Chapter 4

Time and Frequency Domain Analysis of Discrete Time Linear Systems

Most of our attention has so far been devoted to continuous-time systems and signals. These occur naturally in engineering systems implemented using *analog hardware*. Electrical networks, and more generally, electronic circuits fabricated with analog devices such as transistors, diodes, tunnel diodes, and so on, provide good examples of systems which are best described using the continuous-time techniques of the previous chapters. Increasingly however, analog hardware is being supplemented, and sometimes completely replaced, with *digital hardware*, which works most naturally in a discrete-time setting. This is particularly true in areas such as automatic control and signal processing. As a result, discrete-time signals and systems are of increasing importance and deserve a thorough examination in their own right.

The goal of this chapter is to study discrete time linear systems from both the *time domain* and *frequency domain* points of view. The main ideas of this chapter will be discrete time analogs of the concepts and methods learned in Chapters 2 and 3 in a continuous-time setting. That is, we shall study the discrete-time version of the impulse function, impulse response, systems described by linear difference equations with constant coefficients, and the zero-input and zero-state response. Then we shall look at the z-transform, which is a discrete-time analog of the Laplace transform.

4.1 Examples of Discrete Time Signals and Systems

Discrete time signals typically arise in two ways: either the signal is *naturally* discrete, because of the very way in which the signal originates, or the discrete-time signal is obtained by *sampling* a given continuous-time signal. Examples of the first kind of signal arise quite typically in economics. Thus, the monthly statement on your bank account is a discrete-time signal, as are the quarterly reports on the growth-rate of the national gross domestic product. Examples of the second kind of signal occur most commonly in electrical engineering. Here we begin with a given continuous-time signal $x(t)$, such as a time-varying voltage, and pass this through an electronic device called a *sampler*, which produces a train of samples $x(kT)$, $k = \ldots, -2, -1, 0, 1, 2, \ldots$ where $T > 0$ is a constant called the **intersample period**. See Fig. 4.1.

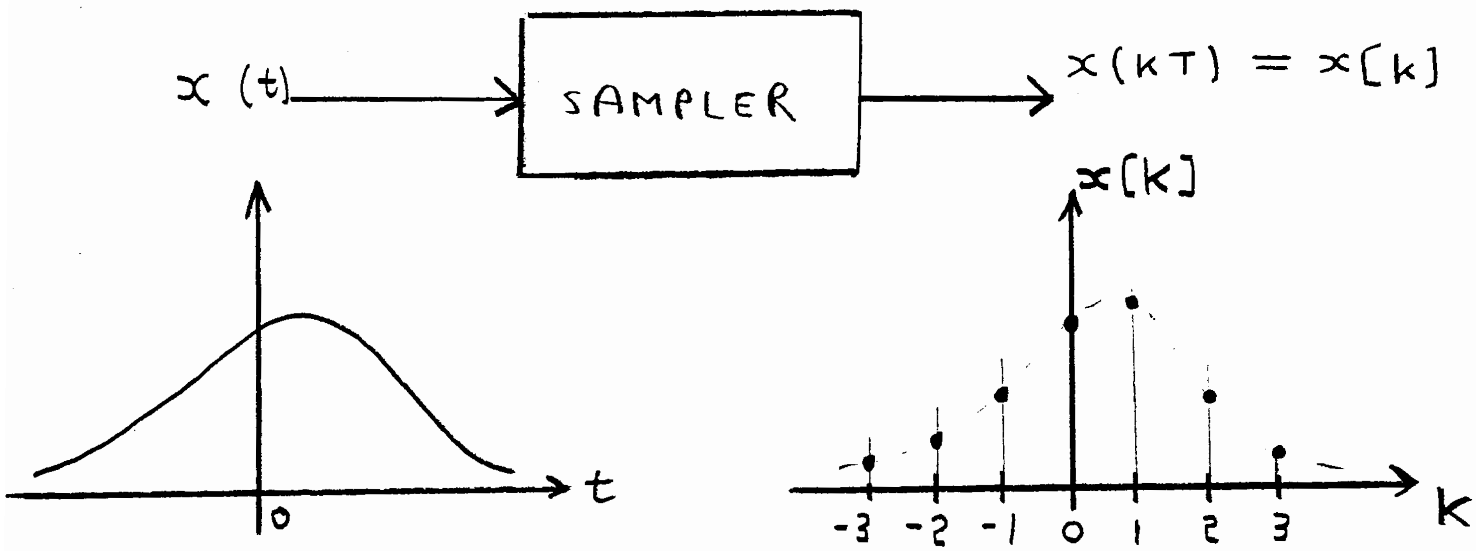

Fig. 4.1

That is, as a result of sampling the continuous-time signal $x(t)$ at every T seconds we get a discrete-time signal $x[k]$ given by

$$x[k] = x(kT), \qquad \text{for all } k = \ldots, -2, -1, 0, 1, 2, \ldots$$

Samplers occur in an enormous range of applications, including CD players, automatic control systems, data transmission equipment, television sets, and measuring instruments, to mention only a few.

We next give examples of discrete-time systems, which accept a discrete-time input signal $x[k]$

and respond by producing a discrete-time output signal $y[k]$. Discrete time systems are typically fabricated from summing and differencing junctions (which we have seen in Chapter 3), amplifiers, and *unit delay blocks*. In a discrete-time setting an amplifier with gain A accepts an input signal $x[k]$ and produces an output signal $y[k]$ given by

$$y[k] = Ax[k].$$

See Fig. 4.2.

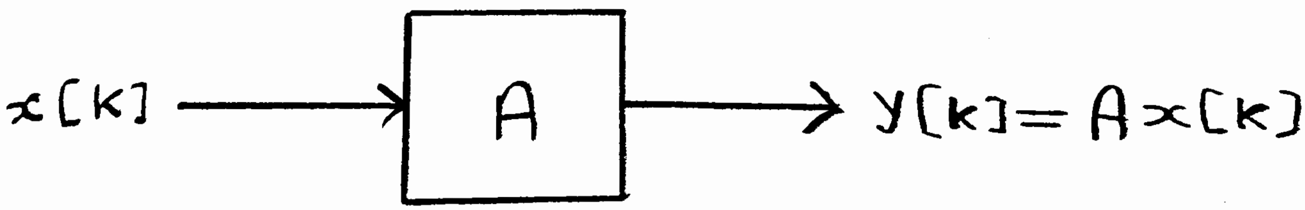

Fig. 4.2

A unit delay block accepts an input signal $x[k]$ and produces an output signal $y[k]$ which is the input signal *delayed by one step*, namely

$$y[k] = x[k-1].$$

See Fig. 4.3.

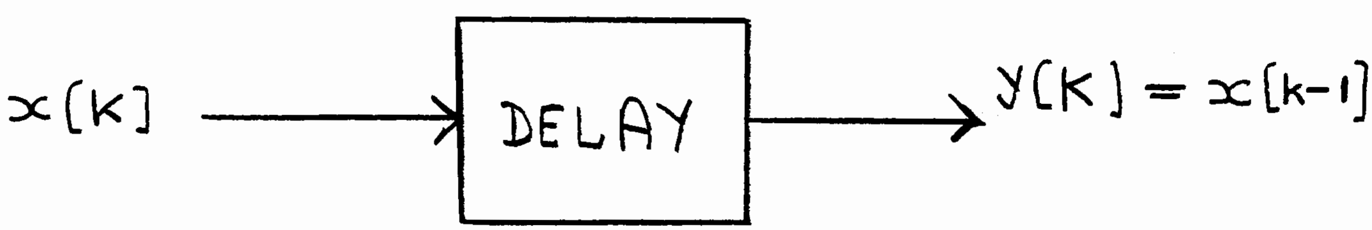

Fig. 4.3

Example 4.1.1 (First order digital filter) A first order digital filter is shown in Fig. 4.4

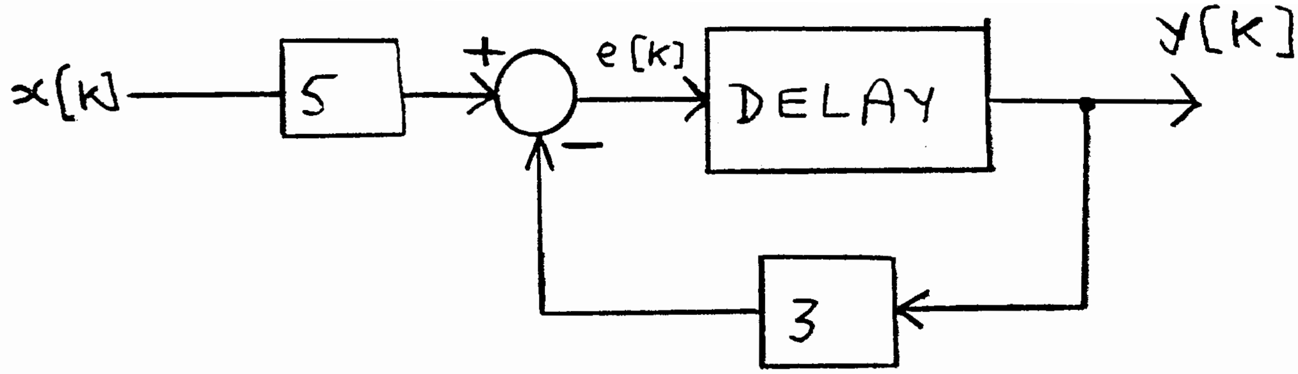

Fig. 4.4

We see that

$$e[k] = 5x[k] - 3y[k],$$

and

$$y[k] = e[k-1],$$

so that

$$y[k] = 5x[k-1] - 3y[k-1],$$

or

$$y[k] + 3y[k-1] = 5x[k-1].$$

First order digital filters are commonly used in CD players.

Example 4.1.2 (Sigma-Delta coders) A Sigma-delta coder is shown in Fig. 4.5.

