ECE 318 - TUTORIAL 6 ANGLE MODULATION

* Review: A general angle modulation signal is given by $\varphi(t) = A\cos(\theta_i(t))$

where A is the amplitude of the carrier, and $\theta_i(t)$ is the instantaneous phase (in rad) of the signal that carries the message signal $m(t)$.

* The instantaneous frequency $f_i(t)$ Hz or $\omega_i(t)$ rad/sec is given by $\omega_i(t) = \frac{d}{dt}\theta_i(t) \Leftrightarrow \theta_i(t) = \int_0^t \omega_i(\tau)\,d\tau + \theta_0$

θ_0 is the instantaneous phase at $t=0$ (usually $\theta_0 = 0$)

$$f_i(t) = \frac{1}{2\pi}\omega_i(t) = \frac{1}{2\pi}\frac{d}{dt}\theta_i(t) \quad \text{(Hz)}$$

Angle Modulation

Phase Modulation (PM)

In phase modulation, the instantaneous phase $\theta_i(t)$ is varied as a linear function of the message $m(t)$.

$$\theta_i(t) = \underbrace{\omega_c}_{\text{carrier freq.}} t + \underbrace{k_p}_{\text{phase constant}} m(t) + \theta_0$$

Thus,

$$\varphi_{PM}(t) = A\cos[\omega_c t + k_p m(t) + \theta_0]$$

* Peak phase deviation $\Delta\theta$: the largest deviation of $\theta_i(t)$ from $(\omega_c t + \theta_0)$. Hence, $\Delta\theta = k_p \max\{|m(t)|\}$

We refer to the peak phase deviation as the modulation index, $\beta_{PM} = \Delta\theta$

Frequency Modulation (FM)

In frequency modulation, the instantaneous frequency $\omega_i(t)$ / $f_i(t)$ is varied as a linear function of the message $m(t)$.

$$\omega_i(t) = \omega_c + k_\omega m(t)$$

or

$$f_i(t) = f_c + k_f m(t)$$

As a result,

$$\theta_i(t) = \omega_c t + k_\omega \int_0^t m(\tau)\,d\tau + \theta_0$$

Thus,

$$\varphi_{FM}(t) = A\cos\left[\omega_c t + k_\omega \int_0^t m(\tau)\,d\tau + \theta_0\right]$$

* Peak frequency deviation $\Delta\omega$, the largest deviation of $\omega_i(t)$ from ω_c. $\Delta\omega = k_\omega \max\{|m(t)|\}$

* Since the instantaneous phase is:

$$\theta_i = \omega_c t + k_\omega \int_0^t m(\tau)\,d\tau + \theta_0$$

the peak phase deviation $(\Delta\theta)$ is given by:

$$\Delta\theta = k_\omega \max\left\{\left|\int_0^t m(\tau)\,d\tau\right|\right\}, \quad \beta_{FM} = \frac{\Delta\omega}{2\pi B} = \frac{\Delta f}{B}$$

B: message bw

For example, let us consider the single tone modulation $m(t) = A_m \cos \omega_m t$

$$\varphi_{PM}(t) = A \cos\left[\omega_c t + K_p A_m \cos \omega_m t + \theta_0\right]$$

$$\Delta\theta = K_p A_m$$

$$\beta = K_p A_m$$

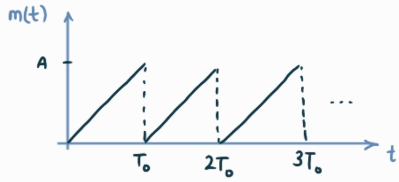
$$\varphi_{FM}(t) = A \cos\left[\omega_c t + K_\omega A_m \int_0^t \cos \omega_m \tau \, d\tau + \theta_0\right]$$

$$\Delta\omega = K_\omega A_m$$

$$\Delta\theta = \max\left\{ \left| \frac{K_\omega A_m}{\omega_m} \sin(\omega_m t) \right| \right\} = \frac{K_\omega A_m}{\omega_m}$$

$$\beta = \frac{K_\omega A_m}{\omega_m} = \frac{\Delta\omega}{\omega_m} = \frac{\Delta f}{f_m}$$

Ex. Sketch the PM and FM waves produced by the sawtooth wave shown below.



