Tutorial 5

- Linearization
- Poles in TF
- 1st order systems

Ex. 5.1 (con't from Ex. 4.1). Linearize the following ODEs: $\dot{i} = \frac{1}{L}(u - Ri) = f_1$, $\ddot{y} = g - \frac{K}{M}\frac{i^2}{y^2} = f_2$ about the following equilibrium point: $i^* = \frac{5}{R}$, $y^* = \frac{5}{R}\sqrt{\frac{K}{M}}$, $u^* = 5$

① $\Delta\dot{i} \approx \frac{\partial f_1}{\partial i}(i^*, y^*, u^*)\,\Delta i + \frac{\partial f_1}{\partial u}(i^*, y^*, u^*)\,\Delta u = -\frac{R}{L}\Delta i + \frac{1}{L}\Delta u$

② $\Delta\ddot{y} \approx \frac{\partial f_2}{\partial i}(i^*, y^*, u^*)\,\Delta i + \frac{\partial f_2}{\partial y}(i^*, y^*, u^*)\,\Delta y + \cancelto{0}{\frac{\partial f_2}{\partial u}(i^*, y^*, u^*)\,\Delta u}$

$$= -2\frac{K}{M}\frac{i}{y^2}\Big|_{\left(\frac{5}{R},\, \frac{5}{R}\sqrt{\frac{K}{Mg}},\, 5\right)} \Delta i + 2\frac{K}{M}\frac{i^2}{y^3}\Big|_{\left(\frac{5}{R},\, \frac{5}{R}\sqrt{\frac{K}{Mg}},\, 5\right)} \Delta y$$

$$= -\frac{2R}{5}g\,\Delta i + \frac{2R}{5}\sqrt{\frac{Mg^3}{K}}\,\Delta y$$

Ex. 5.2 (con't from Ex. 4.2) Linearize the following ODEs: $\ddot{x}_1 = \frac{1}{m}(u_1 + u_2)\cos(x_2) - g = f_1$, $\ddot{x}_2 = \frac{L}{I}(u_2 - u_1) = f_2$, $\ddot{x}_3 = -(u_1 + u_2)\sin(x_2) = f_3$, about the following equilibrium point: $u_1^* = u_2^* = \frac{1}{2}mg$, $x_1^* = x_2^* = x_3^* = 0$

① $\Delta\ddot{x}_1 \approx -\frac{1}{m}(u_1^* + u_2^*)\sin(x_2^*)\,\Delta x_2 + \frac{1}{m}\cos x_2^*\,\Delta u_1 + \frac{1}{m}\cos x_2^*\,\Delta u_2$

$$= \frac{1}{m}\Delta u_1 + \frac{1}{m}\Delta u_2$$

② $\Delta\ddot{x}_2 \approx \frac{L}{I}\Delta u_1 - \frac{L}{I}\Delta u_2$

③ $\Delta\ddot{x}_3 \approx -(u_1^* + u_2^*)\cos(x_2^*)\,\Delta x_2 - u_1^*\sin(x_2^*)\,\Delta u_1 - u_2^*\sin(x_2^*)\,\Delta u_2$

$$= -mg\,\Delta x_2$$

Stability of TFs Assume $G(s)$ is a real, rational, proper. Then $G(s)$ is stable if every pole of $G(s)$ has negative real part. [defn of stability for TF]

Ex. 5.3 Are the following TF stable or unstable?

a) $G(s) = \frac{1}{s+1}$ poles $\{-1\}$ stable $g(t) = e^{-t}$
b) $G(s) = \frac{1}{s-1}$ poles $\{1\}$ unstable $g(t) = e^{t}$

impulse responses

c) $G(s) = \frac{1}{(s+1)(s+2)(s+3)(s-1)}$ poles $\{-1, -2, -3, 1\}$ unstable

d) $G(s) = \frac{1}{(s+1)(s+2)(s+3)(s^2+1)}$ poles $\{-1, -2, -3, \pm i\}$ unstable

e) $G(s) = \frac{1}{(s+1)(s+2)(s+3)(s)}$ poles $\{-1, -2, -3, 0\}$ unstable

f) $G(s) = \dfrac{1}{s^2 + 6s + 25}$ poles $\{-3 \pm 4j\}$ stable

Believe it or not, pole locations affect the response of the system!