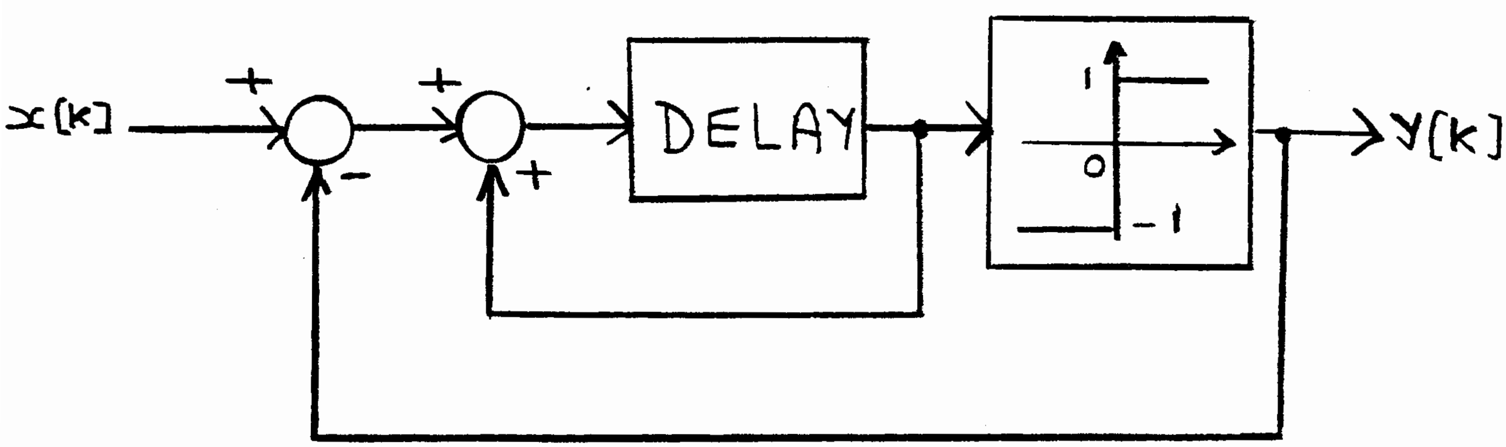

Fig. 4.5

Having motivated the idea of discrete-time systems with some examples, we are now ready to move on to look at the mathematical concepts that are needed to study these systems.

4.2 The Discrete Time Impulse Function

In this section we formulate the discrete-time analog of the continuous-time impulse function defined in Section 2.1.1, and establish a discrete-time analog of the sifting formula, the continuous-time version of which was seen in Section 2.1.2.

4.2.1 Definition of Discrete Time Impulse Functions

We recall from Section 2.1.1 that the continuous-time impulse function $\delta(t)$ is a rather subtle concept which is defined as follows: imagine a "pulse" of height $1/\epsilon$ and width ϵ, and then define $\delta(t)$ as the limit of this pulse when $\epsilon \to 0$. The definition of the **discrete time impulse function** $\delta[k]$ is much easier, namely

$$\delta[k] \triangleq \begin{cases} 1, & \text{when } k = 0, \\ 0, & \text{when } k \neq 0. \end{cases} \tag{4.1}$$

See Fig. 4.6

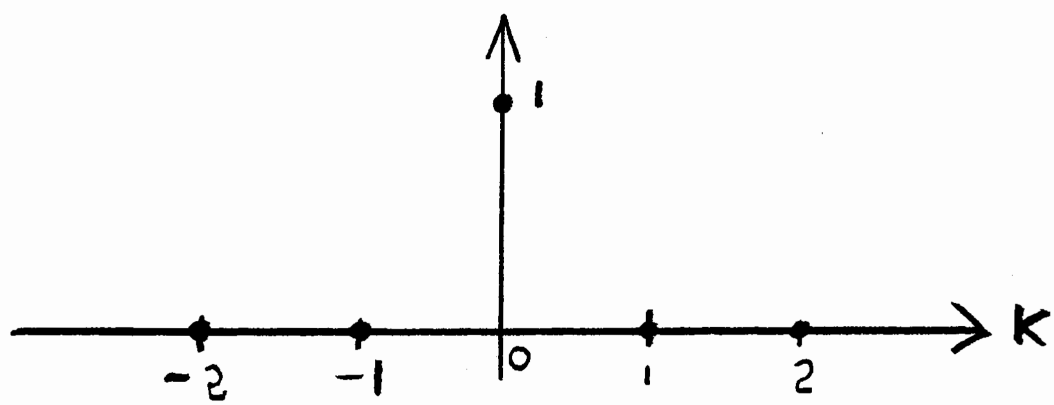

Fig. 4.6

It follows at once from Definition 1.1.7 of the discrete-time unit step function $u[k]$ that

$$u[k] = \sum_{l=-\infty}^{k} \delta[l],$$

which is analogous to the relation (2.2). Taking $k \to \infty$ then gives

$$\sum_{l=-\infty}^{\infty} \delta[l] = 1,$$

which is analogous to the relation (2.5). For a real constant a we define

$$a\delta[k] \triangleq \begin{cases} a, & \text{when } k = 0, \\ 0, & \text{when } k \neq 0. \end{cases}$$

4.2.2 The Discrete Time Sifting Formula

In this section we establish a discrete-time analog of the *sifting formula*, the continuous-time version of which is given by Theorem 2.1.1. To this end fix a discrete-time signal $x[k]$, and fix some integer n. From (4.1) we see that

$$\delta[k-n] = \begin{cases} 1, & \text{when } k = n, \\ 0, & \text{when } k \neq n, \end{cases}$$

and therefore

$$\sum_{n=-\infty}^{\infty} x[n]\delta[k-n] = x[k], \qquad \text{for all integers } n.$$

In view of the importance of this result we restate it in a way which makes clear its similarity to Theorem 2.1.1:

Theorem 4.2.1 Suppose that $x[k]$ is a discrete-time signal. Then we have

$$x[k] = \sum_{n=-\infty}^{\infty} x[n]\delta[k-n], \qquad \text{for each integer } k. \tag{4.2}$$

Remark 4.2.2 From Theorem 2.1.1 we see that the discrete-time impulse function, appropriately shifted by some instant n, "sifts out" the value of a discrete-time signal at that instant.

4.3 The Discrete Time Impulse Response

Definition 4.3.1 A discrete-time system (linear or not) with input $x[k]$ and output $y[k]$ is **initially at rest** when the following holds for each fixed integer-valued instant k_0: if $x[k] = 0$ for all $k \leq k_0$ then $y[k] = 0$ for all $k \leq k_0$.

The intuitive idea of a system being initially at rest is that at $k = -\infty$ it is "standing completely still". In accordance with this interpretation we shall use the notation

$$y(-\infty) = 0$$

to **symbolically denote** the fact that a system is initially at rest in the sense of Definition 4.3.1.

We next define the natural discrete-time analogue of the continuous-time impulse response in Definition 2.2.2 as follows:

Definition 4.3.2 The **discrete time impulse response** of a linear discrete-time system is the output signal when the input signal is the discrete-time impulse function.

Remark 4.3.3 The discrete-time impulse response is denoted by $h[k]$.

Remark 4.3.4 The natural question arises: how do we compute the discrete-time impulse response? Later in this chapter we shall study a discrete-time analog of the Laplace transform, called the z-transform, which provides a powerful method to calculate the discrete-time impulse response. Putting aside the question of how to compute the discrete-time impulse response, we next give a result which shows the usefulness of the impulse response, and which is the discrete-time analog of Theorem 2.2.8:

Theorem 4.3.5 Suppose that $h[k]$ is the impulse response of a discrete-time LTI system. Then, for an input signal $x[k]$, the response of the system is given by

$$y[k] = \sum_{n=-\infty}^{\infty} x[n]h[k-n], \tag{4.3}$$

for each integer k.

Remark 4.3.6 This result shows that if we know the impulse response $h[k]$ of a discrete-time LTI system then we can determine the response of the system to *any* input signal $x[k]$.

It is quite easy to justify Theorem 4.3.5 (in contrast to the continuous-time analog, Theorem 2.2.8, which is rather difficult to establish). The argument is as follows: Let $x[k]$ be a given input signal. Fix some integer n and define the *shifted impulse* $\delta_n[k]$ by

$$\delta_n[k] \triangleq \delta[k-n], \qquad \text{for all } k = \ldots, -2, -1, 0, 1, 2, \ldots$$

See Fig. 4.7

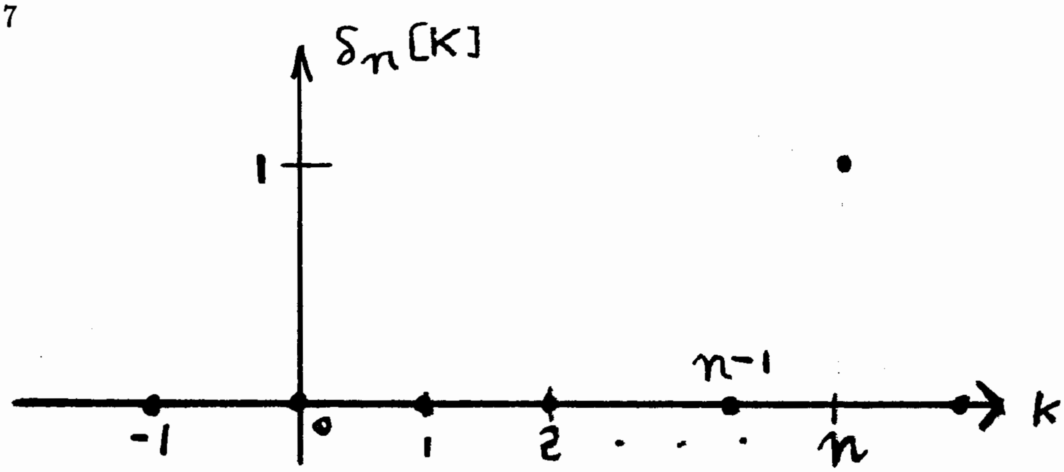

Fig. 4.7

Also, define

$$h_n[k] \triangleq h[k-n], \qquad \text{for all } k = \ldots, -2, -1, 0, 1, 2, \ldots$$

Clearly $h_n[k]$ is the result of shifting the impulse response function by n steps. Now $h[k]$ is the response of the system to the input signal $\delta[k]$. Since the system is time invariant (recall Definition 1.2.10), it follows that the response to the input signal $\delta_n[k]$, which is just the impulse function shifted by n steps, is the impulse response likewise shifted by n steps, namely the function $h_n[k]$. To repeat: time invariance of the system ensures that $h_n[k]$ is the response to the input signal $\delta_n[k]$. It then follows from linearity that $x[n]h_n[k]$ is the response to the input $x[n]\delta_n[k]$, since a rescaling of the input signal by $x[n]$ leads to a corresponding rescaling of the output signal by $x[n]$. Finally, recall from Theorem 4.2.1 that

$$(4.4) \qquad x[k] = \sum_{n=-\infty}^{\infty} x[n]\delta_n[k], \qquad \text{for each integer } k.$$

Since $x[n]h_n[k]$ is the response to the input $x[n]\delta_n[k]$, it follows from linearity and (4.4) that

$$y[k] = \sum_{n=-\infty}^{\infty} x[n]h_n[k]$$

is the response to the input signal $x[k]$, and this of course may be written as

$$y[k] = \sum_{n=-\infty}^{\infty} x[n]h[k-n]$$

as required.

