Imperial College London, Department of Mathematics

M3R Project

The Stubborn-Effect in Opinion Dynamics

Cynthia Zeng
CID:00923820
Under the supervision of Professor Jonathan Haskel

June 19, 2017

Abstract

This project intends to study the dynamics of opinion formation in a well-mixed system. In specific, the effect of a certain type of stubborn agent's effect in altering the dynamics of opinion formation.

We adapt the Galam sequential model and assume two types of agents: the stubborn and the open-minded, and a set of specified interaction rules. In addition, we analyse what level of stubborn-agents may lead to an extreme case, absolute winning situation. In addition, computer simulation results are obtained to verify analytical conclusions.

Two important findings include:
1. According to the level of stubborns, we can separate group dynamics into three cases. Among those cases, low-stubborn absolute winning situation reveals that even a small portion of stubborn agents can strongly influence the dynamics.
2. Higher frequency of information exchange lead to an enhanced stubborn-effect

Acknowledgements

I would like to thank my supervisors Prof. Jonathan Haskel and Prof. Henrik Jensen for providing guidance and feedback throughout the course of my project. In addition, I would like to thank Dr. Lynda White for providing guidance on the topic of voting theory.

In addition, I would like to thank the Mathematics Department of Imperial College London for all the resources provided.

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is my own work unless otherwise stated.

Contents

Chapter 1

Introduction

The initial inspiration for this project was triggered by the "Brexit" vote in 2016. I was intrigued by the sense of surprise in this referendum result. Because intuitively, a voting outcome should be one that reflects the majority's opinion; hence it should be democratic, sensible, and somewhat "expected". Therefore I conducted this project in hope to offer some explanations to the counter-intuitive result by studying how opinions evolve in a social group.

Our society has never been so connected before. As an example of "small world " phenomenon, research shows that any two Facebook users "know" each other through less than 4 intermediaries [1]. As individual, our view points can be easily influenced, if not shaped, by this ever-growing complex network. As Alex Pentland stated: "we are now coming to realise that human behaviour is determined as much by social context as by rational thinking" [2]. Therefore collective behaviour, such as majority opinion, can be viewed as an emergent property of a system that is intrinsically linked to the interactions of its component.

One approach among scientists to such a problem is through social physics, in which mathematical models inspired by physical systems are studied to understand collective behaviour of a crowd [3] [4] [5]. In other words, by studying hypothetical models which reflect some realistic but simplified behaviour rules, we can gain insights into understanding the opinion dynamics of an actual social system.

One important model is the Galam sequential model [6] [7], where two distinctive types of agents are assumed: stubborn agents (or "inflexibles" in his paper) and open-minded agents (or "floaters"). Stubborn agents represent the "loyal" voters in the society; whereas the open-minded represent those who are more willing to change opinions. Interactions among agents are represented by the "grouping process" to reflect debates or social gatherings, during which agents are randomly allocated to small subgroups. Open-minded agents may change their opinion according to the local majority, whereas stubborn agents do no change. This model shows that the "stubborns" have much greater influence over the open-minded on opinion formation in a well-connected social group.

In this project, we adapt the Galam sequential model; we assume the same two agents types and similar interaction rules. But in addition, we focus on some extreme cases in order to explore the extend to which stubborn agents can alter the outcome. We conclude that both high and low levels of stubborn agents can obtain absolute winning situation, and in fact less proportion of stubborn agents can lead to more volatile shifts in formation.

The structure of this report is as follows. In Chapter 2, we establish the model and derive key equations to describe the system. In Chapter 3, by treating the level of stubborns as parameters, we study when does the system obtain a special case: absolute winning situation. Numerical and analytical results reveal that according to the level of stubborns, we can separate the system into three distinctive cases. In Chapter 4, using Python simulations, we verify previous conclusions, and further explore the relationship between group size and the "stubborn-effect". Finally, in Chapter 5 we apply our conclusion into the context of the "Brexit" vote, and propose further research ideas.

Chapter 2

Model Setup

2.1 Background

In social choice theory, various mathematicians and economists have studied the difficulties and challenges in making a collective decision via voting [9] [10]. In specific, Arrow's impossibility theorem states that when three or more alternatives are present, no ranked voting scheme can satisfy a specified set of four criteria to ensure "fairness" [8]. In response, May's theorem [11] states a simple majority voting scheme between two alternatives is a voting scheme satisfying the four criteria set by Arrow to ensure fairness

Hence to avoid having an electoral paradox at any time, we assume two distinctive opinions in this project. Specifically, we assume a well-mixed population of agents (or voters), and each agent holds either opinion A or B at one time. We conduct a simple majority vote at various times to decide on a majority opinion.

Definition 2.1.1. *Simple majority vote consists of two alternative candidates, represented by A and B. Every agent's opinion count as one vote, and the opinion with more votes is the majority opinion or the winner. Local majority is the majority opinion within a specific subgroup of agents. The percentage of agents who hold opinion A (or B) is called the support for A (or B).*

Definition 2.1.2. *Well-mixed population is a system where all agents are allowed to freely interact with any other agent.*

2.2 Key Assumptions

This project intends to explore the bias in opinion formation caused by "stubborn" behaviour among agents. This can be understood as loyalty or conviction towards one party during an election. Hence we distinguish between two types of agents: open-minded and stubborn agents. We represent social interaction by repeatedly arranging agents into subgroups. Each grouping step represents one unit of time. Two types of agents behave differently during grouping processes, and we study the effect of stubborn agents on shaping majority over time.

Definition 2.2.1. *An <u>open-minded agent</u> may hold different opinions at various times, and open-minded agents update its opinion to the local majority when placed in a subgroup. Denote by open-A or open-B.*

A <u>stubborn agent</u> holds one opinion at all times; in addition, a stubborn agent does not change its opinion when placed in a subgroup. Denote by stubborn-A or stubborn-B. Denote the proportion of stubborn-A in the system by a, stubborn-B by b; where $0 \geq a, b < 1$ *and* $a + b < 1$

Here is an example to demonstrate the difference between two types of agents.

Example 2.2.2. *Let* A_o, B_o *denote open-A, open-B, and* A_s, B_s *denote stubborn-A and stubborn-B. Suppose we have a group of 5 agents,* $\{A_o, A_o, A_s, B_s, B_o\}$*, hence a=0.2, b-0.2, local majority is A with 60% support. Let open-minded agents "update" their opinion to the local majority, i.e. open-B becomes open-A, and stubborn-B unchanged. Thus the group becomes* $\{A_o, A_o, A_s, B_s, A_o\}$*, opinion A has 80% majority vote.*

Remark: Open-minded agents can interchange opinions but it would always remain open-minded, i.e. A_o, B_o interchange. Therefore, the level of $A_o + B_o$ remains constant through out. On the other hand, the level of stubborn-A and stubborn-B, denoted by a,b remain unchanged throughout. Hence we view them as parameters in the system, and their effect will be discussed in more detail later in Chapter 2 and Chapter 3.

To represent the act of a small "debate" or "social gathering" among voters, in our model we arrange agents into subgroups. Let us first assume discrete time step, and each grouping step represent one unit of time. We assume the two types of agents have different behaviours, and study the effect of stubborn agents on majority formation over time.

Definition 2.2.3. *Let us formally set up the model in the following steps:*

1. *Distribute a large population of N agents into subgroups of size r, a fixed odd integer.*
2. *Within each group, a local majority opinion emerges because r is odd. Open-minded agents update their opinions to the local majority; stubborn agents unchanged. Record opinion distribution, and move on to the next time step.*
3. *Shuffle and regroup agents; repeat step.2.*

Remarks:
- We assume N is very large, and r divides N.
- We take r to be an odd integer to ensure local majority within subgroups
- This model is symmetric for A and B. Therefore in this project let us choose to study the problem from only A's perspective, i.e. looking at the support change for A throughout the grouping process. Support change for B would be the complement of A.

2.3 Simple Population Model (SPM)

Definition 2.3.1. *A simple population is a population where all agents are open-minded, i.e.* $a = b = 0$*. A mixed population is a population consist of both stubborn agents and open-minded agents.*

In this section, let us first study the problem in a simple population, where all agents are open-minded and act according to rules specified by definition 2.2.3. Hence it is called a Simple Population Model (SPM). The purpose is to derive a discrete-time difference equation to describe opinion change over time, so based on this we can derive a similar equation for a more complicated model later.

By definition, open-minded agents update opinions according local majority within subgroups, and these subgroups are randomly formed at each time step. Hence, due to the stochastic nature of group formation, we study the average behaviour using a statistical mechanics approach [12]. In other words, instead of looking for the exact state (or opinion make up) for a system, we calculate the probability of the system having a certain state at a specific time. This approximation should be realised when the population is sufficiently large. Hence, let us now derive a difference equation to describe the change of A's support, proportion of stubborn-A and open-A, over time.

Definition 2.3.2. *Looking at the problem from A's perspective let:*
p_t= *probability that a randomly chosen agent holds opinion A at time t*
Hence equivalently,
p_t=*proportion of agents that support opinion A at time t*
= *proportion of open-A and stubborn-A at time t*

Proposition 2.3.3. *The difference equation to describe the evolution of A's support, p_t over time under a SPM is*

$$p_{t+1} = \sum_{m=\frac{r+1}{2}}^{r} \binom{r}{m} p_t^m (1-p_t)^{r-m} \tag{2.1}$$

Derivation:
First, by definition, p_{t+1} = probability that a randomly chosen agent holds opinion A at time t+1. Since in SPM all agents are open-minded agents, an agent holds opinion A at time t+1 if and only if it belongs to a subgroups of local majority A at time t.

Hence, the problem reduces to finding the probability that a randomly chosen agent belongs to a subgroup whose local majority is A at time t. In other words, it is the probability that a randomly chosen subgroup has local majority A at time t.

Consider a subgroup consisting of r agents, denote:
M= number of agents who support A in a subgroup

There are only two possible states for an agent: support A or B. At time t, a randomly picked agent has probability p_t of supporting A. Hence, in a randomly picked subgroup of size r, the number of agents supporting A follows a binomial distribution: $M \sim Bino(r, p_t)$

Therefore, the probability of choosing a subgroup whose local majority is A is equivalent as choosing a subgroup where $M > \frac{r}{2}$, hence:

$$\begin{aligned} p_{t+1} &= P(M \geq \frac{r+1}{2}) \\ &= \sum_{m=\frac{r+1}{2}}^{r} P(M=m) \\ &= \sum_{m=\frac{r+1}{2}}^{r} \binom{r}{m} p_t^m (1-p_t)^{r-m} \end{aligned}$$

as required. $\square$

Consider the incrementation of time steps to be very small, and we move from discrete time case into continuous time case. Thus, we view the support for A, p(t) as a continuous function of t, and derive a one-dimensional ODE to describe the evolution of p(t). This allows us to analyse the fixed points of the model.

$$p_{t+1} = \sum_{m=\frac{r+1}{2}}^{r} \binom{r}{m} p_t^m (1-p_t)^{r-m}$$

$$p_{t+1} - p_t = \sum_{m=\frac{r+1}{2}}^{r} \binom{r}{m} p_t^m (1-p_t)^{r-m} - p_t$$

taking small increment of time

$$\frac{dp}{dt} \approx \lim_{\Delta t \to 0} p_{t+1} - p_t \approx \sum_{m=\frac{r+1}{2}}^{r} \binom{r}{m} p^m (1-p)^{r-m} - p$$

Therefore, we arrive at the following differential equation to describe the continuous time evolution of support for A, with this we can study the system by analysing behaviour of its fixed points.

$$\frac{dp}{dt} = \sum_{m=\frac{r+1}{2}}^{r} \binom{r}{m} p^m (1-p)^{r-m} - p \tag{2.2}$$

Now let us take $r = 3$, the smallest odd integer to begin with. Following a standard method [13] to determine fixed points and their stabilities :
Settings equation 2.2 to zero to determine the fixed points:

$$f(p) = \frac{dp}{dt} = p^3 + 3p^2(1-p) - p = 0$$
$$P^*_{1,2,3} = 0, \frac{1}{2}, 1$$

For stability, consider the Jacobian matrix, J. Since this is a one-dimensional ODE, $J = f'$:

$$J = f' = -6p^2 + 6p - 1$$
$$f'(P^*_1) = -1 < 0 \implies stable$$
$$f'(P^*_2) = \frac{1}{2} > 0 \implies unstable$$
$$f'(P^*_2) = -1 < 0 \implies stable$$

p(t) is the proportion of the population that support A, let us denote the three fixed points $P^*_{1,2,3}$ respectively by: $P_A = 1$, all agents support A; $P_B = 0$, all agents support B; $P_C = \frac{1}{2}$, the critical value which separates the direction of the opinion flow.

