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EXISTENCE AND CONCENTRATION FOR KIRCHHOFF TYPE EQUATIONS AROUND TOPOLOGICALLY CRITICAL POINTS OF THE POTENTIAL

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Abstract. We consider the existence and concentration of solutions for the following Kirchhoff Type Equations

$$-\varepsilon^2 M\left(\varepsilon^{2-N}\int_{\mathbb{R}^N}|\nabla v|^2dx\right)\Delta v + V(x)v = f(v), \quad \text{in } \mathbb{R}^N.$$

Under suitable conditions on the continuous functions M, V and f, we obtain a family of positive solutions concentrating around the local maximum or saddle points of V. Moreover with appropriate assumptions on V, we also have multiple solutions clustering respectively around three kinds of critical points of V.

1. **Introduction.** In this paper, we focus our attention on the following Kirchhoff type equations:

$$\begin{cases} -\varepsilon^2 M\left(\varepsilon^{2-N}\displaystyle\int_{\mathbb{R}^N}|\nabla v|^2dx\right)\Delta v + V(x)v = f(v), \quad \text{in } \mathbb{R}^N \\ v \in H^1(\mathbb{R}^N), \quad v > 0 \end{cases} \tag{1.1}$$

where $N \geq 3$, $\varepsilon > 0$ is a small parameter and M is a positive continuous function. our main purpose is to consider the existence and concentration of positive solutions for (1.1) when the potential $V(x)$ possesses local maxima or saddle points. This problem was raised by Figueiredo et al. in [16]. The equation (1.1) is a nonlocal problem due to the presence of the term $M\big(\varepsilon^{2-N}\int_{\mathbb{R}^N}|\nabla v|^2dx\big)$, then (1.1) is not a pointwise identity.

When $\varepsilon = 1$, $V(x) = 0$, $M(t) = a + bt$ and $\mathbb{R}^N$ is replaced by a smooth bounded domain $\Omega \subset \mathbb{R}^N$ in (1.1), it becomes

$$-\left(a + b\int_\Omega |\nabla v|^2dx\right)\Delta v = f(x, v) \quad \text{in} \quad \Omega. \tag{1.2}$$

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This type of equation is related to the stationary analogue of the equation

$$v_{tt}-\left(a+b\int_{\Omega}|\nabla v|^2dx\right)\Delta v=f(x,v)\quad\text{in}\quad\Omega,\tag{1.3}$$

where v and $f(x,v)$ stand for the displacement and the external force respectively, and b is the initial tension meanwhile a refers to the intrinsic properties of the string (such as Young' s modulus). These kinds of problems are proposed by Kirchhoff in [21] as the extension of the classical D'Alembert wave equation for free vibrations of elastic strings, particularly, considering the effect of changes in length of strings produced by oscillation. the equation (1.3) began to attract attention by various researchers mainly after the work of Lions in [22], in which the author introduced an abstract framework to this problem.

During the past years, there have been many methods to investigate the existence of solution for (1.2). For example, in [29] Ma and Rivera use variational methods to obtain the existence and nonexistence results for a class of Kirchhoff type system. Perera and Zhang [30] prove the nontrivial solutions by Yang index and critical groups. Applying minimax methods and invariant sets of descent flow, Zhang and Perera [37] obtain three kinds of solutions (positive, negative and sign-changing solution). In [9] Chen et al. consider the multiplicity of positive solutions for (1.2) with concave and convex nonlinearities by Nehari manifold and fiber map methods. In [26] Liang et al. investigate the bifurcation phenomena for this equation. For the case of unbounded domain, when $\varepsilon=1$ and $M(t)=a+bt$ in (1.1), Wu [34] prove the existence of a sequence of high energy solution via symmetric mountain pass theorem. For $\varepsilon\equiv 1$, $V(x)\equiv 0$ and M is a positive function in (1.1),

$$-M\left(\int_{\mathbb{R}^N}|\nabla v|^2dx\right)\Delta v=f(x,v)\quad\text{in }\mathbb{R}^N.\tag{1.4}$$

One of the first works about these types of equation, which turn out to be a generalization of (1.2) in some sense, was considered in functional analysis setting by Vasconcellos [32]. Afterwards there are many results of existence about this equation, such as [1, 28] et al. Here we mention [1]. Azzollini [1] used a re-scaling argument to establish a sufficient condition on M for the existence of a solution for (1.4), then the author applied Pohozaev manifold to look for ground states.

When $M(t)\equiv 1$ in (1.1), it becomes the well-known nonlinear schrödinger equation

$$-\varepsilon^2\Delta v+V(x)v=f(v)\quad\text{in }\mathbb{R}^N.\tag{1.5}$$

In the past two decades, a great deal of work has been devoted to the study of semiclassical standing waves for (1.5). Here, we are concerned in this part with the types of the potential $V(x)$, which are related closely to our paper. In [11] Delpino and Felmer devised a penalization approach to show the existence of a single spike solution which concentrates around the local minimum of the potential $V(x)$. After that, they extended this approach in [12] to construct a family solutions with several spikes located around the prescribed finite set of local minimum of $V(x)$. Furthermore, this approach was developed in [13] to exhibit a family of solutions concentrating around any topologically nontrivial critical points of the potential $V(x)$ with arbitrary degeneracy. These types of critical points of the potential are those that can be captured through a local minimax argument. Further, in [14], they dealt with a degenerate saddle point situation for the potential $V(x)$ under the pure power nonlinearity $f(v)=v^p$. Some necessary technical conditions were made for this kind of potential. And the existence result was obtained by formulating min-max arguments on an appropriate set and certain levels of energy functional. Afterwards, in order to simplify the assumption on nonlinearities, Ruiz et al. [3] took a different minimax argument involving suitable deformation on topological cone and constraint set with center of mass equal to zero, then they proved the existence of spikes around the non-degenerate saddle critical point or isolated local maximum point of $V(x)$.

However, compared with the assumptions of nonlinearities, Byeon and Jeanjean [6] constructed a localized deformation argument in a neighborhood of the set of approximate solutions to obtain a family of positive solutions for (1.5), which concentrate to local minima of $V(x)$ under the almost optimal conditions on $f \in C(\mathbb{R})$:

(f1) $\lim_{s\to 0^+} \frac{f(s)}{s} = 0$.

(f2) There exist $C > 0$ and $p \in (1, \frac{N+2}{N-2})$, such that $|f(s)| \le C(1 + |s|^p)$.

(f3) There exists a constant $s_0 > 0$ such that $\frac{1}{2} m s_0^2 < F(s_0)$, where $F(s) = \int_0^s f(t)dt$.

This kind of conditions was introduced by Berestycki and Lions in [5]. Using Pohozaev' s identity, they [5] showed that (f2) and (f3) were necessary for the existence of a nontrivial solution for (1.5) with $\varepsilon \equiv 1$ and $V(x) \equiv m$. A few years later, Byeon and Tanaka [7] developed a more complicated approach to deal with $f \in C^1$ satisfying (f1)-(f3) and the potential $V(x)$ possessing an isolated saddle point or isolated set of local maximum points.

On the other hand, when $M(t) \not\equiv$ const, more precisely, take $N = 3$ and $M(t) = a + bt$ in (1.1) as the following form:

$$-\left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla v|^2 dx\right) \Delta v + V(x)v = f(v) \quad \text{in } \mathbb{R}^3. \tag{1.6}$$

He and Zou [19] used Nehari manifold, Lusternik-Schnirelmann theory and minimax methods to prove the least energy solution around the global minima of $V(x)$ and the multiplicity of solutions. Afterwards Wang et al. [33] used the similar methods to consider the case of sobolev critical growth on the nonlinearity $f(v)$, replaced by $\lambda f(v) + |v|^4 v$. They both require $f(s) = o(s^3)$ and $f(s)/s^3$ is strictly increasing, which guarantee that the energy functional has the mountain pass geometry and the mountain pass level has equivalent characterization. In order to eliminate these conditions, Li and Ye [27] considered (1.6) with $f(v) = |u|^{p-1}u$. Applying monotonicity trick and some compactness arguments, they partially extended the result of [19] with $2 < p < 5$ which holds in larger range. Recently, in [25] Liu and Guo investigate (1.6) with the competing potential functions involving critical sobolev exponent, which was introduced in [36], then they prove the ground state solution concentrating around a global minimum point of $V(x)$.

Since the presence of nonlocal term $M(t) \in C\big([0,\infty)\big)$, (1.1) is more delicate than (1.5). We make the following assumptions:

(M1) For any $t \ge 0$, $M(t) \ge m_0 > 0$.

(M2) $\liminf_{t\to\infty} \left\{\widehat{M}(t) - (1 - \frac{2}{N})M(t)t\right\} = \infty$, where $\widehat{M}(t) = \int_0^t M(t)dt$.

(M3) $\frac{M(t)}{t^{2/(N-2)}} \to 0$ as $t \to \infty$.

(M4) The function $M(t)$ is increasing in $[0,\infty)$.

(M5) The function $t \mapsto \frac{M(t)}{t^{2/(N-2)}}$ is strictly decreasing in $(0,\infty)$.

(M6) $M(t)$ is differentiable and $M(t) + (1 - N/2)M'(t)t \ne 0$.

Some part of the assumptions are initially proposed in [1]. From [1], we note that among these conditions, (M3) plays an important role for the existence of nontrivial solutions for (1.1). Moreover (M4) and (M5) in this paper are slightly different from those of [16]. Furthermore, (M6) is provided that Pohozaev manifold is a natural constraint for energy functional. However, it is easy to find an example of $M(t)$ satisfying all conditions (M1)-(M6), i.e. $M(t) = a_0 + \Sigma_{i=1}^{k} a_i t^{s_i}$, where $k \in \mathbb{N}$, $a_i > 0$ for $0 \leq i \leq k$ and $0 < s_i < \frac{2}{N-2}$ (see [1] for more examples and details).

In [16], Figueiredo et al. investigate (1.1) in the case of local minimum of $V(x)$. More precisely, they assume:

(V1) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ and $\underline{V} := \inf_{x\in\mathbb{R}^N} V(x) > 0$.

(V2′) there exists a bounded open set $\Omega \subset \mathbb{R}^N$ such that

$$V_0 := \inf_{\Omega} V(x) < \inf_{\partial\Omega} V(x),$$

and set $\mathcal{M}' := \{x \in \Omega : V(x) = V_0\}$. Under (f1)-(f3), they obtain a family of solutions for (1.1) concentrating around the set $\mathcal{M}'$ by the approach developed in [6]. In our paper, we intend to consider the existence and concentration of positive solutions for (1.1) when the potential $V(x)$ possesses local maxima or saddle points, which was raised in [16]. Inspired by localized deformation argument on the invariant neighborhood of approximate solutions in [7, 8], we attempt to prove our main result:

Theorem 1.1. *Assume $V(x) \in C^1(\mathbb{R}^N)$ and (V1) for $N \geq 3$. Let $\mathcal{M}$ be the set of an isolated saddle point or an isolated set of local maximum points of $V(x)$. In addition, suppose (M1)-(M6) and $f \in C^1(\mathbb{R})$ with (f1)-(f3). Then there exist $\widetilde{\varepsilon} > 0$ and a family v_ε ($0 < \varepsilon < \widetilde{\varepsilon}$) of positive solution of (1.1) satisfying the following: let x_ε be a maximum point of v_ε,*

(i) $dist(x_\varepsilon, \mathcal{M}) \to 0$ as $\varepsilon \to 0$.

(ii) up to subsequence, $v_\varepsilon(\varepsilon x + x_\varepsilon) \to U$ in $H^1(\mathbb{R}^N)$, where U is the positive least energy solution of

$$-M\left(\int_{\mathbb{R}^N} |\nabla v|^2 dx\right) \Delta v + V(x_0)v = f(v) \quad \text{in } \mathbb{R}^N,$$

where $x_\varepsilon \to x_0 \in \mathcal{M}$.

(iii) There exist $C, c > 0$ such that

$$v_\varepsilon(x) \leq C \exp\big(-\frac{c}{\varepsilon}|x - x_\varepsilon|\big).$$

In [35], Wang observed that concentration of any family of solutions with uniformly bounded energy may occur only at critical points of $V(x)$. However, arbitrary critical points of $V(x)$ are not all candidates for concentration (example given in [35]). These two types of critical points we treat with in theorem 1.1 are those that can be captured by a local minimax characterization in the following assumptions:

(V2) For $k \geq 1$, there exist a connected bounded open set $O \subset \mathbb{R}^N$ with smooth boundary and a $k-1$ dimensional compact manifold $L_0 \subset O$ without boundary such that

$$m_* := \max_{x\in L_0} V(x) < m := \inf_{\mathcal{B}\in\mathcal{L}(L_0), \varphi\in\Lambda_{\mathcal{B}}} \max_{x\in\mathcal{B}} V(\varphi(x)), \tag{1.7}$$

where m is given in (f3), $\mathcal{L}(L_0)$ is a set of connected (orientable) k dimensional compact manifolds $\mathcal{B}$ and their boundary $\partial\mathcal{B}$ are homeomorphic to L_0, and $\Lambda_{\mathcal{B}}$ is a set:

$$\Lambda_{\mathcal{B}} := \{\varphi \in C(\mathcal{B}, O) : \varphi \mid \partial\mathcal{B} \to L_0 \quad \text{is} \quad \text{homeomorphic}\}.$$

In fact, we assume without loss of generality that $x_0 = 0$ is a non-degenerate isolated saddle point of $V(x)$ with Morse index $k \geq 1$ and $V(x_0) = m$, taking the set $\mathcal{M} = \{x_0\}$, we can find a k dimensional connected closed set $L \subset \mathbb{R}^N$ with $\partial L := L_0$ such that

$$x_0 \in L \setminus L_0 \quad \text{and} \quad V(x) < m, \text{ for } x \in L \setminus \mathcal{M}. \tag{1.8}$$

Moreover choose neighborhood $O \subset \mathbb{R}^N$ of L and continuous map π: $O \to L$ such that

$$\begin{aligned} \pi(x) = x \quad & \text{for any } x \in L, \\ V(x) \geq m \quad & \text{if } \pi(x) \in L \cap \mathcal{M}. \end{aligned}$$

Then, for any $\mathcal{B} \in \mathcal{L}(L_0)$ and $\varphi \in \Lambda_{\mathcal{B}}$, we consider mod 2 degree of $\pi \circ \varphi$: $deg_2(\pi \circ \varphi, \mathcal{B}, x)$ in [18]. Since φ: $\partial\mathcal{B} \to L_0$ is homeomorphic and $\pi \circ \varphi(L_0) \subset L_0$, $deg_2(\pi \circ \varphi, \mathcal{B}, x)$ is well defined for any $x \in L \setminus L_0$, then $deg_2(\pi \circ \varphi, \mathcal{B}, x) = 1$ for any $x \in L \setminus L_0$ close to L_0. Since L is connected, we have that $deg_2(\pi \circ \varphi, \mathcal{B}, x)$ is independent of $x \in L \setminus L_0$. Hence $\max\limits_{x \in \mathcal{B}} V\big(\varphi(x)\big) \geq m$. Thus we have (1.7) holds.

On the other hand, if $V(x)$ has an isolated set of local maximum points (possibly degenerate), let $\mathcal{M}$ be a compact set which consists of the local maximum points. Without loss of generality, we assume $V(x) = m$ for any $x \in \mathcal{M}$, and take a connected neighborhood $O \subset \mathbb{R}^N$ of $\mathcal{M}$ such that ∂O is smooth,

$$\mathcal{M} \subset O \setminus \partial O \quad \text{and} \quad V(x) < m \text{ for any } x \in O \setminus \mathcal{M}. \tag{1.9}$$

For $\xi > 0$ small, we define $L := \{x \in O : dist(x, \partial O) \geq \xi\}$ with $L_0 := \partial L$. Then, noting that L_0 is a retract of $O \setminus L$, there exists a retract continuous map π: $O \to L$ such that $\pi(x) = x$ for any $x \in L$. Hence (V2) also holds by degree argument.

In [16], from the assumption (V2$'$), we observe that the lower bound estimate for the energy functional corresponding to the limit equation can be obtained by the monotonicity with respect to the potential's value. Yet, this property does not hold for the two types of potentials we will cope with. Thus it suffice to explore the more subtle properties from the potentials. Fortunately, from the saddle point and set of local maximum points in the potential $V(x)$ is isolated, there exists the quantitative deformation flow for potentials near the critical points:

(V3) Taking a compact set $\mathcal{M} \subset \{x \in O : V(x) = m\}$ such that for any $d > 0$, $\mathcal{M}^d \subset O$. Then there exist $\beta > 0$, and an open set Ω with $\mathcal{M} \subset \Omega \subset \Omega^{10\beta} \subset \mathcal{M}^d$. And there exist positive constants $a, \mu, c_1, c_2 > 0$ and a map $\Psi \in C([0,1] \times O, O)$ satisfying:

(i) for each $x \in O$, $V\big(\Psi(t,x)\big)$ is non-increasing for $t \in [0,1]$.

(ii) $\Psi(t,x) = x$ for $t = 0$ or $x \in L_0$.

(iii) $|\Psi(t_1,x) - \Psi(t_2,x)| \leq \mu|t_1 - t_2|$ for $t_1, t_2 \in [0,1]$, $x \in O$.

(iv)

$$\limsup_{h \to 0^+} \frac{V\big(\Psi(t+h,x)+y\big) - V\big(\Psi(t,x)+y\big)}{h} \leq -a,$$

uniformly for $|y| \leq c_2$, $t \in [0,1)$ and $x \in \big(\Omega^{10\beta} \cup V_{m-c_1}^{m+c_1}\big) \setminus \Omega$, where $V_{m-c_1}^{m+c_1} := \{x \in O : m - c_1 \leq V(x) \leq m + c_1\}$ and $\mathcal{M}^d := \{x \in \mathbb{R}^N : \inf_{y \in \mathcal{M}} |x - y| \leq d\}$.

For the existence of Ψ, see [7] for details. Thus we turn to prove the following theorem:

Theorem 1.2. *Assume (M1)-(M6), (V1)-(V3) and $f \in C^1(\mathbb{R})$ with* (f1)-(f3) *for $N \geq 3$. Then there exists $\widetilde{\varepsilon} > 0$ such that for $0 < \varepsilon < \widetilde{\varepsilon}$, the conclusions of theorem 1.1 still hold.*

Due to the above results and the results of [16], we have an interesting consequence of our main theorems:

Corollary 1.3. *Assume (M1)-(M6), $V(x) \in C^1(\mathbb{R}^N)$ with (V1) and $f \in C^1(\mathbb{R})$ with (f1)-(f2) for $N \geq 3$. If there exist mutually disjoint bounded domains Ω_i, $i = 1, ..., k$ and constants $0 < c_1 < c_2 < \cdots < c_k$ such that*

(i) for some $i \in \{1, 2, ..., k\}$, (V2′) hold on Ω_i, i.e.

$$c_i := \inf_{\Omega_i} V(x) < \inf_{\partial\Omega_i} V(x);$$

(ii) for some $i \in \{1, 2, ..., k\}$,

$$c_i := \sup_{\Omega_i} V(x) > \sup_{\partial\Omega_i} V(x);$$

(iii) for some $i \in \{1, 2, ..., k\}$, Ω_i possess a non-degenerate saddle point $x^{(i)} \in \Omega_i$ with $V(x^{(i)}) = c_i$.

