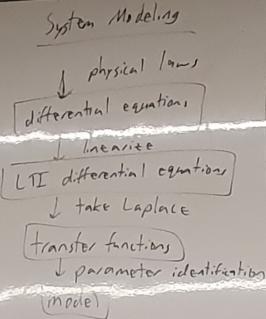


$$v(t) \xrightarrow{P} h(t) \quad P \text{ is LTI}$$

$$y(t) = (h * u)(t)$$

$$e^s \xrightarrow{P} H(s)e^s$$

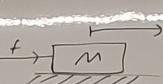
$$u(t) \xrightarrow{\text{transfer function}} H(s) \xrightarrow{\text{Y(s)}} Y(s) = H(s)U(s)$$

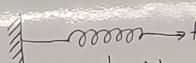


Deriving Differential Equations

Mechanical systems (in translation)

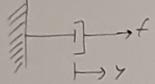
- components:

masses   $f(t) = M \frac{d^2y}{dt^2}$   
Newton's 2nd Law

f is force  
y is displacement  
(linear) springs   $f(t) = -K y(t)$

$y=0 \Rightarrow$  spring is not stretched or contracted (at its natural length)

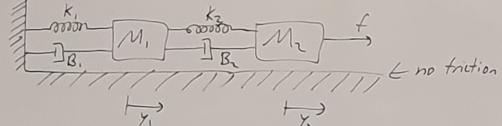
(linear) dashpots (damping elements)



$$f(t) = -B \ddot{y}(t)$$

- Procedure:
1. Draw free-body diagrams for each mass (FBDs)
  2. Apply Newton's 2nd Law
  3. System of differential equations

Ex.



$y_1 = y_2 = D \Rightarrow$  both springs are at their natural lengths

Newton's 2nd Law:

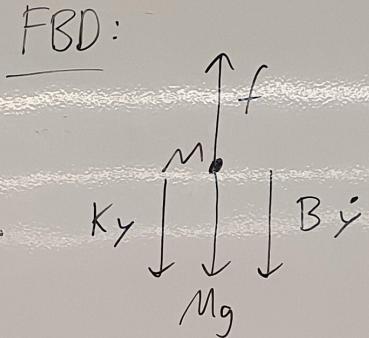
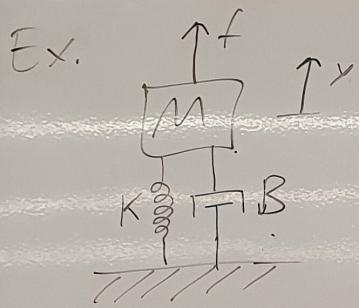
$$\begin{aligned} M_1 \ddot{y}_1 &= K_1(y_2 - y_1) + B_1(\dot{y}_2 - \dot{y}_1) - K_1 y_1 - B_1 \dot{y}_1 \\ M_2 \ddot{y}_2 &= f - K_2(y_2 - y_1) - B_2(\dot{y}_2 - \dot{y}_1) \end{aligned}$$

FBDs:

$$\begin{aligned} \xleftarrow[B_1 \dot{y}_1]{K_1 y_1} M_1 &\xrightarrow[B_2(\dot{y}_2 - \dot{y}_1)]{K_2(y_2 - y_1)} M_2 \\ \xleftarrow[B_2(\dot{y}_2 - \dot{y}_1)]{K_2(y_2 - y_1)} M_2 &\xrightarrow{B_2(\dot{y}_2 - \dot{y}_1) \ddot{y}_2} f \end{aligned}$$

$$f(y_1, \dot{y}_1, \ddot{y}_1, y_2, \dot{y}_2, \ddot{y}_2, f) = 0$$

system of differential equations  
linear, LTI



Newton's 2nd Law:

$$M\ddot{y} = f - K_y y - B\dot{y} - Mg \quad (\Rightarrow) \quad M\ddot{y} + B\dot{y} + K_y y = f - Mg \quad (1)$$

$$\quad (\Rightarrow) \quad 0 = f - K_y y - B\dot{y} - M\ddot{y} - Mg \quad (\Rightarrow) \text{affine (not linear)}$$

$$\quad (\Rightarrow) \quad \text{not LTI}$$

Idea: change variables (coordinates) to cancel out  $\ddot{y}$   
so that this system becomes LTI

→ define  $\Delta y = y - y_0 \rightarrow$  unknown constant

⇒ translation or displacement relative to  $y_0$

$$\Rightarrow \dot{\Delta y} = \dot{y}$$

$$\Rightarrow \ddot{\Delta y} = \ddot{y}$$

- Goal:
1. Choose  $y_0$  so that  $\Delta y$  satisfies an LTI differential equation.
  2. Solve for  $\Delta y$  using LTI methods.
  3. Calculate  $y = \Delta y + y_0$ .

$$M\ddot{\Delta y} + B\dot{\Delta y} + K\Delta y = \underbrace{M\ddot{y} + B\dot{y} + Ky - Ky_0}_{(y-y_0)} = \underbrace{f - Mg - Ky_0}_{\text{Eq. (1)}} = f - \cancel{Mg} + \cancel{Mg} = f$$

Choose  $y_0 \ni -Ky_0 = Mg \Leftrightarrow y_0 = \frac{-Mg}{K}$

$$\Rightarrow M\ddot{\Delta y} + B\dot{\Delta y} + K\Delta y = f$$

$\Rightarrow \Delta y$  satisfies an LTI differential equation.