Figure 2.1 illustrates the opinion flow. It shows that P_C=0.5 is a critical point, if initially if $P_0 < 0.5$, we eventually arrive at P_B; if $P_0 > 0.5$, we arrive at P_A, if $P_0 = 0.5$, we remain there. In other words, if A is the initial majority, the mechanism of the system would convince the whole population to support A, and vice versa.

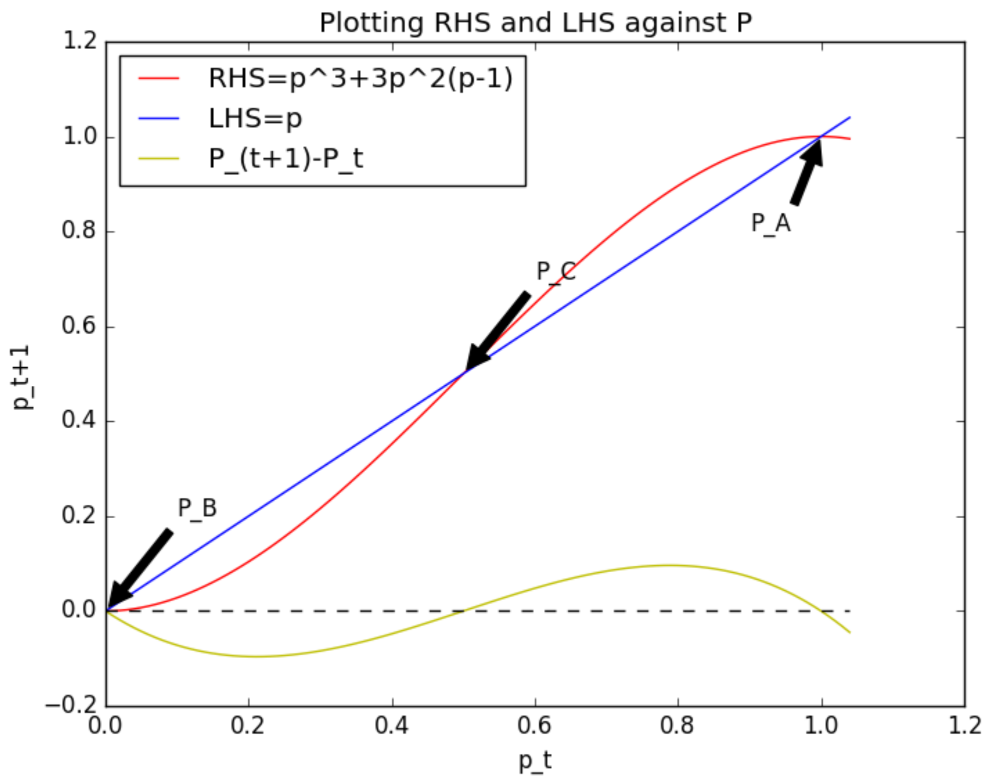


Figure 2.1: The blue curve indicates p_t, red curve indicates p_{t+1}, and yellow curve indicates the direction and velocity of the opinion flow. To the left of P_C, velocity is negative, hence opinion evolves to the left towards P_B. To the right of P_C, velocity is positive, hence opinion evolves to the right towards P_A.

We see that the unstable fixed point P_C serves as a critical value in this system. That is, in order for A to eventually become the winner, A needs $P_0 > P_C$. Let us formally define such critical value:

Definition 2.3.4. *Critical initial support, $P_C \in [0, 1]$, such that if $P_0 > P_C$, the system would approach P_A; if $P_0 < P_C$, the system would shift away from P_A. Or equivalently, A needs to achieve at least P_C amount of initial support in order to become the eventual winner.*

To summarise, in SPM $P_C = 0.5$ represents an unbiased system, that the mechanism of the system does not work in favour of any party. After interactions, initial majority would become the eventual winner. However, surprising results arise when some stubborn agents are introduced to the system. In the next section, we see by introducing stubborn agents, the value of P_C can be drastically altered and even vanish, thus leading to a situation where regardless of P_0 one party would always be the eventual winner.

2.4 Mixed Population Model (MPM)

In reality it is likely some people have conviction in their opinion, and they may not change their stance during these subgroup debates. Hence building on the previous methodology, in this section let us derive a similar difference equation to describe opinion flow in a mixed population system. Mixed population model (MPM) refers to a system where stubborn agents are present, and once again, agents interact according to specified rules described by definition 2.2.3.

In contrast to SPM, here stubborn agents within subgroups do not "update" to local majority opinion. For instance, stubborn-B's do not update to support A even when they are placed in a subgroup where local majority is A. Hence, the strategy is to derive the difference equation is to develop from SPM case, exclude "overcounted" stubborn-Bs, and "add back" stubborn-As.

Proposition 2.4.1. *The time difference equation to describe the evolution of support for A in a mixed population for group size r, a positive odd integer, is:*

$$p_{t+1} = \sum_{m=\frac{r+1}{2}}^{r} \binom{r}{m} p_t^m (1-p_t)^{r-m} - \Sigma_1 + \Sigma_2 \tag{2.3}$$

where

$$\Sigma_1 = \sum_{m=\frac{r+1}{2}}^{r} \sum_{k=0}^{r-m} \binom{r}{m, k, r-m-k} p_t^m b^k (1-p_t-b)^{r-m-k} \frac{k}{r}$$

$$\Sigma_2 = \sum_{m=\frac{r+1}{2}}^{r} \sum_{k=0}^{r-m} \binom{r}{m, k, r-m-k} (1-p_t)^m a^k (p_t-a)^{r-m-k} \frac{k}{r}$$

Remark:
Σ_1 represents the "overcounted" stubborn-Bs which we subtract; and Σ_2 represents "undercounted" stubborn-As which we add back.

Derivation:
First, consider Σ_1:

$$\begin{aligned} &Let: \\ &\quad X_1 = \text{number of agents support A in a subgroup} \\ &\quad recall: \quad X_1 \sim Bino(r, p_t) \text{ at time t} \\ &\quad X_2 = \text{number of stubborn-B in a subgroup} \\ &\quad X_3 = \text{number of open-B in a subgroup} \end{aligned}$$

Since an agent can be only one of three types: support A, stubborn-B or open-B. In other words, X_1, X_2, X_3 are disjoint events whose union is the entire sample space, i.e. $\sum_{i=1}^{3} X_i = r$. Hence (X_1, X_2, X_3) follow a multinomial distribution. Recall that at time t, an agent support A with probability p_t, an agent who is stubborn-B with probability b, and an agent who is open-B with probability $1 - p_t - b$:

$$(X_1, X_2, X_3) \sim Multinomial(p_1 = p_t, p_2 = b, p_3 = 1 - p_t - b)$$

Therefore, in a specific subgroup, the probability that m agents support A, k agents are stubborn-B is:

$$P(X_1 = m, X_2 = k, X_3 = r - m - k) = \frac{n!}{m!k!(r-m-k)!} p_t^m b^k (1 - p_t - b)^{r-m-k}$$

Consider probability of "over-counting" stubborn-B, i.e. the stubborn-B's being placed in a group whose local majority is A. In such a group, since majority is A, hence $X_1 \geq \frac{r+1}{2}$, and $X_2 \leq r - X_1$. Hence summing all possibilities we arrive at:

$$\sum_{m=\frac{r+1}{2}}^{r-1} \sum_{k=0}^{r-m} \binom{r}{m, k, r-m-k} p_t^m b^k (1 - p_t - b)^{r-m-k}$$

Note this is the probability of such a subgroup occurs, but we are interested in the probability of choosing specifically the stubborn-B in such a subgroup. Hence we need to multiply by $\frac{k}{r}$, which is the probability of choose one of the k stubborn-B's in this group. Hence we arrive at:

$$\Sigma_1 = \sum_{m=\frac{r+1}{2}}^{r-1} \sum_{k=0}^{r-m} \binom{r}{m, k, r-m-k} p_t^m b^k (1 - p_t - b)^{r-m-k} \frac{k}{r}$$

The derivation for Σ_2 is almost identical due to the symmetry of the problem. If we switch a and b, p_t and $1 - p_t$, we can arrive at desired Σ_2. □

For r=3, under MPM:

$$p_{t+1} = p_t^3 + 3p_t^2(1 - p_t) - p_t^2 b + (1 - p_t)^2 a \tag{2.4}$$

For r=5, under MPM:

$$p_{t+1} = \sum_{m=3}^{5} \binom{5}{m} p_t^m (1 - p_t)^{5-m} - \Sigma_1 + \Sigma_2 \tag{2.5}$$

$$\text{where} \quad \Sigma_1 = \frac{5!}{3!} p_t^3 b (1 - p_t - b) \frac{1}{5} + \frac{5!}{3!2!} p_t^3 b^2 \frac{2}{5} + \frac{5!}{4!} p_t^4 b \frac{1}{5}$$
$$\Sigma_2 = \frac{5!}{3!} (1 - p_t)^3 a (p_t - a) \frac{1}{5} + \frac{5!}{3!2!} (1 - p_t)^3 a^2 \frac{2}{5} + \frac{5!}{4!} (1 - p_t)^4 a \frac{1}{5}$$

To understand the opinion flow of this system, once again let us analyse the fixed points of the system. Figure 2.2 and 2.3 illustrate that stubborn agents can effectively influence the mechanism of the system; let us call it the "stubborn-effect". It is intuitive when the there are more stubborn-A than stubborn-B in the system, there is a bias towards A. Specifically, in figure 2.3, we arrive at the situation where just one fixed point is present, so the stubborn agents determine the outcome completely; i.e. neither P_0 nor open minded agents matter in the system any more. However, looking at the numbers in this case, 29% of total stubborn agents is not a very high proportion of the population. It is

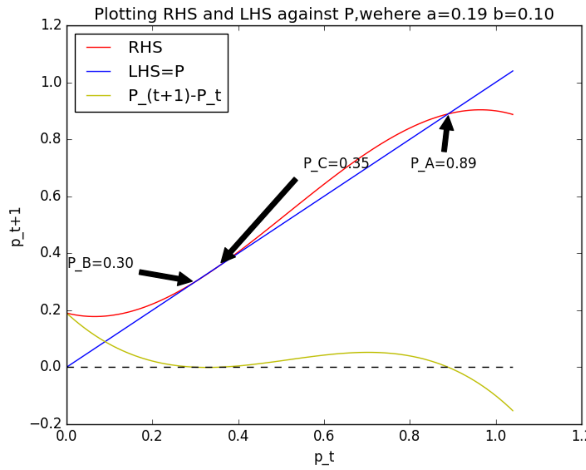


Figure 2.2: Fix points occur when $p_{t+1} = p_t$, or the intersection of the blue and red curves in the figure. At a=0.19, b=0.10, $P_C \approx 0.35$, skewed to the left. If initial support for A, $P_0 > 0.35$, A would become the eventual winner, and vice versa.

