ECE 318 - TUTORIAL 2

Review of :

* Power spectral density
* Energy spectral density
* Linear time invariant system

$x(t)$

Energy signal

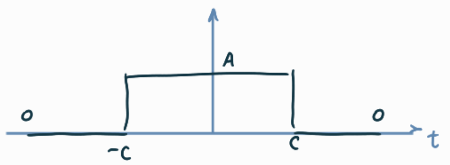
$0 < E_x < \infty$

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Where $|X(\omega)|^2$ is the energy spectral density

<u>Note</u> $X(\omega) = \mathcal{F}\{x(t)\}$

An example for energy signals is bandlimited signals



The signal is limited in time domain between $[-c, c]$

The energy spectral density reveals how much energy is present at each frequency (ω) of the signal.

Power signal

$0 < P_x < \infty$

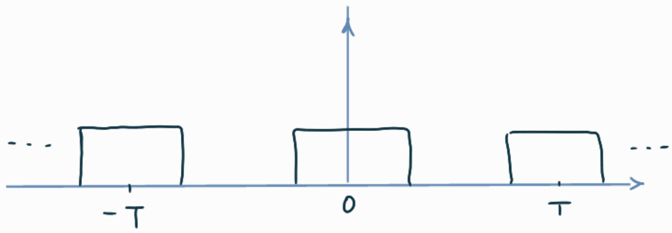
$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |X(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega)\, d\omega$$

where $S_x(\omega)$ is the power spectral density.

<u>Note</u> $S_x(\omega) = \lim_{T \to \infty} \frac{|X_T(\omega)|^2}{T}$

where $X_T(\omega) = \mathcal{F}\left\{x(t)\, \mathrm{rec}\left(\frac{t}{T}\right)\right\}$

An example for power signal is periodic signals



The power spectral density reveals how much power is present at each frequency ω of the signal.

Linear time invariant (LTI) system:

$$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t)$$

LTI systems have some interesting properties:

1/ Linearity

$$a x_1(t) + b x_2(t) \longrightarrow \boxed{LTI} \longrightarrow a y_1(t) + b y_2(t)$$

2/ Time invariant

$$x_1(t-t_0) \longrightarrow \boxed{LTI} \longrightarrow y_1(t-t_0)$$

3/ Its output $y(t)$ is completely defined by the system impulse response $h(t)$ and the input $x(t)$

$$y(t) = h(t) \underbrace{*}_{\text{convolution}} x(t)$$

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau$$

$h(t)$ is the impulse response of the system.

$$x(t) = \delta(t) \longrightarrow \boxed{LTI} \longrightarrow y(t) = h(t)$$

Impulse response means the output of the system when the input is $\delta(t)$.

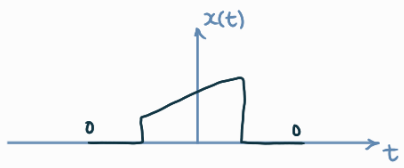
4/ In frequency domain $\mathcal{F}\{y(t)\} = \mathcal{F}\{h(t) * x(t)\} \Rightarrow Y(\omega) = H(\omega) \cdot X(\omega)$

$$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t)$$

$$y(t) = h(t) * x(t)$$
$$Y(\omega) = H(\omega) \cdot X(\omega)$$

$x(t)$ is time band limited signal is "aperiodic"

A special case of energy signals

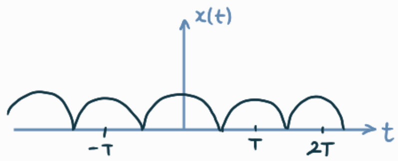


- Since $x(t)$ is any general form or aperiodic signal, we will use Fourier transform to find $X(\omega) = \mathcal{F}\{x(t)\}$
- Energy spectral density of the input is $|X(\omega)|^2$
- Energy spectral density of the output is $|Y(\omega)|^2 = |X(\omega)|^2 |H(\omega)|^2$
- Energy of the output:

$$E_y = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 |X(\omega)|^2 d\omega$$

$x(t)$ is periodic

A special case of power signals



$$Y(\omega) = X(\omega) \cdot H(\omega)$$

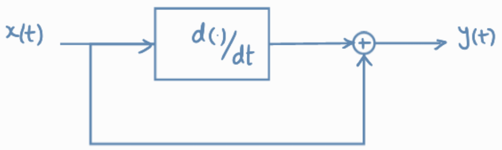
- Since $x(t)$ is periodic we will use <u>Fourier series</u> to find $X(\omega)$

$$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} X_n\, \delta(\omega - \omega_0)$$

, X_n are Fourier series coefficients, $\omega_0 = \frac{2\pi}{T}$

- Power spectral density of the input: $S_x(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |X_n|^2 \delta(\omega - n\omega_0)$
- Power spectral density of the output: $S_y(\omega) = |H(\omega)|^2 S_x(\omega)$
- Power of the output: $P_y = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_y(\omega)\, d\omega$

