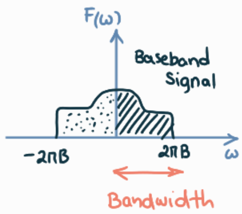
ECE 318 - TUTORIAL 3   AMPLITUDE MODULATION : DOUBLE SIDEBAND

* Definitions & assumptions :

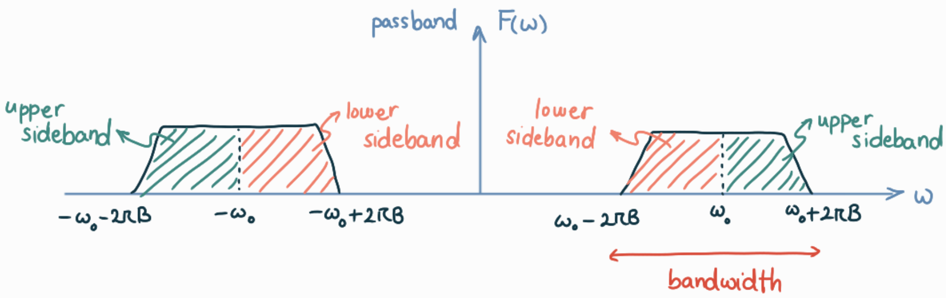
1/ Bandwidth (BW or B) is the range of positive frequencies for which the spectrum is non-zero. BW (or B) can be measured in rad/sec (ω) or cycle/sec (f).

2/ Bandlimited signal is the signal which has a bounded bandwidth, i.e. B is small (depends on application)



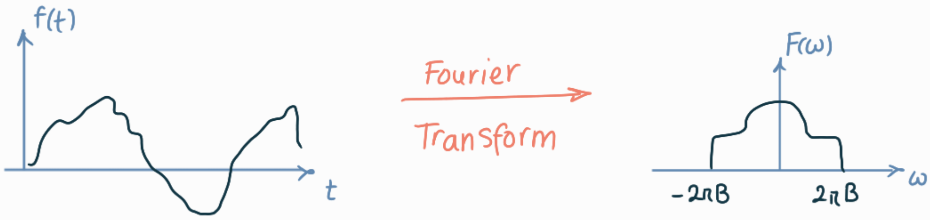
3/ Baseband signal is the signal where its spectrum is located around the zero frequency.

4/ Passband signal is the signal where its spectrum is centered around non-zero frequency ω_0

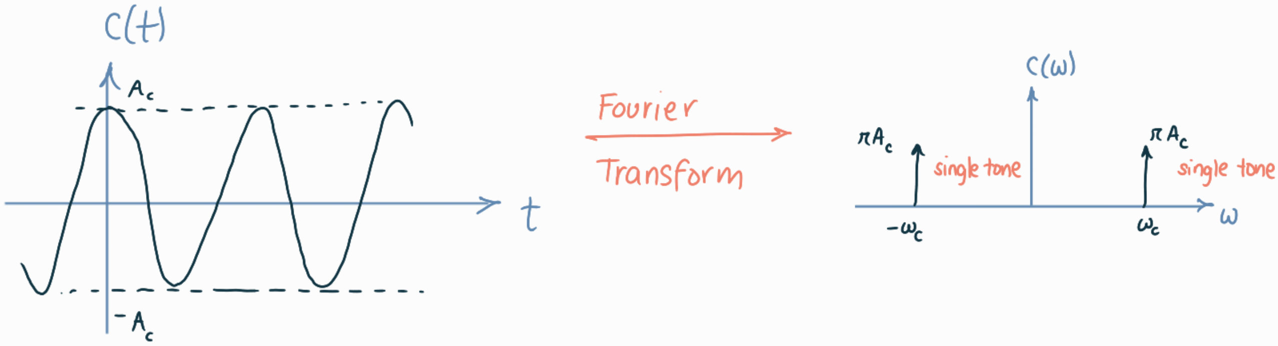


Passband signal has a bandwidth of $2 \times 2\pi B$ rad/sec or $2B$ Hz

5/ Message signal $f(t)$ is a baseband bandlimited signal of bandwidth B.



