↓ physical laws

diff eq.s.

↓ linearization

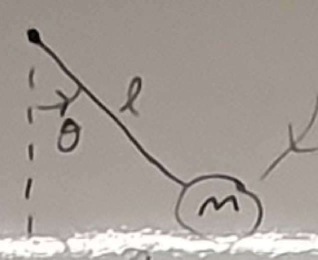
LTI diff eq.s.

↓ Laplace

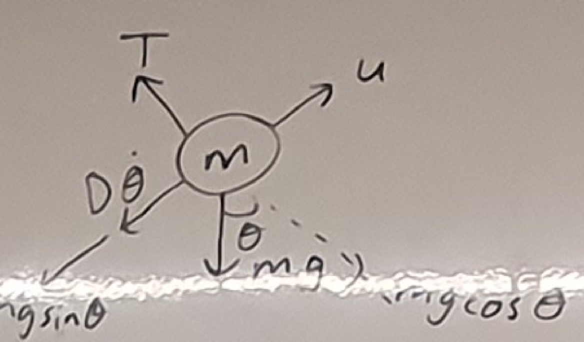
transfer function

Linearization

Ex. Pendulum



fbd



Newton's 2nd Law $\Rightarrow$ $mg\cos\theta - T = m\ddot r \to 0$

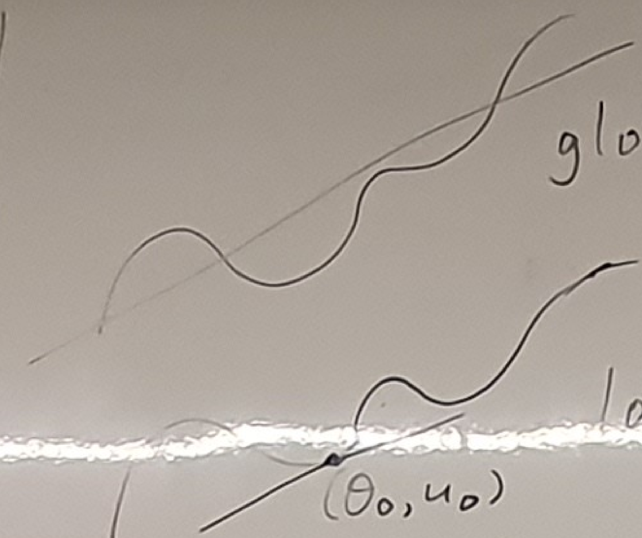
$T = mg\cos\theta$

$\Rightarrow m\ell^2\ddot\theta = u\ell - D\ell\dot\theta - mg\ell\sin\theta$

$\Rightarrow$ nonlinear!

$\Rightarrow m\ell^2\ddot\theta + D\ell\dot\theta + mg\sin\theta - u\ell = 0$

$\Rightarrow f(\theta,\dot\theta,\ddot\theta,u) = 0$

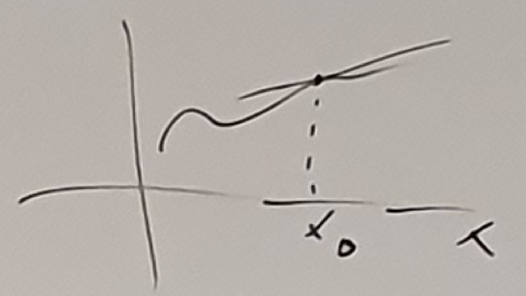


global linear approximation → lots of error

local linear approximation about some point (θ_0, u_0)

Linearizing a function $f(x)$ about a point x_0

$\Rightarrow f(x) \approx f(x_0) + f'(x_0)(x - x_0)$, with $f'(x_0) = \frac{df}{dx}\Big|_{x_0}$ and $x - x_0 = \Delta x$



What if f depends on multiple variables?

