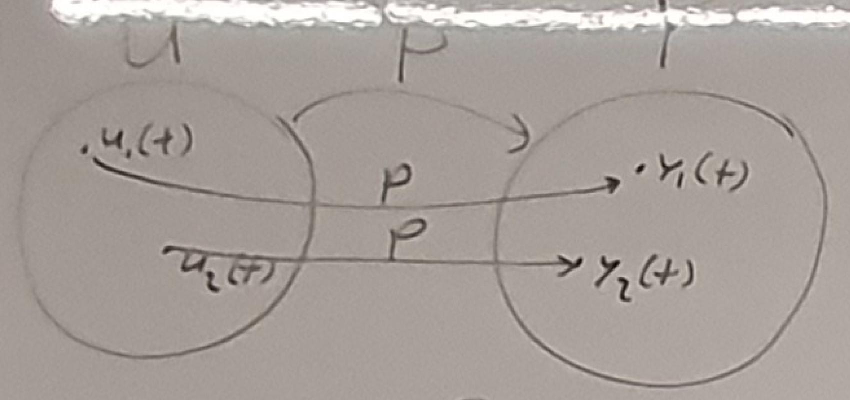
Systems

$$u(t) \xrightarrow{P} y(t)$$



$y_1 = Pu_1$

$y_2 = Pu_2$

Linear combinations

$c_1 f_1(t) + c_2 f_2(t)$

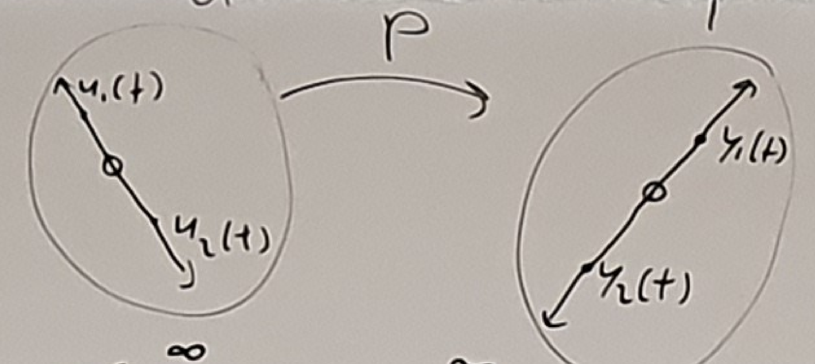
$\sum_{i=1}^{\infty} c_i f_i(t)$

$\int_a^b c(\tau) f(t,\tau)\, d\tau$

linearity

$\forall\, c_1, c_2 \in \mathbb{R},\quad u_1, u_2 \in U$

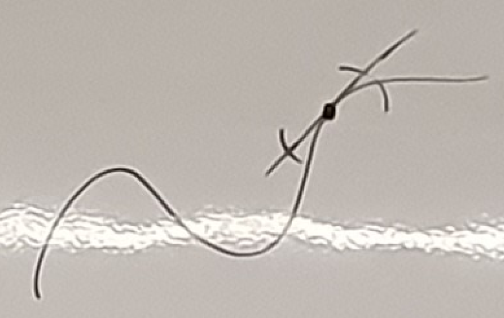
$P(c_1 u_1 + c_2 u_2) = c_1 P(u_1) + c_2 P(u_2)$



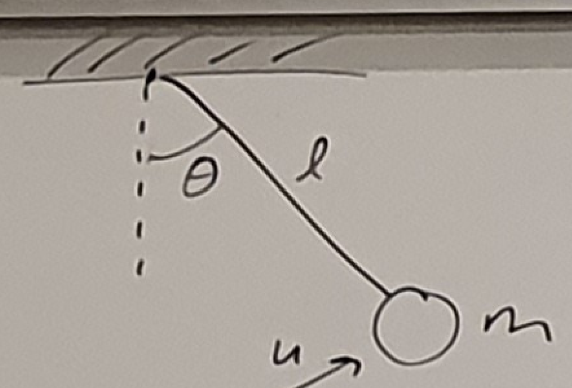
$$P\left(\sum_{i=1}^{\infty} c_i u_i\right) = \sum_{i=1}^{\infty} c_i P(u_i)$$

$$P\left(\int_a^b c(\tau) u(t,\tau)\, d\tau\right) = \int_a^b c(\tau)\, P(u(t,\tau))\, d\tau$$

