University of Waterloo

ECE 380: Analog Control Systems

Problem Set 3

February 6, 2026

Problem 1

A mass-spring-damper system is shown in Figure 1. The input u to the system is an applied force and the output y equals the position q of the mass. The damper is linear but the spring exerts a restoring force given by

$$k(q) = K(1 - a^2q^2)q, \quad |aq| < 1, \quad K > 0.$$

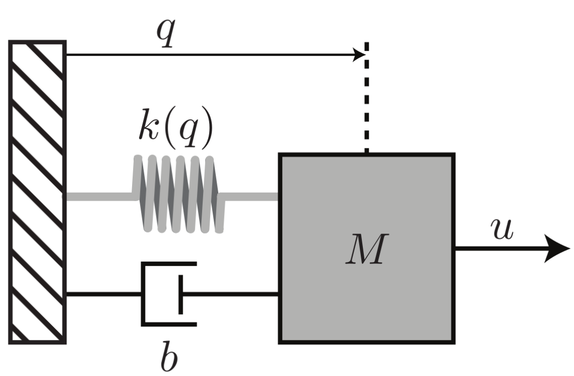

Figure 1: Mass-spring-damper system for Problem 3

(a) Find a differential equation that models this system.

(b) Linearize this model at a controlled equilibrium at which the output equals 0.5 m.

(c) Find the transfer function from Δu to Δq, and its poles and zeros.

Problem 2

Consider the following differential equation, which models the seesaw of Problem 1 on Assignment 2:

$$(M_b + 3M_l)l\ddot{\theta}(t) = 6(f(t) - M_l g)\cos(\theta(t)).$$

(a) Linearize this model at a controlled equilibrium at which $\theta = \frac{\pi}{4}$.

(b) Find the transfer function from Δf to $\Delta\theta$, and its poles and zeros.

(c) Is this transfer function stable?

Problem 3

Consider the following differential equations, which model the electromagnetic relay of Problem 3 on Assignment 2:

$$\begin{aligned}
L\dot{i}_1 &= v - R_1 i_1 - R_2 i_2 \\
R_2 \dot{i}_2 &= \frac{1}{C}(i_1 - i_2) \\
m\ddot{x} &= -K\frac{i_2^2}{x^2} - k(x - 0.1) - B\dot{x}.
\end{aligned}$$

(a) Linearize this model at a controlled equilibrium at which $x = -10m$.

(b) Find the transfer function from Δv to Δx.

Problem 4

Consider the feedback control system shown in Figure 2. Suppose that

$$P(s) = \frac{1}{s-1}, \quad C(s) = K_p$$

where $K_p > 0$ is a constant called the gain. This controller is called a proportional controller because the input $u(t)$ is proportional to the error $e(t)$.

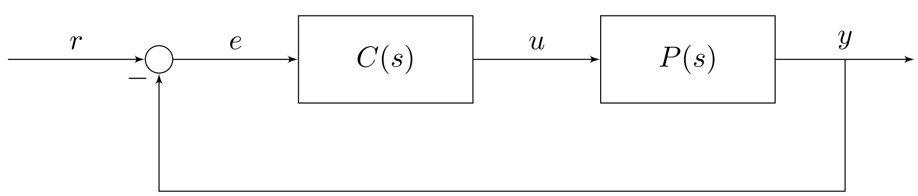

Figure 2: Feedback Control System for Problems 4-5

(a) Is the plant stable?

(b) Find the transfer functions $T_{ry}(s)$ and $T_{ru}(s)$.

(c) For which values of $K_p > 0$ are $T_{ry}(s)$ and $T_{ru}(s)$ stable? For the remainder of this problem, assume that K_p takes values so that $T_{ry}(s)$ and $T_{ru}(s)$ are stable.

(d) What is the time constant for $T_{ry}(s)$ and $T_{ru}(s)$?

(e) Suppose that $r(t) = \mathbf{1}(t)$ the unit step. Find expressions for $y(t)$ and $u(t)$. What is the value of $u(t)$ at $t = 0$, and what is the value of $y(t)$ in the limit as $t \to \infty$?

Hence, while a high gain K_p provides stability, a fast transient, and low steady-state error, it is at the expense of a large initial control input.

Problem 5

Consider the feedback control system shown in Figure 2.

