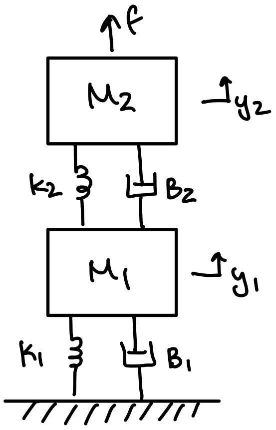
Tutorial 3

Modelling

Ex. 3.1

Consider the following system in vertical translation

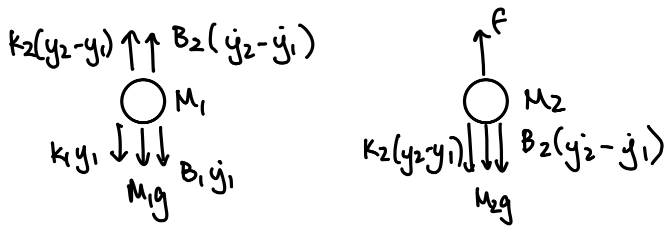


Suppose that when $y_1 = y_2 = 0$, neither spring is stretched nor compressed

Goal: Find the transfer function* of the system

*I'll explain later :)

Start by drawing FBD



Apply Newton's 2nd law to get

① $M_1\ddot{y}_1 = B_2(\dot{y}_2-\dot{y}_1) + k_2(y_2-y_1) - B_1\dot{y}_1 - k_1 y_1 - M_1 g$

② $M_2\ddot{y}_2 = f - B_2(\dot{y}_2-\dot{y}_1) - k_2(y_2-y_1) - M_2 g$

affine functions due to the constant term, not linear

note: suppose $y = ax+b$, $y_1 = ax_1+b$, $y_2 = ax_2+b$

$a(x_1+x_2)+b \neq a(x_1+x_2) + 2b$

⇒ To solve this we can be solved by instead measuring the displacements of the masses away from their equilibrium positions

⇒ To find the equilibrium, set all derivatives to zero, and solve for y_{1e} and y_{2e} (e: equilibrium)

(chose to find equilibrium point for the case where f=0)

① $0 = k_2(y_{2e}-y_{1e}) - k_1 y_{1e} - M_1 g$

② $0 = 0 - k_2(y_{2e}-y_{1e}) - M_2 g \Rightarrow k_2(y_{2e}-y_{1e}) = -M_2 g$

$\Rightarrow k_1 y_{1e} = -M_1 g - M_2 g$ } spring forces counterbalance gravity

This gives us $y_{1e} = \dfrac{-(M_1+M_2)g}{k_1}$ and $y_{2e} = -\dfrac{M_2 g}{k_2} + y_{1e}$

Now introduce new variables

$\Delta y_1 = y_1 - y_{1e}$

$\Delta y_2 = y_2 - y_{2e}$

} measure displacement from equilibrium points

Changing the variables in the original ODE gives us

$M_1\Delta\ddot{y}_1 = B_2(\Delta\dot{y}_2 - \Delta\dot{y}_1) + k_2(\Delta y_2 + y_{2e} - (\Delta y_1 + y_{1e})) - B_1\Delta\dot{y}_1 - k_1(\Delta y_1 + y_{1e}) - M_1 g$

$M_2\Delta\ddot{y}_2 = f - B_2(\Delta\dot{y}_2 - \Delta\dot{y}_1) - k_2(\Delta y_2 + y_{2e} - (\Delta y_1 + y_{1e})) - M_2 g$

note: equilibrium points don't appear in derivative terms because they are constant

⇒ we literally solved for equilibrium points that would cancel out the gravity terms. Simplifying gives us

$$M_1 \Delta\ddot{y}_1 = B_2(\Delta\dot{y}_2 - \Delta\dot{y}_1) + k_2(\Delta y_2 - \Delta y_1) - B_1\Delta\dot{y}_1 - K_1\Delta y_1$$
$$M_2\Delta\ddot{y}_2 = f - B_2(\Delta\dot{y}_2 - \Delta\dot{y}_1) - k_2(\Delta y_2 - \Delta y_1)$$

