Topic 4. Angle Modulation (Chapter 4)

UNIVERSITY OF WATERLOO

FACULTY OF ENGINEERING

1. FM and PM (4.1)
2. Narrowband FM (NBFM, 4.1)
3. Wideband FM, FM Average Power (4.2)
4. Generation and Demodulation of FM Signals (4.3)

Introduction and Objectives

Objectives:

- To study frequency modulation (FM) and phase modulation (PM)
- To analyze the spectrum of FM signals (in the simple case of single-tone message)
- To study the structure of FM modulator and demodulator and their implementation

Amplitude modulation (AM): The amplitude of the carrier varies in accordance with the baseband message signal $m(t)$.

Angle modulation (FM, PM): The angle of the carrier varies in accordance with the message signal while the amplitude of the carrier is constant.

$$\varphi(t) = a(t)\cos(2\pi f_c t + \gamma(t))$$

AM

FM, PM

Amplitude Modulation vs. Angle Modulation

Amplitude Modulation:

- Linear modulation (superposition principle applies): simple transceiver.
- The spectrum of the modulated signal is a shifted/scaled version of the message spectrum.
- Transmission bandwidth (B_T) vs message bandwidth (B): $B \leq B_T \leq 2B$
- The signal to noise ratio (SNR) at the receiver can be increased only by increasing the transmitted signal power.
- AM broadcasting: 535-1,705 kHz (message bandwidth 4 kHz, 10 kHz channel spacing)

Angle Modulation:

- Nonlinear modulation: complicated transceiver.
- The spectrum of the modulated signal is NOT simply related to message signal spectrum.
- Transmission bandwidth $\gg$ Message bandwidth.
- Improvement in SNR without increasing the transmitted signal power (constant envelope). Resistant to interference from adjacent channels (**capture effect**) allowing for frequency reuse.
- FM broadcasting: 87.5-108.0 MHz (message bandwidth 15 kHz, 200 kHz channel spacing)

Some History on FM

- 1920’s (infancy of radio broadcasting): Searching to reduce the utilized bandwidth, superficial reasoning showed that it was possible to largely reduce the bandwidth of AM signals using FM (confusing instantaneous frequency and spectrum).
- 1922: Carson showed the fallacy of this argument: FM requires (much) larger bandwidth than AM.
- Carson didn’t recognize any compensating advantage of FM because the noise power at the receiver increases with the bandwidth ($N_0\ B_T$).
- Edwin H. Armstrong (1890-1954): showed in 1936 that FM was (much) superior to AM in its ability to overcome the effect of additive noise.
- 1948: Shannon-Hartly law (i.e., the channel capacity) corroborated Armstrong result by proving that channel capacity increases with the transmission bandwidth despite the increase in the noise power.

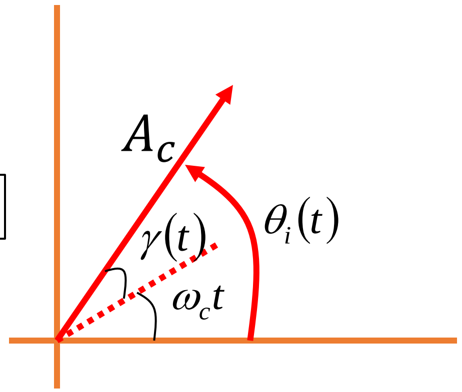
1. FM and PM

Angle modulated signal: $\phi(t) = A_c \cos\theta_i(t) = A_c \cos\left(2\pi f_c t + \gamma(t)\right)$

Phasor representation: The real signal is represented by a rotating vector in the complex plane.

$$\phi(t) = A_c \operatorname{Re}\left\{e^{\left(j2\pi f_c t + j\gamma(t)\right)}\right\}$$

$\boxed{\omega_c = 2\pi f_c}$



A_c: phasor magnitude

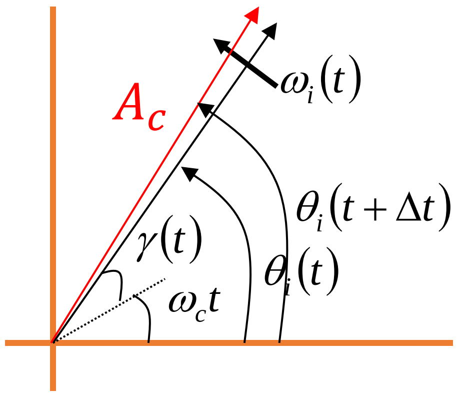
$\theta_i(t)$: instantaneous angle (determines the position of the rotating vector at time t)

$$\theta_i(t) = 2\pi f_c t + \underbrace{\gamma(t)}_{}$$

related to the baseband message signal $m(t)$

Angular velocity (frequency):

$$\phi(t) = A_c \cos\theta_i(t) = A_c \cos\left(2\pi f_c t + \gamma(t)\right)$$



$$\left.\begin{aligned} t_1 &= t \longrightarrow \theta_i(t) \\ t_2 &= t+\Delta t \longrightarrow \theta_i(t+\Delta t) \end{aligned}\right\} \quad \omega_{\Delta t}(t) \overset{def}{=} \frac{\theta_i(t+\Delta t)-\theta_i(t)}{\Delta t}$$

average frequency for duration of Δt
indicates how fast/slow angle changes

$$\omega_i(t) \overset{def}{=} \lim_{\Delta t\to 0} \omega_{\Delta t}(t) = \lim_{\Delta t\to 0} \frac{\theta_i(t+\Delta t)-\theta_i(t)}{\Delta t} = \frac{d\theta_i(t)}{dt}$$

instantaneous frequency at time *t*
also known as "*angular velocity*"

Recall that distance = velocity × time

$$\omega_i(t) = \frac{d\theta_i(t)}{dt} = \omega_c + \frac{d\gamma(t)}{dt}$$

varies with the message signal

Angle: $\theta_i(t) = \int_0^t \omega_i(\tau)d\tau + \theta_0$

Phase Modulation (PM)

$m(t)$ → Phase Modulator → $\phi_{PM}(t) = A_c \cos\left(\omega_c t + k_p m(t) + \theta_0\right)$

$A_c \cos(\omega_c t + \theta_0)$

k_p: phase deviation (or sensitivity) constant (measured in [rad/volt] for a voltage message)

Instantaneous angle $\theta_i(t) = \omega_c t + k_p m(t) + \theta_0$ → Phase $\propto m(t)$ → **PM**

Instantaneous frequency $\omega_i(t) = \dfrac{d\theta_i(t)}{dt} = \omega_c + k_p \dfrac{dm(t)}{dt}$

$\Delta\theta = k_p \max|m(t)|$

Maximum (peak) phase deviation †

†This assumes $\overline{m(t)} = 0$, which is valid for most practical signals. If the time-average of the signal $\overline{m(t)} \neq 0$, then $\overline{m(t)}$ can be added to the initial phase θ_0.

Frequency Modulation (FM)

$m(t) \longrightarrow$ $\boxed{\text{Frequency Modulator}}$ $\longrightarrow$ $\phi_{FM}(t) = A_c \cos\left(\omega_c t + k_\omega \int_0^t m(\tau)d\tau + \theta_0\right)$

$\uparrow$

$A_c \cos(\omega_c t + \theta_0)$

k_ω=$2\pi k_f$: frequency deviation (or sensitivity) constant (k_ω is measured in [rad/volt-sec], k_f is measured in [Hz/volt] for a voltage message)

Instantaneous angle $\qquad \theta_i(t) = \omega_c t + k_\omega \int_0^t m(\tau)d\tau + \theta_0$

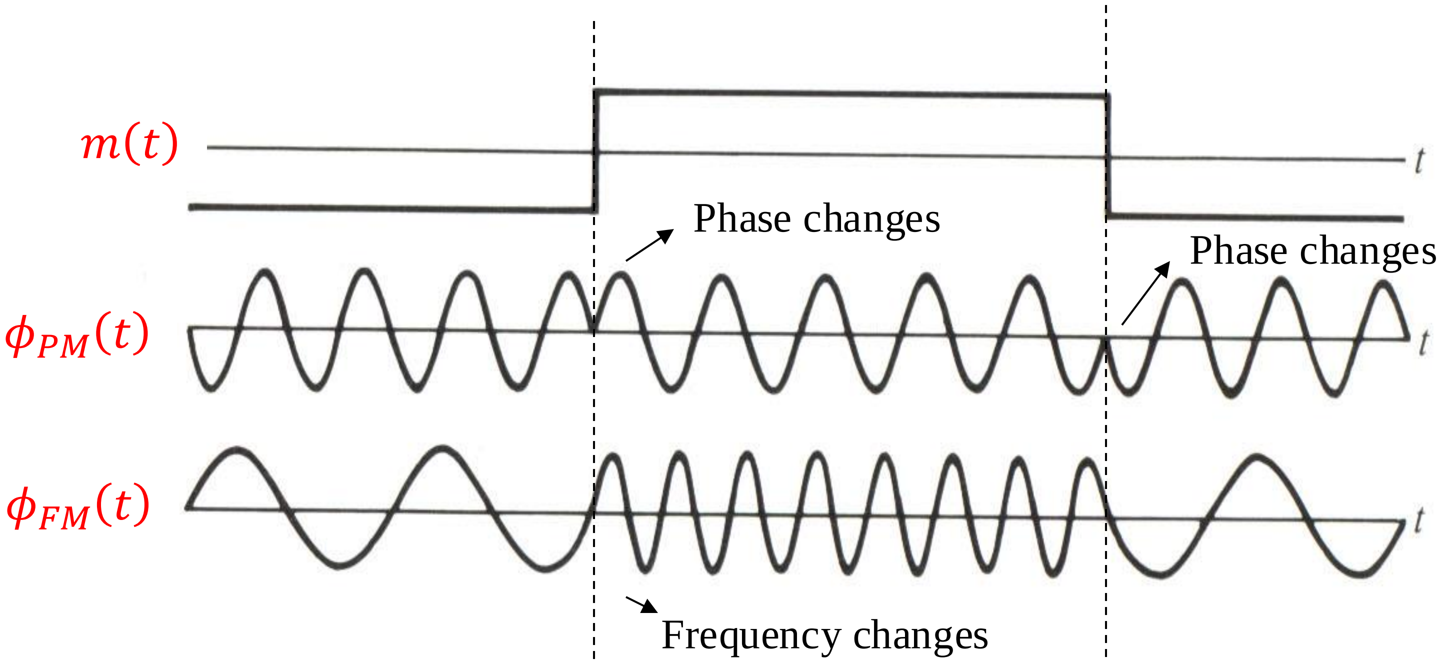
Instantaneous frequency $\qquad \omega_i(t) = \frac{d\theta_i(t)}{dt} = \omega_c + k_\omega m(t)$