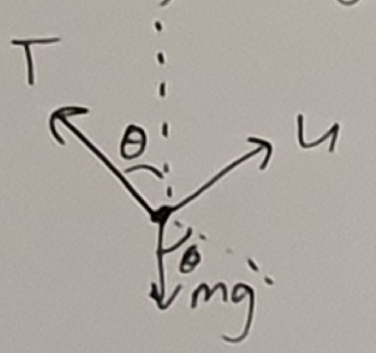
- Linearity greatly simplifies analysis and design.
- Among physical systems, nonlinearity is common
  → we often approximate the operation of a nonlinear system about an "operating point" with a linearized model



Ex.



- free-body diagram



Newton's 2nd law for angular acceleration:

$$\underbrace{m^2 \ell}_{\text{moment of inertia}}\, \ddot{\theta} = \underbrace{u\ell - mg\sin\theta}_{\text{net torque}} \Rightarrow \text{nonlinear!}$$

for sufficiently small θ, $\sin\theta \approx \theta$ $\quad$ $\theta = 0$ "operating point"

$$\Rightarrow m\ell^2 \ddot{\theta} = u\ell - mg\theta \Rightarrow \text{linear!}$$

6. time-invariance

Roughly speaking, a system is time-invariant if its behavior doesn't change with time...

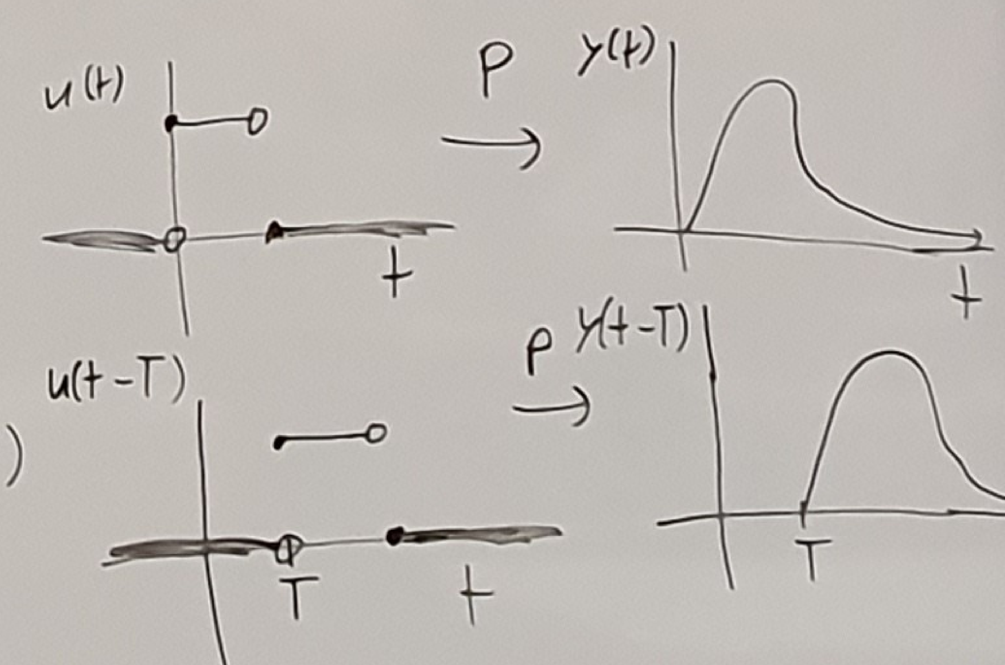
mathematically, if

$$u(t) \xrightarrow{P} y(t)$$

then

$$u(t-T) \xrightarrow{P} y(t-T)$$

delay by time T



Ex. a. $M\ddot{y}(t) = u(t)$, $u(t) = 0 \ \forall\, t \le t_0$, $y(t_0) = \dot{y}(t_0) = 0$

$$\Rightarrow y(t) = \frac{1}{m} \int_{-\infty}^{t} \int_{-\infty}^{\tau} u(\rho)\, d\rho\, d\tau \quad (P) \qquad u(t) \xrightarrow{P} y(t)$$