4.3.1 Causal Discrete Time LTI Systems

In Section 2.2.1 we saw that the impulse response $h(t)$ of a continuous-time **causal** LTI system has the property that

$$h(t) = 0, \qquad \text{for all } t < 0.$$

A precisely analogous result holds in the discrete-time case as well: if $h[k]$ is the impulse response of a discrete-time **causal** LTI system then

$$h[k] = 0, \qquad \text{for all } k < 0. \tag{4.5}$$

We establish this statement in exactly the same way as for the continuous-time case. Thus, fix some integer $k_0 < 0$, let $x_1[k]$ be the zero input signal, and let $y_1[k]$ be the corresponding system response. Then it follows from Remark 1.2.7 that $y_1[k]$ is the zero signal, so that in particular we have

$$y_1[k_0] = 0. \tag{4.6}$$

Next, let $x_2[k] = \delta[k]$ be another input signal, with corresponding output signal $y_2[k]$. Since $\delta[k] = 0$ for all $k < 0$, we see that $x_2[k] = \delta[k] = 0$ for all $k \leq k_0$, so that

$$x_1[k] = x_2[k], \qquad \text{for all } k \leq k_0.$$

It follows from this fact, together with causality (recall Definition 1.2.16) and (4.6) that

$$y_2[k_0] = y_1[k_0] = 0.$$

But $y_2[k]$ is just the impulse response $h[k]$, so we see that

$$h[k_0] = 0.$$

Thus, by the arbitrary choice of $k_0 < 0$ we get that (4.5) holds.

4.3.2 Discrete Time Convolution Identities

The sum which occurs in (4.3) is called a **convolution sum**, and is an obvious discrete-time analog of the convolution integrals which were studied in Section 2.2.2. Adapting the notation of Section 2.2.2, for two discrete-time signal $x[k]$ and $y[k]$ we will write

$$(x * y)[k] \triangleq \sum_{n=-\infty}^{\infty} x[n]y[k-n], \tag{4.7}$$

to denote the **convolution** of $x[k]$ with $y[k]$. Using this notation we can rewrite (4.3) in abbreviated form as follows:

$$y[k] = (x * h)(k).$$

Remark 4.3.7 The symbol $*$ is being used for convolution in both the continuous-time and discrete-time settings. This should cause no confusion, as it will always be clear from the context when we have continuous-time or discrete-time convolution in mind.

We next state analogs of the rules of commutativity, distributivity and associativity which were given earlier for continuous-time convolution: For signals $x[k]$, $y[k]$ and $z[k]$ we have

commutativity: $(x * y)[k] = (y * x)[k]$.

distributivity: $(x * (y + z))[k] = (x * y)[k] + (x * z)[k]$.

associativity: $(x * (y * z))[k] = ((x * y) * z)[k]$.

In Section 2.2.2 we saw how to use indicator functions as a tool for evaluating convolution integrals. The idea was to reduce this problem to the evaluation of a series of ordinary integrals, using the indicator functions to determine the correct limits of integration. One can apply a precisely analogous approach to evaluate convolution sums, using indicator functions to reduce the problem to the evaluation of a series of ordinary summations. However, the practical utility of this method is rather limited by the fact that there are very few rules for ordinary summations, in contrast to the numerous rules available from calculus for evaluating ordinary integrals. For this reason we shall not develop this technique for evaluating convolution sums.

4.4 Linear Difference Equations

Some of the most important examples of discrete-time systems arise from **linear difference equations** with constant coefficients, which have the general form

$$\begin{aligned} y[k+n] + a_{n-1}y[k+n-1] + a_{n-2}y[k+n-2] \ldots + a_1y[k+1] + a_0y[k] \\ = b_mx[k+m] + b_{m-1}x[k+m-1] + b_{m-2}x[k+m-2] \ldots + b_1x[k+1] + b_0x[k]. \end{aligned} \tag{4.8}$$

Here n is a positive integer called the **order** of the equation, m is a nonnegative integer, and $b_m \neq 0$. In this equation we shall regard $x[k]$ as the input signal and $y[k]$ as the output signal. The equation (4.8) is a discrete-time analogue of the constant-coefficient linear differential equation

$$\begin{aligned} \frac{d^ny(t)}{d^nt} + a_{n-1}\frac{d^{n-1}y(t)}{d^{n-1}t} \ldots + a_1\frac{dy(t)}{dt} + a_0y(t) \\ = b_m\frac{d^mx(t)}{d^mt} + b_{m-1}\frac{d^{m-1}x(t)}{d^{m-1}t} \ldots + b_1\frac{dx(t)}{dt} + b_0x(t), \end{aligned} \tag{4.9}$$

which was studied in Section 2.3, and is important for the same reason that (4.9) is important for continuous-time systems: many discrete-time systems are characterized by equations of just this form. Indeed, in Example 4.1.1 we saw a system of genuine engineering importance which is described by an equation having the form of (4.8) (the first-order digital filter). It is therefore important to understand the general properties of the difference equation (4.8) in much the same way that we learned about the differential equation (4.9) in Section 2.3.

Remark 4.4.1 Suppose that $m > n$ in (4.8). Then it follows at once that the output value $y[k+n]$ is determined by the input value $x[k+m]$ (as well as by the earlier input values $x[k+m-1], \ldots, x[k]$). But $k+m > k+n$ so that the input value $x[k+m]$ occurs **later than** the output $y[k+n]$ which is determined by it. Put another way, the output signal anticipates *future* values of the input signal, making the system described by (4.8) a **non-causal** system. Such systems are of little engineering interest, and from now on we shall always suppose that $m \leq n$ in (4.8).

We used the D-operator notation given by (2.41) as a device for simplifying the notation needed to write out the differential equation (2.36). In much the same way we can reduce the notation required for (4.8) by use of the so-called E-operator defined as follows:

$$E^ry[k] \triangleq y[k+r], \qquad E^rx[k] \triangleq x[k+r], \tag{4.10}$$

for nonnegative integers r. Analogously with (2.45) define the following polynomials in E:

$$Q(E) \triangleq E^n + a_{n-1}E^{n-1} + \ldots + a_1 E + a_0, \tag{4.11}$$

$$P(E) \triangleq b_m E^m + b_{m-1}E^{m-1} + \ldots b_1 E + b_0. \tag{4.12}$$

With this notation we have

$$Q(E)y[k] = y[k+n] + a_{n-1}y[k+n-1]\ldots + a_1 y[k+1] + a_0,$$

and similarly for $P(E)x[k]$. Thus (4.8) can be written in compact form as

$$Q(E)y[k] = P(E)x[k], \tag{4.13}$$

and in future it will always be written in this abbreviated way.

In Section 2.3.1 we saw that, in order to regard

$$Q(D)y(t) = P(D)x(t)$$

as a system with input signal $x(t)$ and output signal $y(t)$, we had to ensure that the input signal gives rise to a *uniquely defined* output signal, and this was achieved by imposing auxilary conditions on $y(t)$ and its first $(n-1)$-derivatives at the instant $0-$. An exactly analogous situation holds for the difference equation (4.13): to get a *uniquely defined* output signal $y[k]$ for a given input signal $x[k]$ we must also impose auxiliary conditions. What form should these auxiliary conditions take? To see what is involved here note that we can rewrite (4.8) as follows:

$$\begin{aligned} y[k] &= -a_{n-1}y[k-1] - a_{n-2}y[k-2]\ldots - a_1 y[k-n+1] - a_0 y[k-n] \\ &+ b_m x[k-n+m]\ldots + b_1 x[k-n+1] + b_0 x[k-n]. \end{aligned} \tag{4.14}$$

Thus, in order to compute $y[0]$, in addition to the values of the input signal we must also be given the values of $y[-1], y[-2], \ldots, y[-n]$. We shall regard these as the "initial conditions" for the system (4.13), so that the full description of this system becomes

$$\begin{cases} Q(E)y[k] = P(E)x[k], \\ y[-1] = \alpha_0, \quad y[-2] = \alpha_1, \ldots, \quad y[-n] = \alpha_{n-1}, \end{cases} \tag{4.15}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ are given constants. We are now going to briefly study the response of this system, guided by the notions of zero-input response and zero-state response that were formulated in the continuous-time context in Section 2.3.2. In that section we confined ourselves to input signals $x(t)$ with $x(t) = 0$ for all $t < 0$. Analogously, we shall from now on suppose that the input signals for (4.15) are such that

$$x[k] = 0, \qquad\qquad \text{for all } k < 0.$$

Again analogously with Section 2.3.2, let $y_{zi}[k]$ be the response of (4.15) to the zero input signal, that is, the response of the system

$$\begin{cases} Q(E)y[k] = 0, \\ y[-1] = \alpha_0, \quad y[-2] = \alpha_1, \ldots, \quad y[-n] = \alpha_{n-1}, \end{cases} \tag{4.16}$$

and let $y_{zs}[k]$ be the response of the system

$$\begin{cases} Q(E)y[k] = P(E)x[k], \\ y[-1] = 0, \quad y[-2] = 0, \ldots, \quad y[-n] = 0. \end{cases} \tag{4.17}$$

We call the signal $y_{zi}[k]$ the **zero-input response** of (4.15), and call the signal $y_{zs}[k]$ the **zero-state response** of the system (4.15). Notice that, exactly as in Section 2.3.2, the zero-input response depends **only** on the initial values $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ and has nothing to do with the input signal, while the zero-state response depends **only** on the input signal and has nothing to do with the initial values. Put

$$y[k] \triangleq y_{zi}[k] + y_{zs}[k]. \tag{4.18}$$

By the obvious linearity of the operator E (see (4.10)) we see that the signal $y[k]$ given by (4.18) is indeed the response of (4.15).