$\longrightarrow$ $\boxed{\text{Frequency} \propto m(t)}$ $\longrightarrow$ **FM**

$$\Delta\omega = k_\omega \max|m(t)|$$

$$\Delta f = \frac{\Delta\omega}{2\pi} = k_f \max|m(t)|$$ Maximum (peak) frequency deviation

Example of FM and PM Signals



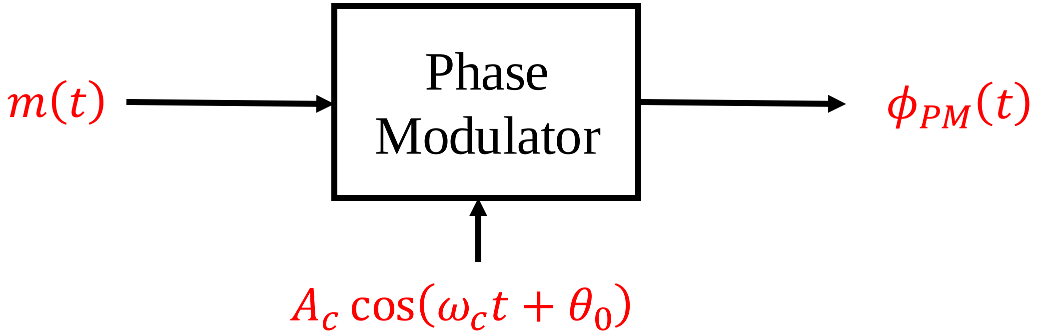
$$\phi_{FM}(t) = A_c \cos\left(\omega_c t + k_\omega \int_0^t m(\tau) d\tau + \theta_0 \right)$$

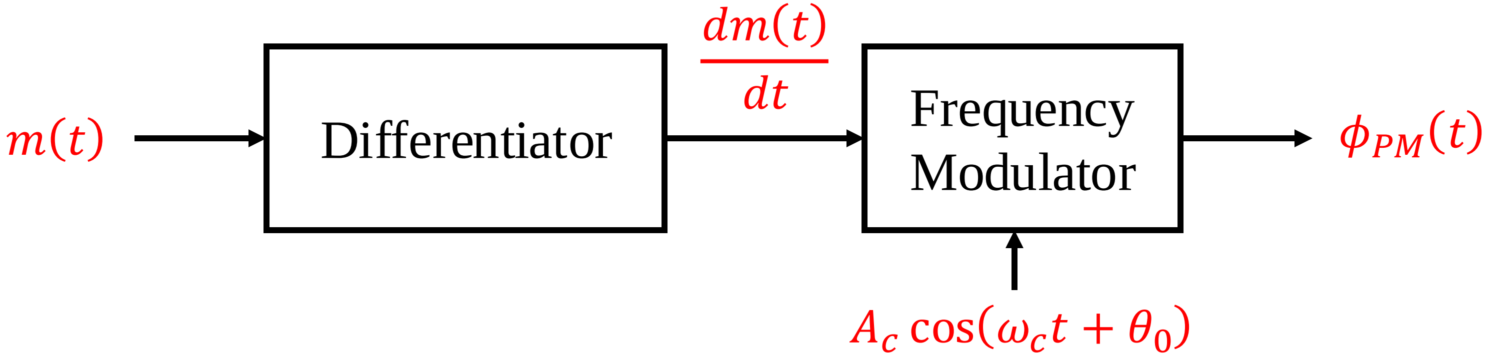
$$\phi_{PM}(t) = A_c \cos\left(\omega_c t + k_p m(t) + \theta_0\right)$$

Relation Between FM and PM Signals

$$\phi_{PM}(t) = A_c \cos\left(\omega_c t + k_p m(t) + \theta_0\right)$$

$$\phi_{FM}(t) = A_c \cos\left(\omega_c t + k_\omega \int_0^t m(\tau)d\tau + \theta_0\right)$$

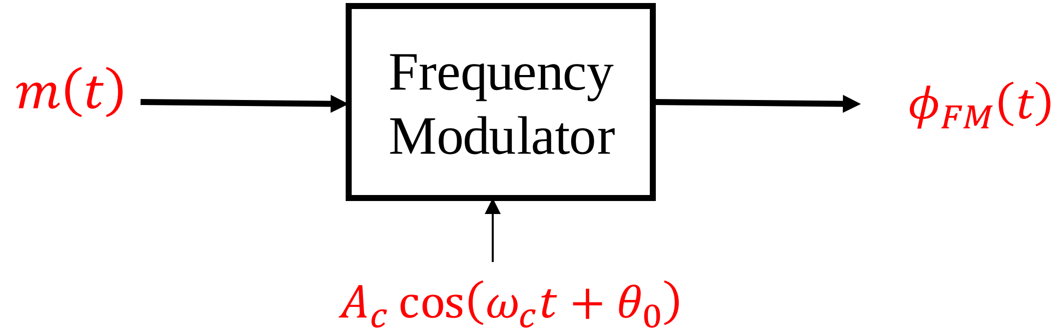


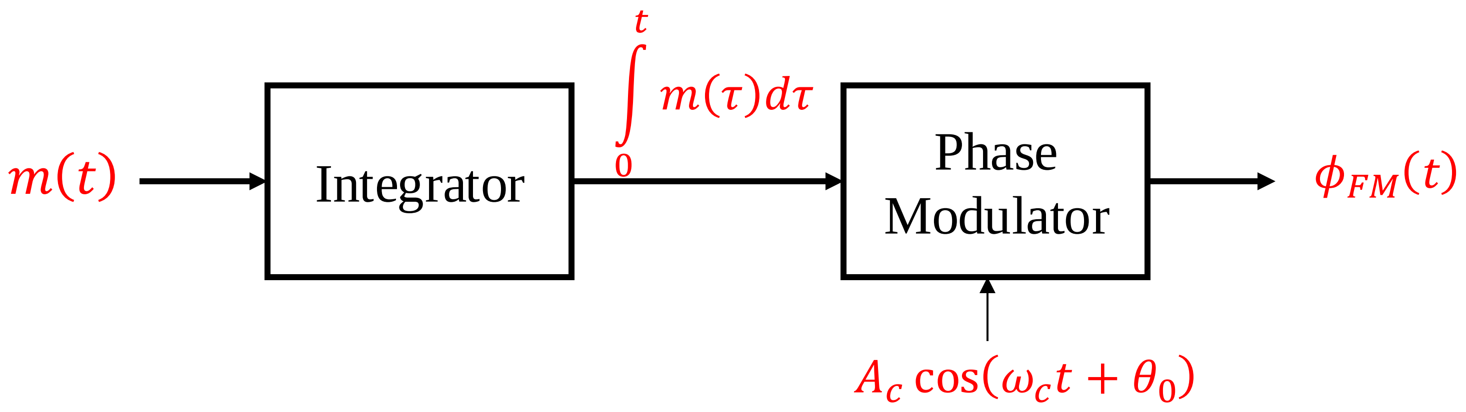


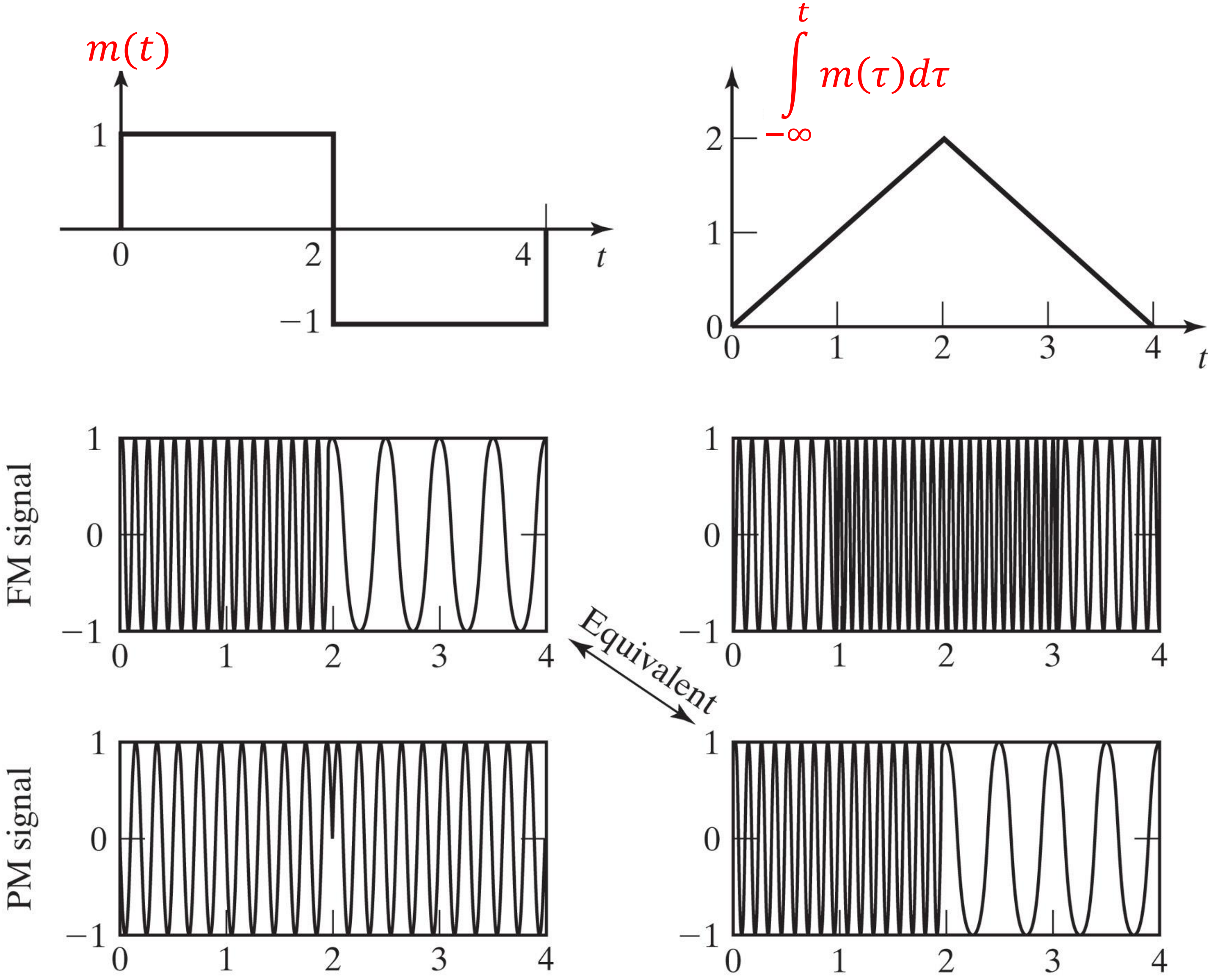
Relation Between FM and PM Signals (continued)

$$\phi_{PM}(t) = A_c \cos\left(\omega_c t + k_p m(t) + \theta_0\right)$$

$$\phi_{FM}(t) = A_c \cos\left(\omega_c t + k_\omega \int_0^t m(\tau)d\tau + \theta_0\right)$$







$m(t)$
1
0
2
4
t
−1
$\int_{-\infty}^{t} m(\tau)d\tau$
2
1
0
0
1
2
3
4
t
FM signal
1
0
−1
0
1
2
3
4
Equivalent
PM signal

Modulation Indexes for PM and FM

Define $\beta = \beta_p = \Delta\theta$ as the modulation index of a PM signal, and

$$\beta = \beta_f = \frac{\Delta\omega}{W} = \frac{k_\omega \max|m(t)|}{2\pi B} = \frac{\Delta f}{B}$$

as the modulation index of an FM signal where B [Hz] (or $\mathrm{W} = 2\pi B$ [rad/s]) is the bandwidth of the message signal $m(t)$

Depending on the values of β, we may distinguish between two cases of frequency (phase) modulations:

- Narrowband FM (PM) when $\beta \ll 1$;
- Wideband FM (PM) when β is large.

