Chapter 3

Frequency Domain Analysis of Continuous Time Linear Systems

The goal of this chapter is to study continuous time linear systems from the *frequency domain* point of view. The main idea of this chapter will be the Laplace transform , an amazingly powerful tool for dealing with linear systems arising from constant coefficient linear differential equations of the kind introduced in Section 2.3. The Laplace transform provides an extremely efficient technique for computing both the zero-input and zero-state response of such systems. This in turn leads to an automatic technique for determining the response of electrical networks. Finally, the Laplace transform viewpoint will allow us to formulate the ideas of *system transfer function* and *system frequency response*, which are crucial for the study of feedback control systems and communication systems.

3.1 Definition of the Laplace Transform

Throughout this course we use s to denote the complex variable

$$s = \sigma + j\omega,$$

where σ and ω are real numbers. Let $x(t)$ be a continuous-time signal, which we recall is a real or complex-valued function defined for all $-\infty < t < \infty$. For each complex number s put

$$X(s) \triangleq \lim_{\epsilon \to 0,\, \epsilon > 0} \int_{-\epsilon}^{\infty} x(\tau) e^{-s\tau}\, d\tau \tag{3.1}$$

2-

provided the quantity on the right exists. The resulting complex-valued function of the complex variable s is called the **Laplace transform** of the signal $x(t)$, and is denoted variously by

$$X(s), \qquad \mathcal{L}\{x\}(s), \qquad \text{or} \qquad \mathcal{L}\{x(t)\}(s).$$

Remark 3.1.1 Many textbooks simply define the Laplace transform of $x(t)$ by

$$X(s) \triangleq \int_0^\infty x(\tau)e^{-s\tau}\,d\tau, \tag{3.2}$$

which looks simpler than our definition (3.1). The trouble with the integral in (3.2) is that it is unclear if the instant $\tau = 0$ is included in the interval of definition or not. This becomes a problem when we want to take $x(\tau)$ to be the impulse function $\delta(\tau)$, since $\tau = 0$ is the instant at which $\delta(\tau)$ "strikes", and including or excluding $\tau = 0$ from the interval of integration makes a radical difference in the value of the integration that we get. For example, if we *exclude* $\tau = 0$ from the interval of definition, then since $x(\tau) = \delta(\tau) = 0$ for all $\tau > 0$ (see (2.8)) we get from (3.2) that

$$X(s) = \int_0^\infty 0e^{-s\tau}d\tau = 0.$$

On the other hand, if we *include* $\tau = 0$ in the interval of definition, then it follows that

$$X(s) = \int_0^\infty \delta(\tau)e^{-s\tau}d\tau = \int_{-\infty}^\infty \delta(\tau)e^{-s\tau}d\tau = e^{s(0)} = 1. \tag{3.3}$$

Here the second equality follows because we know that $\delta(\tau) = 0$ for all $\tau < 0$, hence we can change the lower limit of integration from 0 to $-\infty$ without affecting the value of the integral, and the third equality follows from the Sifting Theorem (take $t = 0$ in Remark 2.1.2). For reasons of consistency of Laplace transforms we will **always** want the interval of integration to *include* the instant $\tau = 0$. Our definition (3.1) makes this unambiguously clear since $\tau = 0$ belongs to each and every interval of integration $(-\epsilon, \infty)$ regardless of how small $\epsilon > 0$ is. The notation in (3.1) is of course rather cumbersome, and henceforth we shall rewrite this definition in the abbreviated form

$$X(s) = \int_{0-}^\infty x(\tau)e^{-s\tau}d\tau, \tag{3.4}$$

where the lower limit $0-$ is shorthand for the limiting process in (3.1).

Remark 3.1.2 The majority of signals $x(t)$ with which we shall be concerned do **not involve** an impulse function $\delta(t)$ or any of the higher order impulse functions $\delta^{(n)}(t)$. We may call these signals "well-behaved" at instant $t = 0$. For such signals there is no distinction between (3.2) and (3.4), and we can use either of these integrals to compute the Laplace transform . In this case we usually use the slightly simpler form given by (3.2). However, there will be a few important applications when we do need to deal with signals that include impulse functions and in this case we must always use the form given by (3.4). Finally, we emphasize that a signal such as the unit step function $u(t)$ is well-behaved, **even though it has a jump or discontinuity** at $t = 0$, and we can compute its Laplace transform using either of the integrals (3.2) or (3.4). Just having a jump at $t = 0$ is a much less serious manifestation of misbehaviour than having an impulse function $\delta(t)$.

We now compute the Laplace transform of some signals:

Example 3.1.3 Take $x(t)$ to be the impulse function $\delta(t)$. The Laplace transform is given by (3.1), namely

(3.5)
$$X(s) \triangleq \lim_{\epsilon\to 0,\, \epsilon>0} \int_{-\epsilon}^{\infty} x(\tau)e^{-s\tau}\, d\tau.$$

Then, for each $\epsilon > 0$ we have

(3.6)
$$\int_{-\epsilon}^{\infty} \delta(\tau)e^{-s\tau}d\tau = \int_{-\infty}^{\infty} \delta(\tau)e^{-s\tau}d\tau = 1,$$

where we have used the fact that $\delta(\tau) = 0$ for all $\tau < 0$ at the first equality in (3.6) and the same reasoning for (3.3) at the second equality. Since (3.6) holds for each $\epsilon > 0$ we get from (3.5) that

$$X(s) = 1, \qquad \text{for all complex numbers } s.$$

Example 3.1.4 Take

$$x(t) = e^{2t}.$$

Since $x(t)$ is well-behaved at $t = 0$ (see Remark 3.1.2) the Laplace transform is given by

(3.7)
$$X(s) \triangleq \int_{0}^{\infty} x(t)e^{-st}\, dt,$$

thus

(3.8)
$$\begin{aligned} X(s) &= \int_0^{\infty} e^{-(s-2)t}\, dt = \frac{-1}{(s-2)}e^{-(s-2)t}\Big|_{t=0}^{\infty} \\ &= \frac{-1}{(s-2)}\left\{ e^{-(s-2)t}\Big|_{t=\infty} - e^{-(s-2)t}\Big|_{t=0}\right\}. \end{aligned}$$

Clearly

$$e^{-(s-2)t}\Big|_{t=0} = e^0 = 1,$$

and

$$e^{-(s-2)t}\Big|_{t=\infty} = \lim_{t\to\infty} e^{-(s-2)t}$$

thus from (3.8) we get

$$X(s) = \frac{1}{(s-2)}\left[1 - \lim_{t\to\infty} e^{-(s-2)t}\right], \qquad \text{for each complex } s. \tag{3.9}$$

Now, for $s = \sigma + j\omega$ we see that

$$\begin{aligned} \lim_{t\to\infty} e^{-(s-2)t} &= \lim_{t\to\infty} e^{-(\sigma-2)t}e^{-j\omega t} \\ &= \lim_{t\to\infty} e^{-(\sigma-2)t}\cos\omega t - j[\lim_{t\to\infty} e^{-(\sigma-2)t}\sin\omega t]. \end{aligned} \tag{3.10}$$

When $\sigma \equiv \mathrm{re}(s) > 2$ then clearly

$$\lim_{t\to\infty} e^{-(\sigma-2)t}\cos\omega t = 0 \qquad \text{and} \qquad \lim_{t\to\infty} e^{-(\sigma-2)t}\sin\omega t = 0,$$

so that, from (3.10), we get

$$\lim_{t\to\infty} e^{-(s-2)t} = 0,$$

and putting this into (3.9) then gives

$$X(s) = \frac{1}{(s-2)}, \qquad \text{for all complex } s \text{ with } \mathrm{re}(s) > 2.$$

On the other hand, when $\mathrm{re}(s) \equiv \sigma = 2$ then (3.10) shows

$$\lim_{t\to\infty} e^{-(s-2)t} = \lim_{t\to\infty} \cos\omega t - j[\lim_{t\to\infty} \sin\omega t],$$

which does not exist! It follows that the Laplace transform $X(s)$ **does not make sense** for complex s with $\mathrm{re}(s) = 2$. In polite mathematical language we say that $X(s)$ is **undefined** when $\mathrm{re}(s) = 2$. Next, suppose that $\mathrm{re}(s) \equiv \sigma < 2$. Then clearly,

$$\lim_{t\to\infty} e^{-(\sigma-2)t} = \infty,$$

so that again the limits

$$\lim_{t\to\infty} e^{-(\sigma-2)t}\cos\omega t \qquad \text{and} \qquad \lim_{t\to\infty} e^{-(\sigma-2)t}\sin\omega t$$

fail to exist, thus the limit in (3.10) fails to exist. Hence it follows from (3.9) that $X(s)$ does not make sense for complex s with $\mathrm{re}(s) < 2$. To conclude, we see that the Laplace transform $X(s)$ of the signal $x(t) = e^{2t}$ exists and is given by

$$X(s) = \frac{1}{(s-2)}$$

for all complex s with $\mathrm{re}(s) > 2$, but is **undefined** for all other complex numbers s. The set of complex numbers for which $X(s)$ is defined is called the **region of existence** of the Laplace transform. See Fig. 3.1.

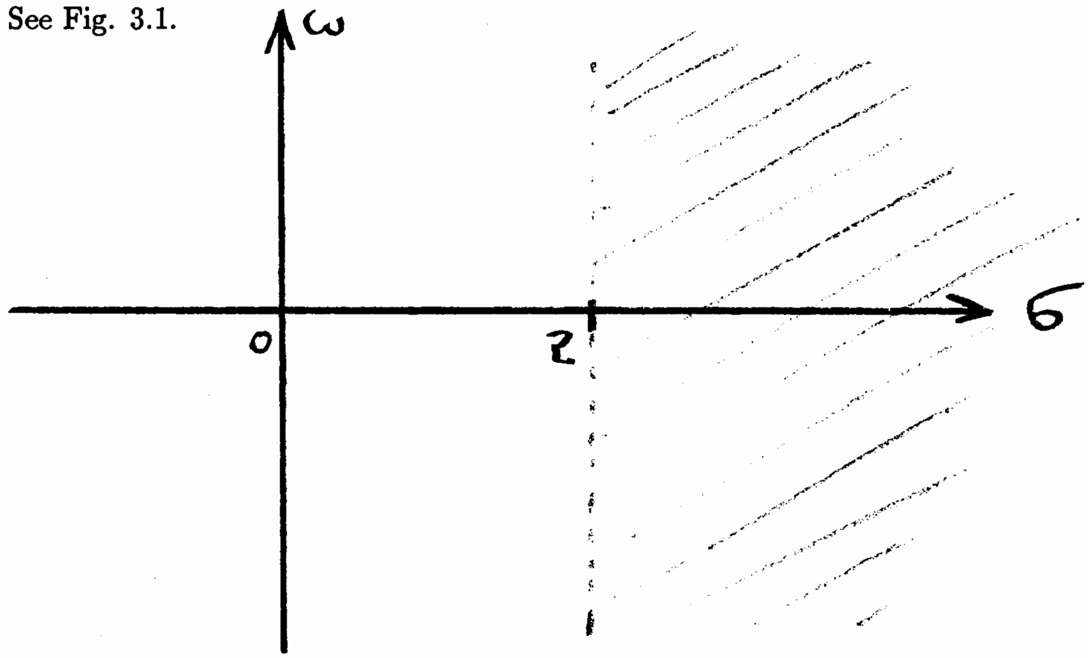

Fig. 3.1.

