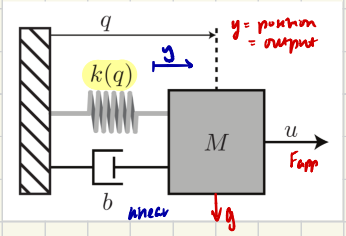
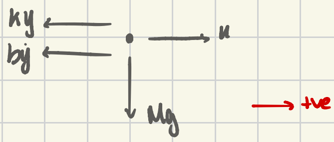
Asn #3

1) DE + Linearization + TF



spring restoring force: $k(q) = K(1-a^2q^2)q$



$F_{net} = ma$

$M\ddot{q} = u - kq - b\dot{q} = u - K(1-a^2q^2)q$

$u = M\ddot{q} + b\dot{q} + K(1-a^2q^2)q$

$u = M\ddot{q} + b\dot{q} + Kq - Ka^2q^3$ (a)

linearize at $q_0 = 0.5m$ & controlled equilibrium

$u = \cancel{M\ddot{q}} + \cancel{b\dot{q}} + Kq - Ka^2q^3$ ⌊ system at rest

$u_0 = \underbrace{Kq_0 - Ka^2q_0^3}_{\text{nonlinear term}}$ ⌊ $\dot{q}, \ddot{q} = 0$

Let $f(q) = Kq - Ka^2q^3$

$f(q) = f(q_0) + \left.\frac{df}{dq}\right|_{q=q_0}(q-q_0)$ → Taylor Series (neglect higher order terms)

$f(q) - f(q_0) = \left.\frac{df}{dq}\right|_{q=q_0}(q-q_0)$

$\Delta f(q) = m|_{q=q_0}\Delta q$

$\left.\frac{df}{dq}\right|_{q=q_0} = K - 3Ka^2q^2 = K - 3Ka^2(0.5)^2$

$= K - Ka^2(0.75)$

$= (1 - 0.75a^2)K$ ⌊ linearized stiffness

$\Delta u = M\Delta\ddot{q} + b\Delta\dot{q} + K(1-0.75a^2)\Delta q$ (b)

transfer function: $\Delta Q(s)/\Delta U(s)$ using Laplace Transform

$U(s) = Ms^2Q(s) - \cancel{sq(0)} - \cancel{\dot{q}(0)} + bsQ(s) - \cancel{q(0)} + K(1-0.75a^2)Q(s)$ → 0 initial conditions

$U(s) = Ms^2Q(s) + bsQ(s) + K(1-0.75a^2)Q(s)$

$G(s) = \frac{Q(s)}{U(s)} = \frac{1}{Ms^2 + bs + K(1-0.75a^2)}$ (c)

2) seesaw: $(M_b + 3M_l)L\ddot{\theta}(t) = 6(f(t) - M_l(g))\cos(\theta(t))$

linearize at $\theta = \pi/4$ & controlled equilibrium

let $\theta = \Delta\theta + \pi/4 \longrightarrow 0 = 6(f_0 - M_l(g))\cos(\pi/4)$

$f_0 = M_l(g)$

nonlinear part: $h(f, \theta) = 6(f - M_l(g))\cos(\theta)$ // 1st order Taylor expansion at (f_0, θ_0)

$$\Delta RHS = \left.\frac{\partial h}{\partial f}\right|_{(f_0,\theta_0)} \Delta f + \left.\frac{\partial h}{\partial \theta}\right|_{(f_0,\theta_0)} \Delta\theta$$

$$= 6\cos(\theta_0)\Delta f + \cancel{[-6(\overbrace{f_0 - M_l(g)}^{= M_l(g) - M_l(g) = 0})\sin(\theta_0)]}\Delta\theta$$

$$= 3\sqrt{2}\,\Delta f$$

$(M_b + 3M_l)L\Delta\ddot{\theta} = 3\sqrt{2}\,\Delta f$ (a)

transfer fn $\Delta\theta/\Delta f$

$(M_b + 3M_l)L\Delta\ddot{\theta} = 3\sqrt{2}\,\Delta f \longrightarrow$ take LT, zero ICs