Example 1:

a) Determine the instantaneous frequency of $\phi(t) = 5\cos(20\pi t + \sin 10\pi t)$ as well as its maximum and minimum values

b) What is the modulation index β if $\phi(t)$ is an FM signal? If it is a PM signal?

Example 2: Let $m(t) = \cos\omega_m t + \frac{1}{2}\cos 3\omega_m t$, and $c(t) = A_c \cos\omega_c t$ be a baseband message and a carrier sinusoidal wave, respectively.

Determine the angle modulated signal $\phi(t)$ (with $m(t)$ modulating the carrier $c(t)$) when:

a) $\phi(t)$ is a PM signal and its modulation index is 3.

b) $\phi(t)$ is an FM signal and its modulation index is $1/4$.

2. Narrowband FM (NBFM)

$$\phi(t) = A_c \cos(\omega_c t + \gamma(t))$$
$$= A_c \cos(\omega_c t)\cos(\gamma(t)) - A_c \sin(\omega_c t)\sin(\gamma(t))$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

NBFM assumption:

$$|\gamma(t)| \ll 1 \implies \cos\gamma(t) \approx 1, \sin\gamma(t) \approx \gamma(t)$$

$$\phi(t) \approx A_c \cos\omega_c t - A_c\gamma(t)\sin\omega_c t$$

NBFM signal

A Case Study: NBFM Single Tone Modulation

Assume $m(t) = a\cos\omega_m t \Rightarrow$

Single tone modulation $\quad \phi(t) = A_c\cos\left(\omega_c t + k_\omega \int_0^t m(\tau)d\tau\right), \theta_0 = 0$

Instantaneous frequency

$$\omega_i(t) = \frac{d\theta_i(t)}{dt} = \omega_c + k_\omega m(t) = \omega_c + a\,k_\omega\cos\omega_m t$$
$$= \omega_c + \Delta\omega\cos\omega_m t$$

$\Delta\omega = k_\omega \max|m(t)| = ak_\omega$ Peak frequency deviation

Instantaneous angle

$$\theta_i(t) = \omega_c t + k_\omega \int_0^t m(\tau)d\tau = \omega_c t + a\,k_\omega \int_0^t \cos\omega_m\tau\, d\tau$$
$$= \omega_c t + \frac{ak_\omega}{\omega_m}\sin\omega_m t$$

$\beta = \dfrac{\Delta\omega}{2\pi B} = \dfrac{ak_\omega}{\omega_m}$ Modulation index of FM signal

A Case Study: NBFM Single Tone Modulation (Cont.)

$$m(t) = a\cos\omega_m t$$

$$\phi(t) = A_c \cos\left(\omega_c t + k_\omega \int_0^t m(\tau)d\tau\right)$$

$$= A_c \cos(\omega_c t + \beta \sin\omega_m t)$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$= A_c \cos\omega_c t\ \cos(\beta\sin\omega_m t) - A_c \sin\omega_c t\ \sin(\beta \sin\omega_m t)$$

NBFM: $\beta \ll 1 \Rightarrow |\beta \sin\omega_m t| \ll 1$

$$\cos[\beta\sin(\omega_m t)] \approx 1 \quad \sin[\beta\sin(\omega_m t)] \approx \beta\sin(\omega_m t)$$

$$\phi(t) \approx A_c\cos(\omega_c t) - A_c\beta\sin(\omega_m t)\sin(\omega_c t)$$

NBFM signal for single tone modulation

A Case Study: NBFM Single Tone Modulation (Cont.)

$$\phi(t) \approx A_c \cos(\omega_c t) - A_c \beta \sin(\omega_m t)\sin(\omega_c t)$$

Rearranging as $\phi(t) \approx k \cos(\omega_c t + \alpha)$

$$k = A_c \sqrt{1 + \beta^2 \sin(\omega_m t)}$$

$$\alpha = \tan^{-1}(\beta \sin(\omega_m t))$$

$$\phi(t) \approx \underbrace{A_c \sqrt{1 + \beta^2 \sin^2(\omega_m t)}} \cos\left\{\omega_c t + \tan^{-1}\left[\beta \sin(\omega_m t)\right]\right\}$$

modulates the amplitude

$\rightarrow$ the envelope suffers distortion, unless β is very small.

3. Wideband FM (WBFM)

3.I The WBFM Signal for Single-Tone Message

For a general FM signal, it is not possible to evaluate the Fourier transform in a closed-form. Here, we again focus on single tone modulation.

$$m(t) = a\cos\omega_m t$$

$$\phi(t) = A_c\cos\left(\omega_c t + k_\omega\int_0^t m(\tau)d\tau\right) = A_c\cos(\omega_c t + \beta\sin\omega_m t) = \mathrm{Re}\{A_c e^{j\beta\sin\omega_m t}e^{j\omega_c t}\}$$

How can we write the above expression in terms of $\cos\big((\omega_c + n\omega_m)t\big)$?

$n = 0, \pm 1, \pm 2, \ldots$

WBFM Signal for Single Tone Message (Cont.)

$$\phi(t) = A_c \cos(\omega_c t + \beta \sin(\omega_m t))$$

$$= A_c \operatorname{Re}\{e^{j[\omega_c t + \beta \sin(\omega_m t)]}\} = A_c \operatorname{Re}\{e^{j\omega_c t} x(t)\}$$

$$x(t) = e^{j\beta \sin(\omega_m t)}$$

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_m t} \longleftrightarrow \boxed{X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jn\omega_m t} dt}$$

$$X_n = \frac{1}{T}\int_{-T/2}^{T/2} e^{j\beta \sin(\omega_m t)} e^{-jn\omega_m t} dt$$

Variable change

$$\xi = \omega_m t = (2\pi/T)t$$

$$X_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta \sin \xi - n\xi)} d\xi$$

This integral can be evaluated numerically in terms of the parameters n and β and tabulated extensively. It is denoted by $J_n(\beta)$ and called the *Bessel function of the first kind*. The Bessel functions are real-valued.

WBFM Signal for Single Tone Message (Cont.)

$$x(t) = e^{j\beta\sin(\omega_m t)} = \sum_{n=-\infty}^{\infty} \underbrace{J_n(\beta)}_{X_n} e^{jn\omega_m t} \quad \text{where} \quad J_n(\beta) \overset{def}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta\sin\xi - n\xi)} d\xi$$

$$\phi(t) = A_c \operatorname{Re}\left\{e^{j[\omega_c t + \beta\sin(\omega_m t)]}\right\} = A_c \operatorname{Re}\left\{e^{j\omega_c t} x(t)\right\}$$

$$= A_c \operatorname{Re}\left\{e^{j\omega_c t} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{jn\omega_m t}\right\} = A_c \sum_{n=-\infty}^{+\infty} J_n(\beta)\cos[(\omega_c + n\omega_m)t]$$

$$\phi(t) = A_c\cos(\omega_c t + \beta\sin(\omega_m t)) = A_c \sum_{n=-\infty}^{+\infty} J_n(\beta)\cos[(\omega_c + n\omega_m)t]$$

$$\Phi(\omega) = \mathcal{F}\{\phi(t)\}$$

$$\Phi(\omega) = A_c\pi \sum_{n=-\infty}^{+\infty} J_n(\beta)[\delta(\omega - \omega_c - n\omega_m) + \delta(\omega + \omega_c + n\omega_m)]$$

$$\Phi(f) = \frac{A_c}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)\big(\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)\big)$$

3.II Basic Properties of Bessel Functions

$$J_n(\beta) = (-1)^n J_{-n}(\beta)$$

$$J_n(\beta) = \begin{cases} J_{-n}(\beta), & \text{if } n \text{ is even} \\ -J_{-n}(\beta), & \text{if } n \text{ is odd} \end{cases}$$