Example 3.1.5 An easy extension of the method used in Example 3.1.4 shows that

$$\mathcal{L}\{e^{\alpha t}\}(s) = \frac{1}{s-a}, \qquad \text{for all complex } s \text{ with } \mathrm{re}(s) > \mathrm{re}(\alpha),$$

where α is a complex constant, and $\mathcal{L}\{e^{\alpha t}\}(s)$ is **undefined** for complex s with $\mathrm{re}(s) \leq \mathrm{re}(\alpha)$.

Remark 3.1.6 The preceding examples show that the Laplace transform $X(s)$ of the signal $x(t) = e^{2t}$ is defined over only a **portion** of the complex plane, which is called its region of existence. In general the Laplace transform of most of the signals we shall encounter will likewise be defined over only a portion of the complex plane. The message of this example is that the Laplace transform is a concept of some subtlety. Indeed, a comprehensive study of the mathematical properties of Laplace transforms requires advanced tools from complex analysis and functional analysis at a level very far beyond this introductory course. On the other hand, a very pleasant feature of Laplace transforms is that we can use them as an **operational device** for solving difficult problems without in any way having to be aware of the rather advanced mathematical theory associated with them.

3.2 Properties of Laplace Transforms

We list the most important properties of Laplace transforms :

(I) LINEARITY: Suppose that $x_1(t)$, $x_2(t)$ are signals and c_1, c_2, are complex constants. Then

$$\mathcal{L}\{c_1x_1 + c_2x_2\}(s) = c_1\mathcal{L}\{x_1\}(s) + c_2\mathcal{L}\{x_2\}(s).$$

Example 3.2.1 Determine $\mathcal{L}\{\cos\omega t\}(s)$.

We know that

$$\cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}.$$

From Example 3.1.5 we know

$$\mathcal{L}\{e^{j\omega t}\}(s) = \frac{1}{s - j\omega}, \qquad \mathcal{L}\{e^{-j\omega t}\}(s) = \frac{1}{s + j\omega},$$

and by linearity we get

$$\begin{aligned}\mathcal{L}\{\cos\omega t\}(s) &= \frac{1}{2}\left[\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right] \\ &= \frac{s}{s^2+\omega^2}.\end{aligned}$$

(II) DERIVATIVES: For a signal $x(t)$ with Laplace transform $X(s)$ we have

$$\mathcal{L}\{x^{(1)}(t)\}(s) = sX(s) - x(0-), \tag{3.11}$$

where $x(0-)$ is the value of $x(t)$ *infinitesimally prior* to instant $t = 0$, that is

$$x(0-) \triangleq \lim_{t\to 0,\ t<0} x(t).$$

Example 3.2.2 Determine $\mathcal{L}\{\delta^{(1)}(t)\}(s)$, where $\delta^{(1)}(t)$ is given in Section 2.1.3.

Put $x(t) = \delta(t)$. Then we know from Example 3.1.3 that the Laplace transform of $x(t)$ is

$$X(s) = 1.$$

Also, since $x(t) = \delta(t) = 0$ for all $t < 0$, we have $x(0-) = 0$, so that

$$\begin{aligned}\mathcal{L}\{\delta^{(1)}(t)\}(s) &= sX(s) - x(0-) \\ &= s. \qquad (3.12)\end{aligned}$$

8-

Remark 3.2.3 Repeating the method of Example 3.2.2 we see that

$$\mathcal{L}\{\delta^{(n)}(t)\}(s) = s^n, \qquad \text{for all } n = 1, 2, \ldots$$

Remark 3.2.4 If $x(t)$ is *continuous* at $t = 0$, that is has no jump at $t = 0$, then of course $x(0-) = x(0) = x(0+)$, and we can then write

$$\mathcal{L}\{x^{(1)}(t)\}(s) = sX(s) - x(0).$$

However, when $x(t)$ has a jump at $t = 0$ then $x(0)$ is *indeterminate* and now we *must* use $x(0-)$ and not $x(0)$ in the right hand side of (3.11).

More generally, the Laplace transform of the n-th derivative of a signal $x(t)$ with Laplace transform $X(s)$ is given by

$$\mathcal{L}\{x^{(n)}(t)\}(s) = s^n X(s) - s^{n-1}x(0-) - \ldots - sx^{(n-2)}(0-) - x^{(n-1)}(0-).$$

(III) INTEGRALS: Suppose that $x(t)$ is a signal with Laplace transform $X(s)$, and put

$$y(t) \triangleq \int_{0-}^{t} x(\tau)d\tau.$$

Then

$$\mathcal{L}\{y(t)\}(s) = \frac{X(s)}{s}.$$

(IV) EXPONENTIAL SHIFT: Suppose $x(t)$ is a signal with Laplace transform $X(s)$ and α is a complex constant. Then

(3.13)
$$\mathcal{L}\{e^{\alpha t}x(t)\}(s) = X(s - \alpha).$$

Example 3.2.5 Determine $\mathcal{L}\{e^{\alpha t}\cos\omega t\}(s)$.

From Example 3.2.1 we know that

$$\mathcal{L}\{\cos\omega t\}(s) = \frac{s}{s^2 + \omega^2},$$

so that (3.13) gives

$$\mathcal{L}\{e^{\alpha t}\cos\omega t\}(s) = \frac{s - \alpha}{(s-\alpha)^2 + \omega^2}.$$

(V) CONVOLUTION: Suppose that $x(t)$ and $y(t)$ are signals with Laplace transforms $X(s)$ and $Y(s)$ respectively. Then

$$\mathcal{L}\{(x * y)(t)\}\,(s) = X(s)Y(s).$$

(VI) FINAL VALUE THEOREM: Suppose that $x(t)$ is a signal with Laplace transform $X(s)$, and tends to a **finite** constant as $t \to \infty$. Then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s). \tag{3.14}$$

Remark 3.2.6 In the final value theorem it is crucial that $x(t)$ tend towards a **finite** value for (3.14) to hold. To see what can go wrong when this condition does not hold, take $x(t) = e^t$. Then we know that

$$X(s) = \frac{1}{s-1},$$

so that

$$\lim_{s\to 0} sX(s) = \lim_{s\to 0} \frac{s}{s-1} = 0.$$

However, clearly

$$\lim_{t\to\infty} x(t) = \infty,$$

so that (3.14) clearly fails.

(VII) INITIAL VALUE THEOREM: Suppose that $x(t)$ is a signal with Laplace transform $X(s)$ and is continuous at $t = 0$. Then

$$x(0) = \lim_{s\to\infty} sX(s).$$

3.3 Inverse Laplace Transforms

The problem of inverse Laplace transforms is the following: given a complex-valued function $X(s)$ of the complex variable s, compute a signal $x(t)$ such that

$$\mathcal{L}\{x(t)\}(s) = X(s).$$

This signal is called the **inverse Laplace transform** of $X(s)$ and is denoted by

$$\mathcal{L}^{-1}\{X(s)\}(t).$$

Basic to determining inverse Laplace transforms is the following result which says that two signals which agree for all $t \geq 0$ must have the same Laplace transform :

Theorem 3.3.1 Suppose that $x(t)$ is a signal with Laplace transform $X(s)$. If $y(t)$ is a signal with the same Laplace transform $X(s)$ then the signals $x(t)$ and $y(t)$ are identical for all $t \geq 0$:

$$x(t) = y(t) \qquad \text{for all instants } t \geq 0.$$

Remark 3.3.2 From the definition of Laplace transform , the only part of a signal $x(t)$ which "goes into" the computation of its Laplace transform $X(s)$ is that for $t \geq 0$. It follows that if two signals $x(t)$ and $y(t)$ agree for all $t \geq 0$, but are distinct for $t < 0$, then these signals will have identical Laplace transforms . Thus, when we compute the inverse Laplace transform $x(t)$ of a function $X(s)$ we may as well take $x(t)$ to be the simplest possible signal for negative values of t, and *by convention* we will always take

$$\mathcal{L}^{-1}\{X(s)\}(t) = 0 \qquad \text{for all instants } t < 0,$$

for the inverse Laplace transform of a function $X(s)$.

In general, determining the inverse Laplace transform of a given function $X(s)$ is a difficult problem. Here we shall take advantage of the fact that we will need the inverse Laplace transforms for very few signals. Using properties (I) to (IV) of Section 3.2 we can easily compute the Laplace transforms $X(s)$ of the signals $x(t)$ listed in the table that follows.

To determine the inverse Laplace transform for a function $X(s)$ listed in the right hand column of the table we merely look up the corresponding signal $x(t)$ in the left hand column.

Signal $x(t)$ for $t \geq 0$	Laplace Transform $X(s)$
$\delta(t)$	1
$\delta^{(n)}(t)$, $n = 1, 2, \ldots$	s^n
$u(t)$	$1/s$
t^n, $n = 0, 1, 2, \ldots$	$n!/s^{n+1}$
$e^{\alpha t}$, α complex	$1/(s-\alpha)$
$t^n e^{\alpha t}$, α complex $n = 0, 1, 2, \ldots$	$n!/(s-\alpha)^{n+1}$
$\sin \omega t$	$\omega/(s^2+\omega^2)$
$\cos \omega t$	$s/(s^2+\omega^2)$
$e^{\alpha t} \sin \omega t$	$\omega/[(s-\alpha)^2+\omega^2]$
$e^{\alpha t} \cos \omega t$	$(s-\alpha)/[(s-\alpha)^2+\omega^2]$

The most important functions whose inverse Laplace transforms we shall need are the so-called rational functions. In the next section we look at these functions and the computation of their inverse Laplace transforms .

3.4 Rational Functions and the Method of Partial Fractions

Recall that a function $P(s)$ of the complex variable s is called a **polynomial in** s when it has the form

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0,$$

where the a_i are complex constants, and n is a positive integer. Assuming that $a_n \neq 0$, the integer n is called the **degree** of $P(s)$, usually written as

$$\deg(P).$$

According to the fundamental theorem of algebra (see Theorem 2.3.5) the polynomial $P(s)$ can always be factorized as follows:

$$P(s) = a_n (s - p_1)^{m_1} (s - p_2)^{m_2} \ldots (s - p_l)^{m_l},$$

where the p_i are complex constants called the **roots** of $P(s)$, amd the m_i are positive integers such that

$$n = m_1 + m_2 + \ldots + m_l.$$

Furthermore, if the constants a_i are *real* then the roots p_i always occur in *conjugate pairs.* The integer m_i is called the **multiplicity** of the root p_i.

If $N(s)$ and $D(s)$ are polynomials in s then the function

$$X(s) \triangleq \frac{N(s)}{D(s)}$$

is called a **rational function** of the complex variable s. When $N(s)$ and $D(s)$ have *no roots in common* then $X(s)$ is called a **coprime rational function**. If a rational function $X(s)$ is not coprime then clearly it can be made coprime by factorizing the numerator and denominator polynomials $N(s)$ and $D(s)$ and cancelling the factors $(s - p_i)$ which give rise to the common roots.

