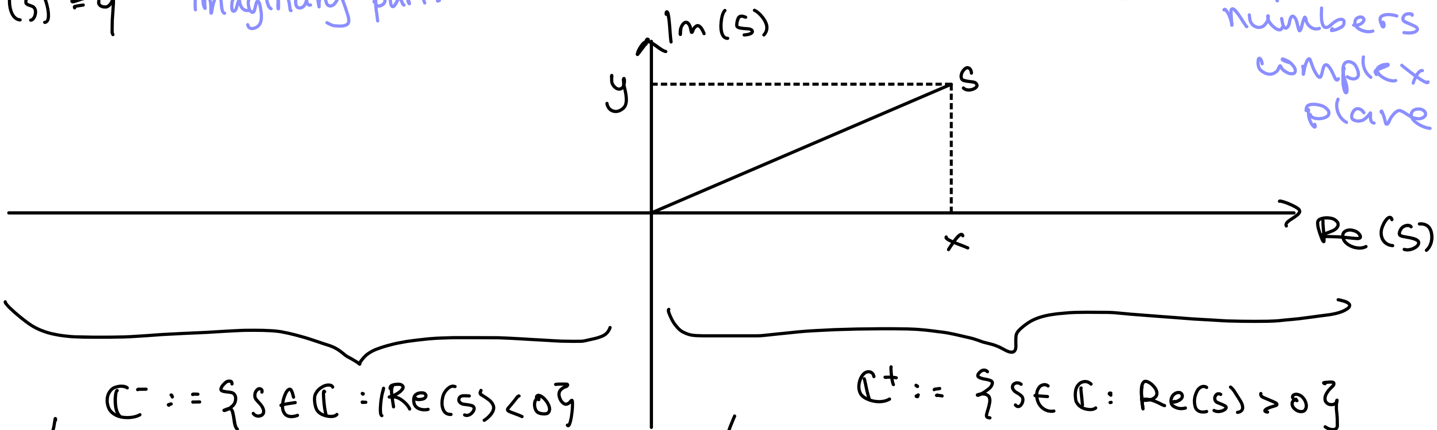
Tutorial 1

Complex Numbers

A **complex number** is an expression of the form $s = x + jy$ with $x, y \in \mathbb{R}$

$\mathrm{Re}(s) = x$ real part
$\mathrm{Im}(s) = y$ imaginary part.

↪ set of all $\mathbb{C}$ complex numbers form complex plane



$\mathbb{C}^- := \{ s \in \mathbb{C} : \mathrm{Re}(s) < 0 \}$
↳ OLHP (open left hand plane)

$\mathbb{C}^+ := \{ s \in \mathbb{C} : \mathrm{Re}(s) > 0 \}$
↳ ORHP (open right hand plane)

The **complex conjugate** of $z = x + jy$ is $\bar{z} = x - jy$ reflection in x axis

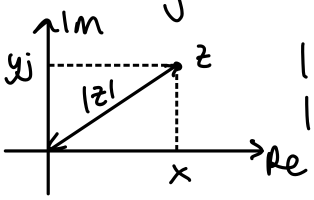
$\overline{z+w} = \bar{z} + \bar{w}$ $\quad (x_1 + x_2) - j(y_1 + y_2) = (x_1 - jy_1) + (x_2 - jy_2)$

$\overline{zw} = \bar{z}\,\bar{w}$ $\quad (x_1x_2 - y_1y_2) - j(x_1y_2 + y_1x_2)$