For $\beta \ll 1 \implies J_0(\beta) \approx 1$

$$J_1(\beta) \approx \beta/2$$

$$J_n(\beta) \approx 0 \quad \text{for } n \geq 2$$

$$\sum_{n=-\infty}^{+\infty} J_n^2(\beta) = 1$$

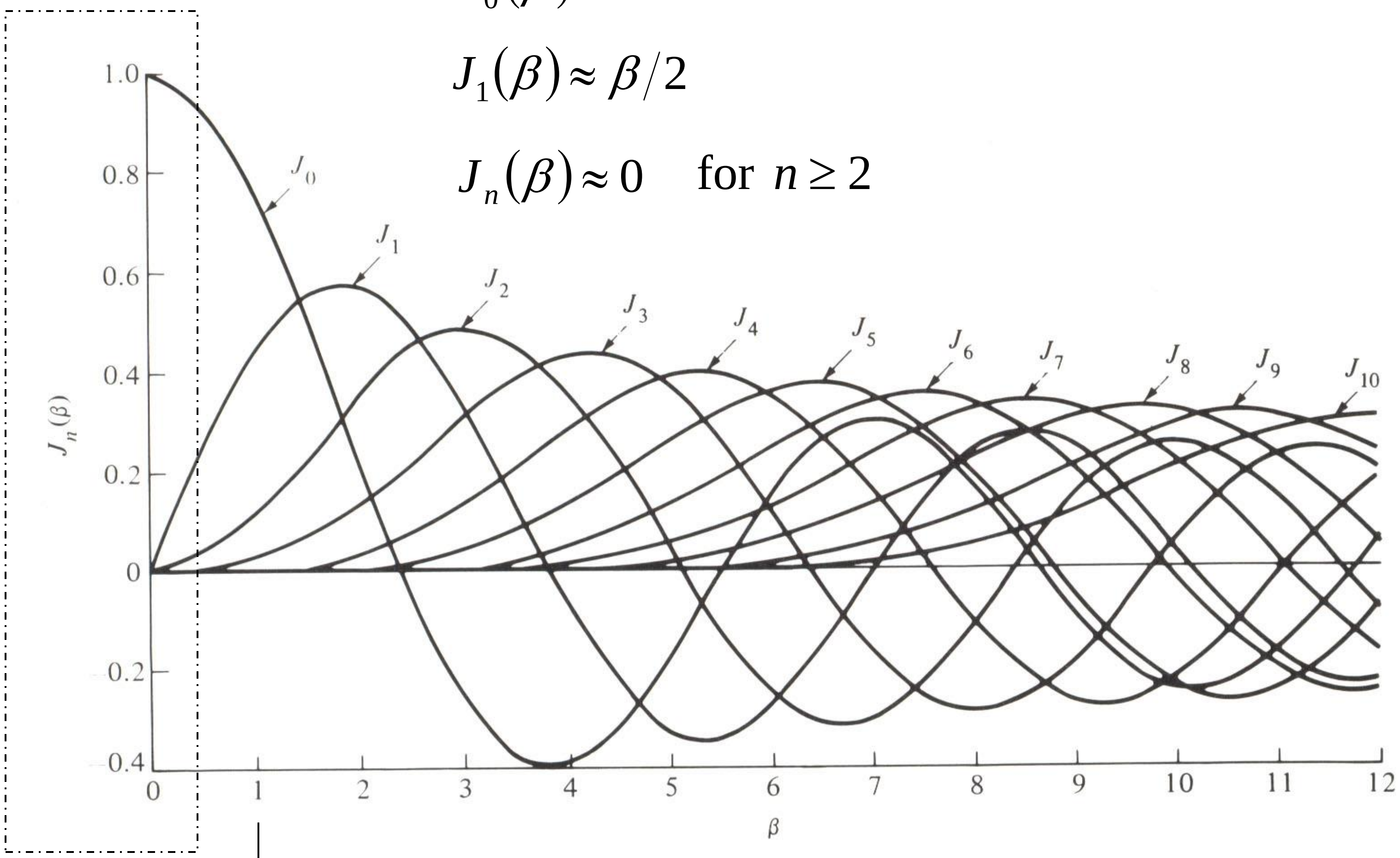
Plots of Bessel function of the first kind

For small β

$$J_0(\beta) \approx 1$$

$$J_1(\beta) \approx \beta/2$$

$$J_n(\beta) \approx 0 \quad \text{for } n \geq 2$$



$\beta \ll 1$

$\beta = 1$

WBFM Signal

$$\phi(t) = A_c \sum_{n=-\infty}^{+\infty} J_n(\beta)\cos[(\omega_c + n\omega_m)t]$$

How can we obtain NBFM signal from WBFM expression?

For $\beta \ll 1 \implies$

$J_n(\beta) \approx 0 \quad \text{for } n \geq 2$

$$\phi(t) \approx A_c J_0(\beta)\cos(\omega_c t) + A_c J_1(\beta)\cos[(\omega_c + \omega_m)t] + A_c J_{-1}(\beta)\cos[(\omega_c - \omega_m)t]$$

$J_0(\beta) \approx 1 \qquad J_1(\beta) \approx \beta/2 \qquad J_{-1}(\beta) \approx -\beta/2$

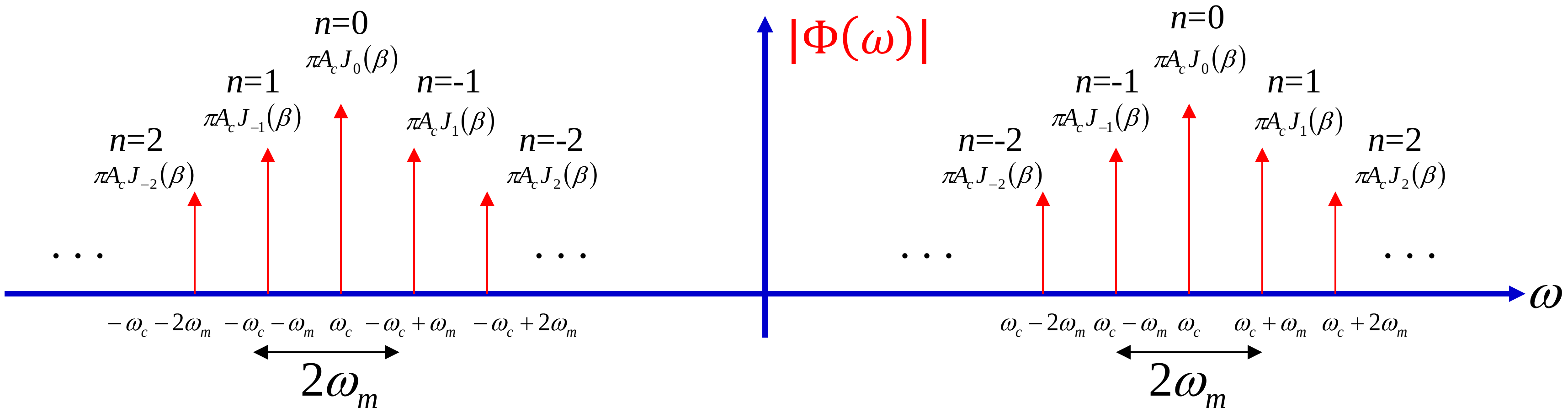
$$\longrightarrow \quad \phi(t) \approx A_c\cos(\omega_c t) - A_c\beta\sin(\omega_m t)\sin(\omega_c t)$$

3.III Bandwidth of FM Signals

$$\phi(t) = A_c \sum_{n=-\infty}^{+\infty} J_n(\beta)\cos[(\omega_c + n\omega_m)t]$$

$$\longleftrightarrow \; \Phi(\omega) = A_c\pi \sum_{n=-\infty}^{+\infty} J_n(\beta)[\delta(\omega - \omega_c - n\omega_m) + \delta(\omega + \omega_c + n\omega_m)]$$

$\Phi(\omega)$ contains a carrier component and an infinite set of side frequencies located symmetrically on either side of carrier frequency (i.e. $\pm n\omega_m$). These frequencies are integer multiples of ω_m , defined as *harmonics*.



In general, $W_T = 2n\omega_m$ [rad/s], where n depends on definition of *significant sidebands*.
When $\beta \ll 1 \implies$ only $n = 0, \pm 1$ terms count, $W_T = 2\omega_m$ [rad/s], or $B_T = 2B$ [Hz]

"Significant Sidebands" Definition for Bandwidth of FM Signals

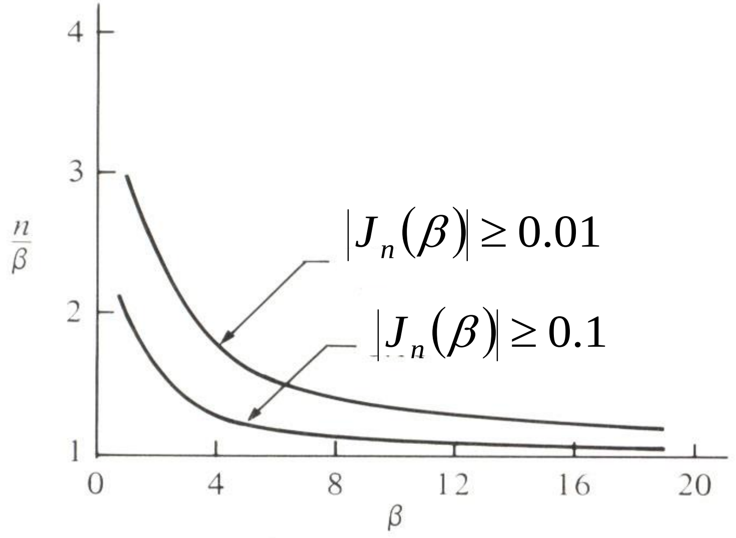
How many sidebands are important to the FM transmission of a signal?

- This depends on the intended application requirements and message signal.
- A rule commonly adopted is that a sideband is significant if its magnitude is equal to or exceeds *1%* of the unmodulated carrier, i.e.,

$$|J_n(\beta)| \geq 0.01$$

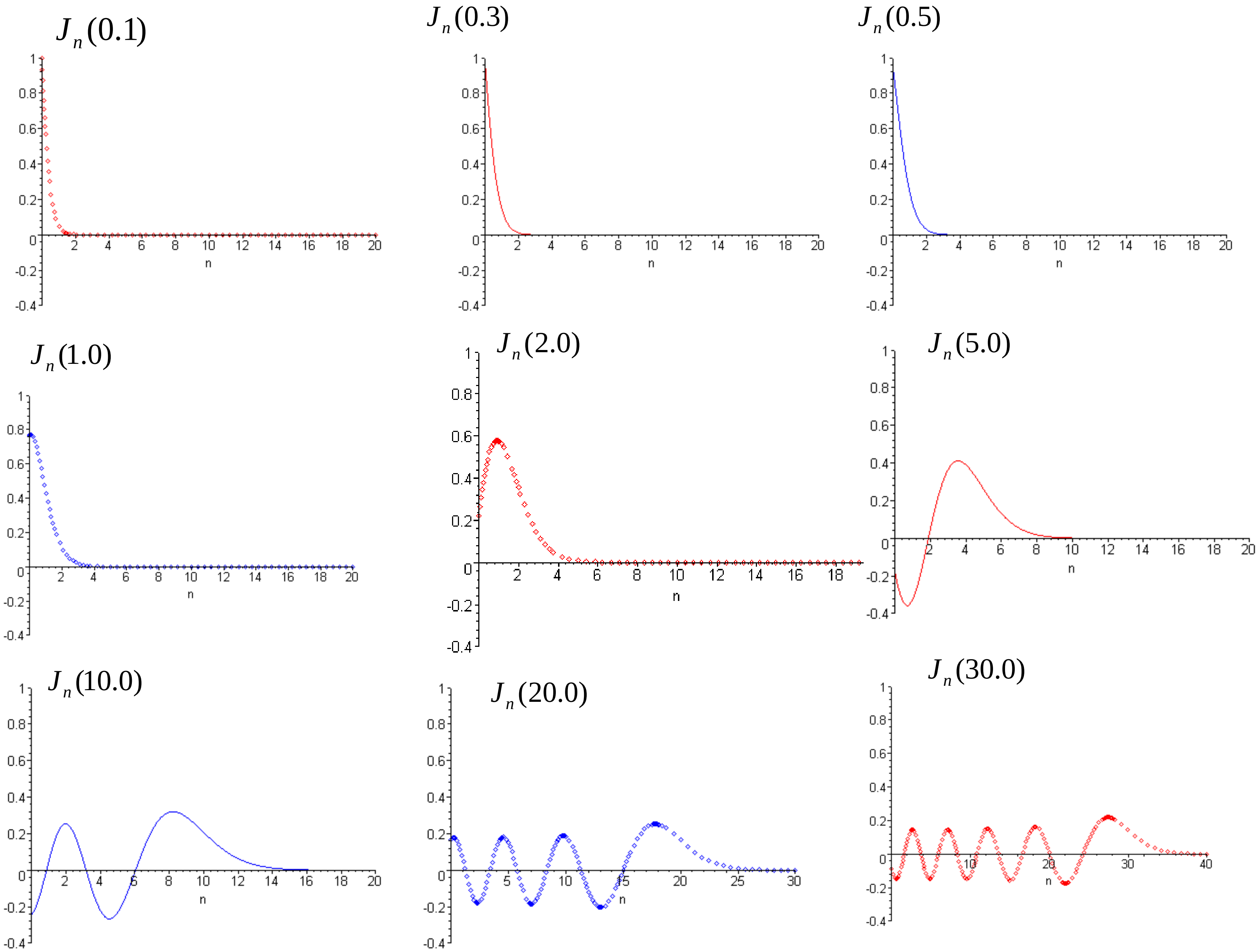
- $J_n(\beta)$ diminishes rapidly and the ratio $n/\beta \rightarrow 1$ as β becomes large.