At this point, we could proceed by direct analogy with Sections 2.3.3 and 2.3.4 to develop direct methods for determining the zero-input and zero-state response, and indeed it is quite possible to do this. However, we saw in the continuous-time context that the methods of Sections 2.3.3 and 2.3.4 are quite uncompetitive compared with those furnished by the powerful method of Laplace transforms . A similar situation holds in the discrete-time case: the obvious discrete-time analogs of the techniques of Sections 2.3.3 and 2.3.4 turn out to be uncompetitive compared with those provided by the discrete-time analog of the Laplace transform , namely the so-called z-transform. Later in this chapter we shall learn the method of z-transforms, and so we shall put off the actual determination of zero-input and zero-state response until this technique is available to us.

4.4.1 Discrete Time Stability

Consider the following discrete time system, where the input signal $x[k]$ is such that $x[k] = 0$ for all $k < 0$.

$$\begin{cases} Q(E)y[k] = P(E)x[k], \\ y[-1] = \alpha_0, \quad y[-2] = \alpha_1, \ldots, \quad y[-n] = \alpha_{n-1}, \end{cases} \tag{4.19}$$

As with the continuous time case, the *stability* of the system concerns the behaviour of the output $y[k]$ as $k \to \infty$. We have seen in Section 4.4 that the response of this system is determined by its *zero-input response* $y_{zi}[k]$ and its *zero-state response* $y_{zs}[k]$. Thus, if we can identify long-run behaviour (as $k \to \infty$) for both the zero-input response $y_{zi}[k]$ and the zero-state response $y_{zs}[k]$, we will then be able to transfer this into knowledge about the long-run behaviour of the output $y[k]$, since

$$y[k] = y_{zi}[k] + y_{zs}[k].$$

Motivated by these considerations, we are going to formulate desirable long-run behaviour (called "stability") for both the zero-input response and the zero-state response of (4.19). We begin with the zero-input response:

Definition 4.4.2 The system (4.19) is called **asymptotically stable** when its zero-input response $y_{zi}[k]$ has the property that

$$\lim_{k \to \infty} y_{zi}[k] = 0$$

for each and every choice of constants $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$.

The intuitive idea of asymptotic stability is the following: when the input signal is the zero signal then the output signal $y[k]$ of (4.19), which is just the zero-input response $y_{zi}[k]$, tends towards the zero value, regardless of the initial conditions α_0, $\alpha_1, \ldots, \alpha_{n-1}$. How can we determine when (4.19) is asymptotically stable? As the next result shows, the essential determining element is the location of the roots λ_i of the polynomial $Q(\lambda)$ in (4.19) (recall Theorem 2.3.5):

Theorem 4.4.3 The system (4.19) is asymptotically stable if and only if

$$|\lambda_i| < 1, \qquad \text{for all } r = 1, 2, \ldots, r.$$

Remark 4.4.4 The theorem holds regardless of whether or not the roots of $Q(\lambda)$ are repeated.

Next, consider the long run behaviour of the zero-state response. As in Section 4.4 we are only interested in input signals which are such that $x[k] = 0$ for all $k < 0$. The desirable long-run behaviour that we want to define precisely is that a *bounded* input signal $x[k]$ should give rise to only a *bounded* zero-state response. To do this, we must first make the notion of a bounded signal precise in the discrete time context:

Definition 4.4.5 A signal $x[k]$ with $x[k] = 0$ for all $k < 0$ is called *bounded* when there is some *finite* number $B > 0$ such that

$$|x[k]| < B, \qquad \text{for all } k \geq 0.$$

Now we can formulate stability of the zero-state response:

Definition 4.4.6 The system (4.19) is called **bounded input bounded output** or *BIBO* stable when every bounded input signal causes a bounded zero-state response $y_{zs}[k]$.

Theorem 4.4.7 The system (4.19) is BIBO stable if and only if ~~$m \leq n$ (see (4.8)) and~~

$$|\lambda_i| < 1, \qquad \text{for all } r = 1, 2, \ldots, r.$$

4.5 The Method of Z-Transforms

In this section we study the so-called z-transform, a tool which enables us to analyze discrete time linear systems in much the same way that Laplace transforms help us to analyze continuous-time linear systems.

4.5.1 Definition of the z-Transform

Let $x[k]$ be a discrete-time signal. For each complex number z put

$$X(z) \triangleq \sum_{k=0}^{\infty} x[k] z^{-k}, \tag{4.20}$$

provided the sum on the right exists. The resulting complex-valued function of the complex variable z is called the **z-transform** of the signal $x[k]$, and is denoted variously by

$$X(z), \qquad \mathcal{Z}\{x\}(z), \qquad \text{or} \qquad \mathcal{Z}\{x[k]\}(z).$$

Remark 4.5.1 The following fact is useful for the computation of z-transforms: Suppose that ρ is a given complex number. If $|\rho| < 1$ then the power series $\sum_{k=0}^{\infty} \rho^k$ **converges** and is given by

$$\sum_{k=0}^{\infty} \rho^k = \frac{1}{1-\rho}.$$

On the other hand, if $|\rho| \geq 1$ then the power series $\sum_{k=0}^{\infty} \rho^k$ makes no sense and is **undefined.**

Example 4.5.2 Suppose a signal $x[k]$ is given by

$$x[k] = \alpha^k,$$

for a complex constant α. Then the z-transform of the signal is

$$X(z) = \sum_{k=0}^{\infty} \alpha^k z^{-k},$$

for all complex numbers z for which the sum on the right converges. We can rewrite this sum as

$$X(z) = \sum_{k=0}^{\infty} (\alpha z^{-1})^k,$$

and it is evident from Remark 4.5.1 that this sum converges if and only if

$$|\alpha z^{-1}| < 1, \qquad \text{or equivalently} \qquad |\alpha| < |z|,$$

in which case

$$X(z) = \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}.$$

To conclude, we see that the z-transform of the signal $x[k] = \alpha^k$ exists and is given by

$$X(z) = \frac{z}{z-\alpha},$$

for all complex z with $|z| > |\alpha|$, but is **undefined** for all other complex z. The set of complex numbers z for which $X(z)$ is defined is called the **region of existence** of the z-transform . See Fig. 4.8.

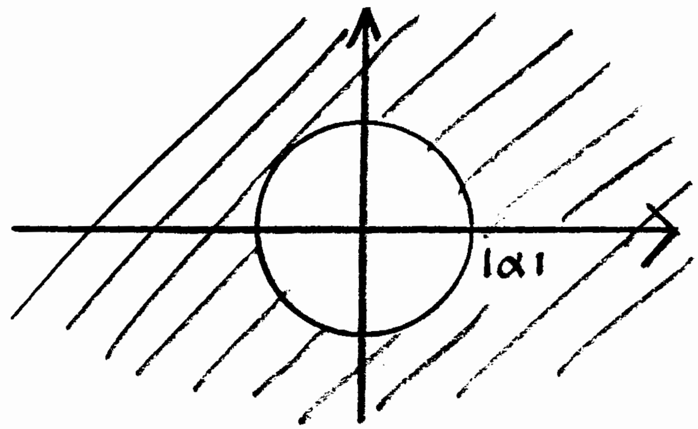

Fig. 4.8

The phenomenon is very similar to that seen in Example 3.1.4 in connection with Laplace transforms which likewise have a region of existence. In exactly the same way that we never had to be concerned about regions of existence when *using* Laplace transforms, so in the same way we will be able to use z-transforms without being concerned about their regions of existence.

4.5.2 Properties of z-Transforms

In this section we list the most important properties of z-transforms. These properties are easy to establish on the basis of the definition of z-transforms, and are partial (but not complete) analogs of the basic properties of Laplace transforms.

(I) LINEARITY: Suppose that $x_1[k]$, $x_2[k]$, are signals and c_1, c_2, are complex constants. Then

$$\mathcal{Z}\{c_1x_1 + c_2x_2\}(z) = c_1\mathcal{Z}\{x_1\} + c_2\mathcal{Z}\{x_2\}(z).$$

Example 4.5.3 Determine $\mathcal{Z}\{x[k]\}(z)$ for $x[k]$ given by

$$x[k] = \sin(k\omega T), \qquad \text{for } T > 0 \text{ a constant.}$$

We know that

$$x[k] = \frac{e^{jk\omega T} - e^{-jk\omega T}}{2j}.$$

From Example 4.5.2 we have

$$\mathcal{Z}\left\{\alpha^k\right\}(z) = \frac{z}{z-\alpha}.$$

Taking $\alpha = e^{j\omega T}$ gives

$$\mathcal{Z}\left\{e^{jk\omega T}\right\}(z) = \frac{z}{z - e^{j\omega T}},$$

and taking $\alpha = e^{-j\omega T}$ gives

$$\mathcal{Z}\left\{e^{-jk\omega T}\right\}(z) = \frac{z}{z - e^{-j\omega T}}.$$

Then, by linearity, we get

$$
\begin{aligned}
\mathcal{Z}\{x[k]\}(z) &= \frac{1}{2j}\left[\frac{z}{z-e^{j\omega T}} - \frac{z}{z-e^{-j\omega T}}\right] \\
&= \frac{[e^{j\omega T}-e^{-j\omega T}]z}{2j[z^2-(e^{j\omega T}+e^{-j\omega T})z+1]} \\
&= \frac{[\sin(\omega T)]z}{z^2-2z\cos(\omega T)+1}.
\end{aligned}
$$

(II) TIME ADVANCE: This property is a partial analog of the Laplace transform of the derivative of a continuous-time signal. Let $x[k]$ be a discrete-time signal with Laplace transform $X(z)$. Put

$$y[k] \triangleq x[k+1], \qquad \text{for all } k.$$

We call the signal $y[k]$ defined in this way the **one-step advance** of the given signal $x[k]$. The z-transform of $y[k]$ is given by

$$\mathcal{Z}\{y[k]\}(z) = zX(z) - zx[0],$$

which is usually written as

$$\mathcal{Z}\big\{x[k+1]\big\}(z) = zX(z) - zx[0].$$

More generally for a fixed *positive* integer N put

$$y[k] \triangleq x[k+N], \qquad \text{for all } k.$$

We call the signal $y[k]$ defined in this way the N**-step advance** of the given signal $x[k]$. See Fig. 4.9.

