Lecture 2 (1.2)

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Contents

1.2 Introduction to Sequences

Sequence: an ordered list of numbers: 1, 3, 4, 5, 7, 0, -1, 20001 (finite!)

An infinite sequence: {2, 4, 8, 16..., 2^n,...} or $\{2^n\}$.
Notation: $\{a_1, a_2, a_3, ..., a_n, ...\}$ or $\{a_n\}$from $n = 1$ to infinite, or simply $\{a_n\}$.

Definition: Explicitly (like above), or recursively: $a_1 = 2, a_n + 1 = 2 \times a_n$ (n is called the index, a_n are called elements or terms)

Subsequences and Tails

Definition:
Let $\{a_n\}$ be a sequence. Let $\{n_1, n_2, n_3, ..., n_k, ...\}$ be a sequence of natural numbers such that $n_1 < n_2 < n_3 < ... < n_k < ...$ A subsequence of $\{a_n\}$ is a sequence of the form $\{a_{n_k}\} = \{a_{n_1}, a_{n_2}, a_{n_3}, ..., a_{n_k}, ...\}$. $\{n1, n2, n3, ..., nk, ...\}$ represents the indexes of the subsequence taken from the original {an} sequence.

A tail of a sequence is when given a sequence $\{a_n\}$ and $k \in \mathbb{N}$, the subsequence $\{a_k, a_{k+1}, a_{k+2}, ...\}$ is called the tail of $\{a_n\}$ with a cutoff k.

Limits of Sequences

Formal definition of the Limit of a Sequence:
We say that L is the limit of the sequence $\{a_n\}$ as n goes to infinite if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then

$$|an - L| < \epsilon$$

If such an L exists, we say that the sequence is convergent and write:

$$\lim_{n\to\infty} a_n = L$$

We may also use the notation $a_n \to L$ to mean $\{a_n\}$ converges to L.
If no such L exists, then we say that the sequence diverges.

Examples:

1) Prove $a_n = \frac{1}{n^2}$, $L = 0$

If $\epsilon = \frac{1}{100}$, will there be some N for which $n \geq N$ and $|\frac{1}{n^2} - 0| < \frac{1}{100}$?

We can rearrange like this: $\frac{1}{n^2} < \frac{1}{100} \to 100 < n^2$
If $n > 10$ then, $N = 10, n \geq 10 \to |\frac{1}{n^2} - 0| < \frac{1}{100}$
Since $n \geq N$ and $|\frac{1}{n^2} - 0| < \epsilon$, we can write this as:
$\frac{1}{n^2} < \epsilon \to n^2 > \frac{1}{\epsilon} \to n > \frac{1}{\sqrt{\epsilon}}$
$N \geq \frac{1}{\sqrt{\epsilon}} \to |\frac{1}{n^2} - 0| < \epsilon$

Essentially, what we've proven here is that if we rearrange n so that we get $n > \frac{1}{\sqrt{\epsilon}}$, then we know that if N is greater or equal to $\frac{1}{\sqrt{\epsilon}}$, then this inequality will be true. So when $N > \frac{1}{\sqrt{\epsilon}}$ then we know that $n^2 > N^2 \geq \frac{1}{\epsilon}$.
Given this fact, we can prove that if we rearrange that inequality, we get $\frac{1}{n^2} < \epsilon$, meaning $|\frac{1}{n^2} - 0| < \epsilon$ if $N > \frac{1}{\sqrt{\epsilon}}$. Essentially, we are rearranging the inequality so we find what N has to be for the formal definition of the limit to be true, and returning to the original statement.

2) Prove $\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2}$

We know $\epsilon > 0, n > N$.
Step by step:

$$|\frac{n}{2n+3} - \frac{1}{2}| < \epsilon \tag{1}$$
$$|\frac{2n - 2n - 3}{4n+6}| < \epsilon \tag{2}$$
$$|\frac{-3}{4n+6}| < \epsilon \tag{3}$$
$$\frac{3}{4n+6} < \epsilon \tag{4}$$
$$\frac{3}{4\epsilon} - \frac{3}{2} < n \tag{5}$$
$$\tag{6}$$