6/ Carrier signal $c(t)$ is mostly a single tone signal that has a relatively high frequency $f_c = \frac{\omega_c}{2\pi}$ compared to the message signal. Example : $A_c \cos \omega_c t$

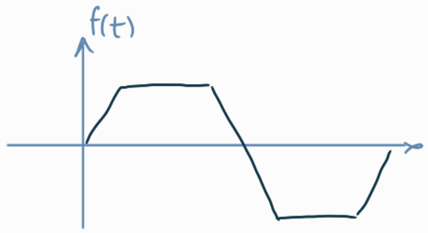


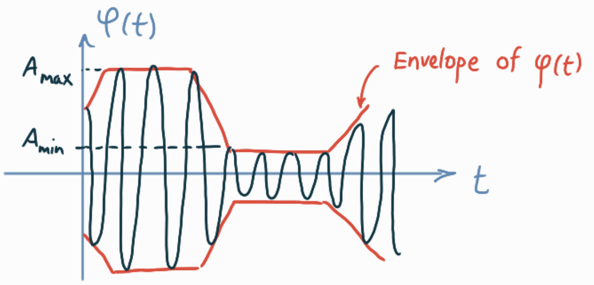
Double sideband (DSB)

Large carrier DSB-LC

* Modulated signal $\varphi(t) = (A_c + f(t)) \cos \omega_c t$

$$= f(t) \cos \omega_c t + \underbrace{A_c \cos \omega_c t}_{LC}$$





* Modulation index $\alpha = \dfrac{A_{max} - A_{min}}{A_{max} + A_{min}}$

only if message is symmetrical: $\max\{m(t)\} = -\min\{m(t)\}$

* Condition to prevent overmodulation is $A_{min} > 0$

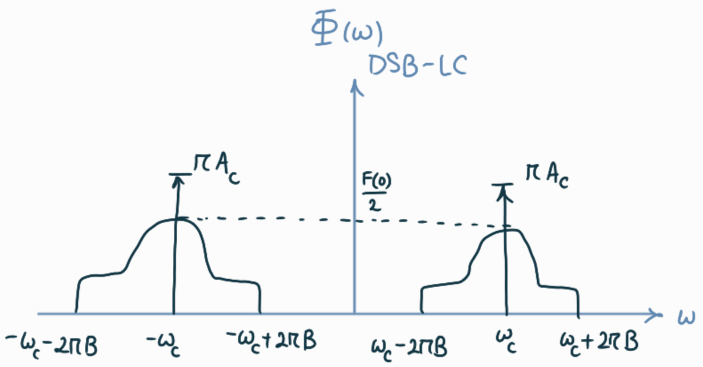
* For normalized message signal

$$\varphi(t) = A_c (1 + \alpha f_n(t)) \cos \omega_c t$$

where $f_n(t) = \dfrac{f(t)}{\max\{f(t)\}}$

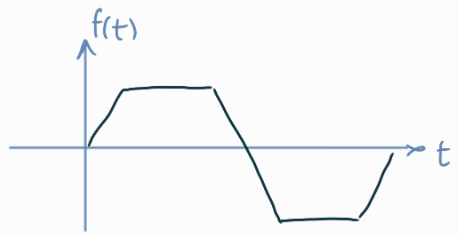
* To prevent overmodulation $\alpha < 1$

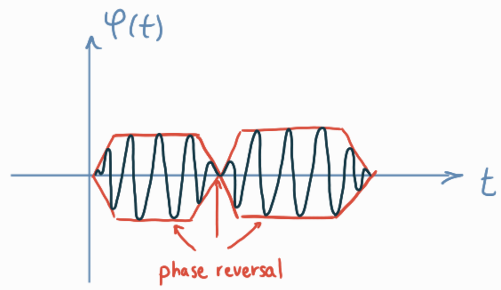
* Frequency domain



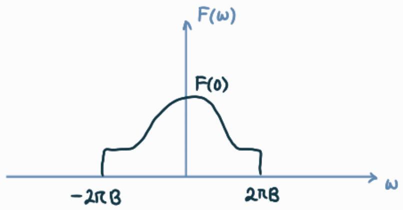
Supressed carrier DSB-SC

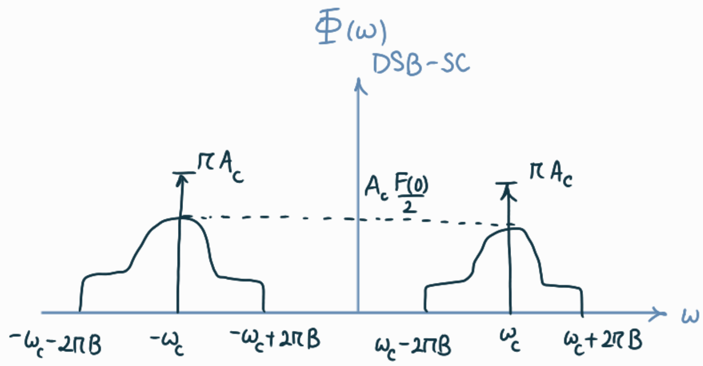
* Modulated signal $\varphi(t) = A_c f(t) \cos \omega_c t$





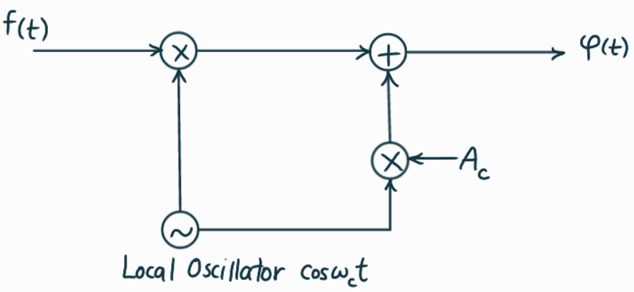
* Given that $F(\omega)$ is :