We next consider how to find the inverse Laplace transform of a given coprime rational function:

Theorem 3.4.1 (Heaviside expansion theorem) Let

$$X(s) \triangleq \frac{N(s)}{D(s)}$$

be a coprime rational function such that

$$\deg(N) < \deg(D),$$

and factorize the denominator polynomial $D(s)$ as follows

$$D(s) = a_n(s - p_1)^{m_1}(s - p_2)^{m_2} \dots (s - p_l)^{m_l}.$$

Then $X(s)$ can be expanded as

$$X(s) = \sum_{i=1}^{l} \left\{ \sum_{j=1}^{m_i} \frac{r_{ij}}{(s - p_i)^j} \right\},$$

where the r_{ij} are complex constants given by

$$r_{ij} = \frac{1}{(m_i - j)!} \frac{d^{m_i - j}}{ds^{m_i - j}} \left[X(s)(s - p_i)^{m_i} \right] \Big|_{s=p_i}, \tag{3.15}$$

for $i = 1, 2, \dots, l$, $j = 1, 2, \dots, m_i$.

Remark 3.4.2 The expansion of the comprime rational function $X(s)$ given by Theorem 3.4.1 is usually called the **method of partial fractions**.

Remark 3.4.3 From the table of inverse Laplace transforms we know that

$$\mathcal{L}^{-1}\left\{ \frac{1}{(s - p_i)^j} \right\}(t) = \frac{1}{(j-1)!} t^{j-1} e^{p_i t}, \qquad \text{for all } t \geq 0,$$

hence the inverse Laplace transform of the coprime rational function $X(s)$ in Theorem 3.4.1 is given by

$$x(t) = \sum_{i=1}^{l} \left\{ \sum_{j=1}^{m_i} \frac{r_{ij}}{(j-1)!} t^{j-1} e^{p_i t} \right\}, \qquad \text{for all } t \geq 0,$$

and recall that we take

$$x(t) = 0 \qquad \text{for all } t < 0.$$

Remark 3.4.4 Theorem 3.4.1 provides us a with a "closed-form" mechanical formula (3.15) for computing the constants r_{ij} in a partial fraction expansion. How is this theorem established? The proof is based on the so-called "covering technique". Rather than developing this idea in full generality we shall illustrate the covering technique by a simple example. The proof of Theorem 3.4.1 merely abstracts this idea into a general context.

Example 3.4.5 Find the partial fraction expansion of

$$X(s) = \frac{1}{(s^2+1)(s-2)^2}$$

and then determine its inverse Laplace transform . We have

$$D(s) = (s^2+1)(s-2)^2 = (s+j)(s-j)(s-2)^2.$$

Using Theorem 3.4.1 we can then write

$$X(s) = \left\{\frac{r_{11}}{(s+j)}\right\} + \left\{\frac{r_{21}}{(s-j)}\right\} + \left\{\frac{r_{31}}{(s-2)} + \frac{r_{32}}{(s-2)^2}\right\}. \tag{3.16}$$

It remains to find the r_{ij}. We begin with r_{11}. Multiply each side of (3.16) by $(s+j)$ to get

$$(s+j)X(s) = r_{11} + r_{21}\frac{(s+j)}{(s-j)} + r_{31}\frac{(s+j)}{(s-2)} + r_{32}\frac{(s+j)}{(s-2)^2}.$$

Now put $s=-j$. Then the last three terms on the right hand side vanish and we get

$$\begin{aligned} r_{11} &= (s+j)X(s)\big|_{s=-j} \\ &= \left.\frac{1}{(s-j)(s-2)^2}\right|_{s=-j} \\ &= 0.08 + 0.06j. \end{aligned}$$

One can find r_{21} in exactly the same way, that is by multiplying each side of (3.16) by $(s-j)$ and taking $s=j$. However, it is easier to note that r_{21} is the complex conjugate of r_{11} since the coefficients in the denominator polynomial of $X(s)$ are real, thus

$$r_{21} = 0.08 - 0.06j.$$

It remains to determine r_{31} and r_{32}. Multiply each side of (3.16) by $(s-2)^2$:

$$(s-2)^2X(s) = r_{11}\frac{(s-2)^2}{(s+j)} + r_{21}\frac{(s-2)^2}{(s-j)} + r_{31}(s-2) + r_{32}. \tag{3.17}$$

Taking $s = 2$ makes the first three terms on the right hand side of (3.17) vanish, giving

$$\begin{aligned} r_{32} &= (s-2)^2 X(s)\big|_{s=2} \\ &= \left.\frac{1}{(s^2+1)}\right|_{s=2} \\ &= \frac{1}{5}. \end{aligned}$$

It remains to determine r_{31}. For this we take derivatives with respect to s on each side of (3.17), namely

$$\frac{d}{ds}\left[(s-2)^2 X(s)\right] = r_{11}\frac{d}{ds}\left[\frac{(s-2)^2}{(s+j)}\right] + r_{21}\frac{d}{ds}\left[\frac{(s-2)^2}{(s-j)}\right] + r_{31}. \tag{3.18}$$

Now

$$\frac{d}{ds}\left[\frac{(s-2)^2}{(s+j)}\right] = \frac{2(s-2)(s+j) - (1)(s-2)^2}{(s+j)^2},$$

so that

$$\left.\frac{d}{ds}\left[\frac{(s-2)^2}{(s+j)}\right]\right|_{s=2} = 0.$$

Similarly,

$$\left.\frac{d}{ds}\left[\frac{(s-2)^2}{(s-j)}\right]\right|_{s=2} = 0.$$

Substituting these into (3.18) then gives

$$\begin{aligned} r_{31} &= \left.\frac{d}{ds}\left[(s-2)^2 X(s)\right]\right|_{s=2} \\ &= \left.\frac{d}{ds}\left[\frac{1}{(s^2+1)}\right]\right|_{s=2} = -0.16. \end{aligned}$$

Substituting the values for r_{ij} into (3.16) then gives

$$X(s) = \left\{\frac{0.08+0.06j}{(s+j)}\right\} + \left\{\frac{0.08-0.06j}{(s-j)}\right\} + \left\{-\frac{0.16}{(s-2)} + \frac{0.2}{(s-2)^2}\right\}.$$

Using the table of inverse Laplace transforms we obtain

$$x(t) = (0.08+0.06j)e^{-jt} + (0.08-0.06j)e^{jt} - 0.16e^{2t} + 0.2te^{2t}, \qquad \text{for all } t \geq 0.$$

Although the preceding expression is the required inverse Laplace transform the presence of the imaginary numbers gives it a rather inconvenient form. We can get rid of these by noting that

$$e^{j\alpha} + e^{-j\alpha} = 2\cos\alpha, \qquad e^{j\alpha} - e^{-j\alpha} = 2j\sin\alpha.$$

so that $x(t)$ can be written in the more convenient form

$$x(t) = 0.16\cos t - 0.12\sin t - 0.16e^{2t} + 0.2te^{2t}, \qquad \text{for all } t \geq 0.$$

Remark 3.4.6 The method of partial fractions given by Theorem 3.4.1 enables us to find the inverse Laplace transform of a coprime rational function

$$X(s) = \frac{N(s)}{D(s)}$$

when $\deg(N) < \deg(D)$. Now suppose we must find the inverse Laplace transform when

$$\deg(N) \geq \deg(D).$$

Put

$$m \triangleq \deg(N) - \deg(D).$$

Then, by simple algebra, we can divide $D(s)$ into $N(s)$ to get

$$X(s) = Q(s) + \frac{R(s)}{D(s)},$$

where $R(s)$ is the **remainder polynomial** with

$$\deg(R) < \deg(D),$$

while $Q(s)$ is the **quotient polynomial**, with $\deg(Q) = m$, namely

$$Q(s) = \gamma_m s^m + \gamma_{m-1} s^{m-1} + \ldots + \gamma_1 s + \gamma_0,$$

where the γ_i are constants. We can then get the inverse Laplace transform of $X(s)$ as follows:

$$\begin{aligned} \mathcal{L}^{-1}\{X(s)\}(t) &= \mathcal{L}^{-1}\left\{Q(s) + \frac{R(s)}{D(s)}\right\}(t) \\ &= \mathcal{L}^{-1}\{Q(s)\}(t) + \mathcal{L}^{-1}\left\{\frac{R(s)}{D(s)}\right\}(t) \end{aligned}$$

Since $\deg(R) < \deg(D)$ we can use partial fractions (i.e. Theorem 3.4.1) to find the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{R(s)}{D(s)}\right\}(t).$$

As for the inverse Laplace transform of $Q(s)$, we note from the table of Laplace transforms that

$$\mathcal{L}^{-1}\{1\}(t) = \delta(t) \qquad \text{and} \qquad \mathcal{L}^{-1}\{s^n\}(t) = \delta^{(n)}(t), \qquad \text{for each } n = 1, 2, \ldots,$$

and thus

$$\begin{aligned} \mathcal{L}^{-1}\{Q(s)\}(t) &= \mathcal{L}^{-1}\{\gamma_m s^m + \gamma_{m-1} s^{m-1} + \ldots + \gamma_1 s + \gamma_0\}(t) \\ &= \gamma_m \mathcal{L}^{-1}\{s^m\}(t) + \gamma_{m-1}\mathcal{L}^{-1}\{s^{m-1}\}(t) + \ldots + \gamma_1 \mathcal{L}^{-1}\{s\}(t) + \gamma_0 \mathcal{L}^{-1}\{1\}(t) \\ &= \gamma_m \delta^{(m)}(t) + \gamma_{m-1}\delta^{(m-1)}(t) + \ldots + \gamma_1 \delta^{(1)}(t) + \gamma_0 \delta(t). \end{aligned}$$

Example 3.4.7 Determine the inverse Laplace transform of the rational function

$$X(s) = \frac{3s^3 + 3s^2 - 5s - 2}{s^2 + s - 2}.$$

By long division we get

$$X(s) = 3s + \frac{s-2}{s^2+s-2}.$$

From the table of Laplace transforms we have

$$\mathcal{L}^{-1}\{3s\}(t) = 3\delta^{(1)}(t),$$

and by the method of partial fractions, we easily see

$$\frac{s-2}{s^2+s-2} = \frac{-1}{3(s-1)} + \frac{4}{3(s+2)},$$

so that

$$\mathcal{L}^{-1}\left\{\frac{s-2}{s^2+s-2}\right\}(t) = \frac{4}{3}e^{-2t} - \frac{1}{3}e^{t},$$

and therefore

$$\mathcal{L}^{-1}\left\{\frac{3s^3 + 3s^2 - 5s - 2}{s^2+s-2}\right\}(t) = 3\delta^{(1)}(t) + \frac{4}{3}e^{-2t} - \frac{1}{3}e^{t}, \qquad \text{for all } t \geq 0.$$

Remark 3.4.8 We see from the preceding discussion and example that whenever $\deg(N) \geq \deg(D)$ then the inverse Laplace transform of the rational function

$$X(s) = \frac{N(s)}{D(s)}$$

always includes "shocks" in the form of impulse functions and high-order impulse functions.

18.

