Chapter 5

Periodic Inputs and Fourier Series

In Chapter 3 we learned how to use Laplace transforms to determine the response of a continuous-time system

$$(5.1) \qquad\qquad Q(D)y(t) = P(D)x(t),$$

for *right-sided* input signals $x(t)$, that is signals $x(t)$ such that

$$(5.2) \qquad\qquad x(t) = 0, \qquad \text{for all } t < 0.$$

There are many applications, for example the modulation and demodulation of signals in radio and television transmission, for which one must determine the response of this system to *periodic* input signals $x(t)$, namely signals with the property

$$x(t) = x(t + T), \qquad \text{for all } t,$$

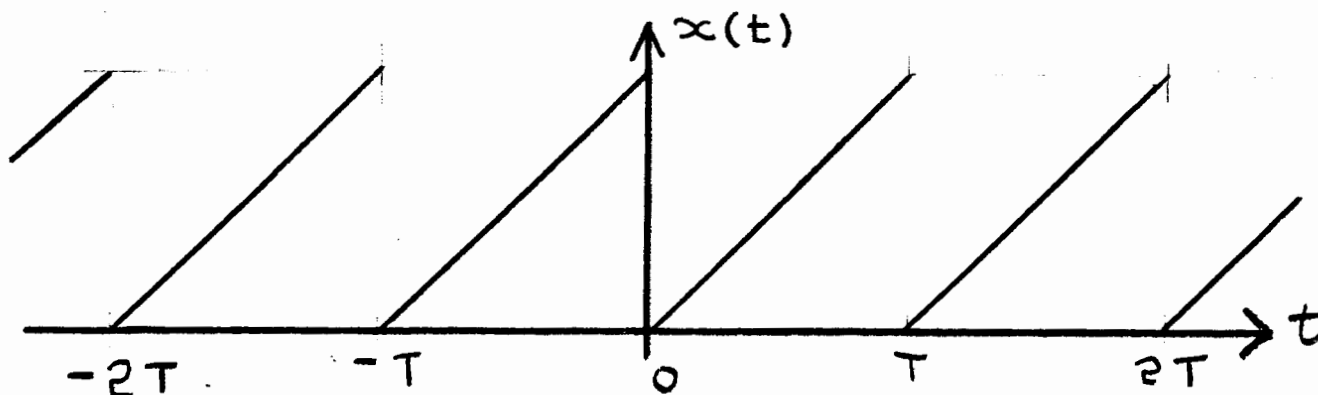where the constant $T > 0$ is a *period* of the signal.

Fig. 5.1.

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Obviously a periodic signal can never satisfy the right-sided condition (5.2) which is vital to applicability of the Laplace transform method that was learned in Chapter 3. In this chapter our goal is to develop a method, based on the use of *Fourier series*, to analyze the response of (5.1) to periodic input signals.

5.1   Exponential Fourier Series

Suppose that $x(t)$ is a complex-valued periodic signal with a period $T > 0$, and put $x(t)$ in the form

(5.3) $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \qquad \text{where} \qquad \omega_0 = \frac{2\pi}{T},$$

and the a_k are complex constants which must be correctly chosen. To see how to determine values for the a_k, fix some positive or negative integer n, and use (5.3) to write

(5.4) $$\int_0^T x(t)e^{-jn\omega_0 t}dt = \int_0^T \left\{ \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right\} e^{-jn\omega_0 t}dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}dt.$$

Now for any integer $N \neq 0$ we clearly have

$$\int_0^T \cos(N\omega_0 t)dt = \int_0^T \sin(N\omega_0 t)dt = 0,$$

so that

$$\int_0^T e^{jN\omega_0 t}dt = \int_0^T \cos(N\omega_0 t)dt + j \int_0^T \sin(N\omega_0 t)dt = 0.$$

Also, when $N = 0$, clearly

$$\int_0^T e^{jN\omega_0 t}dt = \int_0^T 1 dt = T.$$

Thus

$$\int_0^T e^{jN\omega_0 t}dt = \begin{cases} 0, & \text{when } N \neq 0, \\ T, & \text{when } N = 0, \end{cases}$$

and so

(5.5) $$\int_0^T e^{j(k-n)\omega_0 t}dt = \begin{cases} 0, & \text{when } k \neq n, \\ T, & \text{when } k = n. \end{cases}$$

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Using (5.5) in (5.4), we see that

$$(5.6) \qquad a_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt, \qquad n = \ldots, -2, -1, 0, 1, 2, \ldots$$

To conclude: if a periodic signal $x(t)$ is put in the form of (5.3) then the constants a_n are necessarily given by (5.6). This motivates the following

Definition 5.1.1 Suppose that $x(t)$ is a complex-valued periodic signal with period T. Then the function

$$(5.7) \qquad \tilde{x}(t) \triangleq \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \qquad \text{where} \qquad \omega_0 = \frac{2\pi}{T},$$

with

$$(5.8) \qquad a_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt, \qquad k = \ldots, -2, -1, 0, 1, 2, \ldots$$

is called the **exponential Fourier series expansion** of the signal $x(t)$, and the complex constants a_k are the **exponential Fourier coefficients** of the signal $x(t)$.

Remark 5.1.2 The sequence of complex numbers

$$a_k, \qquad k = \ldots, -2, -1, 0, 1, 2, \ldots$$

is called the **spectrum** of the periodic signal $x(t)$, and is usually displayed graphically in the form of a **magnitude plot**, namely

$$|a_k|, \qquad \text{versus} \qquad k = \ldots, -2, -1, 0, 1, 2, \ldots$$

and an **angular plot**, namely

$$\angle a_k, \qquad \text{versus} \qquad k = \ldots, -2, -1, 0, 1, 2, \ldots$$

Remark 5.1.3 Of particular importance in applications is the case where the signal $x(t)$ is **real-valued**. Here a common fallacy is to suppose that the exponential Fourier coefficients a_k must also be real-valued. In general this is not so (see Example 5.1.11 which follows) but in the case of real-valued $x(t)$ there is still a nice relation between the exponential Fourier coefficients which can be seen as follows. For each $k = 1, 2, \ldots$ the exponential Fourier coefficients are

$$(5.9) \qquad a_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$(5.10) \qquad a_{-k} = \frac{1}{T} \int_0^T x(t) e^{jk\omega_0 t} dt,$$