SOL: The mathematical expression of m(t) is :

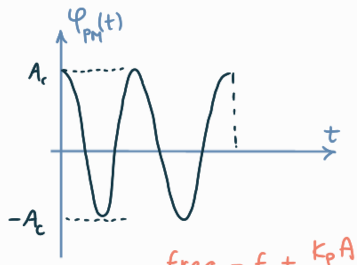
$$m(t) = A/T_0 \,(t - nT_0), \quad nT_0 \le t < (n+1)T_0 \quad \text{for } n = 0, 1, 2, \dots$$

The PM signal is $\varphi(t) = A_c \cos(\omega_c t + K_p m(t))$. In the interval $0 \le t < T_0$, $m(t) = \frac{At}{T_0}$. Hence, in this

$$\varphi_{PM}(t) = A_c \cos\left[\omega_c t + K_p \frac{At}{T_0}\right] = A_c \cos\left[\left(\omega_c + \frac{K_p A}{T_0}\right)t\right], \quad 0 \le t < T$$

Thus, in $0 \le t < T$, $\varphi_{PM}(t)$ is just a cosine signal with frequency $\frac{1}{2\pi}\left(\omega_c + \frac{K_p A}{T_0}\right) = f_c + \frac{K_p A}{2\pi T_0}$ and $\theta_0 = 0$

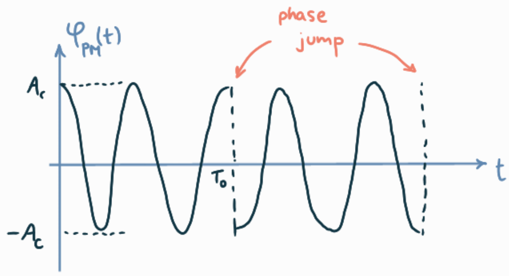
In the next interval, $T_0 \le t < 2T_0$, $m(t) = \frac{At}{T_0} - A$. Thus,



$$\varphi_{PM}(t) = A_c \cos\left[\omega_c t + K_p\left(\frac{At}{T_0} - A\right)\right] = A_c \cos\left[\left(\omega_c + \frac{K_p A}{T_0}\right)t - AK_p\right], \quad T_0 \le t < 2T_0$$

$\text{freq} = f_c + \frac{K_p A}{2\pi T_0}$

Thus, $\varphi_{PM}(t)$ is still a cosine signal with frequency $f_c + \frac{K_p A}{2\pi T_0}$, but at $t = T_0$, there is a phase shift (jump) of (AK_p) rad. Similarly, $\varphi_{PM}(t)$ will still be a cosine signal but at $t = 2T_0, 3T_0, \dots$ the phase will be changed by $-AK_p$ rad. In this case the phase of the signal is discontinuous as the message signal m(t) is discontinuous.

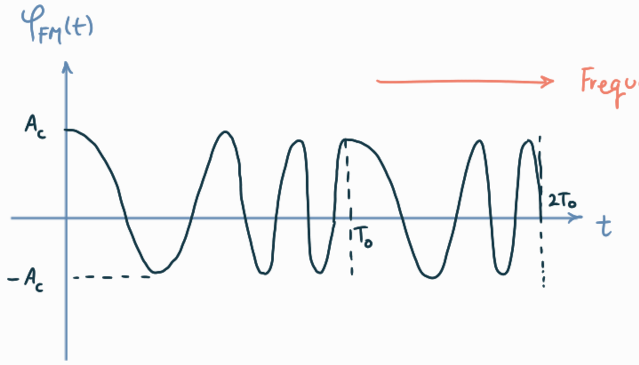


Here we illustrate with $AK_p = \pi$ rad

In the FM signal, the phase is proportional to the integral of m(t), and hence is continuous. In FM, the instantaneous frequency is $\omega_i = \omega_c + K_\omega m(t)$