Now replace $u(t)$ with $\tilde{u}(t) = u(t-T)$

The corresponding response is:

$$\tilde{y}(t) = \frac{1}{m} \int_{-\infty}^{t} \left(\int_{-\infty}^{\tau} u(\rho - T)\, d\rho \right) d\tau$$

$\tilde{\rho} := \rho - T \Rightarrow d\tilde{\rho} = d\rho$

$$\rightarrow \ = \frac{1}{m} \int_{-\infty}^{t} \int_{-\infty}^{\tau - T} u(\tilde{\rho})\, d\tilde{\rho}\, d\tau$$

$\tilde{\tau} := \tau - T \Rightarrow d\tilde{\tau} = d\tau$

$$\rightarrow \ = \frac{1}{m} \int_{-\infty}^{t-T} \int_{-\infty}^{\tilde{\tau}} u(\tilde{\rho})\, d\tilde{\rho}\, d\tilde{\tau}$$

$= y(t-T)$ $\Rightarrow$ time-invariant

b. Sample and hold

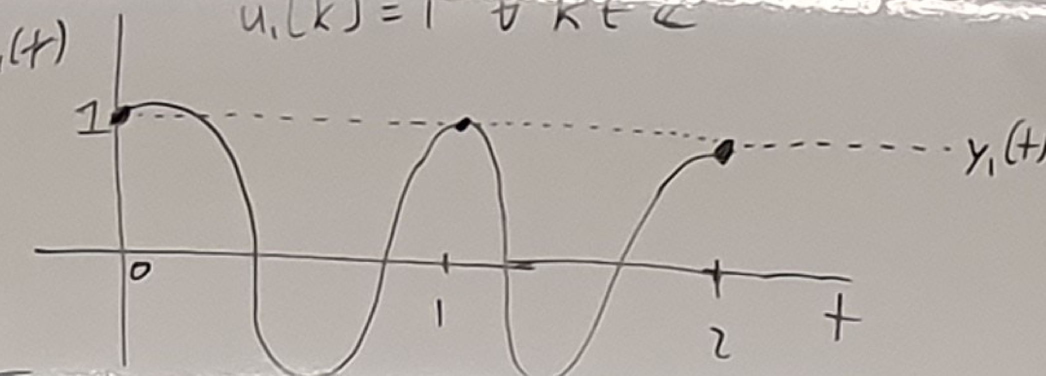
$u(t) \xrightarrow{P} y(t) = u[k]$, $k \le t < k+1$, $k \in \mathbb{Z}$

Ex. $u_1(t) = \cos(2\pi t)$

$u_1[k] = 1 \ \forall\, k \in \mathbb{Z}$ $\xrightarrow{P}$ $y_1(k) = 1 \ \forall\, k \in \mathbb{Z}$