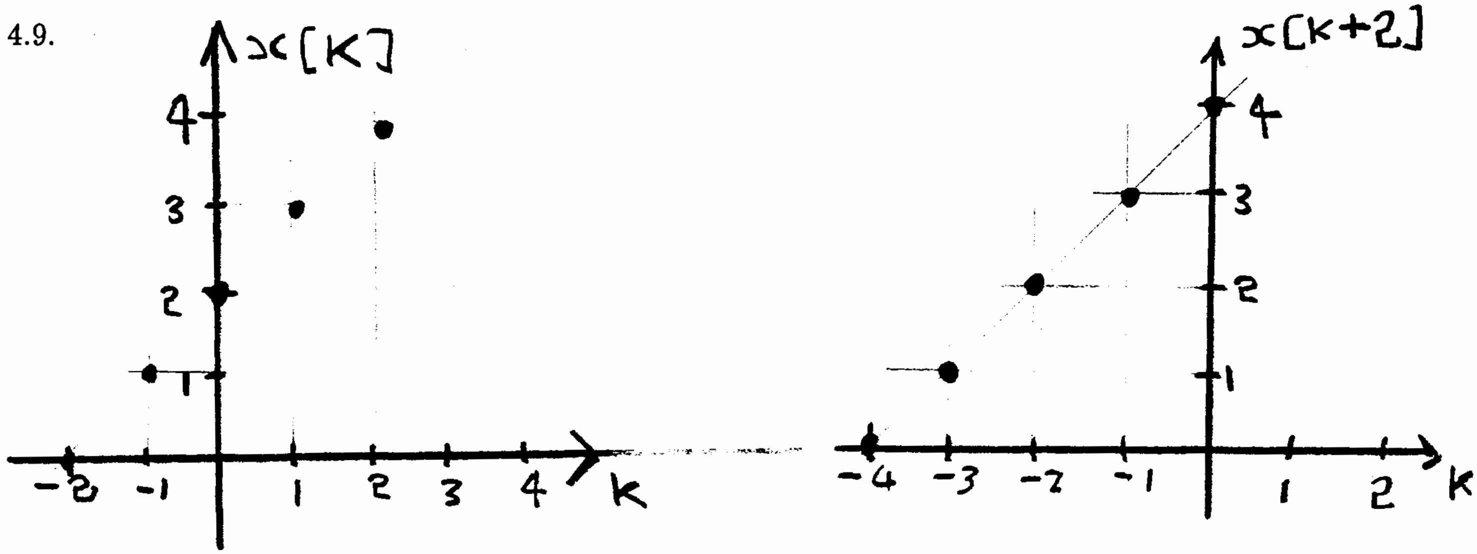

Fig. 4.9

The z-transform of $y[k]$ is given by

$$\mathcal{Z}\{y[k]\}(z) = z^N X(z) - z^N x[0] - z^{N-1}x[1] - \ldots - z^2 x[N-2] - zx[N-1],$$

which is usually written as

$$\mathcal{Z}\{x[k+N]\}(z) = z^N X(z) - z^N x[0] - z^{N-1}x[1] - \ldots - z^2 x[N-2] - zx[N-1].$$

(III) TIME DELAY: This property supplements (II) and is particularly useful for solving linear difference equations with constant coeffficients: Let $x[k]$ be a given signal with z-transform $X(z)$, and define

$$y[k] \triangleq x[k-N], \qquad \text{for all } k.$$

We call the signal $y[k]$ defined in this way the N-**step delay** of the given signal $x[k]$. See Fig. 4.10.

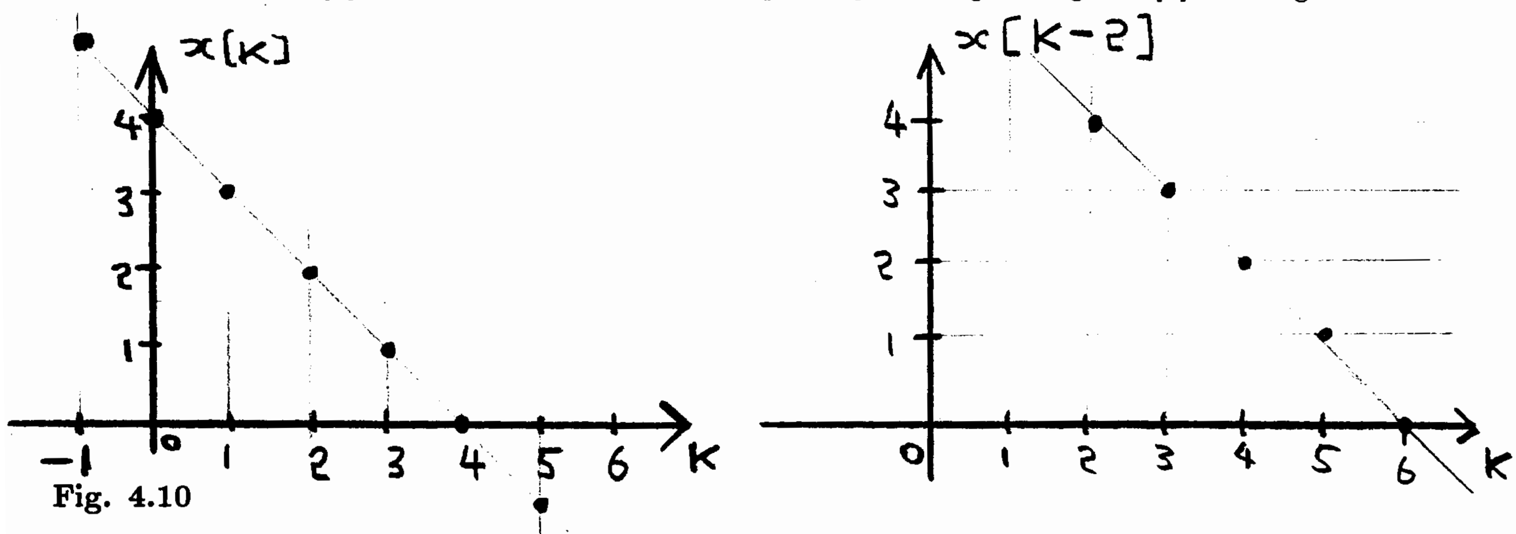

Fig. 4.10

The z-transform of $y[k]$ is given by

$$\mathcal{Z}\{y[k]\}(z) = z^{-N} X(z) + \left\{x[-N] + z^{-1}x[1-N] + z^{-2}x[2-N] + \ldots + z^{1-N}x[-1]\right\},$$

which is usually written as

$$\mathcal{Z}\{x[k-N]\}(z) = z^{-N} X(z) + \sum_{j=0}^{N-1} z^{-j} x[j-N].$$

(IV) MULTIPLICATION BY α^k: This property is a partial analog of the exponential shift property of Laplace transforms. Let α be a complex constant and $x[k]$ be a signal with z-transform $X(z)$. Then

$$\mathcal{Z}\left\{\alpha^k x[k]\right\}(z) = X\left(\frac{z}{\alpha}\right).$$

Example 4.5.4 Determine $\mathcal{Z}\{\alpha^k \sin(k\omega T)\}(z)$. From Example 4.5.3 we have that

$$X(z) \triangleq \mathcal{Z}\{\sin(k\omega T)\}(z) \equiv \frac{z\sin(\omega T)}{z^2 - 2z\cos(\omega T) + 1}.$$

Then, by Property (IV) we have

$$\begin{aligned}\mathcal{Z}\{\alpha^k \sin(k\omega T)\}(z) &= X\left(\frac{z}{\alpha}\right) \\ &= \frac{(z/\alpha)\sin(\omega T)}{(z/\alpha)^2 - 2(z/\alpha)\cos(\omega T) + 1} \\ &= \frac{\alpha z\sin(\omega T)}{z^2 - 2\alpha z\cos(\omega T) + \alpha^2}.\end{aligned}$$

(V) CONVOLUTION: Suppose that $x[k]$ and $y[k]$ are signals with z-transforms $X(z)$ and $Y(z)$ respectively. Then

$$\mathcal{Z}\{x * y\}(z) = X(z)Y(z).$$

(VI) FINAL VALUE THEOREM: Suppose that $x[k]$ is a signal with z-transform $X(z)$, and tends to a *finite* constant as $k \to \infty$. Then

$$\lim_{k\to\infty} x[k] = \lim_{z\to 1}(z-1)X(z).$$

(VII) INITIAL VALUE THEOREM: Suppose that $x[k]$ is a signal with z-transform $X(z)$. Then

$$x[0] = \lim_{z\to\infty} X(z).$$

(VIII) MULTIPLICATION BY k: Suppose that $x[k]$ is a signal with z-transform $X(z)$. Then

$$\mathcal{Z}\{kx[k]\}(z) = -z\frac{dX(z)}{dz}.$$

Example 4.5.5 Let α be a complex number. Determine

(i) $\mathcal{Z}\{k\alpha^k\}(z)$;

(ii) $\mathcal{Z}\{k^2\alpha^k\}(z)$.

