\gamma R_0)^{2\alpha} - C(\gamma R_0)^\alpha \Big) \\ &\geq (\gamma R_0)^{\tau-\alpha} \Big(\frac{1}{C_0} - C_1(\gamma R_0)^\alpha \Big) \geq \frac{1}{2C_0} (\gamma R_0)^{\tau-\alpha}. \end{aligned}$$

Here, the last inequality is obtained by taking $\gamma \leq (2C_0 C_1 R_0^\alpha)^{-1/\alpha}$ and thus (3.18) for $d = 2$ is proved.

Moreover, for $d \geq 3$, combining (3.21), (3.23) and (3.27), we have for fixed $\gamma \leq (2\widetilde{C}_0 \widetilde{C}_1 R_0^\alpha)^{-1/\alpha}$ such that

$$\begin{aligned} \big|b_{i_0}^*[\mathsf{g}]\big| &\geq \zeta_0 (\gamma R_0)^{\tau+d-2-\alpha} \Big(\frac{1}{\widetilde{C}_0} - \widetilde{C}_1 (\gamma R_0)^\alpha \Big) - \widetilde{C}_2 \zeta_0^{1/2} (\gamma R_0)^\tau \\ &\geq \zeta_0 (\gamma R_0)^\tau \Big(\frac{(\gamma R_0)^{d-2-\alpha}}{2\widetilde{C}_0} - \frac{\widetilde{C}_2}{\zeta_0^{1/2}} \Big). \end{aligned}$$

Then, taking large ζ_0 such that $\zeta_0 \geq \big(4\widetilde{C}_0 \widetilde{C}_2 / (\gamma R_0)^{d-2-\alpha}\big)^2$, we have

$$\big|b_{i_0}^*[\mathsf{g}]\big| \geq \frac{1}{4\widetilde{C}_0} \zeta_0 (\gamma R_0)^{\tau+d-2-\alpha} > 0.$$

Hence, (3.18) for $d \geq 3$ is proved.

For (3.19), combining with (3.21), (3.22) and (3.25)-(3.27), we have for fixed small $\gamma \leq (C_2 C_3 R_0^\alpha)^{-1/\alpha}$,

$$\begin{aligned} \Big| \sum_{l=1}^3 b_l^*[\mathsf{g}] \Big| &\geq \big|b_{i_0}^*[\mathsf{g}]\big| - \sum_{l \neq i_0} \big|b_l^*[\mathsf{g}]\big| \\ &\geq \frac{1}{C} \big|b_{i_0}^*[\mathsf{g}]\big| + (\gamma R_0)^{\tau-\alpha} \Big(\frac{1}{C_2} - C_3 (\gamma R_0)^\alpha \Big) \geq \frac{1}{C} \big|b_{i_0}^*[\mathsf{g}]\big|. \end{aligned}$$

For (3.20), using (3.21), (3.23) and (3.25)-(3.27), we have, by taking small $\gamma \leq (2\widetilde{C}_3\widetilde{C}_4R_0^\alpha)^{-1/\alpha}$,

$$\Big|\sum_{l=1}^{d(d+1)/2} b_l^*[\mathsf{g}]\Big| \geq \frac{1}{C}\big|b_{i_0}^*[\mathsf{g}]\big| + \zeta_0(\gamma R_0)^{\tau+d-2-\alpha}\Big(\frac{1}{\widetilde{C}_3} - \widetilde{C}_4(\gamma R_0)^\alpha\Big) - \widetilde{C}_5\zeta_0^{1/2}(\gamma R_0)^\tau$$
$$\geq \frac{1}{C}\big|b_{i_0}^*[\mathsf{g}]\big| + \zeta_0(\gamma R_0)^\tau\Big(\frac{(\gamma R_0)^{d-2-\alpha}}{2\widetilde{C}_3} - \frac{\widetilde{C}_5}{\zeta_0^{1/2}}\Big).$$

Then, (3.20) follows from by taking $\zeta_0 \geq \big(2\widetilde{C}_3\widetilde{C}_5/(\gamma R_0)^{d-2-\alpha}\big)^2$. The proof is completed. $\square$

Now, it is the position to establish the lower bound estimates.

Proof of Theorem 1.2. For $d = 2$, in view of (2.31) and (2.32), it follows from Proposition 3.1 that for small $\varepsilon > 0$,

$$|C^i| \geq \frac{1}{a_{ii}}\Big(|b_i^*[\mathsf{g}]| - \frac{|a_{3i}|}{a_{33}}|b_3^*[\mathsf{g}]|\Big) + o(\varepsilon^{\frac{\alpha}{1+\alpha}}), \quad i = 1,\, 2.$$