... large carrier DSB-LC

* Modulator design:



* Demodulator design "simple envelope detector"

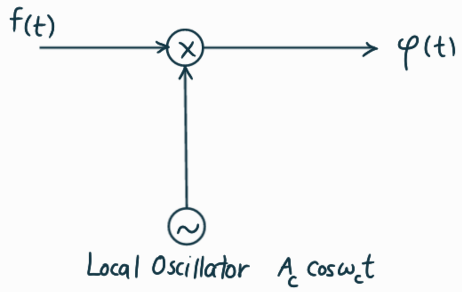
* Power of $\varphi(t)$: $\overline{\varphi^2(t)} = \frac{A_c^2}{2} + \frac{\overline{f^2(t)}}{2}$

* Modulation efficiency: $\mu = \dfrac{\overline{f^2(t)}/2}{A_c^2/2 + \overline{f^2(t)}/2}$

* DSB-LC has simple receiver structure, but wastes some power in the carrier signals.

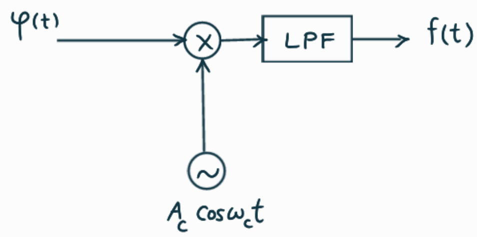
... supressed carrier DSB-SC

* Modulator design:



* Demodulator design "coherent detector"

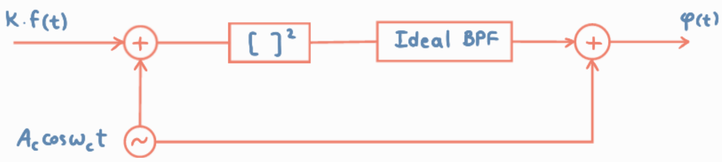
i.e. the local oscillators at both tx & rx are synchronized.



* Power of $\varphi(t)$ $\overline{\varphi^2(t)} = \frac{A_c \overline{f^2(t)}}{2}$

* DSB-SC is more efficient than DSB-LC. But it requires coherent detection.

Ex. Consider the following DSB-LC modulator.



Assume that:

1/ Bandwidth of $f(t) = \frac{B}{2\pi}$ Hz

2/ Carrier frequency $\omega_c \gg 1$

3/ $\max\{f(t)\} = -\min\{f(t)\} = 1$

4/ $K > 0$, 5/ $\overline{f(t)} = 0$ "average = 0"

Find: a) The center frequency and minimum bandwidth of BPF.

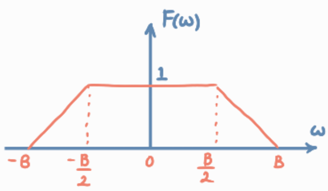
b) $\varphi(t)$ and the modulation index α

c) The maximum K that avoids over-modulation.

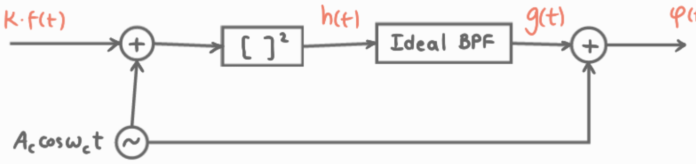
d) The modulation efficiency μ

e) The best possible modulation efficiency to avoid over-modulation if $\overline{f^2(t)} = 1$

f) Sketch the spectrum of $\varphi(t)$ for the given $F(\omega) = \mathcal{F}\{f(t)\}$



Sol a) $h(t) = [K f(t) + A_c \cos \omega_c t]^2$



$$= K^2 f^2(t) + A_c^2 \cos^2(\omega_c t) + 2KA_c f(t) \cos \omega_c t$$

$$= \underbrace{K^2 f^2(t)}_{\text{baseband}} + \underbrace{\frac{A_c^2}{2}}_{\text{DC}} + \underbrace{\frac{A_c^2}{2} \cos(2\omega_c t)}_{\text{centered at } 2\omega_c} + \underbrace{2KA_c f(t) \cos(\omega_c t)}_{\text{centered at } \omega_c}$$

Since we are interested in generating DSB-LC signal, then $\varphi(t) = (A_c + f(t)) \cos \omega_c t$. From the diagram, $\varphi(t) = g(t) + A_c \cos \omega_c t$. As a result, the output of BPF $g(t) = f(t) \cdot \cos \omega_c t$. By comparing the input and output of BPF, i.e. $h(t)$ & $g(t)$, we need to filter out the first three terms of $h(t)$. Therefore the BPF is designed to filter out: ① $K^2 f^2(t) + \frac{A_c^2}{2}$ "baseband component" ② $\frac{A_c^2}{2} \cos(2\omega_c t)$ "passband centered around $2\omega_c$". Thus we need a bandpass filter (BPF) with center frequency ω_c.