Ex.



a) Find the PSD of $x(t)$ and $y(t)$

b) Find the average power of $x(t)$ and $y(t)$ when

$$x(t) = 4\cos(20\pi t) + 2\sin(30\pi t)$$

Sol

a) Since the input signal $x(t)$ is periodic, we will use Fourier series. Moreover, we get the Fourier series coefficients directly by using Euler's formula.

$$x(t) = 4\left\{\frac{e^{j20\pi t}+e^{-j20\pi t}}{2}\right\} + 2\left\{\frac{e^{j30\pi t}+e^{-j30\pi t}}{2j}\right\}$$

$$x(t) = -\frac{1}{j}e^{-j30\pi t} + 2e^{-j20\pi t} + 2e^{j20\pi t} + \frac{1}{j}e^{j30\pi t} \qquad \text{(I)}$$

Using Fourier series:

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \qquad \text{(II)}$$

Comparing (I) & (II):

$$X_n = \left\{-\frac{1}{j},\ 2,\ 2,\ \frac{1}{j}\right\}$$

$$n\omega_0 = \left\{-30\pi,\ -20\pi,\ 20\pi,\ 30\pi\right\}$$

Now, PSD of input $S_x(\omega) = 2\pi\sum_{n=-\infty}^{\infty}|X_n|^2\,\delta(\omega-n\omega_0) = 2\pi\left[\left|\frac{-1}{j}\right|^2\delta(\omega+30\pi) + 2^2\delta(\omega+20\pi) + 2^2\delta(\omega-20\pi) + \left|\frac{1}{j}\right|^2\delta(\omega-30\pi)\right]$

Now, PSD of output $S_y(\omega) = |H(\omega)|^2 S_x(\omega)$, therefore we need to find $|H(\omega)|^2$ first in order to find $S_y(\omega)$

$$H(\omega) = \mathcal{F}\{h(t)\} \quad \text{or} \quad H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$$y(t) = \frac{d}{dt}x(t) + x(t)$$

Using h(t) method

$$h(t) = \frac{d}{dt}\delta(t) + \delta(t)$$

$$H(\omega) = \mathcal{F}\{h(t)\} = \mathcal{F}\left\{\frac{d}{dt}\delta(t)\right\} + \mathcal{F}\{\delta(t)\}$$

$$H(\omega) = j\omega\cdot 1 + 1$$

Taking Fourier transform of both sides of equation

$$Y(\omega) = \mathcal{F}\left\{\frac{d}{dt}x(t)\right\} + \mathcal{F}\{x(t)\}$$

$$Y(\omega) = j\omega X(\omega) + X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = j\omega + 1$$

$\Rightarrow H(\omega) = 1 + j\omega$

Thus $S_y(\omega) = S_x(\omega)\,|H(\omega)|^2$, where $|1+j\omega|^2 = \left(\sqrt{1+\omega^2}\right)^2$

$\Rightarrow S_y(\omega) = (1+\omega^2)\cdot 2\pi\left[\delta(\omega+30\pi) + 4\delta(\omega+20\pi) + 4\delta(\omega-20\pi) + \delta(\omega-30\pi)\right]$

$\Rightarrow S_y(\omega) = 2\pi\left[(1+(-30\pi)^2)\,\delta(\omega+30\pi) + 4\left(1+(-20\pi)^2\right)\delta(\omega+20\pi) + 4\left(1+(20\pi)^2\right)\delta(\omega-20\pi) + \left(1+(30\pi)^2\right)\delta(\omega-30\pi)\right.$

b) Average power in $x(t)$, we have three ways to calculate $P_{avg,x}$:

① Since the input is a periodic signal, $P_{avg,x} = \frac{1}{T}\int_0^T |x(t)|^2\,dt$

② Using Parseval's theorem and Fourier series : $P_{avg,x} = \sum_{n=-\infty}^{\infty} |X_n|^2$

from (a) $X_n = \left\{ \frac{1}{-j}, 2, 2, \frac{1}{j} \right\}$

$$|X_n| = \{1, 2, 2, 1\}$$

$$|X_n|^2 = \{1, 4, 4, 1\}$$

$$P_{avg,x} = 1 + 4 + 4 + 1 = 10$$

③ Using PSD method : $P_{avg,x} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega)\, d\omega$

$$S_x(\omega) = 2\pi \left(\delta(\omega+30\pi) + 4\,\delta(\omega+20\pi) + 4\,\delta(\omega-20\pi) + \delta(\omega-30\pi) \right)$$