$$\beta \gg 1 \qquad n \approx \beta$$



Therefore, the bandwidth for large β can be approximated as

$$W_T = 2n\omega_m \approx 2\beta\omega_m = 2\Delta\omega \text{ [rad/s], or } B_T = 2nB \approx 2\beta B = 2\Delta f \text{ [Hz]}$$



$J_n(0.1)$
$J_n(0.3)$
$J_n(0.5)$
$J_n(1.0)$
$J_n(2.0)$
$J_n(5.0)$
$J_n(10.0)$
$J_n(20.0)$
$J_n(30.0)$

The total number of significant side-bands for different values of the modulation index β, calculated on the *1%* basis

Modulation index	Number of significant side-frequencies
β	n_{max}
0.1	1
0.3	2
0.5	2
1.0	3
2.0	4
5.0	8
10.0	14
20.0	25
30.0	35

3.IV Carson's Rule for Bandwidth of FM Signals

$$\boxed{B_T \approx 2B(1+\beta) = 2(B+\Delta f)}$$

with B [Hz] the baseband message bandwidth.

For a very small $\beta,\ B_T \approx 2B(1+\beta) \approx 2B$

For a very large $\beta,\ B_T \approx 2B(1+\beta) \approx 2B\beta = 2\Delta f$

- Carson's rule agrees with our previous observations for limiting cases, obtained for the special case of modulating signal in the form of a sinusoidal.
- Carson's rule also holds for general modulating signals that are band-limited and have finite power.
- Carson's rule gives less bandwidth than our definition of "significant sidebands" (except for the limiting cases of the modulation index).

Example: A 10 MHz is frequency-modulated by a sinusoidal signal such that the peak frequency deviation is 50 kHz. Determine the bandwidth of the FM signal if the frequency of the modulating sinusoid is

a) 500 kHz b) 500 Hz, and c) 10 kHz.

Solution.

a) $\beta = \frac{\Delta f}{B} = \frac{50 \cdot 10^3}{500 \cdot 10^3} = 0.1$ Narrowband signal

$B_T \approx 2B = 1$ MHz; or $B_T = 2B(1+\beta)|_{\beta=0.1} = 1.1$ MHz (Carson's rule)

b) $\beta = \frac{\Delta f}{B} = \frac{50 \cdot 10^3}{500} = 100$ Wideband signal

$B_T \approx 2\Delta f = 100$ kHz; or $B_T = 2B(1+\beta)|_{\beta=100} = 101$ kHz (Carson's rule)

c) $\beta = \frac{\Delta f}{B} = \frac{50 \cdot 10^3}{10 \cdot 10^3} = 5$

$B_T = 2nB = 2.8.10^4 = 160$ kHz Using definition of "significant sidebands"

$B_T = 2B(1+\beta)|_{\beta=5} = 120$ kHz (Carson's rule)

3.V Average Power of FM Signals

$$\phi(t) = A_c \cos(\omega_c t + \beta \sin(\omega_m t))$$

Total average power $P = \overline{\phi^2(t)} = A_c^2 / 2$

Alternative representation (in terms of Bessel function):

$$\phi(t) = A_c \sum_{n=-\infty}^{+\infty} J_n(\beta) \cos[(\omega_c + n\omega_m)t]$$

$$\longleftrightarrow \quad \Phi(\omega) = 2\pi \sum_{n=-\infty}^{+\infty} \frac{A_c J_n(\beta)}{2} [\delta(\omega - \omega_c - n\omega_m) + \delta(\omega + \omega_c + n\omega_m)]$$

Power spectral density

$$S_\phi(\omega) = 2\pi \sum_{n=-\infty}^{+\infty} \left| \frac{A_c J_n(\beta)}{2} \right|^2 [\delta(\omega - \omega_c - n\omega_m) + \delta(\omega + \omega_c + n\omega_m)]$$

Total average power $P = \overline{\phi^2(t)} = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} S_\phi(\omega) d\omega = 2 \dfrac{A_c^2}{4} \underbrace{\sum_{n=-\infty}^{+\infty} J_n^2(\beta)}_{=1} = \dfrac{A_c^2}{2}$

Example. The message signal $m(t) = \cos 20\pi t$ and the carrier is $10 \cos \omega_c t$

The message is used to frequency modulate the carrier with $k_\omega = 100\pi$. Find the expression for the modulated signal and determine how many harmonics should be selected to contain 99% of the modulated signal power.

Solution: message signal $m(t) = \cos 20\pi t$

modulated signal: $\phi(t) = 10\cos(\omega_c t + 5\sin(20\pi t)) = 10 \sum_{n=-\infty}^{\infty} J_n(5)\cos(\omega_c + 20\pi n)t$

The modulated signal power contained up to the k-th harmonics is

$P_{\le k} = \frac{100}{2}\left[J_0^2(5) + 2\sum_{n=1}^{k} J_n^2(5)\right] \ge 49.5$

⇨ By trial and error, $k = 6$.

TABLE 4.1 REQUIRED NUMBER OF HARMONICS IN FM

Power (%)	$\beta = 0.5$	$\beta = 1$	$\beta = 2$	$\beta = 5$	$\beta = 8$	$\beta = 10$	$\beta = 15$
80	—	1	2	4	7	9	14
90	1	1	2	5	8	10	15
98	1	2	3	6	9	11	16

4. Generation and Demodulation of FM Signals

4.I NBFM Implementation

4.II Generation of WBFM Signal: Armstrong's Indirect Method

4.III Generation of WBFM Signals: Direct Method

4.IV Demodulation of FM Signals:

A. Direct Method (Frequency Discriminator)

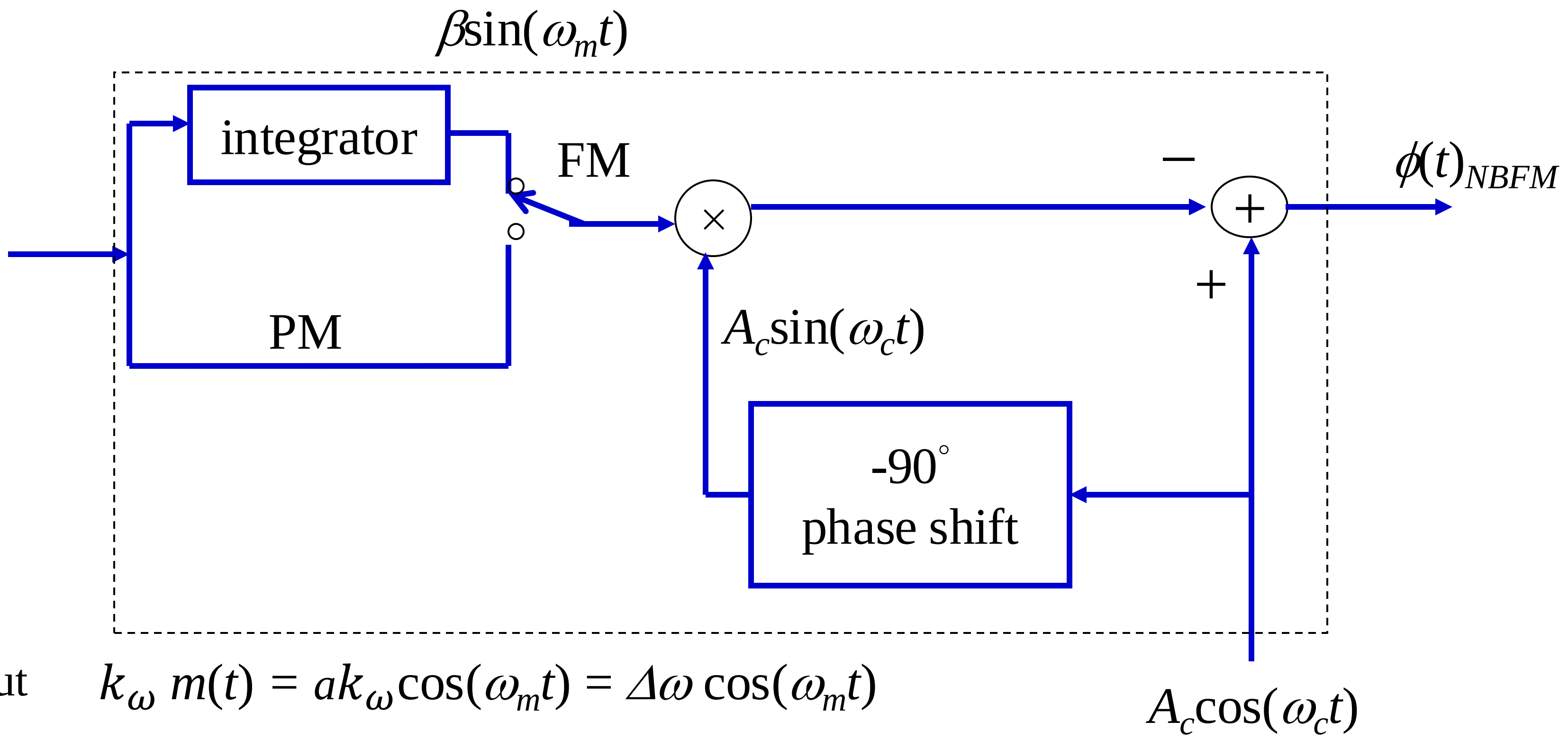
B. Indirect Method (PLL)