3.5 Zero-Input and Zero-State Response by Laplace Transforms

In Section 2.3.2 we looked at the response $y(t)$ of the system

$$(3.19)\qquad \begin{cases} Q(D)y(t) = P(D)x(t), \\ y(0-) = \alpha_0, \quad y^{(1)}(0-) = \alpha_1, \ldots, \quad y^{(n-1)}(0-) = \alpha_{n-1}, \end{cases}$$

to an input signal $x(t)$ such that $x(t) = 0$ for all $t < 0$, and saw that $y(t)$ is the sum of a *zero-input* response $y_{zi}(t)$ and a *zero-state* response $y_{zs}(t)$ (see (2.65)). We also developed some methods to evaluate these. We shall now see how Laplace transforms provide an extremely powerful and simple-to-use technique for determining the zero-input and zero-state response. To this end we first recall in the next example how one uses Laplace transforms to solve linear differential equations with constant coefficients:

Example 3.5.1 Use Laplace transforms to find $y(t)$ such that

$$(3.20)\qquad \begin{cases} (D+3)y(t) = (D-1)x(t), \\ y(0-) = 1, \end{cases}$$

where $x(t)$ is the unit step function. Taking Laplace transforms on each side of the differential equation gives

$$(3.21)\qquad \mathcal{L}\{Dy(t)\}(s) + 3\mathcal{L}\{y(t)\}(s) = \mathcal{L}\{Dx(t)\}(s) - \mathcal{L}\{x(t)\}(s).$$

Now put

$$Y(s) \triangleq \mathcal{L}\{y(t)\}(s), \qquad X(s) \triangleq \mathcal{L}\{x(t)\}(s).$$

Then:

$$\mathcal{L}\{Dy(t)\}(s) = sY(s) - y(0-) = sY(s) - 1;$$
$$\mathcal{L}\{x(t)\}(s) = \frac{1}{s}.$$

Also, since $x(t) = 0$ for all $t < 0$ when $x(t)$ is a step function, we know that $x(0-) = 0$. Thus

$$\begin{aligned} \mathcal{L}\{Dx(t)\}(s) &= sX(s) - x(0-) \\ &= s(1/s) - 0 = 1. \end{aligned}$$

Substituting into (3.21):

$$[sY(s) - 1] + 3Y(s) = 1 - 1/s,$$

thus, using partial fractions,

$$\begin{aligned} Y(s) &= \frac{(2s-1)}{s(s+3)} \\ &= -\frac{1}{3s} + \frac{7}{3(s+3)}. \end{aligned}$$

Taking inverse Laplace transforms gives the solution

$$y(t) = -\frac{1}{3} + \frac{7}{3}e^{-3t},$$

for all $t \geq 0$.

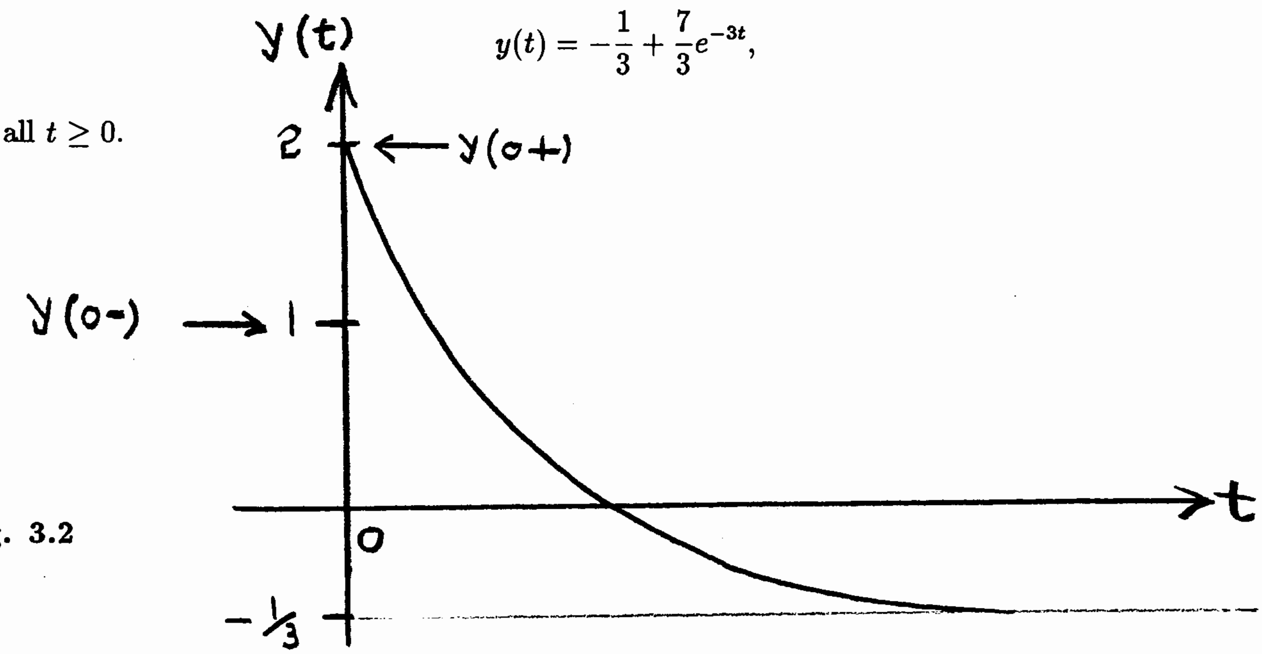

Fig. 3.2

Remark 3.5.2 Observe that the solution $y(t)$ has a jump at $t = 0$ since

$$y(0-) = 1, \qquad y(0+) = 2.$$

Now consider the general system (3.19), where

$$Q(D) \triangleq D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0,$$
$$P(D) \triangleq b_m D^m + b_{m-1}D^{m-1} + \ldots + b_1 D + b_0.$$

We are going to generalize the method used in Example 3.5.1 to determine the zero-input and zero-state response, when the input signal $x(t)$ is such that $x(t) = 0$ for all $t < 0$. Put

$$Y(s) \triangleq \mathcal{L}\{y(t)\}(s), \qquad X(s) \triangleq \mathcal{L}\{x(t)\}(s).$$

20.

Now, by Laplace transforms for derivatives and the auxiliary conditions in (3.19):

$$\mathcal{L}\{D^i y(t)\}(s) = s^i Y(s) - \sum_{j=0}^{i-1} s^j y^{(i-j-1)}(0-) = s^i Y(s) - \sum_{j=0}^{i-1} s^j \alpha_{i-j-1}. \tag{3.22}$$

Also, since $x(t) = 0$ for all $t < 0$, we have $x^{(k)}(0-) = 0$, for all $k = 0, 1, 2, \ldots$, so that

$$\mathcal{L}\{D^i x(t)\}(s) = s^i X(s) - \sum_{j=0}^{i-1} s^j x^{(i-j-1)}(0-) = s^i X(s). \tag{3.23}$$

From (3.22) we get

$$\mathcal{L}\{Q(D)y(t)\}(s) = Q(s)Y(s) - \sum_{i=0}^{n} a_i \sum_{j=0}^{i-1} s^j \alpha_{i-j-1}, \tag{3.24}$$

and from (3.23) we similarly get

$$\mathcal{L}\{P(D)x(t)\}(s) = P(s)X(s). \tag{3.25}$$

Also, from (3.19) we know

$$\mathcal{L}\{Q(D)y(t)\}(s) = \mathcal{L}\{P(D)x(t)\}(s),$$

and combining this with (3.24) and (3.25) we see

$$Q(s)Y(s) - \sum_{i=0}^{n} a_i \sum_{j=0}^{i-1} s^j \alpha_{i-j-1} = P(s)X(s),$$

so that

$$Y(s) = Y_{zi}(s) + Y_{zs}(s),$$

for

$$y_{zi}(s) = \frac{\sum_{i=0}^{n} a_i \sum_{j=0}^{i-1} s^j \alpha_{i-j-1}}{Q(s)}, \tag{3.26}$$

and

$$Y_{zs}(s) = \frac{P(s)}{Q(s)} X(s). \tag{3.27}$$

It is readily apparent that

$$y_{zi}(t) \triangleq \mathcal{L}^{-1}\{Y_{zi}(s)\}(t), \qquad y_{zs}(t) \triangleq \mathcal{L}^{-1}\{Y_{zs}(s)\}(t),$$

are respectively the zero-input response and zero-state response.

Remark 3.5.3 Observe that the method of Laplace transforms reduces the computation of the zero-input and zero-state response to the evaluation of the partial fractions occurring on the right hand side of (3.26) and (3.27). This is much easier than implementing the methods that we looked at in Section 2.3.3 and Section 2.3.4.

Laplace transforms also give us a powerful technique for determining the impulse response of the linear system

$$Q(D)y(t) = P(D)x(t).$$

Recall that the determination of the impulse response in Examples 2.2.4 and 2.2.6 involved a rather laborious procedure in which we approximated the "spike" of the impulse by a tall and narrow pulse of total unit area, and then took limits as the pulse became indefinitely tall and thin. With Laplace transforms we can avoid all of this. In fact, the impulse response $h(t)$ is nothing but the zero-state response $y_{zs}(t)$ in the special case where the input signal is the impulse function $\delta(t)$. Let us write

$$H(s) = \mathcal{L}\{h(t)\}\,. \tag{3.28}$$

When $x(t)$ is the impulse function then

$$\mathcal{L}\{x(t)\}\,(s) = 1,$$

so that (3.27) and (3.7) gives

$$H(s) = \frac{P(s)}{Q(s)}.$$

Thus the impulse response is given by

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\}(t), \qquad \text{for all } t > 0, \\ h(t) &= 0, \qquad \text{for all } t < 0, \end{aligned} \tag{3.29}$$

and $h(0)$ is *indeterminate* at $t = 0$ because of the jump in $h(t)$ at $t = 0$ caused by the impulse function. The inverse Laplace transform $\mathcal{L}^{-1}\{P(s)/Q(s)\}\,(t)$, $t > 0$, is easily found by partial fractions.

3.6 Electrical Networks

Electrical networks are a particularly important class of linear systems and the Laplace transform is a particularly powerful method for dealing with them. The idea is best illustrated by some examples:

Example 3.6.1 In the circuit of Fig. 3.3 the input signal is the voltage $x(t)$ and the output signal is the current $y(t)$.