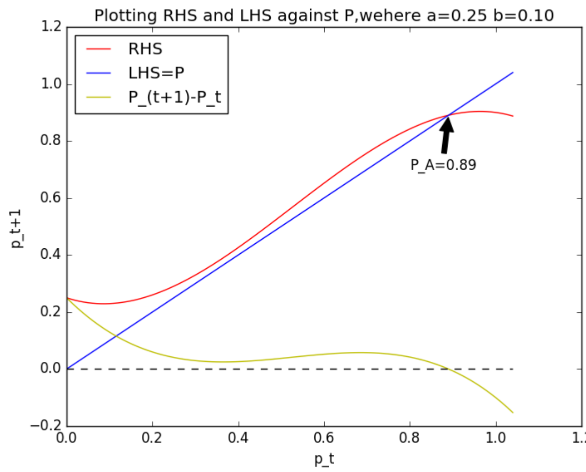


Figure 2.3: a=0.25, b=0.10, P_C vanishes completely.That is to say, regardless of initial support, A would always become eventual winner.

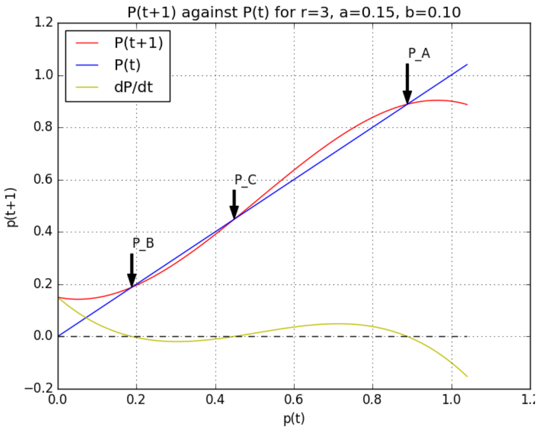
rather surprising that such a relatively low proportion of the population can determine the outcome of the entire system.

Let us formally define the situation in figure 2.3, where a high proportion of stubborn-A makes A the "absolute winner" regardless of opinion of the open-minded agents.

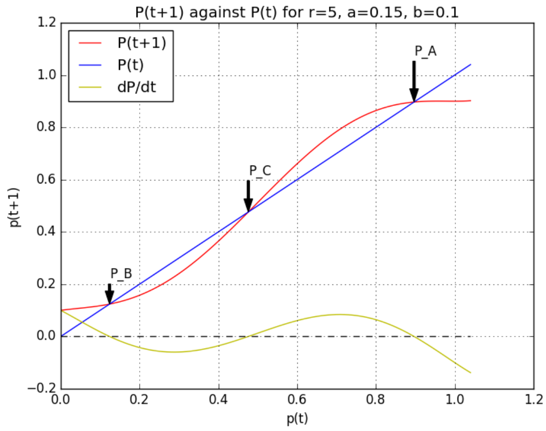
Definition 2.4.2. *Absolute winning (AW), the case when there is only one attractive fixed point in the system; denote the level of stubborn-A and stubborn-B by* a^*, b^* *respectively; winner in this situation is called the winner absolute winner.*

Now let us investigate the stubborn-effect with group sizes r=3 and r=5. Figure 2.4 reinforces the point that when stubborn-A is more than stubborn-B, level of P_C is skewed to the left, hence creating a bias for A in the system. In addition, we see that this stubborn-effect is less strong when group size is bigger.

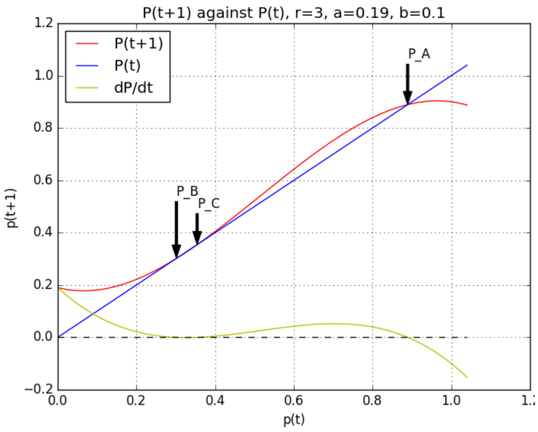
To summarise, by introducing stubborn agents into the system, they create a "stubborn-effect", reflected by a shift in critical initial support P_C. On special cases, we have absolute winning situation, where the outcome is entirely determined by the level of stubborns.



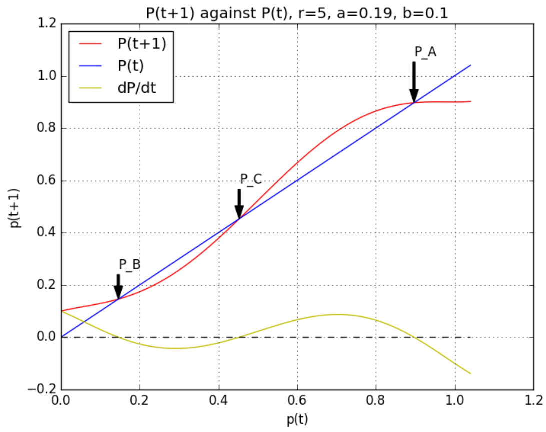
(a)



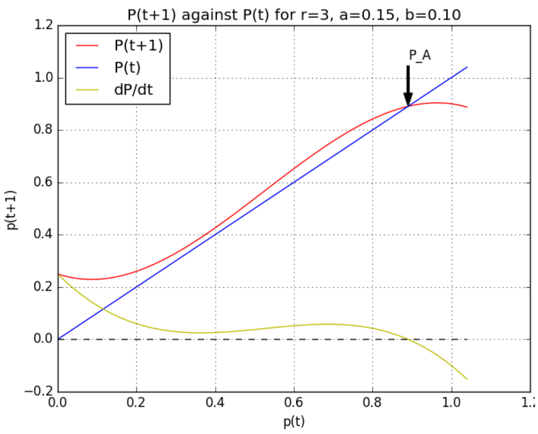
(b)



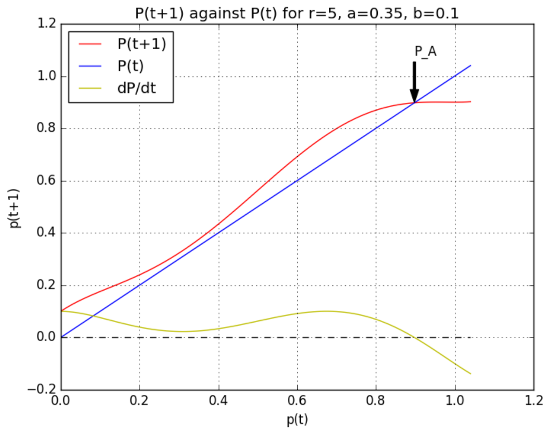
(c)



(d)



(e) A is absolute winner, and a^*=0.15 $b^* = 0.1$



(f) Here A is the absolute winner, but compare to (e) $a^* = 0.35$ requires much higher stubborn-A.

Figure 2.4: Figures on the left all have group size r=3, on the right are r=5. Testing on various different values of a, b, we see that the stubborn-effect, which is reflected shifting by P_C away from 0.5, is less strong for r=5 compared to r=3.

2.5 Semi-Stubborns

So far all agents belong to two categories: open-minded and stubborn. In reality, there may be people who are semi-stubborn, or stubborn only to a certain extent. Even though these agents are not explicitly studied in the model, in fact they can be represented as discounted stubborn-A $c * a$ where $c \in [0, 1]$, similar for semi-stubborn-B.

Definition 2.5.1. *A semi-stubborn agent with flexibility c, $c \in [0, 1]$, is an agent who updates to local majority with a probability c.*

Proposition 2.5.2. *If a population consists of all open-minded agents except for αsemi-stubborn-A with flexibility c, then the difference equation becomes:*

$$p_{t+1} = \sum_{m=3}^{r} \binom{r}{m} p_t^m (1 - p_t)^{r-m} + \Sigma_2' \tag{2.6}$$

where

$$\Sigma_2' = c * \sum_{m=\frac{r+1}{2}}^{r} \sum_{k=0}^{r-m} \binom{r}{m, k, r-m-k} (1 - p_t)^m \alpha^k (p_t - \alpha)^{r-m-k} \frac{k}{r}$$

Remark: Method here is very similar to the derivation outlined in proposition 2.4.1. In fact, looking at equation 2.6, it looks almost identical to equation 2.3. Here Σ_2' is $c * \Sigma_2$ with a substituted by α.

Outline of Derivation:
Let us first consider c=0.5, i.e. a semi-stubborn-A who only updates its opinion to local majority at half of the times. Hence as before, when adding back "undercounted" semi-stubborn-A agents during grouping process, we can treat it as a stubborn-A but multiplied by 50% to account for the fact that it is a semi-stubborn agent. Hence we have $\Sigma_2' = 0.5 * \Sigma_2$

This argument can be extended to any $c \in [0, 1]$, and hence we can treat any semi-stubborn agent as discounted stubborn-A or stubborn-B. Hence we do not need to explicitly study the effect of semi-stubborns; instead, their effect could be incorporated as discounted stubborn-A and stubborn-B and be taken into account by an over all a and b.

Chapter 3

The Stubborn-Effect

In Chapter 2, we understand that stubborn agents effectively inject a bias into the system. Moreover, as figure 2.3 demonstrates, in special cases, the system can become an absolute winning situation, where initial support for A is not relevant to the outcome of the system. That is to say, the opinion of open-minded agents are not relevant at all, and the group dynamics is entirely determined by the level of stubborn-A and stubborn-B.

Therefore, in this chapter, we study the effect of stubborn agents by focusing on this special case. That is to say, we treat a and b as parameters of the system, and investigate at what level of a and b do we obtain an absolute winning situation.

Note that group size determines the leading power of the difference equation. The goal of this chapter is not to obtain solutions for an arbitrary r, but rather to gain insights into how a, b as parameters influence the dynamics of the system. Hence, for simplicity of calculation, we consider only $r = 3$ in this chapter and work with degree three polynomial.

3.1 One-Parameter Problem

Similar as before, let us derive a continuous-time differential equation from for MPM from equation 2.4 by taking small time increment, hence:

$$\frac{dp}{dt} = -2p^3 + (3 - b + a)p^2 - (2a + 1)p + a \tag{3.1}$$

with constraints:

$$0 \leq a \leq 1, \quad 0 \leq b \leq 1, \quad 0 \leq a + b \leq 1$$

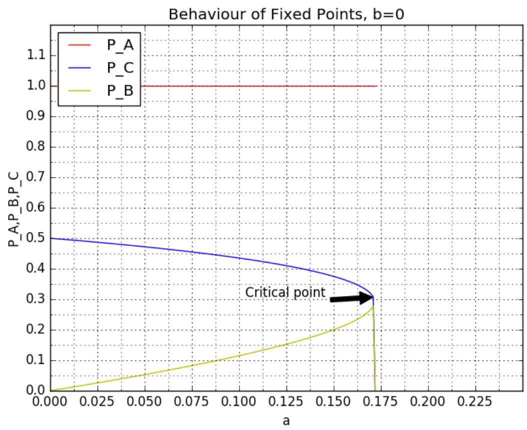
Recall that p(t) is the support for A at time t, so this equation describes the evolution of A's support over time. Fixed points are the roots of this equation. We are trying to understand the relationships between a, b and bifurcation of fixed points in this system.