4.I NBFM Implementation

FM Signal $\phi(t) = A_c \cos(\omega_c t + \beta \sin(\omega_m t))$

Under NB assumptions $\phi(t) \approx A_c \cos(\omega_c t) - A_c \beta \sin(\omega_m t) \sin(\omega_c t)$

e.g., $k_\omega m(t) = a k_\omega \cos \omega_m t$

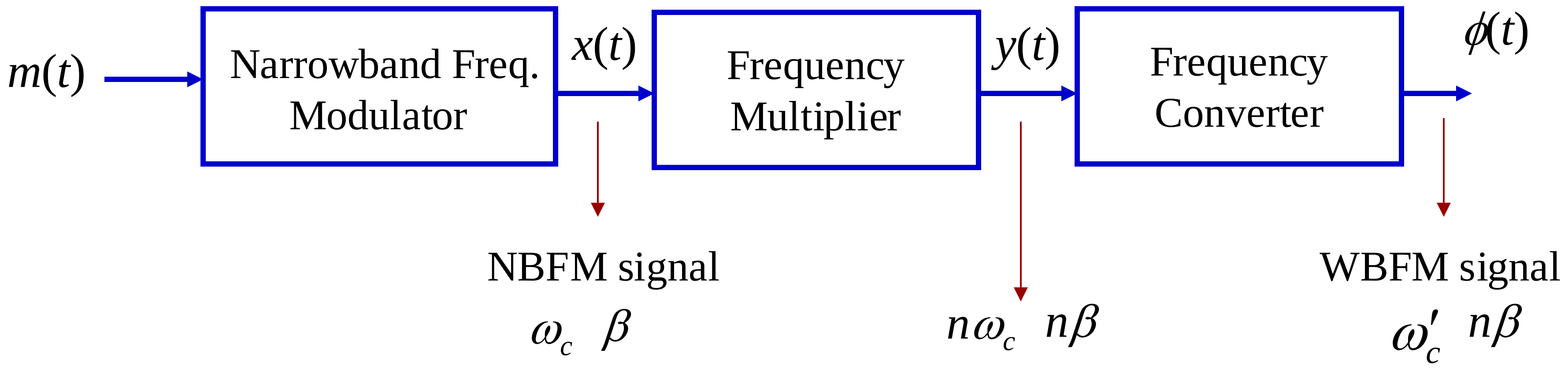


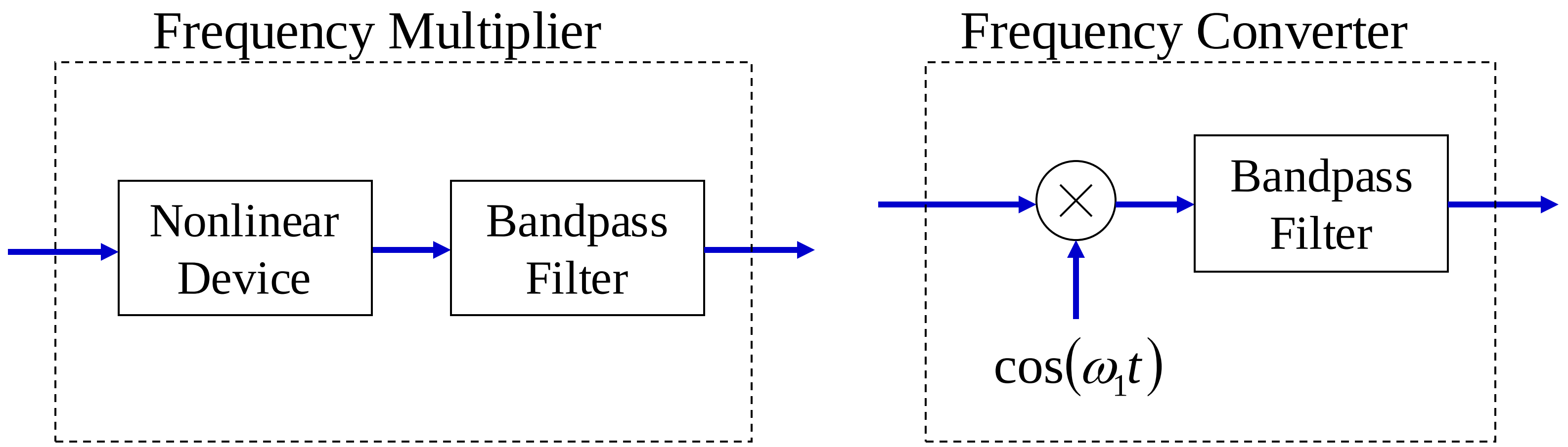
Integrator input $k_\omega\, m(t) = a k_\omega \cos(\omega_m t) = \Delta\omega \cos(\omega_m t)$

Integrator output $(\Delta\omega/\omega_m) \sin(\omega_m t) = \beta \sin(\omega_m t)$

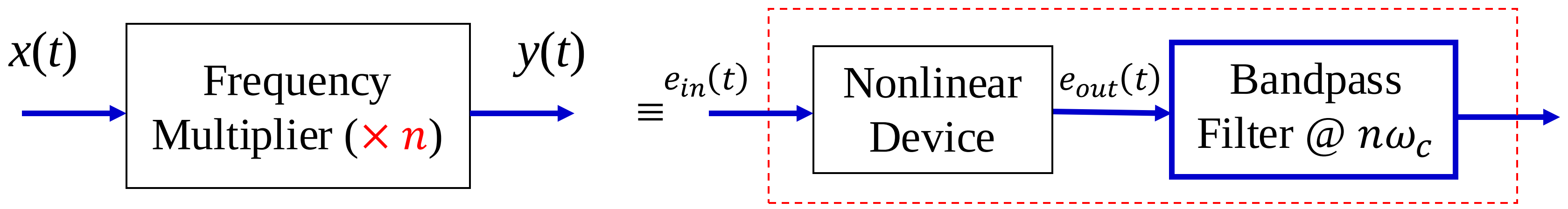
In practice, the circuit also has an amplitude limiter that outputs $A_c\, \mathrm{sgn}(\phi(t)_{NBFM})$ followed by a bandpass filter

4.II Generation of WBFM Signal: Armstrong's Indirect Method





Frequency Multiplier Implementation



Nonlinear device designed to multiply the frequencies of the input signal by a given factor.

$$e_{out}(t) = \sum_{k=0}^{n} c_k e_{in}^{k}(t)$$

For example, assume NBFM has mod. index β and desired WBFM has mod. index 2β.

We need to use $n = 2$ $\quad e_{out}(t) = c_0 + c_1 e_{in}(t) + c_2 e_{in}^{2}(t)$

Input $\quad e_{in}(t) = A\cos(\omega_c t + \beta \sin \omega_m t)$

Output $\quad e_{out}(t) = c_0 + c_1 A\cos(\omega_c t + \beta \sin \omega_m t) + c_2 A^2 \cos^2(\omega_c t + \beta \sin \omega_m t)$

$$= \underbrace{c_0 + \frac{c_2 A^2}{2} + c_1 A\cos(\omega_c t + \beta \sin \omega_m t)}_{\text{Removed by the filter}} + \underbrace{\frac{c_2 A^2}{2}}_{\text{Can be adjusted by the filter gain}} \cos\left(\underline{2\omega_c}\, t + 2\beta \sin \omega_m t \right)$$

Multiplied by 2

Frequency Multiplier Implementation (Cont.)

Considering the requirements on *n*, frequency multiplier might need to be implemented using multiple steps.

The output of the nonlinear device:

$$\cos(n\omega_c t + n\beta \sin \omega_m t) \ + \text{additional terms}$$

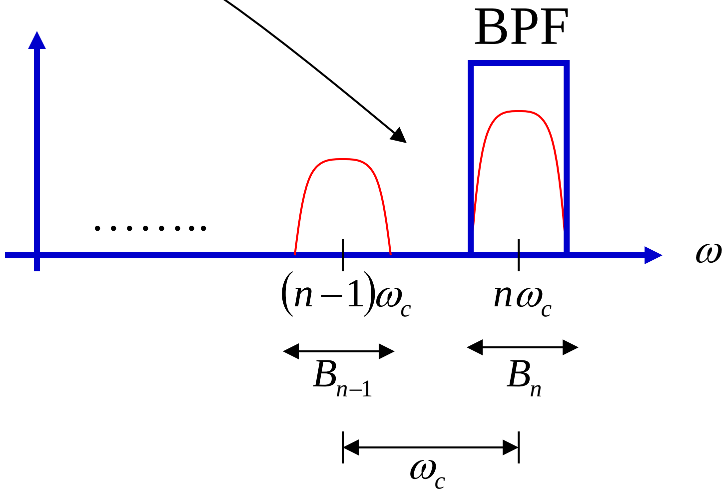
$$B_{n-1} = 2(\omega_m + (n-1)\Delta\omega)$$

$$B_n = 2(\omega_m + n\Delta\omega)$$

Limiting condition (no overlapping)

$$\frac{B_{n-1}}{2} + \frac{B_n}{2} < \omega_c$$

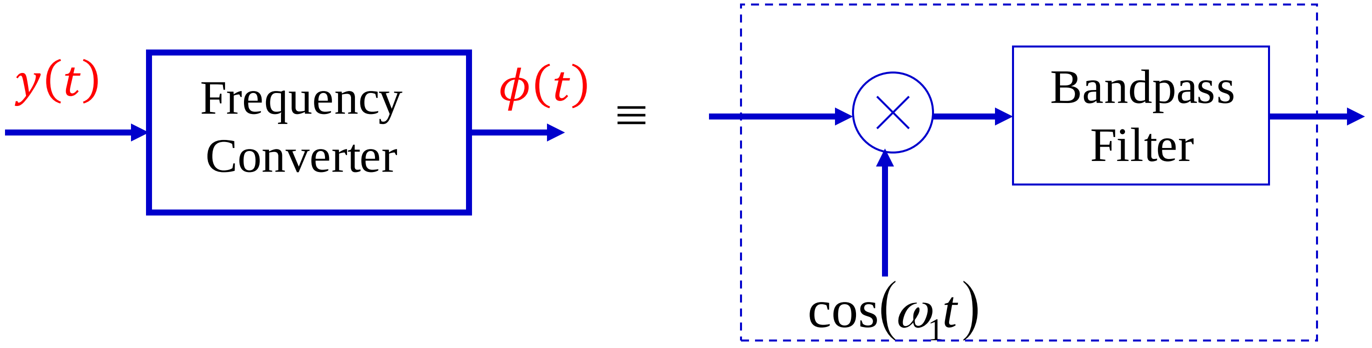
$$\Rightarrow \quad n < \frac{\omega_c - 2\omega_m}{2\Delta\omega} + \frac{1}{2}$$



Frequency Converter: Mixing

Frequency multiplier output: $y(t) = \cos(n\omega_c t + n\beta \sin \omega_m t)$

Frequency multiplier increases the modulation index by a factor of n as desired. This also results in an increase in the carrier frequency, which **might not be sufficient** in many cases. The carrier frequency needs to be shifted to the desired frequency by frequency converter or mixer.



Frequency converter is used to shift the spectrum of the signal by a given amount. It does not change its spectral content.