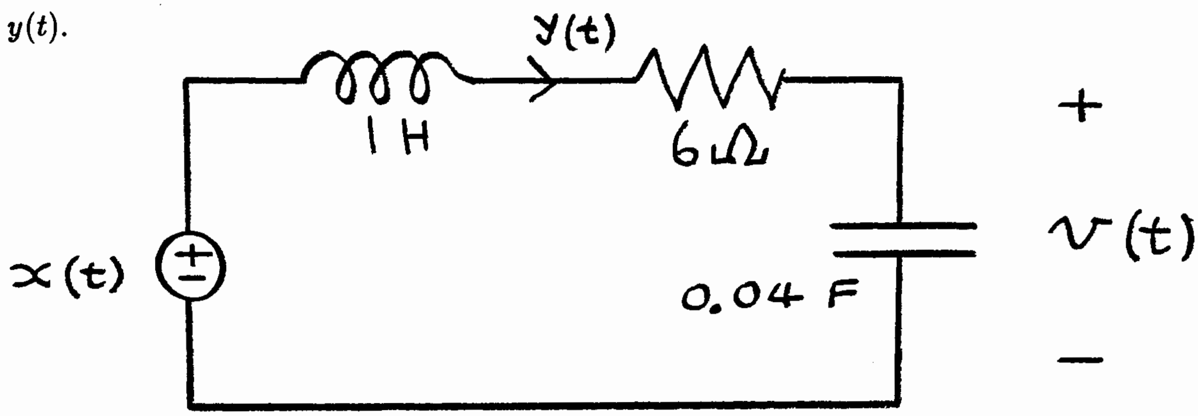

Fig. 3.3

Let $v(t)$ be the voltage across the capacitor. Determine the zero-input and zero-sdtate response when

$$y(0-) = 5 \text{ Amps}, \qquad v(0-) = 1 \text{ volts},$$

and the input signal $x(t)$ is given by

$$x(t) = \begin{cases} 0, & \text{when } t < 0, \\ 12 \sin 5t, & \text{when } t \geq 0. \end{cases} \tag{3.30}$$

By KVL around the loop:

$$x(t) = Dy(t) + 6y(t) + \left\{ v(0-) + \frac{1}{0.04} \int_{0-}^{t} y(\tau)\, d\tau \right\}, \tag{3.31}$$

thus, on taking Laplace transforms we get

$$\mathcal{L}\{x(t)\}(s) = \mathcal{L}\{Dy(t)\}(s) + 6\mathcal{L}\{y(t)\}(s) + \mathcal{L}\{v(0-)\}(s) + 25\mathcal{L}\left\{ \int_{0-}^{t} y(\tau) d\tau \right\}(s). \tag{3.32}$$

Now

$$\begin{aligned} \mathcal{L}\{Dy(t)\}(s) &= sY(s) - y(0-) = sY(s) - 5, \\ \mathcal{L}\{v(0-)\}(s) &= \frac{v(0-)}{s} = \frac{1}{s}, \\ \mathcal{L}\left\{ \int_{0-}^{t} y(\tau) d\tau \right\}(s) &= \frac{Y(s)}{s}. \end{aligned}$$

Substitution into (3.32) then gives

$$X(s) = [sY(s) - y(0-)] + 6Y(s) + \frac{v(0-)}{s} + 25\frac{Y(s)}{s},$$

which, after simplification, yields

$$Y(s) = \frac{(sy(0-) - v(0-))}{[(s+3)^2 + 4^2]} + \frac{s}{[(s+3)^2 + 4^2]}X(s). \tag{3.33}$$

Now put

$$Y_{zi}(s) = \frac{(sy(0-) - v(0-))}{[(s+3)^2 + 4^2]} = \frac{(5s-1)}{[(s+3)^2 + 4^2]}, \tag{3.34}$$

$$Y_{zs}(s) = \frac{s}{[(s+3)^2 + 4^2]}X(s). \tag{3.35}$$

The zero-input response is then given by

$$\begin{aligned} y_{zi}(t) &= \mathcal{L}^{-1}\{Y_{zi}(s)\}(t) \\ &= \mathcal{L}^{-1}\left\{\frac{(5s-1)}{[(s+3)^2 + 4^2]}\right\} \\ &= 5e^{-3t}\cos 4t - 4e^{-3t}\sin 4t. \end{aligned}$$

For the zero-state response we have from (3.30) and a table of transforms

$$X(s) = 12\left(\frac{5}{s^2 + 5^2}\right),$$

thus, from (3.35),

$$Y_{zs}(s) = \frac{60s}{[(s+3)^2 + 4^2](s^2 + 5^2)}.$$

The zero-state response is now given by

$$\begin{aligned} y_{zs}(t) &= \mathcal{L}^{-1}\{Y_{zs}(s)\}(t) \\ &= \mathcal{L}^{-1}\left\{\frac{60s}{[(s+3)^2 + 4^2](s^2 + 5^2)}\right\} \\ &= -2.5e^{-3t}\sin 4t + 2\sin 5t. \end{aligned}$$

Finally, the *complete response* is

$$\begin{aligned} y(t) &= y_{zi}(t) + y_{zs}(t) \\ &= 5e^{-3t}\cos 4t - 4e^{-3t}\sin 4t - 2.5e^{-3t}\sin 4t + 2\sin 5t. \end{aligned}$$

Remark 3.6.2 In Example 3.6.1 we used KVL to get the **integro-differential equation** (3.31) and then took Laplace transforms of this equation. Alternatively, we could have put this equation into the form of the general system (3.19) by taking derivatives with respect to t:

$$D^2y(t) + 6Dy(t) + 25y(t) = Dx(t). \tag{3.36}$$

Next, we must determine $y^{(1)}(0-)$ for the auxiliary conditions. Taking $t = 0-$ in (3.31) gives

$$x(0-) = y^{(1)}(0-) + 6y(0-) + v(0-).$$

Since $x(t) = 0$ for all $t < 0$ we have $x(0-) = 0$, giving

$$0 = y^{(1)}(0-) + 6(5) + 1, \qquad \text{hence} \quad y^{(1)}(0-) = -31\text{Amps per sec.}$$

We have thus established the auxiliary conditions

$$y(0-) = 5\text{Amps} \qquad y^{(1)}(0-) = -31\text{Amps per sec.} \tag{3.37}$$

Now (3.36) and (3.37) comprise a system of the form (3.19). Taking Laplace transforms on both sides of (3.36) gives

$$\mathcal{L}\{D^2y(t)\}(s) + 6\mathcal{L}\{Dy(t)\}(s) + 25\mathcal{L}\{y(t)\}(s) = \mathcal{L}\{Dx(t)\}(s). \tag{3.38}$$

From (3.37) and Laplace transforms of derivatives, we get

$$\begin{aligned} \mathcal{L}\{D^2y(t)\}(s) &= s^2Y(s) - sy(0-) - y^{(1)}(0-) = s^2Y(s) - 5s + 31, \\ \mathcal{L}\{Dy(t)\} &= sY(s) - y(0-) = sY(s) - 5, \end{aligned}$$

and, using $x(0-) = 0$,

$$\mathcal{L}\{Dx(t)\} = sX(s) - x(0-) = sX(s).$$

Substituting into (3.38) gives

$$[s^2Y(s) - 5s + 31] + 6[sY(s) - 5] + 25Y(s) = sX(s)$$

and therefore

$$Y(s) = \frac{(5s-1)}{[(s+3)^2 + 4^2]} + \frac{s}{[(s+3)^2 + 4^2]}X(s),$$

25.

and we now continue as in Example 3.6.1. While this approach certainly correct mathematically, it is a bit unnatural from the physical viewpoint, since the given initial conditions are in terms of $y(0-)$ and $v(0-)$ rather than $y^{(1)}(0-)$. Rather than forcing our problem into the form of (3.19) as we have just done, it is usually easier and preferable to proceed as shown in Example 3.6.1, namely by taking Laplace transforms of the integro-differential equation (3.31) directly, instead of trying to convert it into a differential equation.

Remark 3.6.3 Notice how in Example 3.6.1 we kept the input signal Laplace transform $X(s)$ in *general form* right up until line (3.33). Only after having written out $Y_{zs}(s)$ in (3.35) do we actually substitute the Laplace transform for $X(s)$. It is generally a good idea to do this, since it is then very apparent which part of the right hand side of (3.33) makes up the zero-state response namely the sum of all terms which involve $X(s)$. The remaining terms on the right hand side of (3.33) then give the zero-input response. If we had substituted the Laplace transform for $X(s)$ before line (3.33) then the decomposition of (3.33) into its zero-input and zero-state elements would not be as clearly apparent.

Example 3.6.4 In the network shown in Fig. 3.4, the input signal is the voltage $x(t)$, the output signal is the current $y(t)$, and the initial data is

$$v(0-) = 5 \text{ volts}, \qquad y(0-) = 4 \text{ amps},$$

where $v(t)$ is the voltage across the capacitor.

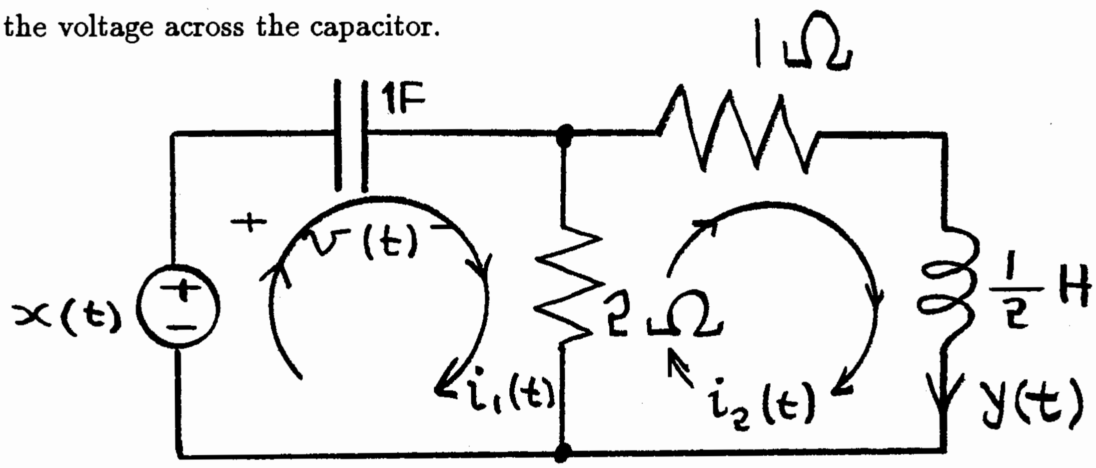

Fig. 3.4

Determine the zero-input and zero-state response when the input signal $x(t)$ is given by

$$x(t) = 20u(t).$$

Introduce mesh currents $i_1(t)$ and $i_2(t)$ as shown. Clearly $y(t) = i_2(t)$. By KVL we have

$$x(t) = \left\{ v(0-) + \int_{0-}^{t} i_1(\tau)d\tau \right\} + 2[i_1(t) - i_2(t)] \tag{3.39}$$

$$0 = 2[i_2(t) - i_1(t)] + i_2(t) + \frac{1}{2}Di_2(t) \tag{3.40}$$

Instead of trying to manipulate this into the form of (3.19) we just take Laplace transforms in (3.39) and (3.40) and use the given initial data to get

$$X(s) = \left\{ \frac{5}{s} + \frac{I_1(s)}{s} \right\} + 2[I_1(s) - I_2(s)] \tag{3.41}$$

$$0 = 2[I_2(s) - I_1(s)] + I_2(s) + \frac{1}{2}[sI_2(s) - 4], \tag{3.42}$$

(in (3.42) we used $i_2(0-) = y(0-) = 4$ amps). Now write (3.41) and (3.42) in matrix form:

$$\begin{bmatrix} (2 + 1/s) & -2 \\ -2 & (3 + s/2) \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} X(s) - 5/s \\ 2 \end{bmatrix}. \tag{3.43}$$

Now the output signal is $Y(s) = I_2(s)$, thus we solve for $I_2(s)$ using Cramer's rule:

$$Y(s) = I_2(s) = \frac{\begin{vmatrix} (2+1/s) & (X(s) - 5/s) \\ -2 & 2 \end{vmatrix}}{\begin{vmatrix} (2+1/s) & -2 \\ -2 & (3+s/2) \end{vmatrix}}$$

giving

$$\begin{aligned} Y(s) &= \frac{(4s-8) + 2sX(s)}{s^2 + (5s/2) + 3} \\ &= Y_{zi}(s) + Y_{zs}(s) \end{aligned}$$

where

$$Y_{zi}(s) \triangleq \frac{(4s-8)}{s^2 + (5s/2) + 3} \qquad Y_{zs}(s) \triangleq \frac{2sX(s)}{s^2 + (5s/2) + 3}.$$

