Tutorial 2

LTI systems

Linear Systems

① Homogeneity $\quad x(t) \to \boxed{\quad} \to y(t) \Rightarrow a\,x(t) \to \boxed{\quad} \to a\,y(t)$

② Additivity / Superposition

$$\left.\begin{array}{l} x_1(t) \to \boxed{\quad} \to y_1(t) \\ x_2(t) \to \boxed{\quad} \to y_2(t) \end{array}\right\} \Rightarrow x_1(t)+x_2(t) \to \boxed{\quad} \to y_1(t)+y_2(t)$$

If ① and ② hold, the system is linear

$$a_n(x)\frac{d^n y(t)}{dx^n} + a_{n-1}(x)\frac{d^{(n-1)} y(t)}{dx^{n-1}} + \dots + a_1(x)\frac{dy(t)}{dx} + a_0(x)\,y(x) + b(x)$$

describes a general linear differential equation

Time Invariance

defn: A system is time-invariant if it is described by differential equations and all the coefficients are constant wrt time

$$x(t) \to \boxed{\quad} \to y(t)$$

$$x(t-t_0) \to \boxed{\quad} \to y(t, t_0) \stackrel{?}{=} y(t-t_0) \quad \begin{array}{l} \to \text{yes} \Rightarrow \text{time invariant} \\ \hookrightarrow \text{no} \Rightarrow \text{time varying} \end{array}$$

highlight time shifted input

Example 2.1

a. $y(t) = 2t^2 x(t)$ time shifted input

$$\left.\begin{array}{l} y(t,t_0) = 2t^2 x(t-t_0) \\ y(t-t_0) = 2(t-t_0)^2\,x(t-t_0) \end{array}\right\} y(t,t_0) \neq y(t-t_0) \Rightarrow \text{time varying}$$

$$\left.\begin{array}{l} a\,y(t) = 2t^2 a\,x(t) \\ \left.\begin{array}{l} y_1(t) = 2t^2 x_1(t) \\ y_2(t) = 2t^2 x_2(t) \end{array}\right\} y_1(t)+y_2(t) = 2t^2\left(x_1(t)+x_2(t)\right) \end{array}\right\} \text{linear}$$

b. $y(t) = 3e^{3x(t)}$

$$\left.\begin{array}{l} y(t,t_0) = 3e^{3x(t-t_0)} \\ y(t-t_0) = 3e^{3x(t-t_0)} \end{array}\right\} y(t,t_0) = y(t-t_0) \Rightarrow \text{time invariant}$$

$3a\,e^{3x(t)} \neq 3e^{3ax(t)} \Rightarrow$ violates homogeneity $\Rightarrow$ easy way to tell: $\left\{\begin{array}{l} \text{if } a<0 \\ 3a\,e^{3x(t)} < 0 \\ 3e^{3ax(t)} > 0 \end{array}\right.$

$\hookrightarrow$ nonlinear

c. $5t^2 \frac{d^2 y(t)}{dt} + 4y(t) = 2x(t) \Rightarrow$ time varying + linear

Laplace Transforms

defn: $F(s) = \int_0^\infty f(t)\,e^{-st}\,dt$

Continous time	Differential Equations
	$\mathcal{L}^{-1}$ Impulse Response $\mathcal{L}$
	Transfer Function

$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - \dot{f}(0)$

$\mathcal{L}\{f^n(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) \cdots - sf^{n-2}(0) - f^{n-1}(0)$

Laplace Transform Table

Example 2.2 Show that the Laplace transform is a linear operator

$$\mathcal{L}\{af(t)\} = \int_0^\infty a f(t) e^{-st}\,dt = a\int_0^\infty f(t)e^{-st}\,dt = a\mathcal{L}\{f(t)\}$$ ✓ homogeneity holds

$$\mathcal{L}\{f_1(t)+f_2(t)\} = \int_0^\infty (f_1(t)+f_2(t))e^{-st}\,dt = \int_0^\infty f_1(t)e^{-st}\,dt + \int_0^\infty f_2(t)e^{-st}\,dt =$$

$$\mathcal{L}\{f_1(t)\} + \mathcal{L}\{f_2(t)\}$$ ✓ superposition holds

Why do we care? → Can be useful when solving ODEs

Example 2.3 Suppose we have the following ODE: $\ddot{x}(t) + 3\dot{x}(t) + 2x(t) = f(t)$
Suppose we are given an input $f(t)$ and zero initial conditions. Find $x(t)$.

