Topic 3. Amplitude Modulation (Chapter 3)

UNIVERSITY OF WATERLOO

FACULTY OF ENGINEERING

3.1 Why is Modulation Required?

3.2 Double Side Band Large Carrier (DSB-LC or Conventional AM) Modulation

3.3 Double Side Band Suppressed Carrier (DSB-SC) Modulation

3.4 Single Side Band (SSB) Modulation

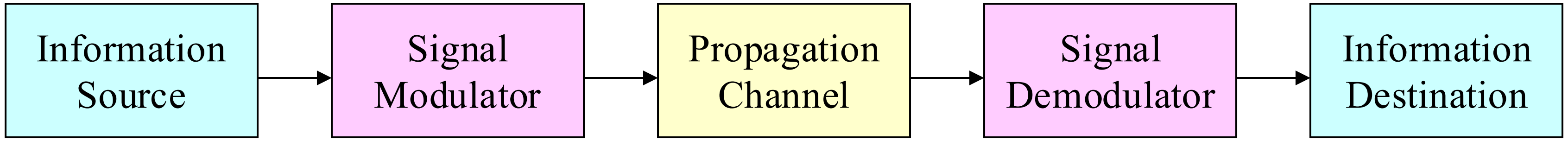
3.5 Vestigial Side Band (VSB) Modulation

3.6 Signal Multiplexing (Using AM)

Objectives of This Topic

- To study different amplitude modulation (AM) schemes
- To investigate generation and detection of AM signals
- To study applications of AM (e.g., radio broadcasting, signal multiplexing)
- To compare different AM systems

Analog Communication System



Analog signals may be transmitted directly via carrier modulation over the propagation channel and then carrier-demodulated at the receiver.

Transmitter → Modulator

Receiver → Demodulator

(Carrier) Modulation: The process by which some characteristics of a carrier passband signal (i.e., modulated signal) are varied in accordance with the message baseband signal (i.e., modulating signal).♀

- $m(t)$: message signal (*baseband* signal)

  Baseband signal: A bandlimited signal whose frequency content is in the neighbourhood of $f = 0$ Hz (DC) is referred to as a baseband signal.

- $c(t)$: the carrier signal, independent of the message $m(t)$

  $c(t) = A_c \cos(2\pi f_c t + \theta_c)$

  A_c : Carrier amplitude

  f_c : Carrier frequency $\omega_c = 2\pi f_c$ (radian frequency)

  θ_c : Carrier phase

- $m(t)$ modulates $c(t)$ in either amplitude, frequency or phase. In effect, the modulation converts $m(t)$ into a *passband* (or bandpass) form, in the neighbourhood of the carrier frequency f_c.

♀ One can also use baseband modulations which generally require wired channels, e.g., telephone landline, coaxial cable, optical fibers, etc. For an example of baseband modulation, see (digital) PCM in Topic 6.

3.1 Why is Modulation Required?

- *To achieve efficient radiation:* If the communication channel consists of free space, antennas are required to transmit and receive the signal. For efficient transmission and reception, the dimension of the antenna is determined by the corresponding wavelength.

Example: Voice signal bandwidth $f = 3$ kHz.

$$\lambda = \frac{c}{f} = \frac{3 \cdot 10^8}{3 \cdot 10^3} = 10^5 \text{ m}$$

$\rightarrow \lambda/4 = 25000\text{m}!!$

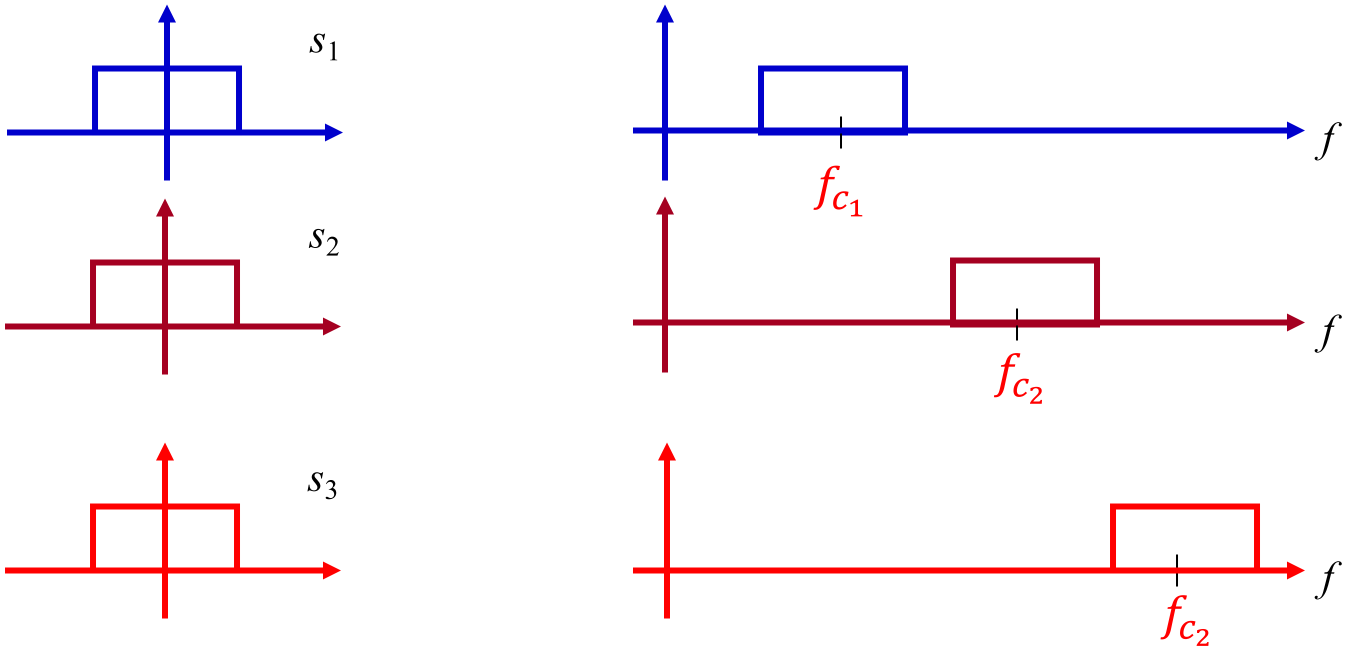
If we modulate a carrier wave at $f_c = 100$ MHz with the voice signal

$$\lambda = \frac{c}{f} = \frac{3 \cdot 10^8}{100 \cdot 10^6} = 3 \text{ m}$$

$\rightarrow \lambda/4 = 75\text{cm}$

3.1 Why is Modulation Required? (Cont.)

- *To accommodate simultaneous transmission of several signals (e.g., different radio stations, or signal multiplexing, etc.)*



Example: Radio/TV broadcasting.

3.1 Why is Modulation Required? (Cont.)

- *To expand the bandwidth of the transmitted signal for better transmission quality* (e.g., reduce the effect of noise and interference).

$$C = B \cdot \log_2(1 + SNR) \qquad \text{(Shannon Theorem)}$$

Channel capacity   Bandwidth   Signal-to-noise ratio

Channel capacity: Maximum achievable information rate that can be transmitted over the channel. We may write the Shannon Theorem as

$$SNR = 2^{C/B} - 1$$

For a given capacity C, as the bandwidth B ↗, the required SNR↘ (or the transmitted signal power decreases for fixed noise level).

3.2 Double Side Band Large Carrier (DSB-LC)

3.2.1 DSB-LC or conventional AM signal form

3.2.2 DSB-LC modulation index

3.2.3 Carrier and sideband power in DSB-LC

3.2.4 Generation of DSB-LC signals

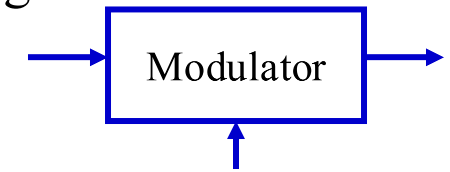
3.2.5 Demodulation of DSB-LC

3.2.6 Application of DSB-LC in AM radio broadcast

3.2.1 DSB-LC Modulated Signal (Conventional AM Signals)

Modulating signal

$m(t)$



Modulated signal

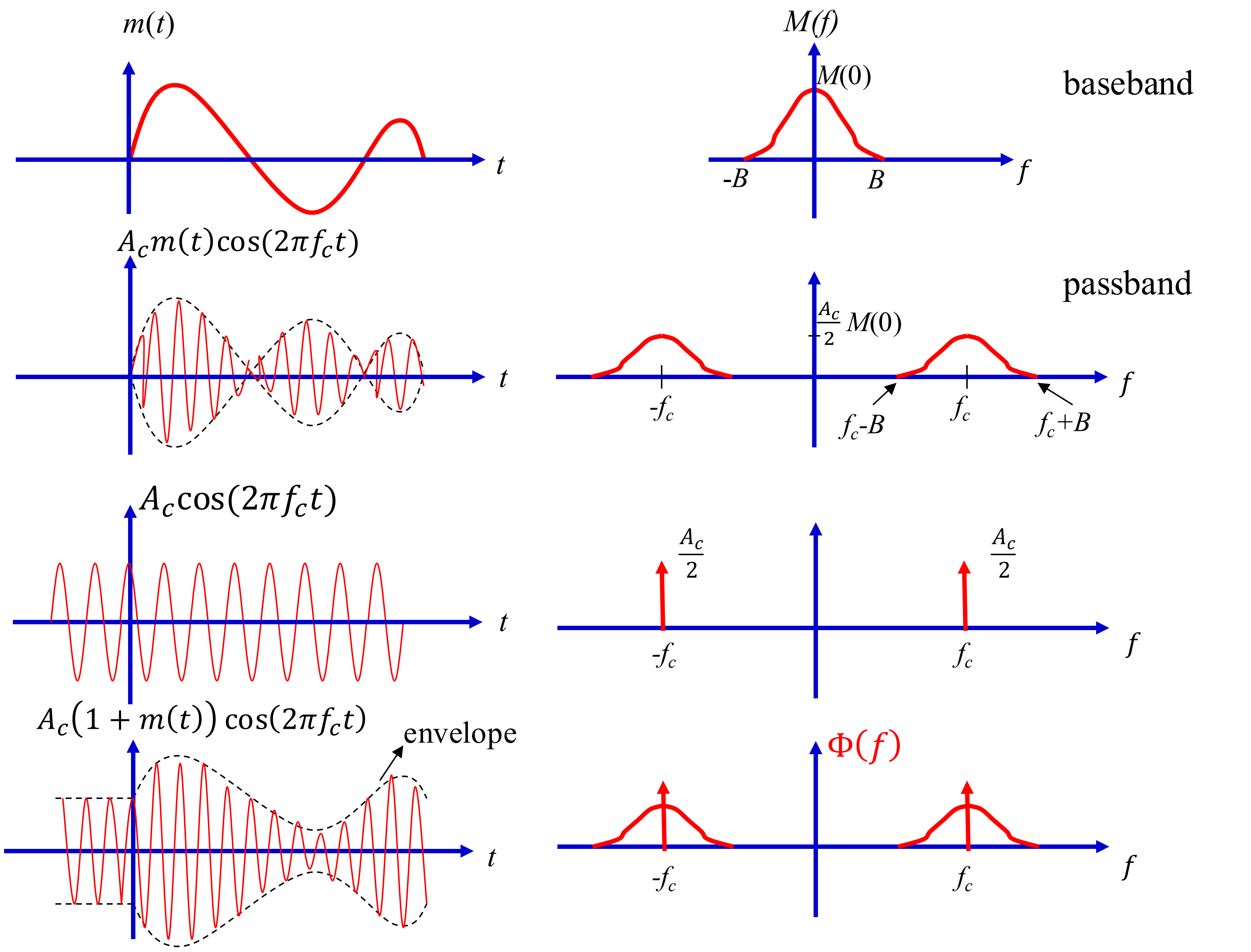
$$\phi(t) = A_c\big(1 + m(t)\big)\cos(2\pi f_c t)$$
$$= A_c\cos(2\pi f_c t) + A_c m(t)\cos(2\pi f_c t)$$

$$c(t) = A_c\cos(2\pi f_c t)$$

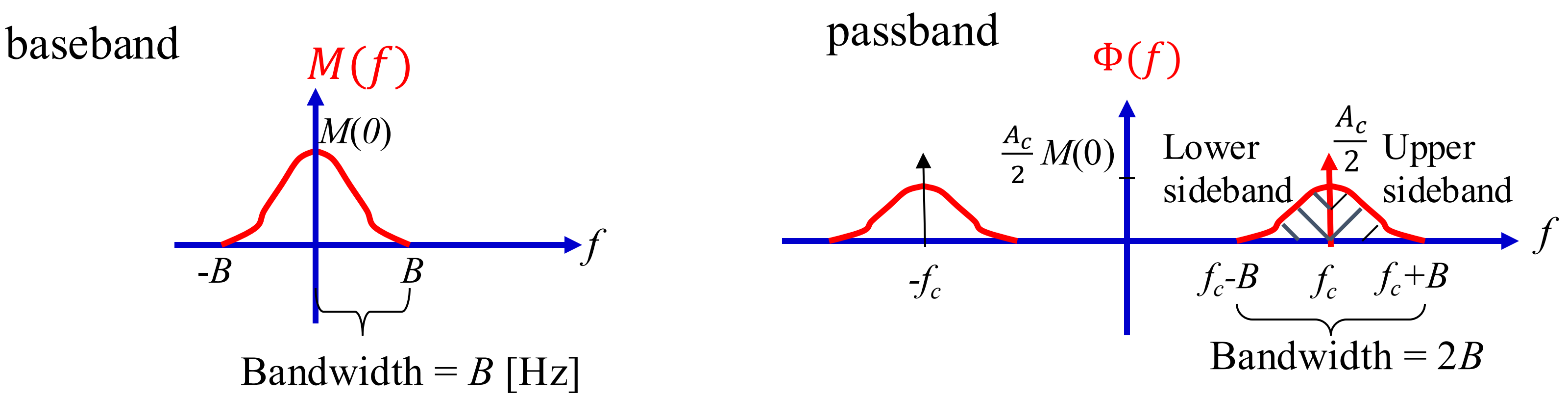
carrier

In the frequency domain:

$$\Phi(f) = \frac{A_c}{2}\big(\delta(f - f_c) + \delta(f + f_c)\big) + \frac{A_c}{2}\big(M(f - f_c) + M(f + f_c)\big)$$



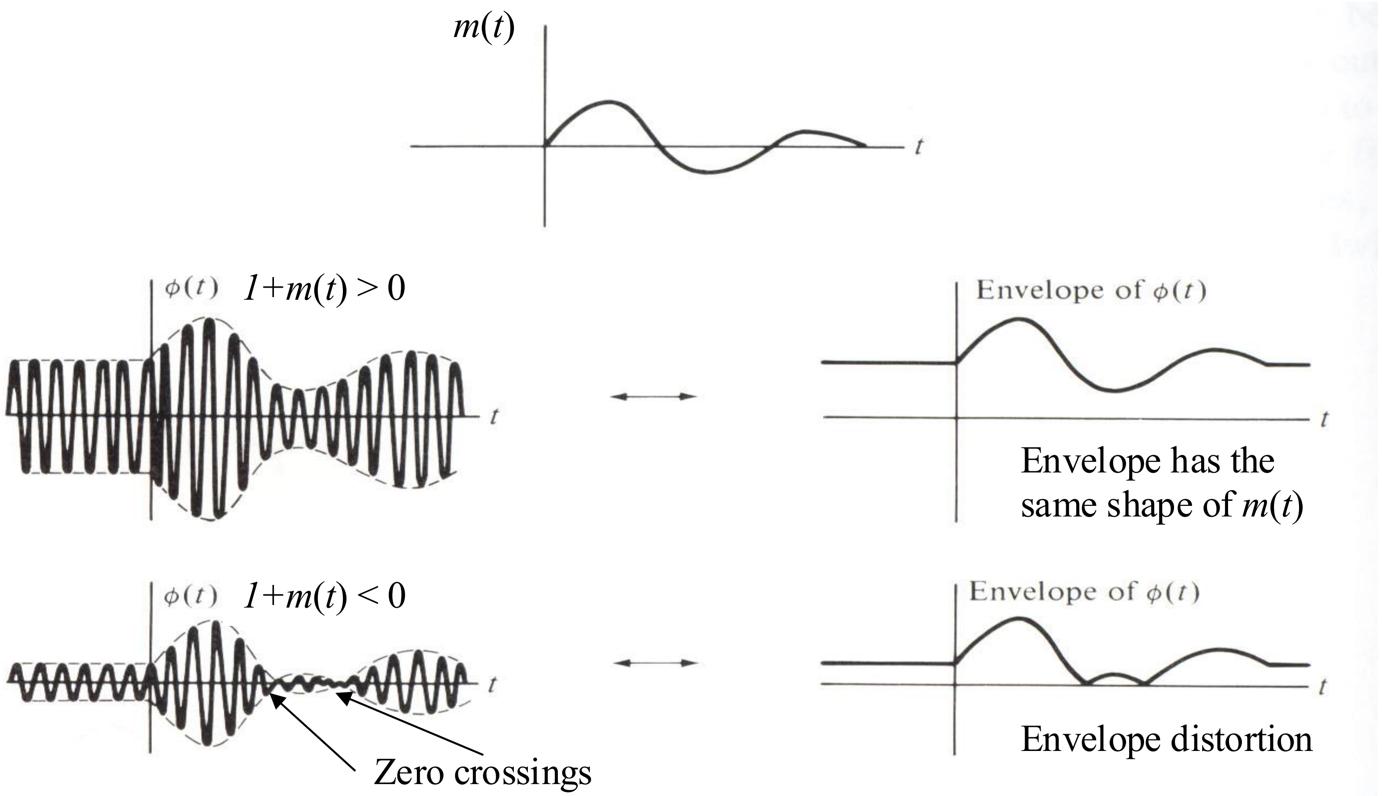
$m(t)$
t
$M(f)$
$M(0)$
$-B$
B
f
baseband
$A_c m(t)\cos(2\pi f_c t)$
t
$\frac{A_c}{2}M(0)$
$-f_c$
f_c-B
f_c
f_c+B
f
passband
$A_c \cos(2\pi f_c t)$
t
$\frac{A_c}{2}$
$\frac{A_c}{2}$
$-f_c$
f_c
f
$A_c(1+m(t))\cos(2\pi f_c t)$
envelope
t
$\Phi(f)$
$-f_c$
f_c
f



Observations

- Modulation shifts the content of $M(f)$ to the neighborhood of f_c.
- $M(f)$ for $f \in [-B, 0]$ is shifted to $\Phi(f)$ for $f \in [f_c - B, f_c]$ and is referred to as the *lower sideband*.
- $M(f)$ for $f \in [0, B]$ is shifted to $\Phi(f)$ for $f \in [f_c, f_c + B]$ and is referred to as the *upper sideband*.
- Let B denote the highest frequency component of the message $m(t)$.

  Assume $f_c \gg B \Rightarrow \phi(t)$ is defined as a *narrowband* signal (i.e., its spectral content is located in the immediate vicinity of some high center frequency f_c).
- The bandwidth of the message signal is B. The transmission bandwidth $\beta_T = 2B$ (i.e., DSB-LC is wasteful of bandwidth).
- The carrier term does not carry any information and hence the carrier power is wasted (from carrying information viewpoint- the large carrier is useful for simplifying the detection).



Observations (cont.)

- If $1+m(t) > 0$ for all t, the envelope of $\phi(t)$ of has essentially the same shape as $m(t)$.
- If $1+m(t\,) < 0$ for any t, the carrier wave becomes *overmodulated*, which results in *envelope distortion*.

3.2.2 DSB-LC Modulation Index

For a general message, to prevent overmodulation (i.e., envelope distortion), we put

$$\phi(t) = A_c(1 + \alpha m_n(t)) \cos 2\pi f_c t$$

Where $m_n(t)$ is normalized such that its minimum is *-1*. For example *,

$$m_n(t) = \frac{m(t)}{|\min m(t)|}$$

and $0 < \alpha < 1$ is defined to be the **modulation index** so chosen to prevent envelope zero-crossing.

Example: Suppose that m(t) = $3\cos(300\pi t)$ is used to modulate $c(t) = \cos(2\pi 10^5 t)$. The modulation index is $\alpha = 0.85$. Find the time and frequency domain forms of the modulated AM signal.