This is a problem of two parameters, namely a and b. To understand the dynamics, let us first study the problem with respect to one variable, and fixing the other parameter to a constant. Note that due to symmetry, we can choose to treat either a or b as a constant. Hence, let us fix b to be constant, and explore the relationship between a and fixed points of the system.

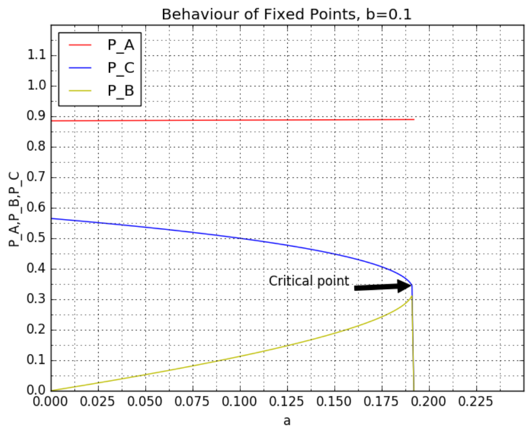
To approach this problem, we should assume b to be relatively low (e.g. < 0.25) so that bifurcation can happen by increasing A. Because if b is too high, it is likely that B is the absolute winner and bifurcation cannot happen regardless of a. Here, numerical results are obtained in Python by treating b as a fixed constant, and solving equation 3.1 at various a values.

Figures 3.1 show that, by fixing stubborn-B to be a relatively low level, at $b = 0, 0.1, 0.15$ respectively. While increasing stubborn-A, P_C should decrease, hence increasing bias towards P_A . Birfucation of system, reflected by the merge of P_B and P_C , occurs at critical level of stubborn-A, denote by a^*. With these specific b values, we obtain $a^* \approx 0.172, 0.192, 0.206$ respectively. Hence, we see that the higher level of stubborn-B, the higher a^* is required to achieve absolute winning situation.

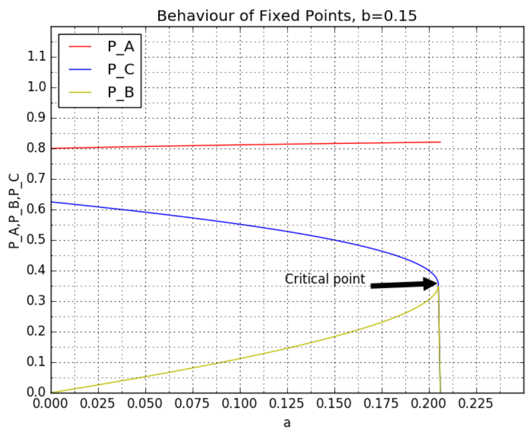
Remark: precision of a^* to three decimal places should suffice, because a^* is an estimation obtained using equation 3.1, which describes the statistical behaviour of the system



(a) b=0, $a^\star \approx 0.172$



(b) $b = 0.1, a^\star \approx 0.192$



(c) $b = 0.15, a^\star \approx 0.206$

Figure 3.1: These results are obtained in Python, red, blue, yellow curve represent respectively the position of three fixed points when level of stubborn-A increases from 0 to 0.25, stubborn-B fixed at b=0, 0.1, 0.15 respectively. As a increase, bifurcation happens at a^* when blue curve and yellow curve merge, hence beyond this point, the system has only one attractive fixed point P_A .

In specific, figure 3.1b explains the bifurcation behind figures 2.2 and 2.3 from Chapter 2. We know $a^* \approx 0.192$ is the exact level of a at which bifurcation happens. Recall in figure 2.2 $a = 0.19 < a^*$, there are three fix points in the system; whereas in figure 2.3 when $a = 0.25 > a^*$, only one fix point exists.

To summarise, in section we study the stubborn-effect by fixing one parameter, b, to be a constant. We can make a conjecture that at a fixed level of stubborn-B (which we assume to be relatively low), there exists a critical level of stubborn-A, a^*, which can be found numerically. Such that if $a > a^*$, the system becomes absolute winning situation where A is the absolute winner. In addition, the total proportion of stubborn agents need not be over 50% to achieve absolute winning situation. This section gives us the intuition that bifurcation should be intrinsically connected to the level of both a and b. Hence in the next section, we study the problem by treating both a, b as parameters.

3.2 Two-Parameter Problem

Recall that fixed points are roots to the differential equation 3.1:

$$f = \frac{dp}{dt} = -2p^3 + (3 - b + a)p^2 - (2a + 1)p + a$$

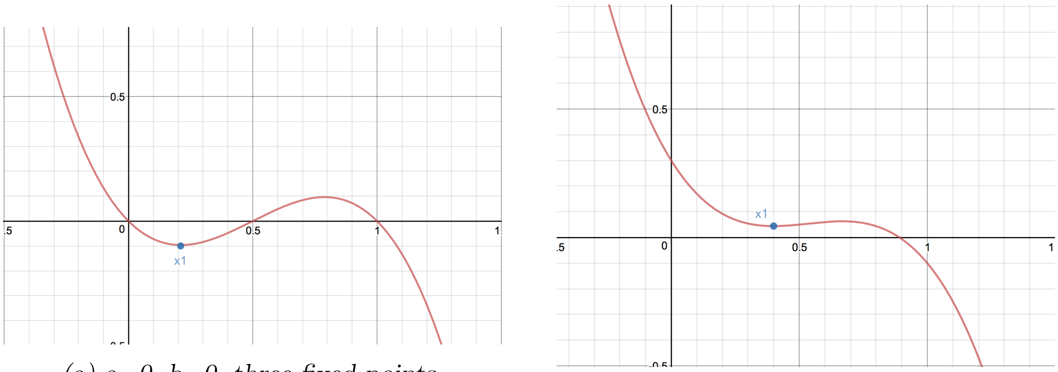
with constraints:

$$0 \le a \le \frac{1}{2}, \quad 0 \le b \le \frac{1}{2}, \quad 0 \le a + b \le 1$$

This is a degree 3 polynomial, with leading coefficient equals to -2, so it has a downward shape. By the fundamental theory of algebra, this equation has three complex roots, with at least one root in the real space. Hence, we can paraphrase the problem to understand at what level of a, b does the system shift from having three real roots to one real root.

Once again, due to symmetry, let us consider the problem from A's perspective. That is, we investigate at what level of a, b does the system become absolute winning situation, and A is the absolute winner.

We know that the number of real roots to polynomial 3.1 is an "indicator" for whether bifurcation happens in the system. Hence the strategy is to find a connection between this indicator and parameters a, b. One approach is analysing the polynomial geometrically. In specific, as illustrated by figure 3.2a and 3.2b, by looking at the position of the local minimum point, denoted by x_1 , we can instantly tell how many real roots this polynomial has. Hence, we can further specify the "indicator" to be the y-value of x_1 , $f(x_1)$. When $f(x_1)$ is negative, there are three fixed points; when $f(x_1)$ is positive there is one fixed point; hence, bifurcation happens when $f(x_1)$ =0. Let us express this indicator in terms of a and b, to investigate how these two parameters act together effect the dynamics.



(a) a=0, b=0, three fixed points, non-AW

(b) a=0.3 b=0.1, one fixed point, AW situation

Figure 3.2

First, note that x_1 is a stationary point of the curve; we can derive an algebraic expression for x_1 by setting f' to zero. Hence:

$$x_1 = \frac{3 - b + a}{6} - \frac{\sqrt{\Delta}}{6}, \text{ where } \Delta = (a-b)^2 - 6(a+b) + 3$$

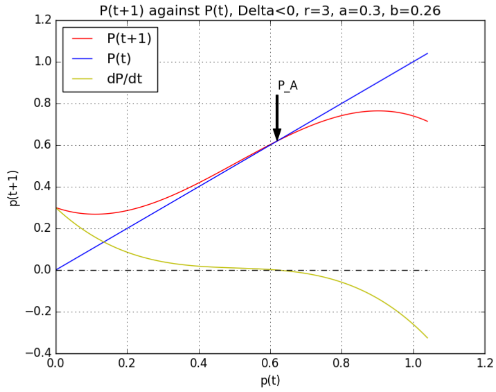
Here the sign of Δ determines the existence of such a local minimum. If $\Delta < 0$, x_1 is not in the real space, the curve has one inflexible point instead of a local minimum or a local maximum; and the system has only one fixed point. If $\Delta > 0$, the curve has a local minimum $f(x_1)$; according to the sign of $f(x_1)$ the system may or may not have three fixed points. Hence according to the sign of Δ we can separate the problem into two cases:

Case I: $\Delta < 0$
Rearranging terms for Δ:

$$\begin{aligned} &\Delta = 3[1 - 2(a+b)] + (a-b)^2 \\ &\text{and since } (a-b)^2 \geq 0 \\ &\text{hence } \Delta < 0 \implies 1 - 2(a+b) < 0 \end{aligned}$$

This implies that $(a+b) > \frac{1}{2}$ is a necessary but not sufficient condition for $\Delta < 0$. In addition, we need $|a-b|$ to be relatively small, in specific $|a-b|^2 < 6(a+b) - 3$. Hence Case I implies that there is a absolute winning situation, when the sum of stubborn-A and stubborn-B is over 50% of the population; in addition, the difference between a and b should be relatively small. The party with slightly more stubborn agents would gain dominance. Let us name this case "high-stubborn AW case".

(a) a=0.3, b=0.26, $\Delta < 0$,



(b)

Figure 3.3: (a) illustrates f has no local minimum or maximum. (b) illustrates the corresponding opinion flow of the system. Note that the yellow curve is the same as f in (a).

Case II: $\Delta > 0$

$\Delta > 0$ implies local minimum x_1 exist. Bifurcation occurs when $f(x_1)$ shifts from negative to positive, hence the question reduces to when does $f(x_1)$ =0. Substitute x_1 into equation 3.1, after simplification, we obtain:

$$f(x_1) = \frac{1}{108} * \{-\Phi^3 - 3 * \Phi * \sqrt{\Delta} + \sqrt{\Delta}[(a-b)^2 + 30a - 6b + 21] + 108a + 3\Delta\} \quad (3.2)$$

$$\text{where } \Phi = 3 + (a-b), \quad \Delta = 3 + (a-b)^2 - 6(a+b)$$

Looking at the expression of this equation 3.2, it is "almost" entirely dependent on (a-b) and (a+b). Hence we can deduce from this that the behaviour of the opinion flow is closely linked to the sum and difference of stubborn-A and stubborn-B.

Figure 3.4 is a visual representation of the system when both a, b are varied. Colour indicates the state of the system: red indicates AW; blue indicates non-AW. The "white band" verifies our conjecture from the previous section, and hence we conclude that for relatively small b ($b \leq 0.25$), there exists an $a^*, a^* > b$ such that A can become the absolute winner, and vice versa. In addition, we can deduce that the level of stubborn does not need to exceed 50% to achieve AW.