Now the zero-input and zero-state response is obtained from the inverse Laplace transforms

$$y_{zi}(t) = \mathcal{L}^{-1}\left\{\frac{(4s-8)}{s^2 + (5s/2) + 3}\right\}(t), \qquad y_{zs}(t) = \mathcal{L}^{-1}\left\{\frac{2sX(s)}{s^2 + (5s/2) + 3}\right\}(t),$$

for all $t \geq 0$, where

$$X(s) = \frac{20}{s}.$$

The method used in Example 3.6.1 and Example 3.6.4 is the basis of a shortcut technique for analysing electrical networks involving sources, resistors, inductors, and capacitors, without even writing down the differential equations of the network. To do this we replace these circuit elements with their **Laplace transform equivalents** as follows:

RESISTORS: Consider the resistor in Fig. 3.5(i):

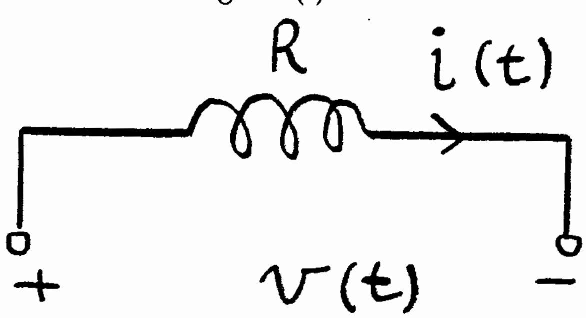

Fig. 3.5(i)

Here we have

$$v(t) = Ri(t),$$

and therefore, after taking Laplace transforms , we have

$$V(s) = RI(s).$$

The Laplace transform equivalent of a resistor is as shown in Fig. 3.5(ii):

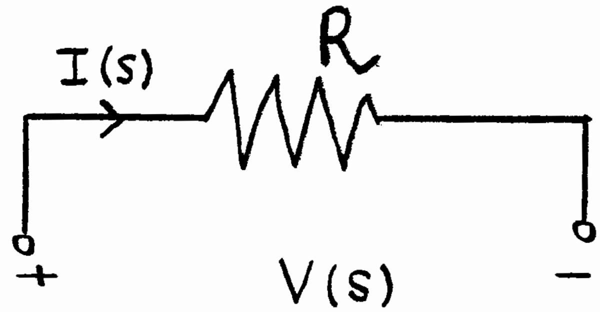

Fig. 3.5(ii)

INDUCTORS: Consider the inductor in Fig. 3.7(i):

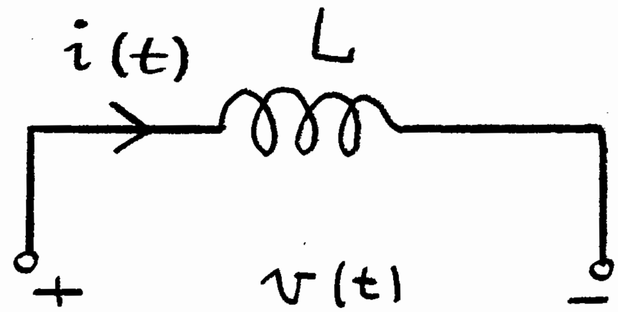

Fig. 3.7(i)

Here we have

$$v(t) = L\frac{di(t)}{dt},$$

and therefore, after taking Laplace transforms , we have

$$V(s) = LI(s) - Li(0-).$$

The Laplace transform equivalent of an inductor is as shown in Fig. 3.7(ii):

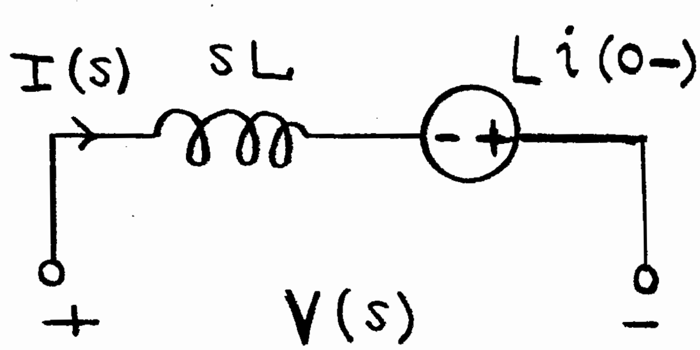

Fig. 3.7(ii)

CAPACITORS: Consider the capacitor in Fig. 3.8(i):

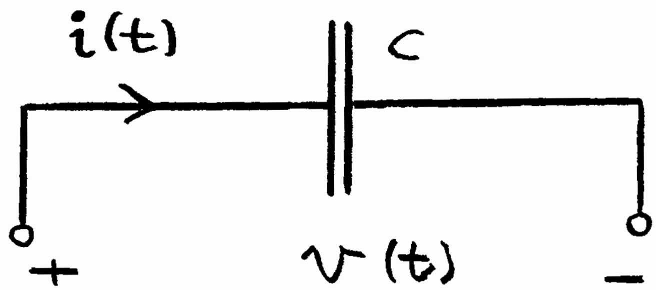

Fig. 3.8(i)

Here we have

$$v(t) = v(0-) + \frac{1}{C}\int_{0-}^{t} i(\tau)d\tau,$$

and therefore, after taking Laplace transforms , we have

$$V(s) = \frac{v(0-)}{s} + \frac{I(s)}{sC}.$$

The Laplace transform equivalent of a capacitor is as shown in Fig. 3.8(ii):

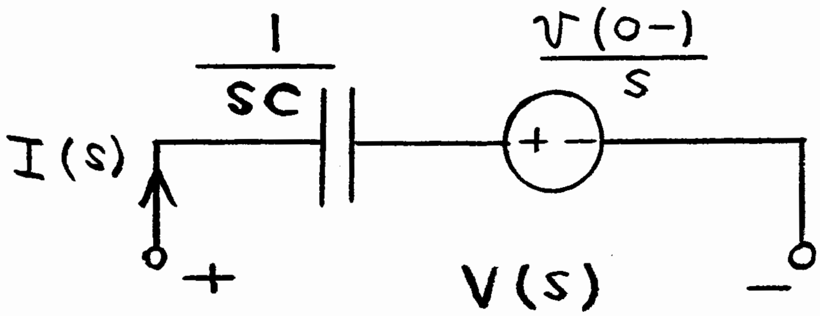

Fig. 3.8(ii)

SOURCES: Consider the voltage and current sources shown in Fig. 3.9(i), where it is assumed that

$$v(t) = 0 \qquad \text{and} \qquad i(t) = 0 \qquad \text{for all } t < 0.$$

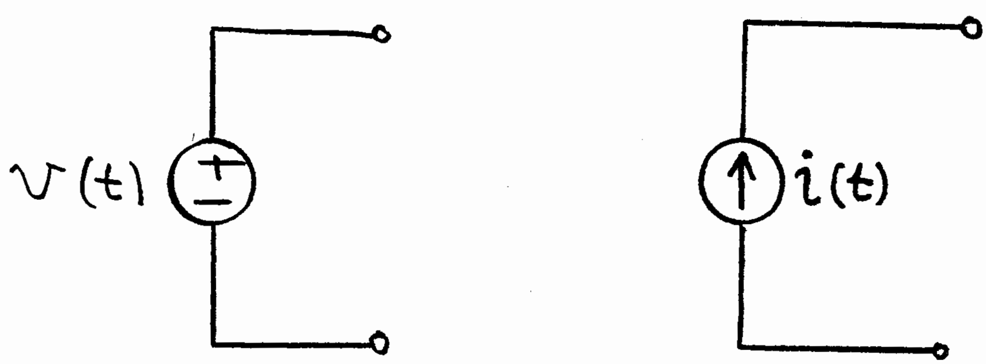

Fig. 3.9(i)

The Laplace transform equivalents of these sources is then shown in Fig. 3.9(ii):

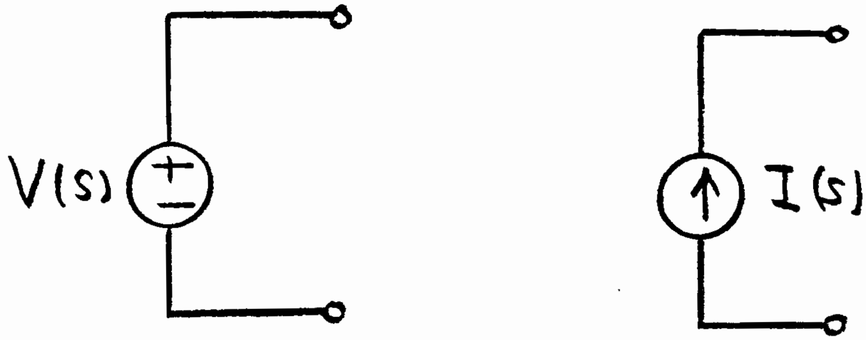

Fig. 3.9(ii)

In dealing with an electrical network we just replace these circuit elements with their Laplace transform equivalents, and then apply the Kirchhoff voltage and current laws in terms of their Laplace transform equivalents. To illustrate the idea we look at Example 3.6.4 again, but now using this approach the determining the zero-input and zero-state response:

Example 3.6.5 Repeat Example 3.6.4 using the Laplace transform equivalents of the circuit elements rather than writing down the differential equations of the circuit. The Laplace transform equivalent of the circuit is shown in Fig. 3.10:

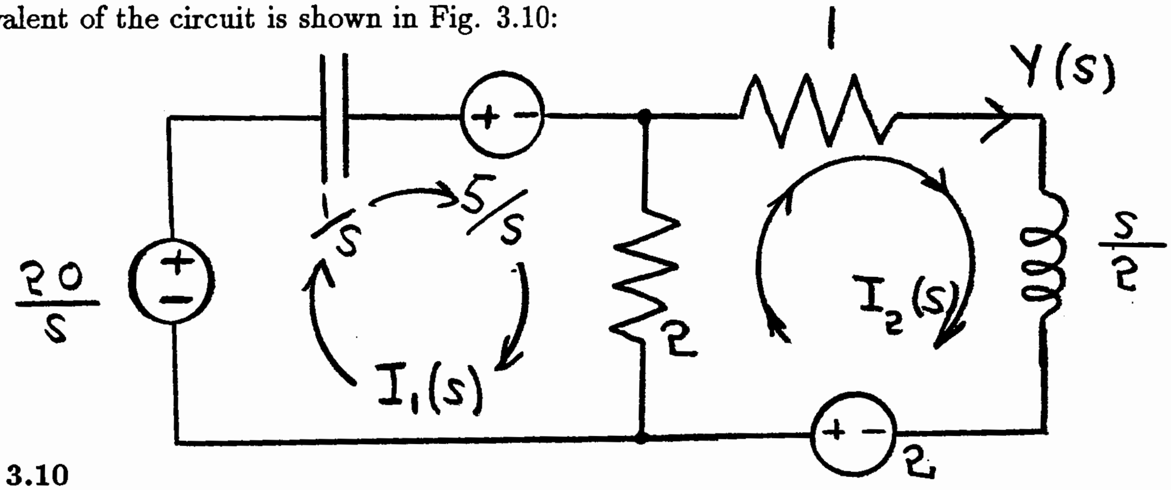

Fig. 3.10

Introduce mesh currents $i_1(t)$ and $i_2(t)$ as shown. By KVL around the meshes we have

$$X(s) = \frac{I_1(s)}{s} + \frac{5}{s} + 2[I_1(s) - I_2(s)] \tag{3.44}$$

$$0 = 2[I_2(s) - I_1(s)] + I_2(s) + \frac{sI_2(s)}{2} - 2. \tag{3.45}$$

Now write (3.44) and (3.45) in matrix form:

$$\begin{bmatrix} (2+1/s) & -2 \\ -2 & (3+s/2) \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} X(s) - 5/s \\ 2 \end{bmatrix}. \tag{3.46}$$