*For most signals of practical interest $|\min m(t)| = \max |m(t)|$

Example (continued):

For a single tone message

$m(t) = A_m \cos(2\pi f_m t)$

with carrier signal $c(t) = A_c \cos(2\pi f_c t)$

the normalized message is $m_n(t) = \cos(2\pi f_m t)$

giving

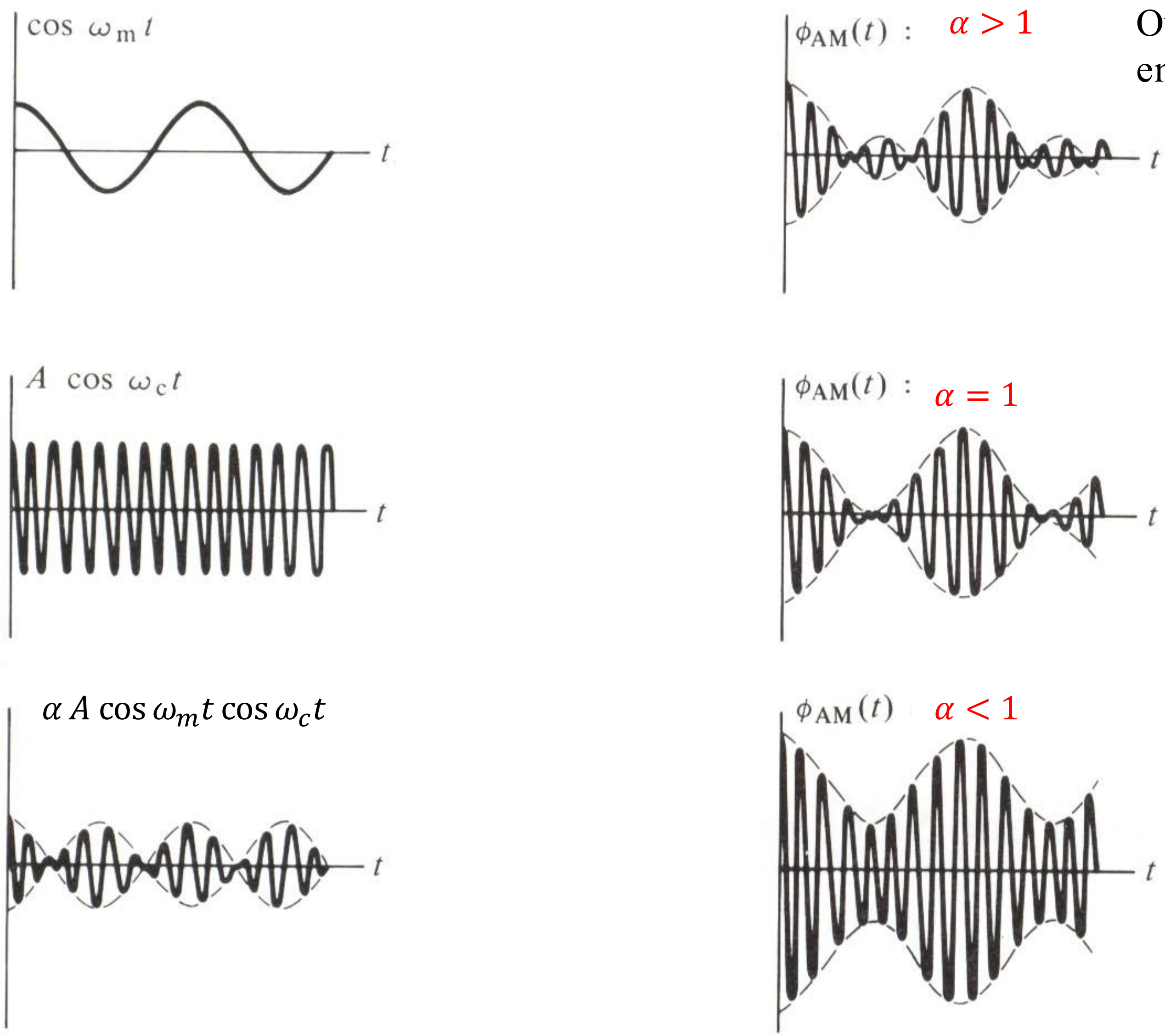
$\phi(t) = A_c(1 + \alpha \cos(2\pi f_m t))\cos(2\pi f_c t)$

Let $A_{\max}$ and $A_{\min}$ denote the maximum and minimum values of the envelope of $\phi(t)$. Then

$$\boxed{A_{\max} = A_c(1+\alpha)} \qquad \boxed{A_{\min} = A_c(1-\alpha)}$$

$$\frac{A_{\max}}{A_{\min}} = \frac{1+\alpha}{1-\alpha} \quad \Rightarrow \quad \boxed{\alpha = \frac{A_{\max} - A_{\min}}{A_{\max} + A_{\min}}}$$

A formula frequently used for reconstruction of the modulated signal from its envelope (valid for $\alpha \leq 1$)



Overmodulated $\rightarrow$ envelope distortion

Effects of varying modulation indexes

3.2.3 Carrier and Sideband Power in DSB-LC

$$\phi(t) = A_c\big(1 + \alpha m_n(t)\big)\cos 2\pi f_c t$$

$$\phi(t) = A_c g(t)\cos 2\pi f_c t\,, g(t) = 1 + \alpha m_n(t)$$

Assume that the message $m(t)$ varies slowly with respect to the carrier $c(t)$

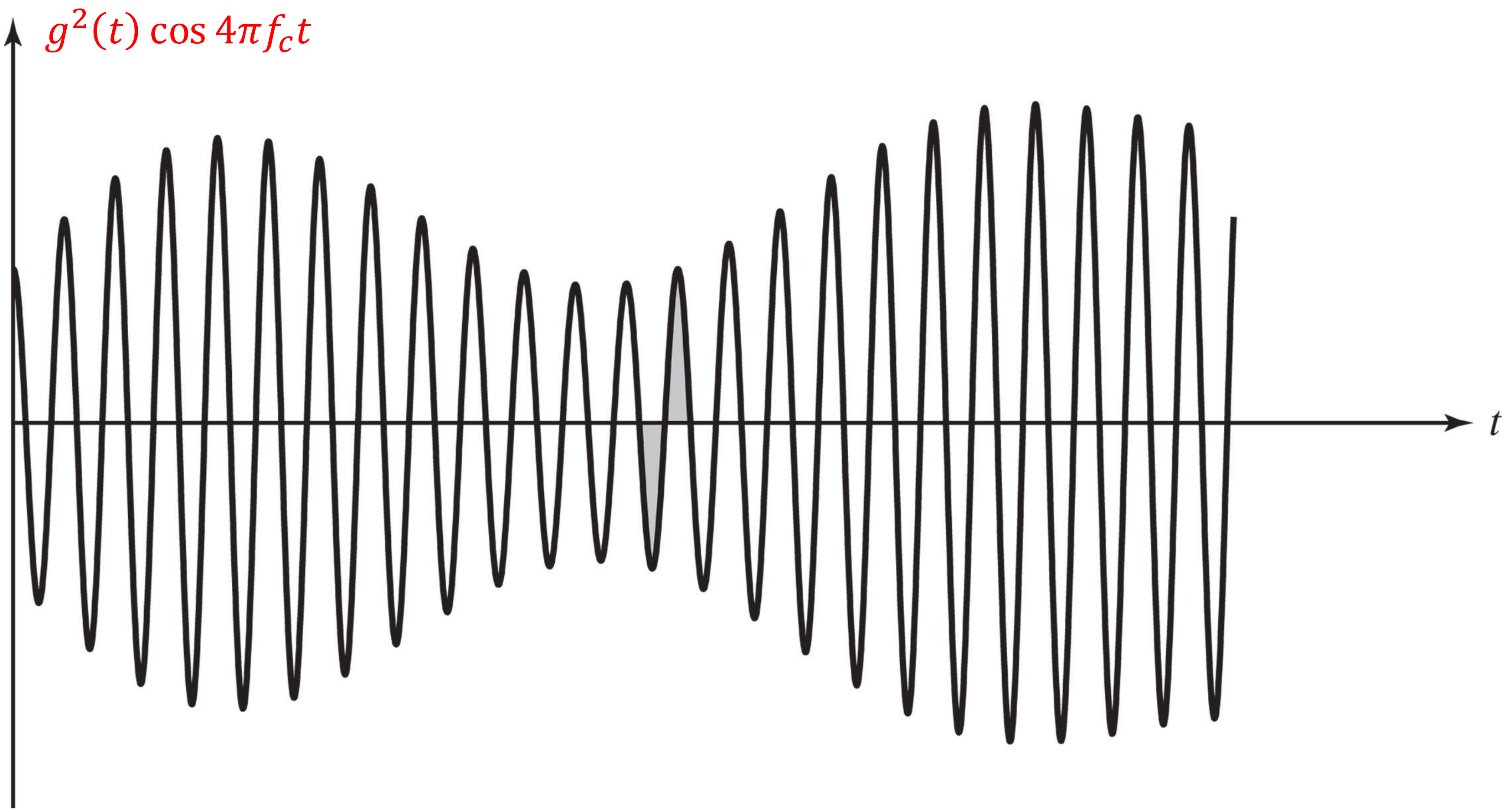
$$\phi^2(t) = A_c^2 g^2(t)\cos^2 2\pi f_c t = \frac{A_c^2}{2} g^2(t)(1 + \cos 4\pi f_c t)$$

$$\overline{\phi^2(t)} = A_c^2\overline{g^2(t)\cos^2(2\pi f_c t)} = \frac{A_c^2}{2}\left(\overline{g^2(t)} + \overline{g^2(t)\cos 4\pi f_c t}\right)$$

$$\overline{\phi^2(t)} = \frac{A_c^2}{2}\overline{g^2(t)}$$

Assume further that the message $m(t)$ has zero mean:

$$\overline{\phi^2(t)} = \frac{A_c^2}{2}(1 + \alpha^2 P_{m_n})$$


$g^2(t)\cos 4\pi f_c t$
t

$$\overline{\phi^2(t)} = \frac{A_c^2}{2} + \alpha^2 \frac{A_c^2}{2} P_{m_n}$$

Carrier Power

Sideband Power
(carries information)

Modulation (Power or transmission) Efficiency

$$\mu = \frac{\text{useful power}}{\text{total power}}$$

$$= \frac{\alpha^2 \frac{A_c^2}{2} P_{m_n}}{\frac{A_c^2}{2} + \alpha^2 \frac{A_c^2}{2} P_{m_n}}$$

$$= \frac{\alpha^2 P_{m_n}}{1 + \alpha^2 P_{m_n}}$$

Example. Continue the example of single-frequency sinusoidal signal. Determine its DSB-LC signal, the power in its upper and lower sidebands, and modulation efficiency, assuming the modulation index is α.

Solution.

$$\phi(t) = A_c \cos\omega_c t + \frac{\alpha A_c}{2}\cos(\omega_c + \omega_m)t + \frac{\alpha A_c}{2}\cos(\omega_c - \omega_m)t$$

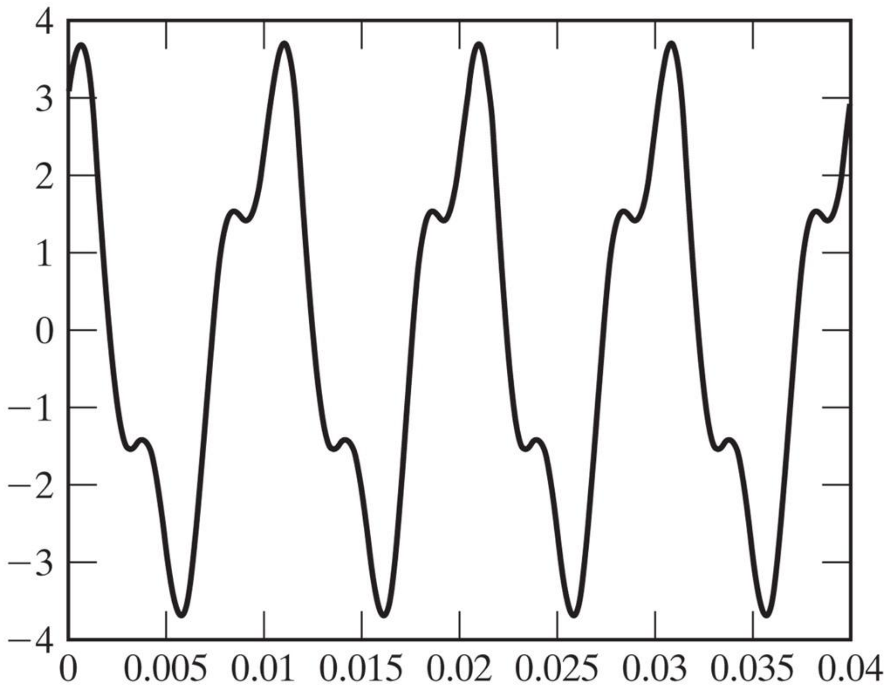
Upper sideband power $\left(\frac{\alpha A_c}{2}\right)^2 \overline{\cos^2(\omega_c + \omega_m)t} = \frac{\alpha^2 A_c^2}{8}$

Lower sideband power $\left(\frac{\alpha A_c}{2}\right)^2 \overline{\cos^2(\omega_c - \omega_m)t} = \frac{\alpha^2 A_c^2}{8}$

$$\mu = \frac{\text{total useful power}}{\text{total power}} = \frac{\alpha^2 A_c^2/4}{A_c^2/2 + \alpha^2 A_c^2/4} = \frac{\alpha^2}{2+\alpha^2}$$

Remark. For $\alpha \le 1 \rightarrow \mu \le$ 33%. Under the best condition, i.e. , α = 1, 67% of the total power is used in the carrier and represents wasted power.

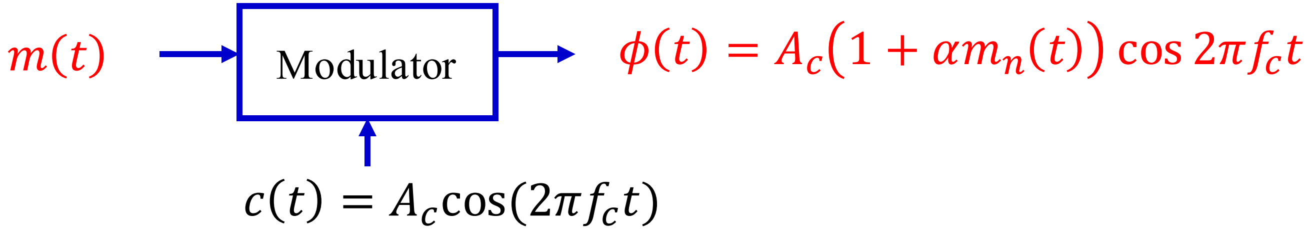
Example: The signal $m(t) = 3\cos 200\pi t + \sin 600\pi t$ is used to modulate the carrier $c(t) = \cos(2\pi 10^5 t)$.
The modulation index is $\alpha = 0.85$. Determine the power in the carrier component and the sideband components of the AM modulated signal.



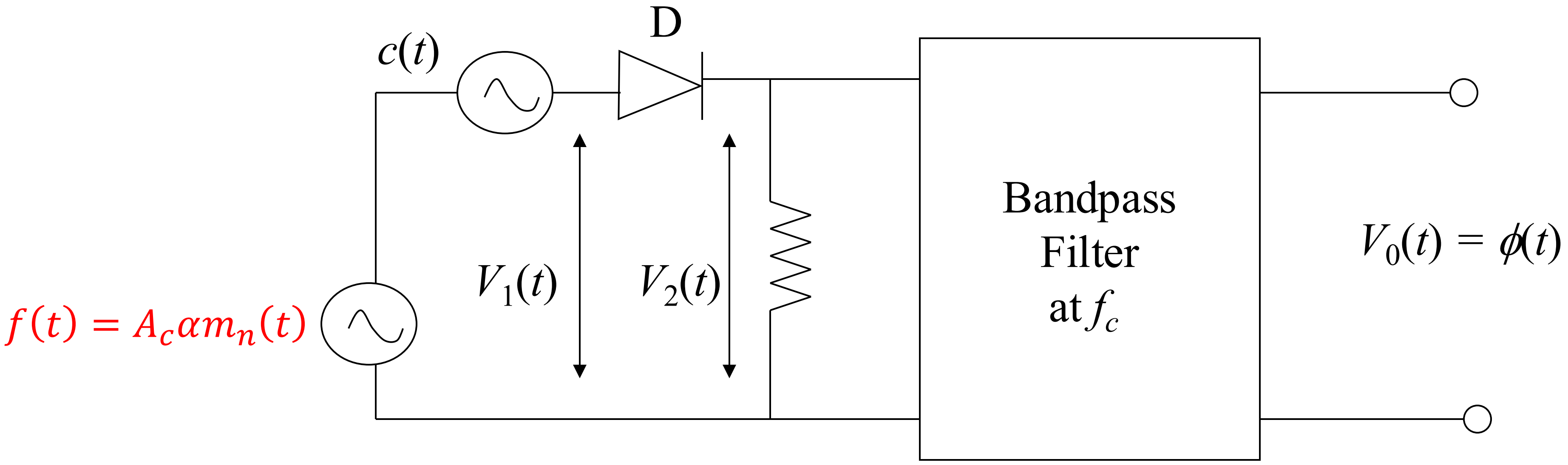
3.2.4 Generation of DSB-LC Signals

Circuitry aspects of this topic will not be tested in any examination, mathematical aspects may be tested.

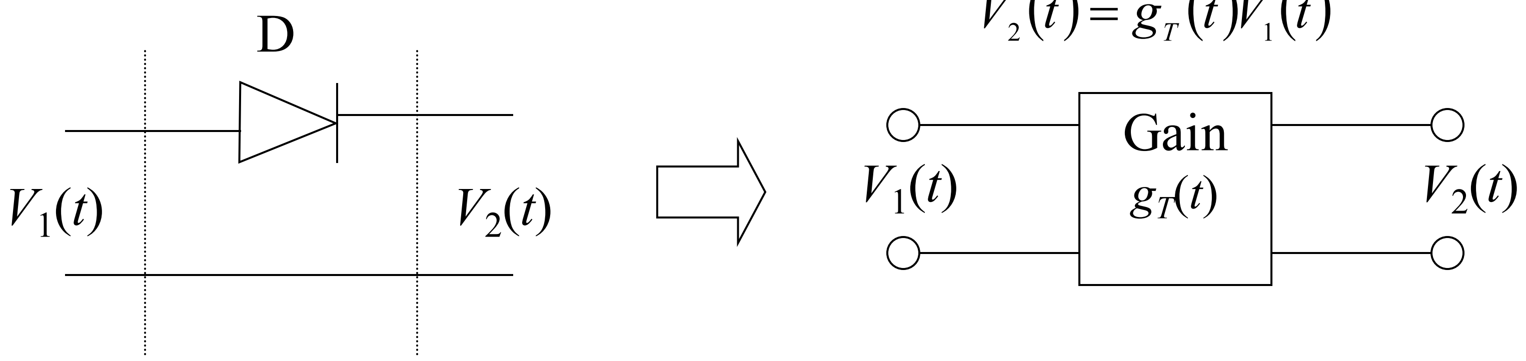
Given $m(t)$, how will be the modulated signal $\phi(t)$ be generated?



Chopper Modulator



$$V_2(t) = g_T(t)V_1(t)$$



$$V_1(t) = f(t) + c(t) = f(t) + A_c \cos(\omega_c t)$$

Assume D is an ideal diode, and $|f(t)| < A_c$, then

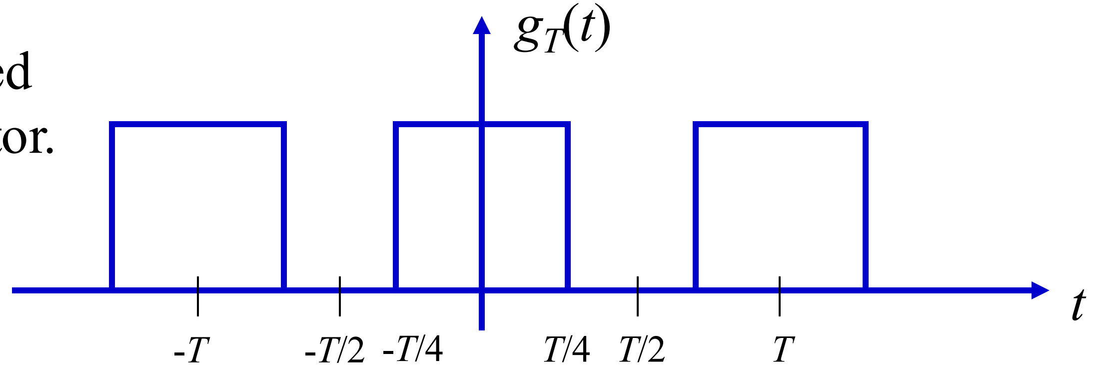
$$V_2(t) = \begin{cases} V_1(t), & \text{if } c(t) > 0 \\ 0, & \text{if } c(t) < 0 \end{cases}$$

i.e. the load voltage $V_2(t)$ varies periodically between the values $V_1(t)$ and 0 at a rate equal to the carrier frequency ω_c .

$$\left.\begin{array}{r} \text{When the diode is on, } g_T(t) = 1 \\ \\ \text{is off, } g_T(t) = 0 \end{array}\right\} \text{chopper modulation}$$

The diode can be represented by a rectangle pulse generator.

