Lecture 3 (1.4 - 2.3)

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Contents

1.4.3 Negation of Quantified statements

Consider: "Everyone in this room is right handed"
Which in symbols is,

$$\forall x \in S, P(x)$$

S = set of people in room
P(x) is "x is right-handed"
The negation is "Someone in the room is not right-handed",

$$\exists x \in S, \neg P(x)$$

Thus,

$$\neg(\forall x \in S, P(x)) = \exists x \in S, \neg P(x)$$

Also since,

$$\neg(\neg P) = P$$
$$\neg(\exists x \in S), P(x)) = \forall x \in S, \neg P(x)$$

Negation of $x^2 - x >= 0$ for all real numbers x (false) is,

$$x^2 - x < 0 \ for\ some\ real\ numbers\ x\ (true)$$
$$\neg(\forall x \in \mathbb{R}, x^2 - x \geq 0) \equiv (\exists x \in \mathbb{R}, x^2 - x < 0)$$

1.5 Nested Quantifiers

Examples:
a) $\exists m \in \mathbb{Z}, \exists n \in \mathbb{Z}, m + n \ is\ even\ (True)$
b) $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, m + n \ is\ odd\ (False)$
c) $\exists m \in \mathbb{Z}, \forall n \in \mathbb{Z}, m + n \ is\ odd\ (False)$
d) $\forall m \in \mathbb{Z}, \exists n \in \mathbb{Z}, m + n \ is\ even\ (True)$
e) $\exists m \in \mathbb{Z}, \forall n \in \mathbb{Z}, m + n \ is\ even\ (False)$
f) $\forall m \in \mathbb{Z}, \exists n \in \mathbb{Z}, m + n \ is\ odd\ (True)$
Variables to the right depend on variables to the left.
From above, A and B are negations of each other,

$$\neg(\exists m \in \mathbb{Z}, \exists n \in \mathbb{Z}, m + n \ is\ even) \equiv \forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, m + n \ is\ odd\ (not\ even)$$

C and D and E and F are negations as well
In general **flip quantifiers, and negate the open sentence P(x).**
The order of quantifiers matters in statements with mixed quantifiers.

2.2 Conjunctions and Disjunctions

Definition: Conjunction (P AND Q) "$P \wedge Q$" is defined as,

P	Q	$P \wedge Q$
True	True	True
True	False	False
False	True	False
False	False	False

Example:

$$(1 + 1 = 3) \wedge (\pi + 2 < 6)$$

is false because $1 + 1 = 3$ is false. However,

$$(1 + 1 = 2) \wedge (\pi + 2 < 6)$$

is true.

Definition: Disjunction (P or Q) "$P \vee Q$" is defined as,

P	Q	$P \vee Q$
True	True	True
True	False	True
False	True	True
False	False	False

$\neg P$, $P \wedge Q$, $P \vee Q$, are examples of compound statements, ie. statements that are composed of one or more component statements (P, Q, ...) via logical operators. ($\neg$, $\wedge, \vee$, ...)

P	Q	$(P \wedge Q)$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$(\neg P) \vee (\neg Q)$
True	True	True	False	False	False	False
True	False	False	True	False	True	True
False	True	False	True	True	False	True
True	True	False	True	True	True	True

As $\neg(P \wedge Q)$ always has the same truth values as $(\neg P) \vee (\neg Q)$, they are logically equivalent.

$$\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$$

2.3 Logical Operators and Algebra

De Morgan's Laws (DML)
1. $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$
2. $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$

Example: The negation of
Both $e + \pi$ and $e\pi$ are rational
is
Either $e + \pi$ is irrational or $e\pi$ is irrational

Practice:
Let P, Q, R be statements

$$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R) -$$

P	Q	R	$Q \wedge R$	$P \wedge (Q \wedge R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
True	True	True	True	True	True	True	True
True	True	False	True	True	True	False	True
True	False	True	True	True	False	True	True
False	True	True	True	False	False	False	False
False	False	True	True	False	False	False	False
False	True	False	True	False	False	False	False
True	False	False	False	False	False	False	False
False	False	False	False	False	False	False	False

The truth table shows that, for every possible assignment of truth values P, Q, R, statement 1 has the same truth value as statement 2. Hence the two statements are logically equivalent.