Now the output signal is $Y(s) = I_2(s)$, thus we solve for $I_2(s)$ using Cramer's rule:

$$Y(s) = I_2(s) = \frac{\begin{vmatrix} (2+1/s) & (X(s)-5/s) \\ -2 & 2 \end{vmatrix}}{\begin{vmatrix} (2+1/s) & -2 \\ -2 & (3+s/2) \end{vmatrix}}$$

giving

$$\begin{aligned} Y(s) &= \frac{(4s-8) + 2sX(s)}{s^2 + (5s/2) + 3} \\ &= Y_{zi}(s) + Y_{zs}(s) \end{aligned}$$

where

$$Y_{zi}(s) \triangleq \frac{(4s-8)}{s^2 + (5s/2) + 3} \qquad Y_{zs}(s) \triangleq \frac{2sX(s)}{s^2 + (5s/2) + 3}.$$

1.

Now the zero-input and zero-state response is obtained from the inverse Laplace transforms

$$y_{zi}(t) = \mathcal{L}^{-1}\left\{\frac{(4s-8)}{s^2+(5s/2)+3}\right\}(t), \qquad y_{zs}(t) = \mathcal{L}^{-1}\left\{\frac{2sX(s)}{s^2+(5s/2)+3}\right\}(t),$$

for all $t \geq 0$, where

$$X(s) = \frac{20}{s}.$$

Remark 3.6.6 Which of the approaches in Examples 3.6.4 and 3.6.5 is the preferable one? My preference is for actually writing out the differential equations of the circuit followed by taking Laplace transforms , rather than using the more automatic technique of just substituting circuit elements with their Laplace transform equivalents. The trouble with the second approach is that it suppresses the physics of the problem and replaces it with a method of algorithmic substitution in which it is quite easy to make mistakes, particularly when inserting the voltage sources corresponding to the initial data.

3.

3.7 System Transfer Function

Consider the system given by (3.19), and let $x(t)$ be an input signal with $x(t) = 0$ for all $t \geq 0$. We define the **transfer function** of the system as the function of s given by the ratio

$$(3.47)\quad H(s) = \frac{\text{Laplace transform of the zero-state response } y_{zs}(t) \text{ to the input signal } x(t)}{\text{Laplace transform of the input signal } x(t)}.$$

In view of (3.27) we see that the transfer function is therefore given by

$$H(s) = \frac{P(s)}{Q(s)}.$$

In E&CE-380 it will be seen that the transfer function of a system is an indispensable idea for studying feedback control systems.

The roots of the numerator polynomial $P(s)$ (namely those values of s such that $P(s) = 0$) are called the **zeros** of the transfer function, and the roots of the denominator polynomial $Q(s)$ are called the **poles** of the transfer function.

How does one determine the transfer function of a system? From (3.29) we see that if $h(t)$ is the impulse response of the system then $h(t)$ and $H(s)$ are related by

$$\mathcal{L}\{h(t)\}(s) = H(s),$$

so that the transfer function $H(s)$ is just the Laplace transform of the system impulse response. The nice thing about the transfer function $H(s)$ is that we can use it to determine the zero-state response $y_{zs}(t)$ to **any** input signal $x(t)$ provided that $x(t) = 0$ for all $t < 0$. In fact, this is immediate from the definition (3.47), which shows that

$$Y_{zs}(s) = H(s)X(s),$$

so that the zero-state response is

$$y_{zs}(t) = \mathcal{L}^{-1}\{H(s)X(s)\}(t).$$

Schematically, we shall respresent a system with input signal $x(t)$, output signal $y(t)$ and transfer function $H(s)$ as in Fig. 3.11:

4

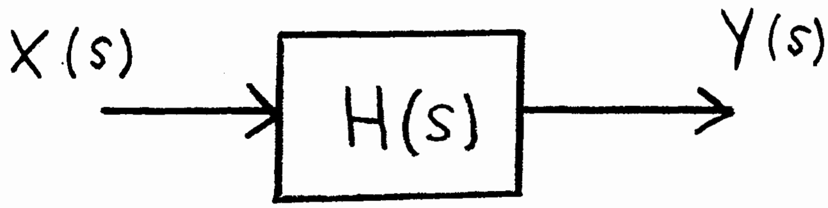

Fig. 3.11

It is evident from Theorems 2.3.8 and 2.3.12 that the information needed to determine asymptotic stability and BIBO stability resides in the location of the poles of the transfer function. Indeed, we can state these two results together as follows:

Theorem 3.7.1 *The system (3.19) is asymptotically stable if and only if all poles of $Q(s)$ have strictly negative real parts (that is < 0). Moreover the system is BIBO stable if and only if*

$$\deg(P) \leq \deg(Q)$$

and all poles of $Q(s)$ have strictly negative real parts.

There are a number of ways in which we can combine systems into larger systems namely:

SERIES COMBINATION: Here we take the output of system with transfer function $H_1(s)$ and use it for the input of a second system with transfer function $H_2(s)$, as shown in Fig. 3.12:

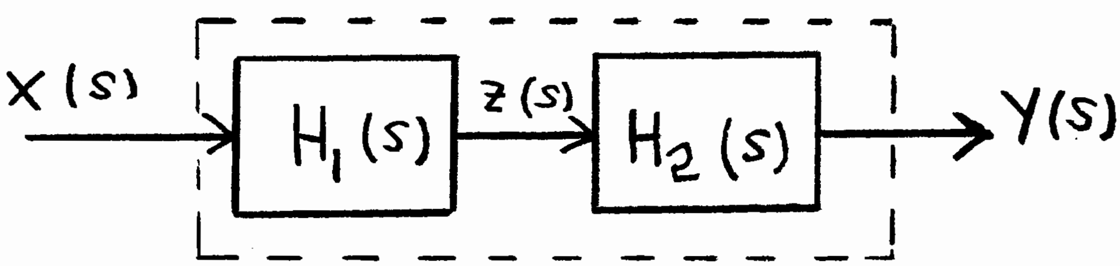

Fig. 3.12

A natural problem is to determine the **equivalent** transfer function of the overall series combination. Fix an input signal $x(t)$, let $z(t)$ be the corresponding zero-state response of the system with transfer function $H_1(s)$, and let $y(t)$ be the corresponding zero-state response of the system with transfer function $H_2(s)$. Then clearly

$$Z(s) = H_1(s)X(s), \qquad Y(s) = H_2(s)Z(s),$$

so that

$$Y(s) = H_2(s)H_1(s)X(s).$$

Therefore, the equivalent transfer function of the series combination is

$$H(s) = \frac{Y(s)}{X(s)} = H_1(s)H_2(s).$$

PARALLEL COMBINATION: To place systems in parallel we need a device called **a summing junction**. This accepts two input signals $x_1(t)$ and $x_2(t)$ and has an output which is the *sum* of these signals, namely $x_(t) + x_2(t)$. The schematic figure for a summing junction is shown in Fig. 3.13:

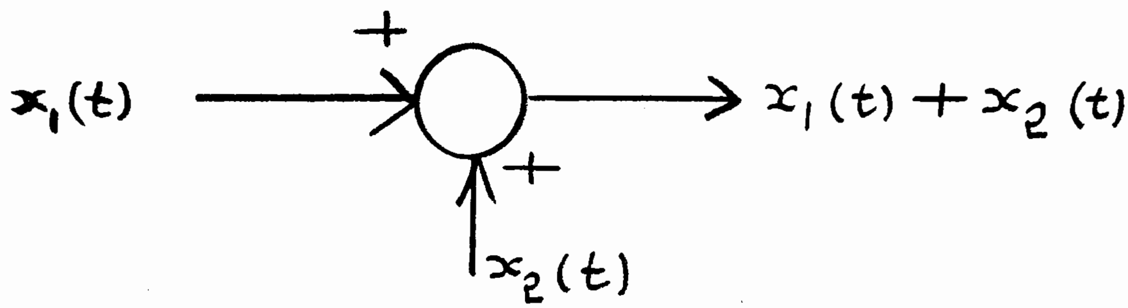

Fig. 3.13

Summing junctions that add signals which are voltages can be fabricated from operational amplifiers.

Using a summing junction we can put systems with transfer functions $H_1(s)$ and $H_2(s)$ in parallel as follows:

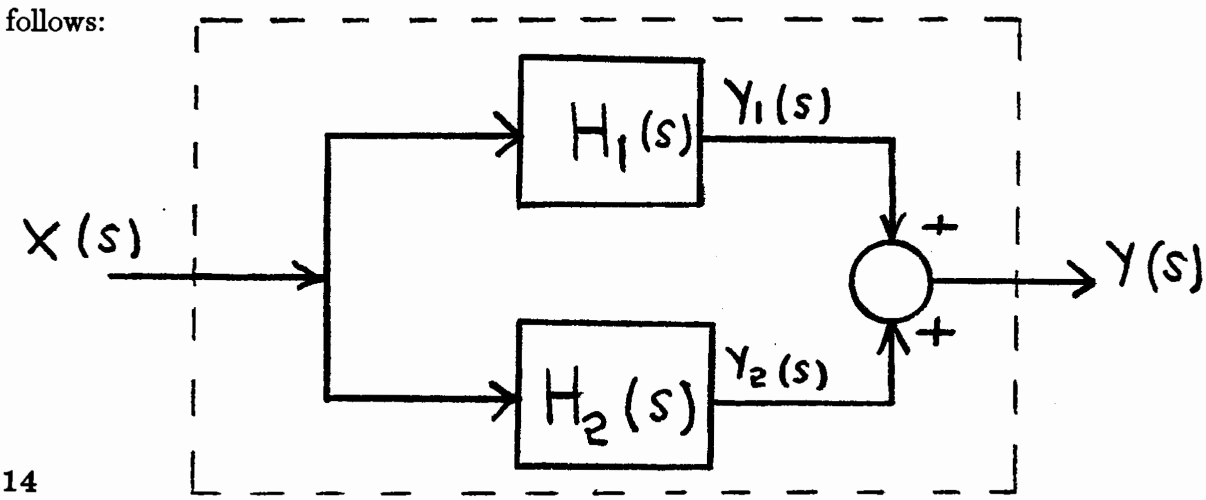

Fig. 3.14

Here $x(t)$ is the input signal for both systems and then the overall output signal $y(t)$ is the sum of the outputs $y_1(t)$ and $y_2(t)$ of the individual systems. We must determinee the equivalent transfer function . Clearly

$$Y_1(s) = H_1(s)X(s), \qquad Y_2(s) = H_2(s)X(s),$$

thus the overall output is

$$\begin{aligned} Y(s) &= Y_1(s) + Y_2(s) \\ &= [H_1(s) + H_2(s)]X(s). \end{aligned}$$