$T = 2\pi/\omega_c = 1/f_c$



Expand $g_T(t)$ using Fourier Series, we have

$$g_T(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos(\omega_c t(2n-1))$$

$$V_2(t) = V_1(t) g_T(t)$$

$$V_2(t) = (f(t) + A_c\cos(\omega_c t))\left\{\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos(\omega_c t(2n-1))\right\}$$

$$= \frac{1}{2}f(t) + \frac{1}{2}A_c\cos(\omega_c t) + \frac{2}{\pi}\cos^2(\omega_c t) + \frac{2}{\pi}f(t)\cos(\omega_c t)$$

$$-\frac{2}{3\pi}A_c\cos(\omega_c t)\cos(3\omega_c t) - \frac{2}{3\pi}f(t)\cos(3\omega_c t) + \text{other terms}$$

Useful components in $V_2(t)$ are $\frac{1}{2}A_c\cos(\omega_c t)$ and $\frac{2}{\pi}f(t)\cos(\omega_c t)$

Use a bandpass filter centered at ω_c with bandwidth $2W$ $(W = 2\pi B)$ to extract the useful components.

$$V_2(t)=\frac{1}{2}f(t)+\frac{1}{2}A_c\cos(\omega_c t)+\frac{2}{\pi}\cos^2(\omega_c t)+\frac{2}{\pi}f(t)\cos(\omega_c t)$$
$$-\frac{2}{3\pi}A_c\cos(\omega_c t)\cos(3\omega_c t)-\frac{2}{3\pi}f(t)\cos(3\omega_c t)+\text{other terms}$$

Only these terms go through the bandpass filters

Output of BPF: $$\boxed{V_0(t)=\frac{1}{2}A_c\cos(\omega_c t)+\frac{2}{\pi}f(t)\cos(\omega_c t)}$$

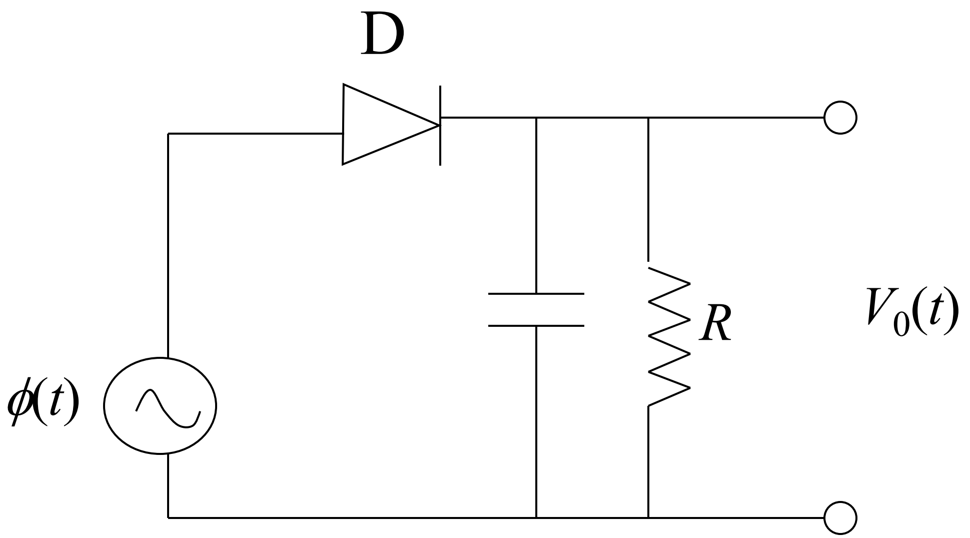
$V_0(t)$ is DSB-LC modulated signal $\phi(t)$ with some scaling factors.

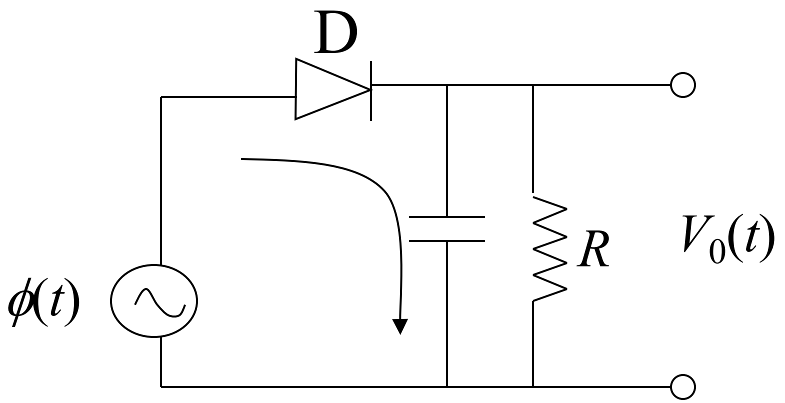
3.2.5 Demodulation of DSB-LC Signals

Circuitry aspects of this topic will not be tested in any examination, mathematical aspects may be tested.

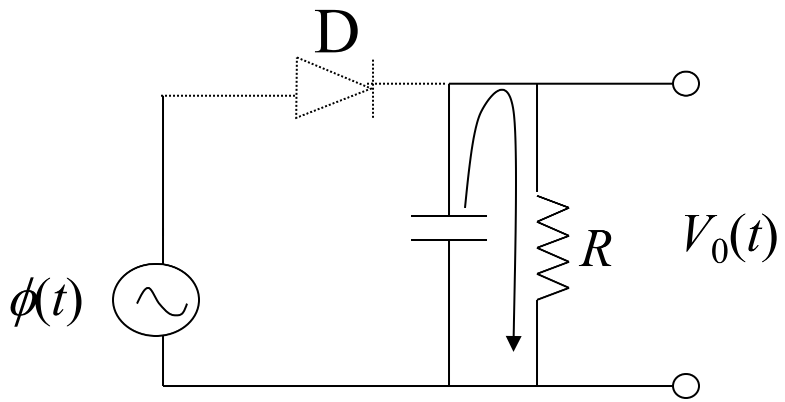
Given $\phi(t)$, how will be the message signal $m(t)$ be recovered?

Envelope Detector



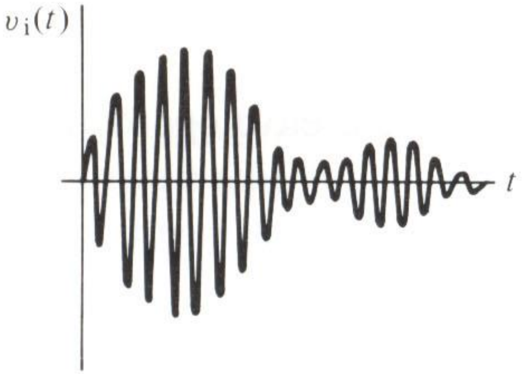


$\phi(t) > V_0(t)$
The diode is forward biased
C charges up: Follows the variation of input signal $\phi(t)$

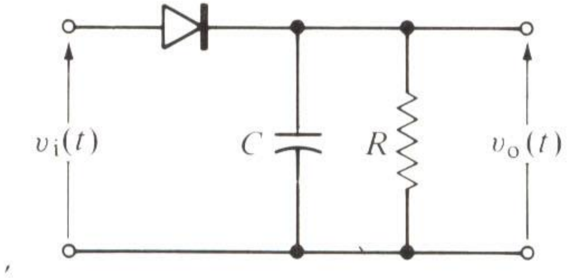


$\phi(t) < V_0(t)$
The diode turns off
C discharges (time constant $\tau = R\,C$)

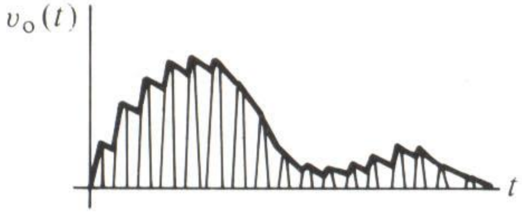
Choice of RC $\quad \frac{1}{f_c} \ll \tau \ll \frac{1}{B}$ $\quad$ B is the bandwidth of $m(t)$.



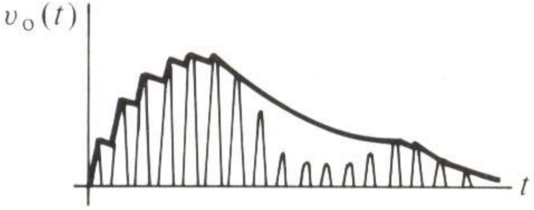
(a)



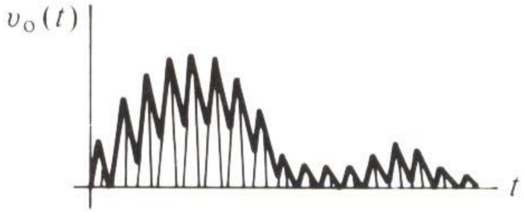
(b)



(c) Correct RC



(d) RC too large



(e) RC too small

RC too large: Envelope detector misses some positive half cycles.

RC too small: Envelope generates a very ragged waveform.

3.2.6 Applications of DSB-LC: Broadcast

Some of the FCC technical standards for AM (i.e., DSB-LC) broadcast stations are shown below.

Item	FCC Technical Standard
Assigned frequency f_c	In 10-kHz increments from 540 to 1700 kHz
Channel bandwidth	10 kHz
Carrier frequency stability	±20 Hz of the assigned frequency
% modulation	Maintain 85-95%
Audio-frequency response	±2 dB from 100 Hz to 5 Hz with 1 kHz being the 0-dB reference
Harmonic distortion	Less than 5% for up to 85% modulation; less than 7.5 form modulation between 85% and 95%
Noise and hum	At least 45 dB below 100% modulation in the band 30 Hz to 20 Hz
Maximum power licensed	10 kW

The Motorola C-QUAM (compatible quadrature amplitude modulation) uses quadrature modulation (QM), i.e., the envelop of modulated signal has the form $A_c[m_1(t)+jm_2(t)]$, which is adopted by the FCC for the AM stereo standards.

Comments on DSB-LC or Conventional AM

Both the modulator and demodulator have simple structures (low cost).
DSB-LC is wasteful of power (carrier does not carry information).
DSB-LC is wasteful of bandwidth (B vs $2B$).

To overcome the drawbacks, we have

- Double Sideband Suppressed Carrier (DSB-SC) Modulation
- Single Sideband (SSB) Modulation
- Vestigial Sideband (VSB) Modulation

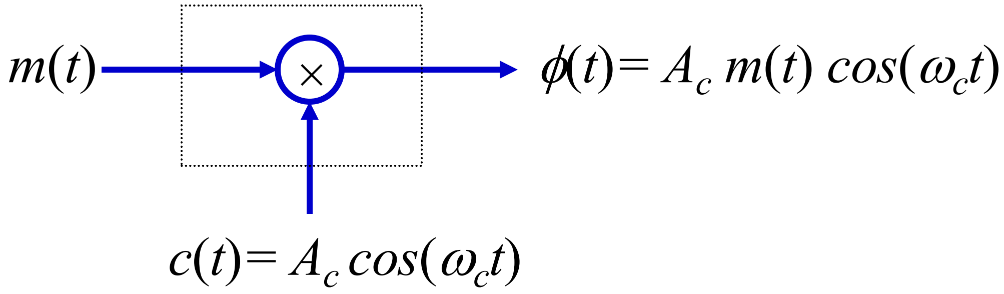
3.3 Double Side Band Suppressed Carrier (DSB-SC)

3.3.1 DSB-SC modulated signal

3.3.2 Demodulation of DSB-SC

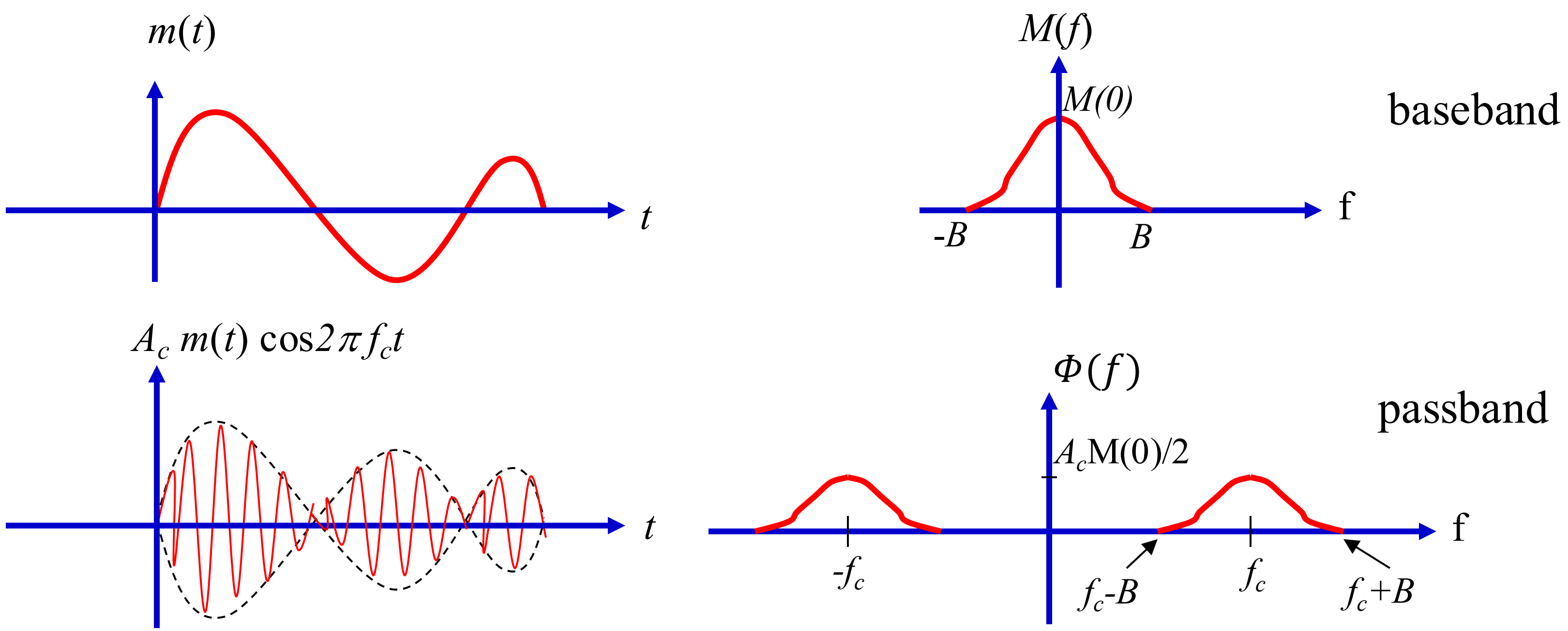
3.3.3 Practical implementation of DSB-SC modulation/demodulation

3.3.1 DSB-SC Modulated Signal



In the frequency domain:

$$\Phi(f) = \frac{A_c}{2}\left(M(f - f_c) + M(f + f_c)\right)$$

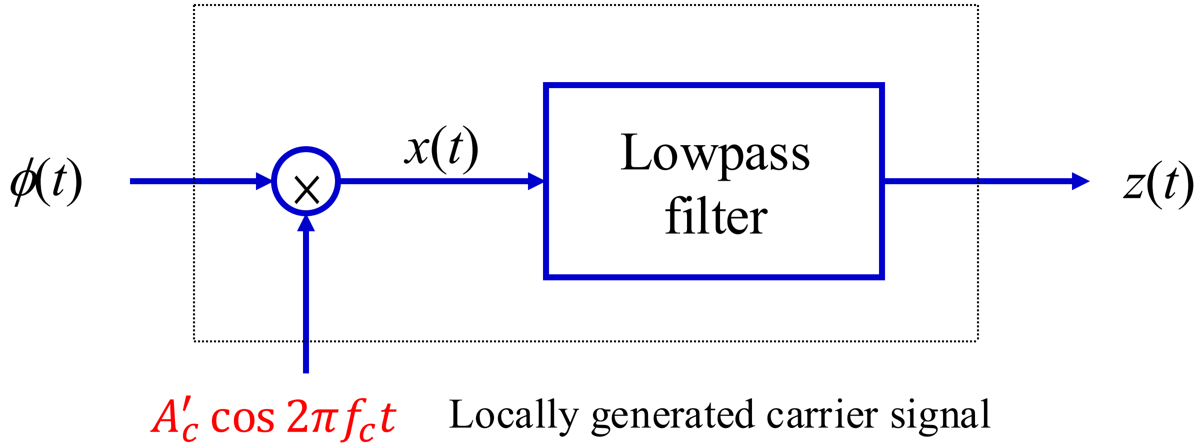


Observations

- The envelope of $\phi(t)$ is different from $m(t)$. Both envelope and phase of $\phi(t)$ carry information of $m(t)$.
- The transmission bandwidth required by DSB-SC is the same as that for DSB-LC, i.e. $\beta_T = 2B$.

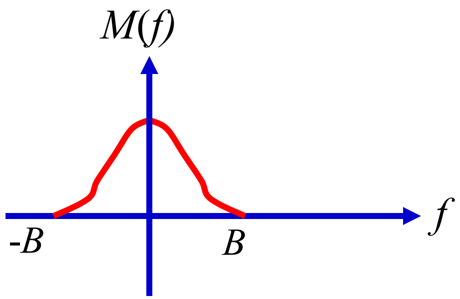
3.3.2 Demodulation of DSB-SC signals

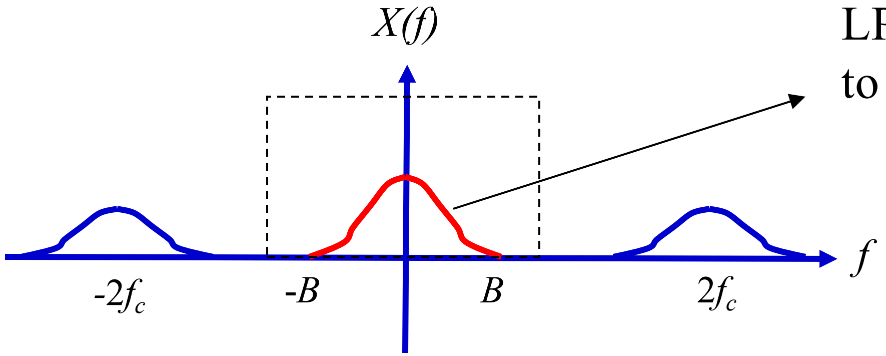
Given $\phi(t)$, how will be the message signal $m(t)$ be recovered?



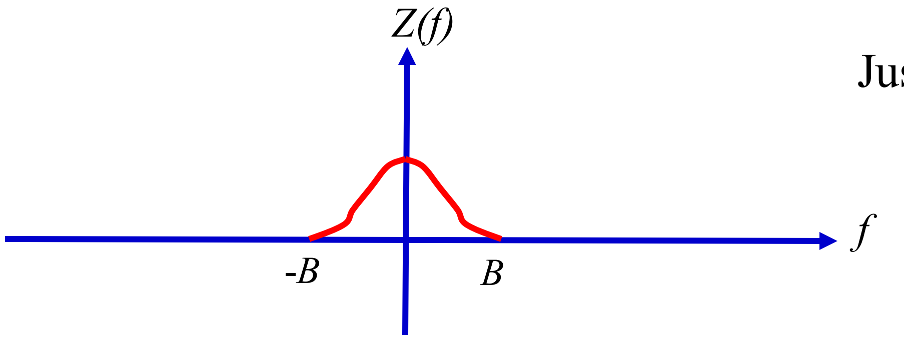
$$x(t) = \phi(t) A'_c \cos 2\pi f_c t = A_c A'_c\, m(\mathrm{t}) \cos^2 2\pi f_c t = \frac{A_c A'_c}{2} m(t) + \frac{A_c A'_c}{2} m(t) \cos 4\pi f_c t$$

$$X(f) = \frac{A_c A'_c}{2} M(f) + \frac{A_c A'_c}{4} \left(M(f - 2f_c) + M(f + 2f_c) \right)$$



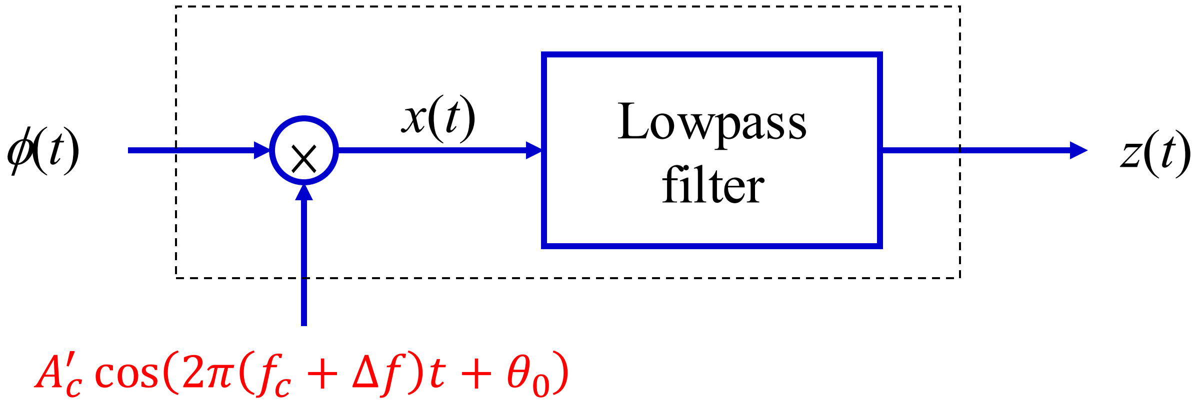


LPF only allows this component to pass, the others are rejected



Just a scaled version of $m(t)$

Now assume a frequency error and a phase error in the locally generated signal at the receiver.



$$x(t) = \phi(t)A_c' \cos 2\pi f_c t = A_c A_c' m(\mathrm{t}) \cos 2\pi f_c t \cos(2\pi(f_c + \Delta f)t + \theta_0)$$

$$= \underbrace{\frac{A_c A_c'}{2} m(t) \cos(2\pi \Delta f t + \theta_0)}_{\text{Only this term goes through LPF}} + \cancel{\frac{A_c A_c'}{2} m(t) \cos(2\pi(2f_c + \Delta f)t + \theta_0)}$$

$$z(t) = \frac{A_c A'_c}{2} m(t) \cos(2\pi\Delta f t + \theta_0)$$

- If $\Delta f = 0$ and $\theta_0 = 0$, the output is $z(t) = \frac{A_c A'_c}{2} m(t)$ → no distortion

- If $\Delta f = 0$, the output is $z(t) = \frac{A_c A'_c}{2} m(t) \cos \theta_0$

  The phase error introduces a variable attenuation factor. For small fixed phase errors, this is quite tolerable. If $\theta_0 = \pm 90^{\circ}$, the received signal is wiped out.