To summarise, we can distinguish the system into AW and non-AW situations. Furthermore, within AW situation, there are two distinctive cases:
case (i): high-stubborn AW situation, one fixed point, and no local minimum, $\Delta < 0$, which implies $a + b > 0.5$. Note the name "high-stubborn" refers to the fact that $a + b > 50\%$.
case (ii): low-stubborn AW situation, one fixed point, $\Delta > 0$, and $f(x_1) > 0$.
In addition, low-stubborn AW situation indicates a volatile system. Because there is a high proportion of open-minded agents, whose opinion does not influence the outcome of the system. Hence, in this case, there could be a "dramatic" opinion shift when initial support for the absolute winner is low.

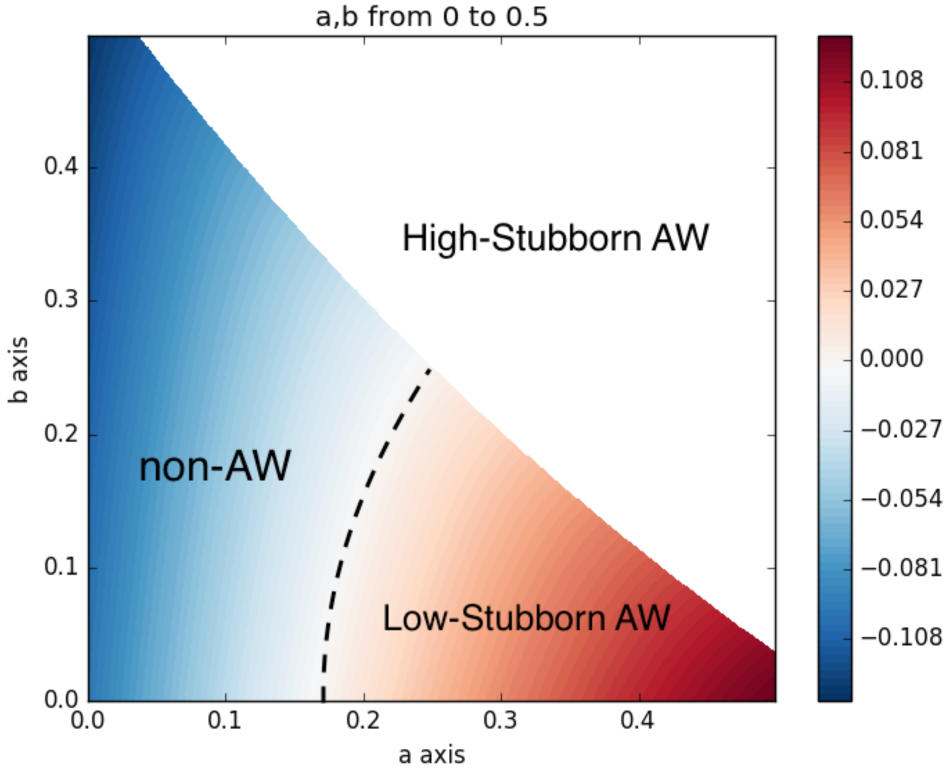


Figure 3.4: Results are obtained in Python by plotting $f(x_1)$ against a, b, where $a, b \in [0, 0.5]$. State of the system is indicated by colour: blue indicates $f(x_1)$ less than 0, that is the system has three fixed points; red indicates $f(x_1)$ greater than 0, that is the system has one fixed point. The white band in the middle represent at those specific a, b values, bifurcation occurs. The "cut-off" of colours correspond to the fact that those values of a and b are in Case I, where $\Delta < 0$.

Chapter 4

Computer Simulation Results

Recall that equations are obtained based on statistical mechanics approach, and at heart it is a probabilistic description of the opinion formation. Hence in this section, we run the actual simulation in Python, letting agents act according to the rules as defined in the model 2.2.3. We aim to achieve two goals in this section: first, verify analysis from previous sections; second, study the effect of different group size r on stubborn-effect. Note that only when first goal is achieved, can we say that the equations are correct, and progress to achieve the second goal.

4.1 Code Explained

Recall from Chapter 1 that the set up of the MPM contains three steps:

1. Distribute population into subgroups of size r, a fixed odd integer.
2. Within each group, open-minded agents update opinions to the local majority.
3. Shuffle and regroup agents; repeat step.2. Stop the process if one party gains absolute dominance (i.e. dominate all open-minded agents).

Hence my Python code very much reflects these two rules. Here is a very brief extract of the code, full code could be found in the appendix.

```
def group_sim(P0,a,b,totalpop,r): #r is group size
        threshold=1-b-0.01 #when all open-minded agents are converted
        np.random.shuffle(opinion0)

        for k in range(iteration): #max iteration = 100
                #exit loop if majority reached threshold
                if ratio > threashold:
                        break
                #otherwise, perform updating within subgroups
                for i in range(num):
                        subgroup=opinion[r*i:r*(i+1)]
                        a2,b2=count(subgroup) #call count function

                        if a2>b2: #A is local majority
                                #replace open-B with open-A
```

```
                for j in range(r):
                    if subgroup[j]==2:
                        subgroup[j]=1
            elif b2>a2: #B is local majority
                for j in range(r):
                    if subgroup[j]==1:
                        subgroup[j]=2
            #update values and reshuffle
            opinion[r*i:r*(i+1)]=subgroup
            np.random.shuffle(opinion)

    fin_a,fin_b=count(opinion)
    #return final support for A and B
    return fin_a,fin_b, iteration
```

4.2 Results

We conduct three different tests specified below to achieve goals set out at the beginning of the chapter.
Test 1: non-AW situation equation: Test 2: AW situation when (i) $\Delta < 0$, (ii) $\Delta > 0$.
Test 3: experiment with different subgroup sizes

Passing test 1-2 implies the validity of statistical mechanics approach on a large population, i.e. our equations and calculations indeed describe a hypothetical social group where agents interact according to specified rules. Hence verifies conclusions made in previous chapters. Test 3 allows us to study how subgroup size, r, affects the stubborn effect. Recall that larger r lead to higher power of polynomials, so previously we could not obtain analytical solutions with large group sizes. Therefore running computer simulation provides advantage to study larger r and reveal some basic correlations between r and stubborn effect.

Test I: Non-AW Situation First let us recall some notations:
a, b are constants (or parameters) in the system, and they denote proportion of stubborn-A and stubborn-B respectively. P_0 is the initial support for A, i.e. the number of agents who hold opinion A at time 0. In a non-AW situation, P_C is the critical initial support, this value is the unstable fixed point of the system. In other words, if $P_0 > P_C$, A wins, if $P_0 < P_C$, B wins.

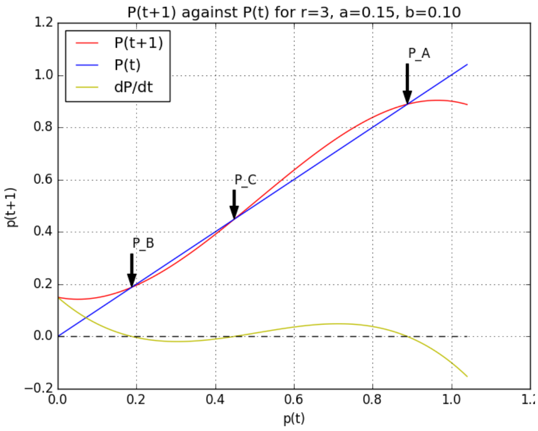
With this understanding, we expect that as we increase P_0 , there should be a "jump" of the final support for A, i.e. mechanism of the system "flips" from working against A to in favour of A. Hence let us take the order parameter to be $|P_{final} - P_0|$, i.e. we measure the difference of A's support between before and after grouping interactions, and a "jump" of this order parameter should happen at the critical initial support, value P_C .

Hence in Test I, let us first verify in the case where P_C exist. Let us pick some random values at a=0.15 b=0.1, and a=0.19, b=0.1, and test on group size r= 3 and 5 respectively.

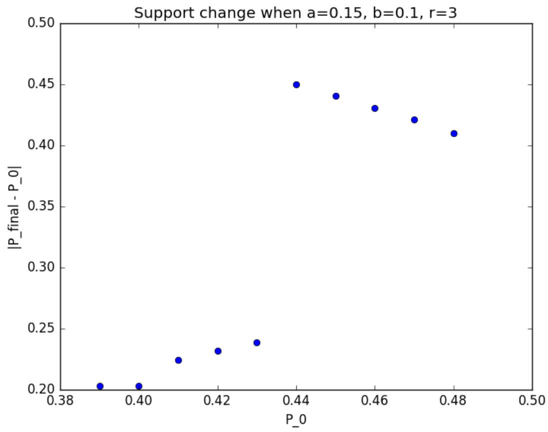
Note that we run the code by varying P_0, initial support for A, at an increment of 0.01. Refining this interval would lead to a more precise result; however, it would also multiply the runtime of the program. Since the goal is to verify analytical results obtained previously, precision to 0.01 should suffice.

Figures 4.1, 4.2, implies that the error between predicted P_C and actual P_C is very small. Hence we can conclude that these experiments verify the validity of equation2.3 to describe the hypothetical system on a sufficiently large population.

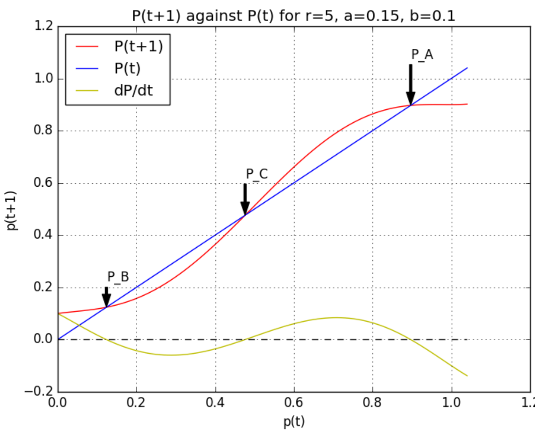
Remark: in those experiments, computer simulations are conducted on a population size of 3000. Recall that due to the stochastic nature of grouping processes, equations are obtained by analysing the statistical behaviour of the system. Hence emergent properties described should emerge only if the population is sufficiently large. After trials and experiments, as demonstrated by figure 4.3, when test population is greater than 1500, group dynamics starts to become more regular.