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Since $x(t)$ is real-valued we clearly have

$$x(t)e^{jk\omega_0 t} = [x(t)e^{-jk\omega_0 t}]^*,$$

and using this in (5.9), (5.10) shows that

$$\begin{aligned}
a_{-k} &= \frac{1}{T}\int_0^T x(t)e^{jk\omega_0 t}\,dt \\
&= \frac{1}{T}\int_0^T [x(t)e^{-jk\omega_0 t}]^*\,dt \\
&= \left\{\frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}\,dt\right\}^* \\
&= a_k^*, \qquad \text{for all } k = 1,2,\dots
\end{aligned}$$

From this we see the following: if $x(t)$ is a real-valued periodic signal, then the magnitude plot of its frequency spectrum is an *even* function of k, namely

$$|a_k| = |a_{-k}|, \qquad \text{for all } k.$$

and the angular plot of its frequency spectrum is an *odd* function of k, namely

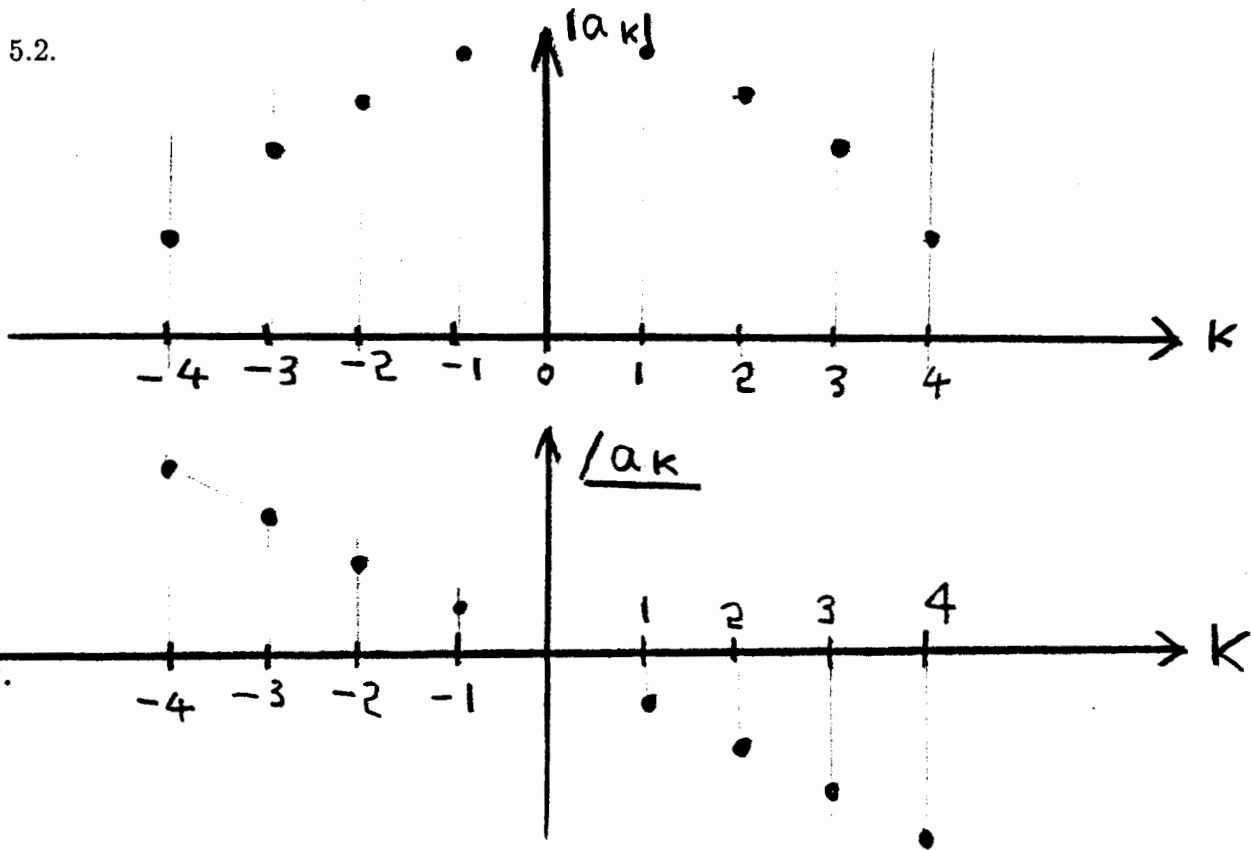$$\angle a_{-k} = \angle a_k^* = -\angle a_k, \qquad \text{for all } k.$$

See Fig. 5.2.

Fig. 5.2.

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Remark 5.1.4 Since

$$e^{-jk\omega_0 t} = \cos(k\omega_0 t) - j\sin(k\omega_0 t)$$

the exponential Fourier coefficients can of course be written as

$$a_k = \frac{1}{T}\int_0^T x(t)\cos(k\omega_0 t)dt - j\frac{1}{T}\int_0^T x(t)\sin(k\omega_0 t)dt,$$

for a periodic signal $x(t)$ with a period $T > 0$. Since $\cos(k\omega_0 t)$ and $\sin(k\omega_0 t)$ are each periodic with period equal to T, and $x(t)$ is periodic with period T, it follows that

$$x(t)\cos(k\omega_0 t) \qquad \text{and} \qquad x(t)\sin(k\omega_0 t)$$

are also periodic with period of T. In view of Remark 1.1.11 (with $t_0 = -T/2$) it then follows that

$$\int_0^T x(t)\cos(k\omega_0 t)dt = \int_{-T/2}^{T/2} x(t)\cos(k\omega_0 t)dt$$

$$\int_0^T x(t)\sin(k\omega_0 t)dt = \int_{-T/2}^{T/2} x(t)\sin(k\omega_0 t)dt$$

and therefore

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\cos(k\omega_0 t)dt - \frac{j}{T}\int_{-T/2}^{T/2} x(t)\sin(k\omega_0 t)dt.$$

Now suppose that the periodic signal $x(t)$ is both real-valued and an *even* function, namely

$$x(t) = x(-t).$$

Then, since $\sin(k\omega_0 t)$ is an odd function it follows that

$$x(t)\sin(k\omega_0 t)$$

is an odd function and thus

$$\int_{-T/2}^{T/2} x(t)\sin(k\omega_0 t)dt = \int_{-T/2}^0 x(t)\sin(k\omega_0 t)dt + \int_0^{T/2} x(t)\sin(k\omega_0 t)dt = 0$$

since the second and third integrals cancel. Thus a_k is real-valued. To summarize; when $x(t)$ is **real-valued** and an **even function** then the exponential Fourier coefficients a_k are **real-valued**. In exactly the same way, we see that when $x(t)$ is **real-valued** and an **odd function**, namely

$$x(t) = -x(-t)$$

then the exponential Fourier coefficients a_k are **imaginary** numbers.