From Lemma 2.3, we notice that $|a_{3i}|/a_{33} \leq C$ independent of g. If there exists $i_0 \in \{1,\, 2\,\}$ such that g satisfies (1.19), Then, from the proof of (3.19), we have for fixed small $\gamma > 0$,

$$|b_{i_0}^*[\mathsf{g}]| - \frac{|a_{3i_0}|}{a_{33}}|b_3^*[\mathsf{g}]| \geq |b_{i_0}^*[\mathsf{g}]| - C|b_3^*[\mathsf{g}]| \geq \frac{1}{C}|b_{i_0}^*[\mathsf{g}]|.$$

Hence, in view of (2.20), (2.25), (2.29), (3.1), and (3.2), we have for small $\varepsilon > 0$,

$$\begin{aligned}|\nabla \mathsf{u}(x)| &\geq \Big|\sum_{l=1}^{2} C^l\nabla\tilde{\mathsf{u}}_l(x)\Big| - \Big|\sum_{l=1}^{2} C^l\nabla(\mathsf{u}_l - \tilde{\mathsf{u}}_l)(x)\Big| - C\\ &\geq |C^{i_0}|\cdot|\partial_2\tilde{\mathsf{u}}_{i_0}^{(i_0)}(x)| - C\varepsilon^{-\frac{1-\alpha}{1+\alpha}}\\ &\geq C\varepsilon^{-\frac{1}{1+\alpha}}\big|b_{i_0}^*[\mathsf{g}]\big|, \qquad x = (0', x_d) \in \overline{P_1P}.\end{aligned}$$

For $d \geq 3$, in view of (2.33), it follows from Lemma 2.3 and Proposition 3.1 that for small $\varepsilon > 0$,

$$|C^i| \geq \frac{|a_{ii}^*|}{\det A}\Big(|b_i^*[\mathsf{g}]| - \sum_{j\neq i}\frac{|a_{ji}^*|}{|a_{ii}^*|}\cdot|b_j^*[\mathsf{g}]|\Big) + O(\varepsilon^{\frac{d-1-\alpha}{(1+2\alpha)(d-\alpha)}}),$$

where $i = 1,\, 2, \cdots, d$, $j = 1, 2, \cdots, d(d+1)/2$ and $j \neq i$, a_{kl}^* is the cofactor of a_{kl} in matrix A, $k, l = 1,\, 2,\, \cdots, d(d+1)/2$. We notice from Lemma 2.3 that $|a_{ji}^*|/|a_{ii}^*| \leq C$ independent of ζ_0. If there exists $i_0 \in \{1,\, 2, \cdots, d\,\}$ such that g satisfies (1.22), then, in view of (3.20), by taking a fixed ζ_0 large enough, we have, for small $\varepsilon > 0$

$$|C^{i_0}| \geq \frac{1}{C}|b_{i_0}^*[\mathsf{g}]|.$$

Thus, similarly, we have for small $\varepsilon > 0$,

$$|\nabla \mathsf{u}(x)| \geq |C^{i_0}|\cdot|\partial_d\tilde{\mathsf{u}}_{i_0}^{(i_0)}(x)| - C\varepsilon^{-\frac{1}{1+\alpha}} \geq C\varepsilon^{-1}\big|b_{i_0}^*[\mathsf{g}]\big|, \quad x \in \overline{P_1P}.$$

The proof of Theorem 1.2 is completed. $\square$

4. Proof of Theorems 1.6 and 1.7

In this section, we focus on the scalar case i.e. perfect conductivity problem. We also assume that $\varphi(P) = 0$. Otherwise, we can replace u by $u - \varphi(P)$. Compared to (1.7), the solution u of (1.25) can be decomposed more concisely

$$u(x) = C_1 v_1(x) + v_0(x),\ x \in \Omega,$$

where the constant C_1 is determined by the third line of (1.25), and v_1 satisfies

$$\begin{cases} \Delta v_1 = 0, & \text{in } D \setminus \overline{D}_1, \\ v_1 = 1, & \text{on } \partial D_1, \\ v_1 = 0, & \text{on } \partial D, \end{cases} \tag{4.1}$$

and v_0 satisfies

$$\begin{cases} \Delta v_0 = 0, & \text{in } D \setminus \overline{D}_1, \\ v_0 = 0, & \text{on } \partial D_1, \\ v_0 = \varphi, & \text{on } \partial D. \end{cases} \tag{4.2}$$

We notice that

$$\nabla u(x) = C_1 \nabla v_1(x) + \nabla v_0(x),\ x \in \Omega. \tag{4.3}$$

To prove Theorem 1.6, we need to estimate each term in (4.3), i.e. $|C_1|$, $|\nabla v_1|$, and $|\nabla v_0|$. We follow the idea developed in [14].