Recall that $\int_{-\infty}^{\infty} \delta(\omega - c)\, d\omega = 1$

$$P_{avg,x} = \frac{1}{\cancel{2\pi}} \int_{-\infty}^{\infty} \cancel{2\pi} \left[\delta(\omega+30\pi) + 4\,\delta(\omega+20\pi) + 4\,\delta(\omega-20\pi) + \delta(\omega-30\pi) \right] d\omega$$

$$= 1 \int_{-\infty}^{\infty} \overbrace{\delta(\omega+30\pi)\, d\omega}^{1} + 4 \int_{-\infty}^{\infty} \overbrace{\delta(\omega+20\pi)\, d\omega}^{1} + 4 \int_{-\infty}^{\infty} \overbrace{\delta(\omega-20\pi)\, d\omega}^{1} + 1 \int_{-\infty}^{\infty} \overbrace{\delta(\omega-30\pi)\, d\omega}^{1} = 1 + 4 + 4 + 1 = 10$$

Ex. A symmetric square wave (with zero average value) with a peak amplitude of 1 and period T_0 has the Fourier series coefficients : $X_0 = 0$, $X_n = \frac{\sin(n\pi/2)}{n\pi/2}$, $n \neq 0$. The signal is applied to an LTI system :

$$x(t) \longrightarrow \boxed{H(\omega)} \longrightarrow y(t) \quad , \quad |H(\omega)| = \begin{cases} \frac{K}{2}(1+\cos\omega) & |\omega| < 4\pi/T_0 \\ 0 & |\omega| \geq 4\pi/T_0 \end{cases}$$

Determine K such that the output power is 1 Watt.

Sol Since $x(t)$ is periodic, $x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$, $\omega_0 = \frac{2\pi}{T_0}$

LTI system,

$$x(t) \longrightarrow \boxed{H(\omega)} \longrightarrow y(t) \quad , \quad y(t) = x(t) * h(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} * h(t)$$

An important property of complex exponentials : $h(t) * e^{jn\omega_0 t} = \int_{-\infty}^{\infty} h(\tau) e^{jn\omega_0 (t-\tau)} d\tau = e^{jn\omega_0 t} \int_{-\infty}^{\infty} h(\tau) e^{-j(n\omega_0)\tau} d\tau = e^{jn\omega_0 t} \cdot H(n\omega_0)$

Thus $y(t) = \sum_{n=-\infty}^{\infty} X_n \cdot H(n\omega_0) \cdot e^{jn\omega_0 t}$

For our example : $|H(\omega)| = \begin{cases} K/2\,(1+\cos(\omega)) , & |\omega| < \frac{4\pi}{T_0} = 2\omega_0 \\ 0 , & |\omega| \geq \frac{4\pi}{T_0} = 2\omega_0 \end{cases} \Rightarrow \begin{array}{ll} H(n\omega_0) = 0 & \text{for } |n| \geq 2 \\ H(n\omega_0) \neq 0 & \text{for } n = 0, 1, -1 \end{array}$

As a result $y(t) = \sum_{n=0,1,-1} X_n \cdot H(n\omega_0) e^{jn\omega_0 t} = X_{-1} H(-\omega_0) e^{-j\omega_0 t} + \underbrace{\cancel{X_0 H(0)}}_{0 \text{ since } X_0 = 0} + X_1 H(\omega_0) e^{j\omega_0 t}$

$$X_{-1} = \frac{\sin(-\pi/2)}{-\pi/2} = \frac{2}{\pi} \quad , \quad X_1 = \frac{\sin(\pi/2)}{\pi/2} = \frac{2}{\pi}$$

$$y(t) = \frac{2}{\pi} \cdot \frac{K}{2} \cdot (1+\cos\omega_0) e^{-j\omega_0 t} + \frac{2}{\pi} \cdot \frac{K}{2} \cdot (1+\cos\omega_0) e^{j\omega_0 t} = \frac{2K}{\pi} (1+\cos(\omega_0)) \cdot \cos(\omega_0 t)$$

The average power in $y(t)$: $P_{av,y} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} y^2(t)\, dt = \left\{ \frac{2K}{\pi} (1+\cos\omega_0) \right\}^2 \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \cos^2(\omega_0 t)\, dt$, Recall $\cos^2\omega_0 t = \frac{1+\cos 2\omega_0 t}{2}$

$$P_{av,y} = \left\{ \frac{2K}{\pi} (1+\cos\omega_0) \right\}^2 \cdot \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left\{ \frac{1}{2} + \frac{1}{2} \cos(2\omega_0 t) \right\} dt$$

$$P_{av,y} = \left\{ \frac{2K}{\pi}(1+\cos\omega_0) \right\}^2 \cdot \left[\lim_{T\to\infty} \frac{1}{2T} \int_{-T/2}^{T/2} 1\, dt + \lim_{T\to\infty} \frac{1}{T} \cdot \frac{1}{2} \left. \frac{\sin 2\omega_0 t}{2\omega_0} \right|_{-T/2}^{T/2} \right]$$

$$P_{av,y} = \left\{ \frac{2K}{\pi}(1+\cos\omega_0) \right\}^2 \cdot \left(\frac{1}{2}+0\right) = \left\{ \frac{2K}{\pi}(1+\cos(\omega_0)) \right\}^2 \times \frac{1}{2}$$