$(M_b + 3M_l)Ls^2\Delta\Theta(s) = 3\sqrt{2}\,\Delta F(s)$

$$\frac{\Delta\Theta(s)}{\Delta F(s)} = \frac{3\sqrt{2}}{(M_b + 3M_l)Ls^2}$$ (b)

poles: double pole at $s = 0$

zeroes: none

stable? no, unstable (c)

BIBO stability: if every bounded input $\rightarrow$ bounded output

└ since pole @ $s = 0$, this yields an unbounded output

(system is BIBO unstable)

3) EM relay: $L\dot{i}_1 = V - R_1 i_1 - R_2 i_2$
$R_2 \dot{i}_2 = \frac{1}{C}(i_1 - i_2)$
$m\ddot{x} = -\frac{K i_2^2}{x^2} - k(x - 0.1) - B\dot{x}$
} already linear

linearize at $x = -10m$ & controlled equilibrium

$\dot{i}_1 = \dot{i}_2 = \dot{x} = \ddot{x} = 0 \longrightarrow$ equilibrium

$0 = V_0 - R_1 i_{1,0} - R_2 i_{2,0} \longrightarrow V_0 = R_1 i_{1,0} + R_2 i_{2,0}$

$0 = \frac{1}{C}(i_{1,0} - i_{2,0}) \longrightarrow i_{1,0} = i_{2,0} \triangleq i_0$

$V_0 = i_0(R_1 + R_2)$

$0 = -\frac{K i_0^2}{x_0^2} - k(x_0 - 0.1) \longrightarrow \frac{K i_0^2}{x_0^2} = -k(x_0 - 0.1)$

for $x_0 = -10$: $x_0^2 = 100$, $x_0 - 0.1 = -10.1$

$i_0^2 = \frac{(10.1k)100}{K} = \frac{1010k}{K} \longrightarrow i_0 = \pm\sqrt{\frac{1010k}{K}}$

$\Delta i_1 = i_1 - i_0$, $\Delta i_2 = i_2 - i_0$, $\Delta x = x - x_0$, $\Delta V = V - V_0$

linearize: $-\frac{K i_2^2}{x^2}$ about (i_0, x_0)

$$\frac{\partial}{\partial i_2}\left(\frac{-K i_2^2}{x^2}\right)\Bigg|_0 = \frac{-2K i_0}{x_0^2} = \frac{-2K i_0}{100} = \frac{-K i_0}{50}$$

$$\frac{\partial}{\partial x}\left(\frac{-K i_2^2}{x^2}\right)\Bigg|_0 = \frac{2K i_0^2}{x_0^3} = -2.02k$$

$$\frac{\partial}{\partial x}\left[-k(x - 0.1)\right] = -k \longrightarrow \text{spring term derivative}$$

$$-2.02k + (-k) = -3.02k$$

$$m\Delta\ddot{x} = -\frac{K i_0}{50}\Delta i_2 - B\Delta\dot{x} - 3.02k\Delta x \quad \text{(a)}$$

transfer fn $X/V \longrightarrow$ take LTs

$(Ls + R_1)I_1 + R_2 I_2 = V$

$(R_2 s + 1/C)I_2 = (1/C)I_1 \longrightarrow I_2 = \frac{1}{1 + CR_2 s} I_1$

$I_1 = \frac{V - R_2 I_2}{Ls + R_1} \longrightarrow I_2 = \frac{V - R_2 I_2}{(Ls + R_1)(1 + CR_2 s)}$

$I_2(Ls + R_1)(1 + CR_2 s) + R_2 I_2 = V$

$I_2\left[(Ls + R_1)(1 + CR_2 s) + R_2\right] = V$

$(ms^2 + Bs + 3.02k)X = -\frac{K i_0}{50} I_2$

$\hookrightarrow -\frac{K i_0}{50}\left[\frac{V}{(Ls + R_1)(1 + CR_2 s) + R_2}\right]$

$$\frac{\Delta X(s)}{\Delta V(s)} = -\frac{K i_0}{50}\left[\frac{1}{((Ls + R_1)(1 + CR_2 s) + R_2)(ms^2 + Bs + 3.02k)}\right] \quad \text{(b)}$$

4) Consider the feedback control system shown in Figure 2. Suppose that

$$P(s) = \frac{1}{s-1}, \quad C(s) = K_p$$

where $K_p > 0$ is a constant called the gain. This controller is called a proportional controller because the input $u(t)$ is proportional to the error $e(t)$.

