L2. Jan 7.

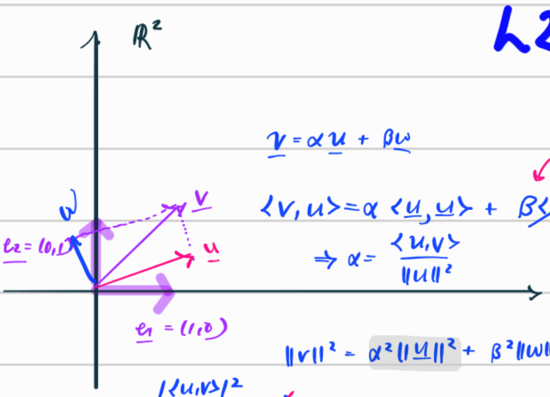
Energy (power signals from a vector space)

$u, v \in \mathbb{R}^N$ $\quad u = (u_1, \dots, u_N)^T$

$\langle u, v \rangle = \sum_{n=1}^{N} u_n v_n$ $\quad v = (v_1, \dots, v_n)^T$

if $u, v \in \mathbb{C}^N$

$\langle u, v \rangle = \sum_{n=1}^{n=1} |u_n|^2$



$v = \alpha u + \beta w$

$\langle v, u \rangle = \alpha \langle u, u \rangle + \beta \cancel{\langle w, u \rangle}$ ← $90°, = 0$.

$\Rightarrow \alpha = \frac{\langle u, v \rangle}{\|u\|^2}$

$\|v\|^2 = \alpha^2 \|u\|^2 + \beta^2 \|w\|^2 \geq \alpha^2 \|u\|^2$

$\frac{|\langle u, v \rangle|^2}{\|u\|^{\cancel{4}}} \|u\|^{\cancel{2}} = \alpha^2 \|u\|^2 \leq \|v\|^2$

$|\langle u, v \rangle| \leq \|v\| \cdot \|u\|$ ← Cauchy-Schwartz inequality

Equality iff $\|w\| = 0$ or u, v are colinear ($\theta = 0$).

if $x[n]$ is a digital signal

$(x[0], \dots, [N-1])^T \in \mathbb{C}^N$

$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x(t) * y(t)\, dt.$

Properties of inner product

inner product = dot product

commutative

bilinear

Fourier series

Let $x(t)$ be a periodic signal, T (1 period): $x(t+T) = x(t)$. $\quad \omega_0 = \frac{2\pi}{T}$ (rad/s).

$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t}$. $\quad f_0 = \frac{1}{T}$ (cycles/sec = Hz).

spectrum → $X_n = \frac{1}{T} \int_T x(t) e^{-jn\omega_0 t} dt. = \frac{1}{T} \langle x(t), e^{jn\omega_0 t} \rangle$

ex. inner prod btw. 2 sigs ↳ $\langle e^{jn\omega_0 t}, e^{jm\omega_0 t} \rangle = \int_T e^{jn\omega_0 t} e^{-jm\omega_0 t} dt = \int_0^T e^{j(n-m)\omega_0 t} dt = \begin{cases} T, & n = m \\ 0, & n \neq m. \end{cases}$

Fourier is a change of bases

↳ write signals in terms of complex exponential

obtain ___ by projecting complex exp.

Spectrum

Analog to spectrum of coloured light that makes up white light.

$e^{st} \longrightarrow \boxed{\begin{matrix} LTI \\ h(t) \end{matrix}} \longrightarrow y(t) = h(t) * e^{st} = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = e^{st} \left(\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \right)$

does not depend on t.

$H(s)$ (transfer func)

$= H(s) e^{st}$

$h(t)$ = impulse resp.

If system is excited by a complex exponential

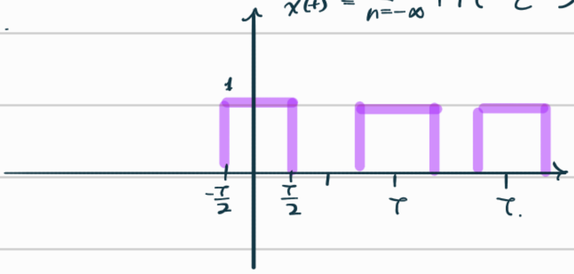
system responds w/ same exp $\cdot$ $H(s)$

Complex exp are eigenvalues of LTI systems.

↳ of oneself.

ex.

$x(t) = \sum_{n=-\infty}^{\infty} \Pi\left(\frac{t-nT}{\tau}\right)$



$X_n = \frac{1}{T}\int_T x(t) e^{-jn\omega_0 t} dt = \frac{1}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} 1 \cdot e^{-jn\omega_0 t} dt$

note $n \neq 0$, need to state.