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Remark 5.1.5 When $k = 0$ then of course $e^{jk\omega_0 t} \equiv 1$ for all t, so it follows from (5.7) and (5.8) that the Fourier series expansion of the signal $x(t)$ is given by

$$\tilde{x}(t) = a_0 + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} a_k e^{jk\omega_0 t}, \qquad \text{where} \qquad a_0 = \frac{1}{T} \int_0^T x(t) \, dt,$$

is the **average value** (also called the **d.c. value**) of the periodic signal $x(t)$. The quantity

$$\omega_0 = \frac{2\pi}{T}$$

is called the **angular frequency** of the signal $x(t)$, while the complex sinusoid

$$e^{j\omega_0 t},$$

corresponding to $k = 1$, is called the **first harmonic component**, and the complex sinusoid

$$e^{-j\omega_0 t}$$

is called the **negative image** of the first harmonic component. Generally, for all $k = 1, 2, \ldots$, the complex sinusoid

$$e^{jk\omega_0 t}$$

is called the k-th **harmonic component** and the complex sinusoid

$$e^{-jk\omega_0 t}$$

is called the **negative image** of the k-th harmonic component.

The question now arises: is it the case that $x(t)$ and its exponential Fourier series expansion (5.7) are identical, namely

$$x(t) = \tilde{x}(t), \qquad \text{for all } t.$$

This question is an issue in the general theory of convergence of Fourier series, an elegant and rather difficult subject. Indeed, it turns out that $\tilde{x}(t)$ is usually not quite identical to the signal $x(t)$, but agrees with $x(t)$ for "nearly all" values of t, provided that $x(t)$ is reasonably well-behaved. To make matters precise we introduce the following terminology: A periodic function $x(t)$ with a period T satisfies the **Dirichlet conditions** when

(a) $x(t)$ has a finite number of discontinuities in one period;

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(b) The real and imaginary parts of $x(t)$ have a finite number of maxima and minima in one period;

(c) $x(t)$ is absolutely integrable over one period in the sense that

$$\int_0^T |x(t)|\, dt < \infty.$$

Then we have the following important result:

Theorem 5.1.6 Suppose that $x(t)$ is a periodic signal with a period T and satisfies the Dirichlet conditions. Then

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \begin{cases} x(t), & \text{when } x(t) \text{ is continuous at } t, \\ (1/2)[x(t+) + x(t-)], & \text{when } x(t) \text{ is discontinuous at } t. \end{cases}$$

Remark 5.1.7 Theorem 5.1.6 is a very useful result which says essentially that one can reconstruct a periodic signal $x(t)$ from its exponential Fourier series at each instant t where $x(t)$ is continuous. On the other hand, at those instants t where $x(t)$ has a jump or discontinuity, the value of the exponential Fourier series is just the *average* of its left and right hand limits at instant t, namely

$$(1/2)[x(t+) + x(t-)].$$

Thus, for a periodic signal $x(t)$ with period $T = 1$ and $x(t)$ given by

$$x(t) = t, \qquad 0 \le t < 1,$$

the exponential Fourier series expansion $\tilde{x}(t)$ is as shown in Fig. 5.3(ii):
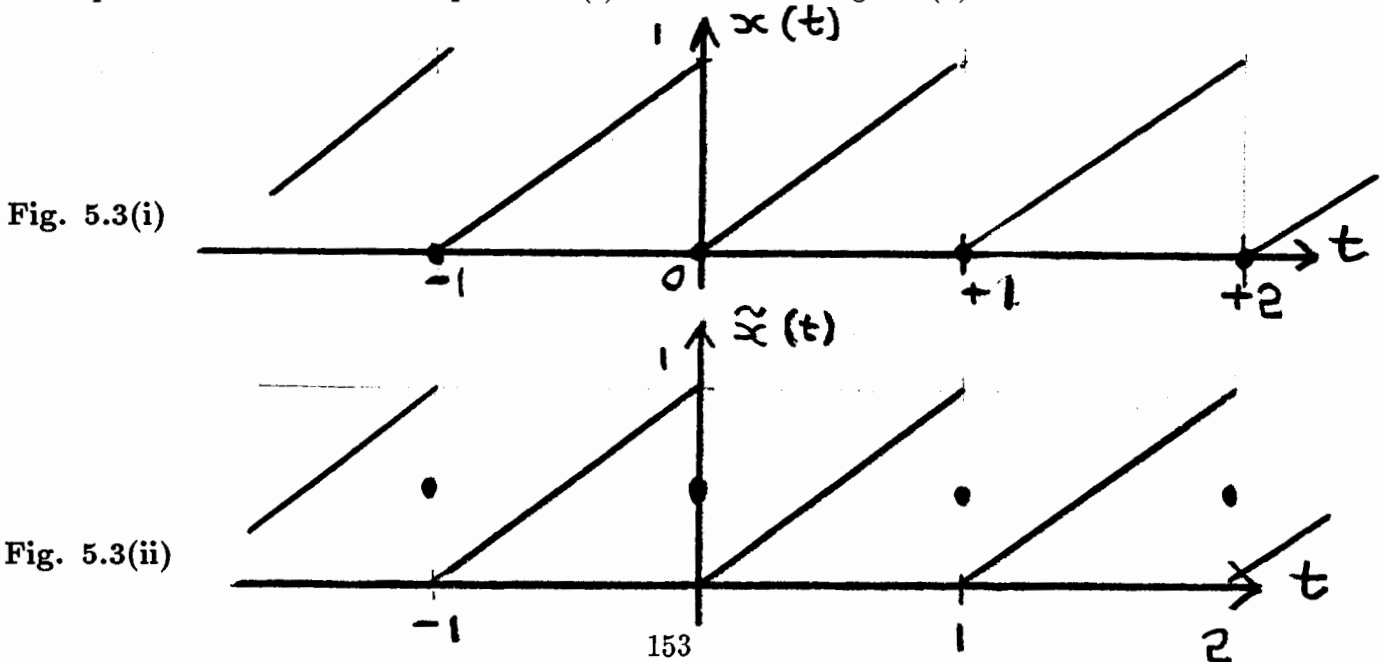
Fig. 5.3(i)

Fig. 5.3(ii)

Notice that $x(t)$ has jumps at $t = \ldots, -2, -1, 0, 1, 2, \ldots$ and the value of the exponential Fourier series at these instants of discontinuity is given by

$$\tilde{x}(t) = \frac{1}{2}[x(t+) + x(t-)] = \frac{1}{2}[0 + 1] = \frac{1}{2}.$$

On the other hand, we see from Fig. 5.3(i) that

$$x(t) = 0, \qquad \text{for all } t = \ldots, -2, -1, 0, 1, 2, \ldots,$$

so that $x(t)$ and $\tilde{x}(t)$ agree everywhere except at the instants $t = \ldots, -2, -1, 0, 1, 2, \ldots$ at which $x(t)$ is discontinuous.