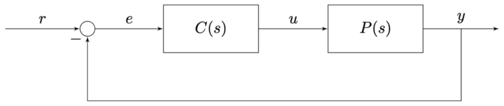


Figure 2: Feedback Control System for Problems 4-5

$U(s) = C(s) \cdot (R(s) - Y(s))$

$U(s) = K_p(R(s) - Y(s))$

$Y(s) = (R(s) - Y(s))\, C(s) \cdot P(s)$

$Y(s) = \dfrac{K_p(R(s) - Y(s))}{s-1}$

↳ $Y(s) = \dfrac{U(s)}{s-1}$ // sub into T_{ry}

a. is plant stable?

pole at $s=1 \rightarrow Re(p) > 0$, growth $\rightarrow$ unstable

pole in RHP

b. transfer fn T_{ry}, T_{ru}

$T_{ry} = Y(s)/R(s) = \dfrac{K_p}{s-1+K_p}$

$$\left\{ \begin{array}{l} Y(s) = \dfrac{K_p(R(s) - Y(s))}{s-1} \\[2ex] \dfrac{Y(s)(s-1)}{K_p} = R(s) - Y(s) \rightarrow Y(s)\left(\dfrac{(s-1)}{K_p} + 1\right) = R(s) \end{array} \right.$$

↳ $\dfrac{s-1+K_p}{K_p}$

$T_{ru} = U(s)/R(s)$

└ $Y(s) = \dfrac{U(s)}{s-1} \longrightarrow U(s) = Y(s)(s-1)$

$T_{ru} = U(s)/R(s) = \dfrac{K_p(s-1)}{s-1+K_p}$

c. values of K_p for stability $\longrightarrow$ stable if pole in LHP

$T_{ry} = \dfrac{K_p}{s-1+K_p}$ $\qquad s + K_p - 1 = 0$

$s = 1 - K_p$

$T_{ru} = \dfrac{K_p(s-1)}{s-1+K_p}$ $\qquad 1 - K_p < 0 \longrightarrow K_p > 1$

d. tradeoff in choosing large K_p

settling time: $p = K_p - 1 \longrightarrow$ as K_p increases, τ decreases

$\tau = \dfrac{1}{K_p - 1}$ $\qquad$ faster response, shorter settling time

steady-state error: DC gain of T_{ry} is $\dfrac{K_p}{K_p - 1}$ $\longrightarrow$ as $K_p \rightarrow \infty$, gain approaches 1

output tracks reference correctly

tradeoff: larger K_p makes system faster and more accurate

requires larger initial control effect $\rightarrow$ could saturate / damage physical actuator

5) Consider the feedback control system shown in Figure 2. Suppose that

$$P(s) = \frac{1}{s-1}, \quad C(s) = K_p$$

where $K_p > 0$ is a constant called the gain. This controller is called a proportional controller because the input $u(t)$ is proportional to the error $e(t)$.

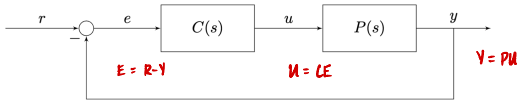


Figure 2: Feedback Control System for Problems 4-5

from P4: $\frac{Y(S)}{R(S)} = \frac{K_p}{S-1+K_p}$

$Y(S) = (R(S) - Y(S))C(S)P(S)$

$Y(S) = E(S)C(S)P(S)$

a. find transfer fn $T_{re}(S)$

$T_{re}(S) \longrightarrow E(S)/R(S)$

$E(S) = R(S) - Y(S)$

$= R(S) - P(S)U(S)$

$= R(S) - P(S)C(S)E(S)$

$E(S)(1 + P(S)C(S)) = R(S)$

$$\frac{E(S)}{R(S)} = \frac{1}{(1+P(S)C(S))}$$

characteristic equation: $1 + P(S)C(S)$

b. stability of PC combos

(i) $P(S) = \frac{1}{S+1}$ $\quad C(S) = \frac{-6}{S+2}$ $\quad T_{re} = \frac{1}{1+PC}$