$f(x,y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}(x-x_0) + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}(y-y_0)$, with $x - x_0 = \Delta x$, $y - y_0 = \Delta y$

$\Rightarrow f(x,y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}\Delta x + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}\Delta y$

$$\boxed{f(x^1,\dots,x^n) \approx f(x_0^1,\dots,x_0^n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\Big|_{(x_0^1,\dots,x_0^n)}\Delta x^i}$$

So linearizing the pendulum diff eq:

$0 = f(\theta,\dot\theta,\ddot\theta,u) \approx f(\theta_0,\dot\theta_0,\ddot\theta_0,u_0) + \frac{\partial f}{\partial\theta}\Big|_{(\theta_0,\dot\theta_0,\ddot\theta_0,u_0)}\Delta\theta + \frac{\partial f}{\partial\dot\theta}\Big|_{(\sim)}\Delta\dot\theta + \frac{\partial f}{\partial\ddot\theta}\Big|_{(\sim)}\Delta\ddot\theta + \frac{\partial f}{\partial u}\Big|_{(\sim)}\Delta u$

$\Rightarrow 0 = f(\theta_0,\dot\theta_0,\ddot\theta_0,u_0) + mg\ell(\cos\theta_0)\Delta\theta + D\ell\Delta\dot\theta + m\ell^2\Delta\ddot\theta - \ell\Delta u$

If $f(\theta_0,\dot\theta_0,\ddot\theta_0,u_0) \neq 0 \Rightarrow$ our linearization is affine but NOT LTI!

How do we choose the point $(\theta_0,\dot\theta_0,\ddot\theta_0,u_0)$ to linearize about?

1. We want to choose a point such that $f(\theta_0,\dot\theta_0,\ddot\theta_0,u_0) = 0 \Rightarrow$ our linearization is LTI

If we linearize about some point and then move far away from that point, the linearization becomes inaccurate

$\Rightarrow$ we want to linearize about a point that the system will ideally stay close to always

2. We want (θ_0,u_0) to be a controlled equilibrium

$\Rightarrow$ if we start near (θ_0,u_0) we will typically stay near (θ_0,u_0) $\forall t$ (assuming the equilibrium is stable)

Def. A controlled equilibrium point is a point (θ_0,u_0) such that $\theta(t) = \theta_0\ \forall t$, $u(t) = u_0\ \forall t$ is a solution to $f(\theta,\dot\theta,\ddot\theta,u) = 0$.

Note that this implies $\dot\theta(t) = 0 = \ddot\theta(t)\ \forall t$

Ex. Finding controlled equilibrium points

$\dot{\theta}(t) = \ddot{\theta}(t) = 0 \ \forall t$

$$\Rightarrow \cancel{ml^2} \cancel{\ddot{\theta}}^{0} + D\cancel{l}\cancel{\dot{\theta}}^{0} + mg\cancel{l}\sin\theta_0 - u_0 \cancel{l} = 0$$

$$\Rightarrow \sin\theta_0 = \frac{u_0}{mg}$$

$$\Rightarrow \theta_0 = \sin^{-1}\left(\frac{u_0}{mg}\right)$$

all of the controlled equilibrium points for the pendulum.

Ex. $u_0 = 0$, $\theta_0 = \sin^{-1}(0) = 0$ or π

$\Rightarrow$ $(0,0)$ and $(\pi, 0)$ are both controlled equilibrium points at $u_0 = 0$

(θ_0, u_0) $\quad$ (θ_0, u_0)

Linearization about $(\theta_0, u_0) = (0,0)$ $\Rightarrow$ $mgl\Delta\theta + Dl\Delta\dot{\theta} + ml^2\Delta\ddot{\theta} - l\Delta u = 0$

$\Rightarrow$ stable

Linearization about $(\theta_0, u_0) = (\pi, 0)$ $\Rightarrow$ $-mgl\Delta\theta + Dl\Delta\dot{\theta} + ml^2\Delta\ddot{\theta} - l\Delta u = 0$

$\Rightarrow$ unstable

$\Rightarrow$ both LTI!