$= (x_1x_2 + jy_1jy_2) - x_1jy_2 - x_2jy_1$

$= (x_1 - jy_1)(x_2 - jy_2)$

The **magnitude** of $z \in \mathbb{C}$ is $|z| := (x^2 + y^2)^{1/2}$

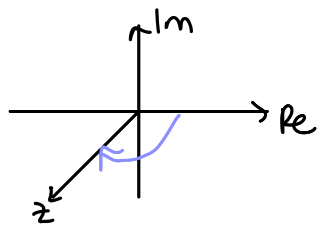


$|z| = |\bar{z}|$

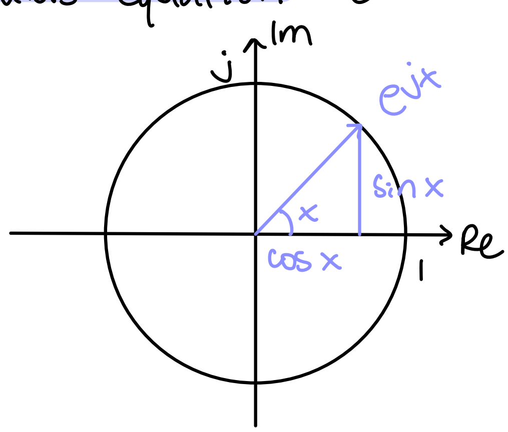
$|z|^2 = z\bar{z}$

$\measuredangle z$ denotes the **angle** of $z \in \mathbb{C} \setminus \{0\}$

Ex. 1 $z = -1 - j$, $\measuredangle z = -\frac{3\pi}{4}$



Eulers equation $e^{jx} = \cos(x) + j\sin(x)$, $x \in \mathbb{R}$. $|e^{jx}| = 1$. $\measuredangle(e^{jx}) = x$



The **polar form** $z = x + jy$, $z \neq 0$, is $|z|\, e^{j\measuredangle z}$

Addition (easier in cartesian)

$$r_1 e^{j\theta_1} + r_2 e^{j\theta_2} = (r_1 \cos\theta_1 + j r_1 \sin\theta_1) + (r_2 \cos\theta_2 + j r_2 \sin\theta_2)$$
$$= (r_1 \cos\theta_1 + r_2 \cos\theta_2) + j(r_1 \sin\theta_1 + r_2 \sin\theta_2)$$

Multiplication (easier in polar)

$$x_1 + jy_1 = \sqrt{x_1^2 + y_1^2}\, e^{j \arctan\left(\frac{y_1}{x_1}\right)} = r_1 e^{j\theta_1}$$
$$x_2 + jy_2 = \sqrt{x_2^2 + y_2^2}\, e^{j \arctan\left(\frac{y_2}{x_2}\right)} = r_2 e^{j\theta_2}$$
$$(x_1 + jy_1)(x_2 + jy_2) = (r_1 e^{j\theta_1})(r_2 e^{j\theta_2}) = r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

Vectors and Matricies

We will work with vectors in $\mathbb{R}^n$. These are collections of numbers represented as a column

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

addition is done component wise

if $\alpha \in \mathbb{R}$, $x \in \mathbb{R}^n$, $\alpha x := \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix}$ (scalar multiplication)

A matrix $\in \mathbb{R}^{m \times n}$ is also a collection of numbers arranged as a rectangular array with m rows and n columns.

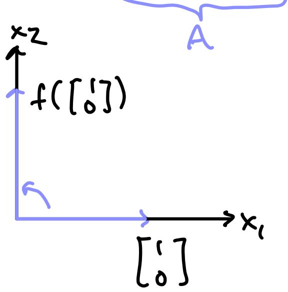
A function $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is linear if $f(c_1 x_1 + c_2 x_2) = c_1 f(x_1) + c_2 f(x_2)$
$c_1, c_2 \in \mathbb{R}$, $x_1, x_2 \in \mathbb{R}^n$

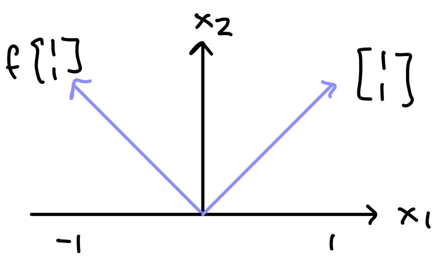
If and only if f is linear, then it can be written as a matrix

$$f(x) = Ax, \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

Ex 2 If $f : \mathbb{R}^2 \to \mathbb{R}^2$ rotates vectors by 90° then f can be represented as

$$f(x) = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A} x$$





$$f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Matrix vector multiplication

When multiplying a matrix and a vector, $(\overset{\text{matrix}}{M}\overset{\text{vector}}{x})$ the number of columns of the matrix need to match the number of rows of the vector

Ex 3

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{2\times 3} \underbrace{\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}}_{3\times 1} = \begin{bmatrix} (1)(7)+(2)(8)+(3)(9) \\ (4)(7)+(5)(8)+(6)(9) \end{bmatrix} = \underbrace{\begin{bmatrix} 50 \\ 122 \end{bmatrix}}_{2\times 1}$$

inner number needs to match

outer numbers form dimensions of final result

★ common mistake made during linear state space section is getting the dimensions wrong ★

