Standard 1st-order systems

$$H(s) = \frac{K}{sT+1}, \quad k, T > 0 \implies \text{pole} = \frac{-1}{T}$$

impulse response:

$$x(t) = \frac{k}{T} e^{-\frac{t}{T}} \, \mathbb{1}(t) \longrightarrow \text{steady-state value} = 0$$
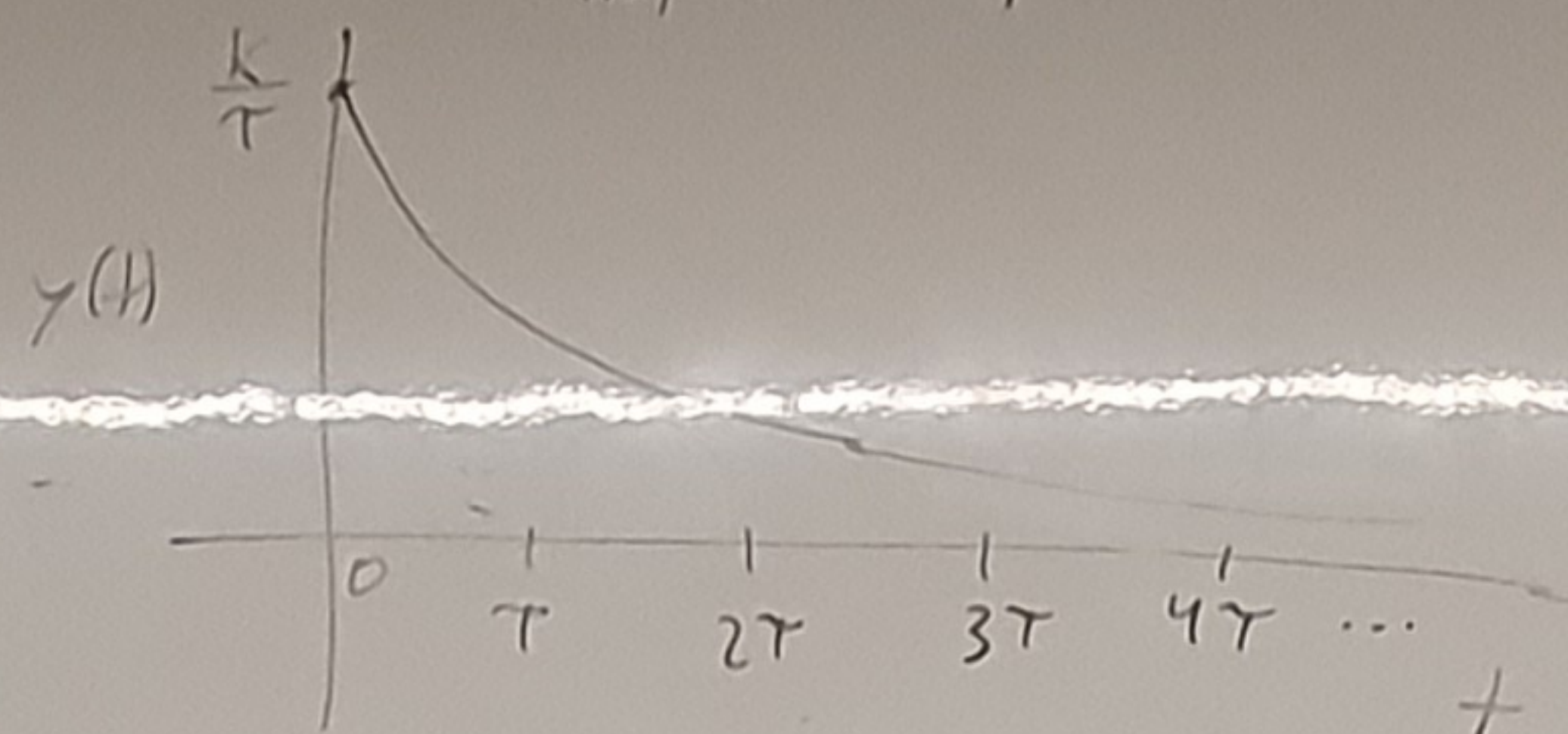
step response:

$$y(t) = K\left(1 - e^{-\frac{t}{T}}\right) \mathbb{1}(t) \longrightarrow \text{steady-state value} = K$$