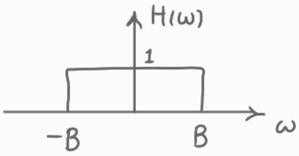
To find K, we need $P_{av,y}=1 \Rightarrow \left[\frac{2K}{\pi}(1+\cos\omega_0) \right]^2 \cdot \frac{1}{2} = 1 \Rightarrow K = \frac{\pi}{\sqrt{2}}(1+\cos\omega_0)$

Ex. Find E_y for the following LTI system and consider $x(t) = \frac{2}{1+t^2}$, $x(t) \rightarrow \boxed{\text{rect}(\frac{\omega}{2B})} \rightarrow y(t)$

Hint: $\mathcal{F}\{e^{-|t|}\} = \frac{2}{1+\omega^2}$

Sol: $E_y = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(\omega)|^2 d\omega$ where $|Y(\omega)|^2$ is the energy spectral density $\triangleq$ Energy/rad

$Y(\omega) = X(\omega)\cdot H(\omega)$, where $H(\omega) = \text{rect}\left(\frac{\omega}{2B}\right)$



$X(\omega) = \mathcal{F}\left\{ \frac{2}{1+t^2} \right\}$

Using the hint and duality property: $\mathcal{F}\{e^{-|t|}\} = \frac{2}{1+\omega^2} \xrightarrow{\text{duality}} \mathcal{F}\left\{\frac{2}{1+t^2}\right\} = 2\pi e^{-|-\omega|} = 2\pi e^{-|\omega|}$

Thus $X(\omega) = 2\pi e^{-|\omega|}$

$$E_y = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 \cdot |X(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-B}^{B} 1\cdot(2\pi)^2 e^{-2|\omega|} d\omega = 2\pi \int_{-B}^{B} e^{-2|\omega|} d\omega = 4\pi \int_0^B e^{-2\omega} d\omega = 4\pi \cdot \frac{-1}{2}(e^{-2B}-1)$$

$$\Rightarrow E_y = 2\pi(1-e^{-2B})$$

Ex. Find K such that $P_{av,y} = 2\ W$. $H(\omega) = K\cdot\text{rect}\left(\frac{\omega}{3}\right)$, $x(t) = \cos^2(t/2)\cdot\sin(t)$

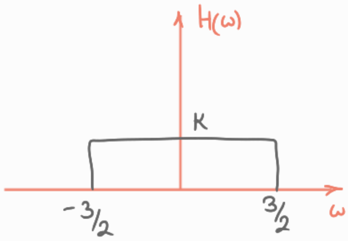
Sol:

$$x(t) = \cos^2(t/2)\sin(t) = \frac{1+\cos(t)}{2}\cdot\sin(t) = \frac{1+\frac{e^{jt}+e^{-jt}}{2}}{2}\cdot\frac{e^{jt}-e^{-jt}}{2j} = \frac{1}{8j}(e^{jt}+e^{-jt}+2)(e^{jt}-e^{-jt})$$

$$\Rightarrow x(t) = \frac{1}{8j}e^{j2t} - \frac{1}{8j} + \frac{1}{8j} - \frac{1}{8j}e^{-j2t} + \frac{1}{4j}e^{jt} - \frac{1}{4j}e^{-jt}, \quad x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

$$X_n = \left\{ -\frac{1}{8j}, -\frac{1}{4j}, \frac{1}{4j}, \frac{1}{8j} \right\}, \quad n\omega_0 = \{-2,-1,1,2\}$$

$H(\omega) = K\cdot\text{rect}(\omega/3)$



Intuitively speaking, $H(\omega)$ is a low-pass filter with cutoff frequency $3/2$, which means that $H(\omega)$ will pass only signals with frequency between $-3/2 \le \omega \le 3/2$. However, $x(t)$ has some harmonics or frequency higher than $3/2$ and lower than $-3/2$. These frequencies will not pass. As a result, $y(t) = K\left(-\frac{1}{4j}e^{-jt} + \frac{1}{4j}e^{jt}\right)$

$$P_{av,y} = \sum_n |Y_n|^2 = K^2\left|\frac{-1}{4j}\right|^2 + K^2\left|\frac{1}{4j}\right|^2 = 2W \Rightarrow K^2 = 16 \Rightarrow K = \pm 4$$

Mathematically speaking: $y(t) = \sum_n X_n H(n\omega_0) e^{jn\omega_0 t}$