Tutorial 6
2nd order systems

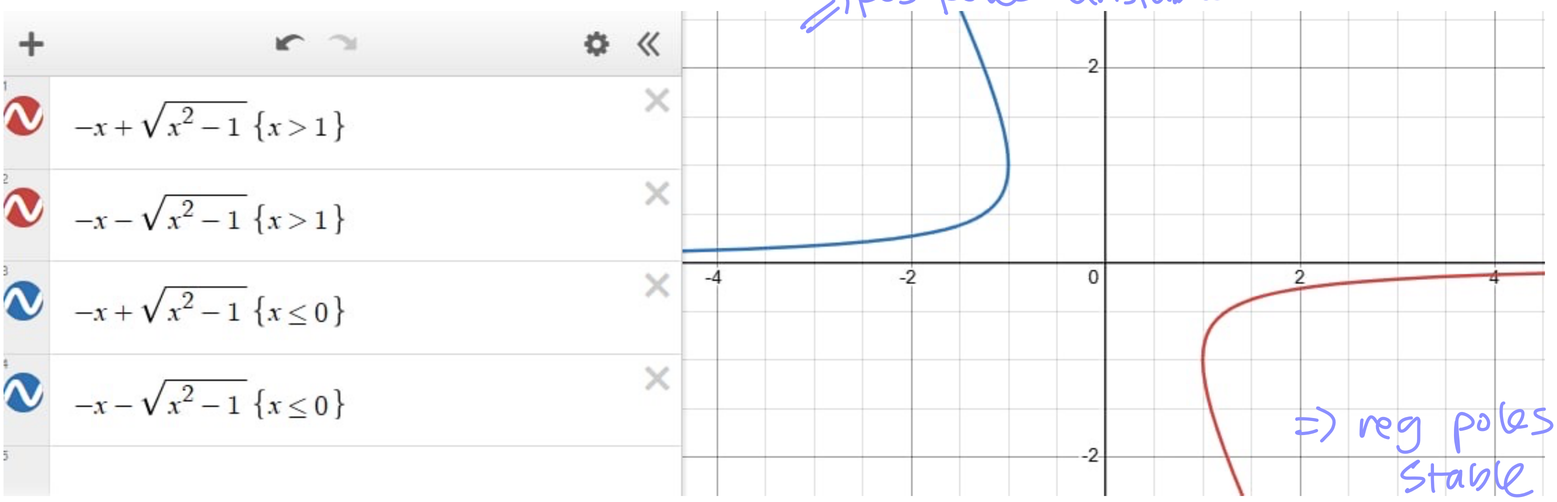
Recall the associated transfer function of the standard second order system

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

ξ ← damping ratio
ω_n ← natural frequency

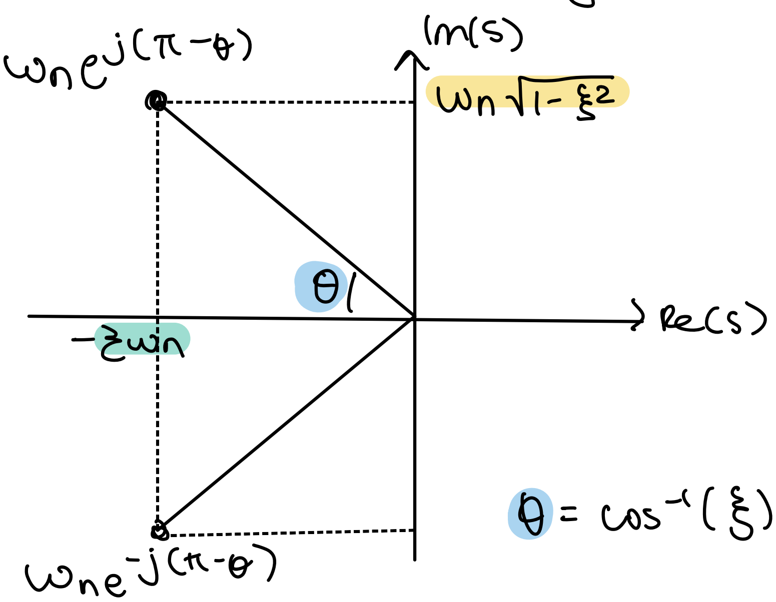
poles of the system : $-\xi\omega_n \pm \sqrt{(\xi\omega_n)^2 - \omega_n^2} = \omega_n\left(-\xi \pm \sqrt{\xi^2-1}\right)$

1. $\xi > 1$, 2 distinct real poles — overdamped
2. $\xi = 1$, 2 repeated real poles — critically damped
3. $0 < \xi < 1$, 2 complex conjugate poles — underdamped.
4. $\xi < 0$, system is unstable



Underdamped 2nd order systems — case 3 only!

poles at $s = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} = \omega_n e^{\pm j(\pi - \cos^{-1}(\xi))}$



$Re(s) = -\xi\omega_n$
(more negative real part ⇒ faster response)

$Im(s) = \omega_n\sqrt{1-\xi^2}$
(greater imaginary part ⇒ more oscillations)

$\theta = \cos^{-1}(\xi)$

Ex 6.1 Consider the following 2^{nd} order system

$$u \rightarrow \boxed{\dfrac{1/LC}{s^2 + \frac{R}{L}s + 1/LC}} \rightarrow y$$

a) Draw the region of the s-plane in which the system's poles must lie so that its step response satisfies
$\%OS \le 0.02$, $T_s \le 1s$, $T_p \le 0.5s$

b) Choose values of R, L, and C so that the poles of the system are in the allowable region from a)

a) **overshoot spec**

$$\xi = -\frac{\ln(\%OS)}{\sqrt{\pi^2 + (\ln(\%OS))^2}}$$ inversely proportional $\xi\uparrow$ $OS\downarrow$, want $\xi \ge -\dfrac{\ln(0.02)}{\sqrt{\pi^2+(\ln(0.02))^2}} = 0.7797$