We choose an auxiliary function $\hat{u} \in C^{1,\alpha}(\mathbb{R}^d)$ such that $\hat{u} = 0$ on ∂D_1, $\hat{u} = \varphi$ on ∂D, and

$$\hat{u}(x) := \big(1 - \bar{u}(x)\big)\varphi(x', h(x')),\ x \in \Omega_{2R_0},$$

and

$$\|\hat{u}\|_{C^{1,\alpha}(\Omega\setminus\Omega_{R_0})} \leq C,$$

where $\bar{u}$ defined in (2.5) and (2.6). Through a direct computation, we obtain that

$$|\nabla \hat{u}| \leq C\Big(\frac{|x'|}{\varepsilon + |x'|^{1+\alpha}} + 1\Big)\|\varphi\|_{C^1(\partial D)}, \quad \text{in } \Omega_{2R_0}.$$

Then we have

Proposition 4.1. *Let $v_1, v_0 \in H^1(\Omega)$ be the weak solutions of (4.1) and (4.2), respectively. Then, for small $\varepsilon > 0$, we have for $x \in \Omega_{R_0}$,*

$$|\nabla(v_0 - \hat{u})(x)| \leq C\|\varphi\|_{C^{1,\alpha}(\partial D)},$$

$$|\nabla(v_1 - \bar{u})(x)| \leq \frac{C}{(\varepsilon + |x'|^{1+\alpha})^{\frac{1}{1+\alpha}}}, \tag{4.4}$$

and

$$|\nabla v_1(x)| \leq C, \quad \text{and} \quad |\nabla v_0(x)| \leq C||\varphi||_{C^{1,\alpha}(\partial D)}, \qquad x \in \Omega \setminus \Omega_{R_0}. \tag{4.5}$$

Consequently,

$$|\nabla v_0(x)| \leq C\Big(\frac{|x'|}{\varepsilon + |x'|^{1+\alpha}} + 1\Big) \cdot \|\varphi\|_{C^{1,\alpha}(\partial D)}, \quad x \in \Omega_{R_0}, \tag{4.6}$$

and

$$\frac{1}{C(\varepsilon + |x'|^{1+\alpha})} \leq |\nabla v_1(x)| \leq \frac{C}{\varepsilon + |x'|^{1+\alpha}}, \quad x \in \Omega_{R_0}, \tag{4.7}$$

where $C > 0$ is universal constant independent of ε.

Since the proof of Proposition 4.1 is similar to that of Proposition 2.1, we omit them and refer to [14] for more details.

To estimate $|C_1|$, denote

$$\tilde{a}_{11} := -\int_{\partial D_1} \frac{\partial v_1}{\partial \mathsf{n}} ds, \quad \text{and} \quad Q[\varphi] := \int_{\partial D_1} \frac{\partial v_0}{\partial \mathsf{n}} ds. \tag{4.8}$$

We have

Lemma 4.2. *Under the assumptions of Theorem 1.6, we have for small $\varepsilon > 0$,*

$$\frac{1}{C\rho_{d,\,\alpha}(\varepsilon)} \leq \tilde{a}_{11} \leq \frac{C}{\rho_{d,\,\alpha}(\varepsilon)}, \tag{4.9}$$

and

$$|Q[\varphi]| \leq C\|\varphi\|_{C^{1,\alpha}(\partial D)}. \tag{4.10}$$

Consequently,

$$|C_1| \leq C\rho_{d,\,\alpha}(\varepsilon)\|\varphi\|_{C^{1,\alpha}(\partial D)}, \tag{4.11}$$

where $\rho_{d,\,\alpha}(\varepsilon)$ is defined in (1.10).

Proof. Since we notice that v_1 satisfies (4.1), by virtue of integrating by parts, we have

$$\tilde{a}_{11} = -\int_{\partial D_1} \frac{\partial v_1}{\partial \mathsf{n}} ds = \int_{\partial \Omega} \frac{\partial v_1}{\partial \mathsf{n}} v_1 ds = \int_{\Omega} |\nabla v_1|^2 dx.$$

Then, by using (4.5) and (4.7), we have

$$\int_{\Omega_{R_0}} \frac{1}{C(\varepsilon + |x'|^{1+\alpha})^2} dx \leq \tilde{a}_{11} \leq \int_{\Omega_{R_0}} \frac{C}{(\varepsilon + |x'|^{1+\alpha})^2} dx + C.$$

Thus, a direct calculation yields to (4.9).

Moreover, noticing v_0 satisfies (4.2), we also have

$$Q[\varphi] = \int_{\partial D_1} \frac{\partial v_0}{\partial \mathsf{n}} ds = -\int_{\partial \Omega} \frac{\partial v_0}{\partial \mathsf{n}} v_1 ds = -\int_{\Omega} \nabla v_0 \cdot \nabla v_1 dx.$$

By using Proposition 4.1, we have

$$\begin{aligned} |Q[\varphi]| &\leq \int_{\Omega_{R_0}} |\nabla v_0||\nabla v_1| dx + C \\ &\leq C\|\varphi\|_{C^{1,\alpha}(\partial D)} \int_{\Omega_{R_0}} \frac{\varepsilon + |x'|}{(\varepsilon + |x'|^{1+\alpha})^2} dx + C \leq C\|\varphi\|_{C^{1,\alpha}(\partial D)}. \end{aligned}$$