Hence, ω_i will increase from ω_c to $\omega_c + K_\omega A$ between $t=0$ and $t<T_0$. Then at $t = T_0$, ω_i goes back down to ω_c

and starts increasing again. An illustrative example is given below:



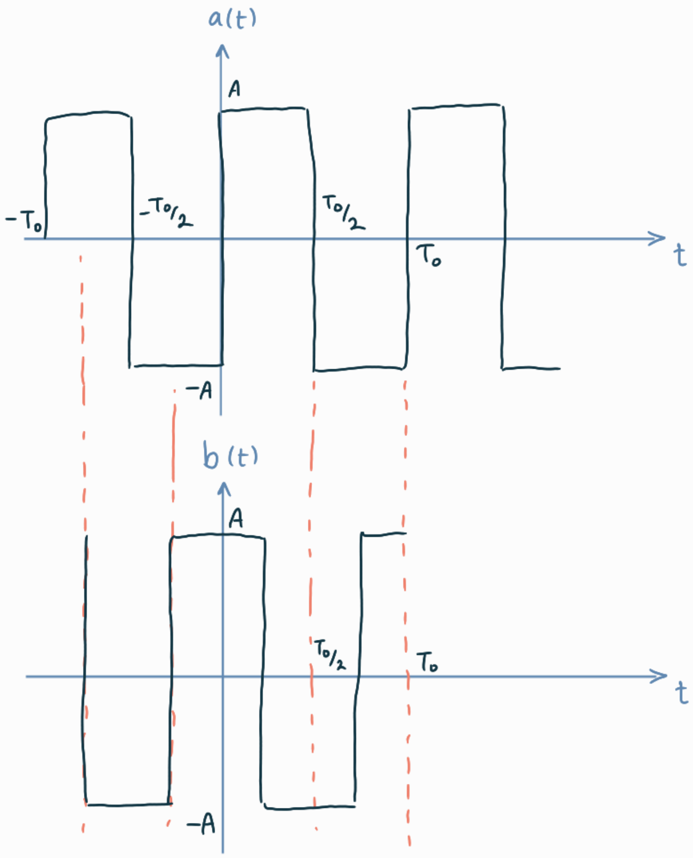
Frequency increases linearly for $t=0$ to $t<T_0$

At $t=T_0$ it goes back to ω_c

Ex. Find the modulation index for each of the following cases. Assume that $k_p = 1$ and $\theta_0 = 0$.

(a) PM, $m(t) = a(t)$

(b) PM, $m(t) = b(t)$



SoL :

(a) In PM, the modulation index (peak phase deviation) β is given by $\beta = K_p \cdot \max\{|m(t)|\}$.

Note that $\max\{|m(t)|\} = A$ for $a(t)$.

Hence, $\beta = K_p \cdot A = A$

(b) Note that $b(t)$ is just a shifted (time) version of $a(t)$, and it does not affect the maximum amplitude. Hence, $\beta = A$.

Ex. Consider the following modulated signal: $\varphi(t) = 5\cos[\omega_c t + \cos\omega_m t]$ for some ω_c, ω_m.

(a) If it is a PM signal with $K_p = 2$, then what is the message $m(t)$?

(b) If it is an FM signal with $K_\omega = 0.2$, then what is $m(t)$?