Remark 5.1.8 We see from Theorem 5.1.6 that if $x(t)$ satisfies the Dirichlet conditions then $x(t)$ and its exponential Fourier series $\tilde{x}(t)$ agree at all instants t except for the occasional instants at which $x(t)$ is discontinuous. This means that we may regard the periodic signal $x(t)$ and its exponential Fourier series expansion $\tilde{x}(t)$ as "the same thing" and use $\tilde{x}(t)$ in place of $x(t)$ whenever convenient. As will be seen in the following secton, this will be the key step in determining the response of a system to a periodic input signal.

Remark 5.1.9 The proof of Theorem 5.1.6 is far beyond the scope of this introductory course.

Remark 5.1.10 One can find pathological periodic functions $x(t)$ which do not satisfy the Dirichlet conditions, and which cannot be reconstructed from their Fourier series in the sense of Theorem 5.1.6. An example is the periodic function $x(t)$ with period $T = 1$ given by

$$x(t) = \begin{cases} 0, & \text{for rational } 0 \le t \le 1, \\ 1, & \text{for irrational } 0 \le t \le 1. \end{cases}$$

This function clearly has infinitely many points of discontinuity over the period $0 \le t \le 1$, and hence does not satisfy the Dirichlet conditions. The convergence of Fourier series for such "strange" periodic functions is a facinating branch of modern mathematics, but is of little relevance to signal analysis in electrical engineering, where essentially all periodic signals satisfy the Dirichlet conditions, and are therefore reconstructable from their Fourier series in the sense of Theorem 5.1.6.

Example 5.1.11 A periodic signal $x(t)$ with period $T = 2$ is given by

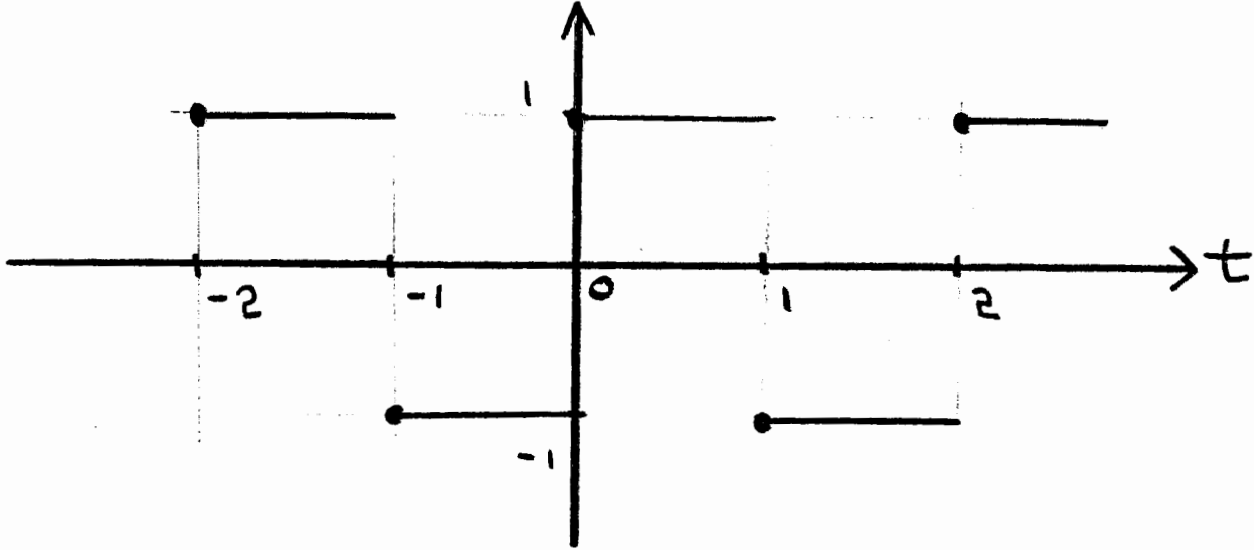$$x(t) = \begin{cases} 1, & \text{when } 0 \le t < 1, \\ -1, & \text{when } 1 \le t < 2. \end{cases}$$

Fig. 5.4.

We next determine the exponential Fourier series expansion of $x(t)$. Since $T = 2$, the fundamental angular frequency is given by

$$\omega_0 = \frac{2\pi}{T} = \pi.$$

Then

$$a_0 = \frac{1}{T} \int_0^T x(t)dt = 0,$$

and, when $k \ne 0$,

$$
\begin{aligned}
a_k &= \frac{1}{T} \int_0^T x(t)e^{-jk\omega_0 t} dt \\
&= \frac{1}{2} \int_0^2 x(t)e^{-jk\pi t} dt \\
&= \frac{1}{2} \left[\int_0^1 e^{-jk\pi t} dt - \int_1^2 e^{-jk\pi t} dt \right] \\
&= \frac{1}{2(-jk\pi)} \left[(e^{-jk\pi} - 1) - (e^{-2jk\pi} - e^{-jk\pi}) \right] \\
&= \frac{-1}{2jk\pi} \left[(-1)^k - 1 - 1 + (-1)^k \right] \\
&= \frac{1}{jk\pi} \left[1 - (-1)^k \right].
\end{aligned}
$$

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Now obviously

$$[1 - (-1)^k] = \begin{cases} 0, & k \text{ even,} \\ 2, & k \text{ odd,} \end{cases}$$

and therefore

$$a_k = \begin{cases} 0, & k \text{ even,} \\ 2/(jk\pi), & k \text{ odd.} \end{cases}$$