(iv) There exist constants $s_i > 0$ such that $\frac{1}{2}c_i s_i^2 < F(s_i)$ for each i.

Then there exists $\widetilde{\varepsilon} > 0$ such that for $0 < \varepsilon < \widetilde{\varepsilon}$,

(i) there exist at least k families of positive solutions $v_\varepsilon^{(i)}$ for (1.1), $i = 1, ..., k$;

(ii) let $x_\varepsilon^{(i)}$ be a maximum point of $v_\varepsilon^{(i)}$,

$$\lim_{\varepsilon \to 0} V(x_\varepsilon^{(i)}) = c_i;$$

(iii) up to subsequence, $v_\varepsilon^{(i)} \to U^{(i)}$ in $H^1(\mathbb{R}^N)$ as $\varepsilon \to 0$, where $U^{(i)}$ is the least energy solution of

$$-M\left(\int_{\mathbb{R}^N} |\nabla v|^2 dx\right) \Delta v + c_i v = f(v) \quad \text{in } \mathbb{R}^N;$$

(iv) there exist $C, c > 0$ such that

$$v_\varepsilon^{(i)}(x) \leq C \exp\big(-\frac{c}{\varepsilon}|x - x_\varepsilon^{(i)}|\big).$$

We remark that the solutions obtained in corollary 1.3 can be separated when ε is small, since Ω_i are mutually disjoint. Moreover corollary 1.3 describes a kind of multiple concentrating phenomena.

Finally, noting that [8, 15], It seems interesting to consider whether one can find a family of positive or sign-changing solutions with multi-peaks which cluster around a local set of maximum points or saddle points of $V(x)$.

The proof of our results relies on variational arguments and is organized as follows. In section 2, we first introduce the center of mass which is used to estimate concentration property of functions and to define the localized neighborhood of approximate solutions in subsection 2.3. Then we estimate the energy function and its norm of gradient in the subsection 2.4. Next in the following subsections, three maps on the localized neighborhood are defined and some related properties are proved. In section 3, we iterate the three maps to obtain the estimates of the energy functional and prove the theorem.

2. **Preliminaries.** We observe that defining $u(x) = v(\varepsilon x)$, the equation (1.1) is equivalent to

$$-M\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u + V(\varepsilon x)u = f(u). \tag{2.1}$$

In what follows we will focus on this equivalent problem. Now we introduce the notation: for each $\varepsilon > 0$, we define H_ε by

$$H_\varepsilon := \left\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon x)u^2 dx < \infty\right\},$$

with its norm:

$$\|u\|_\varepsilon := \left(\int_{\mathbb{R}^N} |\nabla u|^2 + V(\varepsilon x)u^2 dx\right)^{1/2}.$$

Let H_ε^* be the dual space of H_ε with its norm

$$\|v\|_\varepsilon^* := \sup_{u\in H_\varepsilon, \|u\|_\varepsilon\le 1} |\langle v,u\rangle| \quad \text{for} \quad v \in H_\varepsilon^*,$$

where $\langle v,u\rangle$ is the duality product between H_ε^* and H_ε. And we use $\|\cdot\|$ and $\|\cdot\|_r$ to denote the norm of $H^1(\mathbb{R}^N)$ and $L^r(\mathbb{R}^N)$ by

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + m|u|^2 dx\right)^{1/2} \quad \text{and} \quad \|u\|_r := \left(\int_{\mathbb{R}^N} |u|^r dx\right)^{1/r} \quad \text{for } r \in [1,\infty).$$

From (V1), we observe that $H_\varepsilon \subset H^1(\mathbb{R}^N)$. For any set $A \subset \mathbb{R}^N$ and $\varepsilon > 0$, we define $A_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in A\}$. And we define $A^\beta := \{x \in \mathbb{R}^N : \inf_{y\in A}|x-y| \le \beta\}$. Similarly, for $\mathcal{A} \subset H_\varepsilon$, we use the notation:

$$N_\delta(\mathcal{A}) := \{u \in H_\varepsilon : dist_\varepsilon(u,\mathcal{A}) \le \delta\}, \text{ where } \quad dist_\varepsilon(u,\mathcal{A}) = \inf_{v\in\mathcal{A}} \|u-v\|_\varepsilon.$$

We look for critical points of the functional $\Gamma_\varepsilon(u) \in C^1(H_\varepsilon,\mathbb{R})$ defined by

$$\Gamma_\varepsilon(u) = \frac{1}{2}\widehat{M}(\|\nabla u\|_2^2) + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)u^2 dx - \int_{\mathbb{R}^N} F(u)dx \tag{2.2}$$

The critical points of Γ_ε are clearly solutions of (2.1). We assume that $f(t) = 0$ for $t \le 0$. Indeed, it follows from Bony maximum principle [23] that any nontrivial solution of (2.1) is positive.

2.1. **Limit equations.** For any $a > 0$, we define a functional $L_a \in C^1(H^1(\mathbb{R}^N),\mathbb{R})$ by

$$L_a(u) = \frac{1}{2}\widehat{M}(\|\nabla u\|_2^2) + \frac{a}{2}\int_{\mathbb{R}^N} u^2 dx - \int_{\mathbb{R}^N} F(u)dx, \tag{2.3}$$

which is associated to the limit equation

$$-M\left(\|\nabla u\|_2^2\right)\Delta u + au = f(u). \tag{2.4}$$

We note that in (f3), it follows from continuity that there exist $m_1, m_2 > 0$ with $m_1 < m < m_2$ such that for any $m \in [m_1,m_2]$, (f3) still holds. We define the least energy value and the set of least energy solutions for (2.4) by:

$$E_a = \inf\{L_a(u) : L_a'(u) = 0, \quad u \ne 0\}.$$

$$S_a = \left\{U \in H^1(\mathbb{R}^N)\backslash\{0\} : L_a'(U) = 0, \quad L_a(U) = E_a, \quad U(0) = \max_{x\in\mathbb{R}^N} U(x)\right\}.$$

In [16], the existence of the least energy solution of (2.4) is proved, when (M1)-(M5) and (f1)-(f3) are satisfied. It is also showed that any solution of (2.4) satisfies the Pohozaev identity:

$$P(u) := \frac{N-2}{2}M\left(\|\nabla u\|_2^2\right)\|\nabla u\|_2^2 + N\int_{\mathbb{R}^N} \frac{a}{2}u^2 - F(u)dx = 0. \tag{2.5}$$

We define $S_a^m := \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : L_a'(u) = 0,\ E_a \le L_a(u) \le E_m\}$ for $a \in [m_1,m]$ and $S_a^m := S_m$ for $a \in [m,m_2]$. Denoting $\underline{m}(d) := \inf\{V(x) : x \in \mathcal{M}^d\}$ and $\overline{m}(d) := \sup\{V(x) : x \in \mathcal{M}^d\}$, we choose $d > 0$ small enough such that

$$m_1 \le \underline{m}(d) < m \le \overline{m}(d) \le m_2.$$

Then we define

$$\widehat{S} = \bigcup_{a\in[\underline{m}(d),\overline{m}(d)]} S_a^m.$$

From the proof of proposition 2.19 in [16], we see that $\widehat{S}$ is compact in $H^1(\mathbb{R}^N)$ and there exist $C, c > 0$ such that for any $U \in \widehat{S}$,

$$U(x) + |\nabla U(x)| \le C\exp(-c|x|) \quad x \in \mathbb{R}^N. \tag{2.6}$$

We choose a radially symmetric function $\phi_\varepsilon \in C_0^\infty(\mathbb{R}^N)$ such that $\phi_\varepsilon(x) = 1$ for $|x| \le 1/2\varepsilon^{\frac{1}{3}}$, $\phi_\varepsilon(x) = 0$ for $|x| \ge 1/\varepsilon^{\frac{1}{3}}$ and $|\nabla\phi_\varepsilon| \le 3\varepsilon^{\frac{1}{3}}$. Then we define

$$\mathcal{Z}_\varepsilon^{10\beta} = \left\{\phi_\varepsilon(\cdot - \frac{x}{\varepsilon})U(\cdot - \frac{x}{\varepsilon}) : x \in \Omega^{10\beta},\ U \in \widehat{S}\right\}.$$

By the compactness of $\widehat{S}$, $\mathcal{Z}_\varepsilon^{10\beta}$ is also compact in H_ε. For $\delta > 0$, we denote the neighborhood of $\mathcal{Z}_\varepsilon^{10\beta}$ by

$$N_\delta(\mathcal{Z}_\varepsilon^{10\beta}) = \left\{u \in H_\varepsilon : \mathrm{dist}_\varepsilon(u, \mathcal{Z}_\varepsilon^{10\beta}) \le \delta\right\}.$$

2.2. **Center of mass in $N_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$.** Following [7, 8], we introduce a center of mass in $N_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$. Let us define the set: for $z \in \mathbb{R}^N$,

$$S(z) := \left\{(\phi_\varepsilon U)(\cdot - z) : U \in \widehat{S}\right\}.$$

Then it is easy to see that $\mathcal{Z}_\varepsilon^{10\beta} = \bigcup_{z\in\Omega_\varepsilon^{10\beta}} S(z)$. Next we introduce the map of center of mass, which is used to estimate the concentration property of functions.

Lemma 2.1. *There exist $\delta, R_0 > 0$ and a map $\Upsilon_\varepsilon : N_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta}) \to \mathbb{R}^N$ such that $|\Upsilon_\varepsilon(u) - y| \le R_0$, for all $u(x) = (\phi_\varepsilon U)(\cdot - y) + w \in N_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$ with $U \in \widehat{S}$, $y \in \Omega_\varepsilon^{10\beta}$ and $\|w\|_\varepsilon \le 2\delta$.*

Proof. Set $r_* = \min_{u\in\mathcal{Z}_\varepsilon^{10\beta}} \|u\|_\varepsilon$ and choose $R_0 > 1$ such that for $u \in S(0)$,

$$\|u\|_{H_\varepsilon(|x|\le R_0/2)} > \frac{3}{4}r_* \quad \text{and} \quad \|u\|_{H_\varepsilon(|x|\ge R_0/2)} < \frac{1}{8}r_*.$$

This is possible by the uniform exponential decay. We take $\delta > 0$ small such that

$$r_* \ge 16\delta. \tag{2.7}$$

Choose $\sigma(t) \in C_0^\infty(\mathbb{R}, [0, 1])$ such that $\sigma(t) = 1$ for $|t| \le 2\delta$ and $\sigma(t) = 0$ for $|t| \ge 3\delta$. Then we define the map Υ_ε:

$$\Upsilon_\varepsilon(u) = \frac{\displaystyle\int_{\Omega_\varepsilon^{10\beta}} \sigma(dist_\varepsilon(u, S(z)))z\,dz}{\displaystyle\int_{\Omega_\varepsilon^{10\beta}} \sigma(dist_\varepsilon(u, S(z)))\,dz}.$$

For $v_z \in S(z)$ and $|z - y| \ge R_0$, we have

$$\begin{aligned}\|u - v_z\|_\varepsilon &\ge \|(\phi_\varepsilon U)(\cdot - y)\|_{H_\varepsilon(|x-y|\le R_0)} - \|v_z\|_{H_\varepsilon(|x-z|\ge R_0)} - \|w\|_\varepsilon \\ &\ge \frac{3}{4}r_* - \frac{1}{8}r_* - 2\delta \ge 3\delta.\end{aligned}$$

This implies that $\sigma(dist_\varepsilon(u, S(z))) = 0$ for $|y - z| \ge R_0$. Hence $\Upsilon_\varepsilon(u) \in B(y, R_0)$. □

2.3. **New invariant neighborhoods $\widehat{N}_r(\mathcal{Z}_\varepsilon^{10\beta})$.** Now following the idea in [7], we introduce a new invariant neighborhood $\widehat{N}_r(\mathcal{Z}_\varepsilon^{10\beta})$, which is the refinement of $N_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$ and is different from [7] due to the presence of the nonlocal term M. This set is invariant under the maps which will be introduced in the back. We first denote $V_\varepsilon := V(\varepsilon x)$ and

$$|w|_{\varepsilon,u} := \int_{|x-\Upsilon_\varepsilon(u)|\le 1/\sqrt{\varepsilon}} |\nabla w|^2 + V_\varepsilon w^2 dx,$$

for $u \in N_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$ and $w \in H_\varepsilon$. Then we first define, for $r \in (0, 2\delta]$

$$\begin{aligned} G_r(\mathcal{Z}_\varepsilon^{10\beta}) := \Big\{ & u \in N_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta}) : |u - v_z|_{\varepsilon,u} \le \frac{r^2}{2} \quad \text{and} \\ & \int_{D_\varepsilon} |\nabla u|^2 + V_\varepsilon u^2 - 2F(u) dx \le \frac{r^2}{2} \Big\}, \end{aligned}$$

where $v_z \in S(z)$ and $z \in \Omega_\varepsilon^{10\beta}$, $D_\varepsilon = \{x \in \mathbb{R}^N : |x - \Upsilon_\varepsilon(u)| \ge 1/\sqrt{\varepsilon}\}$. Next we also define that

$$\begin{aligned} G_r^M(\mathcal{Z}_\varepsilon^{10\beta}) := \Big\{ & u \in N_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta}) : |u - v_z|_{\varepsilon,u} \le \frac{r^2}{2} \quad \text{and} \\ & \int_{\alpha_{\varepsilon,u}}^{\alpha_{\varepsilon,u} + \int_{D_\varepsilon} |\nabla u|^2 dx} M(t)dt + \int_{D_\varepsilon} V_\varepsilon u^2 dx - 2\int_{D_\varepsilon} F(u) dx \le \frac{r^2}{2} \min\{m_0, 1\} \Big\}, \end{aligned}$$

where $\alpha_{\varepsilon,u} = \int_{|x-\Upsilon_\varepsilon(u)|\le 1/\sqrt{\varepsilon}} |\nabla u|^2 dx$. Thus we denote

$$\widehat{N}_r(\mathcal{Z}_\varepsilon^{10\beta}) := G_r^M(\mathcal{Z}_\varepsilon^{10\beta}) \cup G_r(\mathcal{Z}_\varepsilon^{10\beta}).$$

From the definition, there is some relationship between the neighborhoods $N_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$ and $\widehat{N}_r(\mathcal{Z}_\varepsilon^{10\beta})$. For simplicity, we denote

$$\|u\|_{M,D_\varepsilon}^2 := \int_{\alpha_{\varepsilon,u}}^{\alpha_{\varepsilon,u} + \int_{D_\varepsilon} |\nabla u|^2 dx} M(t)dt + \int_{D_\varepsilon} V_\varepsilon u^2 dx.$$

Lemma 2.2. *For $c, c' \in (0, 1]$ and $\delta > 0$, there exist $q := q(\delta) > 0$ such that $q(\delta) \to 0$ as $\delta \to 0$, and for small $\varepsilon > 0$ independent of δ,*

$$\widehat{N}_{(1-q)c\delta}(\mathcal{Z}_\varepsilon^{10\beta}) \subset N_{c\delta}(\mathcal{Z}_\varepsilon^{10\beta}) \quad \textit{and} \quad N_{c'\delta}(\mathcal{Z}_\varepsilon^{10\beta}) \subset \widehat{N}_{(1+q)\sqrt{2}c'\delta}(\mathcal{Z}_\varepsilon^{10\beta}).$$

Proof. First, we need to verify that there exist $q := q(\delta) > 0$ such that for small $\varepsilon > 0$ and $u \in N_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$, one has

$$\int_{D_\varepsilon} F(u) dx \le \frac{1}{2} q(\delta) \|u\|_{M,D_\varepsilon}^2, \tag{2.8}$$

where $q(\delta) \to 0$ as $\delta \to 0$.

In fact, set $u = (\phi_\varepsilon U)(\cdot - y) + w$ with $U \in \widehat{S}$, $y \in \Omega_\varepsilon^{10\beta}$ and $\|w\|_\varepsilon \le 2\delta$. By (2.6) and lemma 2.1, we have

$$\lim_{R\to\infty} \int_{|x-\Upsilon_\varepsilon(u)|\ge R} |\nabla(\phi_\varepsilon U)(\cdot - y)|^2 + V_\varepsilon (\phi_\varepsilon U)^2(\cdot - y) dx = 0.$$

Hence there exists $R' > R_0$ such that

$$\int_{|x-\Upsilon_\varepsilon(u)|\ge R'} |\nabla u|^2 + V_\varepsilon u^2 dx \le (4\delta)^2.$$

Moreover by (f1) and (f2), one has for any $\eta > 0$, there exists $C_\eta > 0$ such that $F(u) \leq \eta u^2 + C_\eta |u|^{p+1}$. Thus by sobolev inequality and (M1), we have for small $\varepsilon > 0$,

$$\int_{D_\varepsilon} |u|^{p+1} dx \leq C \left(\int_{D_\varepsilon} |\nabla u|^2 + V_\varepsilon u^2 dx \right)^{\frac{p+1}{2}} \leq C \max\{1, m_0^{-1}\} (4\delta)^{p-1} \|u\|^2_{M, D_\varepsilon}.$$

Then we complete the proof of (2.8). Moreover we observe that $\int_{D_\varepsilon} F(u) dx \leq \frac{1}{2} q(\delta) \|u\|^2_{H_\varepsilon(D_\varepsilon)}$ still holds for the same $q(\delta)$ in (2.8). Hence, for $u \in N_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$, one has

$$(1-q) \int_{D_\varepsilon} |\nabla u|^2 + V_\varepsilon u^2 dx \leq \int_{D_\varepsilon} |\nabla u|^2 + V_\varepsilon u^2 - 2F(u) dx,$$

and

$$\begin{aligned} &(1-q) \min\{1, m_0\} \int_{D_\varepsilon} |\nabla u|^2 + V_\varepsilon u^2 dx \\ \leq &\int_{\alpha_{\varepsilon,u}}^{\alpha_{\varepsilon,u} + \int_{D_\varepsilon} |\nabla u|^2 dx} M(t) dt + \int_{D_\varepsilon} V_\varepsilon u^2 dx - 2 \int_{D_\varepsilon} F(u) dx. \end{aligned}$$

Then for $u \in \widehat{N}_{(1-q)c\delta}(\mathcal{Z}_\varepsilon^{10\beta})$, we have $\int_{D_\varepsilon} |\nabla u|^2 + V_\varepsilon u^2 dx \leq \frac{1}{2}(1-q)(c\delta)^2$. Note that $|u - v_z|_{\varepsilon,u} \leq ((1-q)c\delta)^2/2$, where $v_z \in S(z)$ and $z \in \Omega_\varepsilon^{10\beta}$. By lemma 2.1, $|\Upsilon_\varepsilon(u) - z| \leq R_0$. Then for small $\varepsilon > 0$, we have $\text{supp}\{v_z\} \subset B(\Upsilon_\varepsilon(u), 1/\sqrt{\varepsilon})$. Hence for small $\varepsilon > 0$

$$\|u - v_z\|_\varepsilon^2 = |u - v_z|_{\varepsilon,u} + \int_{D_\varepsilon} |\nabla u|^2 + V_\varepsilon u^2 dx \leq \frac{1}{2}((1-q)c\delta)^2 + \frac{1}{2}(1-q)(c\delta)^2 \leq (c\delta)^2.$$

On the other hand for $u \in N_{c'\delta}(\mathcal{Z}_\varepsilon^{10\beta})$, there exist $v_z \in S(z)$, $z \in \Omega_\varepsilon^{10\beta}$ such that $\|u - v_z\|_\varepsilon \leq c'\delta$. Then $|u - v_z|_{\varepsilon,u} \leq (c'\delta)^2$. For $\varepsilon > 0$ small, we have

$$\text{supp}\{v_z\} \subset B(\Upsilon_\varepsilon(u), 1/\sqrt{\varepsilon}) \quad \text{and} \quad \int_{D_\varepsilon} |\nabla u|^2 + V_\varepsilon u^2 dx \leq (c'\delta)^2.$$

Thus

$$\int_{D_\varepsilon} |\nabla u|^2 + V_\varepsilon u^2 - 2F(u) dx \leq \big((1+q)(c'\delta)\big)^2.$$

Hence $u \in \widehat{N}_{(1+q)\sqrt{2}c'\delta}(\mathcal{Z}_\varepsilon^{10\beta})$. □

Here we choose appropriate $\delta > 0$ such that $1 + q < \sqrt{2}$.