SoL: (a) Since $\varphi_{PM}(t) = 5\cos[\omega_c t + 2m(t) + \theta_0] = 5\cos(\omega_c t + \cos\omega_m t)$, we find $\theta_0 = 0$ and $m(t) = \frac{1}{2}\cos\omega_m t$

(b) Since $\varphi_{FM}(t) = 5\cos\left[\omega_c t + 0.2\int_0^t m(\tau)d\tau + \theta_0\right] = 5\cos(\omega_c t + \cos\omega_m t)$, we have $0.2\int_0^t m(\tau)d\tau + \theta_0 = \cos(\omega_m t)$

$$\overset{d/dt}{\Longrightarrow} \quad 0.2\, m(t) = -\omega_m \sin(\omega_m t) \quad \Rightarrow \quad m(t) = -5\,\omega_m \sin(\omega_m t).$$

Also, using this $m(t)$, one can find that $\theta_0 = 1$ rad.

ANGLE MODULATION REVIEW

Recall that an FM signal is given by:

$$\varphi(t) = A\cos\left[\omega_c t + K_\omega \int_0^t m(\tau)\,d\tau\right]$$

Important parameters: Modulation index, Bandwidth

where, A = carrier amplitude, ω_c = carrier frequency (rad/sec), k_ω = frequency sensitivity, $m(\tau)$ = message signal.

Peak phase deviation / modulation index:

$$\beta = k_\omega \cdot \max\left|\int_0^t m(\tau)\,d\tau\right|$$

NBFM : $\beta \ll 1$

(sometimes we take $\beta < 0.3$ as a threshold)

The FM signal can be approximated as:

$$\varphi_{NB}(t) = A\cos\omega_c t - A k_\omega \int_0^t m(\tau)\,d\tau \cdot \sin\omega_c t$$

By inspection we can notice that $\varphi_{NB}(t)$ is similar to DSB-LC where a carrier $A\cos\omega_c t$ is added to/subtracted from function $m(t)$ multiplied by a carrier $\sin(\omega_c t)$.

Based on this, BW of $\varphi_{NB}(t)$ is $2 \times BW(m(t))$

Ex. $m(t) = a\cos\omega_m t$,

$$\varphi_{NB}(t) = A\cos\omega_c t - \frac{A \cdot k_\omega \cdot a}{\omega_m}\sin\omega_m t \cdot \sin\omega_c t,$$

$$\beta = k_\omega \cdot \max\left|\frac{a}{\omega_m}\sin\omega_m t\right| = \frac{a \cdot k_\omega}{\omega_m}$$

WBFM

Roughly $\beta > 0.3$

We do not have a simple approximation of $\varphi(t)$.

Also, analyzing a general message signal $m(t)$ becomes mathematically intractable. We study WBFM for $m(t) = a\cos\omega_m t$, then WBFM can be expressed in its "Bessel form":

$$\varphi_{WB}(t) = A\sum_{n=-\infty}^{\infty} J_n(\beta)\cos\left[(\omega_c + n\omega_m)t\right]$$

Here, $J_n(\beta)$ is the n^{th} order Bessel function of the first kind. We do not need to study it in details. This is sufficient:

(a) $J_n(\beta)$ is real.

(b) $J_n(\beta) = (-1)^n J_{-n}(\beta)$

(c) For $\beta \ll 1$, $J_0(\beta) = +1$, $J_1(\beta) = \beta/2$, $J_n(\beta) = 0 \quad n \geq 2$

(d) $\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$

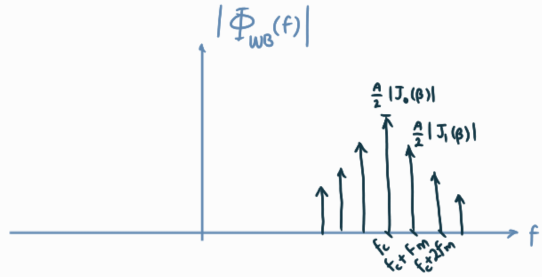
Bandwidth : due to the similarity between the φ_{NB} & φ_{DSB} , the bandwidth of φ_{NB} = $2 \times$ bandwidth of the message signal.