For notational simplicity, we can drop the Δs, provided we keep in mind that y_1 and y_2 are now going to denote the displacement from the equilibrium point

$$M_1\ddot{y}_1 = B_2(\dot{y}_2 - \dot{y}_1) + k_2(y_2 - y_1) - B_1\dot{y}_1 - k_1 y_1$$
$$M_2\ddot{y}_2 = f - B_2(\dot{y}_2 - \dot{y}_1) - k_2(y_2 - y_1)$$

this y_1 is measuring displacement from eqm pt previously y_1 was measuring from when the spring was neither stretched nor compressed

Now, our goal is to find the transfer function from f to y_1* the displacement from eqm pt

* i told you i'd explain later:)

To do this, take the laplace transform with zero initial conditions

$$M_1 s^2 Y_1(s) = B_2(sY_2(s) - sY_1(s)) + k_2(Y_2(s) - Y_1(s)) - B_1 sY_1(s) - K_1 Y_1(s)$$
$$M_2 sY_2(s) = F(s) - B_2(sY_2(s) - sY_1(s)) - k_2(Y_2(s) - Y_1(s))$$

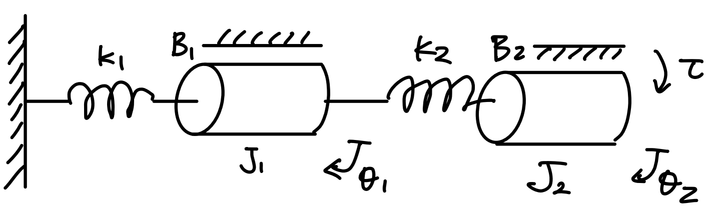
Now substitute and rearrange to get a relationship between $F(s)$ and $Y_1(s)$ only

$$(M_1 s^2 + (B_1+B_2)s + k_1 + k_2)Y_1(s) = (B_2 s + k_2)Y_2(s) \Rightarrow Y_2(s) = \frac{M_1 s^2 + (B_1+B_2)s + k_1 + k_2}{B_2 s + k_2} Y_1(s)$$
$$(M_2 s^2 + B_2 s + k_2)Y_2(s) = (B_2 s + k_2)Y_1(s) + F(s)$$

$$(M_2 s^2 + B_2 s + k_2)\frac{M_1 s^2 + (B_1+B_2)s + k_1 + k_2}{B_2 s + k_2} Y_1(s) = (B_2 s + k_2)Y_1(s) + F(s)$$
$$(M_2 s^2 + B_2 s + k_2)(M_1 s^2 + (B_1+B_2)s + k_1 + k_2)Y_1(s) = (B_2 s + k_2)^2 Y_1(s) + (B_2 s + k_2)F(s)$$

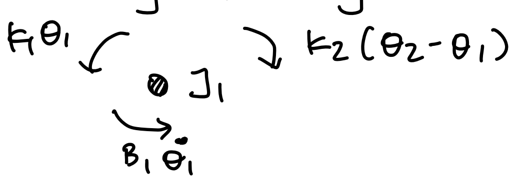
$$\boxed{\frac{Y_1(s)}{F(s)} = \frac{B_2 s + k_2}{(M_2 s^2 + B_2 s + k_2)(M_1 s^2 + (B_1+B_2)s + k_1 + k_2) - (B_2 s + k_2)^2}}$$

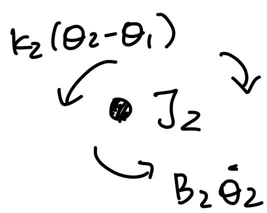
Ex.3.2 Find the model for the following mechanical system in rotation



K_1 and k_2 are torsion-spring coefficients
B_1 and B_2 are kinetic friction torsion coefficients
J_1 and J_2 are moments of inertia
θ_1 and θ_2 are the angular displacements of the rotating masses
τ is the applied torque.

Start by drawing FBD





Apply Newton's 2nd law to get

$$J_1\ddot{\theta}_1 = k_2(\theta_2 - \theta_1) - k_1\theta_1 - B_1\dot{\theta}_1$$
$$J_2\ddot{\theta}_2 = \tau - k_2(\theta_2 - \theta_1) - B_2\dot{\theta}_2$$