One of the output terms should give desired frequency and the other will be removed by the proper choice of BPF: $n\omega_c - \omega_1$ $\quad n\omega_c + \omega_1$

e.g., the final wideband signal is given by (when the input is single tone):

$$\phi(t) = A_c \cos[(n\omega_c - \omega_1)t + n\beta \sin \omega_m t]$$

Example. $\phi_{NBFM}(t) = A_c \cos\left(2\pi 10^5 t + 2\pi 25 \int_0^t m(\tau)d\tau\right)$

Message bandwidth $\omega_m = 2\pi 10^4$ [rad/s] and $\max|m(t)| = 1$

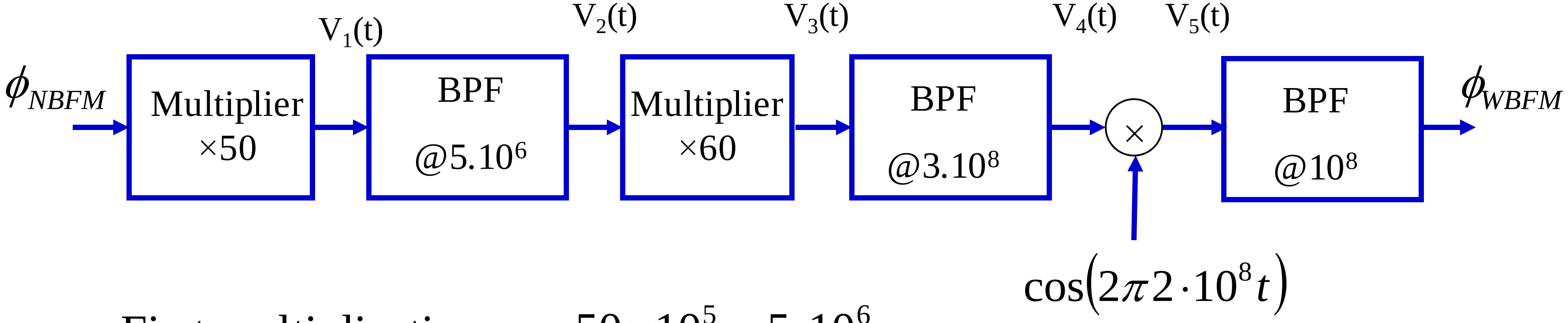
$$\phi_{WBFM}(t) = A_c \cos\left(2\pi 10^8 t + 2\pi 75 \cdot 10^3 \int_0^t m(\tau)d\tau\right)$$

The WBFM can be generated by the following modulator.

$$n < \frac{\omega_c - 2\omega_m}{2\Delta\omega} + \frac{1}{2} \approx 1600$$

$$\left.\begin{array}{l} \Delta\omega = 2\pi 25 \\ \Delta\omega' = 2\pi 75 \cdot 10^3 \end{array}\right\} \quad \frac{\beta'}{\beta} = \frac{\Delta\omega'}{\Delta\omega} = 3000 > 1600$$

Frequency multiplier cannot be implemented in one step!



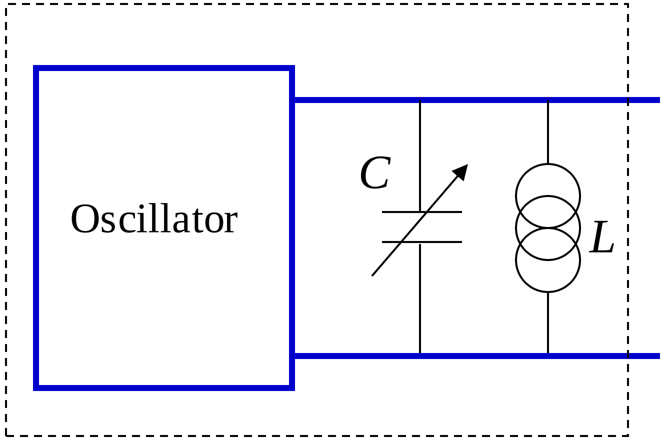
First multiplication $\quad 50 \times 10^5 = 5 \cdot 10^6$

Second multiplication $\quad 60 \times 5 \cdot 10^6 = 3 \cdot 10^8$

$3 \cdot 10^8 \longrightarrow 1 \cdot 10^8$ $\quad$ Carrier frequency needs to be adjusted.

4.IV Generation of WBFM Signals: Direct Method

Voltage-controlled oscillator (VCO) is any oscillator whose frequency is controlled by the modulating-signal voltage.



Oscillation frequency $\omega = \frac{1}{\sqrt{LC}}$

Voltage-variable capacitance $C = C_0 - km(t)$

Recall: $(1-x)^{-\frac{1}{2}} \approx 1 + \frac{x}{2}$

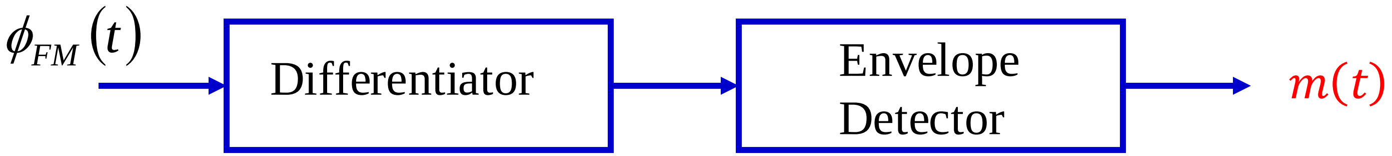
$$\omega = \frac{1}{\sqrt{LC_0}} \frac{1}{\sqrt{1 - \frac{km(t)}{C_0}}} \approx \frac{1}{\sqrt{LC_0}} \left(1 + \frac{k}{2C_0} m(t)\right)$$

The frequency is proportional to the message signal $m(t)$.

The long-term frequency-stability is not as good as in the indirect method.

4.V Demodulation of FM Signals

4.V.A: Direct Method: Frequency Discriminator



$$\phi_{FM}(t) = A_c \cos\left(\omega_c t + k_\omega \int_0^t m(\tau)d\tau\right)$$

$$\frac{d\phi_{FM}(t)}{dt} = -A_c(\omega_c + k_\omega m(t)) \sin\left(\omega_c t + k_\omega \int_0^t m(\tau)d\tau\right)$$

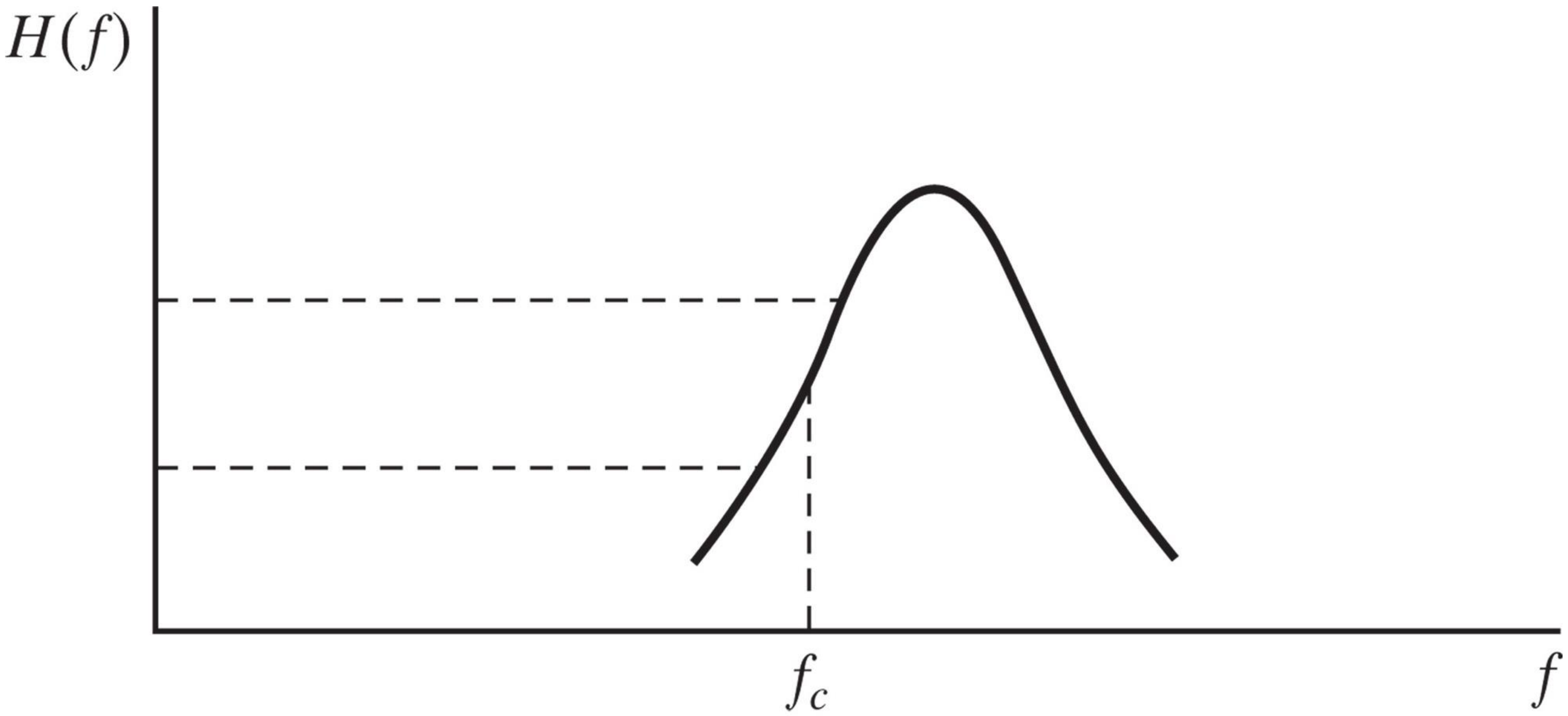
$\frac{k_\omega}{\omega_c} m(t) \ll 1 \implies$ The above expression has the form of DSB-LC signal.