On the other hand, it follows from the third line of (1.25) and (4.3) that

$$C_1 \tilde{a}_{11} = Q[\varphi]. \tag{4.12}$$

This, together with (4.9) and (4.10), leads to (4.11). The proof is completed. □

Proof of Theorem 1.6. In view of the decomposition (4.3) and (4.12), we have

$$\nabla u = C_1 \nabla v_1 + \nabla v_0 = \frac{Q[\varphi]}{\tilde{a}_{11}} \nabla v_1 + \nabla v_0. \tag{4.13}$$

Thus, (1.26) follows from (4.6), (4.7) and (4.11). Theorem 1.6 is proved. □

To obtain the lower bound estimates for $|\nabla u|$, in view of (4.6), (4.7) and (4.9), it suffices to consider the lower bound estimates of $|Q[\varphi]|$. We follow the idea in [28] with some modifications provided to prove these estimates.

Let v_1^* be the solution of

$$\begin{cases} \Delta v_1^* = 0, & \text{in } \Omega^*, \\ v_1^* = 1, & \text{on } \partial D_1^* \setminus \{0\}, \\ v_1^* = 0, & \text{on } \partial D \setminus \{0\}. \end{cases}$$

We have

Proposition 4.3. *Assume that $\varphi \in C^{1,\alpha}(\partial D)$. Let $Q[\varphi]$ and $Q^*[\varphi]$ be defined in (4.8) and (1.27), respectively. Then, we have*

$$|Q[\varphi] - Q^*[\varphi]| \leq C\varepsilon^{\frac{d-1-\alpha}{(1+2\alpha)(d-\alpha)}}. \tag{4.14}$$

where $C > 0$ is a universal constant, independent of ε.

Proof. It follows from the Green's formula that

$$Q[\varphi] = -\int_\Omega \nabla v_0 \cdot \nabla v_1 dx = -\int_{\partial D} \frac{\partial v_1}{\partial \mathsf{n}} \varphi ds.$$

Similarly, one has

$$Q^*[\varphi] := \int_{\partial D_1^*} \frac{\partial v_0^*}{\partial \mathsf{n}} ds = -\int_{\partial \Omega^*} \frac{\partial v_0^*}{\partial \mathsf{n}} v_1^* ds = -\int_{\partial D} \frac{\partial v_1^*}{\partial \mathsf{n}} \varphi ds.$$

Then, we see that

$$Q[\varphi] - Q^*[\varphi] = -\int_{\partial D} \frac{\partial (v_1 - v_1^*)}{\partial \mathsf{n}} \varphi ds. \tag{4.15}$$

To prove (4.14), it suffices to estimate $\nabla(v_1 - v_1^*)$ on ∂D. We can still use the auxiliary function $\bar{u}^*$ defined in (3.6) to estimate v_1^*. Then, (3.9) and (3.10) still hold for $v_1 - v_1^*$. Moreover, applying Proposition 4.1 for v_1^*, we can obtain

$$|\nabla(v_1^* - \bar{u}^*)(x)| \leq \frac{C}{|x'|}, \quad x \in \Omega^*_{R_0}, \tag{4.16}$$

$$|\nabla_{x'} v_1^*(x)| \leq \frac{C}{|x'|}, \quad |\partial_d v_1^*(x)| \leq \frac{C}{|x|^{1+\alpha}}, \quad x \in \Omega^*_{R_0},$$

and

$$|\nabla v_1^*(x)| \leq C, \quad x \in \Omega^* \setminus \Omega^*_{R_0}. \tag{4.17}$$

Moreover, we notice that $v_1 - v_1^*$ satisfies

$$\begin{cases} \Delta(v_1 - v_1^*) = 0, & \text{in } D \setminus (\overline{D_1 \cup D_1^*}), \\ v_1 - v_1^* = 1 - v_1^*, & \text{on } \partial D_1 \setminus D_1^*, \\ v_1 - v_1^* = v_1 - 1, & \text{on } \partial D_1^* \setminus (D_1 \cup \{0\}), \\ v_1 - v_1^* = 0, & \text{on } \partial D. \end{cases}$$

Then, following the same proof in Proposition 3.1, we can obtain that

$$|(v_1 - v_1^*)(x)| \leq C\varepsilon^{\frac{\alpha}{1+2\alpha}}, \quad x \in D \setminus (\overline{D_1 \cup D_1^* \cup \mathcal{C}_{\varepsilon^{1/(1+2\alpha)}}}),$$

where $\mathcal{C}_r$ is defined in (3.13). It follows from the standard interior and boundary estimates for elliptic equation, we have, for any $0 < \tilde{\theta} < \alpha/(1+2\alpha)$,

$$|\nabla(v_1 - v_1^*)(x)| \leq C\varepsilon^{\tilde{\theta}}, \quad x \in D \setminus (\overline{D_1 \cup D_1^* \cup \mathcal{C}_{\varepsilon^{1/(1+2\alpha)-\tilde{\theta}}}}).$$

Thus we have

$$\Big|\int_{\partial D\setminus\mathcal{C}_{\varepsilon^{1/(1+2\alpha)-\tilde{\theta}}}}\frac{\partial(v_1-v_1^*)}{\partial \mathsf{n}}\varphi ds\Big|\leq C\varepsilon^{\tilde{\theta}},\tag{4.18}$$

where $0<\tilde{\theta}<\alpha/(1+2\alpha)$ is to be determined later.