Ex. 5.4 Assume a unit step input.

a) consider the following two transfer functions. Which one would have a faster response?

i) $G_1(s) = \frac{1}{s+1}$ Poles $\{-1\}$ compare e^{-t} vs e^{-2t} $\Rightarrow G_2(s)$ decays faster

ii) $G_2(s) = \frac{1}{s+2}$ poles $\{-2\}$

b) consider the following two transfer functions. Which one would have a response with more overshoot?

i) $G_1(s) = \dfrac{1}{s^2+3s+2}$ poles $\{-1, -2\} \Rightarrow$ 2 real distinct negative poles $\Rightarrow$ overdamped $\Rightarrow$ no overshoot

ii) $G_2(s) = \dfrac{1}{s^2+6s+25}$ poles $\{-3 \pm 4j\} \Rightarrow$ imaginary component $\Rightarrow$ leads to overshoot $\Rightarrow G_2$ has more overshoot

c) consider the following two transfer functions. Which one would have a response with more oscillations?

i) $G_1(s) = \dfrac{1}{s^2+6s+25}$ poles $\{-3 \pm 4j\}$ $\Rightarrow$ greater imaginary component $\Rightarrow$ leads to more oscillations.

ii) $G_2(s) = \dfrac{1}{s^2+6s+10}$ poles $\{-3 \pm j\}$

Where do the poles of the output come from?

Ex. 5.5 Suppose $Y(s) = G(s)U(s)$, where $G(s)$ is the TF of the system and $U(s)$ represents the input. Note where the poles of $Y(s)$ are located.

Suppose: $U(s) = \frac{1}{s}$ (unit step) and $G(s) = \frac{1}{(s+1)(s+2)}$

$$Y(s) = G(s)U(s) = \frac{1}{(s+1)(s+2)(s)}$$ poles: $\{-1, -2, 0\} \subset$ poles $\{U(s)\} \cup$ poles $\{G(s)\}$

Wow, does this happen every time?

Let $U(s) = \dfrac{n_1(s)}{d_1(s)}$, $G(s) = \dfrac{n_2(s)}{d_2(s)}$, where $n_1(s), n_2(s), d_1(s)$, and $d_2(s)$ are polynomials

$$Y(s) = G(s)U(s) = \frac{n_1(s)\,n_2(s)}{d_1(s)\,d_2(s)}$$

poles $\{Y(s)\} \subset$ roots $\{d_1(s)\,d_2(s)\}$ = poles $\{G(s)\} \cup$ poles $\{U(s)\}$

note: contains, not equal, since there could be pole/zero cancellations

1st order systems

Recall the associated transfer function of the standard first order system

$$G(s) = \frac{k}{\tau s + 1}$$

$k \leftarrow$ gain of k

$\tau s + 1 \leftarrow$ pole at $s = -\frac{1}{\tau}$

If our reference signal is a step function,

$$Y(s) = G(s)R(s) = \frac{K}{\tau s+1} \cdot \frac{1}{s} = \frac{A}{s} + \frac{B}{\tau s+1}$$

$$sY(s) = \frac{k}{\tau s+1} = A + \frac{Bs}{\tau s+1}, \text{ evaluate @ } s=0 \Rightarrow A=k$$

$$(\tau s+1)Y(s) = \frac{k}{s} = \frac{A(\tau s+1)}{s} + B. \text{ evaluate @ } s = -\frac{1}{\tau} \Rightarrow B = -k\tau$$

$$\Rightarrow Y(s) = \frac{k}{s} - \frac{k\tau}{\tau s+1} \cdot \frac{1/\tau}{1/\tau} = \frac{k}{s} - \frac{k}{s+\frac{1}{\tau}}$$

assuming zero initial conditions, we have

$$y(t) = k(1 - e^{-t/\tau}), \quad y(4\tau) = k(1-e^{-4}) \approx k(0.98)$$

after 4τ seconds, step response has reached 98% of its steady state value. adjust k, doesnt affect settling time

Ex. 5.6

Consider a first order system in a negative error feedback interconnection with a proportional controller shown in Figure 1. Find conditions on the controller's proportional gain K_p so that,