How much BW do we need in the BPF? The minimum bandwidth of the BPF is the bandwidth of the fourth term "$2KA_c f(t) \cos(\omega_c t)$". As we know if $f(t)$ has a bandwidth B, multiplying $f(t)$ by $\cos \omega_c t$ makes the bandwidth of the $2KA_c f(t) \cos \omega_c t = 2B$ $\Rightarrow$ BPF has a center frequency of ω_c and bandwidth 2B. As a result $g(t) = 2KA_c f(t) \cos \omega_c t$.

b) Clearly, $\varphi(t) = A_c \cos(\omega_c t) + 2KA_c f(t) \cos(\omega_c t) = A_c [1 + 2Kf(t)] \cos \omega_c t$.

Since $f(t)$ is normalized, i.e., $\max\{f(t)\} = -\min\{f(t)\} = 1$, therefore we compare with $\varphi(t) = A_c [1 + \alpha f(t)] \cos \omega_c t$ which means $\alpha = 2K$.

c) To avoid overmodulation $0 < \alpha < 1$, therefore, $K = \alpha/2$. Then the maximum K corresponds to the maximum α and has a value $1/2$. $K_{max} = 1/2$.

d) Modulation efficiency, $\mu = \dfrac{P_{sideband}}{P_{carrier} + P_{sideband}}$

Total power = power in $\varphi(t)$: $\overline{\varphi^2(t)} \triangleq \lim_{T \to \infty} \frac{1}{T} \int_0^T \{\varphi(t)\}^2 dt = \overline{[2KA_c f(t) \cos \omega_c t + A_c \cos \omega_c t]^2} =$

$$\overline{[4K^2A_c^2 f^2(t) \cos^2 \omega_c t + A_c^2 \cos^2 \omega_c t + 4KA_c^2 f(t) \cos^2 \omega_c t]} = \frac{\overline{4K^2A_c^2 f^2(t)}}{2} ① + \frac{\overline{4K^2A_c^2 f^2(t) \cos(2\omega_c t)}}{2} ② + \frac{\overline{A_c^2}}{2} ③ + \frac{\overline{A_c^2 \cancelto{0}{\cos(2\omega_c t)}}}{2} ④ +$$

$$\frac{\overline{4KA_c^2 \cancelto{0}{f(t)}}}{2} ⑤ + \frac{\overline{4KA_c^2 f(t) \cos(2\omega_c t)}}{2} ⑥$$

Since $f(t)$ varies much slower than $\cos(2\omega_c t)$ "$\omega_c \gg B$" : ② = 0 , ⑥ = 0

Average $f(t) = 0$

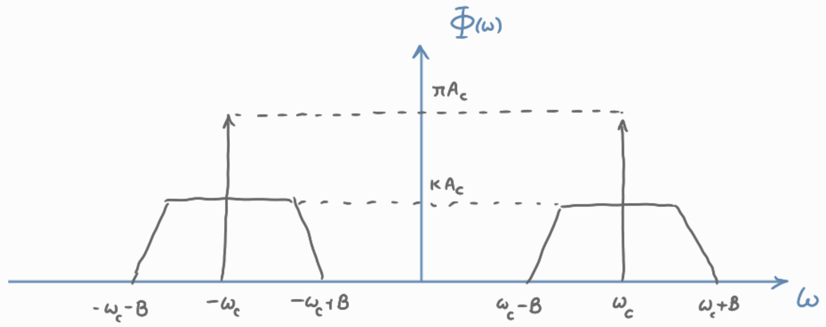
$$\Rightarrow \overline{\varphi^2(t)} = \frac{\overline{4K^2A_c^2 f^2(t)}}{2} + \frac{\overline{A_c^2}}{2} = 2K^2 A_c^2 \overline{f^2(t)} + \frac{A_c^2}{2}$$

$$\Rightarrow P_\varphi = 2K^2A_c^2\overline{f^2(t)} + \frac{A_c^2}{2} \Rightarrow \mu = \frac{P_{sideband}}{P_{sideband}+P_{carrier}} = \frac{2K^2A_c^2\overline{f^2(t)}}{2K^2A_c^2\overline{f^2(t)}+\frac{A_c^2}{2}} = \frac{4K^2\overline{f^2(t)}}{4K^2\overline{f^2(t)}+1}$$