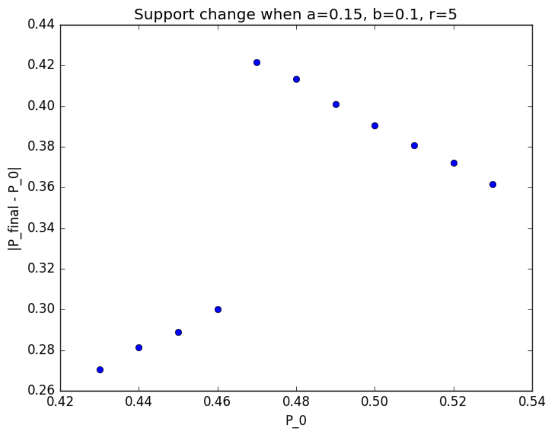
(a) P_C =0.448, P_A =0.888, P_B =0.188



(b) "jump" happens at around P_0 ≈0.44, very close to predicted P_C

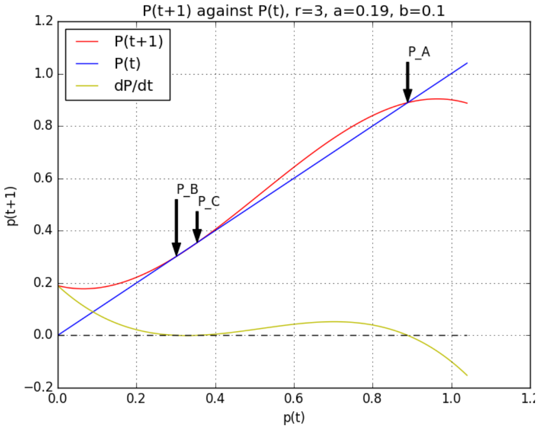


(c) P_C =0.476, P_A =0.897, P_B =0.124

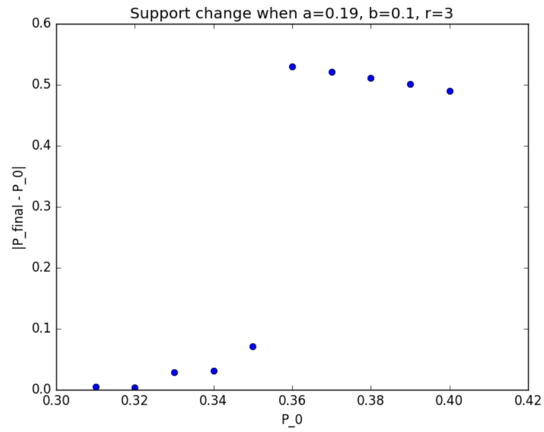


(d) "jump" happens at around P_0 ≈0.47, very close to predicted P_C

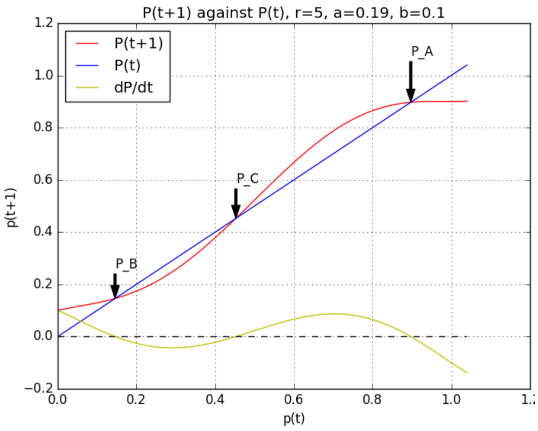
Figure 4.1: Diagrams on the left hand side are statistical predictions using equation 2.1 from Chapter 2. Diagrams on the right hand side are measures of $|P_{final} - P_0|$ obtained from Python simulations. This set of diagrams are tested on population size of 3000, taking a=0.15, b=0.1, and group sizes of r=3 and 5 respectively. (b) verifies (a) with group size r=3; (d) intends to verifies (c) with group size r=5.
Remark: after the "jump" there is a downward slop, this is expected because P_{final} remains relatively constant. Hence $|P_{final} - P_0|$ decreases, when P_0 increases. Intuitively, this can be interpreted as when A initially has more supporters, the effect of stubborn-A is less dramatic.



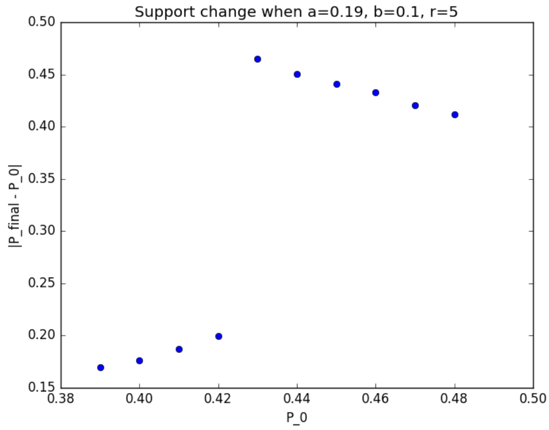
(a) P_C =0.354, P_A =0.889, P_B =0.301



(b) "jump" happens at around P_0 ≈0.35, very close to predicted P_C

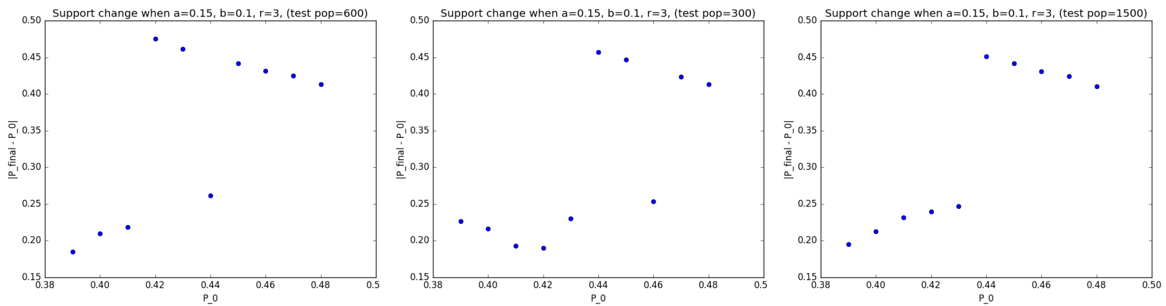


(c) P_C ≈0.453, P_A ≈0.897, P_B ≈0.146



(d) "jump" happens at around P_0 ≈0.43, close to predicted P_C

Figure 4.2: Similar to 4.1, this set of diagrams test on different values for a,b. Here a=0.15, b=0.1, population size= 3000. (b) verifies (a) with group size r=3; (d) intends to verifies (c) with group size r=5.



(a) population=300 (b) population=600 (c) population=1500

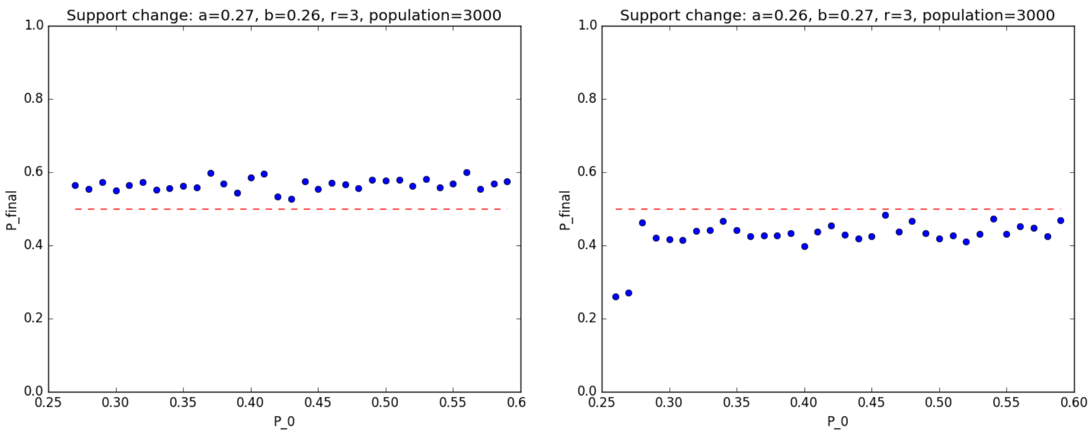
Figure 4.3: During these experiments, a=0.15 b=0.1 fixed. Note figure (a), (b) both contains irregular points, P_0 =0.44 and P_0 =0.46 respectively. Where as in (c), the behaviour becomes more regular,i.e. when P_0 reaches a P_C , $P_{final} - P_0$ remains relatively high.

Test II: AW Situation Recall: absolute winning situation is when there is only one attractive fixed point in the system, and this is achieved when the level of stubborn-A and stubborn-B reach a certain level.

Chapter 3 explores specifically how a,b lead to absolute winning. In specific, we conclude there are two AW cases: (i) when $\Delta < 0$ or $a + b > 0.5$, high-stubborn absolute winning case, where winner is the party with more stubborn agents; (ii) when $\Delta > 0$ or $a+b < 0.5$, low-stubborn case, where AW is determined by (a+b) and (a-b). Let us test those findings respectively.

Case I: $a + b > 0.5$

Figure 4.4 illustrates, that when $a + b > 0.5$, then for all values of P_0 one party with slightly higher stubborn voters is the absolute winner. In figure (a) A is the absolute winner; and in figure (b) B is the absolute winner.



(a) $a > b$ *(b) $a < b$*

Figure 4.4: Here $a + b = 0.53 > 0.5$. Red dotted line is the 50% reference line. In(a): $a > b$ hence A is absolute winner, and all P_{final} are above 50%. In (b) is the opposite situation by switching a, b values.

Case II: $a + b < 0.5$
Recall in this case, for each specific b value, $b < 0.25$, there is an $a^*, a^* > b$, such that A would become the absolute winner. This value is represented as the "white band" in figure 3.4. In specific, when in the case for b=0.1, we calculated $a^* \approx 0.192$. Here figure 4.5 verifies that when a=0.2, b=0.1, A is the absolute winner regardless of P_0.

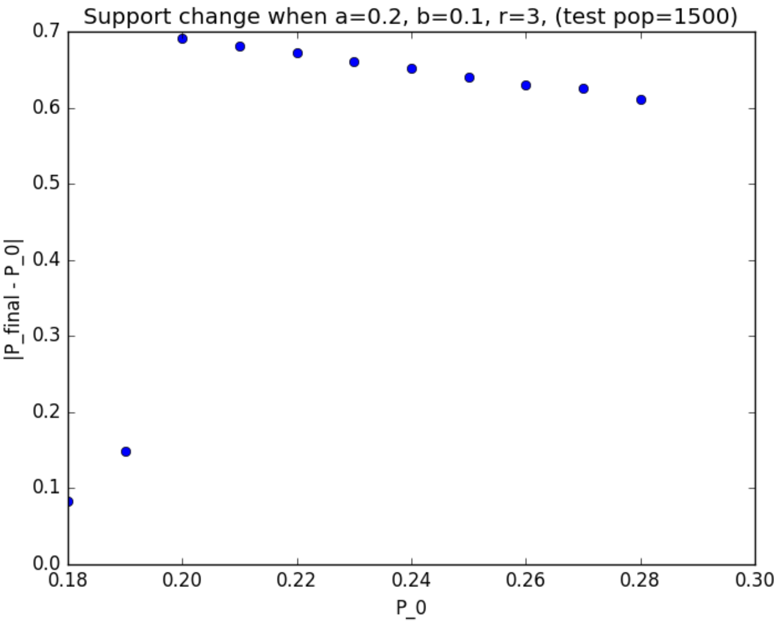


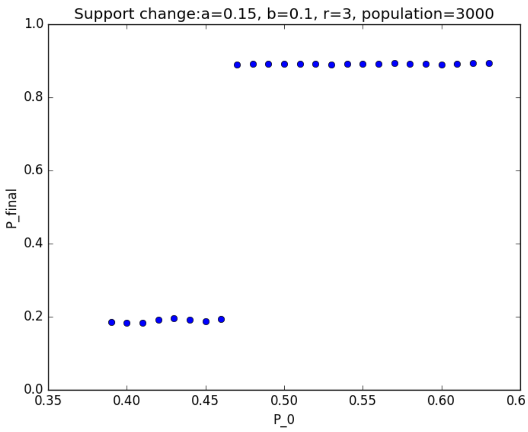
Figure 4.5: A is the winner for when P_0 is greater than 0.2. But the first two points are not relevant, because we can assume $P_0 > a$.

Test III: Various Group Sizes Recall equation 2.1, group size determines the leading power of the difference equation which describes evolution of support for A over time. Hence it is difficult to investigate the effect of different group sizes analytically, due to difficulties in solving high power polynomial. However, changing group size can be easily achieved in computer simulations. As shown in the excerpt of Python code in section 4.1, r is programmed as an input parameter hence can be easily varied.