(a) Find the transfer function $T_{re}(s)$.

(b) For each of the plant-controller combinations below, determine if (1) $P(s)$ is stable, (2) $C(s)$ is stable, and (3) $T_{re}(s)$ is stable.

(i) $P(s) = \frac{1}{s+1}, \quad C(s) = \frac{-6}{s+2}$

(ii) $P(s) = \frac{1}{s+1}, \quad C(s) = \frac{1}{s+2}$

(iii) $P(s) = \frac{1}{s-1}, \quad C(s) = \frac{4s}{s-2}$

Thus, the stability of $P(s)$ and $C(s)$ does not provide any information about the stability of $T_{re}(s)$.

Problem 6

The response of the following transfer functions to a unit step $\mathbf{1}(t)$ is shown in Figures 3-7. Match each transfer function to the figure of its corresponding step response. Do NOT use any step response or performance specification equations from lecture, and do NOT perform any calculations.

(i) $\frac{409}{6}\left(\frac{1}{s+3-20j} + \frac{1}{s+3+20j}\right)$

(ii) $\frac{2.5}{s+5} + \frac{5}{s+10}$

(iii) $\frac{3}{s+3}$

(iv) $\frac{109}{6}\left(\frac{1}{s+3-10j} + \frac{1}{s+3+10j}\right)$

(v) $\frac{-3}{s-3}$

Hint: consider the real and imaginary parts of the poles for each transfer function.

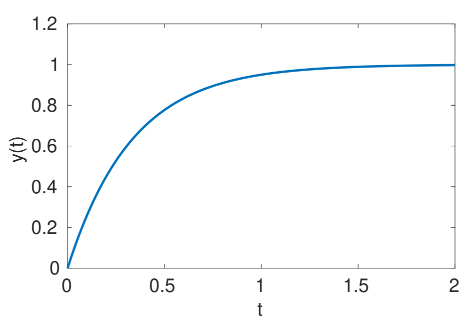

Figure 3: Option (a) for Problem 6

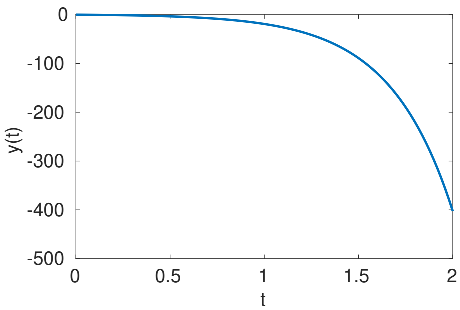

Figure 4: Option (b) for Problem 6

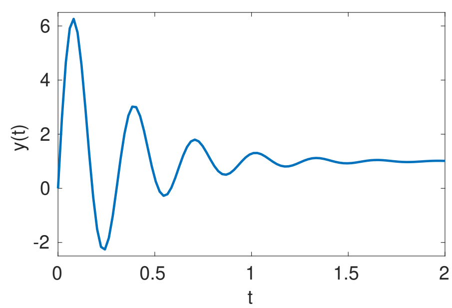

Figure 5: Option (c) for Problem 6

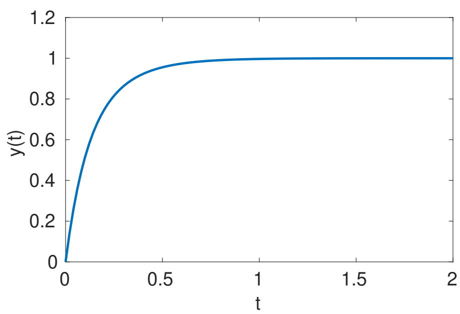

Figure 6: Option (d) for Problem 6

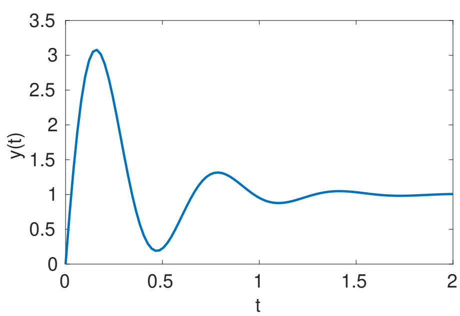

Figure 7: Option (e) for Problem 6