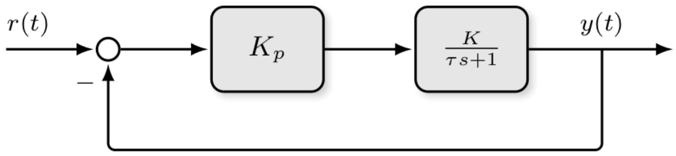

Figure 1: Proportional error feedback on a first order plant.

when the reference r is a step function, the output's settling time[1] is less than or equal to 1 second.

$$Y(s) = K_p \cdot \frac{k}{\tau s+1} (R(s) - Y(s))$$

$$\left(1 + \frac{K_p \cdot k}{\tau s+1}\right) Y(s) = \frac{K_p \cdot k}{\tau s+1} R(s) \qquad G(s) = \frac{PC}{1+PC}$$

$$\frac{(\tau s+1) + K_p \cdot k}{\tau s+1} Y(s) = \frac{K_p \cdot k}{\tau s+1} R(s)$$

$$Y(s) = \frac{K_p \cdot k}{\tau s+1+K_p \cdot k} R(s) = \frac{K_p \cdot k}{1+K_p \cdot k} \cdot \frac{1}{\frac{\tau}{1+K_p k} s+1} R(s)$$

settling time: how long it takes for output to lie within 2% of its final value.

The settling time time for a first order system is around 4 times the time constant. Therefore

$$4 \frac{\tau}{1+K_p k} \leq 1 \Leftrightarrow K_p \geq \frac{4\tau - 1}{k}$$

★ easy, fast ★

An alternative (longer) approach.

If $R(S)$ is a unit step we have

$$Y(s) = \frac{K_p \cdot K}{\tau s + 1 + K_p \cdot K} \cdot \frac{1}{s} = \frac{A}{s} + \frac{B}{\tau s + 1 + K_p \cdot K}$$

$$sY(s) = \frac{K_p \cdot K}{\tau s + 1 + K_p \cdot K} = A + \frac{Bs}{\tau s + 1 + K_p \cdot K}$$, evaluate @ $s=0$ $\Rightarrow A = \frac{K_p \cdot K}{1 + K_p K}$

$$(\tau s + 1 + K_p \cdot K)\, Y(s) = \frac{K_p \cdot K}{s} = \frac{A(\tau s + 1 + K_p \cdot K)}{s} + B$$

evaluate @ $s = -\frac{1 + K_p \cdot K}{\tau}$ $\Rightarrow$ $B = -\frac{K_p \cdot K \tau}{1 + K_p \cdot K}$

$$\Rightarrow Y(s) = \frac{K_p \cdot K}{1 + K_p K}\left(\frac{1}{s} - \frac{1}{\tau s + 1 + K_p K} \cdot \frac{1/\tau}{1/\tau}\right) = \frac{K_p K}{1 + K_p K}\left(\frac{1}{s} - \frac{1}{s + (1 + K_p K)/\tau}\right)$$

$$y(t) = \frac{K_p \cdot K}{1 + K_p \cdot K}\left(1 - e^{-\frac{t(1 + K_p K)}{\tau}}\right)$$

if we want settling time to be ≤ 1 second

$$\left(1 - e^{-\frac{(1 + K_p K)}{\tau}}\right) \geq 0.98$$

$$\boxed{\begin{aligned} -e^{-(1+K_pK)/\tau} &\geq -0.02 \\ e^{-(1+K_pK)/\tau} &\leq 0.02 \\ -(1+K_pK)/\tau &\leq 3.91 \\ (1+K_pK) &\geq 3.91\tau \end{aligned}}$$

$$\boxed{K_p \geq \frac{3.91\tau - 1}{K}}$$

→ this process took way more time

↳ $4 \neq 3.91$?

↳ 4 times time constant is an approximation

if we have $y(t) = K(1 - e^{-t/\tau})$.

$y(4\tau) = K(1 - e^{-4}) \approx \underline{0.98}\, y_{ss}$ → 0.981684361...

↳ if we used this number exactly, we get $K_p \geq (4\tau - 1)/K$