To find the steady-state value,

take $\lim\limits_{t \to \infty}$

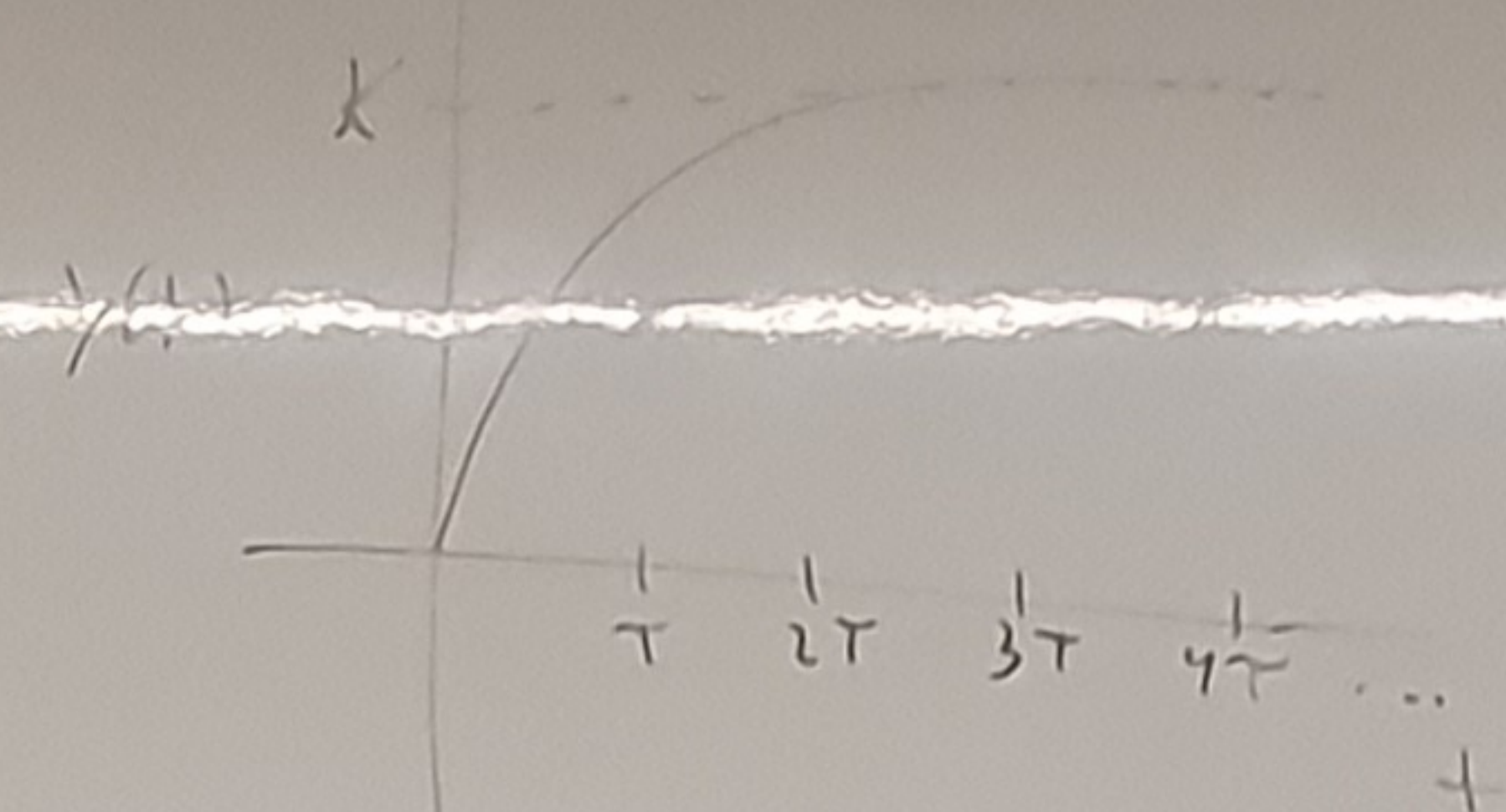* Note: steady-state value is only well-defined for stable systems