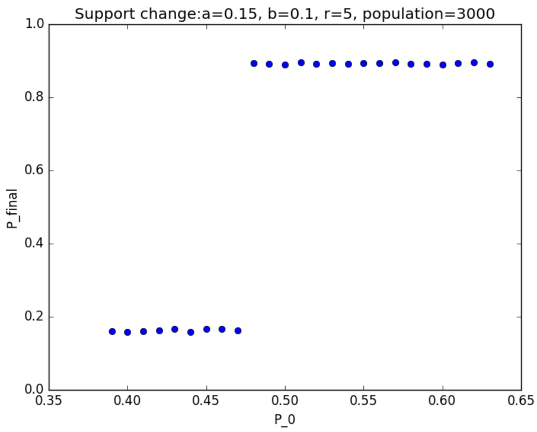
Recall there are three distinctive situations: (i)non-absolute winning (non-AW) and absolute winning (AW) in which we split into two cases: case (ii) when $\Delta < 0$; case (iii) when $\Delta > 0$.

Figure 4.6 tests on (i) non-AW situation, for four different group sizes: $r = 3, 5, 11, 15$ at randomly picked a=0.15, b=0.1. Recall that stubborn-effect can be understood as how far is P_C from 0.5, and we compare P_C for different r. This set of experiment reveals that in non-AW situation, increasing group size reduces stubborn-effect, but by a rather insignificant amount.

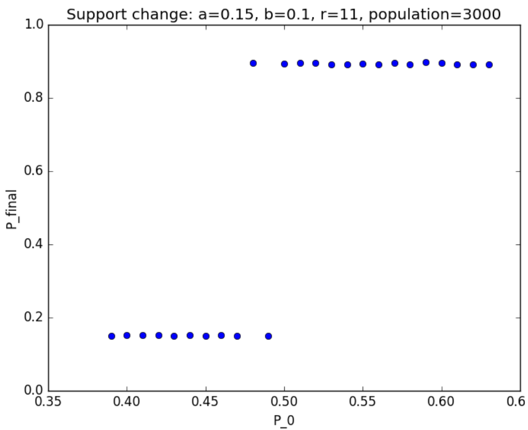
Figure 4.7 tests on case (ii) low-stubborn AW situation, for a=0.27, b=0.26. Figure 4.8 on case (iii) high-stubborn AW situation, for a=0.2, b=0.1. Note that these a, b values are picked to be the same values as in Chapter 2 and Chapter 3 for coherence; in fact any satisfactory a and b value should show a similar result. In both cases, increasing group size can effectively reduce stubborn-effect, i.e. shifting the system to non-AW situation, and P_C towards 0.5.



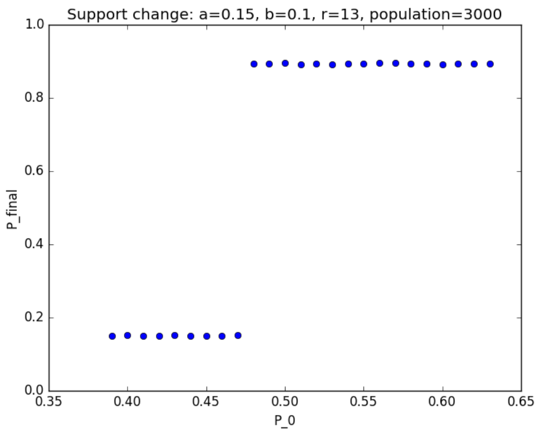
(a) r=3, $P_C \approx$0.46



(b) r=5, $P_C \approx$0.47

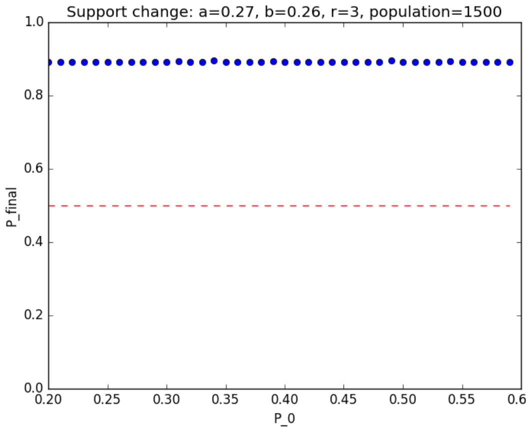


(c) r=11, $P_C \approx$0.47

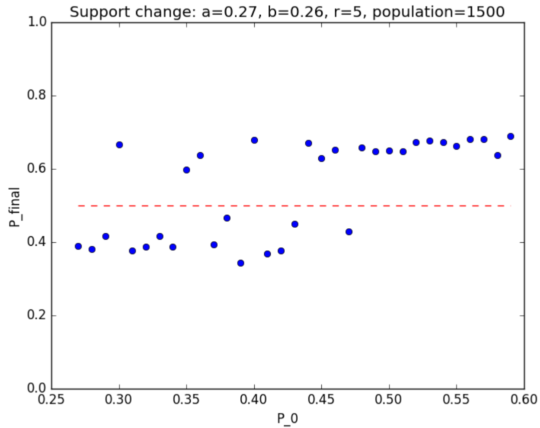


(d) r=13, $P_C \approx$0.47

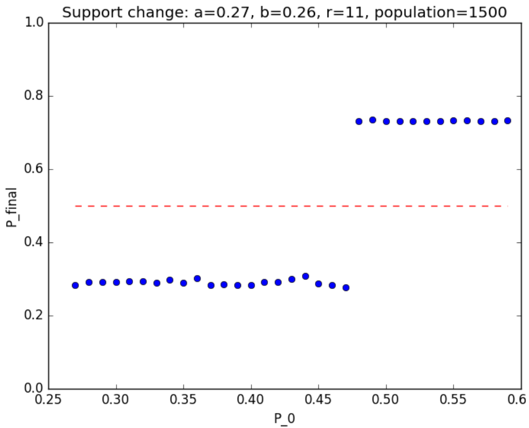
Figure 4.6: "Jump" reflects P_C level of given a,b. This set of figures reveal that at low level of a,b, i.e. in non-absolute winning situations and P_C relatively close to 0.5, increasing r has some little effect on moving P_C toward 0.5.



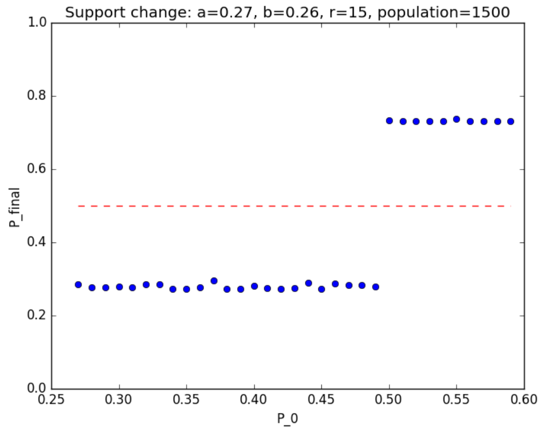
(a) r=3, AW(absolute winning) for A



(b) r=5, non-AW, no clear P_C

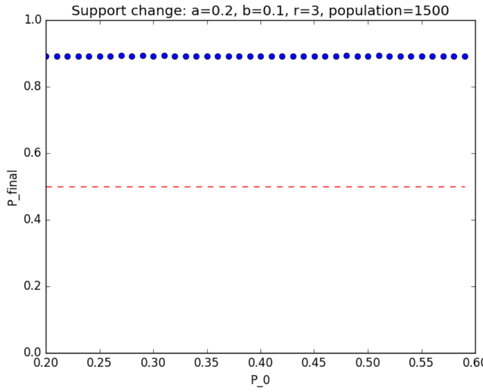


(c) r=11, non-AW , P_C ≈0.47

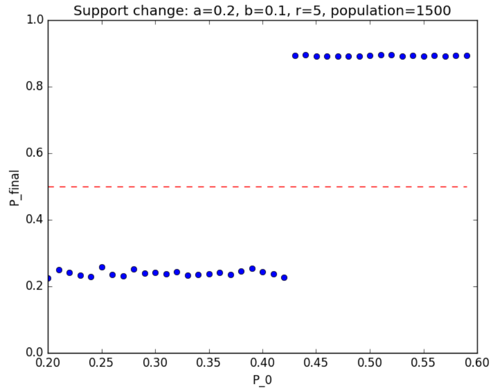


(d) r=15, non-AW, P_C ≈0.49

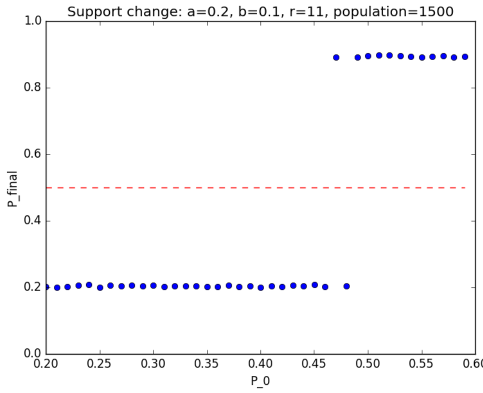
Figure 4.7: Red line is the 50%majority line. When r=5, the system is no longer in AW situation. When r=11, 15, there is clear non-AW behaviour, and P_C emerges. Hence increasing r can effectively reduce the stubborn-effect and bring the system to non-AW, and shift P_C towards 0.5.



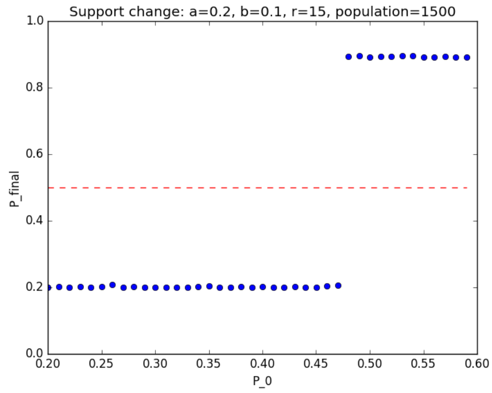
(a) r=3, AW for A



(b) r=5, non-AW, P_C ≈0.43



(c) r=11, non-AW, P_C ≈0.47



(d) r=15, non-AW, P_C ≈0.47

Figure 4.8: Red line is the 50%majority line. When r=5, system shows clear non-AW behaviour and as r increase, P_C shifts towards 0.5. Hence increasing r can effectively reduce the stubborn-effect and bring the system to non-AW, and shift P_C towards 0.5.

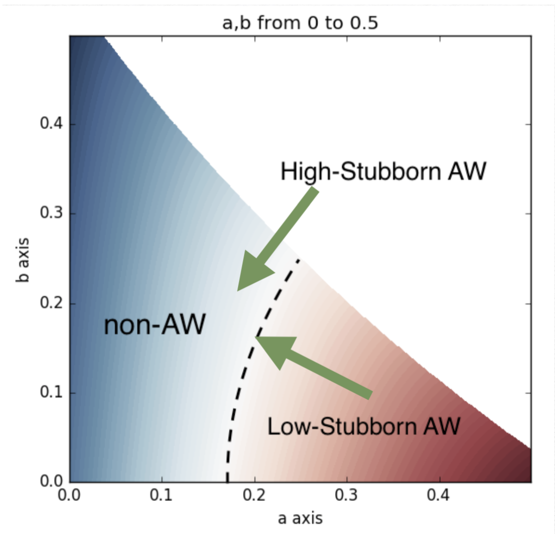


Figure 4.9: Increasing group size r can effectively bring the system from an AW situation to a non-AW situation.