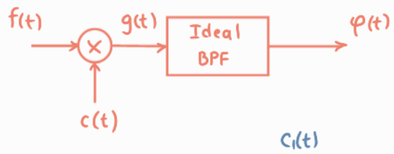
e) If $\overline{f^2(t)} = 1$, $\mu = \frac{4K^2}{4K^2+1}$, the function is an increasing function of K, since $\frac{d\mu}{dk} = \frac{8K}{(1+4K^2)^2} > 0$ for $k>0$. So $\mu(K)$ increases with K. However to avoid overmodulation, $K \leq 1/2$ thus $K_{max} = 1/2 \Rightarrow \mu_{max} = \frac{1}{1+1} = 50\%$.

f) $\Phi(\omega) = \mathcal{F}\{\varphi(t)\} = \mathcal{F}\{2KA_c f(t)\cdot\cos(\omega_c t)\} + \mathcal{F}\{A_c\cos(\omega_c t)\}$

$\mathcal{F}\{A_c\cos\omega_c t\} = A_c\pi\,\delta(\omega+\omega_c) + A_c\pi\,\delta(\omega-\omega_c)$, $\mathcal{F}\{2KA_c f(t)\cos\omega_c t\} = \frac{2KA_c}{2}\left[F(\omega-\omega_c)+F(\omega+\omega_c)\right]$



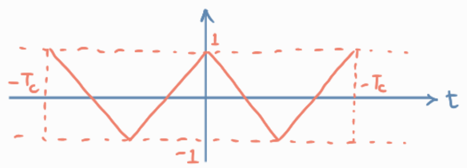
Ex. Using the system, we would like to generate a DSB-SC signal at the carrier frequency $2\omega_c$.



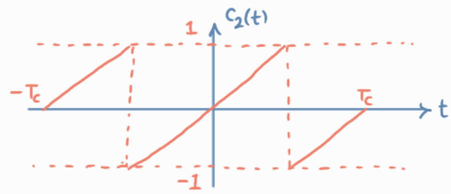
We are given the two choices for $c(t) \in \{c_1(t), c_2(t)\}$.

The respective fourier series are given by

$c_1(t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(n\omega_c t)$, $c_2(t) = \frac{-2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\omega_c t)$.



a) Which carrier should be used as a carrier, $c_1(t)$ or $c_2(t)$?



b) Determine the center frequency and BW of the BPF, assuming that the bandwidth of $f(t)$ is B rad/sec. Find $\varphi(t)$.

c) Design a coherent detector to recover $f(t)$ from $\varphi(t)$.

Sol : a) Observe the Fourier series of $c_1(t)$ and $c_2(t)$: $c_1(t) = \frac{8}{\pi^2}\left[\cos(\omega_c t) + \frac{1}{9}\cos(3\omega_c t) + \frac{1}{25}\cos(5\omega_c t) + \cdots\right]$

$c_2(t) = \frac{2}{\pi}\left[\sin(\omega_c t) - \frac{1}{2}\sin(2\omega_c t) + \frac{1}{9}\sin(3\omega_c t) + \cdots\right]$ and $g(t) = c(t)\cdot f(t)$ while $\varphi(t)$ is the output of the bandpass filter given the input $g(t)$. In the multiplication $c(t)\cdot f(t)$, $c(t)$ is represented by the Fourier series, then $f(t)$ is multiplied by cos or sin, depending on $c(t) = c_1(t)$ or $c_2(t)$. As a result, the signal $g(t)$ has spectrum that is located around ω_c, $3\omega_c$, $5\omega_c$, ... for $c_1(t)$ & located around $\omega_c, 2\omega_c, 3\omega_c, \ldots$ for $c_2(t)$.

Since the carrier frequency is designed at $2\omega_c$, $\varphi(t)$ should have a carrier of $2\omega_c$. As a result, $c_2(t)$ is the carrier since it contains $\sin(2\omega_c t)$.

b) Since the carrier frequency is $2\omega_c$, therefore the center frequency is $2\omega_c$ and the bandwidth is $2B$.

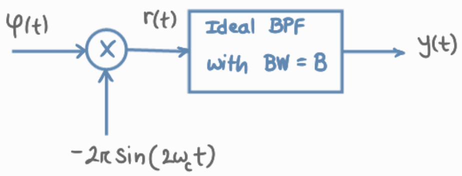
$$g(t) = f(t) \cdot \frac{2}{\pi}\left[\sin(\omega_c t) - \frac{1}{2}\sin(2\omega_c t) + \cdots\right]$$

* In order to keep the component at $2\omega_c$ and filter out all other components, we need the BPF to be:

1 - Centered at $2\omega_c$ 2 - Have a bandwidth of $2B$ rad/sec

Therefore after the filtering $\varphi(t) = -\frac{1}{\pi} f(t)\sin(2\omega_c t)$.

c) Consider the following coherent detector:



Thus, $r(t) = \varphi(t)\cdot(-2\pi)\cdot\sin(2\omega_c t)$

$$= 2f(t)\sin^2(2\omega_c t) = 2f(t)\,\frac{1-\cos(4\omega_c t)}{2}$$

$$= \underbrace{f(t)}_{\text{centered around } \omega=0} - \underbrace{f(t)\cos(4\omega_c t)}_{\text{centered at } 4\omega_c}$$

Therefore, the LPF will filter out the high frequency component and give $y(t) = f(t)$ #

Ex. (Midterm - Fall 2011) $x(t)$ is the output of a DSB-LC modulation.

$$x(t) = 20\cos(300\pi t) + 6\cos(280\pi t) + 6\cos(320\pi t)$$

a) Find the normalized message signal and the carrier signal $c(t)$

b) Find the modulation index

c) Find the modulation efficiency

Sol: $x(t) = 20\cos(300\pi t) + 6\cos(280\pi t) + 6\cos(320\pi t)$, since DSB-LC then $\varphi(t) = A_c[1+\alpha f_n(t)]\cos\omega_c t$, we have to put $x(t)$ in the form $\varphi(t)$.

$$x(t) = 20\cos(300\pi t) + 6\cos(300\pi t - 20\pi t) + 6\cos(300\pi t + 20\pi t) = 20\cos(300\pi t) + 12\cos(300\pi t)\cdot\cos(20\pi t)$$

$$= \underbrace{20}_{A_c}\Big(\underbrace{1 + \tfrac{12}{20}\cos 20\pi t}_{1+\alpha f_n(t)}\Big)\underbrace{\cos(300\pi t)}_{\cos(\omega_c t)}$$

a) The normalized message signal is $\cos(20\pi t)$ & the carrier signal is $c(t) = 20\cos(300\pi t)$

b) The modulation index is $a = \frac{12}{20} = 0.6$

c) Modulation efficiency $\mu = \frac{P_{sidebands}}{P_{carrier} + P_{sidebands}}$

$$x(t) = \underbrace{20\cos(300\pi t)}_{\text{carrier}} + \underbrace{12\cos(300\pi t)\cos(20\pi t)}_{\text{sidebands}}$$

$$P_{carrier} = \frac{(20)^2}{2} = \frac{400}{2} = 200$$

$$P_{sidebands} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} 144\cos^2(300\pi t)\cdot\cos^2(20\pi t)\, dt$$

$$= \lim_{T\to\infty} \frac{144}{4T}\int_{-T/2}^{T/2} (1+\cos 600\pi t)(1+\cos 40\pi t)\, dt$$

$$= \lim_{T\to\infty} \frac{144}{4T}\int_{-T/2}^{T/2} (1+\cos 600\pi t\cos 40\pi t + \cos 600\pi t + \cos 40\pi t)\, dt$$

$$= \lim_{T\to\infty} \frac{144}{4T}\int_{-T/2}^{T/2} 1\, dt + \lim_{T\to\infty} \frac{144}{4T}\int_{-T/2}^{T/2} \cos 600\pi t\cos 40\pi t\, dt + \underbrace{\cancel{\lim_{T\to\infty} \frac{144}{4T}\int_{-T/2}^{T/2} \cos 600\pi t\, dt}}_{0} + \underbrace{\cancel{\lim_{T\to\infty} \frac{144}{4T}\int_{-T/2}^{T/2} \cos 40\pi t\, dt}}_{0}$$

$$= \frac{144}{4} + \lim_{T\to\infty} \frac{144}{8T}\int_{-T/2}^{T/2} \cos(600-40)\pi t + \cos(600+40)\pi t\, dt$$

$$= \frac{144}{4} + \underbrace{\cancel{\lim_{T\to\infty} \frac{144}{8T}\int_{-T/2}^{T/2} \cos(600-40)\pi t\, dt}}_{0} + \underbrace{\cancel{\lim_{T\to\infty} \frac{144}{8T}\int_{-T/2}^{T/2} \cos(600+40)\pi t\, dt}}_{0}$$

$$= \frac{144}{4}$$

$$\mu = \frac{144/4}{200 + 144/4} = \frac{36}{236} = 0.1525 = 15.25\ \%$$

Problem I: [9 points]