Linearization

Suppose we have a nonlinear function. $y = f(x)$ which we want to linearize about a point x^*. In this course we'll use a 1st order approximation

$$f(x) \approx f(x^*) + f'(x^*)(x - x^*)$$

let $y^* := f(x^*)$

$$\underbrace{y - y^*}_{\Delta y} \approx \cancel{f(x^*)} + f'(x^*)\underbrace{(x - x^*)}_{\Delta x} - \left(\cancel{f(x^*)} + f'(x^*)\overset{0}{\cancel{(x^* - x^*)}}\right)$$

$$\Delta y = f'(x^*)\Delta x \Rightarrow y = f'(x^*)\Delta x + y^*$$

Ex 4

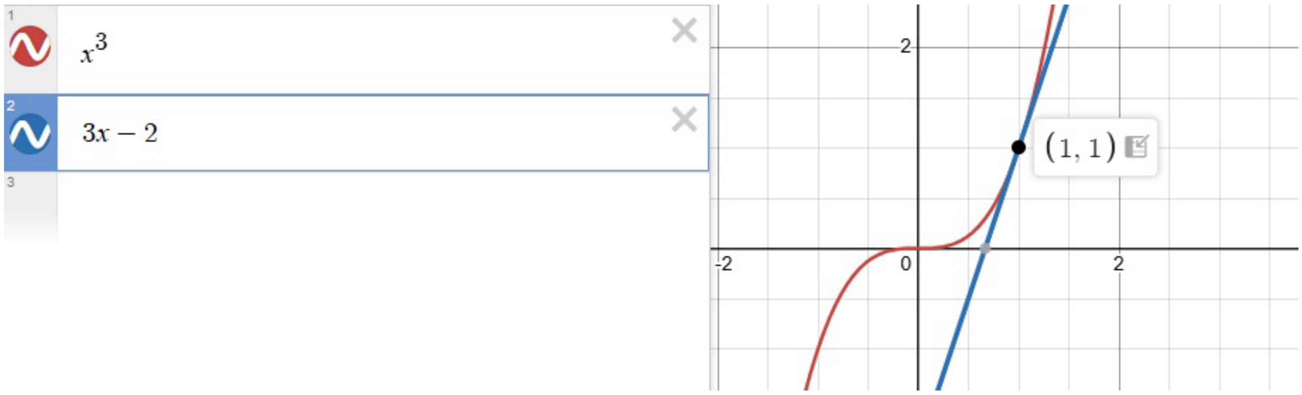
Linearize $y = f(x) = x^3$ about $x^* = 1$

we get $\Delta y = f'(x^*)\Delta x$

$$\Delta y = 3x^2\big|_{x=1} \cdot \Delta x$$

$$\Delta y = 3\Delta x$$

$$y = 3(x-1)+1 = 3x - 2$$



Multivariable case

Suppose we have $f: \mathbb{R}^n \rightarrow \mathbb{R}$ (f takes in a vector and spits out a scalar)

we approximate

$$f(x) \approx f(x^*) + \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x^*}}_{\text{Jacobian}} (x - x^*) \quad , \quad \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots \end{bmatrix}$$

Ex. 5

$f: \mathbb{R}^2 \rightarrow \mathbb{R}$. $(x_1, x_2) \mapsto [x_1^2 - x_2]$

Lets linearize at $x^* = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\left.\frac{\partial f}{\partial x}\right|_{x^*} = \left.\begin{bmatrix} 2x_1 & -1 \end{bmatrix}\right|_{x^* = \begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \begin{bmatrix} 2 & -1 \end{bmatrix}$$

$$\Delta y = \left.\frac{\partial f}{\partial x}\right|_{x^*} \Delta x \Rightarrow y = \underbrace{\begin{bmatrix} 2 & -1 \end{bmatrix}}_{\left.\frac{\partial f}{\partial x}\right|_{x^*}} \underbrace{\left(x - \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)}_{\Delta x} + \underbrace{[2]}_{y^*}$$

Partial Fraction Decomposition (everyone's favourite pasttime!)