To prove (4.14), in view of (4.15) and (4.18), it suffices to estimate

$$\begin{aligned}&\int_{\partial D\cap\mathcal{C}_{\varepsilon^{1/(1+2\alpha)-\tilde{\theta}}}}\frac{\partial(v_1-v_1^*)}{\partial \mathsf{n}}\varphi ds\\&=\int_{\partial D\cap\mathcal{C}_{\varepsilon^{1/(1+2\alpha)-\tilde{\theta}}}}\frac{\partial(\bar{u}-\bar{u}^*)}{\partial \mathsf{n}}\varphi ds+\int_{\partial D\cap\mathcal{C}_{\varepsilon^{1/(1+2\alpha)-\tilde{\theta}}}}\frac{\partial(w-w^*)}{\partial \mathsf{n}}\varphi ds=:\mathrm{I}_1+\mathrm{I}_2,\end{aligned}$$

where $w:=v_1-\bar{u}$ and $w^*:=v_1^*-\bar{u}^*$.

Noticing that $\varphi\in C^{1,\alpha}(\partial D)$ and $\varphi(P)=0$, one has

$$|\varphi(x',h(x'))|\leq C\|\varphi\|_{C^1(\partial D)}|x'|\quad\text{on }\overline{\Omega}_{R_0}\cap\partial D.$$

Combining with (3.15), we have

$$\begin{aligned}|\mathrm{I}_1|&=\Big|\int_{\partial D\cap\mathcal{C}_{\varepsilon^{1/(1+2\alpha)-\tilde{\theta}}}}\sum_{i=1}^{d}\partial_i(\bar{u}-\bar{u}^*)n_i\varphi ds\Big|\\&\leq C\|\varphi\|_{C^1(\partial D)}\int_{|x'|\leq\varepsilon^{1/(1+2\alpha)-\tilde{\theta}}}\frac{1}{|x'|^{\alpha}}dx'\leq C\|\varphi\|_{C^1(\partial D)}\varepsilon^{(\frac{1}{1+2\alpha}-\tilde{\theta})(d-1-\alpha)}.\end{aligned}$$

By virtue of (4.4) and (4.16), we have

$$|\mathrm{I}_2|\leq C\|\varphi\|_{C^1(\partial D)}\varepsilon^{(\frac{1}{1+2\alpha}-\tilde{\theta})(d-1)}.$$

Thus,

$$\Big|\int_{\partial D\cap\mathcal{C}_{\varepsilon^{1/(1+2\alpha)-\tilde{\theta}}}}\frac{\partial(v_1-v_1^*)}{\partial \mathsf{n}}\varphi ds\Big|\leq|\mathrm{I}_1|+|\mathrm{I}_2|\leq C\|\varphi\|_{C^1(\partial D)}\varepsilon^{(\frac{1}{1+2\alpha}-\tilde{\theta})(d-1-\alpha)}.$$

Then, combing with (4.18), we obtain (4.14) by taking $\tilde{\theta}=(\frac{1}{1+2\alpha}-\tilde{\theta})(d-1-\alpha)$, i.e. $\tilde{\theta}=\frac{d-1-\alpha}{(1+2\alpha)(d-\alpha)}$. The proof in completed. $\square$

Using (2.7), (4.16) and (4.17), and following the same way of Proposition 3.2, we obtain that

Proposition 4.4. *Under the assumptions of Theorem 1.6, for $d=2$, we assume that $\varphi\in C^{1,\alpha}(\partial D)$ satisfies* (1.15) *and* (1.16)*, then, we have*

$$\big|Q^*[\varphi]\big|>0.\tag{4.19}$$

For $d\geq3$, we assume that $\varphi\in C^{1,\alpha}(\partial D)$ satisfies (1.17) *and* (1.18)*, then we have* (4.19) *provided $\zeta_0>0$ large enough.*

Then, we are in a position to give the lower bound estimates.

Proof of Theorem 1.7. From (4.13), we see that

$$|\nabla u(x)|\geq\frac{|Q[\varphi]|}{\tilde{a}_{11}}\cdot|\nabla v_1(x)|-|\nabla v_0(x)|,\quad x\in\Omega_{R_0}.$$

Then, it follows from Proposition 4.1, Lemma 4.2, Propositions 4.3 and 4.4 that for small $\varepsilon>0$,

$$|\nabla u(x)|\geq\frac{\rho_{d,\alpha}(\varepsilon)}{C\varepsilon}\cdot|Q^*[\varphi]|,\qquad x\in\overline{PP_1}.$$

The proof of Theorem 1.6 is completed. □

5. Appendix: the proof of Proposition 2.1

This section is devoted to the proof of Proposition 2.1. The main difference with [7,36] is that h_1, h are of $C^{1,\alpha}$ in this paper, the auxiliary function $\bar{u}$ in (2.5) is not of $C^{2,\alpha}$ as before. To adapt the iteration technique, we first calculate the Hölder semi-norm in the following set.