Recall that in DSB-LC,

$$\phi_{\mathsf{DSB-LC}}(t) = (A_c + kf(t))\cos\omega_c t,$$

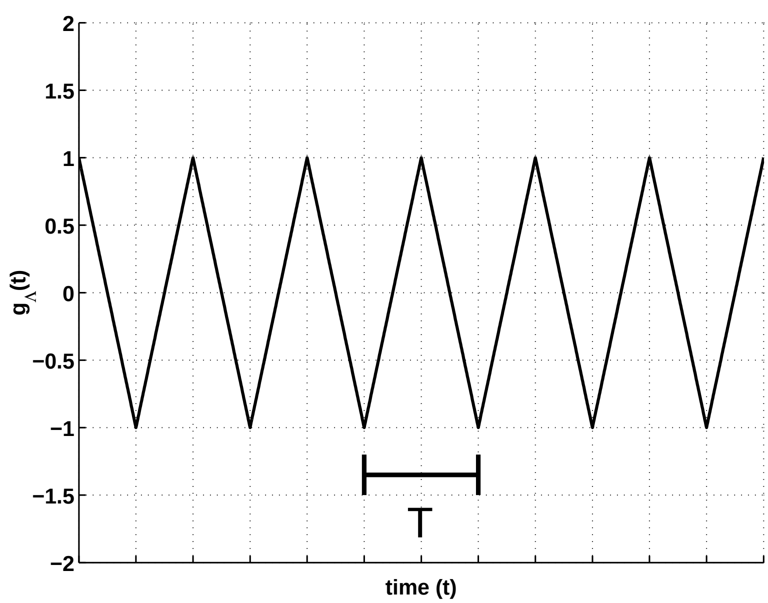
where $A_c > 0$ is the carrier amplitude, $\omega_c > 0$ is the carrier frequency, and $k > 0$ is some modulation parameter.

Since the patents on DSB-LC have expired decades ago, a friend of yours proposes the following modification that they refer to as Triangle Modulation (TM) and hope to patent:

$$\phi_{\mathsf{TM}}(t) = (A_c + kf(t))g_\Lambda(t),$$

where $g_\Lambda(t)$ is the periodic triangle wave with period T and unit amplitude sketched below. The Fourier cosine series is given by (where $\omega_0 = 2\pi/T$):

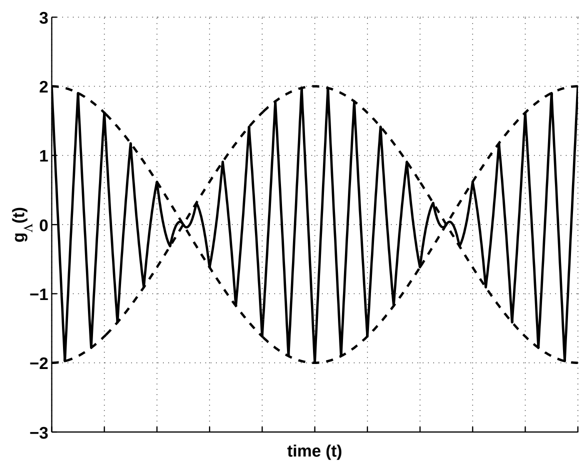
$$g_\Lambda(t) = \frac{8}{\pi^2}\sum_{n=1,3,5,7,\ldots}\frac{1}{n^2}\cos(n\omega_0 t)$$



[3] a) For $A_c = 0$, $k = 2$ and $f(t) = \cos\omega_m t$, qualitatively sketch $\phi_{\mathsf{TM}}(t)$ over one period of $f(t)$. Assume $T \ll 1/\omega_m$. Indicate key values on your graph.

Solutions:

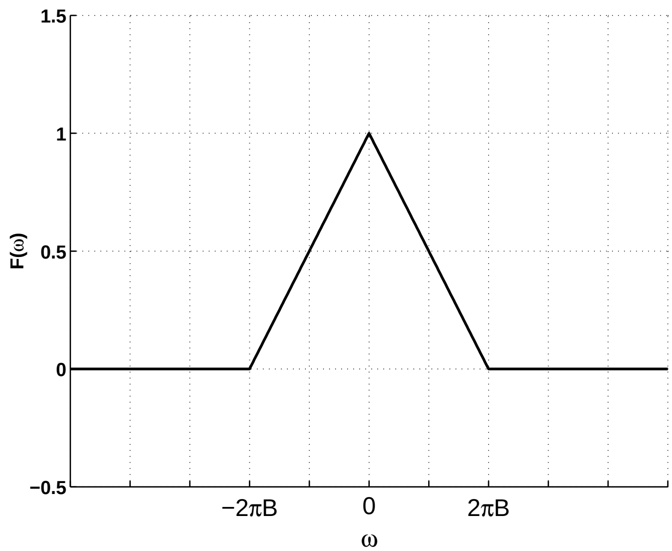
The functions $2\cos\omega_m t$ and $-2\cos\omega_m t$ provide the upper and lower limits that the triangle wave takes. The figure below shows the result in the solid line. Essentially, we draw (almost) straight lines between these upper and lower limits. [Drawing straight lines is a good enough approximation for a qualitative sketch and is fine. Drawing a sinusoid with varying amplitude is not since the "carrier" is not a sinusoid here.]



[3] b) Under what conditions on k, if any, would an ideal envelope detector be able to recover $f(t)$? Justify your answer. You may assume $\max_t f(t) = -\min_t f(t) = A > 0$. Do <u>not</u> assume $A_c = 0$.