$\Rightarrow y_1(t) = 1 \ \forall\, t \in \mathbb{R}$

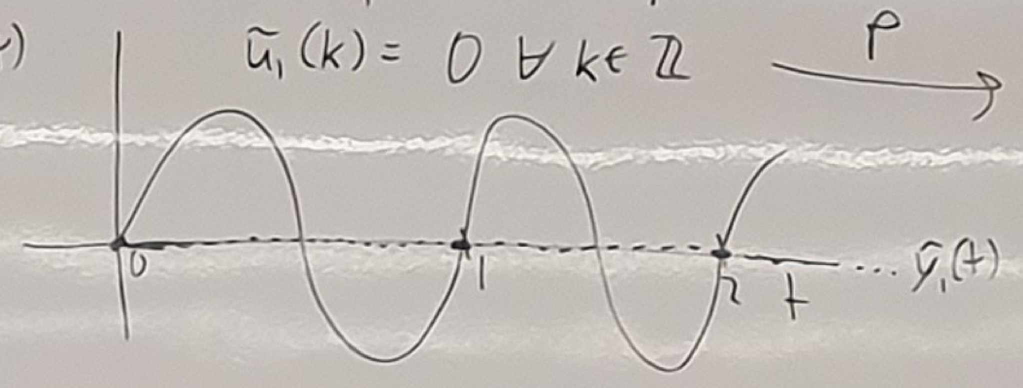
$y_1(t - \frac{1}{4}) = 1 \ \forall\, t \in \mathbb{R}$



$T = \frac{1}{4} \rightarrow \tilde{u}_1(t) = u(t - \frac{1}{4}) = \cos(2\pi(t - \frac{1}{4})) = \cos(2\pi t - \frac{\pi}{2}) = \sin(2\pi t)$

$\tilde{u}_1(k) = 0 \ \forall\, k \in \mathbb{Z}$ $\xrightarrow{P}$ $\tilde{y}_1(k) = 0 \ \forall\, k \in \mathbb{Z}$

$\Rightarrow \tilde{y}_1(t) = 0 \ \forall\, t \in \mathbb{R}$



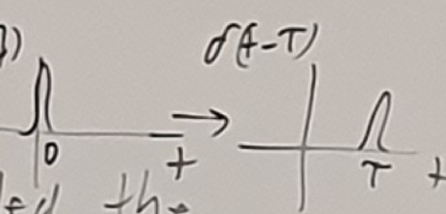
$\Rightarrow \tilde{y}_1(t) = 0 \neq 1 = y_1(t - \frac{1}{4})$

$\Rightarrow$ NOT time-invariant

Focus: systems that are linear and time invariant (LTI)

- Impulse response and convolution

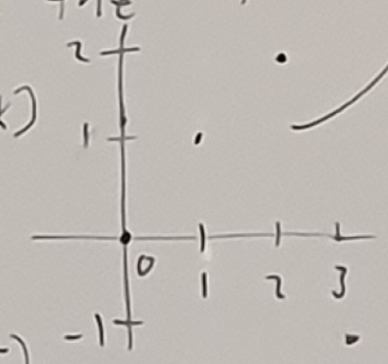
The property $f(t_0) = \int_{-\infty}^{\infty} f(\tau)\, \delta(t_0 - \tau)\, d\tau$



(for a suitably "well behaved" function $f(t)$) is called the sifting or sampling property of the impulse $\delta(t)$

Dropping the subscript, we write

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\, \delta(t - \tau)\, d\tau \quad (1)$$



$\Rightarrow$ $f(t)$ is a linear combination of time-delayed impulses $\delta(t-\tau)$ with weights $f(\tau)$

$f[k] = 0 \cdot \delta[0] + 1 \cdot \delta[1] + 2 \cdot \delta[2] + -1 \cdot \delta[3]$

Assume P is LTI

$\delta(t) \xrightarrow{P} h(t)$ $\qquad$ $h(t)$ is called the impulse response

Since P is time-invariant $\delta(t-\tau) \xrightarrow{P} h(t-\tau) \ \forall\, \tau$

So, by linearity,

$$u(t) = \underbrace{\int_{-\infty}^{\infty} u(\tau)\, \delta(t-\tau)\, d\tau}_{\text{linear combination}} \xrightarrow[\text{by Eq. (1)}]{P} y(t) = \int_{-\infty}^{\infty} u(\tau)\, P(\delta(t-\tau))\, d\tau$$

$$= \int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\, d\tau$$

$$=: (h * u)(t)$$

- the output $y(t)$ is the convolution of the input $u(t)$ with the impulse response $h(t)$
- this key property of LTI systems reduces the analysis of their time-domain responses to convolution