- If $\theta_0 = 0$, the output is $z(t) = \frac{A_c A'_c}{2} m(t) \cos(2\pi\Delta f t)$

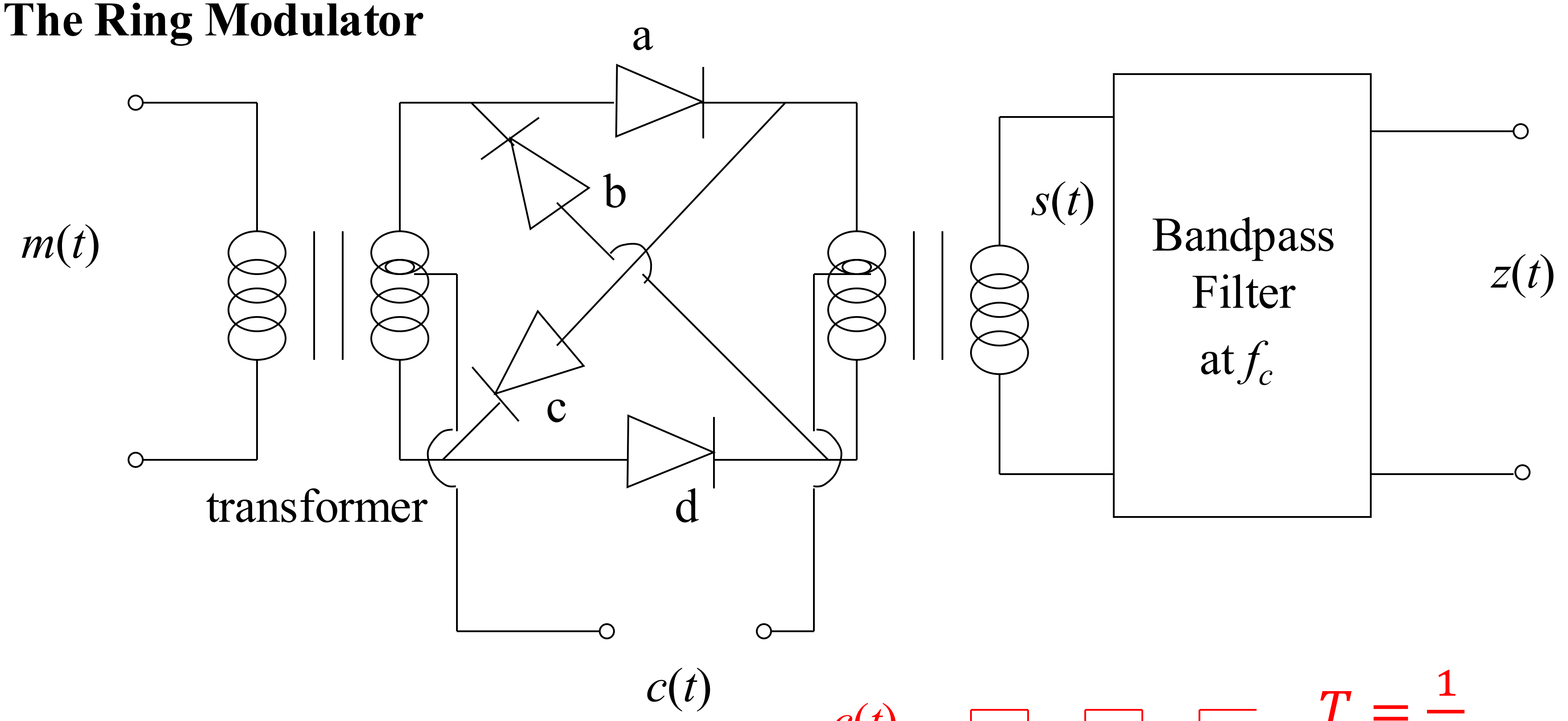
To recover $m(t)$ accurately from $\phi(t)$, we need to use a synchronized oscillator
→ *Coherent detection* (synchronous detection) is required.

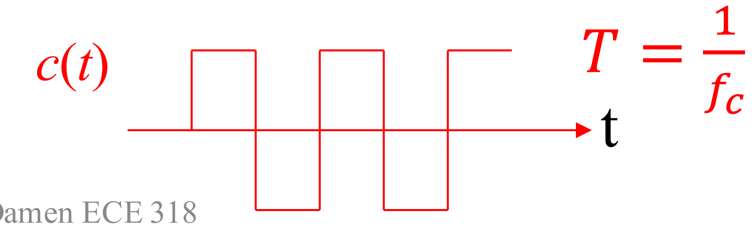
3.3.3 *Practical Implementation* *For the* Generation (Modulation) and Demodulation of DSB-SC signals

3.3.3.A Generation of DSB-SC Signals

Circuitry aspects of this topic will not be tested in any examination, mathematical aspects may be tested.

The Ring Modulator

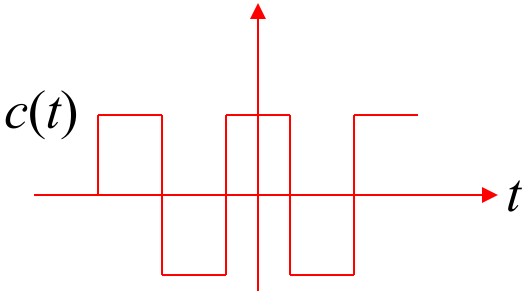




The diodes are controlled by a square-wave carrier $c(t)$ of frequency f_c, assuming $|c(t)| \gg |m(t)|$

$c(t) > 0$ Diodes a and d conduct $s(t) = m(t)$

$c(t) < 0$ Diodes b and c conduct $s(t) = -m(t)$



Expressing in terms of Fourier Series

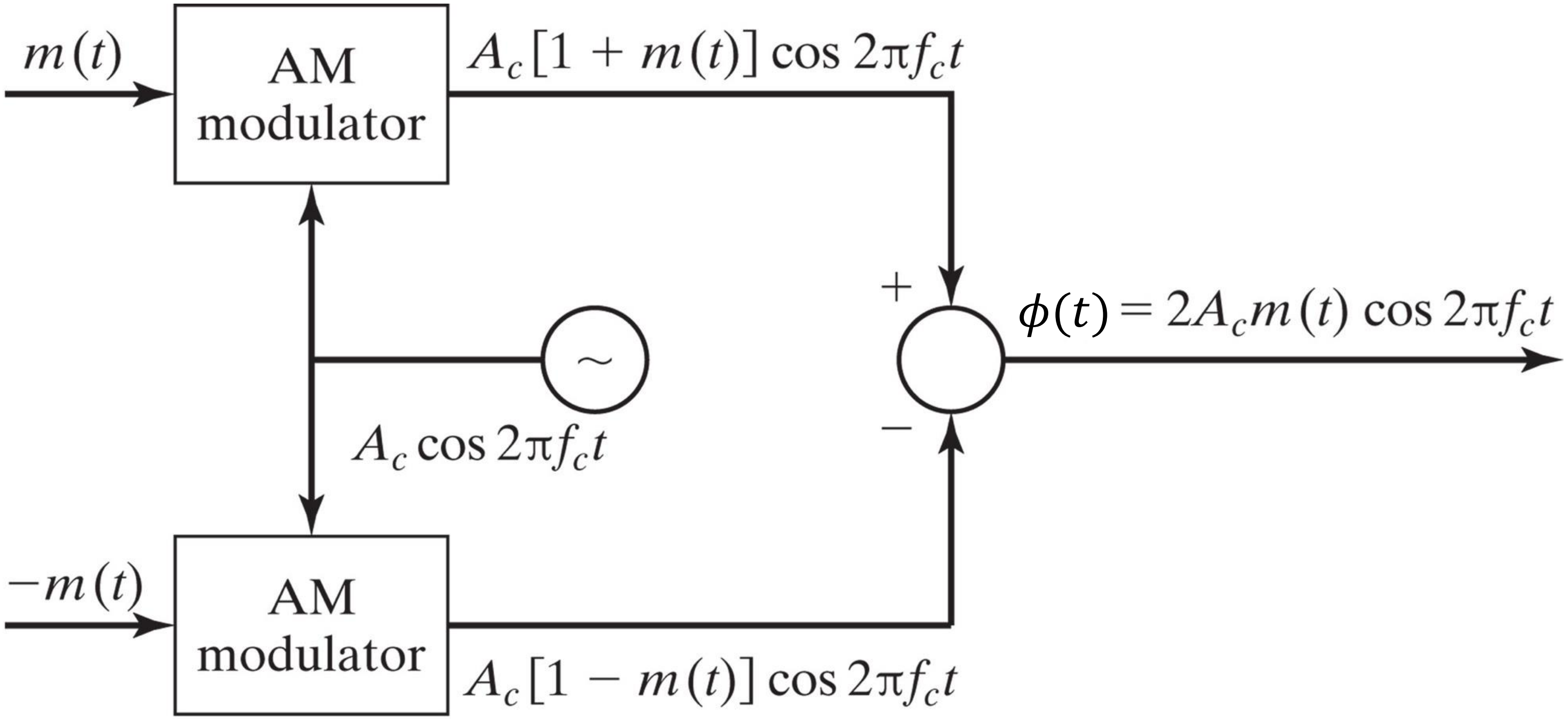
$$c(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos\left(2\pi f_c t\,(2n-1)\right)$$

$$s(t) = c(t)m(t)$$

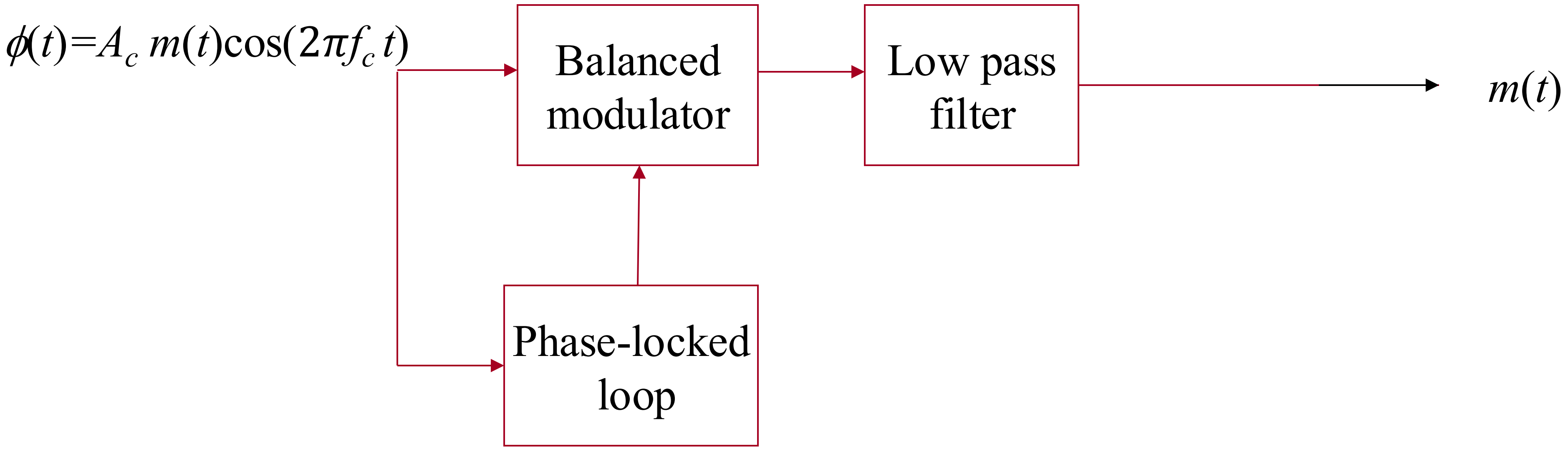
$$s(t) = m(t)\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos\left(2\pi f_c t\,(2n-1)\right)$$

Output of BPF $z(t) = \frac{4}{\pi}m(t)\cos(2\pi f_c t)$

The Balanced Modulator

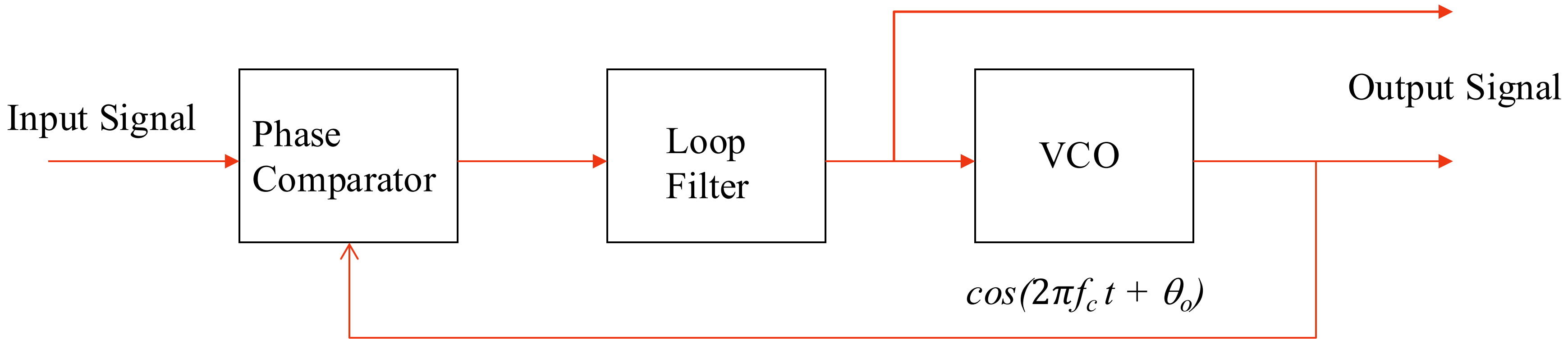


3.3.3.B Demodulation of DSB-SC Signals (Practical Implementation)



Phase Locked Loop (PLL)

- A feedback circuit containing a VCO, phase comparator (or detector) and a loop filter. The phase detector measures the difference in phase between VCO output and the input signal. The loop filter controls the VCO to track the phase of the input signal until it *locks* on it.
- PLL is widely used in modern communications: phase tracking for coherent detection, FM demodulation, frequency synthesizers, and bit/symbol synchronization.
- PLL to be seen again when studying FM demodulation.



3.4 Single Sideband (SSB) Modulation

3.4.1 Generation of SSB signals

3.4.2 The Hilbert transform

3.4.3 Analytic signals

3.4.4 Representation of SSB signals

3.4.5 Demodulation of SSB signals

3.4.6 SSB-LC signals

Motivation: Both DSB-LC and DSB-SC occupy a bandwidth of $2B$. How can we reduce the bandwidth requirements?

$m(t) \leftrightarrow M(f)$

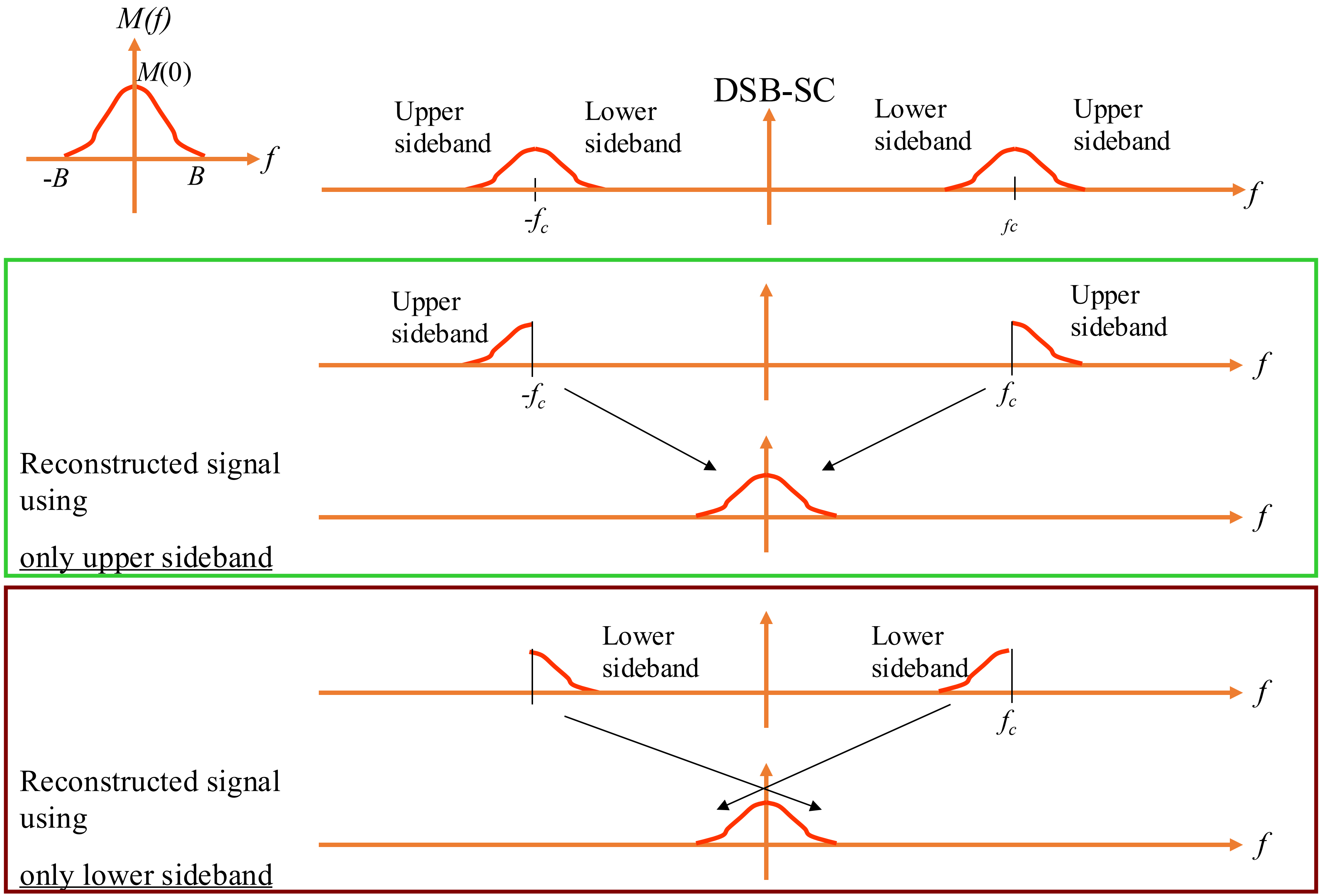
$m^*(t) \leftrightarrow M^*(-f)$

Complex conjugate property of Fourier Transform

For real signals, $m(t) = m^*(t) \;\Rightarrow\; M(-f) = M^*(f)$

The spectral density of any real-valued signal exhibits the (Hermitian) symmetry around $f=0$ (i.e., even spectrum magnitude and odd spectrum phase).

Due to this symmetry, one of the sidebands is sufficient to provide the complete information in original signal.


M(f)
M(0)
-B
B
f
DSB-SC
Upper sideband
Lower sideband
Lower sideband
Upper sideband
-fc
fc
f
Upper sideband
Upper sideband
-fc
fc
f
Reconstructed signal using
only upper sideband
f
Lower sideband
Lower sideband
fc
f
Reconstructed signal using
only lower sideband
f

How to Get Rid of One of the Sidebands?

- To have only one sideband (positive or negative) in the baseband spectrum, the signal **must** be complex.

- We can construct a baseband *complex signal* that has the same upper sideband as the message *f(t)* in baseband and zero in the negative frequencies:

$$Z(f) = \begin{cases} 2M(f), f \geq 0 \\ 0, \qquad f < 0 \end{cases}$$

- Then, we shift the spectrum of the signal around f_c by multiplying it by $\exp(j2\pi f_c t)$, i.e., we modulate the complex signal representation of *M(t)*.

- Finally, taking the real part of the modulated complex signal $z(t)\exp(j2\pi f_c t)$ gives a real signal which has only one sideband on the positive frequencies and its mirror image on the negative frequencies.

Example: $m(t) = \cos 2\pi f_m t$

- Let $z(t) = \exp j2\pi f_m t$

Verify that $Z(f)$ has the same upper sideband as $2M(f)$ and zero elsewhere.