(i) Put $x[k] \triangleq \alpha^k$. From Example 4.5.2 we know that its z-transform is given by

$$X(z) = \frac{z}{z-\alpha}.$$

Thus, by (VIII),

$$\begin{aligned}\mathcal{Z}\{k\alpha^k\}(z) &= -z\frac{dX(z)}{dz} \\ &= -z\frac{d}{dz}\left(\frac{z}{z-\alpha}\right) \\ &= \frac{\alpha z}{(z-\alpha)^2}.\end{aligned}$$

(ii) Put

$$y[k] \triangleq k\alpha^k,$$

so that (i) gives

$$Y(z) \triangleq \mathcal{Z}\{y[k]\}(z) = \frac{\alpha z}{(z-\alpha)^2}.$$

Then of course

$$\mathcal{Z}\{k^2\alpha^k\}(z) = \mathcal{Z}\{ky[k]\}(z),$$

hence (VIII) gives

$$\begin{aligned}\mathcal{Z}\{k^2\alpha^k\}(z) &= -z\frac{dY(z)}{dz} \\ &= -z\frac{d}{dz}\left(\frac{\alpha z}{(z-\alpha)^2}\right) \\ &= \frac{\alpha z(z+\alpha)}{(z-\alpha)^3}.\end{aligned}$$

Remark 4.5.6 From Example 4.5.5(i) with $\alpha \triangleq 1$, we find

$$\mathcal{Z}\{k\} = \frac{z}{(z-1)^2}.$$

This is the z-transform if the so-called *ramp sequence.* Similarly, from Example 4.5.5(ii), we have

$$\mathcal{Z}\{k^2\}(z) = \frac{z(z+1)}{(z-1)^3},$$

which is the z-transform of the *parabolic sequence.*

4.5.3 Inverse z-Transforms

The problem of inverse z-transforms is similar to the problem of inverse Laplace transforms, namely given a complex-valued function $X(z)$ of the complex variable z, compute a discrete-time signal $x[k]$ such that

$$\mathcal{Z}\{x[k]\} = X(z).$$

This signal is called the **inverse z-transform** of $X(z)$ and is denoted by

$$\mathcal{Z}^{-1}\{X(z)\}[k].$$

Remark 4.5.7 From the definition of the z-transform, the only part of a signal $x[k]$ which "goes into" the computation of its z-transform $X(z)$ is that for $k \geq 0$. It follows that if two signals $x[k]$ and $y[k]$ agree for all $k \geq 0$, but are distinct for $k < 0$, then these signals will have identical z-transforms. Thus, when we compute the inverse z-transform $x[k]$ of a function $X(z)$ we may as well take $x[k]$ to be the simplest possible signal for negative values of k, and *by convention* we will always take

$$x[k] = 0 \qquad \text{for all } k < 0,$$

for the inverse z-transform of a function $X(z)$.

By analogy with inverse Laplace transforms, we shall determine inverse z-transforms using the method of table look-up. We have already computed most of the important z-transforms in the examples of the previous section. However, there is one more z-transform that we need in order to have a really useful table. Recall from Section 3.4 that the Laplace transform

$$\mathcal{L}\left\{\frac{1}{(n-1)!}t^{n-1}e^{\alpha t}\right\}(s) = \frac{1}{(s-\alpha)^n}$$

is often useful for determining inverse Laplace transforms by the method of partial fractions. In order to use partial fractions to find inverse z-transforms we will need an analogous result for z-transforms. To state this result we introduce the following notation: put

$$(k)_0 \triangleq 1, \qquad \text{for all } k = 0, 1, 2, 3, \ldots$$

and, for a fixed integer $n \geq 1$, put

$$(k)_n = k(k-1)(k-2)\ldots(k-n+1), \qquad \text{for all } k = 0, 1, 2, 3, \ldots$$

Thus, for example,

$$\begin{aligned}(k)_1 &= k,\\ (k)_2 &= k(k-1),\\ (k)_3 &= k(k-1)(k-2),\end{aligned}$$

and so on. Then, for complex constant α and fixed integer $n = 0, 1, 2, \ldots$, a laborious computation shows

$$\mathcal{Z}\left\{\frac{(k)_{n-1}\alpha^{k-n+1}}{(n-1)!}\right\}(z) = \frac{z}{(z-\alpha)^n}.$$

We can now make up the following table of z-transforms:

Signal $x[k]$ for $k \geq 0$	z-transform $X(z)$
$\delta[k]$	1
$\delta_n[k]$, $n = 1, 2, \ldots$	z^{-n}
$u[k]$	$z/(z-1)$
α^k, α complex	$z/(z-\alpha)$
$(k)_{n-1}\alpha^{k-n+1}/(n-1)!$ α complex, $n = 0, 1, 2, \ldots$	$z/(z-\alpha)^n$
$\sin(k\omega T)$	$[z\sin(\omega T)]/[z^2 - 2z\cos(\omega T) + 1]$
$\cos(k\omega T)$	$[z^2 - z\cos(\omega T)]/[z^2 - 2z\cos(\omega T) + 1]$
$\alpha^k \sin(k\omega T)$	$[\alpha z\sin(\omega T)]/[z^2 - 2\alpha z\cos(\omega T) + \alpha^2]$
$\alpha^k \cos(k\omega T)$	$[z^2 - \alpha z\cos(\omega T)]/[z^2 - 2\alpha z\cos(\omega T) + \alpha^2]$

To determine the inverse z-transform of a function $X(z)$ using the method of table look-up we must search through the table for a signal $x[k]$ whose z-transform , in the second column of the table, is equal to the given function $X(z)$. For this method to work we must be sure that one never gets *different* signals with the same z-transform. To highlight the difficulty if this were to be the case, suppose we must determine the inverse z-transform of

$$X(z) = \frac{z}{z-2}.$$

By consulting the table we see that a signal having $X(z)$ for its z-transform is

$$x[k] = 2^k.$$

If there existed another signal $y[k]$ different from $x[k]$ but with the same z-transform $X(z)$, then we could not be sure if the inverse z-transform of $X(z)$ that we want is the signal $x[k] = 2^k$ given by the table, or the "other signal" $y[k]$. The following result clears away all such doubts and says that this type of situation can never happen:

Theorem 4.5.8 Suppose that $x[k]$ is a signal with z-transform $X(z)$. If $y[k]$ is a signal with the same z-transform $X(z)$ then the signals $x[k]$ and $y[k]$ are identical for all $k \geq 0$:

$$x[k] = y[k], \qquad \text{for all } k \geq 0.$$

In view of this theorem we can be sure that the signal $x[k]$ in the same row of the table as a given function $X(z)$ is **the only** signal whose z-transform is $X(z)$.

Exactly as for inverse Laplace transforms the functions $X(z)$ whose inverse z-transforms we usually want are rational functions in z. We now turn to the problem of finding the inverse z-transform of a rational function using the method of partial fractions.

4.5.4 Partial Fractions and Inverse z-Transforms

We look at the problem of determining the inverse z-transform of the rational function

$$X(z) = \frac{N(z)}{D(z)},$$

assuming to begin with that

$$\deg(N) \leq \deg(D).$$

The basic approach is very similar to that used in Section 3.4 for determining inverse Laplace transforms; make a partial fraction expansion using the Heaviside expansion theorem, to get a sum of terms of the form occurring in the second column of the fourth and fifth rows of the table of z-transforms, then use the table to take individual inverse z-transforms . The one possible difficulty is that the functions of z appearing in the second column of the fourth and fifth rows of the table involve the variable z occurring in the numerator. This suggests that we should instead take a partial fraction expansion, not of the rational function $X(z)$ but rather of the rational function

$$\frac{X(z)}{z} = \frac{N(z)}{zD(z)}.$$

An example illustrates the approach:

Example 4.5.9 Determine the inverse z-transform of the rational function

$$X(z) = \frac{z^2(z+1)}{(z+2)^3}.$$

We begin by taking a partial fraction expansion of

$$Y(z) \triangleq \frac{X(z)}{z},$$

namely

$$Y(z) = \frac{z(z+1)}{(z+2)^3}.$$

We now apply Theorem 3.4.1 with

$$N(z) = z(z+1), \qquad D(z) = (z+2)^3,$$

so that

$$l = 1, \qquad p_1 = -2, \qquad m_1 = 3.$$

By Theorem 3.4.1 we can write

$$Y(z) = \left[\frac{r_{11}}{(z+2)} + \frac{r_{12}}{(z+2)^2} + \frac{r_{13}}{(z+2)^2}\right],$$

where the constants r_{ij} are as follows:

$$\begin{aligned}
r_{13} &= \frac{1}{(m_1-3)!}\frac{d^{m_1-3}}{dz^{m_1-3}}\left[Y(z)(z+2)^{m_1}\right]\Big|_{z=-2} \\
&= \left[Y(z)(z+2)^3\right]\Big|_{z=-2} \\
&= \left[z(z+1)\right]\Big|_{z=-2} = 2.
\end{aligned}$$

$$\begin{aligned}
r_{12} &= \frac{1}{(m_1-2)!}\frac{d^{m_1-2}}{dz^{m_1-2}}\left[Y(z)(z+2)^{m_1}\right]\Big|_{z=-2} \\
&= \frac{d}{dz}\left[Y(z)(z+2)^3\right]\Big|_{z=-2} \\
&= \frac{d}{dz}\left[z(z+1)\right]\Big|_{z=-2} \\
&= -3.
\end{aligned}$$

$$\begin{aligned}
r_{11} &= \frac{1}{(m_1-1)!}\frac{d^{m_1-1}}{dz^{m_1-1}}\left[Y(z)(z+2)^{m_1}\right]\Big|_{z=-2} \\
&= \frac{1}{2}\frac{d^2}{dz^2}\left[Y(z)(z+2)^3\right]\Big|_{z=-2} \\
&= \frac{1}{2}\frac{d^2}{dz^2}\left[z(z+1)\right]\Big|_{z=-2} \\
&= 1.
\end{aligned}$$

Thus, we have

$$Y(z) = \left[\frac{1}{(z+2)} - \frac{3}{(z+2)^2} + \frac{2}{(z+2)^2}\right],$$

or

$$X(z) = \left[\frac{z}{(z+2)} - \frac{3z}{(z+2)^2} + \frac{2z}{(z+2)^2}\right].$$

Use the table to get inverse z-transforms for each term on the right side:

$$\mathcal{Z}^{-1}\left\{\frac{z}{(z+2)}\right\}[k] = (-2)^k,$$

$$\begin{aligned}\mathcal{Z}^{-1}\left\{\frac{z}{(z+2)^2}\right\}[k] &= (k)_1(-2)^{k-1}\\ &= \frac{-k}{2}(-2)^k,\end{aligned}$$

$$\begin{aligned}\mathcal{Z}^{-1}\left\{\frac{z}{(z+2)^3}\right\}[k] &= \frac{(k)_2}{2!}(-2)^{k-2}\\ &= \frac{k(k-1)}{8}(-2)^k.\end{aligned}$$

Putting these into the expansion for $X(z)$ gives

$$\begin{aligned}x[k] &= (-2)^k + \frac{3k}{2}(-2)^k + \frac{k^2-k}{4}(-2)^k\\ &= (1 + 1.25k + 0.25k^2)(-2)^k.\end{aligned}$$

4.6 Zero-Input and Zero-State Response by z-Transforms

In Section 4.4 we looked at the response $y[k]$ of the system

$$\left\{ \begin{array}{l} Q(E)y[k] = P(E)x[k], \\ y[-1] = \alpha_0, \quad y[-2] = \alpha_1, \ldots, \quad y[-n] = \alpha_{n-1}, \end{array} \right. \tag{4.21}$$

to an input signal $x[k]$ such that $x[k] = 0$ for all $k < 0$, and saw that $y[k]$ is the sum of a *zero-input* response $y_{zi}[k]$ and a *zero-state* response $y_{zs}[k]$ (see (4.18)). We shall now see how z-transforms provide an extremely powerful and simple-to-use technique for determining the zero-input and zero-state response. The approach is very similar to that seen in Section 3.5 for the evaluation of zero-input and zero-state response by Laplace transforms in continuous-time systems, and is best illustrated by an example:

Example 4.6.1 Use z-transforms to determine the zero-input and zero-state response of the system

$$\left\{ \begin{array}{l} 2y[k+2] + 3y[k+1] + y[k] = x[k+2] + x[k+1] - x[k], \\ y[-1] = 2, y[-2] = -1, \end{array} \right. \tag{4.22}$$

where $x[k]$ is the discrete-time unit step function.