With these values of a_k the exponential Fourier series expansion of $x(t)$ becomes

$$\begin{aligned} \tilde{x}(t) &= \frac{2}{j\pi} \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} \frac{1}{k} e^{jk\pi t} \\ &= \frac{4}{\pi} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{k} \frac{[e^{jk\omega_0 t} - e^{-jk\omega_0 t}]}{2j} \\ &= \frac{4}{\pi} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{k} \sin(k\omega_0 t). \end{aligned}$$

Now we see from Theorem 5.1.6 that

$$\tilde{x}(t) = \begin{cases} 0, & \text{when } t = \ldots, -2, -1, 0, 1, 2, \ldots, \\ x(t), & \text{otherwise,} \end{cases}$$
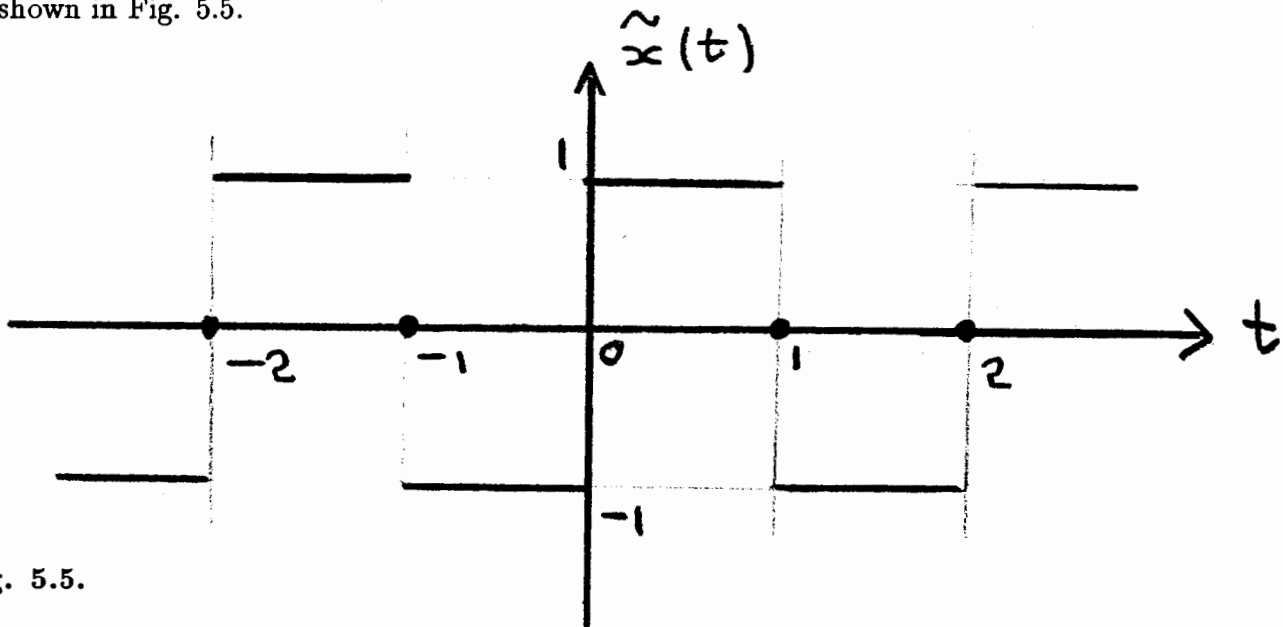
as shown in Fig. 5.5.

Fig. 5.5.

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Remark 5.1.12 The magnitude and angular plots of the spectrum of the square wave found in Example 5.1.11 is shown in Fig. 5.6:
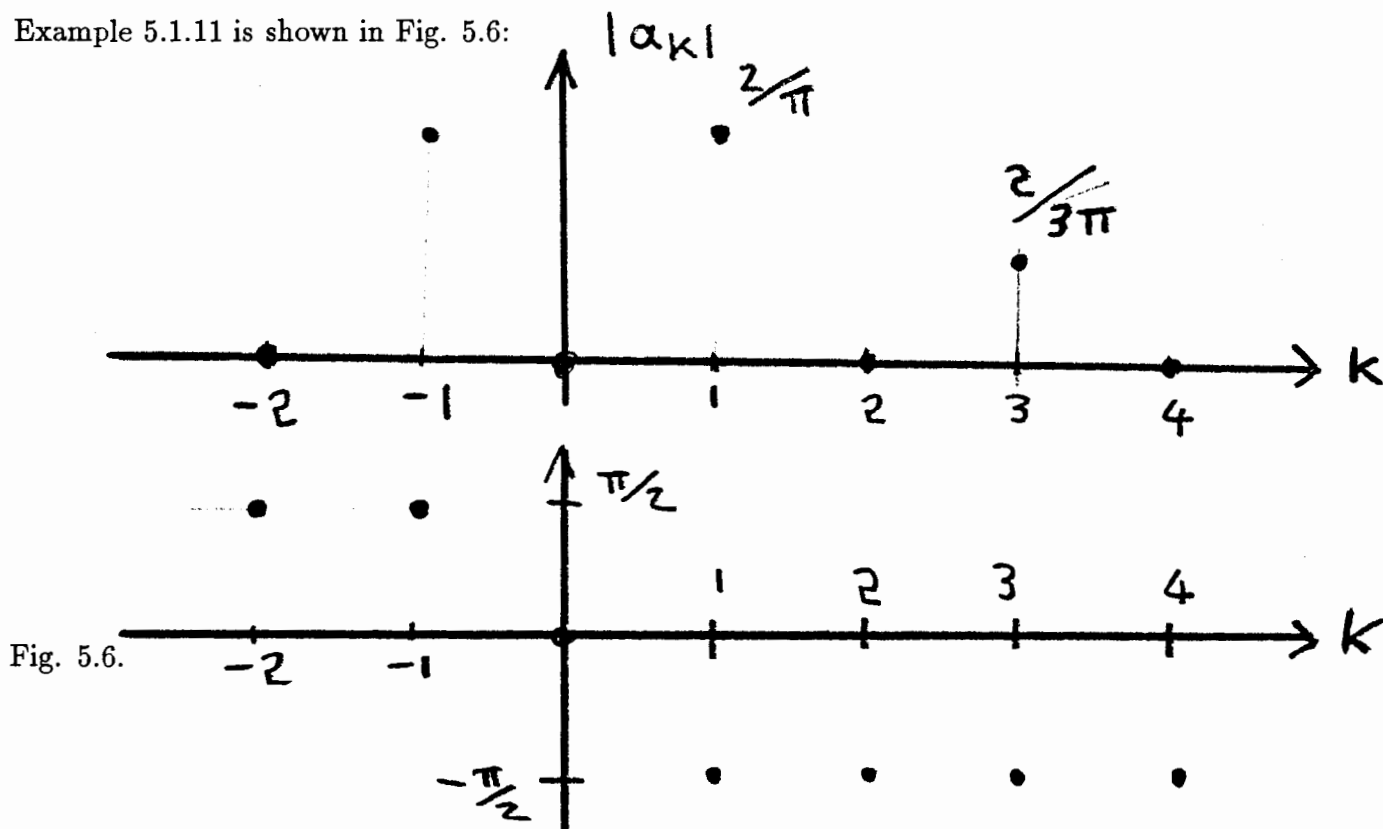

Fig. 5.6.