⇒ Use Laplace Transform

$$\mathcal{L}\{\ddot{x}(t) + 3\dot{x}(t) + 2x(t) = f(t)\} \Rightarrow s^2X(s) + 3sX(s) + 2X(s) = F(s)$$

$$\Rightarrow \underbrace{X(s)}_{\text{output}} = \underbrace{\frac{1}{s^2+3s+2}}_{\text{transfer function}} \underbrace{F(s)}_{\text{input}}$$

★ can find TF by taking laplace transform of ODE ★

$$\Rightarrow f(t) \longrightarrow \boxed{\frac{1}{s^2+3s+2}} \longrightarrow x(t)$$

Suppose we are given $f(t) = \mathbb{1}(t)$ → unit step $\begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$

⇒ From Laplace Transform table $F(s) = \frac{1}{s}$

$$X(s) = \frac{1}{(s^2+3s+2)} \cdot \frac{1}{s} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

⇒ use partial fraction decomposition

↳ multiply both sides by s and evaluate @ $s=0$ ⇒ $A = \frac{1}{2}$
↳ multiply both sides by $s+1$ and evaluate @ $s=-1$ ⇒ $B = -1$
↳ multiply both sides by $s+2$ and evaluate @ $s=-2$ ⇒ $C = \frac{1}{2}$

$$X(s) = \frac{1/2}{s} + \frac{-1}{s+1} + \frac{1/2}{s+2}$$

⇒ take inverse laplace transform

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1/2}{s} + \frac{-1}{s+1} + \frac{1/2}{s+2}\right\} = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

Example 2.4 Suppose we are given the impulse response of a system is $y(t) = t^2 + e^{2t} + 1$, $t \geq 0$. Find the transfer function of the system.

$$\underbrace{Y(s)}_{\text{output}} = \overbrace{G(s)}^{\text{system TF}}\underbrace{U(s)}_{\text{input}} \quad \text{if } u(t) = \underbrace{\delta(t)}_{\text{impulse}} \text{ then } U(s) = \mathcal{L}\{\delta(t)\} = 1$$

then $Y(s) = G(s)\cdot 1$. So how do we find $G(s)$? Find $Y(s)$ by taking Laplace transform of $y(t)$

$$G(s) = Y(s) = \mathcal{L}\{y(t)\} = \mathcal{L}\{t^2 + e^{2t} + 1\} = \frac{2}{s^3} + \frac{1}{s-2} + \frac{1}{s} = \boxed{\frac{2s^3 - 2s^2 + 2s - 4}{s^3(s-2)}}$$

Impulse response / convolution

let $u(t)$ describe the input signal
let $g(t)$ describe the impulse response of the system
let $y(t)$ describe the output

then
$$y(t) = (g * u)(t) \quad \leftarrow \text{convolution yucky}$$
$$= \int_0^t g(t)\, u(t-\tau)\, d\tau$$

$\Rightarrow$ given the impulse response of a system, to find the output we can either

- Use the $\mathcal{L}\{\}$ to find TF and find $Y(s) = G(s)U(s)$ and then take $\mathcal{L}^{-1}\{\}$ to find $y(t)$
- Convolution :(
  (I just dont like integrals unless its e^x)

Example 2.5 Suppose the impulse response of the system is given by $g(t) = \mathcal{L}^{-1}\{G(s)\} = e^{2t}$. Suppose $u(t) = \mathbb{1}(t)$. Find an expression for $y(t)$

$\Rightarrow$ use convolution
$$y(t) = (g * u)(t) = \int_0^t e^{2\tau} \cdot 1\, d\tau = \frac{1}{2} e^{2\tau}\Big|_0^t = \frac{1}{2}\left(e^{2t} - 1\right)$$

(I'm telling you, e^x is the only good integral out there)