The first step is to rewrite the difference equation of (4.22) with k replaced by $k-2$ so that

$$2y[k] + 3y[k-1] + y[k-2] = x[k] + x[k-1] - x[k-2].$$

Now take z-transforms of each side:

$$2\mathcal{Z}\{y[k]\}(z) + 3\mathcal{Z}\{y[k-1]\}(z) + \mathcal{Z}\{y[k-2]\}(z) = \mathcal{Z}\{x[k]\}(z) + \mathcal{Z}\{x[k-1]\}(z) - \mathcal{Z}\{x[k-2]\}(z). \tag{4.23}$$

Now put

$$Y(z) \triangleq \mathcal{Z}\{y[k]\}, \qquad X(z) \triangleq \mathcal{Z}\{x[k]\}.$$

By Property (III) of z-transforms we have

$$\begin{aligned} \mathcal{Z}\{y[k-1]\}(z) &= z^{-1}Y(z) + \{y[-1]\}, \\ \mathcal{Z}\{y[k-2]\}(z) &= z^{-2}Y(z) + \{z^{-1}y[-1] + y[-2]\}, \\ \mathcal{Z}\{x[k-1]\}(z) &= z^{-1}X(z) + \{x[-1]\} = z^{-1}X(z), \\ \mathcal{Z}\{x[k-2]\}(z) &= z^{-2}X(z) + \{z^{-1}x[-1] + x[-2]\} = z^{-2}X(z), \end{aligned}$$

where we have used the fact that

$$x[-1] = x[-2] = 0$$

in the last two z-transforms. Substituting these z-transforms into (4.23) then yields

$$2Y(z) + 3\left[z^{-1}Y(z) + y[-1]\right] + \left[z^{-2}Y(z) + z^{-1}y[-1] + y[-2]\right] = X(z) + z^{-1}X(z) - z^{-2}X(z).$$

Rearranging shows that

$$Y(z) = \frac{-\left(3y[-1] + z^{-1}y[-1] + y[-2]\right)}{2 + 3z^{-1} + z^{-2}} + \frac{\left(1 + z^{-1} - z^{-2}\right)X(z)}{2 + 3z^{-1} + z^{-2}}.$$

It is evident that the first term on the right involves only the given auxiliary conditions on $y[-2]$ and $y[-1]$, hence must be the z-transform of the zero-input response, and the second term on the right involves only the input signal $x[k]$, hence must be the z-transform of the zero-state response. Thus we have

$$Y(z) = Y_{zi}(z) + Y_{zs}(z)$$

where, using the given auxiliary conditions, we get

$$\begin{aligned} Y_{zi}(z) &\triangleq \frac{-\left(3y[-1] + z^{-1}y[-1] + y[-2]\right)}{2 + 3z^{-1} + z^{-2}} \\ &= \frac{-\left(3(2) + z^{-1}(2) - 1\right)}{2 + 3z^{-1} + z^{-2}} \\ &= \frac{-\left(5z^2 + 2z\right)}{2z^2 + 3z^1 + 1}, \end{aligned} \tag{4.24}$$

for the zero-input response. Since $x[k] = u[k]$, we see from the table of z-transforms that

$$X(z) = \frac{z}{z-1},$$

so that

$$\begin{aligned} Y_{zs}(z) &\triangleq \frac{\left(1 + z^{-1} - z^{-2}\right)X(z)}{2 + 3z^{-1} + z^{-2}} \\ &= \frac{z\left(z^2 + z - 1\right)}{(z-1)\left(2z^2 + 3z + 1\right)}. \end{aligned}$$

The zero-input response $y_{zi}[k]$ and zero-state response $y_{zs}[k]$ are now easily found by computing the inverse z-transforms.

It is easy to repeat the main steps of this example in the general case. To this end, we first write the linear difference equation in (4.15) in the form

$$
\begin{aligned}
(4.25) \qquad & y[k]+a_{n-1}y[k-1]+a_{n-2}y[k-2]\ldots+a_1y[k-n+1]+a_0y[k-n] \\
&= b_m x[k-n+m]+\ldots+b_1x[k-n+1]+b_0x[k-n].
\end{aligned}
$$

From Property (III) of z-transforms we know

$$
\mathcal{Z}\{y[k-i]\}(z) = z^{-i}Y(z) + \left\{\sum_{j=0}^{i-1} z^{-j}y[j-i]\right\},
$$

for each $i=1,2,\ldots,n$. But the auxiliary conditions tell us that

$$
y[j-i] = \alpha_{i-j-1},
$$

so that we get

$$
\mathcal{Z}\{y[k-i]\}(z) = z^{-i}Y(z) + \left\{\sum_{j=0}^{i-1} z^{-j}\alpha_{i-j-1}\right\},
$$

for each $i=1,2,\ldots,n$. Thus, for $a_n \triangleq 1$,

$$
\begin{aligned}
(4.26) \qquad & \mathcal{Z}\{y[k]+a_{n-1}y[k-1]+a_{n-2}y[k-2]\ldots+a_1y[k-n+1]+a_0y[k-n]\}(z) \\
&= \left\{\textstyle\sum_{i=0}^{n} a_{n-i}z^{-i}\right\}Y(z) + \textstyle\sum_{i=0}^{n} a_{n-i}\left\{\textstyle\sum_{j=0}^{i-1} z^{-j}\alpha_{i-j-1}\right\}.
\end{aligned}
$$

Similarly from Property (III) of z-transforms we have

$$
\mathcal{Z}\{x[k-i]\}(z) = z^{-i}X(z),
$$

using that fact that $x[j]=0$ when $j<0$, so that

$$
(4.27)\,\mathcal{Z}\{b_m x[k-n+m]+\ldots+b_1x[k-n+1]+b_0x[k-n]\}(z) = \left\{\sum_{r=0}^{m} b_{m-r}z^{m-n-r}\right\}X(z).
$$

Combining (4.26) and (4.27)

$$
\left\{\sum_{i=0}^{n} a_{n-i}z^{-i}\right\}Y(z) + \sum_{i=0}^{n} a_{n-i}\left\{\sum_{j=0}^{i-1} z^{-j}\alpha_{i-j-1}\right\} = \left\{\sum_{r=0}^{m} b_{m-r}z^{m-n-r}\right\}X(z),
$$

so that

$$
Y(z) = Y_{zi}(z) + Y_{zs}(z)
$$

for

$$Y_{zi}(z) \triangleq -\frac{\sum_{i=0}^{n} a_{n-i}\left\{\sum_{j=0}^{i-1} z^{-j}\alpha_{i-j-1}\right\}}{\left\{\sum_{i=0}^{n} a_{n-i}z^{-i}\right\}}$$

$$= -\frac{\sum_{i=0}^{n} a_{n-i}\left\{\sum_{j=0}^{i-1} z^{n-j}\alpha_{i-j-1}\right\}}{\left\{\sum_{i=0}^{n} a_{n-i}z^{n-i}\right\}}$$

$$= -\frac{\sum_{i=0}^{n} a_{n-i}\left\{\sum_{j=0}^{i-1} z^{n-j}\alpha_{i-j-1}\right\}}{Q(z)},$$

and

$$Y_{zs}(z) \triangleq \frac{\left\{\sum_{r=0}^{m} b_{m-r}z^{m-n-r}\right\} X(z),}{\left\{\sum_{i=0}^{n} a_{n-i}z^{-i}\right\}} \tag{4.28}$$

$$= \frac{\left\{\sum_{r=0}^{m} b_{m-r}z^{m-r}\right\} X(z),}{\left\{\sum_{i=0}^{n} a_{n-i}z^{n-i}\right\}}$$

$$= \frac{P(z)}{Q(z)}X(z).$$

It is readily apparent that

$$y_{zi}[k] \triangleq \mathcal{Z}^{-1}\{Y_{zi}(z)\}[k], \qquad y_{zs}[k] \triangleq \mathcal{Z}^{-1}\{Y_{zs}(z)\}[k],$$

give respectively the zero-input response and the zero-state response.

The z-transform method also give us a powerful technique for determining the impulse response of the linear system

$$Q(E)y[k] = P(E)x[k].$$

In fact, the impulse response $h[k]$ is defined to be the zero-state response $y_{zs}[k]$ in the special case where the input signal is the impulse function $\delta[k]$. When $x[k]$ is the impulse function then

$$\mathcal{Z}\{x[k]\}(z) = 1,$$

so that (4.28) gives

$$\mathcal{Z}\{h[k]\}(z) = \frac{P(z)}{Q(z)}.$$

Thus the impulse response is given by

$$h[k] = \begin{cases} \mathcal{Z}^{-1}\{P(z)/Q(z)\}[k], & \text{for all } k \geq 0, \\ 0, & \text{for all } k < 0. \end{cases} \tag{4.29}$$

The inverse z-transform $\mathcal{Z}^{-1}\{P(z)/Q(z)\}[k]$ is easily found by partial fractions.

4.7 Discrete Time System Transfer Function

Here we look at the discrete time analog of the notion of a transfer function, that was introduced for continuous-time systems in Section 3.7. Consider the system given by

$$\begin{cases} Q(E)y[k] = P(E)x[k], \\ y[-1] = \alpha_0, \quad y[-2] = \alpha_1, \ldots, \quad y[-n] = \alpha_{n-1}, \end{cases}$$

and let $x[k]$ be an input signal with $x[k] = 0$ for all $k \geq 0$. By analogy with the continuous-time case, we define the **discrete-time transfer function** of the system as the function of z given by the ratio

$$H(z) = \frac{z\text{-transform of the zero-state response } y_{zs}[k] \text{ to the input signal } x[k]}{z\text{-transform of the input signal } x[k]}. \tag{4.30}$$

In view of (4.28) we see that the transfer function is therefore given by

$$H(z) = \frac{P(z)}{Q(z)}.$$

In E&CE-481 it will be seen that the discrete-time transfer function of a system is an indispensable idea for studying *digital* feedback control systems.