Spectrum of WBFM :

$$\Phi_{WB}(f) = \mathcal{F}\left\{ A \sum_{n=-\infty}^{\infty} J_n(\beta) \cdot \cos\left[(\omega_c + n\omega_m)t\right]\right\}$$

$$= A \sum_{n=-\infty}^{\infty} J_n(\beta)\, \mathcal{F}\left\{\cos\left[(\omega_c + n\omega_m)t\right]\right\}$$

$$= \frac{A}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)\left[\delta(\omega - \omega_c - n\omega_m) + \delta(\omega + \omega_c + n\omega_m)\right]$$



@ f_c : $\frac{A}{2}|J_0(\beta)|$ @ $f_c \pm f_m$: $\frac{A}{2}|J_1(\beta)|$

Bandwidth of WBFM :

(a) The 1% Rule : Among all the delta functions in $|\Phi_{WB}(\omega)|$, keep the ones that have a magnitude ≥ 0.01, and ignore the rest. Specially choose n_0 such that $|J_n(\beta)| \geq 0.01$ for $n \leq n_0$ and $|J_n(\beta)| < 0.01$ for $n > n_0$. Then the bandwidth is : $BW = 2 n_0 \omega_m$ rad/sec

(b) Carson's Rule : $BW = 2\omega_m(1+\beta)$ rad/sec OR $BW = 2 f_m (1+\beta)$ This works well for large β.

Ex. A carrier $c(t) = 100 \cos \omega_c t$ is frequency modulated by the message $m(t) = 100 \cos(\omega_m t)$ where $f_c = 100\,KHz$, $f_m = 1\,KHz$. The frequency sensitivity $K_\omega = 32\pi$.

(a) Using the 1% rule, find the approximate BW of $\varphi(t)$. Which frequency components fall within this BW?

(b) Sketch the magnitude spectrum of $\varphi(t)$ [only positive frequencies]. Include only the frequency components that are centered within the approximation.

(c) Find the BW of $\varphi(t)$ using Carson's rule. Which frequency components fall within this BW? How do they compare to those in part (a)?

Sol : (a) According to the 1% rule, we need to find n_0 such that $|J_0(\beta)| \geq 0.01$ for $|n| \leq n_0$ and $|J_n(\beta)| < 0.01$

for $|n| > n_0$. First we find β.

$$\beta = \frac{\Delta\omega}{\omega_m} = \frac{a \cdot k_\omega}{\omega_m} = \frac{32\pi \times 100}{2\pi \times 10^3} = 1.6$$

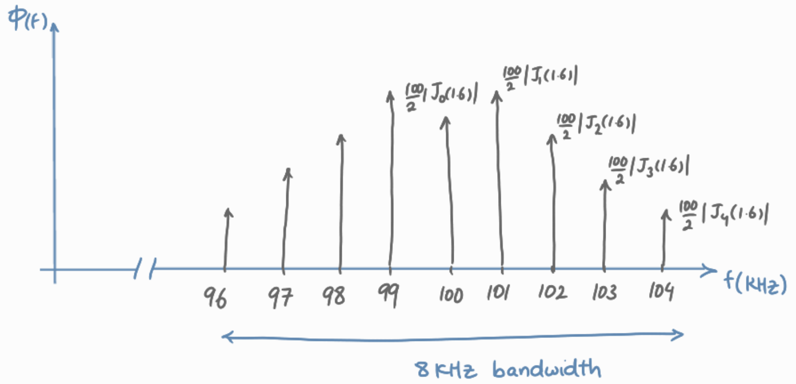
Now, let's look at the Bessel function table for $\beta = 1.6$:

β	$J_0(\beta)$	$J_1(\beta)$	$J_2(\beta)$	$J_3(\beta)$	$J_4(\beta)$	$J_5(\beta)$
1.6	0.455	0.570	0.257	0.073	0.015	0.002
Frequency	f_c	$f_c \pm f_m$	$f_c \pm 2f_m$	$f_c \pm 3f_m$	$f_c \pm 4f_m$	$f_c \pm 5f_m$

We note that $n_0 = 4$, since $J_4(\beta) \geq 0.01$ but $J_5(\beta) < 0.01$. Thus the BW is $2 \times n_0 \times \omega_m = 2\pi \times 8$ krad/sec or 8 KHz.