- Shifting the spectrum of $z(t)$ around f_c gives

$$\exp j2\pi f_m t \, \exp j2\pi f_c t = \exp j2\pi (f_m + f_c) t$$

- Taking the real part gives

$$\phi(t) = \cos 2\pi (f_m + f_c) t = \cos 2\pi f_m t \cos 2\pi f_c t - \sin 2\pi f_m t \sin 2\pi f_c t$$

which has a spectrum that comprises only the (shifted) upper sideband of $m(t)$ and its mirror image:

$$\Phi(f) = \frac{1}{2}\left(\delta(f - f_c - f_m) + \delta(f + f_c + f_m)\right)$$

Compare with the spectrum of $m(t) \cos(2\pi f_c t)$.

So What Have We Done?

- The message, $m(t) = \cos 2\pi f_m t$ is sent in phase (i.e., over $\cos 2\pi f_c t$)

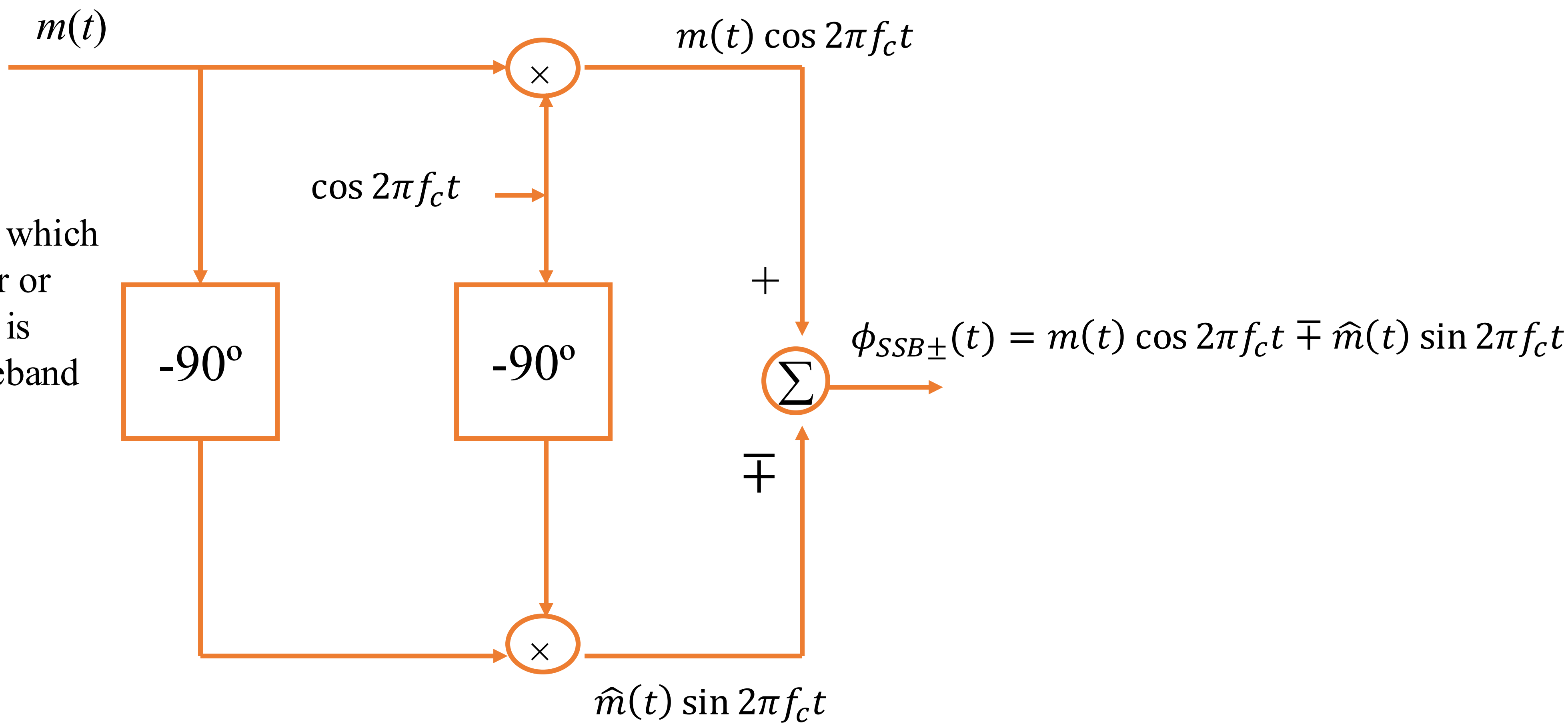
- A *transformation of the message*, $\hat{m}(t) = \sin 2\pi f_m t$, is sent in quadrature (i.e., over $\cos\left(2\pi f_c t + \frac{\pi}{2}\right) = -\sin 2\pi f_c t$)

- The transformation is obtained by phase-shifting the message by *-π/2*.

- The *-π/2* phase shifting, referred to as the **Hilbert Transform**, can be applied to any message as illustrated in the sequel.

3.4.1 Generation of SSB Signals

The modulation method which transmits only the upper or lower pair of sidebands is referred to as single-sideband (SSB) modulation.



$\hat{m}(t)$ is called the *quadrature function* of $m(t)$, or the *Hilbert transform* of $m(t)$

3.4.2 The Hilbert Transform

Phase shifter $m(t) \longrightarrow \boxed{-90^o} \longrightarrow \hat{m}(t)$ (does not change the domain of the independent variable of the signal)

Frequency-domain description $M(f) \longrightarrow \boxed{-j\mathrm{sgn}(f)} \longrightarrow \mathcal{F}\{\hat{m}(t)\} = -j\,\mathrm{sgn}(f)\,M(f)$

$$-j\,\mathrm{sgn}(f) = \begin{cases} -j, & f > 0 \\ 0, & f = 0 \\ j, & f < 0 \end{cases} = e^{-\frac{j\pi}{2}}\mathrm{sgn}\,(f)$$

This means that the Hilbert transform is equivalent to a $-\pi/2$ phase shift for positive frequencies and $+\pi/2$ phase shift for negative frequencies.

Time-domain description $m(t) \longrightarrow \boxed{1/(\pi t)} \longrightarrow \hat{m}(t)$

$$\frac{1}{\pi t} \leftrightarrow -j\,\mathrm{sgn}(f)$$

$$\hat{m}(t) = \frac{1}{\pi t} * m(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{m(\tau)}{t-\tau}\,d\tau$$

Properties of the Hilbert Transform

- $m(t)$ and $\hat{m}(t)$ have the same amplitude spectral density.

$$\hat{M}(f) = \mathcal{F}\{\hat{m}(t)\} = -j\,\mathrm{sgn}(f)\,M(f) \longrightarrow |\hat{M}(f)| = |M(f)|$$

- If $\hat{m}(t)$ is the Hilbert transform of $m(t)$, then the Hilbert transform of $\hat{m}(t)$ is $\hat{\hat{m}}(t) = -m(t)$

- $\hat{m}(t)$ and $m(t)$ are orthogonal: $\int_{-\infty}^{\infty} m(t)\hat{m}(t)dt = 0$

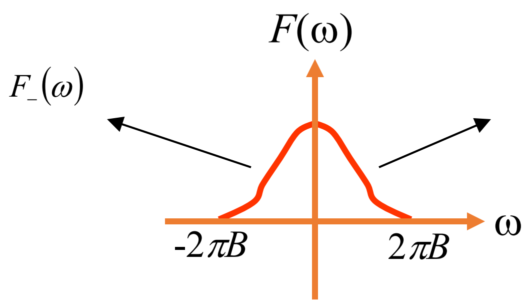
- The Hilbert transform of an even signal is odd and the Hilbert transform of an odd signal is even

- The energy content of the Hilbert transform of a signal is equal to the energy content of the signal.

3.4.3 Analytic Signals

Definition: A real-valued signal can be represented in terms of complex valued signals with one-sided spectral density. Such signals are called *analytic signals*. The real part of the analytic signal gives the original signal.

$$f(t) \longleftrightarrow F(\omega)$$



$$z_+(t) \longleftrightarrow \qquad Z_+(\omega) = \begin{cases} 2F(\omega), & \omega > 0 \\ 0 \quad , & \omega < 0 \end{cases}$$

$$z_-(t) \longleftrightarrow \qquad Z_-(\omega) = \begin{cases} 0 \quad , & \omega > 0 \\ 2F(\omega), & \omega < 0 \end{cases}$$

$$2F(\omega) = Z_-(\omega) + Z_+(\omega) = 2F(\omega)u(\omega) + 2F^*(-\omega)u(-\omega)$$

$$2f(t) = z_+(t) + z_-(t) = z_+(t) + z_+^*(t) = 2\,\mathrm{Re}\{\, z_+(t) \,\}$$

analytic signal

3.4.3 Analytic Signals (Cont.)

$$2F(\omega) = Z_+(\omega) + Z_-(\omega)$$

Spectrum of Hilbert transform

$$2\hat{F}(\omega) = -jZ_+(\omega) + jZ_-(\omega)$$

Inverse Fourier Transform

$$2\hat{f}(t) = -j\ z_+(t) + j\ z_-(t)$$

$$z_-(t) = 2f(t) - z_+(t)$$

$$= -2j\ z_+(t) + 2j\ f(t)$$

$$\longrightarrow \quad z_+(t) = \underbrace{f(t)}_{\text{Real part}} + j\ \underbrace{\hat{f}(t)}_{\text{Imaginary part}}$$

$$f(t) = \mathrm{Re}\{z_+(t)\} = \mathrm{Re}\{f(t) + j\ \hat{f}(t)\}$$

3.4.4 Representations for SSB Signals

$$\phi_{\text{SSB}+} = \text{Re}\left\{z_+(t)e^{j\omega_c t}\right\}$$

$$\phi_{\text{SSB}-} = \text{Re}\left\{z_-(t)e^{j\omega_c t}\right\}$$

+ Upper sideband

- Lower sideband

$$\phi_{\text{SSB}+} = \text{Re}\left\{\left[f(t) + j\hat{f}(t)\right] e^{j\omega_c t}\right\}$$

$$\phi_{\text{SSB}-} = \text{Re}\left\{\left[f(t) - j\hat{f}(t)\right] e^{j\omega_c t}\right\}$$

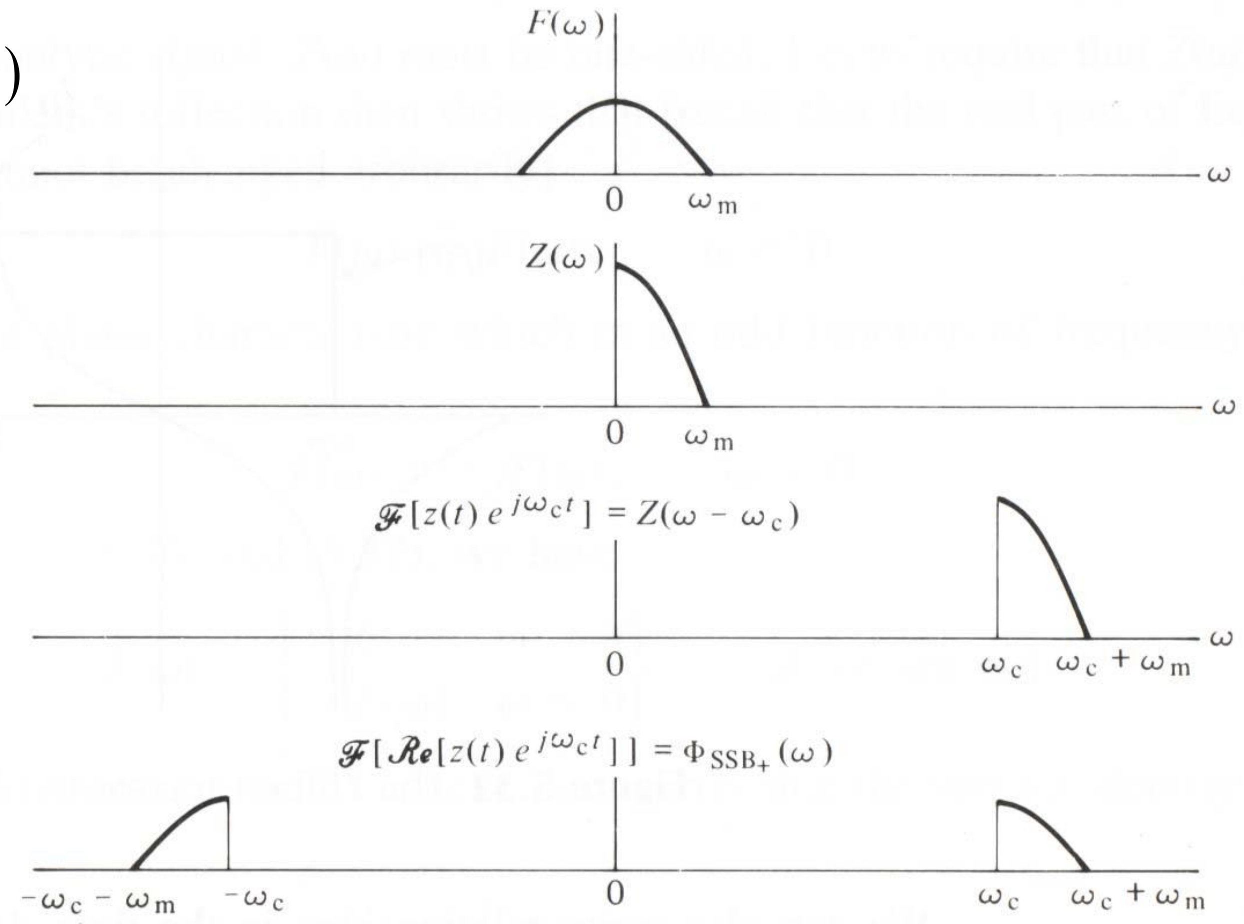
$$\phi_{\text{SSB}+} = f(t)\cos(\omega_c t) - \hat{f}(t)\sin(\omega_c t)$$

$$\phi_{\text{SSB}-} = f(t)\cos(\omega_c t) + \hat{f}(t)\sin(\omega_c t)$$

SSB (Upper Sideband):

$$z(t) = f(t) + j\hat{f}(t)$$

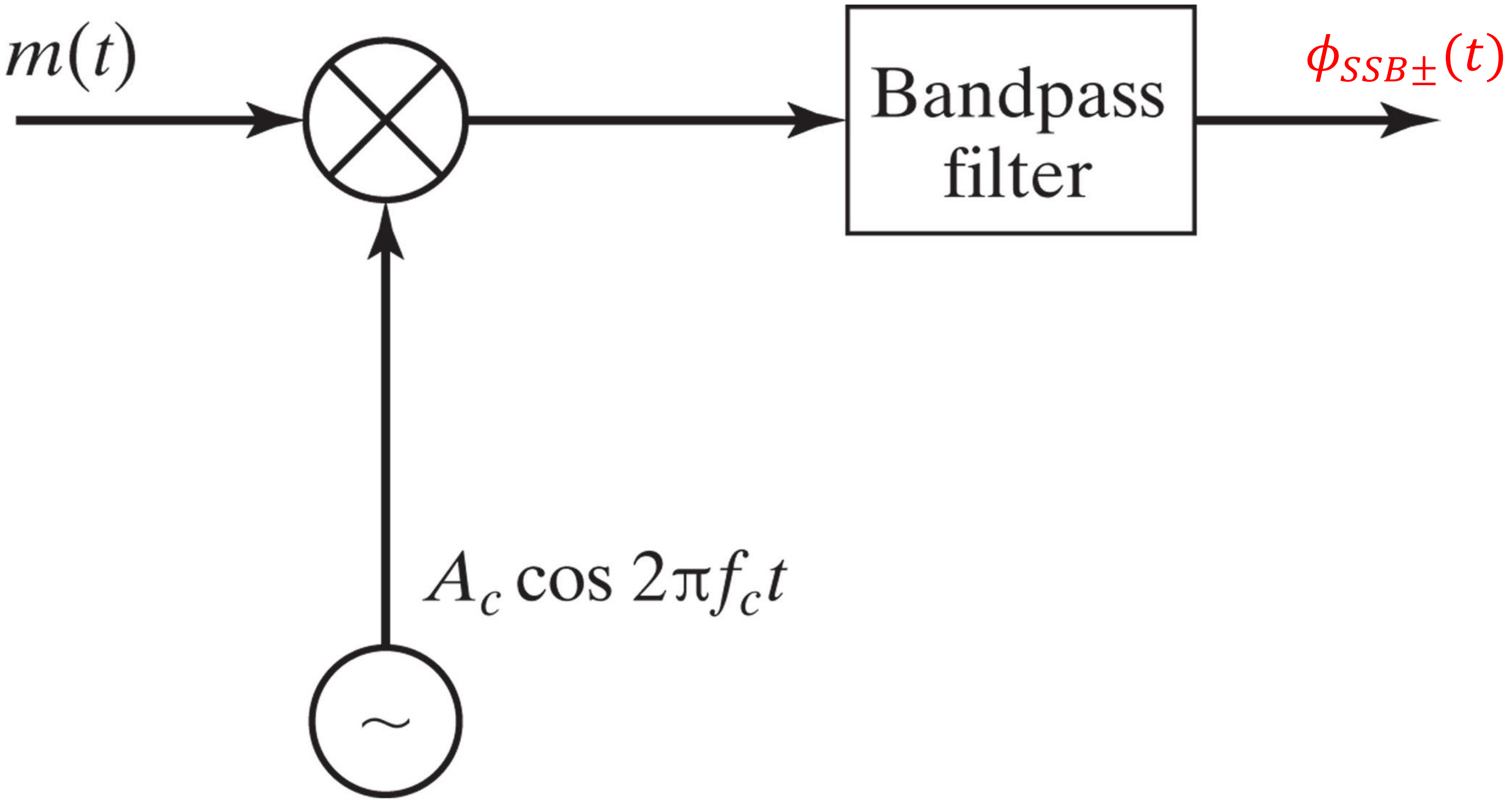
$$z(t) \longleftrightarrow Z(\omega)$$



$$\phi_{\text{SSB+}}(t) = \operatorname{Re}\left\{\, z(t)\; e^{j\omega_c t} \right\} = f(t)\cos(\omega_c t) - \hat{f}(t)\sin(\omega_c t)$$

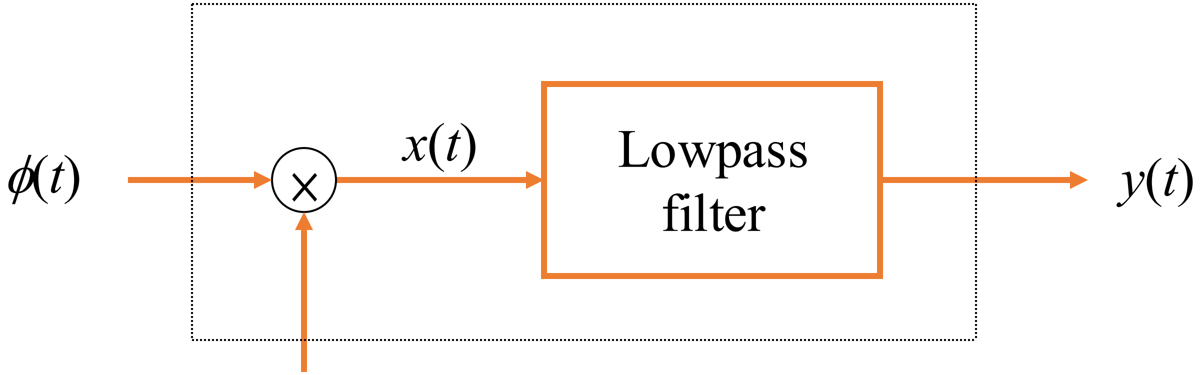
NB: For simple notations, under the assumption of using the upper sideband, the *analytic signal* is denoted by *z(t).*

Generation of an SSB-AM signal by filtering one of the sidebands of a DSB-SC AM signal.



3.4.5 Demodulation of SSB Signals

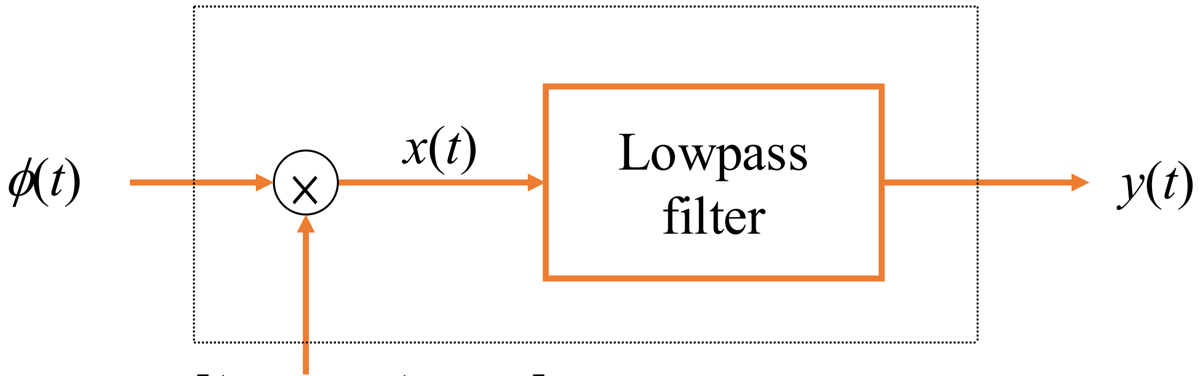
Given $\phi(t)$, how will be the message signal $f(t)$ be recovered?