Solutions: Since $\Lambda(t)$ takes values between -1 and 1, we must have $A_c + kf(t) \geq 0$ for all t. Equivalently, $\min_t[A_c + kf(t)] \geq 0$, and since $k > 0$, this is assured if $A_c + k\min_t f(t) \geq 0$. Hence, provided $A_c - kA \geq 0$ or equivalently $k \leq A_c/A$.

[3] c) If $f(t)$ has the spectrum $F(\omega)$ shown below, sketch the spectrum $\Phi_{\mathsf{TM}}(\omega)$ of $\phi_{\mathsf{TM}}(t)$ over the interval $-6\omega_0 < \omega < 6\omega_0$, making sure to label key values, amplitudes, and features. Take $A_c = 1$ and $k = 2$. Also, comment on the spectrum usage of Triangle Modulation compared to DSB-LC.



Solutions: Since

$$\Lambda(t) = \frac{8}{\pi^2} \sum_{n=1,3,5,7,\ldots} \frac{1}{n^2} \cos(n\omega_0 t),$$

our modulated signal $\phi_{\mathsf{TM}}(t)$ is

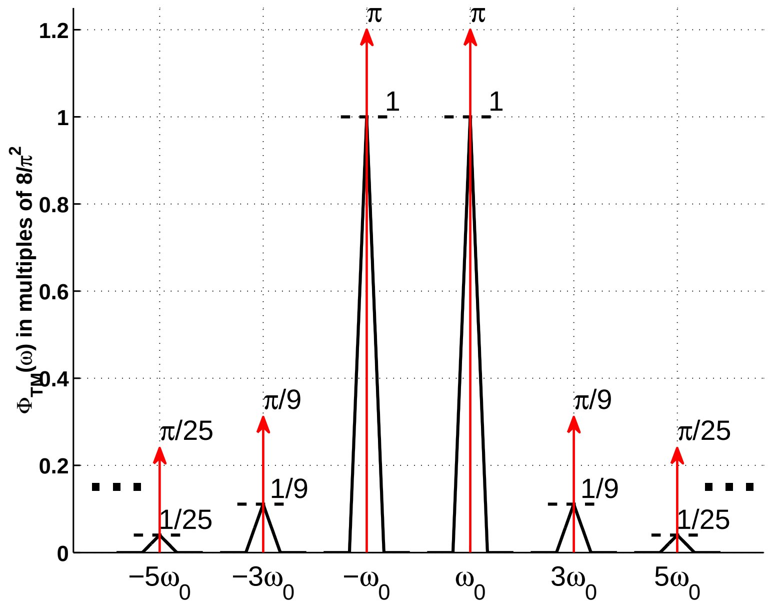
$$\phi_{\mathsf{TM}}(t) = \frac{8}{\pi^2} \sum_{n=1,3,5,7,\ldots} \frac{1}{n^2} (A_c + kf(t)) \cos(n\omega_0 t),$$

and taking the Fourier transform on both sides results in:

$$\Phi_{\mathsf{TM}}(\omega) = \frac{8}{\pi^2} \sum_{n=1,3,5,7,\ldots} \frac{1}{n^2} \Big(\pi A_c \delta(\omega - n\omega_0) + \pi A_c \delta(\omega + n\omega_0) + \frac{k}{2} F(\omega - n\omega_0) + \frac{k}{2} F(\omega + n\omega_0) \Big).$$

Taking $A_c = 1$ and $k = 2$:

$$\Phi_{\mathsf{TM}}(\omega) = \frac{8}{\pi^2} \sum_{n=1,3,5,7,\ldots} \frac{1}{n^2} \Big(\pi \delta(\omega - n\omega_0) + \pi \delta(\omega + n\omega_0) + F(\omega - n\omega_0) + F(\omega + n\omega_0) \Big).$$



The triangles above have bandwidth $4\pi B$. For DSB-LC, there would only be spectrum content of width $4\pi B$ centered at $\pm\omega_c$ and large carrier components at $\pm\omega_c$. Here, we have content centered at $\pm\omega_0$, $\pm 3\omega_0$, $\pm 5\omega_0$, etc of width $4\pi B$, with peaks of $\frac{8}{\pi^2}$, $\frac{1}{3^2}\frac{8}{\pi^2}$, $\frac{1}{5^2}\frac{8}{\pi^2}$, etc. We also have large carrier components at $\pm\omega_0$, $\pm 3\omega_0$, $\pm 5\omega_0$, with magnitudes of $\pi \times \frac{8}{\pi^2}$, $\pi \times \frac{1}{3^2}\frac{8}{\pi^2}$, $\pi \times \frac{1}{5^2}\frac{8}{\pi^2}$, etc.

Thus, the spectrum usage is much greater than that of DSB-LC.