Recall from Remark 1.1.22 that the power in a periodic signal with period T is given by

$$P = \frac{1}{T} \int_0^T |x(t)|^2 dt.$$

In view of Remark 5.1.8 we can replace $x(t)$ with $\tilde{x}(t)$ without changing the value of the integral, so that

(5.11)
$$P = \frac{1}{T} \int_0^T |\tilde{x}(t)|^2 dt.$$

Now

$$\tilde{x}(t) \overset{\Delta}{=} \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

and therefore

$$
\begin{aligned}
|\tilde{x}(t)|^2 &= \tilde{x}(t)\tilde{x}(t)^* \\
&= \left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right] \left[\sum_{n=-\infty}^{\infty} a_n e^{jk\omega_0 t} \right]^* \\
&= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_k (a_n)^* e^{jk\omega_0 t} e^{-jn\omega_0 t}.
\end{aligned}
$$

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Using this is (5.11) gives

$$P = \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_k(a_n)^* \frac{1}{T} \int_0^T e^{j(k-n)\omega_0 t} dt,$$

and, on using (5.5), this becomes

$$
\begin{aligned}
P &= \sum_{k=-\infty}^{\infty} a_k(a_k)^* \\
&= \sum_{k=-\infty}^{\infty} |a_k|^2.
\end{aligned}
$$

In the special case where $x(t)$ is real-valued we have seen that $a_k = a_{-k}$ so that the preceding expression becomes

$$P = a_0^2 + 2 \sum_{k=1}^{\infty} |a_k|^2.$$

We have established the following result which is called **Parseval's theorem:**

Theorem 5.1.13 Suppose that $x(t)$ is a periodic signal satisfying the Dirichlet conditions. Then the average power in $x(t)$ is given by

$$P = \sum_{k=-\infty}^{\infty} |a_k|^2,$$

where the a_k are the exponential Fourier coefficients of $x(t)$.

Example 5.1.14 Determine the average power in the periodic signal

$$x(t) = 1 - \cos(\pi t) + 2\sin(\pi t) + \cos(3\pi t).$$

One can directly use the formula

$$P = \frac{1}{T} \int_0^T |x(t)|^2 dt$$

but the integrations are rather tedious and it is simpler to use Theorem 5.1.13. We first determine the exponential Fourier coefficients of $x(t)$. These could be found using the defining relation for a_k in Definition 5.1.1, but in this case it is easier to observe that

$$
\begin{aligned}
x(t) &= 1 - \frac{e^{j\pi t} + e^{-j\pi t}}{2} + 2\frac{e^{j\pi t} - e^{-j\pi t}}{2j} + \frac{e^{j3\pi t} + e^{-j3\pi t}}{2} \\
&= 0.5e^{-j3\pi t} + (-0.5 + j)e^{-j\pi t} + 1 + (-0.5 - j)e^{j\pi t} + 0.5e^{j3\pi t}.
\end{aligned}
$$

(5.12)

By Theorem 5.1.13 the average power is

$$P = (0.5)^2 + |-0.5 + j|^2 + 1 + |-0.5 - j|^2 + (0.5)^2 = 4.$$

Remark 5.1.15 Theorem 5.1.13 can be used to derive several unexpected results about series. For example, recalling the exponential Fourier series expansion of the square-wave signal in Example 5.1.11, we showed that the exponential Fourier coefficients are

$$a_k = \begin{cases} 0, & k \text{ odd}, \\ 2/(jk\pi), & k \text{ even}. \end{cases}$$

Now the average power in the square wave of Example 5.1.11 is

$$P = \frac{1}{T} \int_0^T |x(t)|^2 dt = \frac{1}{2} \int_0^2 1 . dt = 1.$$

Using Theorem 5.1.13 we then see that

$$1 = \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} |a_k|^2.$$

Thus,

$$\begin{aligned} 1 &= \frac{4}{\pi^2} \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} \frac{1}{k^2} \\ &= \frac{8}{\pi^2} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{k^2} \end{aligned}$$

from which we get the following series expansion for π:

$$\pi^2 = 8 \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{k^2}.$$

5.2 Periodic Inputs

We begin by studying the reponse of (5.1) to the simplest possible periodic input signal, namely a *sinusoid*, which was defined in Section 1.1.3 as a signal of the form

$$x(t) = A\cos(\omega_0 t + \theta),$$

for some real constant ω_0. There will be considerable technical advantages in the use of exponential Fourier series if we adopt a slightly more abstract viewpoint and regard the sinusoidal input signal as a *complex-valued* function

(5.13)
$$x(t) = e^{j\omega_0 t} \equiv \cos(\omega_0 t) + j\sin(\omega_0 t).$$

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Then the output is likewise regarded a complex-valued signal

$$y(t) = y_r(t) + jy_i(t),$$

where $y_r(t)$ and $y_i(t)$ are the response to the individual input signals

$$x_r(t) = \cos(\omega_0 t) \qquad \text{and} \qquad x_i(t) = \sin(\omega_0 t)$$

respectively, that is

$$Q(D)y_r(t) = P(D)x_r(t) \qquad \text{and} \qquad Q(D)y_i(t) = P(D)x_i(t).$$

Before going any further we must settle the following issues:

(i) What are the auxiliary conditions to be used with (5.1) for a sinusoidal input signal?

(ii) Are any stability properties needed for the response of (5.1) to a sinusoidal input signal to even make sense?