$\cos(\omega_c t)$: Locally generated carrier signal

$$
\begin{aligned}
x(t) &= \phi_{\pm}(t)\cos(\omega_c t) \\
&= \left(f(t)\cos(\omega_c t) \pm \hat{f}(t)\sin(\omega_c t)\right)\cos(\omega_c t) \\
&= \underbrace{\frac{1}{2} f(t)}_{\substack{\text{Only this term} \\ \text{passes through LPF}}} + \frac{1}{2} f(t)\cos(2\omega_c t) \pm \frac{1}{2}\hat{f}(t)\sin(2\omega_c t)
\end{aligned}
$$

Now assume a frequency error and a phase error in the locally generated signal at the receiver.



$\cos\left[(\omega_c + \Delta\omega)t + \theta_0\right]$: Locally generated carrier signal

$$x(t) = \left[f(t)\cos(\omega_c t) \pm \hat{f}(t)\sin(\omega_c t)\right]\cos\left[(\omega_c + \Delta\omega)t + \theta_0\right]$$

$$= \frac{1}{2} f(t)\left\{\cos\left[(\Delta\omega)t + \theta_0\right] + \cos\left[(2\omega_c + \Delta\omega)t + \theta_0\right]\right\}$$

$$\pm \frac{1}{2}\hat{f}(t)\left\{\sin\left[(\Delta\omega)t + \theta_0\right] - \sin\left[(2\omega_c + \Delta\omega)t + \theta_0\right]\right\}$$

Only these two terms pass through LPF

$$\cos\alpha \cdot \cos\beta = (1/2)\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right]$$

$$\sin\alpha \cdot \cos\beta = (1/2)\left[\sin(\alpha - \beta) + \sin(\alpha + \beta)\right]$$

Thus

$$y(t)=\frac{1}{2}f(t)\cos[(\Delta\omega)t+\theta_0]\pm\frac{1}{2}\hat{f}(t)\sin[(\Delta\omega)t+\theta_0]$$

- If $\Delta\omega = 0$ and $\theta_0 = 0$, the output is $y(t) = \frac{1}{2} f(t)$ $\longrightarrow$ no distortion

- If $\Delta\omega = 0$, the output is

$$y(t)=\frac{1}{2}f(t)\cos(\theta_0)\pm\frac{1}{2}\hat{f}(t)\sin(\theta_0)$$

$$=\frac{1}{2}\mathrm{Re}\left\{\left[f(t)\pm j\,\hat{f}(t)\right]e^{j\theta_0}\right\}$$

Human ear can interpret speech despite phase changes.

- If $\theta_0 = 0$, the output is

$$y(t)=\frac{1}{2}f(t)\cos[(\Delta\omega)t]\pm\frac{1}{2}\hat{f}(t)\sin[(\Delta\omega)t]$$

$$=\frac{1}{2}\mathrm{Re}\left\{\left[f(t)\pm j\,\hat{f}(t)\right]e^{j(\Delta\omega)t}\right\}$$

Frequency errors result in spectral shifts. Small errors can be tolerated in voice communication.

Conclusion. Similar to DSB-SC, a coherent detector must be used for SSB-SC demodulation.

3.4.6 Single Sideband Large Carrier (SSB-LC)

Coherent (synchronous) detection is required for SSB. However, by inserting a carrier, envelope detection can be used.

SSB: $$\phi_{\pm}(t) = f(t)\cos(\omega_c t) \pm \hat{f}(t)\sin(\omega_c t)$$

SSB-LC $$\begin{aligned}\phi(t) &= A_c\cos(\omega_c t) + f(t)\cos(\omega_c t) \pm \hat{f}(t)\sin(\omega_c t)\\ &= [A_c + f(t)]\cos(\omega_c t) \pm \hat{f}(t)\sin(\omega_c t)\end{aligned}$$

Envelope of $\phi(t)$

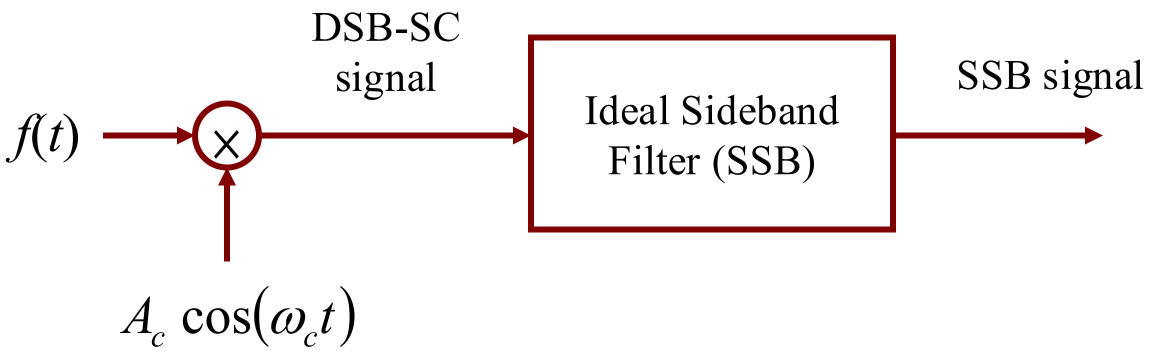
$$r(t) = \sqrt{[A_c + f(t)]^2 + \left[\hat{f}(t)\right]^2} = \sqrt{A_c^2 + 2f(t)A_c + f^2(t) + \hat{f}^2(t)}$$

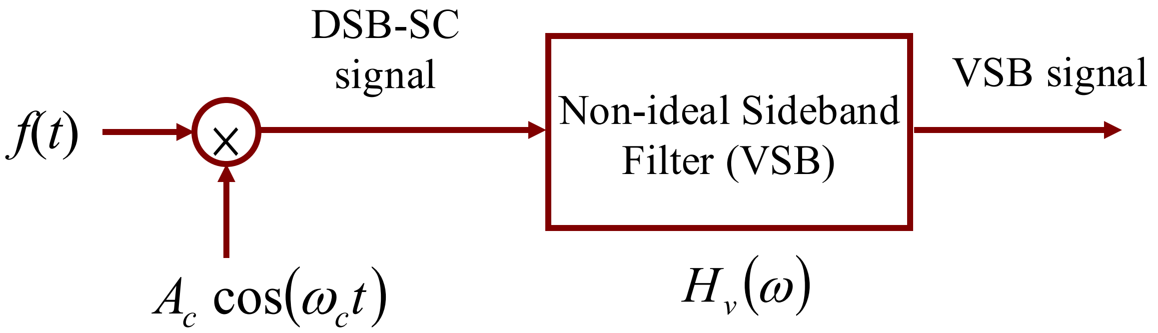
$$\approx A_c\sqrt{1 + \frac{2f(t)}{A_c}} \approx A_c\left[1 + \frac{f(t)}{A_c}\right] = A_c + f(t)$$

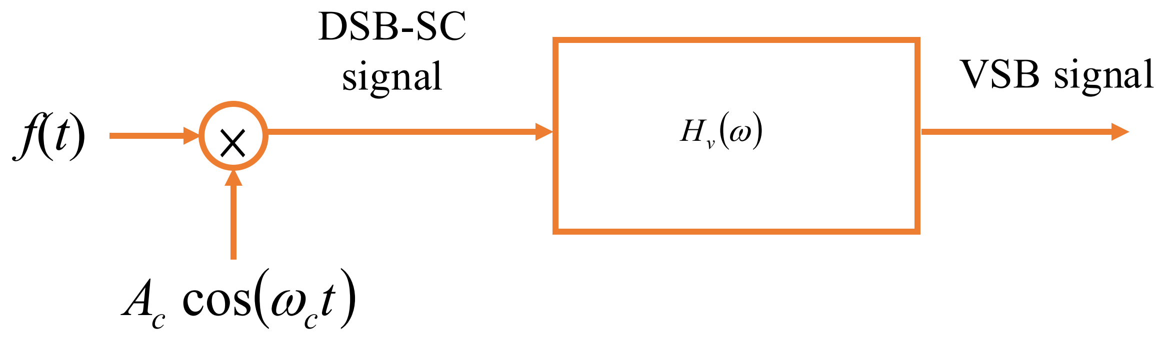
For very large A_c $\sqrt{1+x} \approx 1 + x/2 \quad |x| \ll 1$

3.5 Vestigial Sideband (VSB) Modulation

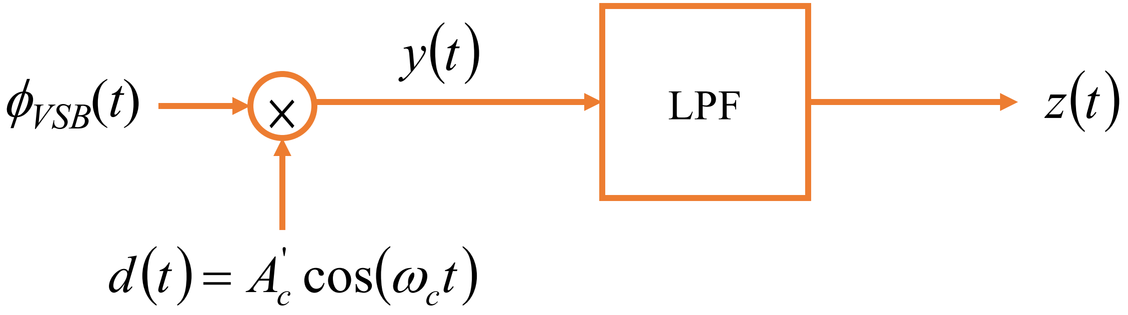
As a compromise between bandwidth efficiency and hardware cost (between DSB and SSB), we can use VSB modulation. In this modulation type, one sideband is transmitted, also allowing a portion of the unwanted sideband.







To determine the frequency-response characteristics of the filter, let us consider the demodulation for VSB signal.



$$\phi_{VSB}(t) = [A_c f(t)\cos\omega_c t] \otimes h_v(t)$$

$$\Phi_{VSB}(\omega) = \frac{A_c}{2}[F(\omega-\omega_c) + F(\omega+\omega_c)]H_v(\omega)$$

$$y(t) = \phi_{VSB}(t)d(t)$$

$$Y(\omega) = \frac{A_c'}{2}[\Phi_{VSB}(\omega-\omega_c) + \Phi_{VSB}(\omega+\omega_c)]$$

$$Y(\omega) = \frac{A_c'}{2}\left[\Phi_{VSB}(\omega - \omega_c) + \Phi_{VSB}(\omega + \omega_c)\right]$$

$$= \frac{A_c A_c'}{4} F(\omega)\left[H_v(\omega - \omega_c) + H_v(\omega + \omega_c)\right] \quad + \cancelto{0}{\frac{A_c A_c'}{4} F(\omega - 2\omega_c) H_v(\omega - \omega_c)}$$

$$+ \cancelto{0}{\frac{A_c A_c'}{4} F(\omega + 2\omega_c) H_v(\omega + \omega_c)}$$

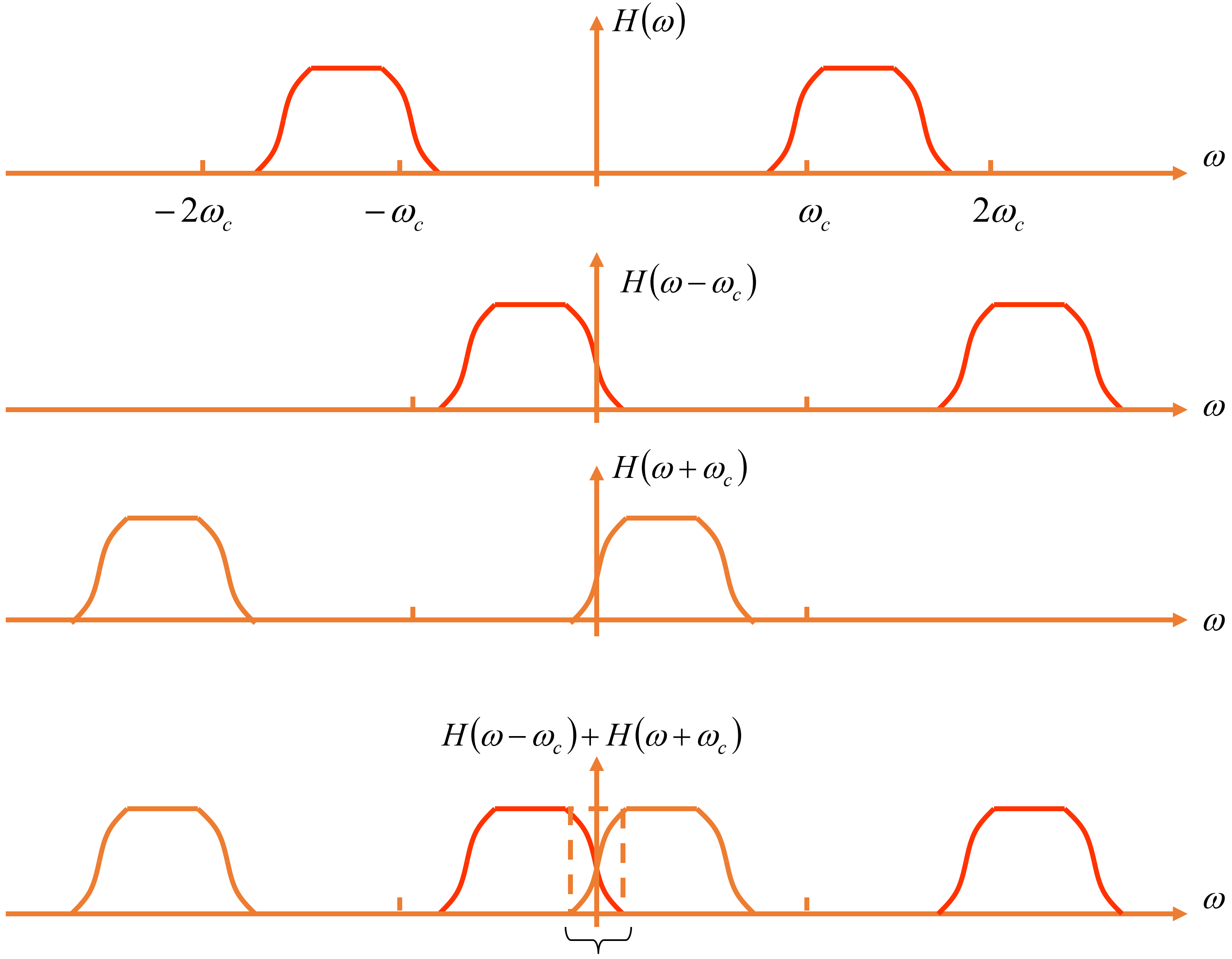
Output of LPF

$$Z(\omega) = \frac{A_c A_c'}{4} F(\omega)\left[H_v(\omega - \omega_c) + H_v(\omega + \omega_c)\right]$$

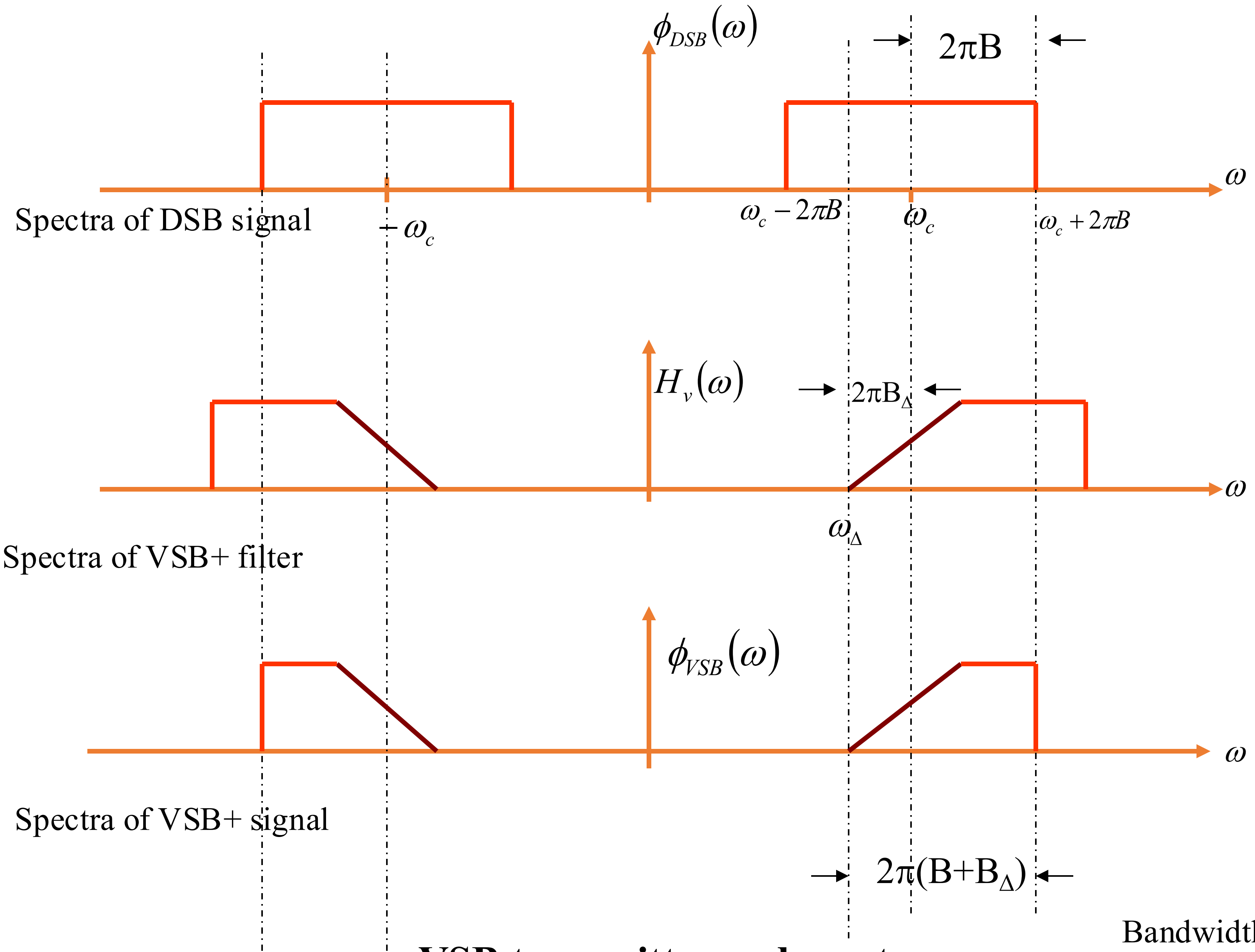
We require that the message signal at the output of LPF be undistorted.

$$H_v(\omega - \omega_c) + H_v(\omega + \omega_c) = \text{constant} \qquad |\omega| \leq \text{Bandwidth of the message signal}$$

Example of VSB Filter



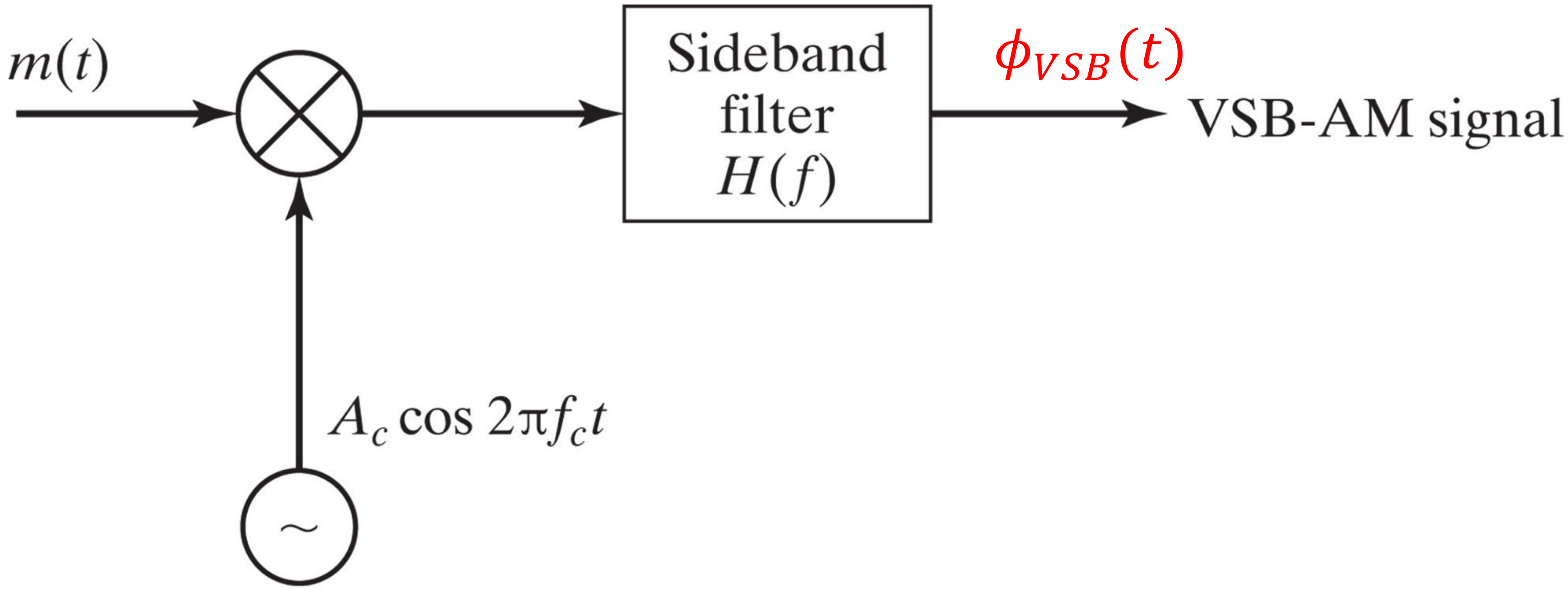
VSB filter constraint: constant → Satisfies the requirement for distortionless demodulation



VSB transmitter and spectra

Bandwidth of VSB signal: $B + B_\Delta$

Generation of VSB-AM signal.