2.4. **Energy and gradient estimates.** In this part, we first use the least energy solution of limit equation to construct a starting surface, which is utilized to estimate the energy functional (2.2). From (1.7), there exist a manifold $\mathcal{L}_\varepsilon \in \mathcal{L}(L_0)$ and a map $\varphi_\varepsilon \in \Lambda_{\mathcal{L}_\varepsilon} \subset C(\mathcal{L}_\varepsilon, O)$ such that $V\big(\varphi_\varepsilon(z)\big) \leq m + \varepsilon$ for any $z \in \mathcal{L}_\varepsilon$. From the definition of $\Lambda_{\mathcal{B}}$ in (V2), we can assume $\partial \mathcal{L}_\varepsilon = L_0$ and $\varphi_\varepsilon(z) = z$, $\forall z \in L_0$. From (V3), there exist $0 < a_0 < \frac{1}{2}(m - m_*)$ and a continuous map $\Psi : [0,1] \times O \mapsto O$ such that for $\varepsilon > 0$ small

$$\Psi(1, \varphi_\varepsilon(z)) = \Psi(1, z) = z, \quad \text{for any } z \in L_0;$$

$$V\big(\Psi(1, \varphi_\varepsilon(z))\big) \leq m + \varepsilon, \quad \text{for any } z \in \mathcal{L}_\varepsilon; \tag{2.9}$$

$$V\big(\Psi(1, \varphi_\varepsilon(z))\big) \leq m - a_0, \quad \text{if } \Psi(1, \varphi_\varepsilon(z)) \notin \Omega^\beta. \tag{2.10}$$

Then we define the starting surface by $\gamma_\varepsilon(z) := \Psi(1, \varphi_\varepsilon(z))$ for $z \in \mathcal{L}_\varepsilon$. Furthermore we define an initial path map $A_\varepsilon(t, z) : (0, \infty) \times \mathcal{L}_\varepsilon \mapsto H_\varepsilon$ by

$$A_\varepsilon(t, z)(x) := (\phi_\varepsilon U)\Big(\frac{x}{t} - \frac{\gamma_\varepsilon(z)}{t\varepsilon}\Big),$$

where $U \in S_m$. Next we have the following energy estimate for $\Gamma_\varepsilon\big(A_\varepsilon(t, z)\big)$:

Lemma 2.3. *Assume (M1)-(M5) hold. (i) There exists $T > 1$ such that $\Gamma_\varepsilon\big(A_\varepsilon(T, z)\big) < 0$.*
(ii) $\lim\limits_{\varepsilon\to 0} \max\limits_{t\in[0,T],\ z\in\mathcal{L}_\varepsilon} \Gamma_\varepsilon\big(A_\varepsilon(t, z)\big) \le E_m$.
(iii) For any $\xi > 0$, $\limsup\limits_{\varepsilon\to 0} \max\limits_{z\in\mathcal{L}_\varepsilon} \{\Gamma_\varepsilon(A_\varepsilon(t, z)) : t \in [0, T] \setminus (1 - \xi, 1 + \xi)\} < E_m$.
(iv) $\limsup\limits_{\varepsilon\to 0} \max\limits_{t\in[0,T]} \big\{\Gamma_\varepsilon(A_\varepsilon(t, z)) : z \in \mathcal{L}_\varepsilon,\ \gamma_\varepsilon(z) \notin \Omega^\beta\big\} < E_m$.

Proof. Through changes of variable, we have

$$\begin{aligned}\Gamma_\varepsilon(A_\varepsilon\big(t, z)\big) =& \frac{1}{2}\widehat{M}\big(t^{N-2}\|\nabla(\phi_\varepsilon U)\|_2^2\big) + \frac{t^N}{2}\int_{\mathbb{R}^N} V\big(\varepsilon t x + \gamma_\varepsilon(z)\big)(\phi_\varepsilon U)^2 dx\\ &- t^N \int_{\mathbb{R}^N} F(\phi_\varepsilon U)dx.\end{aligned}$$

It follows from exponential decay of U and (2.9) that for ε small,

$$\begin{aligned}\Gamma_\varepsilon(A_\varepsilon\big(t, z)\big) &= \frac{1}{2}\widehat{M}\big(t^{N-2}\|\nabla U\|_2^2\big) + t^N \int_{\mathbb{R}^N} \frac{V\big(\gamma_\varepsilon(z)\big)}{2} U^2 - F(U)dx + o(1)\\ &\le t^N \left(\frac{\widehat{M}\big(t^{N-2}\|\nabla U\|_2^2\big)}{2t^N} + \int_{\mathbb{R}^N} \frac{m}{2} U^2 - F(U)dx\right) + o(1).\end{aligned}$$

By (f3), (M3) and changes of variable, we note that $\Gamma_\varepsilon(A_\varepsilon\big(t, z)\big) \to -\infty$, as $t \to +\infty$. Then there exists $T > 1$ such that $\Gamma_\varepsilon(A_\varepsilon\big(T, z) < 0$ for any $z \in \mathcal{L}_\varepsilon$.

Moreover, we denote

$$L(t) := \frac{1}{2}\widehat{M}\big(t^{N-2}\|\nabla U\|_2^2\big) + t^N \int_{\mathbb{R}^N} \frac{m}{2} U^2 - F(U)dx.$$

Differentiating $L(t)$, we have

$$\frac{d}{dt}L(t) = t^{N-1}\left\{\frac{N-2}{2}\cdot\frac{M\big(t^{N-2}\|\nabla U\|_2^2\big)\|\nabla U\|_2^2}{t^2} + N\int_{\mathbb{R}^N} \frac{m}{2}U^2 - F(U)dx\right\}.$$

By (M5), (f3), (2.5) and changes of variable, one has that

$$\frac{dL(1)}{dt} = 0, \quad \frac{dL(t)}{dt} > 0 \ for\ t \in (0, 1) \quad \text{and} \quad \frac{dL(t)}{dt} < 0 \ for\ t \in (1, +\infty).$$

Thus from $U \in S_m$ and (2.3), we have

$$\max_{t\in[0,\infty)} L(t) = L(1) = L_m(U) = E_m.$$

Hence (ii) and (iii) holds.

If $\gamma_\varepsilon(z) \notin \Omega^\beta$, it follows from (2.10) that for ε small,

$$\Gamma_\varepsilon(A_\varepsilon\big(t, z)\big) \le L(t) - \frac{\alpha_0}{2}t^N \int_{\mathbb{R}^N} U^2 dx + o(1) < \max_{t\in[0,\infty)} L(t) = E_m.$$

Above all, we complete the proof of this lemma. □

From the proof of this lemma, we define $A_\varepsilon(0,z)=0$. Then $A_\varepsilon(t,z)$: $[0,T]\times\mathcal{L}_\varepsilon\to H_\varepsilon$ is continuous. Next we estimate the gradient's lower bound of the energy functional on the annular neighborhood of approximate solution. We define

$$C_\varepsilon=\max_{t\in[0,T],z\in\mathcal{L}_\varepsilon}\Gamma_\varepsilon\big(A_\varepsilon(t,z)\big). \tag{2.11}$$

From lemma 2.3, we observe that $\lim\limits_{\varepsilon\to0}C_\varepsilon\le E_m$.

Lemma 2.4. *For some $\delta>0$ and any $\delta'\in(0,\delta)$, there exists $\mu=\mu(\delta,\delta')>0$ such that for small $\varepsilon>0$ independent of μ,*

$$\inf\Big\{\|\Gamma_\varepsilon'(u)\|_\varepsilon^*:\ u\in\Gamma_\varepsilon^{C_\varepsilon}\cap\big(\widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})\setminus\widehat{N}_{\delta'}(\mathcal{Z}_\varepsilon^{10\beta})\big),\ \Upsilon_\varepsilon(u)\in\Omega_\varepsilon^{9\beta}\Big\}>\mu(\delta,\delta').$$

Proof. Assume on the contrary that for some $\delta'\in(0,\delta)$, there exists $u_\varepsilon\in\Gamma_\varepsilon^{C_\varepsilon}\cap\big(\widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})\setminus\widehat{N}_{\delta'}(\mathcal{Z}_\varepsilon^{10\beta})\big)$ with $\Upsilon_\varepsilon(u_\varepsilon)\in\Omega_\varepsilon^{9\beta}$ such that $\Gamma_\varepsilon'(u_\varepsilon)\to0$ as $\varepsilon\to0$. From lemma 2.1, there exist $x_\varepsilon\in\Omega^{10\beta}$ and $U\in\widehat{S}$ such that $|\Upsilon_\varepsilon(u_\varepsilon)-x_\varepsilon/\varepsilon|\le R_0$ and $\|u_\varepsilon-(\phi_\varepsilon U)(\cdot-x_\varepsilon/\varepsilon)\|\le2\delta$. Hence $dist(x_\varepsilon,\Omega^{9\beta})\le\varepsilon R_0$ and $x_\varepsilon\to x_0\in\overline{\Omega^{9\beta}}$.

Let $k_\varepsilon^2\le\frac13\varepsilon^{-1/3}$ and $k_\varepsilon\to\infty$ as $\varepsilon\to0$. It follows from $supp\phi_\varepsilon\subset B(0,\varepsilon^{-1/3})$ that

$$\sum_{j=0}^{k_\varepsilon-1}\int_{D_{\varepsilon,j}}|\nabla u_\varepsilon|^2+V_\varepsilon u_\varepsilon^2dx\le\int_{\mathbb{R}^N\setminus B(\frac{x_\varepsilon}{\varepsilon},\varepsilon^{-\frac13})}|\nabla u_\varepsilon|^2+V_\varepsilon u_\varepsilon^2dx\le4\delta^2,$$

where $D_{\varepsilon,j}=\{x\in\mathbb{R}^N:\varepsilon^{-\frac13}+3jk_\varepsilon\le|x-\frac{x_\varepsilon}{\varepsilon}|\le\varepsilon^{-\frac13}+3(j+1)k_\varepsilon\}$ for $j=0,1,...,k_\varepsilon-1$. Hence there exist $j_\varepsilon\in\{0,1,...,k_\varepsilon-1\}$ such that

$$\int_{D_{\varepsilon,j_\varepsilon}}|\nabla u_\varepsilon|^2+V_\varepsilon u_\varepsilon^2dx\le\frac{4\delta^2}{k_\varepsilon}\to0. \tag{2.12}$$

We choose $\chi_\varepsilon(x)\in C_0^\infty(\mathbb{R}^N,[0,1])$ such that $\chi_\varepsilon(x)=1$ for $|x|\le\varepsilon^{-\frac13}+(3j_\varepsilon+1)k_\varepsilon$ and $\chi_\varepsilon(x)=0$ for $|x|\ge\varepsilon^{-\frac13}+(3j_\varepsilon+2)k_\varepsilon$ and $|\nabla\chi_\varepsilon(x)|\le\varepsilon^{\frac13}$. Then we define $u_\varepsilon^{(1)}(x)=\chi_\varepsilon(x-\frac{x_\varepsilon}{\varepsilon})u_\varepsilon(x)$ and $u_\varepsilon^{(2)}(x)=u_\varepsilon(x)-u_\varepsilon^{(1)}(x)$.

Next we prove that $\|u_\varepsilon^{(2)}\|_\varepsilon\to0$. In fact from $\Gamma_\varepsilon'(u_\varepsilon)\to0$, (2.12) and the proof of (2.8), we have

$$\begin{aligned}\Gamma_\varepsilon'(u_\varepsilon)u_\varepsilon^{(2)}&=M\big(\|\nabla u_\varepsilon\|_2^2\big)\int_{\mathbb{R}^N}|\nabla u_\varepsilon^{(2)}|^2dx+\int_{\mathbb{R}^N}V_\varepsilon\big(u_\varepsilon^{(2)}\big)^2dx-\int_{\mathbb{R}^N}f(u_\varepsilon^{(2)})u_\varepsilon^{(2)}dx+o(1)\\&\ge\min\{1,m_0\}\big(1-C(2\delta)^{p-1}\big)\|u_\varepsilon^{(2)}\|_\varepsilon^2,\end{aligned}$$

then for some $\delta>0$ small enough, we have $\|u_\varepsilon^{(2)}\|_\varepsilon\to0$. Thus by $\Gamma_\varepsilon\in C^1$ and (2.12), we can have $\Gamma_\varepsilon'(u_\varepsilon^{(1)})\to0$. Denote $\widetilde{u}_\varepsilon^{(1)}(x)=u_\varepsilon^{(1)}(x+\frac{x_\varepsilon}{\varepsilon})$. From the boundedness of u_ε, we have $\widetilde{u}_\varepsilon^{(1)}\rightharpoonup W$ in $H^1(\mathbb{R}^N)$ and W solves the equation

$$-\alpha_0\Delta W+V(x_0)W=f(W)\quad\text{in }\mathbb{R}^N, \tag{2.13}$$

where $\alpha_0=\lim\limits_{\varepsilon\to0}M(\|\nabla u_\varepsilon\|_2^2)=\lim\limits_{\varepsilon\to0}M(\|\nabla u_\varepsilon^{(1)}\|_2^2)$.

Then we prove that

$$\lim_{\varepsilon\to0}\sup_{z\in\mathbb{R}^N}\int_{B_1(z)}|\widetilde{u}_\varepsilon^{(1)}-W|^2dx=0. \tag{2.14}$$

We suppose on the contrary that there exist $\{z_\varepsilon\}\subset\mathbb{R}^N$ such that $\lim_{\varepsilon\to0}\int_{B_1(z_\varepsilon)}|\widetilde{u}_\varepsilon^{(1)}-W|^2dx>0$. It follows from $\widetilde{u}_\varepsilon^{(1)}\to W$ in $L^2_{loc}(\mathbb{R}^N)$ that

$|z_\varepsilon| \to \infty$. Moreover, noting that $B_1(z_\varepsilon) \subset supp\{\widetilde{u}_\varepsilon^{(1)}\} = B(0, 2\varepsilon^{-\frac{1}{3}})$ and denoting $v_\varepsilon(x) := \widetilde{u}_\varepsilon^{(1)}(x + z_\varepsilon)$, $v_\varepsilon \rightharpoonup \widetilde{W} \neq 0$ in $H^1(\mathbb{R}^N)$ and $\widetilde{W}$ satisfies

$$-\alpha_0 \Delta \widetilde{W} + V(x_0)\widetilde{W} = f(\widetilde{W}) \quad \text{in} \quad \mathbb{R}^N.$$

Set

$$L_{\alpha_0, V(x_0)}(u) = \frac{\alpha_0}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{V(x_0)}{2} \int_{\mathbb{R}^N} u^2 dx - \int_{\mathbb{R}^N} F(u) dx,$$

$$E_{\alpha_0}(V(x_0)) := \inf \left\{ L_{\alpha_0, V(x_0)}(u) : u \in H^1(\mathbb{R}^N) \setminus \{0\}, \quad L'_{\alpha_0, V(x_0)}(u) = 0 \right\}.$$

By (f1)-(f3) and the result of [5, 20], $E_{\alpha_0}(V(x_0)) > 0$ is well defined and has monotonicity with respect to α_0 and $V(x_0)$. From the Pohozaev identity, we also have $L_{\alpha_0, V(x_0)}(u) = \frac{\alpha_0}{N} \|\nabla u\|_2^2$ for $L'_{\alpha_0, V(x_0)}(u) = 0$. For $u \in N_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$, we denote $M_0 := \sup \left\{ M(\|\nabla u\|_2^2) : u \in N_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta}) \right\}$. Then we note that $M_0 \geq \alpha_0 \geq m_0 > 0$. Moreover we take large $R_1 > 0$ such that

$$\int_{B_{R_1}(0)} |\nabla \widetilde{W}|^2 dx \geq \frac{N}{2\alpha_0} E_{\alpha_0}(V(x_0)) \geq \frac{N}{2\alpha_0} E_{m_0}(V(x_0)).$$

On the other hand, it follows from $|z_\varepsilon| \to \infty$ and the exponential decay of the element in $\widehat{S}$ that $\|U\|_{H^1(B_{R_1}(z_\varepsilon))} \to 0$ uniformly for $U \in \widehat{S}$. Then from $u_\varepsilon \in N_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$, we have that for ε small

$$\int_{B_{R_1}(z_\varepsilon)} |\nabla u_\varepsilon(x + x_\varepsilon/\varepsilon)|^2 + V(\varepsilon x + x_\varepsilon) u_\varepsilon^2(x + x_\varepsilon/\varepsilon) dx \leq 8\delta^2.$$

It follows from the weak lower semi-continuity of norms that $\int_{B_{R_1}(0)} |\nabla \widetilde{W}|^2 dx \leq 8\delta^2$. Then we can take $\delta >$ small such that $\delta^2 M_0 < \frac{N}{32} E_{m_0}(V(x_0))$ to get a contradiction.