The frequency components contained within this bandwidth: f_c, $f_c \pm f_m$, $f_c \pm 2f_m$, $f_c \pm 3f_m$ and $f_c \pm 4f_m$.

(b)



(c) Using Carson's rule, $BW = 2f_m(1+\beta) = 2(1+1.6)$ KHz $= 5.2$ KHz. The largest n_0 that satisfies $2n_0 f_m \leq 5.2$ KHz is $n_0 = 2$.

Hence, the frequency components that fall within this bandwidth are f_c, $f_c \pm f_m$ and $f_c \pm 2f_m$.

Thus Carson's rule includes only 5 frequency components whereas the 1% rule includes 9.

Ex. A certain sinusoid of frequency f_m (Hz) is used as the message in a DSB-LC and an FM signal. When modulated the peak frequency deviation of the FM signal is set to 3 times the bandwidth of the AM signal. Also, the magnitudes of the sidebands spaced at $\pm f_m$ (Hz) from the carrier are equal in both systems. Determine the modulation index of the FM and AM systems.

FM signal: $\varphi_f(t) = A_1 \sum J_n(\beta) \cos[(\omega_c + n\omega_m)t]$

Assume that the AM signal has form: $\varphi_A(t) = (A_2 + A_m \cos(\omega_m t)) \cos \omega_c t$

Sol : We know that Δf is 3 times the bandwidth of the AM system:

$\Delta f = 3 \cdot BW(AM)$, BW of AM $= 2f_m \Rightarrow \Delta f = 3 \times 2f_m = 6f_m$, by definition $\beta = \frac{\Delta f}{f_m} = \frac{6f_m}{f_m} = 6 \Rightarrow \boxed{\beta = 6}$

Next, we know that the power in the two systems are equal. The power of the FM signal is $A_1^2/2$.

Calculating the power of AM : $\varphi_{AM}(t) = \left[A_2 + A_M \cos(\omega_m t)\right] \cos(\omega_c t) = A_2 \cos \omega_c t + \frac{A_M}{2} \cos(\omega_c - \omega_m)t + \frac{A_M}{2} \cos(\omega_c + \omega_m)t$

Power $\varphi_{AM}(t) = \frac{A_2^2}{2} + \frac{A_M^2}{8} + \frac{A_M^2}{8} = \frac{A_2^2}{2} + \frac{A_M^2}{4} \quad \Rightarrow A_1^2 = A_2^2 + \frac{A_M^2}{2}$ (1)

Now the modulation index of the AM signal is $\mu = \frac{A_M}{A_2}$. Hence from (1), $A_1^2 = A_2^2 \left(1 + \mu^2/2\right)$

* We are also given that the magnitude of sidebands at frequencies spaced $\pm f_m$ from the carrier frequency are equal in both systems.

$$\varphi_{AM}(t) = A_2 \cos(\omega_c t) + \frac{\mu A_2}{2} \cos\left[(\omega_c - \omega_m)t\right] + \frac{\mu A_2}{2} \cos\left[(\omega_c + \omega_m)t\right]$$

Thus, the magnitude in question is at frequency $\pm f_m$: $\frac{\mu A_2}{4}$

The magnitude of the frequency components located at $f_c \pm f_m$ is: $\frac{A_1}{2} |J_1(\beta)| = \frac{A_1}{2} |J_1(6)|$.

As a result, $\frac{A_1}{2} |J_1(6)| = \frac{\mu A_2}{4} \Rightarrow A_1^2 J_1^2(6) = \frac{\mu^2 A_2^2}{4}$ (2)

Thus from (1) and (2) $\mu \cong 0.6 A_M$

END OF TUTORIAL