Since a sinusoidal input signal has effectively been applied since the "beginning of time", namely $t_0 = -\infty$, it makes sense to postulate the auxiliary conditions at $t_0 = -\infty$, and the most natural auxiliary condition to assume is that the system is initially at rest (recall Definition 2.2.1). Indeed, with this auxiliary condition in place, Theorem 2.3.4 ensures that we will be dealing with a linear, causal, and time-invariant system. Thus, from now on, we shall suppose that we are dealing with a system which is initially at rest, namely

(5.14)
$$\begin{cases} Q(D)y(t) = P(D)x(t), \\ y(-\infty) = 0, \quad y^{(1)}(-\infty) = 0, \ldots, \quad y^{(n-1)}(-\infty) = 0, \end{cases}$$

To get some idea of (ii) consider the system

$$(D^2 + 1)y(t) = x(t), \qquad y(t_0) = 0,$$

for an input signal

$$x(t) = \cos(t - t_0),$$

where t_0 is some fixed instant. Notice that this is an unstable system, since the roots of $Q(\lambda) = \lambda^2 + 1$ are $\lambda = j$ and $\lambda = -j$, which fail to have strictly negative real parts. Now one easily checks by

substitution that a response to this input signal is

$$y(t) = \frac{t - t_0}{2} \cos(t - t_0).$$

It follows that the limit

$$\lim_{t - t_0 \to \infty} y(t)$$

fails to exist. Since we are effectively going to take $t_0 = -\infty$, it follows that $t - t_0 = +\infty$ for any finite value of t, and hence the output $y(t)$ fails to exist for any finite value t. This simple example illustrates a general difficulty: If a system fails to be BIBO-stable and is driven by a periodic input signal $x(t)$ then the output $y(t)$ may "blow-up" for any finite value of t. Clearly we want to avoid difficulties of this kind, and hence we shall always suppose that the system is BIBO stable.

Having settled these issues we now look at the response of the system (5.14) when the input signal is the complex-valued sinusoid (5.13). From Theorem 2.2.8 we know that the corresponding output signal is

$$
\begin{aligned}
y(t) &= (x * h)(t) \\
&= (h * x)(t) \quad \text{by commutativity of convolution} \\
&= \int_{-\infty}^{\infty} h(\tau)x(t - \tau)\,d\tau
\end{aligned}
$$

where $h(t)$ denotes the impulse response of the system (5.14). With $x(t)$ given by (5.13), we find

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau)e^{j\omega_0(t-\tau)}d\tau \\
&= e^{j\omega_0 t} \int_{-\infty}^{\infty} h(\tau)e^{-j\omega_0 \tau}d\tau
\end{aligned}
$$

Now Theorem 2.3.4 ensures that the system (5.14) is linear time-invariant and causal, and therefore Section 2.2.1 in turn says that

$$h(\tau) = 0, \qquad \text{for all } \tau < 0,$$

so that

(5.15)
$$y(t) = e^{j\omega_0 t} \int_{0-}^{\infty} h(\tau)e^{-j\omega_0 \tau}\,d\tau.$$

Now we recall from Section 3.7 that, if $H(s)$ is the transfer function of the system, then

$$
\begin{aligned}
H(s) &= \mathcal{L}\{h(\tau)\}(s) \\
&= \int_{0-}^{\infty} h(\tau)e^{-s\tau}d\tau.
\end{aligned}
$$

It follows that the integral in (5.15) is just the frequency response function $H(j\omega_0)$, so that the output corresponding to the sinusoidal complex-valued input signal (5.13) is the complex-valued signal

$$
y(t) = H(j\omega_0)e^{j\omega_0 t}.
$$

Clearly, we can rewrite $y(t)$ in the form

$$
\begin{aligned}
y(t) &= |H(j\omega_0)|e^{j\angle H(j\omega_0)}e^{j\omega_0 t} \\
&= |H(j\omega_0)|e^{j[\omega_0 t + \angle H(j\omega_0)]} \\
&= |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0)) + j|H(j\omega_0)|\sin(\omega_0 t + \angle H(j\omega_0)),
\end{aligned}
$$

which clearly displays the output as a complex-valued sinusoidal signal, whose magnitude is scaled by a factor $|H(j\omega_0)|$ and whose phase is shifted by the angle $\angle H(j\omega_0)$. Notice that both the input and output signals have a common period given by

$$
T = \left| \frac{2\pi}{\omega_0} \right|.
$$

We have thus established the following:

Theorem 5.2.1 Suppose that the system (5.14) is BIBO stable with transfer function $H(s)$. Then the response to the complex-valued sinusoidal input signal

$$
x(t) = e^{j\omega_0 t}
$$

is the complex-valued sinusoid

$$
y(t) = H(j\omega_0)e^{j\omega_0 t}.
$$

Remark 5.2.2 Suppose that (5.14) is BIBO stable and the input signal is

$$
x(t) = e^{jk\omega_0 t}
$$

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where k is some fixed positive or negative integer. Then, taking $k\omega_0$ in place of ω_0 in Theorem 5.2.1, one sees that the corresponding output signal is

$$y(t) = H(jk\omega_0)e^{jk\omega_0 t}.$$

In view of this observation and superposition for linear systems, we have the following generalization of Theorem 5.2.1 which gives the response to a periodic input signal:

Theorem 5.2.3 Suppose that the system (5.14) is BIBO stable with transfer function $H(s)$, and the input signal $x(t)$ is periodic with some period T and exponential Fourier series expansion

$$(5.16) \qquad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \qquad \omega_0 \stackrel{\triangle}{=} \frac{2\pi}{T}.$$

Then the corresponding output is the signal

$$(5.17) \qquad y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0)e^{jk\omega_0 t}.$$

Remark 5.2.4 Theorem 5.2.3 says that the response to a periodic input signal of period T is likewise also periodic with the same period T and also gives the exponential Fourier series expansion of the output in terms of the exponential Fourier coefficients a_k of the input signal and the values $H(jk\omega_0)$ of the transfer function. Indeed, it follows from (5.17) that the exponential Fourier coefficients of the output signal are

$$c_k = a_k H(jk\omega_0), \qquad k = \ldots, -2, -1, 0, 1, 2, 3, \ldots$$

Then it follows at once from Theorem 5.1.13 that the average power in the output signal $y(t)$ is

$$P_y = \sum_{k=-\infty}^{\infty} |a_k|^2 \, |H(jk\omega_0)|^2.$$

Example 5.2.5 A a *band-limited differentiator* is a BIBO stable system such that $H(j\omega)$ versus ω is as follows:

$$(5.18) \qquad |H(j\omega)| = \begin{cases} 12|\omega|/8000\pi, & \text{for all } |\omega| \leq 8000\pi, \\ 0, & \text{for all } |\omega| > 8000\pi, \end{cases}$$

and

$$(5.19) \qquad \angle H(j\omega) = \begin{cases} \pi/2, & \text{for all } \omega \geq 0, \\ -\pi/2, & \text{for all } \omega < 0. \end{cases}$$

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The input $x(t)$ is the square wave shown in Fig. 5.7 with $T = 10^{-3}$ secs.