Comparison of Amplitude Modulation Techniques

	Modulator	**Demodulator**	**Power Efficiency**	**Band-width**
DSB-LC (audio broadcasting)	Simple	Simple (envelope det.)	1/3 (single tone)	2B
DSB-SC	Simple	Complex (coherent)	1	2B
SSB-SC (telephony-wirelines and cables)	Complex	Complex (coherent)	1	B
SSB-LC	Complex	Simple (envelope det.)	Worst (very large carrier)	B
VSB-SC	Easier than SSB	Complex (coherent)	1	$B + B_\Delta$ ($B_\Delta < B$)
VSB-LC (TV broadcasting)	Easier than SSB	Simple (envelope det.)	2nd Worst (very large carrier)	$B + B_\Delta$ ($B_\Delta < B$)

Example. A DSB-LC signal has the form

$$\phi(t) = [20 + 2\cos(3000\pi t) + 10\cos(6000\pi t)]\cos(2\pi f_c t) \qquad f_c = 10^5 \text{ Hz}$$

a) Determine the Fourier transform of $\phi(t)$

b) Determine the power in each of the frequency components.

c) Determine the modulation index.

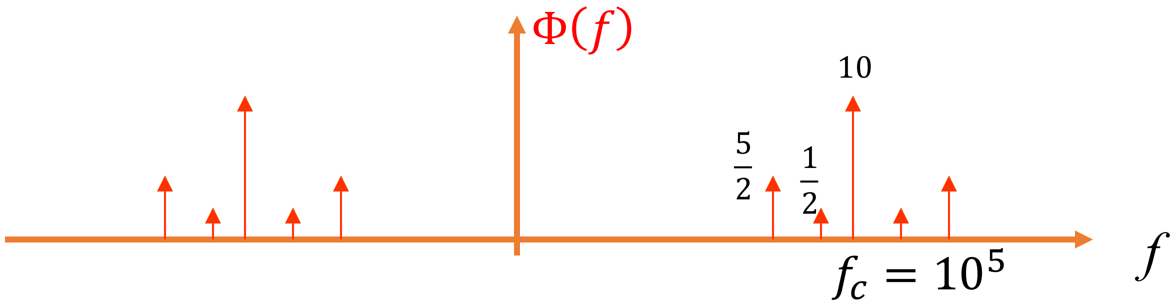
d) Determine the modulation efficiency.

Solution.

a) Determine the Fourier transform of $\phi(t)$

$$\phi(t) = 20\cos(2\pi f_c t) + \cos(2\pi f_c t - 3000\pi t) + \cos(2\pi f_c t + 3000\pi t)$$
$$+ 5\cos(2\pi f_c t - 6000\pi t) + 5\cos(2\pi f_c t + 6000\pi t)$$

$$\Phi(f) = 10\big(\delta(f - f_c) + \delta(f + f_c)\big)$$
$$+\frac{1}{2}\big(\delta(f - f_c + 1500) + \delta(f + f_c - 1500)\big)$$
$$+\frac{1}{2}\big(\delta(f - f_c - 1500) + \delta(f + f_c + 1500)\big)$$
$$+\frac{5}{2}\big(\delta(f - f_c + 3000) + \delta(f + f_c - 3000)\big)$$
$$+\frac{5}{2}\big(\delta(f - f_c - 3000) + \delta(f + f_c + 3000)\big)$$



b) Determine the power in each of the frequency components.

$$\phi(t) = 20\cos(2\pi f_c t) + \cos(2\pi f_c t - 3000\pi t) + \cos(2\pi f_c t + 3000\pi t) + 5\cos(2\pi f_c t - 6000\pi t) + 5\cos(2\pi f_c t + 6000\pi t)$$

$P_1 = 20^2/2 = 200$

$P_2 = (1)^2/2 = 0.5$

$P_3 = (1)^2/2 = 0.5$

$P_4 = (5)^2/2 = 12.5$

$P_5 = (5)^2/2 = 12.5$

c) Determine the modulation index

$$\phi(t) = 20\left(1 + \frac{2}{20}\cos 3000\pi t + \frac{10}{20}\cos 6000\pi t\right)\cos 2\pi f_c t$$

$$\phi(t) = A_c\left(1 + \alpha m_n(t)\right)\cos 2\pi f_c t = A_c\left(1 + \frac{\alpha m(t)}{\max|m(t)|}\right)\cos 2\pi f_c t$$

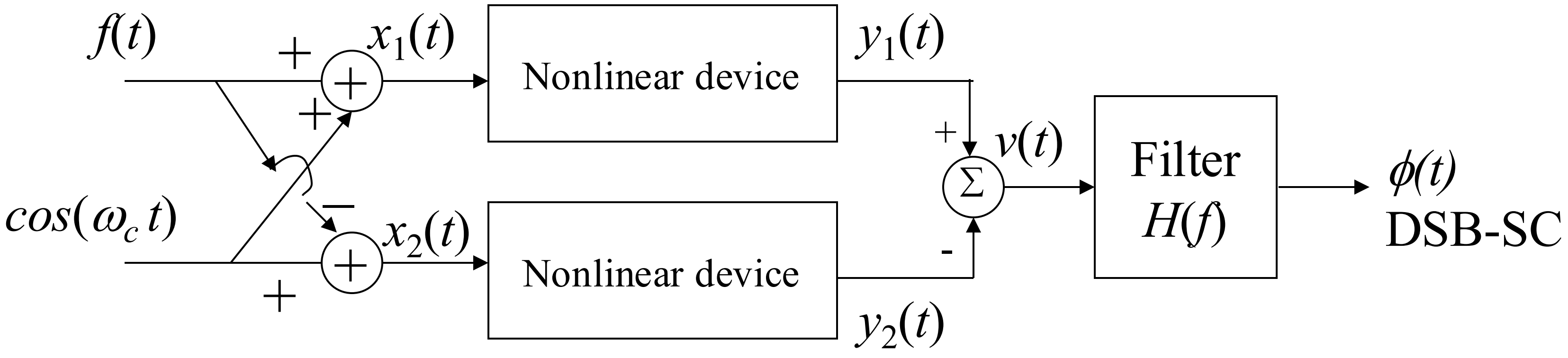
$$m_n(t) = \frac{2}{12}\cos 3000\pi t + \frac{10}{12}\cos 6000\pi t$$

$$\alpha = \frac{12}{20} = 0.6$$

d) Determine the modulation efficiency

$$\mu = \frac{\text{useful power}}{\text{total power}} = \frac{P_2 + P_3 + P_4 + P_5}{P_1 + P_2 + P_3 + P_4 + P_5} = \frac{26}{226} = 0.115$$

Example. A DSB-SC is generated as follows.



Non-linear devices are described by the following input-output relations:

$$y_1(t) = ax_1(t) + bx_1^2(t) \qquad y_2(t) = ax_2(t) + bx_2^2(t)$$

a) What is the input signal of the filter?

b) Specify the transfer function of the filter.

c) What is the output signal of the filter?

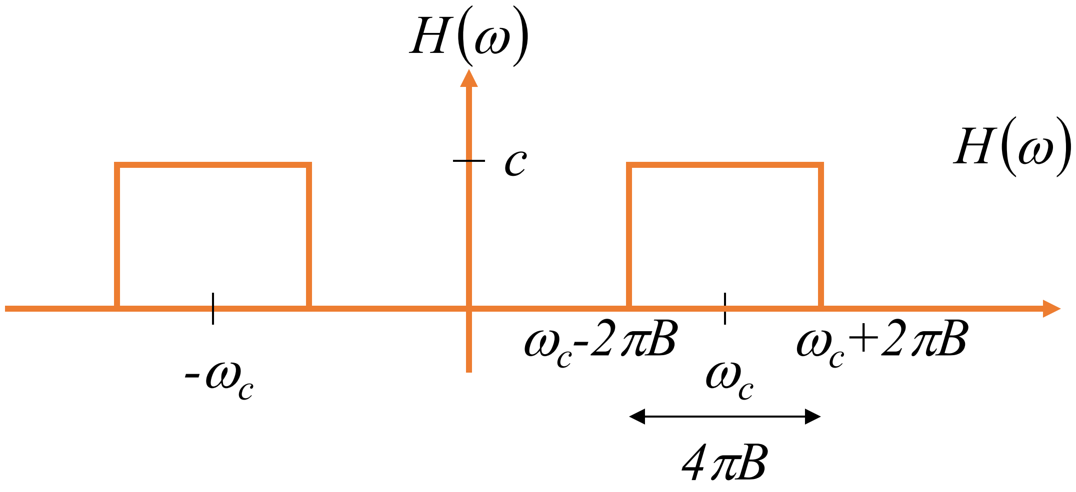
Solution. a)

$$\left.\begin{aligned} x_1(t) &= f(t) + \cos(\omega_c t) \\ x_2(t) &= -f(t) + \cos(\omega_c t) \end{aligned}\right\} \quad \begin{aligned} v(t) &= y_1(t) - y_2(t) \\ &= 2af(t) + 4bf(t)\cos(\omega_c t) \end{aligned}$$

$$V(\omega) = 2aF(\omega) + 2b[F(\omega - \omega_c) + F(\omega + \omega_c)]$$

b) In order to obtain the DSB-SC signal, we need a band-pass filter which will only allow the second term to pass.

$$v(t) = 2af(t) + 4bf(t)\cos(\omega_c t)$$
$$V(\omega) = 2aF(\omega) + 2b[F(\omega - \omega_c) + F(\omega + \omega_c)]$$



$$H(\omega) = \begin{cases} c, & |\omega - \omega_c| \le 2\pi B \\ 0, & \text{otherwise} \end{cases}$$

c) $\varphi(t) = 4bcf(t)\cos(\omega_c t)$

3.6 Signal Multiplexing

Multiplexing: Combining separate message signals into a composite signal for transmission over a common channel so that the different messages do not interfere with each others.

3.6.1 Quadrature Carrier Multiplexing (QCM)

3.6.2 Frequency-Division Multiplexing (FDM)

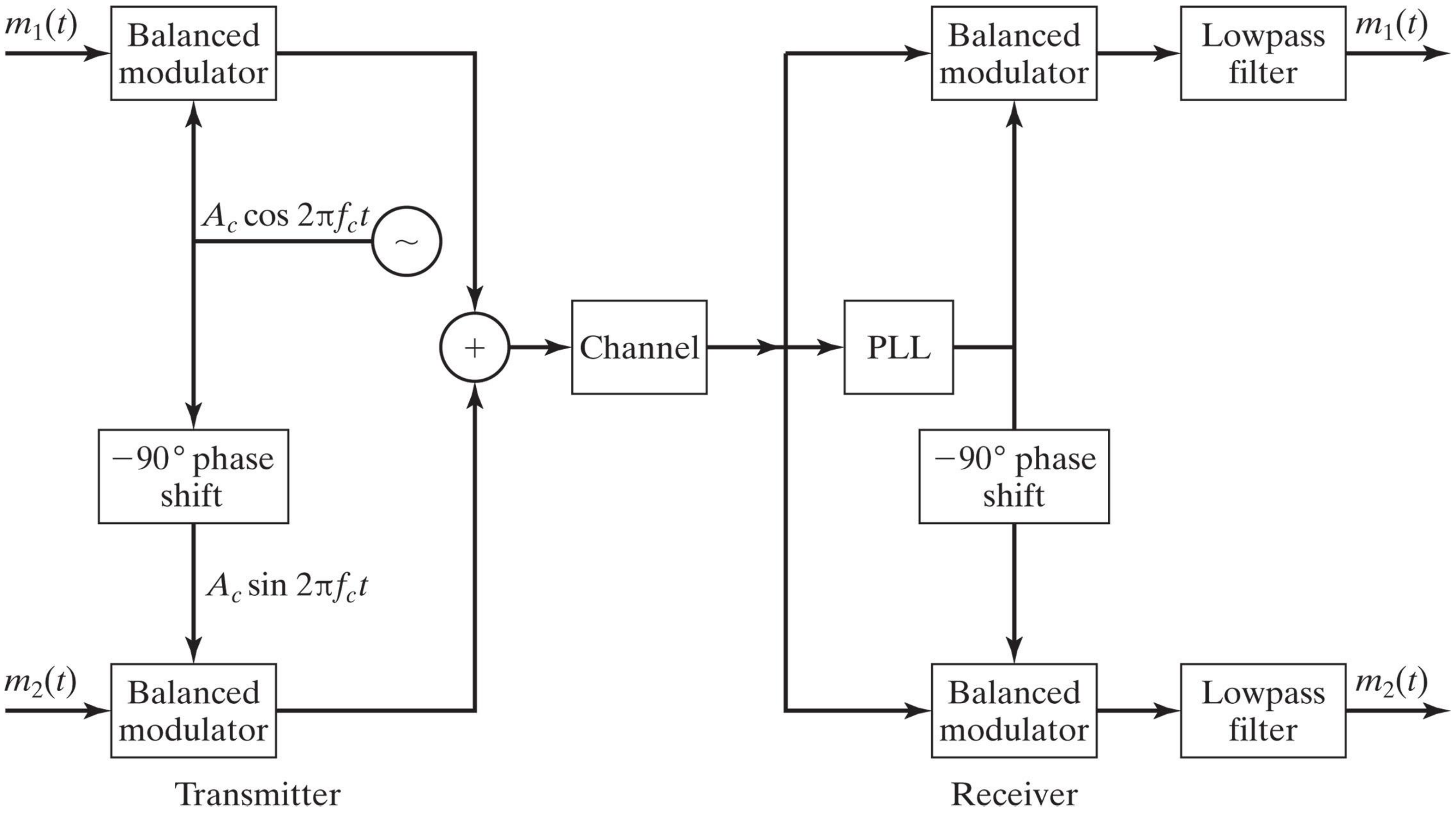
3.6.1 Quadrature Carrier Multiplexing (QCM)

QCM allows the transmission of two message signals on the same carrier frequency using two DSB-SCs on the in-phase and the quadrature channels, respectively. The bandwidth efficiency of QCM is similar to SSB-SC.

$$\phi(t) = A_c m_1(t) \cos 2\pi f_c t + A_c m_2(t) \sin 2\pi f_c t$$

QCM demodulation of $m_1(t)$ consists of multiplying $\phi(t)$ by $\cos 2\pi f_c t$ followed by LPF.
Similarly, the demodulation of $m_2(t)$ consists of multiplying $\phi(t)$ by $\sin 2\pi f_c t$ followed by LPF.

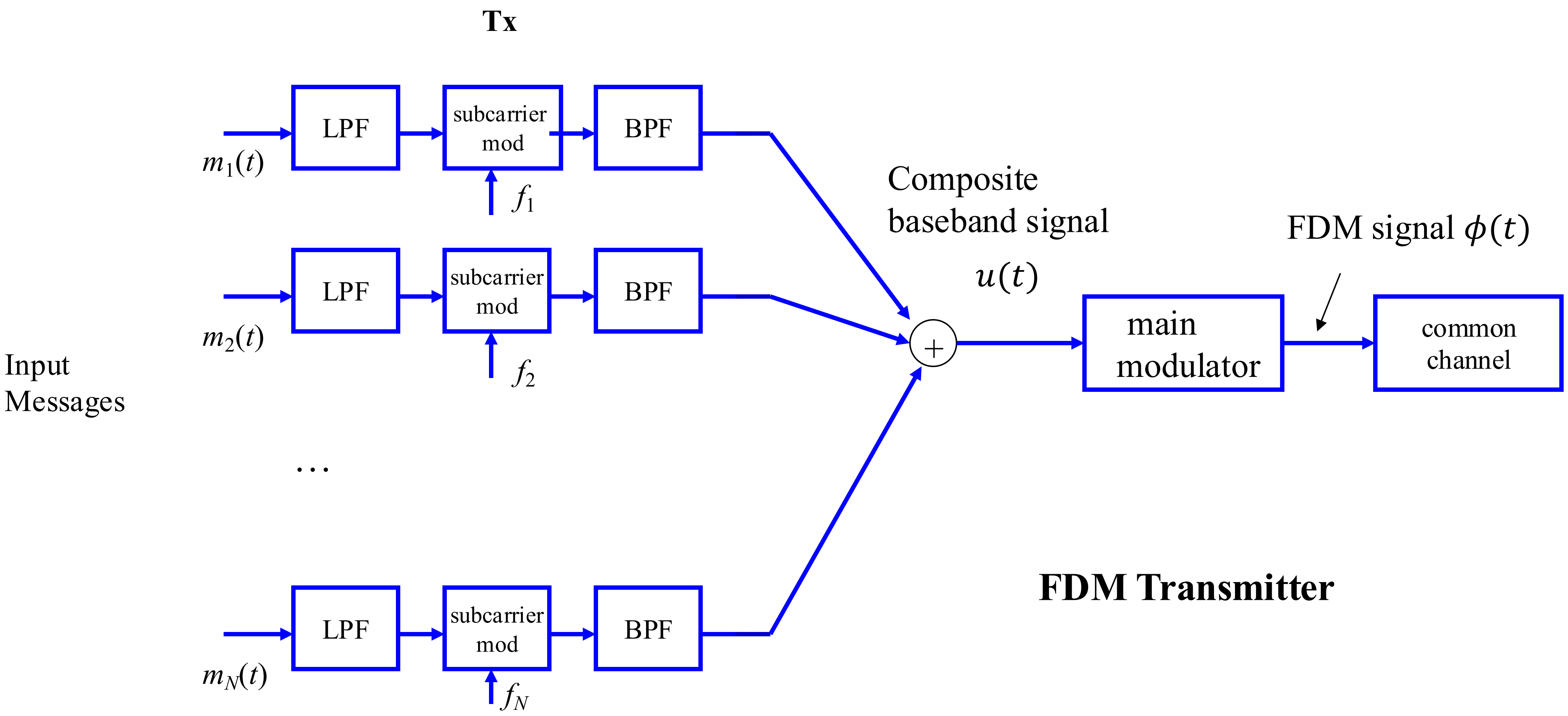
QCM block diagram



3.6.2 Frequency-Division Multiplexing (FDM) and FM Stereo Broadcasting

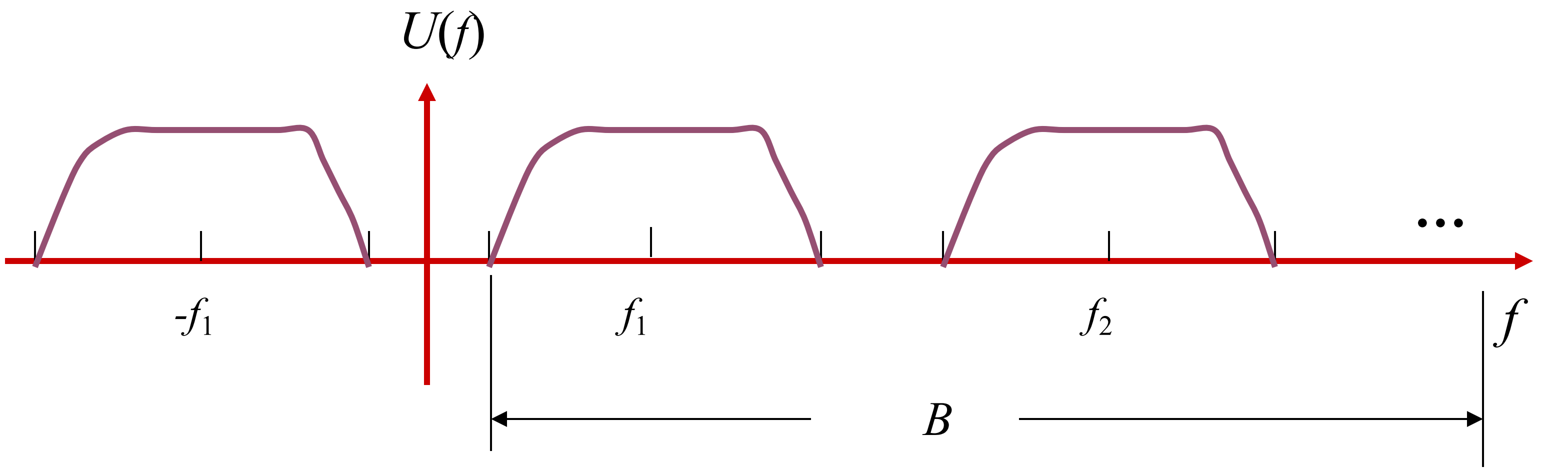
Objective: To transmit a number of independent signals over a common channel so that they do not interfere

This can be accomplished by separating the signals in frequency or in time. The technique of separating the signals in frequency is referred to as *frequency-division multiplexing (FDM)*, whereas the technique of separating the signals in time is called *time-division multiplexing* (TDM) which is used more in digital communications.