Ex. 6

$$G(s) = \frac{s+1}{(s-1)^2(s-0.8)(s-0.9)} = \frac{A_1}{s-1} + \frac{A_2}{(s-1)^2} + \frac{B}{s-0.8} + \frac{C}{s-0.9}$$

Fisher's Preferred Method

1. multiply both sides by $(s-1)^2(s-0.8)(s-0.9)$

$$s+1 = A_1(s-1)(s-0.8)(s-0.9) + A_2(s-0.8)(s-0.9) + B(s-1)^2(s-0.9) + C(s-1)^2(s-0.8)$$

$$s+1 = A_1(s^3 - 2.7s^2 + 2.42s - 0.72) + A_2(s^2 + 1.7s + 0.72) + B(s^3 - 2.9s^2 + 2.8s - 0.9) + C(s^3 - 2.8s^2 + 2.6s - 0.8)$$

2. match coefficients

$$\left.\begin{aligned} s^3&: A_1 + B + C = 0 \\ s^2&: -2.7A_1 + A_2 - 2.9B - 2.8C = 0 \\ s^1&: 2.42A_1 - 1.7A_2 + 2.8B + 2.6C = 1 \\ s^0&: -0.72A_1 + 0.72A_2 - 0.9B - 0.8C = 1 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} A_1 &= -1450 \\ A_2 &= 100 \\ B &= -450 \\ C &= 1900 \end{aligned}\right.$$

Heaviside Coverup Method (my personal preferred method)

$$F(x) = \frac{P(x)}{Q(x)} = \frac{P(x)}{(x-a_1)^n(x-a_2)\cdots(x-a_r)} = \frac{A_1}{(x-a_1)^n} + \cdots + \frac{A_n}{x-a_1} + \frac{B_1}{x-a_2} + \cdots + \frac{B_r}{x-a_r}$$

$$B_i = \left.\left[(x-a_i)\frac{P(x)}{(x-a_1)^n(x-a_2)\cdots(x-a_r)}\right]\right|_{x=a_i}$$

looks yucky but trust me bro its actually better

$$A_k = \frac{1}{(k-1)!}\left[\frac{d^{k-1}}{dx^{k-1}}\left\{(x-a_1)^n F(x)\right\}\right]\Bigg|_{x=a_1}$$

where does this come from?

Ex. 7 $G(s) = \frac{s+2}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$ (no repeated roots!)

so much easier than expanding polynomials then solving a system of equations am I right? (I'm bad at math so I like this method better)

1. multiply both sides by s

$$sG(s) = \frac{\cancel{s}(s+2)}{\cancel{s}(s+1)} = A + \frac{Bs}{s+1}$$

evaluate at $s=0 \Rightarrow 2 = A + B\cdot 0 \Rightarrow A=2$

2. multiply both sides by s+1

$$(s+1)G(s) = \frac{\cancel{(s+1)}(s+2)}{s\,\cancel{(s+1)}} = \frac{A(s+1)}{s} + B$$

evaluate at $s=-1 = \frac{(-1+2)}{(-1)} = -1 = A\cdot 0 + B \Rightarrow B=-1$

$$G(s) = \frac{2}{s} + \frac{-1}{s+1}$$

$$= \frac{2(s+1)-s}{s(s+1)} = \frac{s+2}{s(s+1)}$$

!! **repeated roots annoying (watch out)** !!

★ if there are repeated roots I'd recommend Fisher's method ★ please watch out!! We'll both be sad if I taught a method that ended up tripping people up because they didn't notice the repeated roots part

Ex. 6 revisited

$$A_1 = \frac{1}{0!}\left[\cancel{(s-1)^2}\frac{s+1}{\cancel{(s-1)^2}(s-0.8)(s-0.9)}\right]\Bigg|_{s=1} = \frac{2}{(0.2)(0.1)} = 100$$

$$A_2 = \frac{1}{0!}\left[\frac{d}{ds}\left\{\cancel{(s-1)^2}\frac{s+1}{\cancel{(s-1)^2}(s-0.8)(s-0.9)}\right\}\right]\Bigg|_{s=1} = -\frac{50(50x^2+100x-121)}{(5x-4)^2(10x-9)^2}\Bigg|_{s=1} = -1450$$

$$B = \left[\cancel{(s-0.8)}\frac{(s+1)}{(s-1)^2\cancel{(s-0.8)}(s-0.9)}\right]\Bigg|_{s=0.8} = -450$$

$$C = \left[\cancel{(s-0.9)}\frac{(s+1)}{(s-1)^2(s-0.8)\cancel{(s-0.9)}}\right]\Bigg|_{s=0.9} = 1900$$

★ Either method is a valid approach ☺ ★