For fixed $|z'| \leq R_0 < 1$, we define

$$\widehat{\Omega}_s(z') := \left\{(x', x_d) \in \mathbb{R}^d \mid h(x') < x_d < \varepsilon + h_1(x'),\ |x' - z'| < s\ \right\},$$

for $0 < s < R_0$. Since $|z'| \leq C\delta(z')^{1/(1+\alpha)}$, it follows that for any $x = (x', x_d) \in \widehat{\Omega}_s(z')$,

$$|x'| \leq |x' - z'| + |z'| < s + |z'| \leq C\delta(z')^{1/(1+\alpha)}. \tag{5.1}$$

This, together with (2.11), yields that for any $x = (x', x_d) \in \widehat{\Omega}_s(z')$,

$$|g^{(l)}(x', h(x'))| \leq C\|g^{(l)}\|_{C^1(\partial D)}|x'| \leq C\|g^{(l)}\|_{C^1(\partial D)}\delta(z')^{1/(1+\alpha)}. \tag{5.2}$$

Moreover, we first notice that for any $x, \tilde{x} \in \widehat{\Omega}_s(z')$ with $x \neq \tilde{x}$,

$$\frac{|g^{(l)}(x', \varepsilon + h(x')) - g^{(l)}(\tilde{x}', \varepsilon + h(\tilde{x}'))|}{|x' - \tilde{x}'|^\alpha} \leq \|g^{(l)}\|_{C^1(\partial D)}|x' - \tilde{x}'|^{1-\alpha}.$$

In view of $|x' - \tilde{x}'| \leq |x' - z'| + |z' - \tilde{x}'| \leq 2s$ and

$$[g^{(l)}]_{\alpha, \overline{\widehat{\Omega}_s(z')}\cap\partial D} := \sup_{\substack{x',\, \tilde{x}' \in \overline{\widehat{\Omega}_s(z')}\cap\partial D \\ x' \neq \tilde{x}'}} \frac{|g^{(l)}(x', \varepsilon + h(x')) - g^{(l)}(\tilde{x}', \varepsilon + h(\tilde{x}'))|}{|x' - \tilde{x}'|^\alpha},$$

we have

$$[g^{(l)}]_{\alpha, \overline{\widehat{\Omega}_s(z')}\cap\partial D} \leq C\|g^{(l)}\|_{C^1(\partial D)} s^{1-\alpha}. \tag{5.3}$$

Through the same calculation as (3.1) in [15], we have, for a fixed constant $s \leq C\delta(z')$,

$$[\nabla\bar{u}]_{\alpha, \widehat{\Omega}_s(z')} \leq C\Big(\delta(z')^{-1-\frac{1}{1+\alpha}} s^{1-\alpha} + \delta(z')^{-1-\frac{\alpha^2}{1+\alpha}}\Big). \tag{5.4}$$

Then, by virtue of (5.2), (5.3) and (5.4), we have

$$\begin{aligned}
[\partial_d \tilde{u}_{0l}^{(l)}]_{\alpha, \widehat{\Omega}_s(z')} &= [g^{(l)}\partial_d\bar{u}]_{\alpha, \widehat{\Omega}_s(z')} \\
&\leq \|g^{(l)}\|_{L^\infty\left(\overline{\widehat{\Omega}_s(z')}\cap\partial D\right)}[\partial_d\bar{u}]_{\alpha,\widehat{\Omega}_s(z')} + [g^{(l)}]_{\alpha,\overline{\widehat{\Omega}_s(z')}\cap\partial D}\|\partial_d\bar{u}\|_{L^\infty(\widehat{\Omega}_s(z'))} \\
&\leq C(\delta(z')^{-1}s^{1-\alpha} + \delta(z')^{-\alpha})\|g^{(l)}\|_{C^1(\partial D)}.
\end{aligned} \tag{5.5}$$

By the same reason, we have

$$[\nabla_{x'}\tilde{u}_{0l}^{(l)}]_{\alpha,\widehat{\Omega}_s(z')} \leq C(\delta(z')^{-1}s^{1-\alpha} + \delta(z')^{-\alpha})\|g^{(l)}\|_{C^{1,\alpha}(\partial D)}. \tag{5.6}$$

Thus, it follows from (2.10), (5.5), and (5.6) that

$$[\nabla\tilde{u}_{0l}]_{\alpha, \widehat{\Omega}_s(z')} \leq C(\delta(z')^{-1}s^{1-\alpha} + \delta(z')^{-\alpha})\|g^{(l)}\|_{C^{1,\alpha}(\partial D)}, \tag{5.7}$$

where s is a fixed constant with $s \leq C\delta(z')$.