Hence by (2.14) and lemma I.1 in [24], we have

$$\widetilde{u}_\varepsilon^{(1)} \to W \quad \text{in} \quad L^q(\mathbb{R}^N) \quad \text{for} \quad 2 < q < 2^*. \tag{2.15}$$

Next we prove that $\limsup_{\varepsilon \to 0} \int_{\mathbb{R}^N} f(\widetilde{u}_\varepsilon^{(1)}) \widetilde{u}_\varepsilon^{(1)} dx \leq \int_{\mathbb{R}^N} f(W) W dx$. Let $\eta > 0$, by (f1), (f2), (2.15) and the boundedness of $\widetilde{u}_\varepsilon^{(1)}$ in H_ε. We have for any $\xi > 0$,

$$\limsup_{\varepsilon \to 0} \int_{|x| \geq \xi} f(\widetilde{u}_\varepsilon^{(1)}) \widetilde{u}_\varepsilon^{(1)} dx \leq \eta C + C_\eta \int_{|x| \geq \xi} |W|^{p+1} dx.$$

we take $\xi_\eta > 0$ large such that

$$\limsup_{\varepsilon \to 0} \int_{|x| \geq \xi_\eta} f(\widetilde{u}_\varepsilon^{(1)}) \widetilde{u}_\varepsilon^{(1)} dx \leq (C+1)\eta \quad \text{and} \quad \left| \int_{|x| \geq \xi_\eta} f(W) W dx \right| \leq \eta. \tag{2.16}$$

Moreover by Strauss lemma in [31], we have

$$\lim_{\varepsilon \to 0} \int_{|x| \leq \xi_\eta} f(\widetilde{u}_\varepsilon^{(1)}) \widetilde{u}_\varepsilon^{(1)} dx = \int_{|x| \leq \xi_\eta} f(W) W dx. \tag{2.17}$$

Then it follows from (2.16) and (2.17) that

$$\limsup_{\varepsilon \to 0} \int_{\mathbb{R}^N} f(\widetilde{u}_\varepsilon^{(1)}) \widetilde{u}_\varepsilon^{(1)} dx \leq (C+1)\eta + \int_{|x| \leq \xi_\eta} f(W) W dx \leq (C+2)\eta + \int_{\mathbb{R}^N} f(W) W dx.$$

Letting $\eta \to 0$, we have

$$\limsup_{\varepsilon \to 0} \int_{\mathbb{R}^N} f(\widetilde{u}_\varepsilon^{(1)}) \widetilde{u}_\varepsilon^{(1)} dx \leq \int_{\mathbb{R}^N} f(W) W dx. \tag{2.18}$$

Noting that $\Gamma'_\varepsilon(u_\varepsilon^{(1)}) \to 0$, we have

$$M\big(\|\nabla \widetilde{u}_\varepsilon^{(1)}\|_2^2\big)\int_{\mathbb{R}^N}|\nabla \widetilde{u}_\varepsilon^{(1)}|^2dx+\int_{\mathbb{R}^N}V(\varepsilon x+x_\varepsilon)\big(\widetilde{u}_\varepsilon^{(1)}\big)^2dx=\int_{\mathbb{R}^N}f(\widetilde{u}_\varepsilon^{(1)})\widetilde{u}_\varepsilon^{(1)}dx+o(1). \tag{2.19}$$

Then by (2.18), (2.19) and the weak semi-continuity of norms, one has

$$\begin{aligned}&\alpha_0\int_{\mathbb{R}^N}|\nabla W|^2dx+\int_{\mathbb{R}^N}V(x_0)W^2dx\\ &\le \liminf_{\varepsilon\to 0}\Big(M\big(\|\nabla \widetilde{u}_\varepsilon^{(1)}\|_2^2\big)\int_{\mathbb{R}^N}|\nabla \widetilde{u}_\varepsilon^{(1)}|^2dx+\int_{\mathbb{R}^N}V(\varepsilon x+x_\varepsilon)\big(\widetilde{u}_\varepsilon^{(1)}\big)^2dx\Big)\\ &\le \limsup_{\varepsilon\to 0}\int_{\mathbb{R}^N}f(\widetilde{u}_\varepsilon^{(1)})\widetilde{u}_\varepsilon^{(1)}dx\le\int_{\mathbb{R}^N}f(W)Wdx.\end{aligned}$$

Thus, since W solves the equation (2.13), we have that, up to subsequence,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\Big(M\big(\|\nabla \widetilde{u}_\varepsilon^{(1)}\|_2^2\big)\int_{\mathbb{R}^N}|\nabla \widetilde{u}_\varepsilon^{(1)}|^2dx+\int_{\mathbb{R}^N}V(\varepsilon x+x_\varepsilon)\big(\widetilde{u}_\varepsilon^{(1)}\big)^2dx\Big)\\ =&\alpha_0\int_{\mathbb{R}^N}|\nabla W|^2dx+\int_{\mathbb{R}^N}V(x_0)W^2dx.\end{aligned}$$

Next we prove that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}|\nabla \widetilde{u}_\varepsilon^{(1)}|^2dx=\int_{\mathbb{R}^N}|\nabla W|^2dx. \tag{2.20}$$

On the contrary, we assume that $h_0 := \lim_{\varepsilon\to 0}\big(\int_{\mathbb{R}^N}|\nabla \widetilde{u}_\varepsilon^{(1)}|^2dx-\int_{\mathbb{R}^N}|\nabla W|^2dx\big)>0$. Take $R>0$ large enough such that $\int_{|x|\ge R}V(x_0)W^2dx<\alpha_0h_0/2$. Thus from (2.19), one has

$$\begin{aligned}&\alpha_0\int_{\mathbb{R}^N}|\nabla W|^2dx+\int_{\mathbb{R}^N}V(x_0)W^2dx\le\alpha_0\int_{\mathbb{R}^N}|\nabla W|^2dx+\int_{|x|\le R}V(x_0)W^2dx+\frac{\alpha_0h_0}{2}\\ &\le\liminf_{\varepsilon\to 0}\Big(M\big(\|\nabla \widetilde{u}_\varepsilon^{(1)}\|_2^2\big)\int_{\mathbb{R}^N}|\nabla \widetilde{u}_\varepsilon^{(1)}|^2dx+\int_{|x|\le R}V(\varepsilon x+x_\varepsilon)\big(\widetilde{u}_\varepsilon^{(1)}\big)^2dx\Big)-\frac{\alpha_0h_0}{2}\\ &\le\int_{\mathbb{R}^N}f(W)Wdx-\frac{\alpha_0h_0}{2}.\end{aligned}$$

This is a contradiction. Hence we have (2.20), which implies that $\alpha_0=M(\|\nabla W\|_2^2)$ and

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}|\nabla \widetilde{u}_\varepsilon^{(1)}|^2+V(\varepsilon x+x_\varepsilon)\big(\widetilde{u}_\varepsilon^{(1)}\big)^2dx=\int_{\mathbb{R}^N}|\nabla W|^2+V(x_0)W^2dx \tag{2.21}$$

and W solves the equation

$$-M(\|\nabla W\|_2^2)\Delta W+V(x_0)W=f(W)\quad\text{in }\mathbb{R}^N.$$

Then, by (2.21), $\|u_\varepsilon^{(2)}\|_\varepsilon\to 0$, (2.11) and lemma 2.3 (ii), one has $L_{V(x_0)}(W)\le E_m$. Noting that the definition of $\widehat{S}$, we have $V(x_0)\le m$. Set $y_0\in\mathbb{R}^N$ such that $W(y_0)=\max_{x\in\mathbb{R}^N}W(x)$, then $\widehat{W}:=W(x+y_0)\in\widehat{S}$. Hence we have

$$\|u_\varepsilon-(\phi_\varepsilon\widehat{W})(\cdot-y_0-\frac{x_\varepsilon}{\varepsilon})\|_\varepsilon\le\|\widetilde{u}_\varepsilon^{(1)}-W\|_\varepsilon+\|\widetilde{u}_\varepsilon^{(2)}\|_\varepsilon+\|(\phi_\varepsilon-1)W\|_\varepsilon\to 0.$$

Consequently, we have for $\varepsilon>0$ small, $u_\varepsilon\in\widehat{N}_{\delta'}(\mathcal{Z}_\varepsilon^{10\beta})$ which leads to a contradiction. $\square$

2.5. **A map on $\widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$ to estimate tails.** In this part, we introduce a map on $\widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$ to estimate functions away from the center of mass, which is inspired in [7, 10]. And this map has some good properties: it does not increase the energy of the functional (2.2) and makes functions possess exponential decay away from the center of mass.

For $u \in H_\varepsilon$, some $y \in O_\varepsilon$ and $b \in (0, 2]$, there exists $R > 0$ such that

$$\int_{\mathbb{R}^N \setminus B(y,R)} |\nabla u|^2 + V_\varepsilon u^2 dx \le b^2/2. \tag{2.22}$$

Now, define

$$H^R_{y,b}(u) := \left\{ v \in H_\varepsilon : \int_{\mathbb{R}^N \setminus B(y,R)} |\nabla v|^2 + V_\varepsilon v^2 dx \le b^2,\ v = u \text{ in } B(y,R) \right\}.$$

We consider the minimization problem on $H^R_{y,b}(u)$:

$$\begin{aligned} I^R_{y,b}(u) = \inf \Bigg\{ &\frac{1}{2}\widehat{M}\Big(\int_{B(y,R)} |\nabla u|^2 dx + \int_D |\nabla v|^2 dx \Big) + \frac{1}{2}\int_D V_\varepsilon v^2 dx \\ &- \int_D F(v) dx \ :\ v \in H^R_{y,b}(u) \Bigg\}, \end{aligned}$$

where $D := \mathbb{R}^N \setminus B(y,R)$. Arguing as in the proof in [10], we have the following lemma:

Lemma 2.5. *There exist $b \in (0, 2]$ and a unique minimizer $v_\varepsilon = v(u, y, R) \in H^R_{y,b}(u)$ of $I^R_{y,b}(u)$ and v_ε solves, for some $\alpha_0 > 0$,*

$$-\alpha_0 \Delta v + V_\varepsilon v = f(v) \quad \text{in} \quad \mathbb{R}^N \setminus B(y,R) \quad \text{and} \quad v = u \quad \text{in } B(y,R). \tag{2.23}$$

Moreover there exist $C,\ c > 0$ such that $v_\varepsilon(x) \le C exp(-c|x - y| - R - 1)$ for $|x - y| \ge R + 1$.

Proof. For any $v \in H^R_{y,b}(u)$,

$$M(\|\nabla v\|_2^2) = M\Big(\int_{B(y,R)} |\nabla u|^2 dx + \int_D |\nabla v|^2 dx \Big) \le M\Big(\int_{B(y,R)} |\nabla u|^2 dx + 4 \Big) := M_1.$$

Then combining with (M1), one has the following inequality: denoting $\alpha := \int_{B(y,R)} |\nabla u|^2 dx$.

$$\begin{aligned} \min\{1, m_0\} \int_D |\nabla u|^2 + V_\varepsilon u^2 dx \le \|u\|^2_{M,D} &:= \int_\alpha^{\alpha + \int_D |\nabla u|^2 dx} M(t) dt + \int_D V_\varepsilon u^2 dx \\ &\le \max\{1, M_1\} \int_D |\nabla u|^2 + V_\varepsilon u^2 dx. \end{aligned} \tag{2.24}$$

For simplicity, we denote

$$I_{\varepsilon,D}(v) := \frac{1}{2}\widehat{M}\Big(\alpha + \int_D |\nabla v|^2 dx \Big) + \frac{1}{2}\int_D V_\varepsilon v^2 dx - \int_D F(v) dx.$$

By (f1) and (f2), there exist $C_{\underline{V}/16} > 0$ such that

$$\int_D F(v) dx \le \frac{\underline{V}}{16} \int_D v^2 dx + C_{\underline{V}/16} \int_D |v|^{p+1} dx.$$

By sobolev inequality,

$$\int_D |v|^{p+1}dx \le C\Big(\int_D |\nabla v|^2 + V_\varepsilon v^2 dx\Big)^{(p+1)/2}.$$

Hence for $v \in H^R_{y,b}(u)$ with $\int_D |\nabla v|^2 + V_\varepsilon v^2 dx = b^2$, by (2.24), there exists $c > 0$ such that $\|v\|^2_{M,D} = cb^2$. Then for $b > 0$ small, we have

$$I_{\varepsilon,D}(v) \ge \frac{1}{2}\widehat{M}(\alpha) + \Big(\frac{7}{16} - C\big(cb^2\big)^{\frac{p-1}{2}}\Big)\|v\|^2_{M,D} > \frac{1}{2}\widehat{M}(\alpha) + \frac{3}{8}cb^2,$$

here we use that $\widehat{M}(s+t) = \widehat{M}(s) + \int_s^{s+t} M(t)dt$. On the other hand, for $v \in H^R_{y,b}(u)$ with $\|v\|^2_{M,D} \le cb^2/2$, one has

$$I_{\varepsilon,D}(v) \le \frac{1}{2}\widehat{M}(\alpha) + \Big(\frac{9}{16} + C\big(\frac{cb^2}{2}\big)^{\frac{p-1}{2}}\Big)\|v\|^2_{M,D} < \frac{1}{2}\widehat{M}(\alpha) + \frac{3}{8}cb^2,$$

for $b > 0$ small. Consequently, we observe that the minimizer of $I_{\varepsilon,D}(v)$ is obtained in the interior of $H^R_{y,b}(u)$.

Let v_n be the minimizing sequence, then $\int_D |\nabla v_n|^2 + V_\varepsilon v_n^2 dx \le b^2$. Hence up to subsequence, $v_n \rightharpoonup v_\varepsilon$ in H_ε. Similar with the proof of (2.18), we can obtain $\limsup_{n\to\infty} \int_D F(v_n)dx \le \int_D F(v_\varepsilon)dx$. Combining with weakly lower semi-continuity, v_ε minimizes $I_{\varepsilon,D}$, i.e. solves (2.23), where $\alpha_0 := \lim_{n\to\infty} M\big(\alpha + \int_D |\nabla v_n|^2 dx\big)$.

Next we prove the minimizer is unique. We assume that v^*_ε is the other solution, then we denote $Z = v^*_\varepsilon - v_\varepsilon$. Through calculating, we have for some $c > 0$ and $\lambda \in [0, 1]$,

$$c\int_D |\nabla Z|^2 dx + \int_D V_\varepsilon Z^2 dx \le \int_D f'(Z^\lambda)Z^2 dx,$$

where $Z^\lambda = \lambda v^*_\varepsilon + (1-\lambda)v_\varepsilon$. Then by (f1), (f2) and remark 2.6, for any $\eta > 0$, there exists $C_\eta > 0$ such that $|f'(t)t^2| \le \eta t^2 + C_\eta t^{2^*}$. Then we have for some $C > 0$ such that

$$\int_D |\nabla Z|^2 + V_\varepsilon Z^2 dx \le C\int_D \big(|v^*_\varepsilon|^{\frac{4}{N-2}} + |v_\varepsilon|^{\frac{4}{N-2}}\big)Z^2 dx.$$

By Hölder inequality, one has

$$\int_D \big(|v^*_\varepsilon|^{\frac{4}{N-2}} + |v_\varepsilon|^{\frac{4}{N-2}}\big)Z^2 dx \le \Big(\|v^*_\varepsilon\|^{4/(N-2)}_{2^*} + \|v_\varepsilon\|^{4/(N-2)}_{2^*}\Big)\|Z\|^2_{2^*}.$$

Hence, by sobolev inequality, we have for some $C > 0$

$$\int_D |\nabla Z|^2 + V_\varepsilon Z^2 dx \le C\big(b^2 \max\{1, m_0^{-1}\}\big)^{4/(N-2)} \int_D |\nabla Z|^2 + V_\varepsilon Z^2 dx.$$

Consequently for small $b > 0$, we have $Z \equiv 0$.

Finally, we prove that v_ε has the exponential decay. First we claim that for any fixed $\varepsilon > 0$, $\lim_{|x|\to\infty} v_\varepsilon(x) = 0$. In fact, by absolute continuity of integral, it follows that

$$\lim_{R\to\infty} \int_{|x|\ge R} v_\varepsilon^2 + v_\varepsilon^{2^*} dx = 0. \tag{2.25}$$

Noting that $m_0 \le \alpha_0 \le M_0$ and

$$-\Delta v_\varepsilon + \frac{V_\varepsilon}{\alpha_0}v_\varepsilon = \frac{1}{\alpha_0}f(v_\varepsilon) \quad \text{in } \mathbb{R}^N \setminus B(y, R).$$

In proposition 4.1, we take $b(x) = V_\varepsilon/\alpha_0 \in L^\infty_{loc}$, $q(x) = 0$ and $h(x, u) = \frac{1}{\alpha_0}f(v_\varepsilon) \le C(|v_\varepsilon| + |v_\varepsilon|^{2^*-1})$, then $v_\varepsilon \in L^s(\mathbb{R}^N)$ and $\|v_\varepsilon\|_s \le C_s\|v_\varepsilon\|_\varepsilon$ for any $s \ge 2$. Thus for some $t > N$, $\|f\big(v_\varepsilon(x)\big)\|_{t/2} \le C$. By proposition 4.2, taking $g(x) = \frac{1}{\alpha_0} f\big(v_\varepsilon(x)\big)$, we have for any $B_2(z) \subset \mathbb{R}^N \setminus B(y, R+1)$,

$$\sup_{x \in B_1(z)} v_\varepsilon(x) \le C\big(\|v_\varepsilon\|_{L^2(B_2(z))} + \|g\|_{t/2}\big). \tag{2.26}$$

Hence by (2.25) and (2.26), we have $\lim_{|x|\to\infty} v_\varepsilon(x) = 0$ independent of small $\varepsilon > 0$. Then for $y \in O$, there exist $R > 0$ such that

$$\frac{f(v_\varepsilon)}{\alpha_0} \le \frac{C}{\alpha_0}(|v_\varepsilon| + |v_\varepsilon|^p) < \frac{V}{2\alpha_0} \quad \text{for} \quad x \in \mathbb{R}^N \setminus B(y, R+1).$$

Thus, through calculating, one has

$$-\Delta v_\varepsilon + \frac{V}{2\alpha_0} v_\varepsilon \le 0.$$

Moreover, take a function $d(x) \in C^2(\mathbb{R}^N \setminus B(y, R))$ such that for $r(x) := |x - y|$,

$$\|d(x) - r(x)\|_{C^2(\mathbb{R}^N \setminus B(y,R))} \le \frac{1}{10}. \tag{2.27}$$

Then we choose $c > 0$ independent of $\varepsilon > 0$ such that

$$\begin{aligned} &\Delta exp\big(-c(d(x) - R - 1)\big) - \frac{V}{2\alpha_0} exp\big(-c(d(x) - R - 1)\big) \\ \le &\big(c\Delta d + c^2|\nabla d|^2 - \frac{V}{2\alpha_0}\big) exp\big(-c(d(x) - R - 1)\big) < 0. \end{aligned}$$

And we take $C > 0$ such that

$$v_\varepsilon(x) \le Cexp\big(-c(d(x) - R - 1)\big) \quad \text{on} \quad \partial B(y, R+1).$$