To summarise, as illustrated by figure 4.9, increasing r can effectively reduce the stubborn-effect when the system is in AW situation. That is to say, smaller r enhances the stubborn-effect. On the other hand, in non-AW situation, increasing r has little effect. Note that in non-AW situation, level of a,b is relatively low, and P_C is close to 0.5; hence it is expected that the effect of r is less significant.

Group size can be interpreted as frequency of information exchange in a social group, and small r corresponds to high frequency. Test III reveals that a high frequency exchange enhances the stubborn-effect in a social group. This result is in agreement to a research done on simulating minority opinion spread using agent-based models [14], which concludes that minority opinion spread benefits from high connectivity in a network.

In conclusion, based on results from this chapter, we can obtain the following two conclusions:

1. When population size is large enough (in this case, population =3000 is sufficiently large), Python simulation results are highly close to numerical results obtained from Chapter 1, 2. In other words, the statistical mechanics approach is verified, and hence equations are reliable.
2. By experimenting with different group size, we discover that small group size, corresponding to high frequency of information exchange, enhances the stubborn-effect.

Chapter 5

Conclusion

5.1 Summary

In this report, we first established the model and derived a general difference equation 2.3 to describe the system under a mixed population model. We focused our analysis on a special r=3 case, where the system can be described by equation 3.1:

$$\frac{dp}{dt} = -2p^3 + (3 - b + a)p^2 - (2a + 1)p + a$$

Then, treating a and b as parameters of the system, we explored the effect of the stubborns, or the "stubborn-effect". In specific, we studied at what levels of a and b does the system obtain an absolute winning situation, i.e. only one possible winner in the system. We concluded that according to the level of a and b, we can separate the system into three cases:

1. low-stubborn non-AW: no absolute winner, there exists P_C , the outcome depends on P_0 relative to P_C .

2. low-stubborn AW: $a + b < 50\%$, and there is absolute winner, depending on (a+b) and (a-b)

3. high-stubborn AW: $a + b > 50\%$, absolute winner is the party with more stubborn agents.

Finally, we conducted computer simulations in Python to verify these cases, and further concluded that high frequency of information exchange enhances the stubborn-effect.

5.2 Application to Social Context

To apply these findings into the context of today's society, and remark on the "Brexit" vote, we can infer three points:

1. The population is shifting towards a higher proportion of open-minded agents. Compare to 25 years ago, without internet, voters are more likely to be “stubborn” voters, who form opinions independently and have more conviction in their opinions. In contrast, a recent survey conducted Ipsos Mori shows that more than 30% of voters between

18-24 years old think social media influenced their political opinion [15].

Hence as illustrated by low-stubborn AW situation in the report, a high proportion of open-minded agents means that the dynamics of opinion formation is more volatile under the influence of stubborn-effect.

2. Higher connectivity in the system. More than half of the world population are online today [16]. Technology makes gaining knowledge and exchanging information faster. Therefore, as demonstrated in Chapter 4, the stubborn-effect is enhanced and dynamical changes in public opinion is accelerated.

3. Emotional campaigns during the "Brexit" vote can effectively turn agents into stubborn agents, and enhances the stubborn-effect. In specific, campaigning on topics such as foreigners "stealing" jobs and social benefits, or fear for immigration and terrorism can make voters become more emotional, hence more likely to become stubborn and express opinions outwardly to influence open-minded voters.

Hence we can deduce that in a well-connected society where high ratio of open-minded agents are present, a strong stubborn-effect may be an explanation to the surprising "Brexit" referendum outcome.

5.3 Further Research Suggestions

Throughout the project, we assumed a well-mixed population, where interaction between any two agent is possible and equally likely. In reality, agents are likely to interact among a specific social group, according to age, social background or geographic location. Hence a further research direction could be understanding the stubborn-effect within the a structured network. In specific, some possible network structure to study include: the Erdős–Rényi model and Barabási–Albert model [17].

In addition, within a structured network, we can investigate the effect of stubborns being a well-connected node compared to being a less connected node. This would help us understand the role of influential figures in the society, such as politicians or celebrities, and how their "stubbornness" influence the public opinion.

Bibliography

[1] Lars Backstrom, Paolo Boldi, Marco Rosa, Johan Ugander, and Sebastiano Vigna. Four degrees of separation. 2011. ID: TN_arxiv1111.4570.

[2] Alex Pentland. Social physics : how good ideas spread-the lessons from a new science, 2014. ID: 869880567.

[3] Claudio Castellano, Santo Fortunato, and Vittorio Loreto. Statistical physics of social dynamics. *Reviews of Modern Physics*, 81(2):591–646, 2009. ID: TN_crossref10.1103/RevModPhys.81.591.

[4] Serge Galam. *Sociophysics : a physicist's modeling of psycho-political phenomena.* Springer, New York ; London, 2012. ID: 44IMP_ALMA]_DS2148552600001591; Includes bibliographical references and index.

[5] Thomas M. Liggett. Stochastic models of interacting systems. *The Annals of Probability*, 25(1):1–29, 1997. ID: TN_jstor_archive_302959527.

[6] Serge Galam. Collective beliefs versus individual inflexibility: The unavoidable biases of a public debate. *Physica A: Statistical Mechanics and its Applications*, 390(17):3036–3054, 2011. ID: TN_sciversesciencedirect_elsevierS0378-4371(11)00234-2.

[7] Serge Galam. Application of statistical physics to politics. *Physica A: Statistical Mechanics and its Applications*, 274(1):132–139, 1999. ID: TN_sciversesciencedirect_elsevierS0378-4371(99)00320-9.

[8] Kenneth J. Arrow. *Social choice and individual values.* Wiley, New York ; London, 2nd edition, 1963. ID: 44IMP_ALMA_DS2146716080001591.

[9] Mancur Olson. *The logic of collective action : public goods and the theory of groups.* Harvard University Press, Cambridge, Mass. ; London, 1971. ID: 44IMP_ALMA_DS2142000650001591; Includes bibliographical references and index.

[10] Donald G. Saari. *Geometry of voting*, volume 3. Springer-Verlag, Berlin ; London, 1994. Donald G. Saari.; Includes bibliographical references (p. [365]-368) and index.

[11] Kenneth O. May. A set of independent necessary and sufficient conditions for simple majority decision. *Econometrica*, 20(4):680–684, 1952. ID: TN_jstor_archive_110.2307/1907651.

[12] Richard C. Tolman. *The principles of statistical mechanics.* Dover Publications etc.] ; Constable, New York : London ;, 1979. ID: 44IMP_ALMA_DS2140523110001591; Includes index.

[13] Eugene M. Izhikevich. Equilibrium, 2007.

[14] J. Alvarez-Galvez. Network models of minority opinion spreading: Using agent-based modeling to study possible scenarios of social contagion. *Social Science Computer Review*, 34(5):567–581, 2016. ID: TN_crossref10.1177/0894439315605607.

[15] Ipsos Mori. A third of young people think social media will influence their vote, 12 March 2015 2015.

[16] Simon Kemp. Digital in 2017: Global review, 2017.

[17] Reuven Cohen 1974. *Complex networks : structure, robustness, and function*. Cambridge University Press, Cambridge, 2010. ID: 44IMP_ALMA_DS2142939500001591; Includes bibliographical references and index.

Chapter 6

Appendix

This the full code of the computer simulation of the grouping process
input value P0=initial-A, a=stubborn-A, b=stubborn-B
output value is final ratio of supporters for A and B (regardless if they are open or stubborn)
the function also outputs number of iterations, which is the grouping iterations needed

```

import numpy as np
import matplotlib.pyplot as plt
def group_sim(P0,a,b,totalpop,r): #r is group size

    threashold=1-b-0.01 #when all open-minded agents are converted
    iteration=100 #max number of iterations

    #because I am more comfortable working with numbers, I will assign them numerical values
    #0=A-stubborn,1=A-open,3=B-open,4=B-stubborn (intensity towards hatred for A)

    opinion0=np.zeros(totalpop) #here I choose 3000 agents to have whole number division
    #calculate proportions for each type
    As=np.int(totalpop*a)
    Ao=np.int(totalpop*(P0-a))
    Bs=np.int(totalpop*b)
    Bo=totalpop-As-Ao-Bs #to ensure they add up to totalpop

    opinion0[0:As]=0
    opinion0[As:Ao+As]=1
    opinion0[Ao+As:Ao+As+Bo]=2
    opinion0[Ao+As+Bo:]=3
    #so opinion0 is an array like this [00001111222233333],
    #size is correct but need to shuffle
    np.random.shuffle(opinion0)

    #make it into a list of integers (instead of array)
    opinion= [int(x) for x in opinion0]

    #determine number of groups
```

```
    num=np.int(totalpop/r)

    for k in range(iteration): #max iteration = 100

        #see if majority reached threashold
        a1,b1=count(opinion)
        ratio=max(a1,b1)
        if ratio > threashold:
            iteration=k+1
            #print 'end iteration, # of iterations=', iteration
            break

        #perform updating within each subgroup
        for i in range(num):
            #extract each subgroup
            subgroup=opinion[r*i:r*(i+1)]
            a2,b2=count(subgroup) #call count function

            if a2>b2: #A is local majority
                #replace open-B with open-A
                for j in range(r):
                    if subgroup[j]==2:
                        subgroup[j]=1
            elif b2>a2: #B is local majority
                for j in range(r):
                    if subgroup[j]==1:
                       subgroup[j]=2
            else:
                pass

            #insert values back to array opinion, and shuffle
            opinion[r*i:r*(i+1)]=subgroup
            np.random.shuffle(opinion)

    fin_a,fin_b=count(opinion)

    return fin_a,fin_b, iteration

#write a function that counts local majority
def count(x):#input is array of entries 0,1,2,3
    a1=x.count(0)+x.count(1)
    b1=x.count(2)+x.count(3)
    a2=float(a1)/float(a1+b1)  #a1 is number, a2 is ratio
    b2=float(b1)/float(a1+b1)
    #ratio=max(a2,b2)
    return a2,b2


 #let's define a function that returns a scatter plot of before and after
```

```
def plot(a,b,A,totalpop,r): #A is the theoretical level which I want to test
    #A0 is before iteration
    #A0=np.arange(A-0.,A+0.2,0.01)
    A0=np.arange(a,0.6,0.01)
    n=np.size(A0)
    A1=np.zeros(n)
    for i in range(n):
        A1[i]=group_sim(A0[i],a,b,totalpop,r)[0]
    A2=abs(A1-A0)
    plt.figure()
    ref=0.5*np.ones(n)
    plt.plot(A0,A1,"o",A0,ref,"r--")
    plt.title('Support change: a=0.27, b=0.26, r=3, population=1500')
    plt.xlabel('P_0')
    plt.ylabel('P_final')
    plt.ylim([0,1])
    plt.show()

if __name__ == '__main__':
    totalpop=1500
    A=0.21#initial-A
    B=1-A #initial-B
    a=0.2 #stubborn-A
    b=0.1 #stubborn-B
    r=3 #group size

    #print 'initial distribution'
    #print 'initialA, initialB, A-stubborn,B-stubborn=', [A,B,a,b]

    #call function
    #print 'after interation'
    #print 'supportA, supportB, #iteration=', group_sim(A,B,a,b,totalpop,r)

    #call plot
    plot(a,b,0.44,totalpop,r)
```