Before we prove Proposition 2.1, we recall some properties of tensor $\mathbb{C}^0$. For the isotropic elastic material, the components C_{ijkl} satisfy symmetric condition:

$$C_{ijkl} = C_{klij} = C_{klji}, \quad i,\, j,\, k,\, l = 1,\, 2, \cdots, d.$$

Thus, $\mathbb{C}^0$ satisfies the ellipticity condition: for every $d \times d$ real symmetric matrix $A = (A_{ij})_{i,j=1}^d$,

$$\min\{2\mu,\, d\lambda + 2\mu\}|A|^2 \leq (\mathbb{C}^0 A, A) \leq \max\{2\mu, d\lambda + 2\mu\}|A|^2, \tag{5.8}$$

where $|A|^2 = \sum\limits_{i,j=1}^{d} A_{ij}^2$.

Proof of Proposition 2.1. We only prove the case of $l = 1$ for instance, since the proof is the same as the other cases. For simplicity, we denote

$$\mathsf{w} := \mathsf{u}_{01} - \tilde{\mathsf{u}}_{01} \quad \text{and} \quad \tilde{\mathsf{u}} := \tilde{\mathsf{u}}_{01}.$$

Thus, w satisfies

$$\begin{cases} -\mathcal{L}_{\lambda,\mu}\mathsf{w} = \nabla \cdot \Big(\mathbb{C}^0 e(\tilde{\mathsf{u}})\Big), & \text{in } \Omega, \\ \mathsf{w} = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.9}$$

Clearly, w still satisfies

$$-\mathcal{L}_{\lambda,\mu}\mathsf{w} = \nabla \cdot \Big(\mathbb{C}^0 e(\tilde{\mathsf{u}}) - \mathcal{M}\Big), \quad \text{in } \Omega, \tag{5.10}$$

for any constant matrix $\mathcal{M} = (\mathfrak{a}_{ij})_{i,j=1}^d$.

Step 1. We claim that

$$\int_\Omega |\nabla \mathsf{w}|^2 dx \leq C\|g^{(1)}\|^2_{C^{1,\alpha}(\partial D)}. \tag{5.11}$$

Indeed, multiplying (5.9) by w and integrating by parts, we have

$$\int_\Omega \Big(\mathbb{C}^0 e(\mathsf{w}), e(\mathsf{w})\Big) dx = -\int_\Omega \Big(\mathbb{C}^0 e(\tilde{\mathsf{u}}), e(\mathsf{w})\Big) dx.$$

It follows from (5.8), the first Korn's inequality and Hölder inequality that

$$C\int_\Omega |\nabla \mathsf{w}|^2 dx \leq \Big|\int_\Omega \Big(\mathbb{C}^0 e(\tilde{\mathsf{u}}), e(\mathsf{w})\Big) dx\Big| \leq C\Big(\int_\Omega |\nabla \tilde{\mathsf{u}}|^2 dx\Big)^{\frac{1}{2}} \cdot \Big(\int_\Omega |\nabla \mathsf{w}|^2 dx\Big)^{\frac{1}{2}}. \tag{5.12}$$

In view of (2.6), (2.12) and (2.13), we have

$$\int_\Omega |\nabla \tilde{\mathsf{u}}|^2\ dx \leq \int_{\Omega_{R_0}} C|\partial_d \tilde{\mathsf{u}}(x)|^2 dx + C\|g^{(1)}\|^2_{C^{1,\alpha}(\partial D)} \leq C\|g^{(1)}\|^2_{C^{1,\alpha}(\partial D)}.$$

Combining with (5.12), we obtain

$$\int_\Omega |\nabla \mathsf{w}|^2 dx \leq C\|g^{(1)}\|_{C^{1,\alpha}(\partial D)}\Big(\int_\Omega |\nabla \mathsf{w}|^2 dx\Big)^{\frac{1}{2}}.$$

This implies (5.11).

Step 2. We assert that

$$\int_{\widehat{\Omega}_{\delta(z')}(z')} |\nabla \mathsf{w}|^2 dx \leq C\delta(z')^d \|g^{(1)}\|^2_{C^{1,\,\alpha}(\partial D)}, \tag{5.13}$$

where $\delta(z')$ is defined in (2.8).