Setting $\varphi := Cexp\big(-c(d(x) - R - 1)\big) - v_\varepsilon(x)$, we have

$$\begin{cases} -\Delta \varphi + \frac{V}{2\alpha_0}\varphi \ge 0 & \text{in} \quad \mathbb{R}^N \setminus B(y, R+1); \\ \varphi \ge 0 & \text{on} \quad \partial B(y, R+1); \\ \lim\limits_{|x|\to\infty} \varphi(x) = 0. & \end{cases}$$

According to maximum principle, one has $\varphi \ge 0$ in $\mathbb{R}^N \setminus B(y, R+1)$. Consequently, from (2.27), there exist $C, c > 0$ such that $v_\varepsilon(x) \le Cexp\big(-c(|x - y| - R - 1)\big)$ for $x \in \mathbb{R}^N \setminus B(y, R+1)$ independent of small $\varepsilon > 0$. □

Remark 2.6. *We first assume that there exists a solution u_ε in $\widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$ for some $\delta > 0$ and small $\varepsilon > 0$. Then it follows from Moser's iteration that there exists $C_0 > 0$ such that $\|u_\varepsilon\|_{L^\infty} \le C_0$ for small $\varepsilon > 0$. Thus we can choose $\widetilde{f} \in C^1(\mathbb{R})$ such that $\widetilde{f}(t) = f(t)$ for $t \le 2C_0$, $\widetilde{f}(t) = Ct^p$ for $t \ge 3C_0$ and $\widetilde{f}$ satisfies (f1)-(f3). If we replace f in (2.1) by $\widetilde{f}$, any solution $u_\varepsilon \in \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$ also satisfies the original equation (2.1). Hence without loss of generality, we can assume that*

$$|f'(t)t| \le C(1 + t^p) \quad \textit{for } t \ge 0.$$

For any $u \in \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$, by lemma 2.1, there exist $U \in \widehat{S}$ and $y \in \Omega_\varepsilon^{10\beta}$ such that $|\Upsilon_\varepsilon(u) - y| \le R_0$ and $\|u - \big(\phi_\varepsilon U\big)(\cdot - y)\|_\varepsilon \le 2\delta$. Then we note that for $\varepsilon > 0$ small,

$$supp\,\{(\phi_\varepsilon U)(\cdot - y)\} \subset B(\Upsilon_\varepsilon(u), \frac{1}{\sqrt{\varepsilon}}). \tag{2.28}$$

Thus

$$\int_{\mathbb{R}^N \setminus B(\Upsilon_\varepsilon(u), 1/\sqrt{\varepsilon})} |\nabla u|^2 + V_\varepsilon u^2 dx \le (2\delta)^2.$$

Consequently, we take $b = 2\sqrt{2}\delta$, $y = \Upsilon_\varepsilon(u)$ and $R = 1/\sqrt{\varepsilon}$ in lemma 2.5, then $v(u, \Upsilon_\varepsilon(u), 1/\sqrt{\varepsilon}) \in \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$ for any $u \in \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$. For simplicity, we denote $\tau_\varepsilon(u) = v(u, \Upsilon_\varepsilon(u), 1/\sqrt{\varepsilon})$. It is clear that $\Gamma_\varepsilon(\tau_\varepsilon(u)) \leq \Gamma_\varepsilon(u)$ for $u \in \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$.

From the next lemma, we can see that the center of mass of $\tau_\varepsilon(u)$ does not go far away.

Lemma 2.7. *For $\varepsilon > 0$ small, $|\Upsilon_\varepsilon(\tau_\varepsilon(u)) - \Upsilon_\varepsilon(u)| \leq 2R_0$ for any $u \in \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$.*

Proof. Like (2.28), For $u \in \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$, there exist $U \in \widehat{S}$ and $y \in \Omega_\varepsilon^{10\beta}$ such that $|y - \Upsilon_\varepsilon(u)| \leq R_0$ and $\|u - \big(\phi_\varepsilon U\big)(\cdot - y)\|_\varepsilon \leq 2\delta$. Then we note that for $\varepsilon > 0$ small,

$$supp\,\{(\phi_\varepsilon U)(\cdot - y)\} \subset B(\Upsilon_\varepsilon(u), \frac{1}{\sqrt{\varepsilon}}).$$

On the other hand, from $\tau_\varepsilon(u) \in \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$, if $\tau_\varepsilon(u) \in G_{2\delta}^M(\mathcal{Z}_\varepsilon^{10\beta})$, it follows from (2.8), we have,

$$\int_{\alpha_\varepsilon}^{\alpha_\varepsilon + \int_{D_\varepsilon} |\nabla \tau_\varepsilon(u)|^2 dx} M(t)dt + \int_{D_\varepsilon} V_\varepsilon \tau_\varepsilon^2(u) dx \leq \frac{(2\delta)^2 \min\{1, m_0\}}{2(1-q)} \leq 2(2\delta)^2 \min\{1, m_0\},$$

where $\alpha_\varepsilon = \int_{B(\Upsilon_\varepsilon(u), 1/\sqrt{\varepsilon})} |\nabla u|^2 dx$ and $D_\varepsilon = \mathbb{R}^N \setminus B(\Upsilon_\varepsilon(u), 1/\sqrt{\varepsilon})$. Hence for $\varepsilon > 0$ small,

$$\begin{aligned}
&\|\tau_\varepsilon(u) - (\phi_\varepsilon U)(\cdot - y)\|_\varepsilon^2 \\
&= \int_{B(\Upsilon_\varepsilon(u), 1/\sqrt{\varepsilon})} |\nabla\big(u - (\phi_\varepsilon U)(\cdot - y)\big)|^2 + V_\varepsilon\big(u - (\phi_\varepsilon U)(\cdot - y)\big)^2 dx \\
&\quad + \int_{\mathbb{R}^N \setminus B(\Upsilon_\varepsilon(u), 1/\sqrt{\varepsilon})} |\nabla \tau_\varepsilon(u)|^2 + V_\varepsilon \tau_\varepsilon^2(u) dx \\
&\leq (2\delta)^2 + \frac{1}{\min\{1, m_0\}} \int_{\alpha_\varepsilon}^{\alpha_\varepsilon + \int_{D_\varepsilon} |\nabla \tau_\varepsilon(u)|^2 dx} M(t)dt + \int_{D_\varepsilon} V_\varepsilon \tau_\varepsilon^2(u) dx \\
&\leq (2\delta)^2 + 2(2\delta)^2 \leq (4\delta)^2.
\end{aligned}$$

If $\tau_\varepsilon(u) \in G_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$, similarly, we have

$$\int_{D_\varepsilon} |\nabla \tau_\varepsilon(u)|^2 + V_\varepsilon \tau_\varepsilon(u)^2 dx \leq \frac{(2\delta)^2}{2(1-q)} \leq \frac{(2+\sqrt{2})(2\delta)^2}{4} \leq 2(2\delta)^2.$$

Then one also has

$$\|\tau_\varepsilon(u) - (\phi_\varepsilon U)(\cdot - y)\|_\varepsilon^2 \leq (4\delta)^2.$$

Moreover, for $|z - \Upsilon_\varepsilon(u)| \geq 2R_0$, one has that $|y - z| \geq |z - \Upsilon_\varepsilon(u)| - |y - \Upsilon_\varepsilon(u)| \geq R_0$. Meanwhile, from (2.7), we have, for any $\widetilde{U} \in \widehat{S}$,

$$\begin{aligned}
\|\tau_\varepsilon(u) - (\phi_\varepsilon \widetilde{U})(\cdot - z)\|_\varepsilon &\geq \|(\phi_\varepsilon U)(\cdot - y) - (\phi_\varepsilon \widetilde{U})(\cdot - z)\|_\varepsilon - \|\tau_\varepsilon(u) - (\phi_\varepsilon U)(\cdot - y)\|_\varepsilon \\
&\geq \frac{5}{8} r_* - 4\delta \geq 6\delta,
\end{aligned}$$

then for $|z - \Upsilon_\varepsilon(u)| \geq 2R_0$, $\sigma\big(\mathrm{dist}_\varepsilon(\tau_\varepsilon(u), S(z))\big) = 0$. Hence $\Upsilon_\varepsilon(\tau_\varepsilon(u)) \in B(\Upsilon_\varepsilon(u), 2R_0)$. □

Since there exist $(t, z) \in [0, T] \times \mathcal{L}_\varepsilon$ such that $A_\varepsilon(t, z) \notin \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$ and we will use the initial path to begin with the iteration, thus we extend continuously the center of mass Υ_ε on $\widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$ to $\widetilde{\Upsilon}_\varepsilon$ on $\widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta}) \cup \{A_\varepsilon(t,z) : (t,z) \in [0,T] \times \mathcal{L}_\varepsilon\}$ such that for any $A_\varepsilon(t,z) \notin \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$,

$$\left|\widetilde{\Upsilon}_\varepsilon\big(A_\varepsilon(t,z)\big) - \frac{\gamma_\varepsilon(z)}{\varepsilon}\right| \le 3R_0 \tag{2.29}$$

and

$$\widetilde{\Upsilon}_\varepsilon\big(A_\varepsilon(t,z)\big) = \frac{\gamma_\varepsilon(z)}{\varepsilon} \quad \text{for } (t,z) \in [0,T] \times \mathcal{N}(L_0),$$

where $\mathcal{N}(L_0) \subset \mathcal{L}_\varepsilon$ is a neighborhood of L_0. Hence we note that if $|x - \widetilde{\Upsilon}_\varepsilon\big(A_\varepsilon(t,z)\big)| = 1/\sqrt{\varepsilon}$, from the definition of ϕ_ε, we have $A_\varepsilon(t,z)(x) = 0$ for $\varepsilon > 0$ small. And hence we can also extend τ_ε continuously on $\widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta}) \cup \{A_\varepsilon(t,z) : (t,z) \in [0,T] \times \mathcal{L}_\varepsilon\}$ such that for $\varepsilon > 0$ small,

$$\tau_\varepsilon\big(A_\varepsilon(t,z)\big) = A_\varepsilon(t,z) \quad \text{for } A_\varepsilon(t,z) \notin \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta}).$$

2.6. **Translation operator through the deformation flow of $V(x)$.** Take large $R_1 > \max\{1, 10\beta\}$ such that $O \subset B(0,R_1)$ and define

$$w := \frac{\max_{|x|\le 10R_1} V(x)}{\min_{|x|\le 10R_1} V(x)} > 1. \tag{2.30}$$

Let us choose three cut-off functions for the translation operator:

$\zeta_\varepsilon \in C_0^\infty(\mathbb{R}^N, [0,1])$ with $\zeta_\varepsilon(x) = 1$ for $|x| \le \frac{7R_1}{\varepsilon}$, $\zeta_\varepsilon(x) = 0$ for $|x| \ge \frac{8R_1}{\varepsilon}$ and $|\nabla\zeta_\varepsilon(x)| \le 2\varepsilon$;

$\kappa_1 \in C_0^\infty(\mathbb{R}^N, [0,1])$ with $\kappa_1(x) = 1$ for $x \in \Omega^{6\beta} \setminus \Omega^{3\beta}$ and $\kappa_1(x) = 0$ for $x \in \Omega^{2\beta} \cup (\mathbb{R}^N \setminus \Omega^{7\beta})$;

$\kappa_2 \in C^2(\widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta}), [0,1])$ with $\kappa_2(u) = 1$ for $u \in \widehat{N}_{\delta/6w}(\mathcal{Z}_\varepsilon^{10\beta})$ and $\kappa_2(u) = 0$ for $u \notin \widehat{N}_{\delta/4w}(\mathcal{Z}_\varepsilon^{10\beta})$.

Then we define a function $\Phi : [0,\infty) \times \mathbb{R}^N \times H_\varepsilon \to \mathbb{R}^N$ by

$$\Phi(l,x,u) = \Psi(\kappa_1(x)\kappa_2(u)l, x),$$

where from (iii) of (V3), we take $l_s \in (0,1)$ such that

$$|\Psi(l,x) - x| \le \frac{\beta}{10} \quad \text{for} \quad \text{any} \quad l \in [0,l_s] \quad \text{and} \quad x \in \Omega^{8\beta}. \tag{2.31}$$

Now following [8], we define translation operator $\mathcal{T}_\varepsilon : [0,l_s] \times \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta}) \to H_\varepsilon$ by

$$\mathcal{T}_\varepsilon(l,u)(x) := \big(1 - \zeta_\varepsilon(x)\big)u(x) + (\zeta_\varepsilon u)\Big(x - \frac{\Phi(l, \varepsilon\Upsilon_\varepsilon(u), u)}{\varepsilon} + \Upsilon_\varepsilon(u)\Big).$$

From this definition, we note that $\mathcal{T}_\varepsilon(l,u) = u$ for any $l \in [0,l_s]$ if $\Upsilon_\varepsilon(u) \in \Omega_\varepsilon^{2\beta} \cup (\mathbb{R}^N \setminus \Omega_\varepsilon^{7\beta})$ or $u \notin \widehat{N}_{\delta/4w}(\mathcal{Z}_\varepsilon^{10\beta})$. In we following we take δ small such that $1 - q > 1/\sqrt{2}$ and $(1+q)/\sqrt{w} \le 1$.

Lemma 2.8. *For $u \in \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$,*

$$\left|\Upsilon_\varepsilon(\mathcal{T}_\varepsilon(l,u)) - \frac{\Phi(l, \varepsilon\Upsilon_\varepsilon(u), u)}{\varepsilon}\right| \le 2R_0.$$

Proof. Since if $\Upsilon_\varepsilon(u) \in \Omega^{2\beta} \cup (\mathbb{R}^N \setminus \Omega^{7\beta})$ or $u \notin \widehat{N}_{\delta/4w}(\mathcal{Z}_\varepsilon^{10\beta})$, $\Phi(l, \varepsilon\Upsilon_\varepsilon(u), u) = \varepsilon\Upsilon_\varepsilon(u)$ and $\mathcal{T}_\varepsilon(l,u)) = u$. Then the conclusion follows. Next for $u \in \widehat{N}_{\delta/4w}(\mathcal{Z}_\varepsilon^{10\beta})$, by lemma 2.1 and lemma 2.2, we have $u \in N_{\delta/4(1-q)w}(\mathcal{Z}_\varepsilon^{10\beta})$ and there exist $U \in \widehat{S}$ and $z \in \Omega_\varepsilon^{10\beta}$ such that

$$|\Upsilon_\varepsilon(u) - z| < R_0 \quad \text{and} \quad \|u - (\phi_\varepsilon U)(\cdot - z)\|_\varepsilon \le \frac{\delta}{4(1-q)w} \le \frac{\sqrt{2}\delta}{4w} \tag{2.32}$$

From $supp\{(\phi_\varepsilon U)(\cdot - z)\} \subset B(0, 7R_1/\varepsilon)$ for ε small, one has $\int_{|x|\ge 7R_1/\varepsilon} |\nabla u|^2 + V_\varepsilon u^2 dx \le \delta^2/8w^2$. For simplicity, denote $d_\varepsilon(l,u) = \frac{\Phi(l,\varepsilon\Upsilon_\varepsilon(u),u)}{\varepsilon} - \Upsilon_\varepsilon(u)$. Then for ε small,

$$\begin{aligned}
&\|\mathcal{T}_\varepsilon(l,u) - (\phi_\varepsilon U)\big(\cdot - z - d_\varepsilon(l,u)\big)\|_\varepsilon \\
&\quad \le \|(1-\zeta_\varepsilon)u\|_\varepsilon + \|(\zeta_\varepsilon u)\big(\cdot - d_\varepsilon(l,u)\big) - (\phi_\varepsilon U)\big(\cdot - z - d_\varepsilon(l,u)\big)\|_\varepsilon \\
&\quad \le \frac{\sqrt{2}\delta}{4w} + O(\varepsilon) + \sqrt{w}\|u - (\phi_\varepsilon U)\big(\cdot - z\big)\|_\varepsilon + \sqrt{w}\|(\zeta_\varepsilon - 1)u\|_\varepsilon \\
&\quad \le \frac{\sqrt{2}\delta}{4w} + \frac{\sqrt{2}\delta}{2\sqrt{w}} + O(\varepsilon) \le 2\delta,
\end{aligned}$$

where we use that $V(\varepsilon x + d_\varepsilon(l,u)) \le wV(\varepsilon x)$ if $|x| \le 8R_1/\varepsilon$ and $|d_\varepsilon(l,u)| \le \beta/10\varepsilon$. Then by lemma 2.1,

$$|\Upsilon_\varepsilon(\mathcal{T}_\varepsilon(l,u)) - z - d_\varepsilon(l,u)| \le R_0.$$

Combining the first inequality in (2.32), we get the conclusion. □

Next we prove that $\mathcal{T}_\varepsilon(l, \tau_\varepsilon(u))$ is invariant in $\widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$. For simplicity, we denote

$$d_\varepsilon(l,\tau_\varepsilon) = \frac{\Phi(l,\varepsilon\Upsilon_\varepsilon(\tau_\varepsilon(u)),\tau_\varepsilon(u))}{\varepsilon} - \Upsilon_\varepsilon(\tau_\varepsilon(u)).$$

Lemma 2.9. *For $u \in \widehat{N}_{\delta/4w}(\mathcal{Z}_\varepsilon^{10\beta})$, $\mathcal{T}_\varepsilon(l,\tau_\varepsilon(u)) \in \widehat{N}_{5\delta/8}(\mathcal{Z}_\varepsilon^{10\beta})$ for $l \in [0, l_s]$. Hence, $\mathcal{T}_\varepsilon(l,\tau_\varepsilon(u)) \in \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$ for $u \in \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$.*

Proof. For $u \in \widehat{N}_{\delta/4w}(\mathcal{Z}_\varepsilon^{10\beta})$, it follows from the definition of $\tau_\varepsilon(u)$ that $\tau_\varepsilon(u) \in \widehat{N}_{\delta/4w}(\mathcal{Z}_\varepsilon^{10\beta}) \subset N_{\delta/4w(1-q)}(\mathcal{Z}_\varepsilon^{10\beta})$. By (2.31), lemma 2.7 and lemma 2.8, we have $\Upsilon_\varepsilon(\tau_\varepsilon(u)) \in \Omega_\varepsilon^{8\beta}$ and $\Upsilon_\varepsilon(\mathcal{T}_\varepsilon(l,\tau_\varepsilon(u))) \in \Omega_\varepsilon^{8\beta}$. From $\tau_\varepsilon(u) \in N_{\delta/4w(1-q)}(\mathcal{Z}_\varepsilon^{10\beta})$, there exist $U \in \widehat{S}$ and $z \in \Omega_\varepsilon^{9\beta}$ such that $\|\tau_\varepsilon(u) - (\phi_\varepsilon U)(\cdot - z)\|_\varepsilon \le \delta/4w(1-q)$.