The equivalent transfer function $H(s)$ is then

$$H(s) = \frac{Y(s)}{X(s)} = H_1(s) + H_2(s).$$

FEEDBACK COMBINATION: To create a feedback combination of systems we need a variant of a summing junction, namely a **differencing junction** which accepts two signals $x_1(t)$, and $x_2(t)$, and whose output is the *difference* of the signals, namely $x_1(t) - x_2(t)$. A differencing junction is represented as in Fig. 3.15:

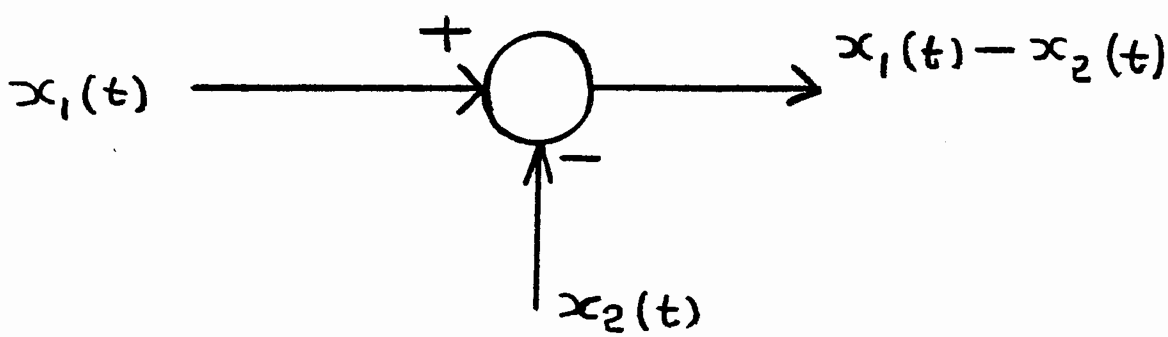

Fig. 3.15

Differencing junctions for signals represented by voltages are easily fabricated using operational amplifiers.

Using a differencing junction we can put two systems in a **feedback combination** as follows:

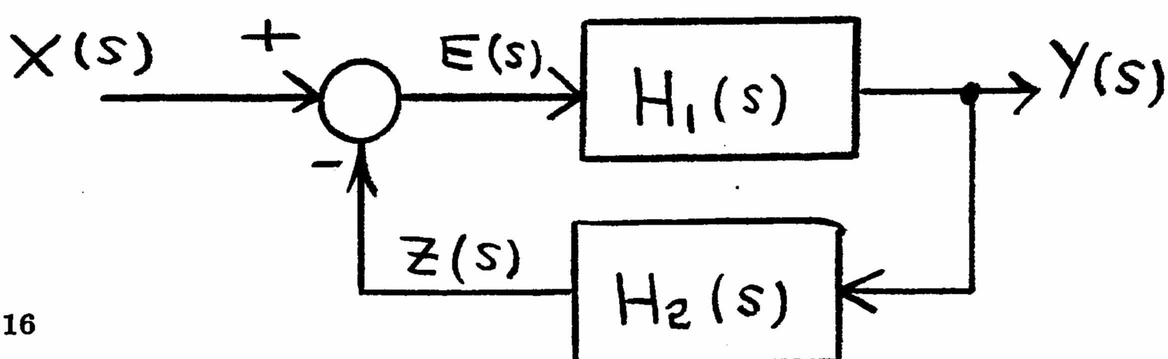

Fig. 3.16

We must find the transfer function

$$H(s) = \frac{Y(s)}{X(s)}.$$

7.

From Fig. 3.16

$$Y(s) = H_1(s)E(s) \tag{3.48}$$

$$Z(s) = H_2(s)Y(s). \tag{3.49}$$

Also, since $e(t) = x(t) - y(t)$, it follows that

$$E(s) = X(s) - Z(s). \tag{3.50}$$

From (3.48), (3.49), and (3.50),

$$\begin{aligned} Y(s) &= H_1(s)[X(s) - Z(s)] \\ &= H_1(s)[X(s) - H_2(s)Y(s)] \end{aligned}$$

so that

$$[1 + H_1(s)H_2(s)]Y(s) = H_1(s)X(s),$$

giving the transfer function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}.$$

8-

3.8 Frequency Response

Consider the system

$$
\begin{cases} Q(D)y(t) = P(D)x(t), \\ y(0-) = \alpha_0, \quad y^{(1)}(0-) = \alpha_1, \ldots, \quad y^{(n-1)}(0-) = \alpha_{n-1}, \end{cases} \tag{3.51}
$$

for which the input is the sinusoid

$$
x(t) = \begin{cases} A\cos(\omega t + \theta), & \text{for all } t \geq 0, \\ 0, & \text{for all } t < 0. \end{cases} \tag{3.52}
$$

If the system is BIBO stable (see Theorem 3.7.1) then it turns out that the corresponding output $y(t)$ settles down to a periodic signal $y_{ss}(t)$ for large values of t, whatever the values of the constants α_i in the auxiliary condition of (3.52). The precise statement of this fact is as follows:

Theorem 3.8.1 *Suppose the system (3.51) is BIBO stable with transfer function $H(s)$. If the input signal $x(t)$ is given by (3.52) with corresponding output $y(t)$ then*

$$
\lim_{t\to\infty} y(t) = y_{ss}(t)
$$

where

$$
y_{ss}(t) = A|H(j\omega)|\cos(\omega t + \theta + \angle H(j\omega)).
$$

Remark 3.8.2 The signal $y_{ss}(t)$ is called the **steady state response** of the system to the sinusoidal input signal (3.52).

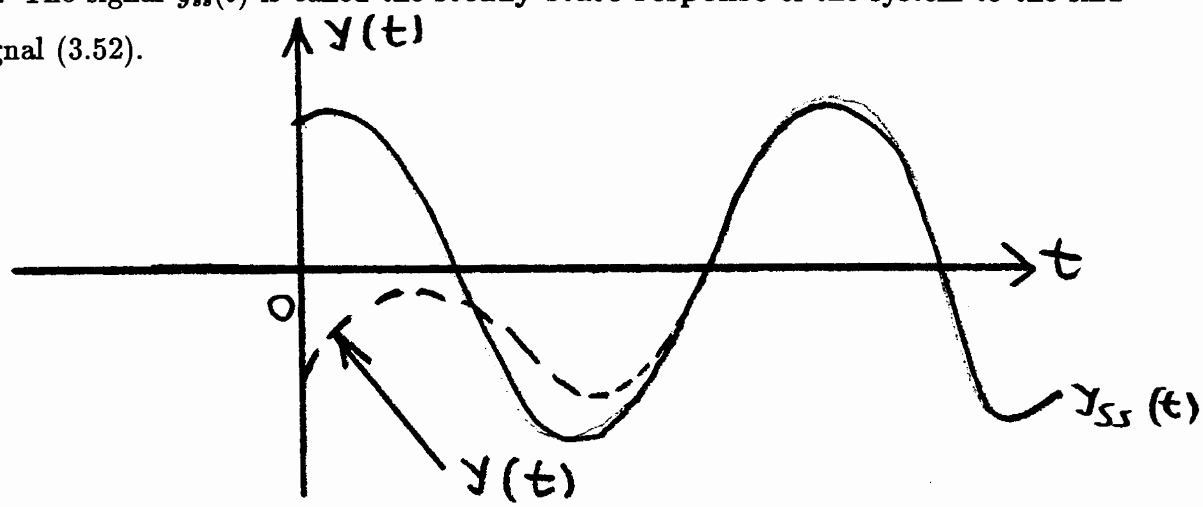

Fig. 3.17

Notice that $y_{ss}(t)$ does not depend in any way on the constants α_i in the auxiliary conditions of (3.51). The theorem is basically saying that BIBO stability of the system causes all initial transients in the response $y(t)$ to decay to zero, leaving us with only the steady-state response. Moreover, $|H(j\omega)|$ gives the **amplification** and $\angle H(j\omega)$ gives the **phase shift** of the steady state response relative to the input signal. The function $H(j\omega)$ of ω is called the **frequency response** of the system 3.51. The frequency response is an essential notion in the study of communication systems.

Example 3.8.3 Consider the system

$$\begin{cases} (D+1)y(t) = x(t), \\ y(0-) = 5, \end{cases} \tag{3.53}$$

where the input signal is

$$x(t) \triangleq \begin{cases} 0, & \text{for all } t < 0, \\ \cos(t), & \text{for all } t \geq 0. \end{cases}$$

Now calculate the response of the system. First take Laplace transforms to get

$$[sY(s) - 5] + Y(s) = X(s),$$

so that

$$\begin{aligned} Y(s) &= \frac{5}{(s+1)} + \frac{X(s)}{(s+1)} \\ &= Y_{zi}(s) + Y_{zs}(s). \end{aligned}$$

Then

$$y_{zi}(t) = \mathcal{L}^{-1}\{Y_{zi}(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{5}{(s+1)}\right\}(t) = 5e^{-t}.$$

Also, by easy partial fractions,

$$\begin{aligned} Y_{zs}(s) &= \frac{X(s)}{(s+1)} \\ &= \frac{s}{(s^2+1)(s+1)} \\ &= \frac{1}{2(1-j)}\frac{1}{(s+j)} + \frac{1}{2(1+j)}\frac{1}{(s-j)} - \frac{1}{2(s+1)}, \end{aligned}$$

so that

$$\begin{aligned} y_{zs}(t) &= \mathcal{L}^{-1}\{Y_{zs}(s)\}(t) \\ &= \frac{1}{\sqrt{2}}\cos(t - \pi/4) - \frac{1}{2}e^{-t}. \end{aligned}$$

10-

Then the complete response of the system is

$$\begin{aligned} y(t) &= y_{zi}(t) + y_{zs}(t) \\ &= 5e^{-t} + \frac{1}{\sqrt{2}}\cos(t - \pi/4) - \frac{1}{2}e^{-t}. \end{aligned}$$

From this it follows that

$$y(t) \to \frac{1}{\sqrt{2}}\cos(t - \pi/4)$$

as $t \to \infty$. Let us now check this result using Theorem 3.8.1. Clearly the transfer function of the system is

$$H(s) = \frac{1}{(s+1)}, \qquad \text{thus} \qquad H(j\omega) = \frac{1}{(1+j\omega)},$$

so that

$$|H(j)| = \frac{1}{\sqrt{2}} \qquad \text{and} \qquad \angle H(j) = -\pi/4.$$

According to Theorem 3.8.1 we must have

$$y(t) \to y_{ss}(t)$$

as $t \to \infty$, for

$$\begin{aligned} y_{ss}(t) &\triangleq |H(j)|\cos(t + \angle H(j)) \\ &= \frac{1}{\sqrt{2}}\cos(t - \pi/4) \end{aligned} \tag{3.54}$$

which confirms the result of our direct calculation.