$S = -1$ stable $\quad S = -2$ stable

$1 + \frac{1}{S+1} \cdot \frac{-6}{S+2} = 1 + \frac{-6}{(S+1)(S+2)} = 1 + \frac{-6}{S^2+3S+2} = 0$

$1 = \frac{6}{S^2+3S+2} \longrightarrow S^2 + 3S + 2 = 6$

$S^2 + 3S - 4 = 0$

$S^2 + 3S - 4 = 0$

$(S+4)(S-1) = 0$

$S = -4, +1 \rightarrow$ unstable

(ii) $P(S) = \frac{1}{S+1}$ $\quad C(S) = \frac{1}{S+2}$ $\quad T_{re} = \frac{1}{1+PC}$

$S = -1$ stable $\quad S = -2$ stable

$1 + PC = 1 + \frac{1}{S^2+3S+2} = 0$

$S^2 + 3S + 3 = 0$

$S = \frac{-3 \pm \sqrt{9-12}}{2} = -1.5 \pm \frac{j\sqrt{3}}{2}$

Re(S) < 0 stable

(iii) $P(S) = \frac{1}{S-1}$ $\quad C(S) = \frac{4S}{S-2}$ $\quad T_{re} = \frac{1}{1+PC}$

$S = 1$ unstable $\quad S = 2$ unstable

$1 + \frac{1}{S-1} \cdot \frac{4S}{S-2} = 1 + \frac{4S}{S^2-3S+2} = 0$

$S^2 - 3S + 2 = -4S$

$S^2 + S + 2 = 0$

$S = \frac{-1 \pm \sqrt{1-8}}{2} = -0.5 \pm \frac{j\sqrt{7}}{2}$

Re(S) < 0 stable

6) unit step response transfer fn matching

c (i) $\frac{409}{6}\left(\frac{1}{s+3-20j}+\frac{1}{s+3+20j}\right)$ (i) $p = -3 \pm 20j$

$Re(p) = -3 \longrightarrow$ decay, stable

$Im(p) = \pm 20 \longrightarrow$ osc, $f = 20$

a (ii) $\frac{2.5}{s+5}+\frac{5}{s+10}$

b (iii) $\frac{3}{s+3}$ pole $s=-3$

$Re(p) < 0 \rightarrow$ decay

e (iv) $\frac{109}{6}\left(\frac{1}{s+3-10j}+\frac{1}{s+3+10j}\right)$ (iv) $p = -3 \pm 10j$

$Re(p) = -3 \longrightarrow$ decay, stable

$Im(p) = \pm 10 \longrightarrow$ osc, $f = 10$

d (v) $\frac{-3}{s-3}$ pole $s=3$

(v) $Re(p) > 0 \longrightarrow$ growth, unstable

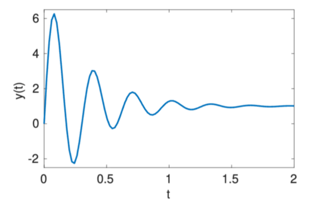


Figure 5: Option (c) for Problem 6

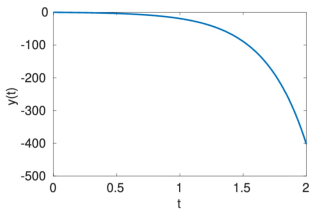


Figure 4: Option (b) for Problem 6

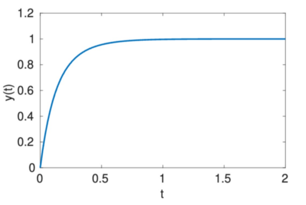


Figure 6: Option (d) for Problem 6

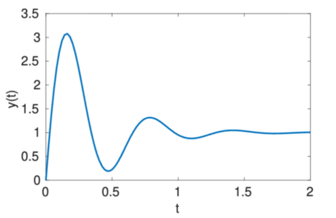


Figure 7: Option (e) for Problem 6

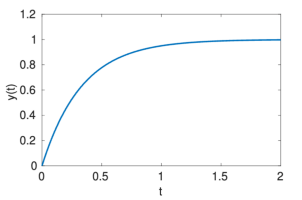


Figure 3: Option (a) for Problem 6