Noting that $z + d_\varepsilon(l,\tau_\varepsilon) \in \Omega_\varepsilon^{10\beta}$, then it follows from lemma 2.5 that for some $C, c > 0$

$$\int_{|x|\ge \frac{5R_1}{\varepsilon}} |\nabla\big((1-\zeta_\varepsilon)\tau_\varepsilon(u)\big)|^2 + V_\varepsilon\big((1-\zeta_\varepsilon)\tau_\varepsilon(u)\big)^2 dx \le C\exp(-\frac{c}{\varepsilon}). \tag{2.33}$$

Then for small $\varepsilon > 0$

$$\begin{aligned}
&\|\mathcal{T}_\varepsilon(l,\tau_\varepsilon(u)) - (\phi_\varepsilon U)\big(\cdot - z - d_\varepsilon(l,\tau_\varepsilon)\big)\|_\varepsilon \\
&\quad \le \|(1-\zeta_\varepsilon)\tau_\varepsilon(u)\|_\varepsilon + \|(\zeta_\varepsilon\tau_\varepsilon(u))\big(\cdot - d_\varepsilon(l,\tau_\varepsilon)\big) - (\phi_\varepsilon U)\big(\cdot - z - d_\varepsilon(l,\tau_\varepsilon)\big)\|_\varepsilon \\
&\quad \le o(1) + \sqrt{w}\|(\zeta_\varepsilon\tau_\varepsilon(u))(\cdot) - (\phi_\varepsilon U)\big(\cdot - z\big)\|_\varepsilon \\
&\quad \le \sqrt{w}\|\tau_\varepsilon(u) - (\phi_\varepsilon U)\big(\cdot - z\big)\|_\varepsilon + o(1) \\
&\quad \le \delta/4\sqrt{w}(1-q) + \delta/8\sqrt{2}(1+q).
\end{aligned}$$

Hence it follows from lemma 2.2 that

$$\sqrt{2}(1+q)\Big(\frac{\delta}{4\sqrt{w}(1-q)} + \frac{\delta}{8\sqrt{2}(1+q)}\Big) \le \frac{5}{8}\delta \quad \text{and} \quad \mathcal{T}_\varepsilon(l,\tau_\varepsilon(u)) \in \widehat{N}_{5\delta/8}(\mathcal{Z}_\varepsilon^{10\beta}).$$

On the other hand for $\tau_\varepsilon(u)\notin \widehat{N}_{\delta/4w}(\mathcal{Z}_\varepsilon^{10\beta})$ and $\Upsilon_\varepsilon(\tau_\varepsilon(u))\in\Omega_\varepsilon^{7\beta}$, $\kappa_2(\tau_\varepsilon(u))=0$, then $\Phi(l,\varepsilon\Upsilon_\varepsilon(\tau_\varepsilon(u)),\tau_\varepsilon(u))=\varepsilon\Upsilon_\varepsilon(\tau_\varepsilon(u))$ and $\mathcal{T}_\varepsilon(l,\tau_\varepsilon(u))=\tau_\varepsilon(u)\in\widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$. Moreover if $\Upsilon_\varepsilon(\tau_\varepsilon(u))\notin\Omega_\varepsilon^{7\beta}$ and $\tau_\varepsilon(u)\in\widehat{N}_{\delta/4w}(\mathcal{Z}_\varepsilon^{10\beta})$, $\kappa_1(\Upsilon_\varepsilon(\tau_\varepsilon(u)))=0$, hence $\mathcal{T}_\varepsilon(l,\tau_\varepsilon(u))=\tau_\varepsilon(u)\in\widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$. Above all we can see that $\mathcal{T}_\varepsilon(l,\tau_\varepsilon(u))$ is a map from $\widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$ to itself. $\square$

Lemma 2.10. *For small $\varepsilon>0$,*

(i) if $u\in\widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$ and $\Upsilon_\varepsilon(u)\in\Omega_\varepsilon^{8\beta}$, $\Gamma_\varepsilon\big(\mathcal{T}_\varepsilon(l,\tau_\varepsilon(u))\big)$ is non-increasing on $[0,l_s]$.

(ii) if $u\in\widehat{N}_{\delta/10w}(\mathcal{Z}_\varepsilon^{10\beta})$ and $\Upsilon_\varepsilon(u)\in\Omega_\varepsilon^{5\beta}\setminus\Omega_\varepsilon^{4\beta}$, there exists $\mu_0>0$ such that $\Gamma_\varepsilon\big(\mathcal{T}_\varepsilon(l_s,\tau_\varepsilon(u))\big)-\Gamma_\varepsilon\big(\mathcal{T}_\varepsilon(0,\tau_\varepsilon(u))\big)\le-\mu_0$.

Proof. Set $0\le l<l+h\le l_s$. For simplicity, we denote $x_\tau:=\varepsilon\Upsilon_\varepsilon(\tau_\varepsilon(u))$ and $\zeta_\varepsilon^C:=1-\zeta_\varepsilon$. Then through calculating, we have

$$\begin{aligned}
&\Gamma_\varepsilon\big(\mathcal{T}_\varepsilon(l+h,\tau_\varepsilon(u))\big)-\Gamma_\varepsilon\big(\mathcal{T}_\varepsilon(l,\tau_\varepsilon(u))\big)\\
=&\frac12\int_{\alpha_1+\alpha_2}^{\alpha_1+\alpha_2+\alpha_3}M(t)dt-\frac12\int_{\alpha_1+\alpha_2}^{\alpha_1+\alpha_2+\alpha_3'}M(t)dt\\
&+\int_{\mathbb{R}^N}V_\varepsilon\cdot\big(\zeta_\varepsilon^C\tau_\varepsilon(u)\big)(x)\cdot\big[\big(\zeta_\varepsilon\tau_\varepsilon(u)\big)(x-d_\varepsilon(l+h,\tau_\varepsilon))-\big(\zeta_\varepsilon\tau_\varepsilon(u)\big)(x-d_\varepsilon(l,\tau_\varepsilon))\big]\,dx\\
&+\frac12\int_{\mathbb{R}^N}\big[V(\varepsilon x+\Phi(l+h,x_\tau,\tau_\varepsilon(u)))-V(\varepsilon x+\Phi(l,x_\tau,\tau_\varepsilon(u)))\big]\big(\zeta_\varepsilon\tau_\varepsilon(u)\big)^2(x+\frac{x_\tau}{\varepsilon})dx\\
&-\int_{\mathbb{R}^N}\big[F(\mathcal{T}_\varepsilon(l+h,\tau_\varepsilon(u)))-F(\mathcal{T}_\varepsilon(l,\tau_\varepsilon(u)))\big]dx\\
&:=T_1-T_2+T_3+T_4-T_5,
\end{aligned}$$

where $\alpha_1=\int_{\mathbb{R}^N}|\nabla(\zeta_\varepsilon^C\tau_\varepsilon(u))|^2dx$, $\alpha_2=\int_{\mathbb{R}^N}|\nabla\big(\zeta_\varepsilon\tau_\varepsilon(u)\big)|^2dx$,
$\alpha_3=\int_{\mathbb{R}^N}\nabla\big(\zeta_\varepsilon^C\tau_\varepsilon(u)\big)(x)\cdot\nabla\big(\zeta_\varepsilon\tau_\varepsilon(u)\big)(x-d_\varepsilon(l+h,\tau_\varepsilon))dx$ and
$\alpha_3'=\int_{\mathbb{R}^N}\nabla\big(\zeta_\varepsilon^C\tau_\varepsilon(u)\big)(x)\cdot\nabla\big(\zeta_\varepsilon\tau_\varepsilon(u)\big)(x-d_\varepsilon(l,\tau_\varepsilon))dx$.

Since $\Phi(l,x_\tau,\tau_\varepsilon(u))=\Psi(\kappa_1(x_\tau)\kappa_2(\tau_\varepsilon(u))l,x_\tau)$, if $x_\tau\in\Omega^{2\beta}\cup(\mathbb{R}^N\setminus\Omega^{7\beta})$ or $\tau_\varepsilon(u)\in\widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})\setminus\widehat{N}_{\delta/4w}(\mathcal{Z}_\varepsilon^{10\beta})$, $\mathcal{T}_\varepsilon(l,\tau_\varepsilon(u))=\tau_\varepsilon(u)$ for all $l\in[0,l_s]$. Thus we consider the case that $\kappa_1(x_\tau)\kappa_2(\tau_\varepsilon(u))>0$. We denote $K=\kappa_1(x_\tau)\kappa_2(\tau_\varepsilon(u))h$ for simplicity. Then it follows from the definition of ζ_ε and (2.33), we have that for $\varepsilon>0$ small,

$$\frac{|T_1-T_2|}{K}\le M_0\frac{|\alpha_3-\alpha_3'|}{K}\le o(1)\quad\text{and}\quad\frac{|T_3|}{K}\le o(1).$$

Since $|d_\varepsilon(l,\tau_\varepsilon)|$, $|d_\varepsilon(l+h,\tau_\varepsilon)|\le R_1/\varepsilon$ and $\zeta_\varepsilon=1$ in $B(0,7R_1/\varepsilon)$,

$$\begin{aligned}
T_5=&\int_{\mathbb{R}^N\setminus B(0,6R_1/\varepsilon)}F\left(\big(\zeta_\varepsilon^C\tau_\varepsilon(u)\big)(x+d_\varepsilon(l+h,\tau_\varepsilon))+\big(\zeta_\varepsilon\tau_\varepsilon(u)\big)(x)\right)dx\\
&-\int_{\mathbb{R}^N\setminus B(0,6R_1/\varepsilon)}F\left(\big(\zeta_\varepsilon^C\tau_\varepsilon(u)\big)(x+d_\varepsilon(l,\tau_\varepsilon))+\big(\zeta_\varepsilon\tau_\varepsilon(u)\big)(x)\right)dx\\
=&\int_{\mathbb{R}^N\setminus B(0,6R_1/\varepsilon)}f\left(\big(\zeta_\varepsilon^C\tau_\varepsilon(u)\big)(x+d_\varepsilon(l,\tau_\varepsilon))+\theta g_\varepsilon(l,h,\tau_\varepsilon)\right)g_\varepsilon(l,h,\tau_\varepsilon)dx,
\end{aligned}$$

where $\theta\in(0,1)$ and $g_\varepsilon(l,h,\tau_\varepsilon)=\big(\zeta_\varepsilon^C\tau_\varepsilon(u)\big)(x+d_\varepsilon(l+h,\tau_\varepsilon))-\big(\zeta_\varepsilon^C\tau_\varepsilon(u)\big)(x+d_\varepsilon(l,\tau_\varepsilon))$. By (f1), (f2), (2.33) and (iii) of (V3), we can obtain that $|T_5|/K\le o(1)$.

Next we consider the term T_4. Denote $\widehat{V_\varepsilon} := V(\varepsilon x+\Phi(l+h,x_\tau,\tau_\varepsilon(u))) - V(\varepsilon x+\Phi(l,x_\tau,\tau_\varepsilon(u)))$, then

$$\begin{aligned}\frac{T_4}{K} =&\frac{1}{2K}\int_{|x|\le c_2/\varepsilon}\widehat{V_\varepsilon}\big(\zeta_\varepsilon\tau_\varepsilon(u)\big)^2(x+\frac{x_\tau}{\varepsilon})dx+\frac{1}{2K}\int_{|x|\ge c_2/\varepsilon}\widehat{V_\varepsilon}\big(\zeta_\varepsilon\tau_\varepsilon(u)\big)^2(x+\frac{x_\tau}{\varepsilon})dx\\ :=&T_{V_1}+T_{V_2}.\end{aligned}$$

We claim that $|T_{V_2}|\le o(1)$. In fact, denoting $x_{l+h} := x-\Phi(l+h,x_\tau,\tau_\varepsilon(u))/\varepsilon+x_\tau/\varepsilon$ and $x_l := x-\Phi(l,x_\tau,\tau_\varepsilon(u))/\varepsilon+x_\tau/\varepsilon$,

$$\begin{aligned}|T_{V_2}| =& \Bigg|\frac{1}{2K}\int_{|x-\frac{\Phi(l+h,x_\tau,\tau_\varepsilon(u))}{\varepsilon}|\ge\frac{c_2}{\varepsilon}} V_\varepsilon\big(\zeta_\varepsilon\tau_\varepsilon(u)\big)^2(x_{l+h})dx\\ &-\frac{1}{2K}\int_{|x-\frac{\Phi(l,x_\tau,\tau_\varepsilon(u))}{\varepsilon}|\ge\frac{c_2}{\varepsilon}} V_\varepsilon\big(\zeta_\varepsilon\tau_\varepsilon(u)\big)^2(x_l)dx\Bigg|\\ \le& \frac{1}{2K}\int_{D_1}V_\varepsilon\big(\zeta_\varepsilon\tau_\varepsilon(u)\big)^2(x_{l+h})dx+\frac{1}{2K}\int_{D_2}V_\varepsilon\big(\zeta_\varepsilon\tau_\varepsilon(u)\big)^2(x_l)dx\\ &+\frac{1}{2K}\int_{|x-\frac{\Phi(l,x_\tau,\tau_\varepsilon(u))}{\varepsilon}|\ge\frac{c_2}{\varepsilon}} V_\varepsilon\left|\big(\zeta_\varepsilon\tau_\varepsilon(u)\big)^2(x_{l+h})-\big(\zeta_\varepsilon\tau_\varepsilon(u)\big)^2(x_l)\right|dx\\ :=& A_1+A_2+A_3,\end{aligned}$$

where

$$D_1=\left\{x\in\mathbb{R}^N:\left|x-\frac{\Phi(l+h,x_\tau,\tau_\varepsilon(u))}{\varepsilon}\right|\ge\frac{c_2}{\varepsilon}\quad\text{and}\quad\left|x-\frac{\Phi(l,x_\tau,\tau_\varepsilon(u))}{\varepsilon}\right|\le\frac{c_2}{\varepsilon}\right\},$$

$$D_2=\left\{x\in\mathbb{R}^N:\left|x-\frac{\Phi(l,x_\tau,\tau_\varepsilon(u))}{\varepsilon}\right|\ge\frac{c_2}{\varepsilon}\quad\text{and}\quad\left|x-\frac{\Phi(l+h,x_\tau,\tau_\varepsilon(u))}{\varepsilon}\right|\le\frac{c_2}{\varepsilon}\right\}.$$

From the definition of D_1, D_2 and (iii) of (V3), there exist $M_1>0$ only dependent of N such that

$$|D_1|+|D_2|\le\frac{M_1}{\varepsilon^N}\left|\Phi(l+h,x_\tau,\tau_\varepsilon(u))-\Phi(l,x_\tau,\tau_\varepsilon(u))\right|\le\frac{M_1\mu K}{\varepsilon^N}. \tag{2.34}$$

Moreover, since $h\ge K>0$, we take $h>0$ small enough such that for any $x\in D_1$,

$$\begin{aligned}&\left|x-\frac{\Phi(l,x_\tau,\tau_\varepsilon(u))}{\varepsilon}\right|\\ \ge&\left|x-\frac{\Phi(l+h,x_\tau,\tau_\varepsilon(u))}{\varepsilon}\right|-\frac{1}{\varepsilon}\left|\Phi(l+h,x_\tau,\tau_\varepsilon(u))-\Phi(l,x_\tau,\tau_\varepsilon(u))\right|\\ \ge&\frac{c_2}{\varepsilon}-\frac{\mu K}{\varepsilon}\ge\frac{c_2}{2\varepsilon}.\end{aligned}$$

Then for h small, we have

$$\left|x-\frac{\Phi(l,x_\tau,\tau_\varepsilon(u))}{\varepsilon}\right|\ge\frac{c_2}{2\varepsilon},\quad\forall\, x\in D_1\cup D_2.$$

Hence combining with (2.33) and (2.34), we have that $|A_1|+|A_2|\le o(1)$.

For the term A_3, it follows from (2.33), standard C^2-estimate for the solution of (2.23) in [17] and (iii) of (V3) that

$$\limsup_{h\to0}|A_3|\le\frac{\mu}{\varepsilon}\int_{|x-\frac{\Phi(l,x_\tau,\tau_\varepsilon(u))}{\varepsilon}|\ge\frac{c_2}{\varepsilon}}V_\varepsilon\left|\big(\zeta_\varepsilon\tau_\varepsilon(u)\big)(x_l)\right|\cdot\left|\nabla\big(\zeta_\varepsilon\tau_\varepsilon(u)\big)(x_l)\right|dx\le o(1).$$

On the other hand for T_{V_1}, by (iv) in (V3) and $x_\tau \in \Omega^{7\beta} \setminus \Omega^{2\beta}$, we have that

$$\begin{aligned}
&T_{V_1}\\
&= \frac{1}{2K}\int_{|x|\le \frac{c_2}{\varepsilon}} [V(\varepsilon x+\Psi(K_0+K,x_\tau))-V(\varepsilon x+\Psi(K_0,x_\tau))]\left(\zeta_\varepsilon\tau_\varepsilon(u)\right)^2(x+\frac{x_\tau}{\varepsilon})dx\\
&\le -\frac{a}{2}\int_{|x|\le \frac{c_2}{\varepsilon}} \left(\zeta_\varepsilon\tau_\varepsilon(u)\right)^2(x+\frac{x_\tau}{\varepsilon})dx,
\end{aligned} \tag{2.35}$$

where $K_0 = \kappa_1(x_\tau)\kappa_2(\tau_\varepsilon(u))l$. Noting that $|x_\tau/\varepsilon - \Upsilon_\varepsilon(u)| \le 2R_0$ by lemma 2.7, then we have

$$B_0 := \liminf_{\varepsilon\to 0} \inf_{u\in \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})} \int_{|x-x_\tau/\varepsilon|\le c_2/\varepsilon} u^2 dx.$$

Hence $T_{V_1} \le -aB_0/2$. Above all, we have that for h and ε small enough,

$$\frac{1}{K}\left[\Gamma_\varepsilon\big(\mathcal{T}_\varepsilon(l+h,\tau_\varepsilon(u))\big) - \Gamma_\varepsilon\big(\mathcal{T}_\varepsilon(l,\tau_\varepsilon(u))\big)\right] \le -\frac{aB_0}{4}.$$

This implies $\Gamma_\varepsilon\big(\mathcal{T}_\varepsilon(l,\tau_\varepsilon(u))\big)$ is decreasing on l. For (ii) of this lemma, from lemma 2.7, $x_\tau \in \Omega^{6\beta} \setminus \Omega^{3\beta}$. By taking $h = l_s$, $l = 0$, $K = \kappa_1(x_\tau)\kappa_2(\tau_\varepsilon(u))l_s$ and the estimate for $T_1 - T_2$, T_3, T_4 and T_5, we can obtain the conclusion. $\square$

2.7. **Gradient flow of the energy functional Γ_ε.** Next we look for the critical points in the following set:

$$\Omega(\varepsilon,\nu,\delta) := (\Gamma_\varepsilon^{C_\varepsilon} \setminus \Gamma_\varepsilon^{E_m-\nu}) \cap \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta}),$$

for some positive constants ν and δ to be determined later. Arguing on the contrary, we assume that $\Gamma_\varepsilon(u)$ does not have the critical points in the set $\Omega(\varepsilon,\nu,\delta)$ for $\varepsilon > 0$ small.