Indeed, for $0 < t < s < R_0$, let η be a smooth cut-off function satisfying

$$\eta(x') = \begin{cases} 1 & \text{if } |x' - z'| < t, \\ 0 & \text{if } |x' - z'| > s, \end{cases} \quad \text{and} \quad |\nabla_{x'}\eta(x')| \leq \frac{2}{s - t}.$$

Multiplying (5.10) by $\eta^2\mathsf{w}$ and using the integration by parts, one has

$$\int_{\widehat{\Omega}_s(z')}\Big(\mathbb{C}^0e(\mathsf{w}),e(\eta^2\mathsf{w})\Big)\,dx=-\int_{\widehat{\Omega}_s(z')}\Big(\mathbb{C}^0e(\tilde{\mathsf{u}})-\mathcal{M},e(\eta^2\mathsf{w})\Big)\,dx.$$

It follows from the first Korn's inequality and Young's inequality that

$$\int_{\widehat{\Omega}_t(z')}|\nabla\mathsf{w}|^2\,dx\leq\frac{C}{(s-t)^2}\int_{\widehat{\Omega}_s(z')}|\mathsf{w}|^2\,dx+C\int_{\widehat{\Omega}_s(z')}|\mathbb{C}^0e(\tilde{\mathsf{u}})-\mathcal{M}|^2\,dx. \tag{5.14}$$

Here, in (5.14), we take the constant matrix $\mathcal{M}=\mathcal{M}_1$ which is defined by

$$\mathcal{M}_1:=⨍_{\widehat{\Omega}_s(z')}\mathbb{C}^0e(\tilde{\mathsf{u}}(y))\,dy:=\frac{1}{|\widehat{\Omega}_s(z')|}\int_{\widehat{\Omega}_s(z')}\mathbb{C}^0e(\tilde{\mathsf{u}}(y))\,dy.$$

Then, we have

$$|\mathbb{C}^0e(\tilde{\mathsf{u}})-\mathcal{M}_1|\leq\frac{C[\nabla\tilde{\mathsf{u}}]_{\alpha,\widehat{\Omega}_s(z')}}{|\widehat{\Omega}_s(z')|}\int_{\widehat{\Omega}_s(z')}|x-y|^\alpha dy\leq C[\nabla\tilde{\mathsf{u}}]_{\alpha,\widehat{\Omega}_s(z')}\big(s^\alpha+\delta(z')^\alpha\big).$$

Combining with (5.7), we have

$$\begin{aligned}&\int_{\widehat{\Omega}_s(z')}|\mathbb{C}^0e(\tilde{\mathsf{u}})-\mathcal{M}_1|^2\,dx\\&\leq C\Big(\delta(z')^{-1}s^{d+1-2\alpha}+\delta(z')^{1-2\alpha}s^{d-1}\Big)(s^{2\alpha}+\delta(z')^{2\alpha})\|g^{(1)}\|^2_{C^{1,\alpha}(\partial D)}.\end{aligned}$$

Then, (5.13) follows from the iteration technique (see proposition 2.1 in [15]). Here, we omit the details.

Step 3. By using the change of variables on $\widehat{\Omega}_{\delta(z')}(z')$ as in [10], and applying $W^{1,p}$ estimates and Schauder's estimates, see theorems 2.3 and 2.4 in [15], then, through rescaling back to the original region $\widehat{\Omega}_{\delta(z')}(z')$, we can obtain that

$$\|\nabla\mathsf{w}\|_{L^\infty(\widehat{\Omega}_{\delta(z')/4}(z'))}\leq\frac{C}{\delta(z')}\left(\delta(z')^{1-\frac{d}{2}}\|\nabla\mathsf{w}\|_{L^2(\widehat{\Omega}_{\delta(z')}(z'))}+\delta(z')^{1+\alpha}[\nabla\tilde{\mathsf{u}}]_{\alpha,\widehat{\Omega}_{\delta(z_1)}(z_1)}\right). \tag{5.15}$$

Substituting (5.13) and (5.7) into (5.15), we have for $(z',x_d)\in\widehat{\Omega}_{\delta(z')/4}(z')$ and $|z'|\leq R_0$,

$$|\nabla\mathsf{w}(z',x_d)|\leq\|\nabla\mathsf{w}\|_{L^\infty(\widehat{\Omega}_{\delta(z')/4}(z'))}\leq C\|g^{(1)}\|_{C^{1,\alpha}(\partial D)}.$$

Thus, (2.14) for $l=1$ is proved.

Combining (2.13) and (2.14), it follows that for $x\in\Omega_{R_0}$,

$$|\partial_d\mathsf{u}_{01}(x)|\leq|\partial_d\tilde{\mathsf{u}}_{01}(x)|+|\nabla\mathsf{w}(x)|\leq\frac{C|\nabla_{x'}g^{(1)}(0')|\cdot|x'|}{\varepsilon+|x'|^{1+\alpha}}+\|g^{(1)}\|_{C^{1,\alpha}(\partial D)},$$

and

$$|\partial_d\mathsf{u}_{01}(x)|\geq|\partial_d\tilde{\mathsf{u}}_{01}(x)|-|\nabla\mathsf{w}(x)|\geq\frac{|g^{(1)}(x',h(x'))|}{C(\varepsilon+|x'|^{1+\alpha})}.$$

Thus, we obtain (2.16). Similarly, (2.15) follows by using (2.12) and (2.14). The proof is completed. □

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