impulse response

T is called the <u>time constant</u> $\to$ determines the rate of decay

If $t = T \implies e^{-t/T} = e^{-1}$

If $t = 3T \implies e^{-t/T} = e^{-3} \approx 0.05$
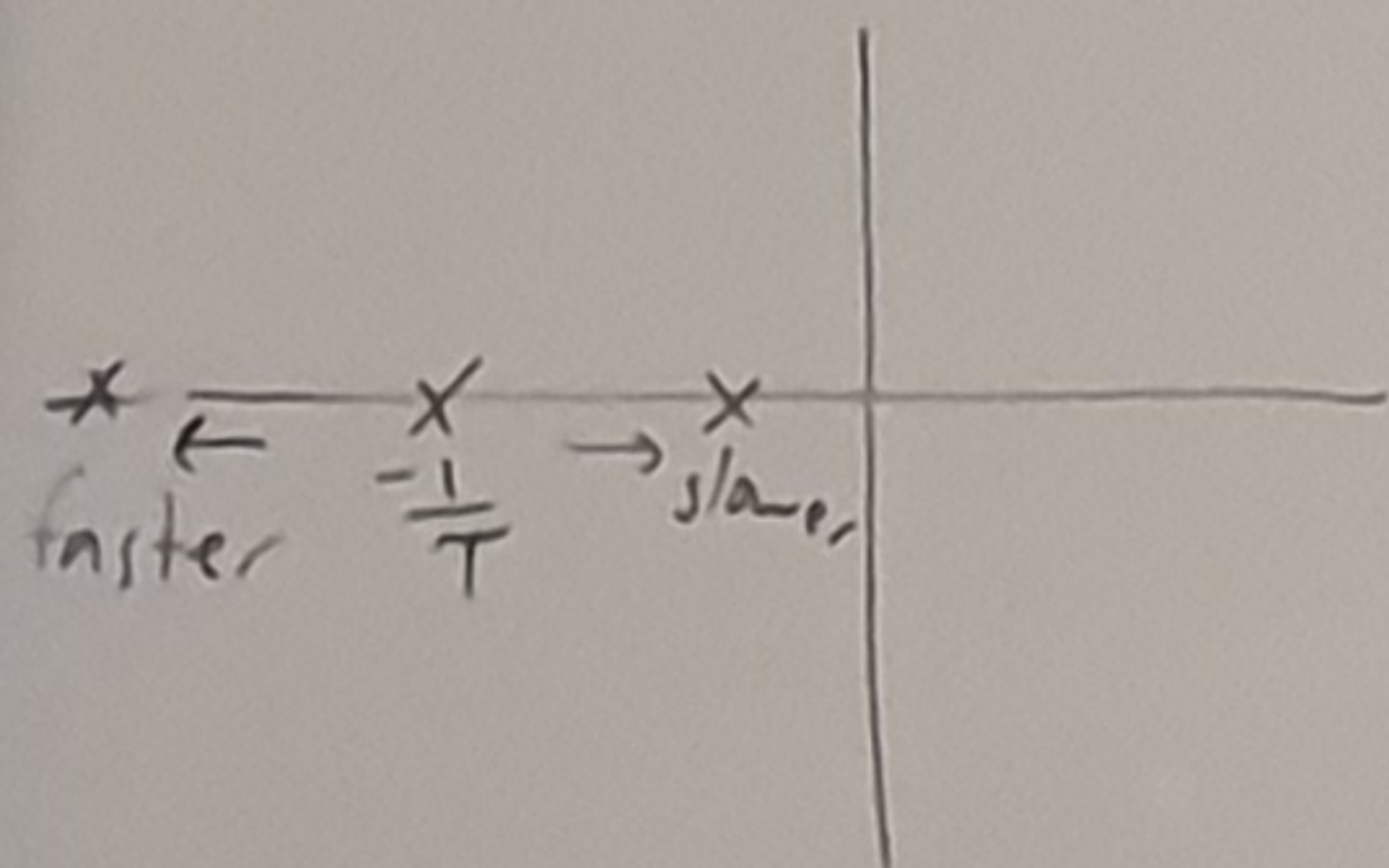
If $t = 4T \implies e^{-t/T} = e^{-4} \approx 0.02$

step response

Position of the pole in $\mathbb{C}$:

- 1 real pole at $s = \frac{-1}{T}$

- as pole $\to 0$, $T \to \infty \implies$ response slows (transient decays more slowly)
- as pole $\to -\infty$, $T \to 0 \implies$ response speeds up (transient decays faster)

potential issues with fast responses:

- larger overshoot
- larger oscillations
- larger control effort
- more vulnerable to disturbances and noise (less inertia)

When designing controllers, we'll try to keep poles away from the imaginary axis to ensure acceptable speed
- but also we will try to avoid potential issues with being too fast

Standard 2nd order system

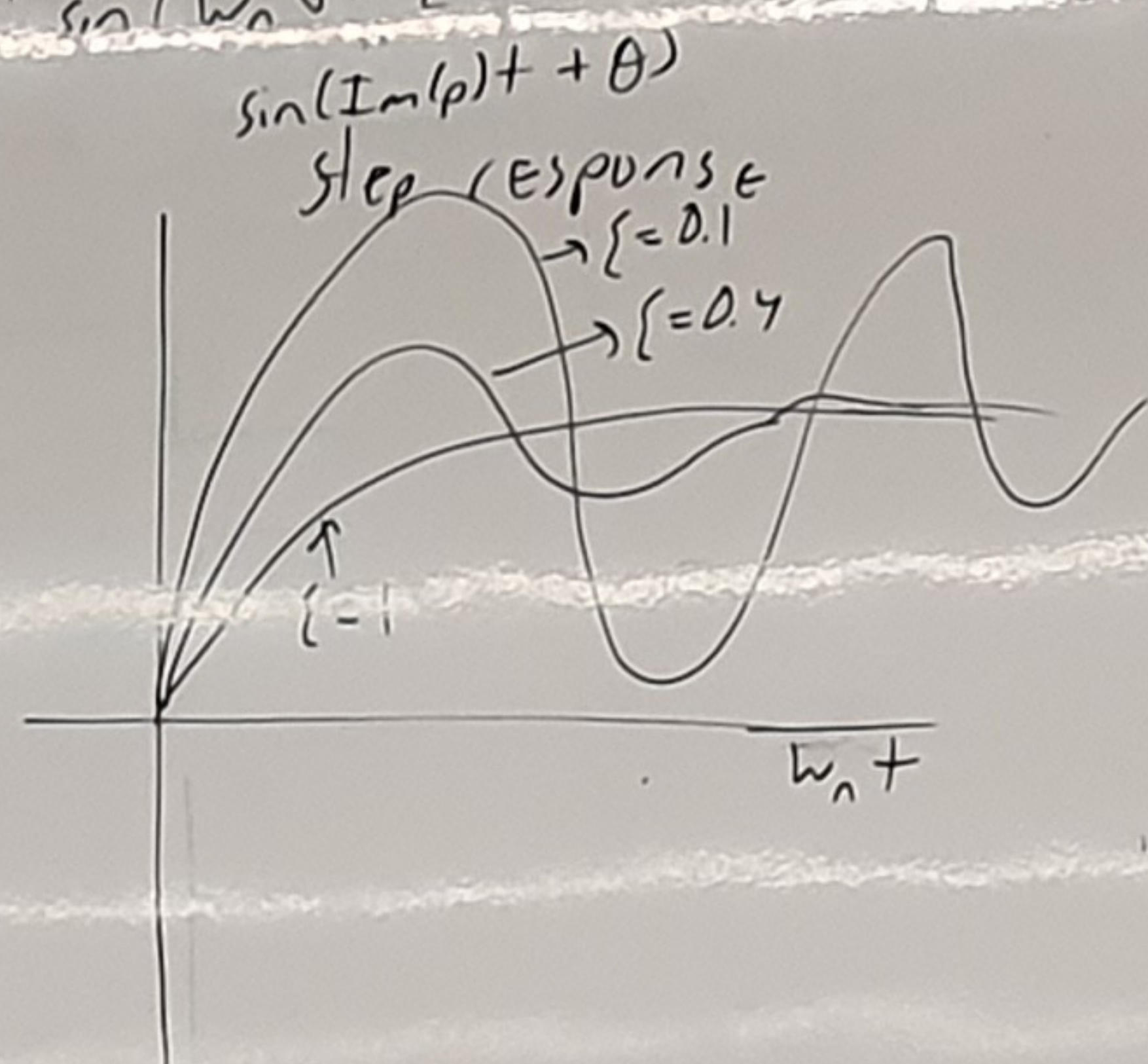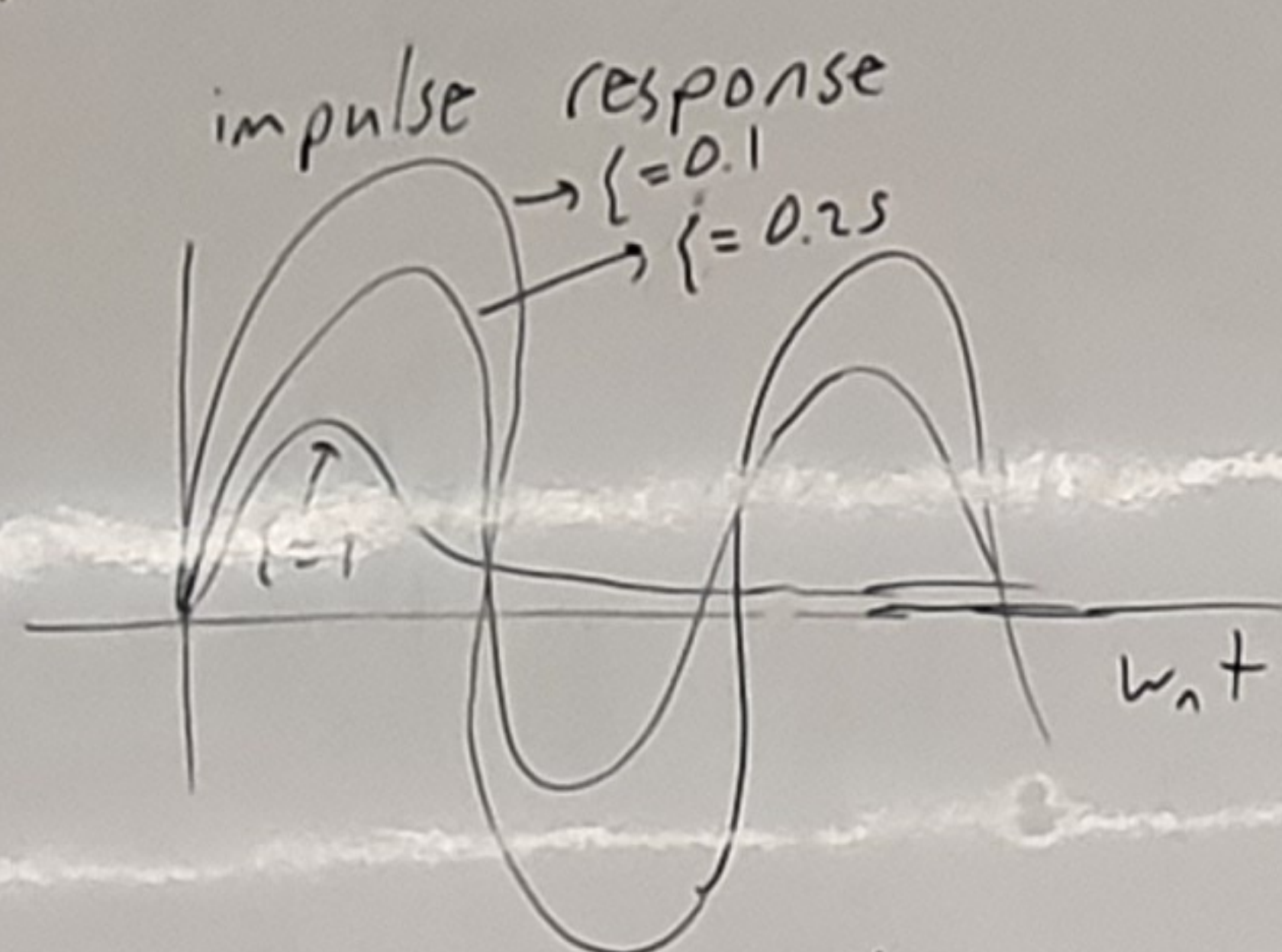$$H(s) = \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}, \quad w_n, \zeta > 0$$