It follows from lemma 2.3 (iv) and lemma 2.4 that for some $\nu,\ \delta > 0$

$$\Gamma_\varepsilon\big(A_\varepsilon(t,z)\big) \le E_m - \nu \quad \text{if } A_\varepsilon(t,z) \notin \widehat{N}_{\delta/40w}(\mathcal{Z}_\varepsilon^{10\beta}) \text{ or } \Upsilon_\varepsilon(A_\varepsilon(t,z)) \notin \Omega_\varepsilon^{2\beta}. \tag{2.36}$$

Taking two cut-off functions $\psi_\nu(l)$ and $\psi_\delta(u)$ in the following:
$\psi_\nu(l) = 1$ for $|l - E_m| \le \nu/2$ and $\psi_\nu(l) = 0$ for $|l - E_m| \ge \nu$;
$\psi_\delta(u) = 1$ for $u \in \widehat{N}_{5\delta/4}(\mathcal{Z}_\varepsilon^{10\beta})$ and $\psi_\delta(u) = 0$ for $u \notin \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$.

Now we consider the following ordinary differential equation:

$$\begin{cases} \dfrac{d\eta_\varepsilon(s,u)}{ds} = -\psi_\nu\big(\Gamma_\varepsilon(\eta_\varepsilon(s,u))\big)\psi_\delta\big(\eta_\varepsilon(s,u)\big)\dfrac{\Gamma_\varepsilon'(\eta_\varepsilon(s,u))}{\|\Gamma_\varepsilon'(\eta_\varepsilon(s,u))\|_\varepsilon^*}, \\ \eta_\varepsilon(0,u) = u. \end{cases} \tag{2.37}$$

Then there exists a unique solution $\eta_\varepsilon(s,u)$ for $s \in [0,\infty)$ such that $\eta_\varepsilon(s,u) \in \Omega(\varepsilon,\nu,\delta)$ for $u \in \Omega(\varepsilon,\nu,\delta)$. Moreover, observing the following lemma, the center of mass for $\eta_\varepsilon(s,u)$ does not go far away when s varies.

Lemma 2.11. *Let $u \in \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$ and $\Upsilon_\varepsilon(u) \in \Omega_\varepsilon^{9\beta}$. Assume that for $0 \le s_1 = s_1(\varepsilon) < s_2 = s_2(\varepsilon)$, there is some $c > 0$ independent of ε such that $|\Upsilon_\varepsilon(\eta_\varepsilon(s_1,u)) - \Upsilon_\varepsilon(\eta_\varepsilon(s_2,u))| \ge c/\varepsilon$, then $\lim_{\varepsilon\to 0}|s_2(\varepsilon) - s_1(\varepsilon)| = \infty$.*

Proof. First for $[s_1,s_2]$, we take interval division into $s_1 = t_0 < t_1 < t_2 < \cdots < t_k = s_2$ such that

$$|\Upsilon_\varepsilon(\eta_\varepsilon(t_{i+1},u)) - \Upsilon_\varepsilon(\eta_\varepsilon(t_i,u))| \ge \frac{c}{k\varepsilon} \quad \text{for} \quad i = 0,1,2,...,k-1.$$

Since $\eta_\varepsilon(s,u)\in N_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$ for any $s\ge 0$, by lemma 2.1 for each $i=0,1,2,...,k$, there exist $z_i\in\Omega_\varepsilon^{10\beta}$ and $v_i\in S(z_i)$ such that $\|\eta_\varepsilon(t_i,u)-v_i\|_\varepsilon\le 2\delta$ and $|\Upsilon_\varepsilon(\eta_\varepsilon(t_i,u))-z_i|\le R_0$. Then we have that $|z_{i+1}-z_i|\ge c/k\varepsilon-2R_0\ge R_0$ for $\varepsilon>0$ small. Hence, by (2.7) we have

$$\begin{aligned}\|\eta_\varepsilon(t_{i+1},u)-\eta_\varepsilon(t_i,u)\|_\varepsilon &\ge \|v_{i+1}-v_i\|_\varepsilon-\|\eta_\varepsilon(t_{i+1},u)-v_{i+1}\|_\varepsilon-\|\eta_\varepsilon(t_i,u)-v_i\|_\varepsilon\\ &\ge \|v_{i+1}\|_{H_\varepsilon(|x-z_i|\le R_0/2)}-\|v_i\|_{H_\varepsilon(|x-z_i|\ge R_0/2)}-4\delta\\ &\ge \frac{5}{8}r_*-4\delta\ge 6\delta.\end{aligned}$$

Thus it follows from $\|\frac{\partial\eta_\varepsilon}{\partial s}\|_\varepsilon^*\le 1$ and mean value theorem that there exists $C'>0$ such that $|t_{i+1}-t_i|>C'$ for each $i=0,1,...,k-1$. Consequently, we have $|s_1-s_2|>kC'$. Taking $k=[1/\sqrt{\varepsilon}]$, this completes the proof of this lemma. □

Lemma 2.12. *Taking appropriate $\delta>0$, for $u_\varepsilon\in\widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$, if $\limsup_{\varepsilon\to0}\Gamma_\varepsilon(u_\varepsilon)\le E_m$, $\eta_\varepsilon(l,u_\varepsilon)\in\widehat{N}_{5\delta/8}(\mathcal{Z}_\varepsilon^{10\beta})\setminus\widehat{N}_{\delta/10w}(\mathcal{Z}_\varepsilon^{10\beta})$ and $\Upsilon_\varepsilon(\eta_\varepsilon(l,u_\varepsilon))\in\Omega_\varepsilon^{8\beta}$ for some $l\in[0,\frac{\delta}{30w}]$, then $\Gamma_\varepsilon(\eta_\varepsilon(\delta/30w,u_\varepsilon))\le E_m-\nu/2$.*

Proof. On one hand we consider that if $l\in[\frac{\delta}{60w},\frac{\delta}{30w}]$, $\eta_\varepsilon(l,u_\varepsilon)\in\widehat{N}_{5\delta/8}(\mathcal{Z}_\varepsilon^{10\beta})\setminus\widehat{N}_{\delta/10w}(\mathcal{Z}_\varepsilon^{10\beta})$ and $\Upsilon_\varepsilon(\eta_\varepsilon(l,u_\varepsilon))\in\Omega_\varepsilon^{8\beta}$. By lemma 2.2, we also have that $\eta_\varepsilon(l,u_\varepsilon)\in N_{5\delta/8(1-q)}(\mathcal{Z}_\varepsilon^{10\beta})\setminus N_{\delta/10\sqrt{2}(1+q)w}(\mathcal{Z}_\varepsilon^{10\beta})$. Taking $s\in[l-\frac{\delta}{60w},l]$, it follows from $\|\frac{\partial\eta_\varepsilon}{\partial s}\|_\varepsilon^*\le 1$ that

$$\|\eta_\varepsilon(s,u_\varepsilon)-\eta_\varepsilon(l,u_\varepsilon)\|_\varepsilon\le\|\frac{d\eta_\varepsilon}{ds}\|_\varepsilon^*|l-s|\le\frac{\delta}{60w}. \tag{2.38}$$

Then noting that $\lim_{\delta\to0}q(\delta)=0$ in lemma 2.2, we take $\delta>0$ small such that

$$\frac{5}{4\sqrt{2}(1+q)}-\frac{5}{8(1-q)}>\frac14>\frac{1}{60w}\quad\text{and}\quad\frac{1}{10\sqrt{2}(1+q)w}-\frac{\sqrt{2}}{60(1-q)w}>\frac{1}{30w}>\frac{1}{60w}.$$

Hence it follows from (2.38) and lemma 2.2 that

$$\eta_\varepsilon(s,u_\varepsilon)\in N_{5\delta/4\sqrt{2}(1+q)}(\mathcal{Z}_\varepsilon^{10\beta})\setminus N_{\sqrt{2}\delta/60(1-q)w}(\mathcal{Z}_\varepsilon^{10\beta})\subset\widehat{N}_{5\delta/4}(\mathcal{Z}_\varepsilon^{10\beta})\setminus\widehat{N}_{\sqrt{2}\delta/60w}(\mathcal{Z}_\varepsilon^{10\beta}).$$

Moreover we take $\delta'=\frac{\sqrt{2}\delta}{60w}$ in lemma 2.4 and $\nu<\frac{\delta}{60w}\mu(\delta,\frac{\sqrt{2}\delta}{60w})$ in (2.36). Next we assume on the contrary that $\Gamma_\varepsilon(\eta_\varepsilon(s,u_\varepsilon))\ge E_m-\nu/2$ for any $s\in[l-\frac{\delta}{60w},l]$, then from (2.37), one has that for $\varepsilon>0$ small,

$$\begin{aligned}\Gamma_\varepsilon(\eta_\varepsilon(l,u_\varepsilon))-E_m+o(1)&\le\Gamma_\varepsilon(\eta_\varepsilon(l,u_\varepsilon))-\Gamma_\varepsilon(\eta_\varepsilon(l-\frac{\delta}{60w},u_\varepsilon))\\ &=\int_{l-\delta/60w}^{l}\Gamma_\varepsilon'(\eta_\varepsilon(t,u_\varepsilon))\frac{d\eta_\varepsilon(t,u_\varepsilon)}{dt}dt\\ &=-\int_{l-\delta/60w}^{l}\|\Gamma_\varepsilon'(\eta_\varepsilon(t,u_\varepsilon))\|_\varepsilon^*dt\\ &\le-\frac{\delta}{60w}\mu(\delta,\frac{\sqrt{2}\delta}{60w})<-\nu,\end{aligned}$$

which lead to a contradiction with the assumption.

On the other hand, if $l\in[0,\frac{\delta}{60w}]$, $\eta_\varepsilon(l,u_\varepsilon)\in\widehat{N}_{5\delta/8}(\mathcal{Z}_\varepsilon^{10\beta})\setminus\widehat{N}_{\delta/10w}(\mathcal{Z}_\varepsilon^{10\beta})$ and $\Upsilon_\varepsilon(\eta_\varepsilon(l,u_\varepsilon))\in\Omega_\varepsilon^{8\beta}$. Similarly, we can claim that there exists $s\in[l,l+\frac{\delta}{60w}]$ such that $\Gamma_\varepsilon(\eta_\varepsilon(s,u_\varepsilon))\le E_m-\nu/2$. Otherwise, we can also get a contradiction like the preceding part. □

3. Proof of the main results.

3.1. Iteration procedure by translation and gradient flow of Γ_ε.

we define a map $I : \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta}) \cap \Gamma_\varepsilon^{C_\varepsilon} \to \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta}) \cap \Gamma_\varepsilon^{C_\varepsilon}$ by

$$I(u) = \mathcal{T}_\varepsilon\left(l_s, \tau_\varepsilon\Big(\eta_\varepsilon(\frac{\delta}{30w}, u)\Big)\right).$$

Then, noting that Γ_ε is non-increasing under the maps τ_ε, $\mathcal{T}_\varepsilon$ and η_ε, one has $\Gamma_\varepsilon(I(u)) \le \Gamma_\varepsilon(u)$ for any $u \in \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$ and $I(u) = u$ if $u \notin \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$. Now we define the iteration I^k by

$$I^k = I^{k-1} \circ I, \qquad k = 2, 3, \cdots.$$

Since we assume that there is no critical point in $\Omega(\varepsilon, v, \delta)$, then there exists $k_\varepsilon > 0$ such that

$$\|\Gamma_\varepsilon'(u)\|_\varepsilon^* \ge k_\varepsilon \quad \text{for } u \in \Omega(\varepsilon, v, \delta). \tag{3.1}$$

We consider the iteration map I^k on $\Omega(\varepsilon, v, \delta)$ in the following lemma:

Lemma 3.1. *Assume (3.1) holds, taking $j_\varepsilon = \left[\frac{30w\nu}{k_\varepsilon\delta}\right] + 1$, then for $\varepsilon > 0$ small,*

$$\Gamma_\varepsilon\big(I^{j_\varepsilon}(A_\varepsilon(t,z))\big) \le E_m - \frac{1}{2}\min\{\nu, \mu_0\}, \quad \textit{for any } (t,z) \in [0,T] \times \mathcal{L}_\varepsilon.$$

Proof. We consider a set of sequences:

$$\left\{\eta_\varepsilon\left(\frac{\delta}{30w}, I^j\Big(A_\varepsilon(t,z)\Big)\right) : \ j = 0, 1, 2, \cdots, j_\varepsilon - 1\right\}.$$

From (2.36) and lemma 2.12, one has that if $\eta_\varepsilon(\frac{\delta}{30w}, I^j(A_\varepsilon(t,z))) \notin \widehat{N}_{\delta/10w}(\mathcal{Z}_\varepsilon^{10\beta})$ and $\Upsilon_\varepsilon\big(\eta_\varepsilon(\frac{\delta}{30w}, I^j(A_\varepsilon(t,z)))\big) \in \Omega_\varepsilon^{5\beta}$ for some $j = 0, 1, 2, \cdots, j_\varepsilon - 1$, we have

$$\Gamma_\varepsilon\left(I^{j_\varepsilon}\big(A_\varepsilon(t,z)\big)\right) \le \Gamma_\varepsilon\left(\eta_\varepsilon\Big(\frac{\delta}{30w}, I^j\big(A_\varepsilon(t,z)\big)\Big)\right) \le E_m - \frac{\nu}{2}.$$

And from lemma 2.10 (ii), if $\eta_\varepsilon(\frac{\delta}{30w}, I^j(A_\varepsilon(t,z))) \in \widehat{N}_{\delta/10w}(\mathcal{Z}_\varepsilon^{10\beta})$ and $\Upsilon_\varepsilon\big(\eta_\varepsilon(\frac{\delta}{30w}, I^j(A_\varepsilon(t,z)))\big) \notin \Omega_\varepsilon^{4\beta}$ for some $j = 0, 1, 2, \cdots, j_\varepsilon - 1$, one has that for $\varepsilon > 0$ small,

$$\begin{aligned}\Gamma_\varepsilon\left(I^{j_\varepsilon}\big(A_\varepsilon(t,z)\big)\right) \le& \Gamma_\varepsilon\left(I^{j+1}\big(A_\varepsilon(t,z)\big)\right)\\ \le& \Gamma_\varepsilon\left(\tau_\varepsilon\Big(\eta_\varepsilon(\frac{\delta}{30w}, I^j(A_\varepsilon(t,z)))\Big)\right) - \mu_0 \le E_m - \frac{\mu_0}{2}.\end{aligned}$$

Hence we note that there is a rest case: $\eta_\varepsilon(\frac{\delta}{30w}, I^j(A_\varepsilon(t,z))) \in \widehat{N}_{\delta/10w}(\mathcal{Z}_\varepsilon^{10\beta})$ and $\Upsilon_\varepsilon\big(\eta_\varepsilon(\frac{\delta}{30w}, I^j(A_\varepsilon(t,z)))\big) \in \Omega_\varepsilon^{4\beta}$ for all $j = 0, 1, 2, \cdots, j_\varepsilon - 1$. If this case takes place, we still have $\Gamma_\varepsilon\left(I^{j_\varepsilon}\big(A_\varepsilon(t,z)\big)\right) \le E_m - \nu/2$. In fact, we assume that for all $j = 0, 1, 2, \cdots, j_\varepsilon - 1$, $\Gamma_\varepsilon\left(\eta_\varepsilon\Big(\frac{\delta}{30w}, I^j\big(A_\varepsilon(t,z)\big)\Big)\right) \ge E_m - \nu/2$. Then

$$\begin{aligned}\Gamma_\varepsilon\left(I^{j_\varepsilon}\big(A_\varepsilon(t,z)\big)\right) =& \Gamma_\varepsilon\big(A_\varepsilon(t,z)\big) + \sum_{j=0}^{j_\varepsilon-1}\left[\Gamma_\varepsilon\Big(I^{j+1}\big(A_\varepsilon(t,z)\big)\Big) - \Gamma_\varepsilon\Big(I^j\big(A_\varepsilon(t,z)\big)\Big)\right]\\ \le& \Gamma_\varepsilon\big(A_\varepsilon(t,z)\big) + \sum_{j=0}^{j_\varepsilon-1}\left[\Gamma_\varepsilon\left(\eta_\varepsilon\Big(\frac{\delta}{30w}, I^j\big(A_\varepsilon(t,z)\big)\Big)\right) - \Gamma_\varepsilon\Big(I^j\big(A_\varepsilon(t,z)\big)\Big)\right],\end{aligned}$$

where

$$\Gamma_\varepsilon\left(\eta_\varepsilon\Big(\frac{\delta}{30w}, I^j\big(A_\varepsilon(t,z)\big)\Big)\right) - \Gamma_\varepsilon\Big(I^j\big(A_\varepsilon(t,z)\big)\Big) = \int_0^{\delta/30w} \Gamma'_\varepsilon(\eta_\varepsilon)\frac{d\eta_\varepsilon}{ds}ds$$
$$= -\int_0^{\delta/30w} \|\Gamma'_\varepsilon(\eta_\varepsilon)\|_\varepsilon^* ds \le -\frac{k_\varepsilon\delta}{30w} \le -\frac{\nu}{j_\varepsilon - 1}.$$

Thus for ε small, one has

$$\Gamma_\varepsilon\left(I^{j_\varepsilon}\big(A_\varepsilon(t,z)\big)\right) \le \Gamma_\varepsilon\big(A_\varepsilon(t,z)\big) - \frac{j_\varepsilon}{j_\varepsilon - 1}\nu \le E_m - \frac{\nu}{2}.$$

Above all, we have that $\Gamma_\varepsilon\left(I^{j_\varepsilon}\big(A_\varepsilon(t,z)\big)\right) \le E_m - \frac{1}{2}\min\{\mu_0, \nu\}$. $\square$

3.2. **Proof of Theorem 1.2.** We denote $B_\varepsilon(t,z) := \tau_\varepsilon\Big(I^{j_\varepsilon}\big(A_\varepsilon(t,z)\big)\Big)$, then it is clear that

$$\Gamma_\varepsilon\left(B_\varepsilon(t,z)\right) \le \Gamma_\varepsilon\left(I^{j_\varepsilon}\big(A_\varepsilon(t,z)\big)\right) \le E_m - \frac{1}{2}\min\{\mu_0, \nu\}, \tag{3.2}$$

for any $(t,z) \in [0,T] \times \mathcal{L}_\varepsilon$. Moreover, we claim that for $\varepsilon > 0$ small,

$$\begin{aligned} \Gamma_\varepsilon\left(B_\varepsilon(t,z)\right) \ge \frac{1}{2}\widehat{M}\big(\|\nabla B_\varepsilon(t,z)\|_2^2\big) + \frac{1}{2}\int_{\mathbb{R}^N} V\big(\varepsilon\widetilde{\Upsilon}_\varepsilon(B_\varepsilon(t,z))\big)B_\varepsilon^2(t,z)dx \\ - \int_{\mathbb{R}^N} F(B_\varepsilon(t,z))dx + o(1) \\ := J\big(B_\varepsilon(t,z)\big) + o(1). \end{aligned} \tag{3.3}$$

Indeed, if $A_\varepsilon(t,z) \notin \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$ for $(t,z) \in [0,T] \times \mathcal{L}_\varepsilon$, $B_\varepsilon(t,z) = A_\varepsilon(t,z)$. Then, from (2.9) and (2.29), we obtain (3.3). On the other hand, if $A_\varepsilon(t,z) \in \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$ for $(t,z) \in [0,T] \times \mathcal{L}_\varepsilon$, $B_\varepsilon(t,z) \in \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$. It follows from lemma 2.1, 2.7, 2.8, 2.11 and the property (i) of (V3) that (3.3) holds.