The roots of the numerator polynomial $P(z)$ (namely those values of z such that $P(z) = 0$) are called the **zeros** of the transfer function, and the roots of the denominator polynomial $Q(z)$ are called the **poles** of the transfer function.

How does one determine the transfer function of a discrete-time system? From (4.29) we see that if $h[k]$ is the impulse response of the system then $h[k]$ and $H(z)$ are related by

$$\mathcal{Z}\{h[k]\}(z) = H(z),$$

so that, just as in the continuous-time case, the transfer function $H(z)$ is the z-transform of the system impulse response. The nice thing about the transfer function $H(z)$ is that we can use it to determine the zero-state response $y_{zs}[k]$ to **any** input signal $x[k]$ provided that $x[k] = 0$ for all $k < 0$. In fact, this is immediate from the definition (4.30), which shows that

$$Y_{zs}(z) = H(z)X(z),$$

so that the zero-state response is

$$y_{zs}[k] = \mathcal{Z}^{-1}\{H(z)X(z)\}\,[k].$$

Schematically, we shall respresent a system with input signal $x[k]$, output signal $y[k]$ and transfer function $H(z)$ as in Fig. 4.11:

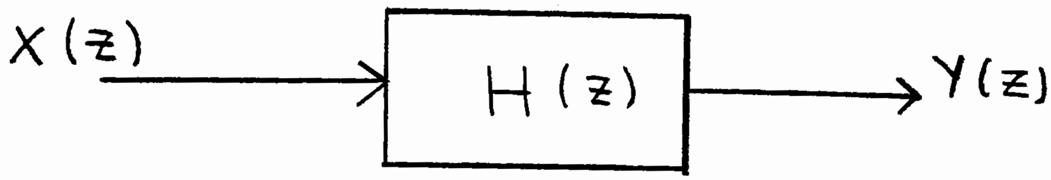

Fig. 4.11

It is evident from Theorems 4.4.3 and 4.4.7 that the information needed to determine asymptotic stability and BIBO stability of a discrete-time system is given by the location of the poles of the transfer function. Indeed, we can state these two results together as follows:

Theorem 4.7.1 *The system (4.21) is asymptotically stable if and only if all poles of* $H(z)$ *have magnitude strictly less than one (that is* < 1*). Moreover the system is BIBO stable if and only if* $\cancel{m \leq n \text{ and}}$ *all poles of the system have magnitude strictly less than one.*

Exactly as in the continuous-time case, there are a number of ways in which we can combine discrete-time systems into larger discrete-time systems namely:

SERIES COMBINATION: Here we take the output of system with transfer function $H_1(z)$ and use it for the input of a second system with transfer function $H_2(z)$, as shown in Fig. 4.12:

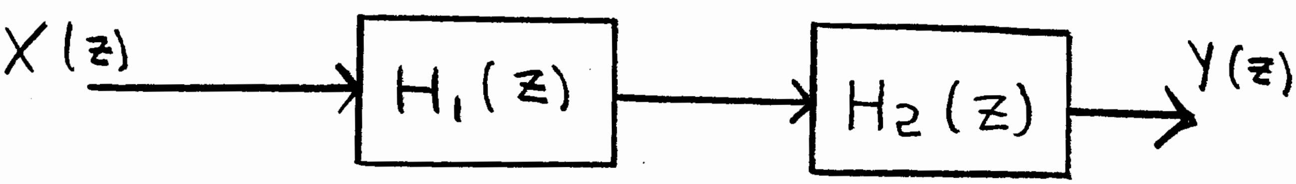

Fig. 4.12

Exactly as in the continuous-time case (but with z replacing s) we see that

$$Y(z) = H_2(z)H_1(z)X(z).$$

Therefore, the equivalent transfer function of the series combination is

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z)H_2(z).$$

PARALLEL COMBINATION: We can put systems with transfer functions $H_1(z)$ and $H_2(z)$ in parallel as follows:

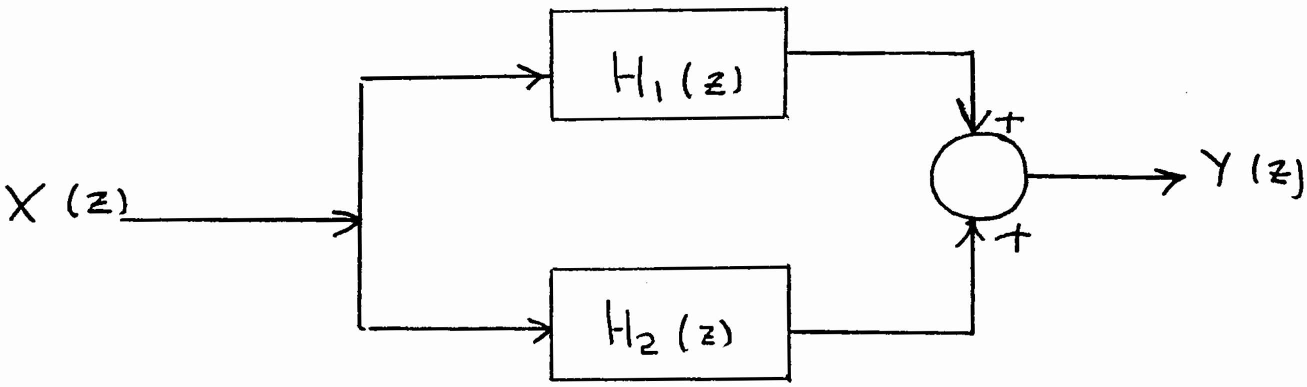

Fig. 4.13

Here $x[k]$ is the input signal for both systems and then the overall output signal $y[k]$ is the sum of the outputs $y_1[k]$ and $y_2[k]$ of the individual systems. Exactly as in the continuous-time case we have

$$Y(z) = [H_1(z) + H_2(z)]X(z),$$

so that the equivalent transfer function $H(z)$ is

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z) + H_2(z).$$

FEEDBACK COMBINATION: We create a feedback combination of discrete-time systems exactly as in the continuous-time case. See Fig. 4.14.

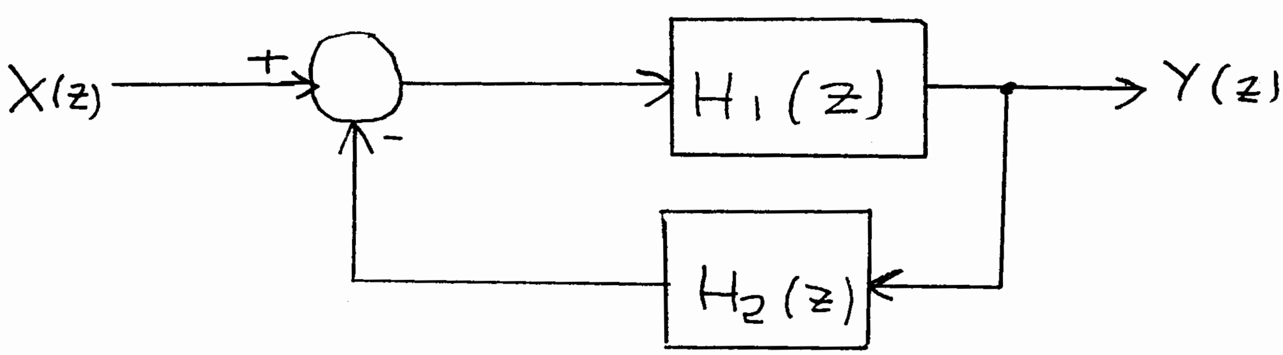

Fig. 4.14

Repeating the algebra that gave us the transfer function of the continuous-time system shown in

Fig. 3.16, but replacing s with z, shows that the transfer function of the system in Fig. 4.14 is given by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 + H_1(z)H_2(z)}.$$

4.8 Discrete-Time Frequency Response

Consider the usual discrete-time system

$$(4.31) \qquad \begin{cases} Q(E)y[k] = P(E)x[k], \\ y[-1] = \alpha_0, \quad y[-2] = \alpha_1, \ldots, \quad y[-n] = \alpha_{n-1}, \end{cases}$$

for which the input is the following sampled sinusoid with sampling period $T > 0$:

$$(4.32) \qquad x[k] = \begin{cases} A\cos(k\omega T + \theta), & \text{for all } k \geq 0, \\ 0, & \text{for all } k < 0. \end{cases}$$

We are now going to formulate a discrete-time analogue of Theorem 3.8.1. Indeed, if the discrete-time system is BIBO stable (see Theorem 4.7.1) then it turns out that the corresponding output $y[k]$ settles down to a sampled sinusoid $y_{ss}[k]$ for large values of k, whatever the values of the constants α_i in the auxiliary condition of (4.31). The precise statement of this fact is as follows:

Theorem 4.8.1 *Suppose the system (4.31) is BIBO stable with discrete-time transfer function $H(z)$. If the input signal $x[k]$ is given by (4.32) with corresponding output $y[k]$ then*

$$\lim_{k\to\infty} y[k] = y_{ss}[k]$$

where

$$y_{ss}[k] = A|H(e^{j\omega T})|\cos(k\omega T + \theta + \angle H(e^{j\omega T})).$$

Remark 4.8.2 The signal $y_{ss}[k]$ is called the **discrete-time steady state response** of the system to the sinusoidal input signal (4.32). Notice that $y_{ss}[k]$ does not depend in any way on the constants α_i in the auxiliary conditions of (4.31). The theorem is basically saying that BIBO stability of the system causes all initial transients in the response $y[k]$ to decay to zero, leaving us with only the steady-state response. Moreover, $|H(e^{j\omega T})|$ gives the **amplification** and $\angle H(e^{j\omega T})$ gives the **phase shift** of the steady state response relative to the input signal. The function $H(e^{j\omega T})$ of ω is called the **discrete-time frequency response** of the system 4.31. The discrete-time frequency response is an essential notion in the study of digital communication systems.