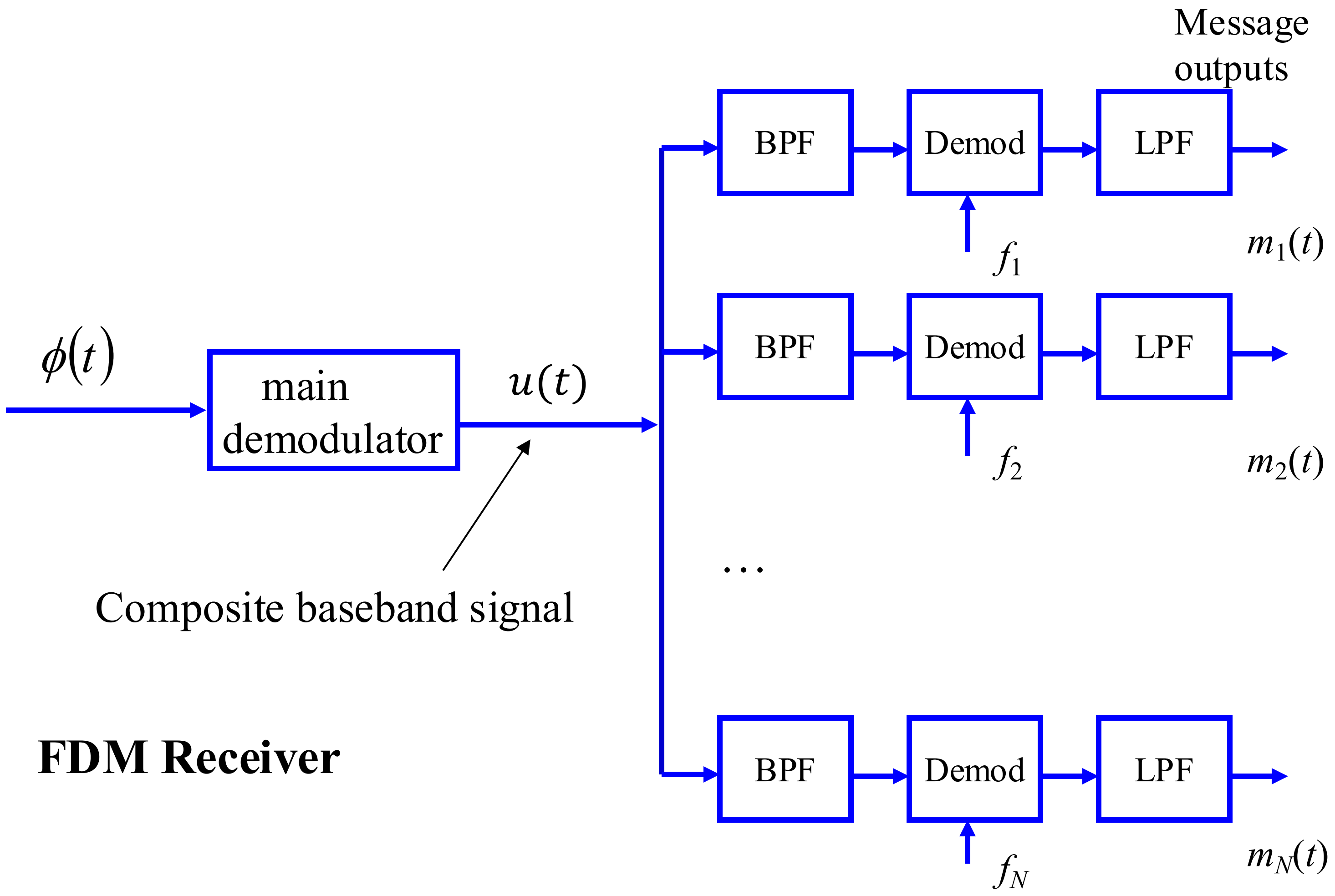


FDM Transmitter

The LPF's may be omitted only if the input signals are sufficiently band-limited initially. The modulators shift the frequency ranges of the signals so as to occupy mutually exclusive frequency intervals. Any one of the modulation methods, introduced before, DSB-LC, DSB-SC, SSB, VSB, may be used here (the most widely used method is SSB).



Spectrum of the composite baseband signal

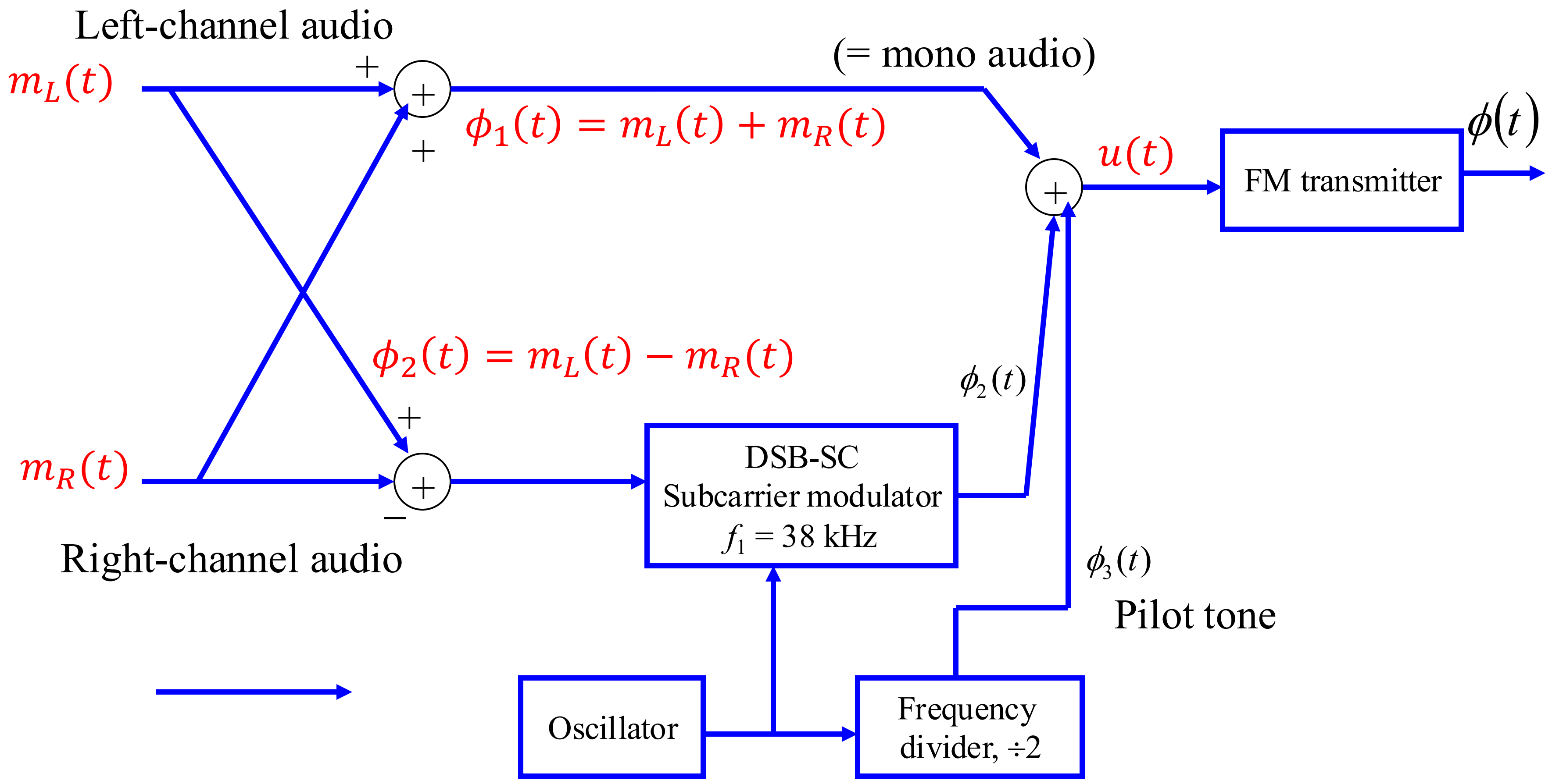


FDM Receiver

The received FDM signal is first demodulated to reproduce the composite baseband signal that is passed through filters to separate the individual modulated subcarriers. The subcarriers are then demodulated to reproduce the message signals $m_1(t)$, …, $m_N(t)$.

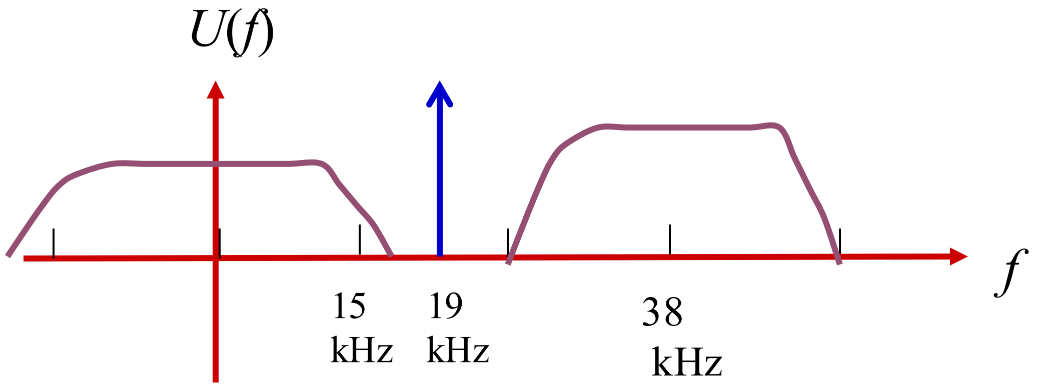
Example. FM stereo broadcasting system in North America

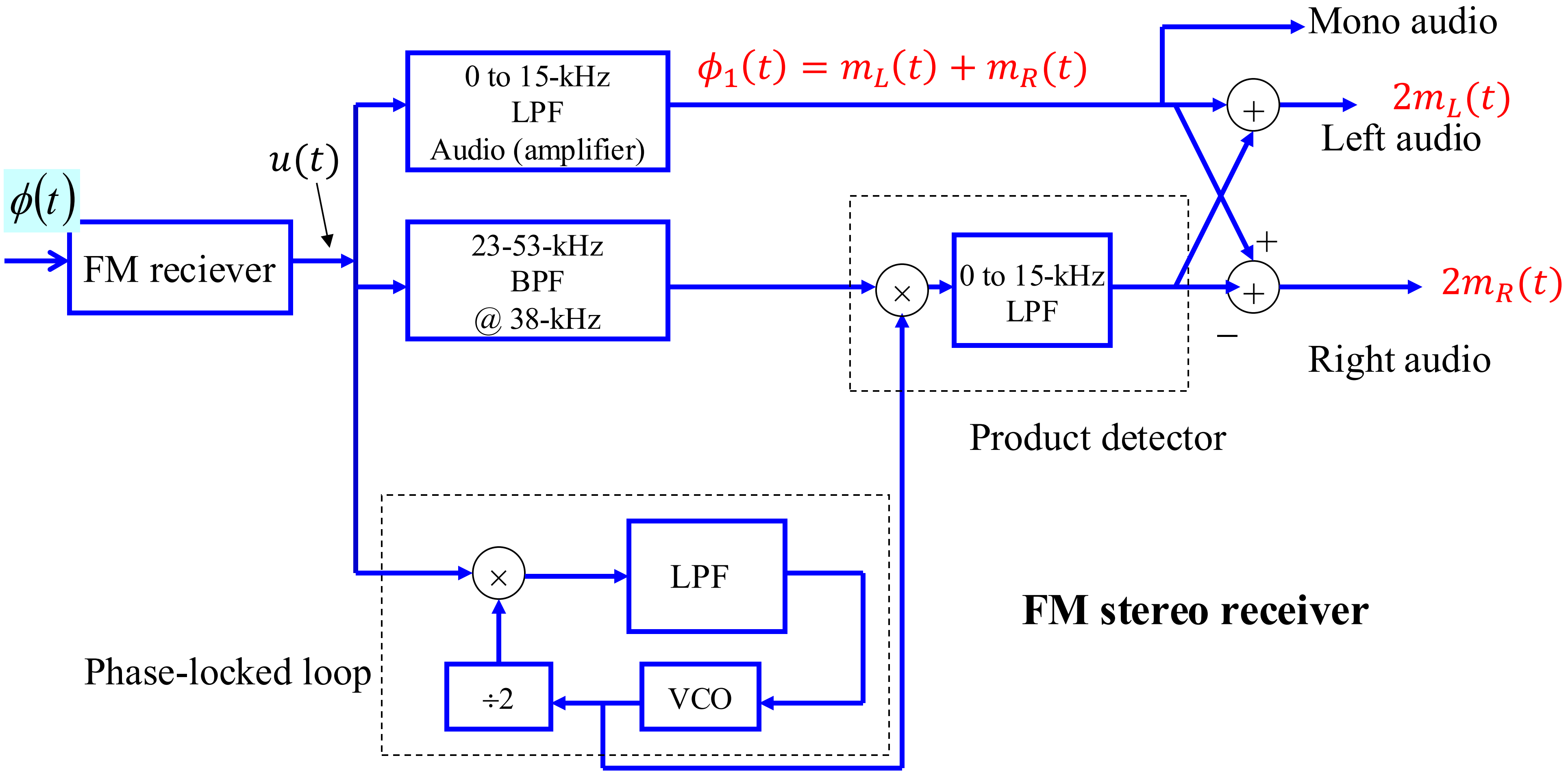
It is compatible with the monaural FM system that has existed since the 1940s. The difference audio is used to modulate a 38 kHz carrier to produce a DSB-SC signal. A 19-kHz pilot signal is added to the composite baseband signal $f(t)$ to provide a reference signal for coherent detector in the receiver.



FM stereo transmitter

Spectrum of composite baseband signal





FM stereo receiver

Appendix: Bandpass Signals and the Complex Envelope

- Recall that given a real signal $x(t)$, we defined the analytic signal associated with it as

$$Z(f) = \begin{cases} 2X(f) & f > 0 \\ 0 & \text{otherwise} \end{cases}$$

Implying that $z(t) = x(t) + j\hat{x}(t)$ with $\hat{x}(t)$ the Hilbert transform of $x(t) = \mathrm{Re}\{z(t)\}$.

The notion of analytic signals is valid for both baseband and passband (or bandpass) signals and allows for the simplification of their study given the Hermitian symmetry property of real signals (i.e., either the positive or negative part of its spectrum contains all the information about the real signal).

- For bandpass signals, however, we can further simplify their study by considering the positive part of the spectrum *shifted around* 0 Hz*,* which gives us the complex envelope of $x(t)$, denoted here by $\tilde{x}(t)$ with

$\tilde{X}(f) = 2X(f + f_c)$ for $f > -f_c$ (and *0* otherwise); or $\tilde{x}(t) = z(t)e^{-j2\pi f_c t}$ giving

$$x(t) = \mathrm{Re}\{z(t)\} = \mathrm{Re}\{\tilde{x}(t)e^{j2\pi f_c t}\}, \text{ where}$$

$$\tilde{x}(t) = x_I(t) + jx_Q(t) = \left(x(t) + j\hat{x}(t)\right) e^{-j2\pi f_c t}$$

which allows for the in-phase and quadrature decomposition of the bandpass signal $x(t)$ as

$$x(t) = x_I(t) \cos 2\pi f_c t \ - \ x_Q(t) \sin 2\pi f_c t$$

- The real envelope (i.e., the smooth curve connecting the peaks of the bandpass signal) is given by

$$A_x(t) = |\tilde{x}(t)| = \sqrt{x_I(t)^2 + x_Q(t)^2}$$

Complex Envelope, In-phase and Quadrature Components of AM Signals

- DSB-SC: $x_I(t) = A_c m(t), x_Q(t) = 0$
- DSB-LC: $x_I(t) = A_c(1 + \alpha m_n(t)),\ x_Q(t) = 0$
- SSB-SC: $x_I(t) = A_c m(t),\ x_Q(t) = A_c \hat{m}(t)$
- VSB-SC: $x_I(t) = \frac{A_c}{2} K\, m(t),\ x_Q(t) = \frac{A_c}{2} m_\star(t),$

where $m_\star(t) = h_\star(t) * m(t)$

$$H_\star(f) = \frac{1}{j}\big(H(f + f_c) - H(f - f_c)\big)$$

such that $H(f)$ is the VSB passband filter satisfying the vestigial symmetry property: $H(f - f_c) + H(f + f_c) = K$

Derivation of the Complex Envelope of VSB Signals

The mathematical derivation here is given for interested students only and will not be tested in any examinations

- Recall that VSB (and SSB) signals can be obtained by appropriately bandpass filtering a DSB-SC signal. In the following we relate bandpass filtering of a signal to baseband filtering of its complex envelope, which allows us to represent the in-phase and quadrature components of VSB (or SSB) signals in terms of the passband filter. Let $x(t)$ denote a passband signal and let $y(t) = x(t) * h(t)$. Then, the complex envelope of $y(t)$ is related to the complex envelope of $x(t)$ by
  $\tilde{y}(t) = \frac{1}{2}\tilde{x}(t) * \tilde{h}(t)$ since $\tilde{Y}(f) = 2Y(f+f_c) = 2X(f+f_c)H(f+f_c) = \frac{1}{2}\tilde{X}(f)\tilde{H}(f)$ for $f + f_c > 0$

- Writing $\tilde{y}(t) = y_I(t) + j\,y_Q(t)$ gives $y_I(t) = \frac{1}{2}\left(\tilde{y}(t) + \tilde{y}^*(t)\right)$ and $y_Q(t) = \frac{1}{2j}\left(\tilde{y}(t) - \tilde{y}^*(t)\right)$.

  Taking the Fourier transform yields:

$$\begin{aligned} Y_I(f) &= H(f+f_c)X(f+f_c) + H^*(-f+f_c)\,X^*(-f+f_c) \\ &= H(f+f_c)X(f+f_c) + H(f-f_c)X(f-f_c) \qquad (\star) \end{aligned}$$

and $$Y_Q(f) = \frac{1}{j}\left(H(f+f_c)X(f+f_c) - H(f-f_c)X(f-f_c)\right)$$

since both $x(t)$ and $h(t)$ are real signals satisfying the Hermitian symmetry property (i.e., $X^*(-f+f_c) = X(f-f_c)$ and $H^*(-f+f_c) = H(f-f_c)$).

The Complex Envelope of VSB Signals (continued)

- For VSB-SC $y(t) = x(t) * h(t)$ with $x(t)$ a DSB-SC signal and the passband filter $H(f)$ satisfying the vestigial symmetry property:

$$H(f + f_c) + H(f - f_c) = K$$

- The spectrum of the DSB-SC signal $x(t)$ is given by

$X(f) = \frac{A_c}{2}\big(M(f - f_c) + M(f + f_c)\big)$ and $\tilde{X}(f) = A_c M(f)$ with $m(t)$ the baseband message. This yields:

$$Y_I(f) = \big(H(f + f_c) + H(f - f_c)\big)\frac{A_c M(f)}{2} = \frac{A_c}{2} K\, M(f)$$

or $y_I(t) = \frac{A_c}{2} K m(t)$ and

$$Y_Q(f) = \frac{1}{j}\big(H(f + f_c) - H(f - f_c)\big)\frac{A_c M(f)}{2},$$

or $y_Q(t) = \frac{A_c}{2} m(t) * h_\star(t)$ with

$$H_\star(f) = \frac{1}{j}\big(H(f + f_c) - H(f - f_c)\big)$$

The Complex Envelope of SSB Signals

- For SSB-SC $y(t) = x(t) * h(t)$ with $x(t)$ a DSB-SC signal and the passband filter $H(f)$ ideally completely removes one of sidebands and passes the other intact. For example, for upper SSB signal:

$$H(f) = K\big(u(f - f_c) + u(-f - f_c)\big)$$

with K a constant representing the filter gain, and $u(f)$ the unit step function (i.e., the filter passes all frequencies above f_c and below $-f_c$ with a gain K and filters out all other frequencies).

- Substituting in equation $(\star)$ in slide 83, gives (over the message bandwidth $-B \leq f \leq B$):

$$Y_I(f) = K\big(u(f) + u(-f)\big)\frac{A_c M(f)}{2} = \frac{A_c}{2} K\, M(f)$$

or $y_I(t) = \frac{A_c}{2} K m(t)$ and

$$Y_Q(f) = \frac{1}{j} K\big(u(f) - u(-f)\big)\frac{A_c M(f)}{2} = -j\,\mathrm{sgn}(f)\frac{A_c K M(f)}{2},$$

or $y_Q(t) = \frac{A_c K}{2} \hat{m}(t)$ with

$\hat{m}(t) = \frac{1}{\pi t} * m(t)$ the Hilbert transform of the message $m(t)$.