Next we prove that the following proposition:

Proposition 3.2. *For $\varepsilon > 0$ small, there exists $(t_\varepsilon, z_\varepsilon) \in (0,T) \times \mathcal{L}_\varepsilon \setminus L_0$ such that $J\big(B_\varepsilon(t_\varepsilon, z_\varepsilon)\big) \ge E_m$.*

Proof. Denote

$$\begin{aligned} D_\varepsilon(t,z) :=& \frac{N-2}{2}M\big(\|\nabla B_\varepsilon(t,z)\|_2^2\big)\int_{\mathbb{R}^N} |\nabla B_\varepsilon(t,z)|^2 dx \\ &+ N\int_{\mathbb{R}^N}\left[\frac{V\big(\varepsilon\widetilde{\Upsilon}_\varepsilon(B_\varepsilon(t,z))\big)}{2}B_\varepsilon^2(t,z) - F\big(B_\varepsilon(t,z)\big)\right]dx, \end{aligned}$$

for $(t,z) \in [0,T] \times \mathcal{L}_\varepsilon$. From the definition of $A_\varepsilon(t,z)$, taking $T_0 > 0$ and $\delta > 0$ small enough, we can have that $A_\varepsilon(T_0, z) \notin \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$ for any $z \in \mathcal{L}_\varepsilon$. Similarly, for $T > 0$ large, we also have $A_\varepsilon(T,z) \notin \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$ for any $z \in \mathcal{L}_\varepsilon$. Then for any $z \in \mathcal{L}_\varepsilon$, $B_\varepsilon(t,z) = A_\varepsilon(t,z)$ if $t = T_0$ or T. Hence it follows from (M3) and (M5) that for any $z \in \mathcal{L}_\varepsilon$

$$D_\varepsilon(T_0, z) > 0 \quad \text{and} \quad D_\varepsilon(T,z) < 0, \tag{3.4}$$

if $T_0 > 0$ small enough and $T > 0$ large enough.

On the other hand for any $z \in L_0$, since $\Upsilon_\varepsilon(A_\varepsilon(t,z)) = z/\varepsilon \notin \Omega_\varepsilon^{10\beta}$, then for ε small we have $A_\varepsilon(t,z) \notin \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$ for $(t,z) \in (T_0, T) \times L_0$. Hence by (M3) and

(M5), there exists unique $t(z) > 0$ such that

$$D_\varepsilon(t(z), z) = 0 \quad \text{and} \quad \frac{\partial D_\varepsilon(t(z), z)}{\partial t} \neq 0 \quad \text{for any} \quad (t, z) \in (T_0, T) \times L_0. \tag{3.5}$$

Now we claim that for small $\varepsilon > 0$, there exists $(t_\varepsilon, z_\varepsilon) \in (T_0, T) \times \mathcal{L}_\varepsilon \setminus L_0$ such that

$$D_\varepsilon(t_\varepsilon, z_\varepsilon) = 0 \quad \text{and} \quad V\Big(\varepsilon \widetilde{\Upsilon}_\varepsilon\big(B_\varepsilon(t_\varepsilon, z_\varepsilon)\big)\Big) = m.$$

In fact, we use the sequences $D_\varepsilon^{(l)} \in C^{N+1}\big((T_0, T) \times \mathcal{L}_\varepsilon\big)$ to approximate $D_\varepsilon(t, z) \in C\big((T_0, T) \times \mathcal{L}_\varepsilon\big)$ and satisfy $D_\varepsilon^{(l)}(t, z) = D_\varepsilon(t, z)$ for $(t, z) \in (T_0, T) \times L_0$.

For each l, it follows from sard theorem that we let $b_i^{(l)}$ be regular value of $D_\varepsilon^{(l)}$ with $b_i^{(l)} \to 0$ as $i \to \infty$. By choosing appropriate subsequences, we take subsequences l_i such that $l_i \to \infty$ and $b_i^{(l_i)} \to 0$ as $i \to \infty$. Since $b_i^{(l_i)}$ is regular value of $D_\varepsilon^{(l_i)}$ for each i, then $\big(D_\varepsilon^{(l_i)}\big)^{-1}(b_i^{(l_i)})$ is union of finitely many $k-$dimensional compact connected sub-manifold of $(T_0, T) \times \mathcal{L}_\varepsilon$.

Set $\mathcal{B}^i$ be the connected component which $\big(D_\varepsilon^{(l_i)}\big)^{-1}(b_i^{(l_i)})$ belongs to and intersects with $(T_0, T) \times L_0$. Then, setting $\pi_\varepsilon : (T_0, T) \times \mathcal{L}_\varepsilon \to \mathcal{L}_\varepsilon$ be natural projection, it follows from (3.5) that $\pi_\varepsilon : \partial\mathcal{B}^i \to \pi_\varepsilon(\partial\mathcal{B}^i) \subset L_0$ is homeomorphic. Moreover, for any $z \in \mathcal{L}_\varepsilon \backslash L_0$, the mod 2 degree $deg_2(\pi_\varepsilon, \mathcal{B}^i, z)$ is well defined, and $deg_2(\pi_\varepsilon, \mathcal{B}^i, z) = 1$ for any z close to L_0. Since $\mathcal{L}_\varepsilon$ is connected and $\pi_\varepsilon(\partial\mathcal{B}^i) \subset L_0$, $deg_2(\pi_\varepsilon, \mathcal{B}^i, z)$ is independent of $z \in \mathcal{L}_\varepsilon \backslash L_0$. Hence $\pi_\varepsilon(\mathcal{B}^i) = \mathcal{L}_\varepsilon$ and $\pi_\varepsilon(\partial\mathcal{B}^i) = L_0$. Then we have $\mathcal{B}^i \in \mathcal{L}(L_0)$. From (1.7), we have

$$\max_{(t,z)\in\mathcal{B}^{(i)}} V\big(\varepsilon \widetilde{\Upsilon}_\varepsilon(B_\varepsilon(t, z))\big) \geq m.$$

Moreover for $(t, z) \in (T_0, T) \times L_0$,

$$V\big(\varepsilon \widetilde{\Upsilon}_\varepsilon(B_\varepsilon(t, z))\big) = V(z) \leq m_* < m.$$

From (3.4), we observe that

$$D_\varepsilon^{(l_i)}(T_0, z) > 0 \quad \text{and} \quad D_\varepsilon^{(l_i)}(T, z) < 0,$$

for i large and $z \in \mathcal{L}_\varepsilon$. Thus there exist $(t_i, z_i) \in (T_0, T) \times \mathcal{L}_\varepsilon \setminus L_0$ such that

$$V\big(\varepsilon \widetilde{\Upsilon}_\varepsilon(B_\varepsilon(t_i, z_i))\big) = m \quad \text{and} \quad D_\varepsilon^{(l_i)}(t_i, z_i) = b_i^{(l_i)}.$$

Letting $i \to \infty$, we have that $(t_i, z_i) \to (t_\varepsilon, z_\varepsilon) \in (T_0, T) \times \mathcal{L}_\varepsilon \backslash L_0$, $D_\varepsilon(t_\varepsilon, z_\varepsilon) = 0$ and $V\big(\varepsilon \widetilde{\Upsilon}_\varepsilon(B_\varepsilon(t_\varepsilon, z_\varepsilon))\big) = m$. By proposition 4.3 in Appendix, $J\big(B_\varepsilon(t_\varepsilon, z_\varepsilon)\big) \geq E_m$. $\square$

Combining with (3.3), proposition 3.2 leads to a contradiction with (3.2). In summary, for any $d > 0$ in (V3), there exists $\varepsilon_d > 0$ such that for $\varepsilon \in (0, \varepsilon_d)$, Γ_ε has critical point $u_\varepsilon \in \Omega(\varepsilon, \nu, \delta)$. Since $u_\varepsilon \in \widehat{N}_{2\delta}(\mathcal{Z}_\varepsilon^{10\beta})$, $\varepsilon\Upsilon_\varepsilon(u_\varepsilon) \in \Omega^{10\beta} \subset \mathcal{M}^d$. Let x_ε be a maximum point of u_ε. By lemma 2.1, one has $\lim_{\varepsilon\to 0} dist(\varepsilon x_\varepsilon, \mathcal{M}) = 0$. Moreover, noting that $\limsup_{\varepsilon\to 0} \Gamma_\varepsilon(u_\varepsilon) \leq E_m$ and $\Gamma'_\varepsilon(u_\varepsilon) = 0$, it follows from the proof of lemma 2.4, up to subsequence, $u_\varepsilon(\cdot + x_\varepsilon)$ converges to $U \in S_m$ as $\varepsilon \to 0$. For theorem 1.2 (iii), we refer to the proof of lemma 2.5 or [16]. Consequently, we complete the proof of the theorem.

4. **Appendix.** In the following, we introduce two propositions to treat with the regularization of positive solutions, which play an important role in the exponential decay of $\tau_\varepsilon(u)$.

Proposition 4.1 ([4]). *Let $u \in H^1(\mathbb{R}^N)$ solve*

$$-\Delta u + \big(b(x) - q(x)\big)u = h(x,u) \quad in \quad \mathbb{R}^N,$$

where $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^+$ is a Caratheodory function with

$$0 \le h(x,v) \le C_h(v + v^r) \quad for\ all\ v > 0,\ x \in \mathbb{R}^N,\ 1 < r < 2^* - 1,$$

and $b : \mathbb{R}^N \to \mathbb{R}^+$ is a $L^\infty_{loc}(\mathbb{R}^N)$ function, $q(x) \in L^{N/2}(\mathbb{R}^N)$. Then, $u \in L^s(\mathbb{R}^N)$ for all $s \ge 2$. Moreover, there is a positive constant C_s depending on s, $q(x)$ and C_h, such that $\|u\|_s \le C_s\|u\|$. In addition, the dependence of C_s on $q(x)$ can be given uniformly on a Cauchy sequence $\{q_k(x)\}$ in $L^{N/2}(\mathbb{R}^N)$.

Proposition 4.2 ([17]). *For an open set $\Omega \subset \mathbb{R}^N$, assume that $t > N$, $g \in L^{t/2}(\Omega)$ and $u \in H^1(\Omega)$ solves in the weak sense*

$$-\Delta u \le g(x) \quad in\ \Omega.$$

Then, for any ball $B_{2R}(z) \subset \Omega$,

$$\sup_{x \in B_R(z)} u(x) \le C\big(\|u^+\|_{L^2(B_{2R}(z))} + \|g\|_{t/2}\big),$$

where C depends on N, t and R.

In the next proposition, we consider the least energy value of L_a on the following set:

$$\mathcal{P} := \left\{u \in H^1(\mathbb{R}^N) \setminus \{0\} : P(u) = 0\right\}.$$

Proposition 4.3. *Assume (f1)-(f3) and (M1)-(M6) hold. Then there exists a minimizer U of $\inf_{u \in \mathcal{P}} L_a(u)$ such that U is a ground state of the equation (2.4). Hence*

$$E_a = \inf_{u \in \mathcal{P}} L_a(u).$$

Proof. For simplicity we assume $a = 1$ in (2.4). Take $\{u_n\} \subset \mathcal{P}$ be the minimizing sequences. Let $\{u_n^*\}$ be a radially symmetric decreasing rearrangement of u_n (see [2]), and denote $u_n^{*,s} = u_n^*(\frac{x}{s})$. Then from $\|\nabla u_n^*\|_2 \le \|\nabla u_n\|_2$, we note that for $s \in (0,1]$

$$\begin{aligned} M\big(\|\nabla u_n^{*,s}\|_2^2\big)\int_{\mathbb{R}^N} |\nabla u_n^{*,s}|^2 dx &= M\big(s^{N-2}\|\nabla u_n^*\|_2^2\big)s^{N-2}\int_{\mathbb{R}^N} |\nabla u_n^*|^2 dx \\ &\le M\big(s^{N-2}\|\nabla u_n\|_2^2\big)s^{N-2}\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \\ &\le M\big(\|\nabla u_n\|_2^2\big)\int_{\mathbb{R}^N} |\nabla u_n|^2 dx, \end{aligned}$$

on the other hand,

$$\begin{aligned} \int_{\mathbb{R}^N} \big[\frac{1}{2}(u_n^{*,s})^2 - F(u_n^{*,s})\big]dx &= s^N \int_{\mathbb{R}^N} \big[\frac{1}{2}(u_n^*)^2 - F(u_n^*)\big]dx \\ &\le \int_{\mathbb{R}^N} \big[\frac{1}{2}u_n^2 - F(u_n)\big]dx. \end{aligned}$$

Thus we have

$$s^N\left\{\frac{N-2}{2}\cdot\frac{M\big(s^{N-2}\|\nabla u_n^*\|_2^2\big)}{s^2}\int_{\mathbb{R}^N}|\nabla u_n^*|^2dx+N\int_{\mathbb{R}^N}\big[\frac{1}{2}(u_n^*)^2-F(u_n^*)\big]dx\right\}\le 0.$$

By (M3) and (M5), there exists $s_n\in(0,1]$ such that $u_n^{*,s_n}\in\mathcal{P}$. Then

$$\begin{aligned}L_1(u_n^{*,s_n})&=\frac{1}{2}\widehat{M}\big(s_n^{N-2}\|\nabla u_n^*\|_2^2\big)+s_n^N\int_{\mathbb{R}^N}\big[\frac{1}{2}(u_n^*)^2-F(u_n^*)\big]dx\\&\le\frac{1}{2}\widehat{M}\big(s_n^{N-2}\|\nabla u_n\|_2^2\big)-s_n^N\frac{N-2}{2N}M\big(\|\nabla u_n\|_2^2\big)\|\nabla u_n\|_2^2:=g(s_n).\end{aligned}$$

Noting that

$$\frac{dg(s_n)}{ds_n}=\frac{N-2}{2}s_n^{N-3}\|\nabla u_n\|_2^2\left\{M\big(s_n^{N-2}\|\nabla u_n\|_2^2\big)-s_n^2M\big(\|\nabla u_n\|_2^2\big)\right\},$$

from (M5), we have

$$\frac{dg(t)}{dt}\ge 0\text{ for }t\in(0,1]\quad\text{and}\quad\frac{dg(t)}{dt}\le 0\text{ for }t\in[1,\infty).$$

Hence

$$\max_{t\in[0,\infty)}g(t)=g(1)=\frac{1}{2}\left\{\widehat{M}\big(\|\nabla u_n\|_2^2\big)-\big(1-\frac{2}{N}\big)M\big(\|\nabla u_n\|_2^2\big)\|\nabla u_n\|_2^2\right\}=L_1(u_n).$$

Thus we can assume the minimizing sequence u_n is radially symmetric decreasing. From (M2), we have $\|\nabla u_n\|_2$ is bounded.

Next we prove $\|u_n\|_2$ is bounded. Since $u_n\in\mathcal{P}$, it follows from (f1) and (f2) that

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{u_n^2}{2}dx\le&\frac{N-2}{2N}M\big(\|\nabla u_n\|_2^2\big)\|\nabla u_n\|_2^2+\int_{\mathbb{R}^N}\frac{u_n^2}{2}dx\\=&\int_{\mathbb{R}^N}F(u_n)dx\le\frac{1}{4}\int_{\mathbb{R}^N}u_n^2dx+C_{1/4}\int_{\mathbb{R}^N}|u_n|^{2^*}dx.\end{aligned}$$

Then, by sobolev imbedding $D^{1,2}(\mathbb{R}^N)\hookrightarrow L^{2^*}(\mathbb{R}^N)$, we have

$$\frac{1}{4}\int_{\mathbb{R}^N}u_n^2dx\le C\int_{\mathbb{R}^N}|\nabla u_n|^2dx.$$

Consequently $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$.

Moreover, noting that $H_r^1\hookrightarrow L^q(\mathbb{R}^N)$ is compact for $q\in(2,\frac{2N}{N-2})$, we have $\lim\limits_{n\to\infty}\int_{\mathbb{R}^N}F(u_n)dx=\int_{\mathbb{R}^N}F(u)dx$. It follows from weakly lower semi-continuity that

$$\frac{N-2}{2}M\big(\|\nabla u\|_2^2\big)\int_{\mathbb{R}^N}|\nabla u|^2dx+N\int_{\mathbb{R}^N}\big[\frac{u^2}{2}-F(u)\big]dx\le 0.$$

As before denoting $u^s(x)=u(x/s)$, there exists $t\in(0,1]$ such that $u^t\in\mathcal{P}$ and $L_1(u^t)\le L_1(u)\le\lim\limits_{n\to\infty}L_1(u_n)$. Hence u^t is a minimizer of $\inf\limits_{u\in\mathcal{P}}L_1(u)$.

Next we prove that this minimizer is the least energy solution of the equation (2.4). Assume U is the minimizer. Then there exists a lagrange multiplier $\lambda\in\mathbb{R}$ satisfying

$$\begin{aligned}&-\Big(\big(1+\lambda(N-2)\big)M(\|\nabla U\|_2^2)+\lambda(N-2)M'(\|\nabla U\|_2^2)\|\nabla U\|_2^2\Big)\Delta U\\&+(1+\lambda N)\big(U-f(U)\big)=0\text{ in }\mathbb{R}^N.\end{aligned}$$

The corresponding pohozaev identity of this equation is

$$\frac{\big(1+\lambda(N-2)\big)(N-2)}{2}M\big(\|\nabla U\|_2^2\big)\|\nabla U\|_2^2+\frac{\lambda(N-2)^2}{2}M'(\|\nabla U\|_2^2)\|\nabla U\|_2^4$$
$$+(1+\lambda N)N\int_{\mathbb{R}^N}\big[\frac{U^2}{2}-F(U)\big]dx=0.$$

Since $U\in\mathcal{P}$, we have

$$\lambda\Big(M\big(\|\nabla U\|_2^2\big)\|\nabla U\|_2^2+\frac{2-N}{2}M'(\|\nabla U\|_2^2)\|\nabla U\|_2^4\Big)=0.$$

It follows from (M6) that $\lambda=0$. Thus the minimizer U satisfies (2.4). Since any solution of (2.4) satisfies pohozaev identity, the minimizer is the least energy solution of (2.4). □

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