$\theta = \cos^{-1}(\xi)$ inversely proportional, want $\theta \le \cos^{-1}(0.7797) = 38.8°$

settling time

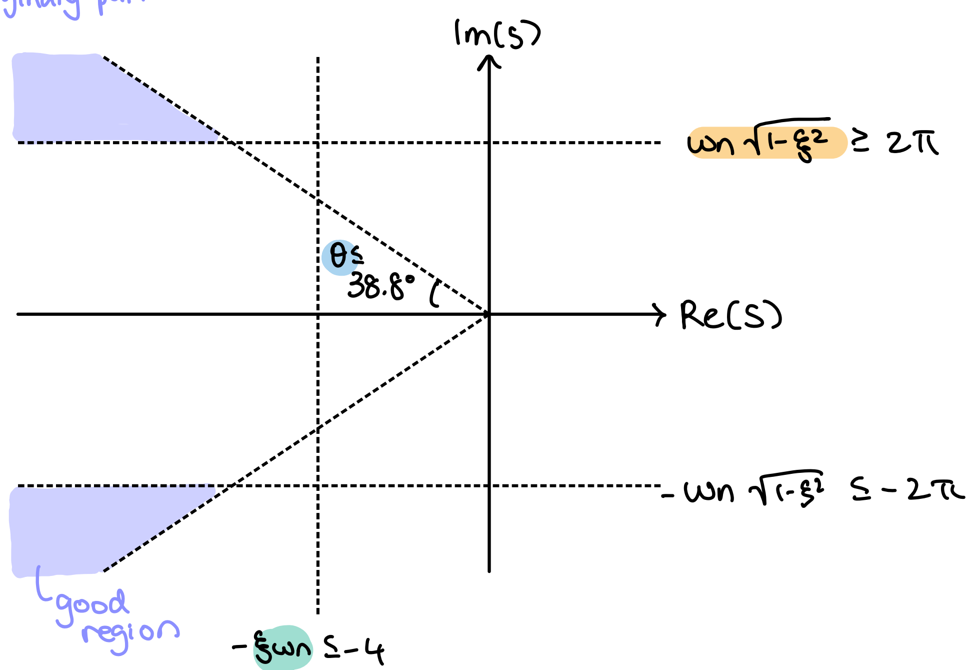
$T_s \approx \dfrac{4}{\xi\omega_n}$ inversely proportional, want $\xi\omega_n \ge \dfrac{4}{T_{s\max}} = \dfrac{4}{1} = 4$

real part

peak time

$T_p = \dfrac{\pi}{\omega_n\sqrt{1-\xi^2}}$ inversely proportional, want $\omega_n\sqrt{1-\xi^2} \ge \dfrac{\pi}{T_{p\max}} = \dfrac{\pi}{0.5} = 2\pi$

imaginary part

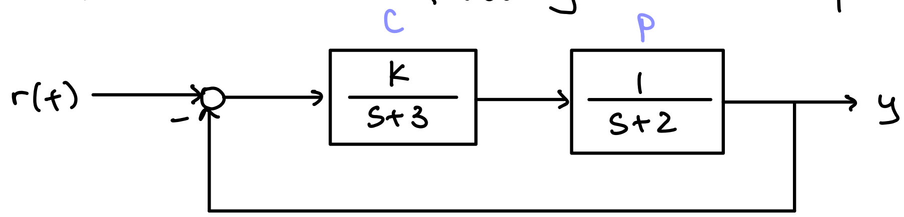


b) pick poles in the good region Ex. $s = -10 \pm 7j$

$(s+10+7j)(s+10-7j) = s^2 + 20s + 149 \Rightarrow \frac{R}{L} = 20, \ \frac{1}{LC} = 149$

$\Rightarrow$ 2 eqn 3 unknowns $\Rightarrow$ pick $R = 20$, $L = 1$, $C = \frac{1}{149}$

Ex. 3.2 Consider the following feedback system.



a) Suppose $k>1$ what is the settling time?
b) For what value of k is the system critically damped?

a) overall transfer function for a negative feedback system helps us find the tf from r to y below

$$\frac{Y}{R} = \frac{PC}{1+PC} = \frac{\frac{K}{(s+3)(s+2)}}{1+\frac{K}{(s+3)(s+2)}} \times \frac{(s+2)(s+3)}{(s+2)(s+3)} = \frac{K}{(s+2)(s+3)+K} = \frac{K}{s^2+5s+(6+K)}$$

$$= \frac{K}{6+K}\,\frac{6+K}{s^2+5s+(6+K)} = \frac{K}{6+K}\,\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

overall gain

$\omega_n^2 = 6+K, \quad \xi\omega_n = \frac{5}{2} \Rightarrow \xi = \frac{5}{2\omega_n} = \frac{5}{2\sqrt{(6+K)}}$

$K>1 \Rightarrow 0<\xi<1 \Rightarrow$ underdamped $\Rightarrow T_s \approx \frac{4}{\xi\omega_n} = \frac{4}{5/2} = \frac{8}{5}\,s$

b) critically damped $\Rightarrow$ want $\xi=1 \Rightarrow$ want $\frac{5}{2\sqrt{6+K}}=1 \Rightarrow 2\sqrt{6+K}=1 \Rightarrow K=0.25$