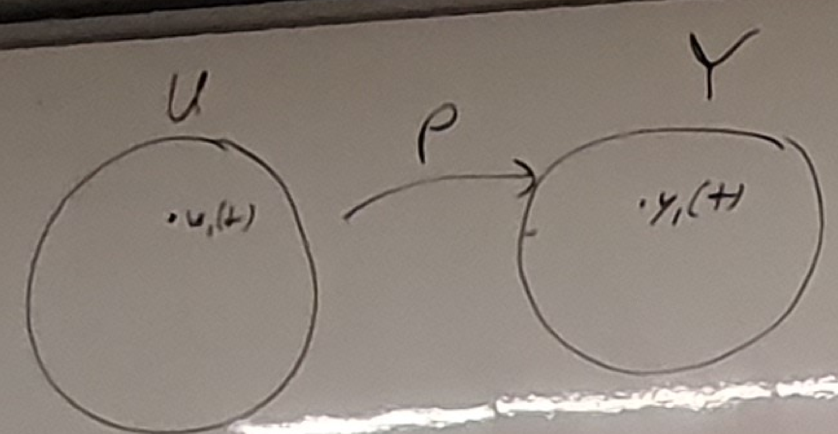

Focus: LTI

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\,\delta(t-\tau)\,d\tau$$

$$\delta(t) \xrightarrow{P} h(t)$$

$h(t)$ ↓ impulse response

Assume P is LTI $\qquad P(\delta(t)) = h(t)$

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\,\delta(t-\tau)\,d\tau \xrightarrow{P} y(t) = \int_{-\infty}^{\infty} u(\tau)\, P(\delta(t-\tau))\,d\tau \quad \text{[linearity]}$$

$$= \int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\,d\tau \quad \text{[time-invariance]}$$

$$=: (h * u)(t)$$

↓ convolution

$$= (u * h)(t)$$

Assume P is LTI.

Fix a constant $s \in \mathbb{C}$.

$$u(t) = e^{st} \xrightarrow{P} y(t)$$

Then, by time-invariance, for any $\tau \in \mathbb{R}$,

$$u(t-\tau) = e^{s(t-\tau)} \xrightarrow{P} y(t-\tau)$$

but $e^{s(t-\tau)} = \underbrace{e^{-s\tau}}_{\text{constant}}\ \underbrace{e^{st}}_{\text{function}} \xrightarrow{P} e^{-s\tau}\, y(t)$ [by linearity]

same input → same output $\Rightarrow y(t-\tau) = e^{-s\tau}\, y(t)$

P LTI, $\quad \delta(t) \xrightarrow{P} h(t), \quad H(s) = \mathcal{L}(h(t))$

Since this holds for any $\tau \in \mathbb{R}$, we can in particular set $\tau = t$ for any given $t \in \mathbb{R}$. Then

$$e^{-st}\, y(t) = y(0)$$

$$\Leftrightarrow \quad y(t) = y(0)\, e^{st}$$

... so the response is just the input e^{st} multiplied by a constant $y(0)$.

What's the value of the constant $y(0)$?

By the convolution integral,

$$y(t) = \int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, u(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\,d\tau$$

$$= e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\,d\tau}_{H(s)}$$

$\Rightarrow \quad y(t) = H(s)\, e^{st}$

where $H(s)$ is the (2-sided) Laplace transform of $h(t)$ and is independent of $u(t)$

$$\Rightarrow \quad e^{st} \xrightarrow{P} H(s)\, e^{st} \qquad [P \text{ is LTI}]$$

So, for exponential inputs we don't even have to convolve – just multiply the input by $H(s)$.

Ex. $s = j\omega$

$\Rightarrow y(t) = H(j\omega)\, e^{j\omega t}$

Ex. $s = 2 + 3j$

$\Rightarrow y(t) = H(2+3j)\, e^{(2+3j)t}$

This suggests that analysis would be simplified if we could express input signals as linear combinations of exponentials

→ leads us to the Laplace transform

(One-sided) Laplace transform:

$$F(s) = \int_{0^-}^{\infty} \underbrace{f(t)}_{\text{weights}} \underbrace{e^{-st}}_{\text{functions of } s} dt = \mathcal{L}(f(t))$$

inverse Laplace transform:

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \underbrace{F(s)}_{\text{weights}} \underbrace{e^{st}}_{\text{functions of } t} ds = \mathcal{L}^{-1}(F(s))$$

A linear combination of exponentials e^{-st} / e^{st} with weights $f(t)$ / $F(s)$

$$u(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \underbrace{U(s)}_{\text{weights}} \underbrace{e^{st}}_{\text{functions}} ds \xrightarrow{P} y(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \underbrace{U(s)}_{\text{weights}} \underbrace{P(e^{st})}_{\text{functions}} ds$$

where $U(s) = \mathcal{L}(u(t))$

$$= \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} U(s) H(s) e^{st} ds$$

$U(s) = \mathcal{L}(u(t))$

$H(s) = \mathcal{L}(h(t))$

$Y(s) = H(s) U(s)$

$\Rightarrow y(t) = \mathcal{L}^{-1}(Y(s))$

$$U(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s)$$

We call $H(s)$ the <u>transfer function</u> from $U(s)$ to $Y(s)$ or from $u(t)$ to $y(t)$

$$y(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} Y(s) e^{st} ds$$

where $Y(s) = \mathcal{L}(y(t))$

$\Rightarrow \boxed{Y(s) = H(s) U(s)}$ [same input → same output]

[linear combinations that are equal and have the same functions must have the same weights]

(functions: linearly independent)

→ we replace time-domain convolution with Laplace-domain multiplication

Common Laplace transforms:

Time Domain $f(t)$	Laplace Domain $F(s)$
$\delta(t)$	1
$\mathbb{1}(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$

Properties:

Linearity: $L(c_1 f_1(t) + c_2 f_2(t)) = c_1 F_1(s) + c_2 F_2(s)$

Derivatives: $L(f'(t)) = sF(s) - f(0^-)$

$L(f''(t)) = s^2 F(s) - sf(0^-) - f'(0^-)$

Integrals: $L\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s} F(s)$

Frequency shift: $L(e^{at} f(t)) = F(s-a)$

Convolution: $L\{(f * g)(t)\} = F(s)\,G(s)$

System Modeling

General modeling procedure:

Apply physical laws
e.g., Newton's 2nd law, KCL, KVL, etc

↓

System of differential equations

↓ "linearize" about a desired "operating point"

System of LTI differential equations

↓ take Laplace transform and solve for the relationship between inputs and outputs

transfer functions → "model"

identify unknown parameters experimentally

we'll do in the lab