We'll consider only this case

e.g. mass-spring-damper system, RLC circuit   (underdamped)

system has 2 poles ~ $0 < \zeta < 1 \implies$ 1 pair of complex conjugate poles
- $\zeta = 1 \implies$ 2 real poles at the same location $\to$ critically damped
- $\zeta > 1 \implies$ 2 distinct real poles $\to$ overdamped

impulse response:

$$y(t) = L^{-1}(H(s)) = \frac{\omega_n}{\sqrt{1-\zeta^2}} \underbrace{e^{-\zeta\omega_n t}}_{e^{Re(p)t}} \underbrace{\sin(\omega_n\sqrt{1-\zeta^2}\,t)}_{\sin(Im(p)t)} \mathbb{1}(t) \longrightarrow \text{steady-state value} = 0$$

step response

$$y(t) = L^{-1}(H(s)\tfrac{1}{s}) = 1 - \underbrace{\frac{1}{\sqrt{1-\zeta^2}}}_{} \underbrace{e^{-\zeta\omega_n t}}_{e^{Re(p)t}} \underbrace{\sin(\omega_n\sqrt{1-\zeta^2}\,t + \cos^{-1}\zeta)}_{\sin(Im(p)t+\theta)} \mathbb{1}(t) \longrightarrow \text{steady-state value} = 1$$

$$\underset{=\theta}{\overset{}{\cos^{-1}\zeta}}$$

impulse response

→ ζ=0.1
→ ζ=0.25
ζ=1
$\omega_n t$

step response
→ ζ=0.1
→ ζ=0.4
ζ=1
$\omega_n t$

It is convenient to plot the responses
as functions of $\omega_n t$ => ω_n acts as a time-scale factor:
$\omega_n \uparrow$ => response speeds up, $\omega_n \downarrow$ => response slows down

The plots show that
as ζ↑ => response is
less oscillatory

We call ω_n the natural
frequency
and
ζ the damping ratio

Positions of poles in ℂ:

$$0 < \zeta < 1 \implies s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

Re(p)   Im(p)
↓   ↓
determine: rate of decay   frequency of oscillations

faster   more
X ·····   ·····X oscillatory
  ····· ·····
less X   slower
oscillatory: ω_n
           θ
$-\zeta\omega_n$

$\omega_n\sqrt{1-\zeta^2}$

X $-\omega_n\sqrt{1-\zeta^2}$

The magnitude of the pole is $\sqrt{(-\zeta\omega_n)^2 + (\pm\omega_n\sqrt{1-\zeta^2})^2} = \sqrt{\omega_n^2(\zeta^2 + 1 - \zeta^2)} = \sqrt{\omega_n^2} = \omega_n$

The phase of the pole is $\theta = \cos^{-1}\frac{\zeta\omega_n}{\omega_n} = \cos^{-1}\zeta$

So we have the following relationship:
magnitude ↑ => ω_n ↑ => response speeds up
  θ ↓ => ζ ↑ => response less oscillatory

Again, we find that the further the poles from the
imaginary axis, the faster the response

Also, the closer the poles are to the real axis, the less
oscillatory the response

To quantify the effects of ζ and ω_n on the speed and oscillation
of the transient response, let's apply some standard control
engineering performance specifications (specs).