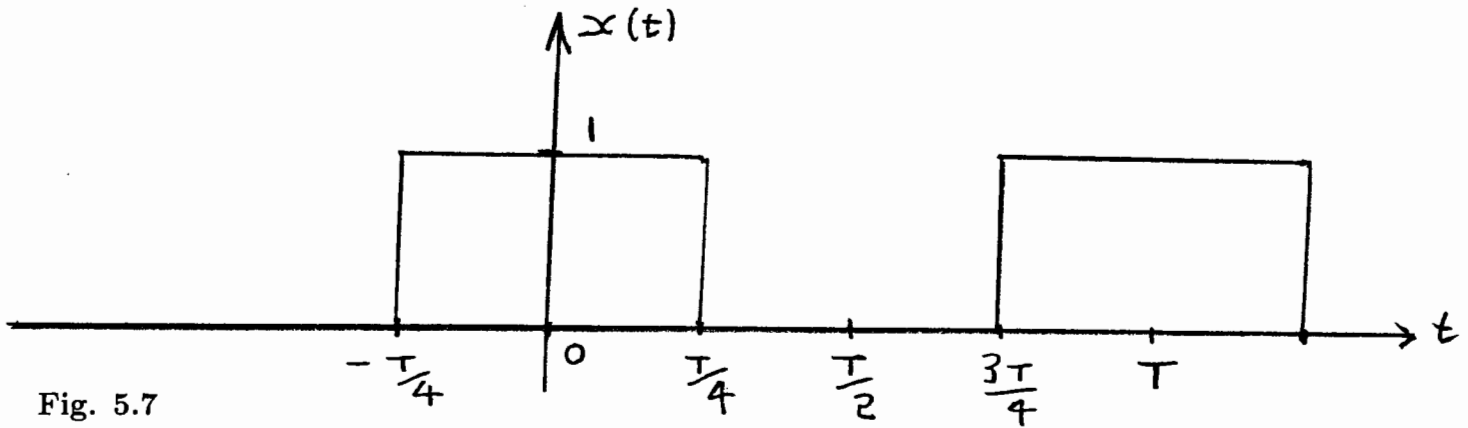
Fig. 5.7

Determine the output signal. Clearly

$$\omega_0 = \frac{2\pi}{T} = 2000\pi.$$

Now determine the Fourier coefficients of the signal $x(t)$.

$$a_0 = \frac{1}{T} \int_{-T/4}^{T/4} 1 dt = \frac{1}{2},$$

and, for all $k \neq 0$,

$$
\begin{aligned}
a_k &= \frac{1}{T} \int_{-T/4}^{T/4} e^{-jk\omega_0 t} dt \\
&= \frac{1}{T} \left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0} \right]_{-T/4}^{T/4} \\
&= \frac{-1}{jk\omega_0 T} \left[e^{-jk\omega_0 T/4} - e^{jk\omega_0 T/4} \right] \\
&= \frac{-1}{2\pi jk} \left[e^{-jk\pi/2} - e^{jk\pi/2} \right] \\
&= \frac{1}{k\pi} \sin(k\pi/2).
\end{aligned}
$$

Clearly

$$a_k = 0 \quad \text{for all even integers } k.$$

Then by Theorem 5.2.3 the output signal is given by

$$
\begin{aligned}
y(t) &= \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t} \\
&= \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} c_k e^{jk\omega_0 t}
\end{aligned}
$$

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where

$$c_k = a_k H(jk\omega_0).$$

Now, from (5.18), we see that

$$H(jk\omega_0) = 0, \qquad \text{for all odd integers } k \text{ with } |k| \geq 5.$$

Thus

$$y(t) = c_{-3}e^{-3j\omega_0 t} + c_{-1}e^{-j\omega_0 t} + c_1 e^{j\omega_0 t} + c_3 e^{3j\omega_0 t}.$$

But

$$
\begin{aligned}
a_1 &= \frac{1}{\pi}\sin(\pi/2) = \frac{1}{\pi}, \\
a_{-1} &= \frac{1}{-\pi}\sin(-\pi/2) = \frac{1}{\pi}, \\
a_3 &= \frac{1}{3\pi}\sin(3\pi/2) = \frac{-1}{3\pi}, \\
a_{-3} &= \frac{1}{-3\pi}\sin(-3\pi/2) = \frac{-1}{3\pi},
\end{aligned}
$$

so that, using (5.18) and (5.19),

$$
\begin{aligned}
c_1 &= a_1 H(j\omega_0) = \frac{1}{\pi}H(2000\pi j) = \frac{1}{\pi}(3e^{j\pi/2}) = \frac{3j}{\pi}, \\
c_{-1} &= a_{-1}H(-j\omega_0) = \frac{1}{\pi}H(-2000\pi j) = \frac{1}{\pi}(3e^{-j\pi/2}) = \frac{-3j}{\pi}, \\
c_3 &= a_3 H(3j\omega_0) = \frac{-1}{3\pi}H(6000\pi j) = \frac{-1}{3\pi}(9e^{j\pi/2}) = \frac{-3j}{\pi}, \\
c_{-3} &= a_{-3}H(-3j\omega_0) = \frac{-1}{3\pi}H(-6000\pi j) = \frac{-1}{3\pi}(9e^{-j\pi/2}) = \frac{3j}{\pi}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
y(t) &= \frac{3j}{\pi}e^{-3j\omega_0 t} - \frac{3j}{\pi}e^{-j\omega_0 t} + \frac{3j}{\pi}e^{j\omega_0 t} - \frac{3j}{\pi}e^{3j\omega_0 t} \\
&= \frac{-6}{\pi}\sin(\omega_0 t) + \frac{6}{\pi}\sin(3\omega_0 t).
\end{aligned}
$$