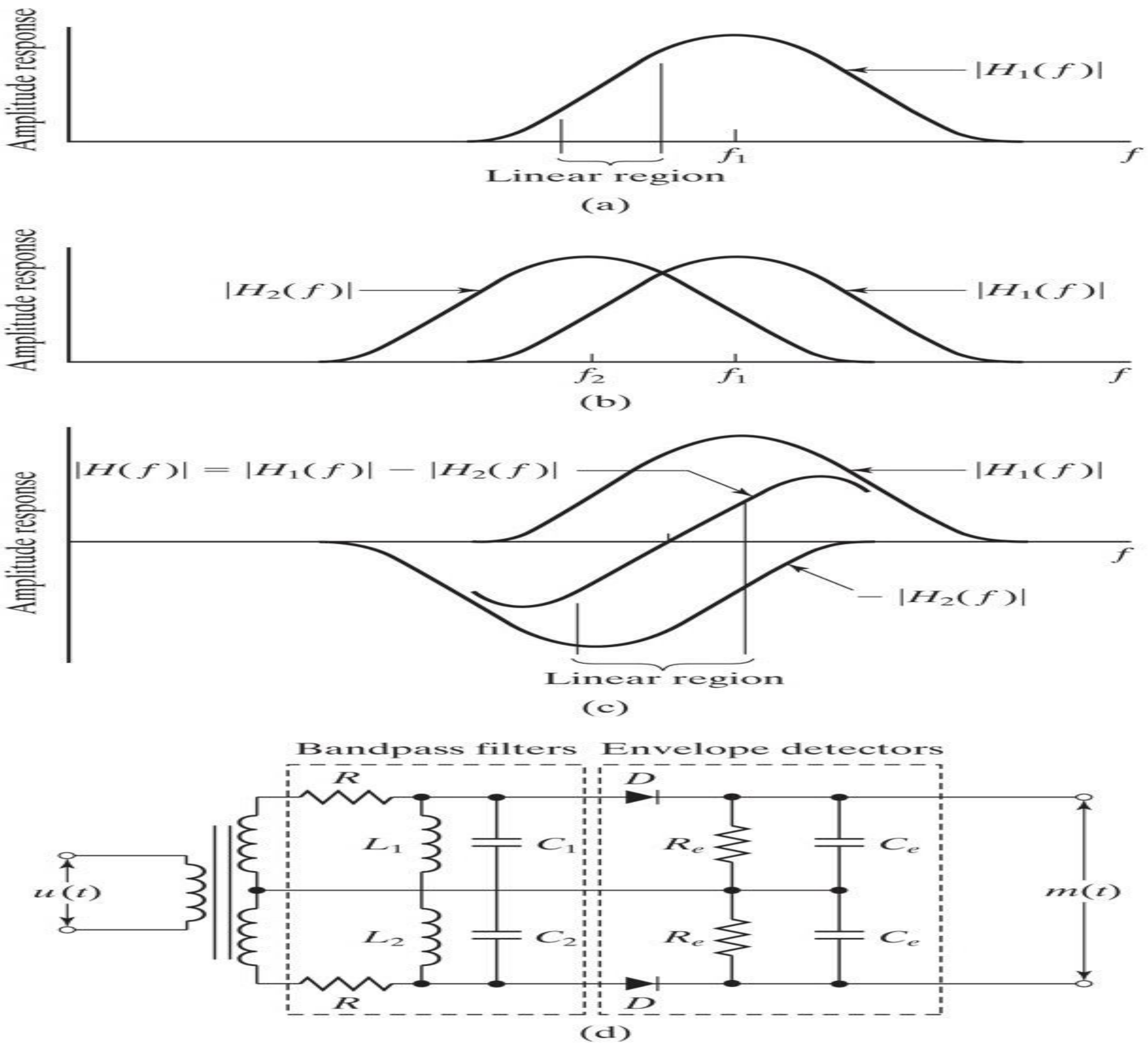
Envelope of the signal at the output of differentiator: $A\omega_c\left[1 + \frac{k_\omega}{\omega_c} m(t)\right]$

There is a slight variation in the frequency. However, the envelope detector can be still used to detect the message $m(t)$.

The ideal differentiator can be approximated by any device whose magnitude transfer function is reasonably linear within the range of frequencies of interest.

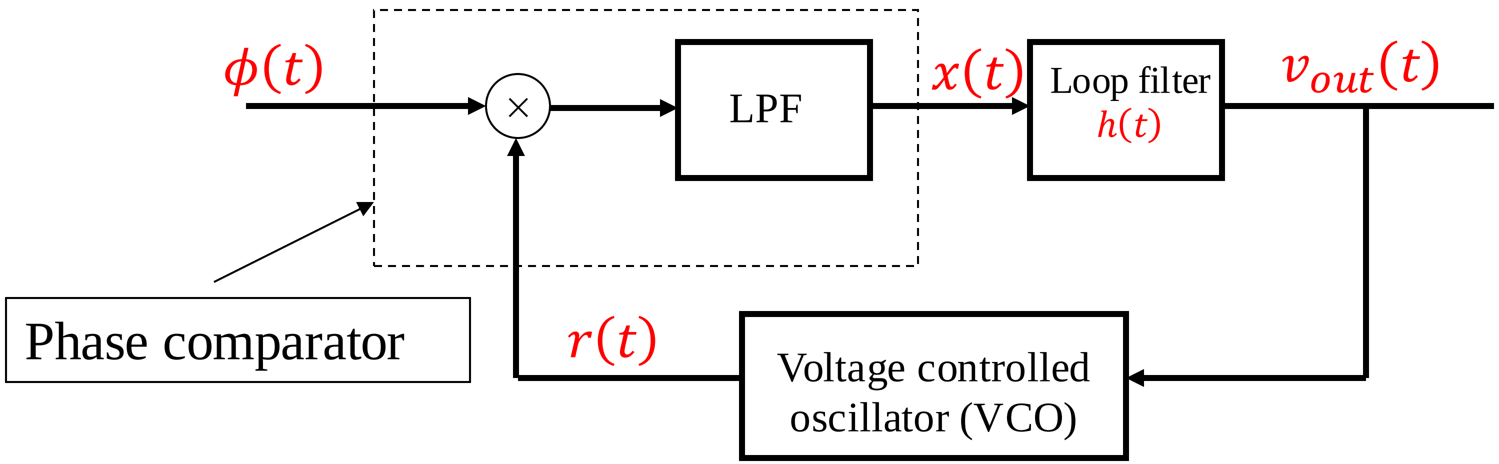
Example of Simple Differentiator: Linear region in a tuned (RLC) circuit





Amplitude response
$|H_1(f)|$
f_1
f
Linear region
(a)
Amplitude response
$|H_2(f)|$
$|H_1(f)|$
f_2
f_1
f
(b)
Amplitude response
$|H(f)| = |H_1(f)| - |H_2(f)|$
$|H_1(f)|$
f
$|H_2(f)|$
Linear region
(c)
Bandpass filters
Envelope detectors
R
D
L_1
C_1
R_e
C_e
$u(t)$
$m(t)$
L_2
C_2
R_e
C_e
R
D
(d)

4.V.B Indirect Method: Phase Locked Loop (PLL)



Assume that the VCO is adjusted so that when the control voltage is zero, two conditions are satisfied:

(1) The frequency of the VCO is precisely set at the unmodulated carrier frequency ω_c.

(2) The VCO output has a 90 degree phase shift with respect to the carrier wave.

The input signal $\phi(t) = A_c \cos[\omega_c t + \gamma(t)] = A_c \cos\left[\omega_c t + k_\omega \int_0^t m(\tau)d\tau\right]$

The voltage-controlled oscillator (VCO) produces an instantaneous frequency of

$$\omega_c + k_r v_{out}(t)$$

which is proportional to $v_{out}(t)$ (note that $\theta_r(t) = \ k_r \int_0^t v_{out}(\tau)d\tau$)

Assume that the VCO output is

$$r(t) = A_r \sin[\omega_c t + \theta_r(t)] = A_r \sin\left[\omega_c t + k_r \int_0^t v_{out}(\tau)d\tau\right]$$

Then, the phase comparator output is

$$\phi(t)r(t) = \underbrace{\frac{A_c A_r}{2}\sin[\theta_r(t) - \gamma(t)]}_{\text{Only this term passes through the filter}} + \frac{A_c A_r}{2}\sin[2\omega_c t + \theta_r(t) + \gamma(t)]$$

Thus, the input signal to the loop filter is given by $x(t) = b\sin\theta_e(t)$

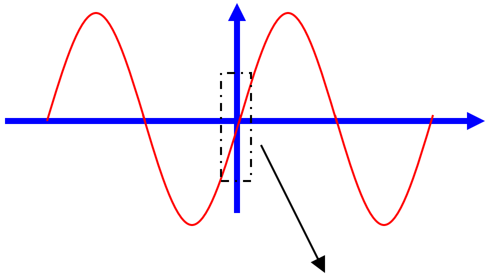
where $\theta_e(t)$ is the phase error defined by $\theta_e(t) = \theta_r(t) - \gamma(t)$

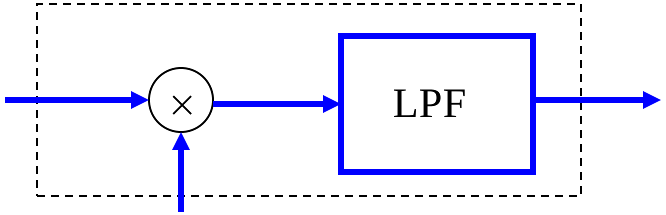
The loop filter operates on the input $x(t)$ to produce the output

$$v_{out}(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} b\sin\theta_e(t)\, h(t-\tau)d\tau$$

Linearized Model:

For small $\theta_e(t) = \theta_r(t) - \gamma(t)$, $\sin(\theta_e(t)) \approx \theta_e(t)$





In the linearized region, this scheme can be used as a phase comparator.

A general phase comparator:

A phase comparator detects the timing difference between the two signals and produces an output voltage which is proportional to this difference. For sinusoidal inputs, this timing difference can be expressed as a phase difference, as shown above.

The output of the phase comparator is proportional to the average of the product of the two inputs as a function of their relative time displacement. If the two inputs are $x(t)$ and $y(t)$, the output of the phase comparator is given by

$$\frac{1}{T}\int_0^T x(t)^* y(t+\tau)dt = R_{xy}(\tau)$$

where T is the period of the input waveform.

Observations:

- When the input signal is applied, phase comparison with VCO generates error voltage. In turn, this forces the VCO to synchronize itself to the input frequency.

- In the "lock" position (i.e., VCO is synchronized) the VCO frequency becomes identical to input frequency.

- As the input frequency varies slowly with the message signal, the PLL is able to track the input frequency through changes in the error voltage.

$$\gamma(t) \approx \theta_r(t)$$

$$k_\omega \int_0^t m(\tau)d\tau \approx k_r \int_0^t v_{out}(\tau)d\tau \Longrightarrow \qquad v_{out}(t) \propto m(t)$$

Example. Assume that a PLL is in lock, i.e., the phase error is small.

(1) Show that $v_{out}(t) \approx \frac{k_\omega}{k_r} m(t)$

(2) Find the condition for the loop filter such that $v_{out}(t)$ is proportional to $m(t)$, i.e., the approximation in (1) above is true.

Some remarks on PLL:

A. Applications of PLL in communication systems:

- FM detection
- Generation of highly stable FM signals
- Coherent AM detection
- Frequency multiplexing
- Frequency synthesis
- Used as a building block within complicated digital systems to provide bit synchronization and data detection

B. PLL's performance degrades rapidly when SNR below a certain threshold value.