$= \frac{-1}{jn\omega_0 T} \cdot e^{-jn\omega_0 t}\Big|_{-\tau/2}^{\tau/2}$

$\sin(n) = \frac{e^{jn\omega} - e^{-j\omega n}}{2}$ $\Rightarrow$ $= \frac{1}{jn\omega_0 t}\left(e^{jn\omega_0(\frac{\tau}{2})} - e^{-jn\omega_0(\frac{\tau}{2})}\right)$

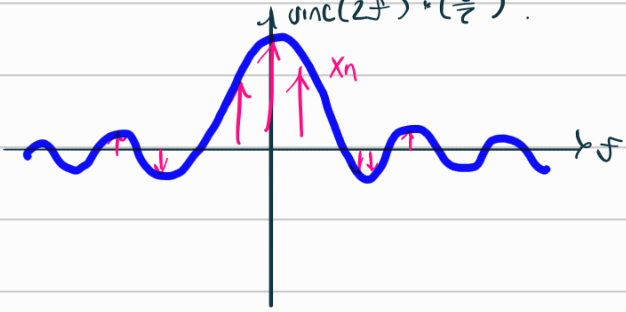
$= \frac{2\sin(n\omega_0 \frac{\tau}{2})\frac{\tau}{2}}{n\omega_0 T \frac{\tau}{2} \cdot T}$

$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$.

$= \frac{\tau}{T}\text{sinc}(n\frac{2}{T})$ ← this case still valid for $n=0$ (smooth func)

$\text{sinc}(0) = 1$.

$X_n = \frac{2}{T}\text{sinc}(n\frac{\tau}{T}) = \frac{\tau}{T}\text{sinc}(n\frac{\tau}{T})$



for real signals:

$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} = X_0 + \sum_{n=1}^{\infty} X_n e^{j\omega_0 t} + X_n^* e^{-jn\omega_0 t}$.

$X_{-n} = X_n^*$

$= X_0 + 2\sum_{n=1}^{\infty} |X_n| \cos(n\omega_0 t + \theta_n)$.

$X_n = |X_n| e^{j\theta_n}$ mag. angle.

Im cancel, only Re.

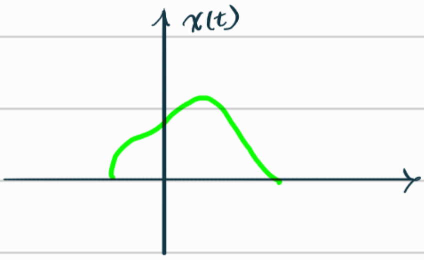
$\omega_0 \rightarrow$ fundamental

$n\omega_0 \rightarrow$ n-th harmonic

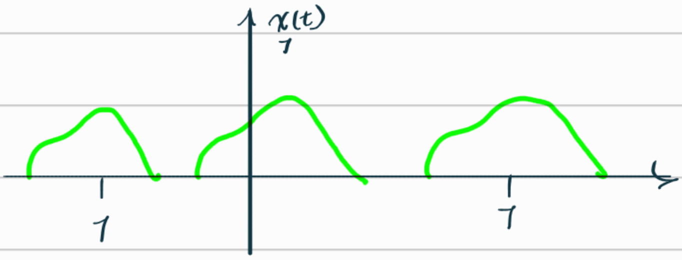
} standing wave harmonics → allegory.

consonance, dissonance

Fourier Transform



periodize.



$$x_T(t) = \sum_{n=-\infty}^{\infty} x_n e^{jnw_0 t}.$$

$$x_n = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jnw_0 t}.$$

← can do instead of $(-\frac{T}{2}, \frac{T}{2})$ since $x(t) = 0$ when $t \notin (-\frac{T}{2}, \frac{T}{2})$.

$$X(w) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt.$$

$T \to \infty$, $w_0 = \frac{2\pi}{T} \to 0$.

$$x_T(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} w_0 \left(X(w) e^{jwt}\right)\Big|_{nw_0}$$

$$x(t) = \lim_{T\to\infty} x_T(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w) e^{jwt} dt.$$

Riemann sum.

Fourier Analysis: $\boxed{X(w) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt.}$

Fourier synthesis.: $\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w) e^{-jwt} dw.}$

↓

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt.$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} dt.$$

$$\therefore x(t) = \Pi(t) \to X(f) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi f t} dt = \frac{1}{\pi f}\left(\frac{e^{j2\pi f\frac{1}{2}} - e^{-j2\pi f \frac{1}{2}}}{2j}\right)$$

$$= \frac{\sin(2\pi f \cdot \frac{1}{2})}{\pi f}$$

$$= \operatorname{sinc}(f).$$

$X(f) = \operatorname{sinc}(f)$

$x(t) = \delta(t)$

$$X(f) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t} dt = 1.$$

Linear superposition

* $\mathcal{F}\{a x_1(t) + b x_2(t)\} = a X_1(f) + b X_2(f).$
* $\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-j2\pi f t} dt$ ($at = \tau$)

$$= \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi \frac{f}{a}\tau} \cdot \frac{d\tau}{|a|} = \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

* mult by a in time domain = div by a in freq. domain (adjusted for same power level)