Topic 2. Signals and Linear Systems, Spectra and Noise (Chapter 2)

UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING

2.1 Basic Concepts from Signals and Systems

- A signal, $x(t)$ is a function of the independent time variable t, e.g., speech, audio, image, video…

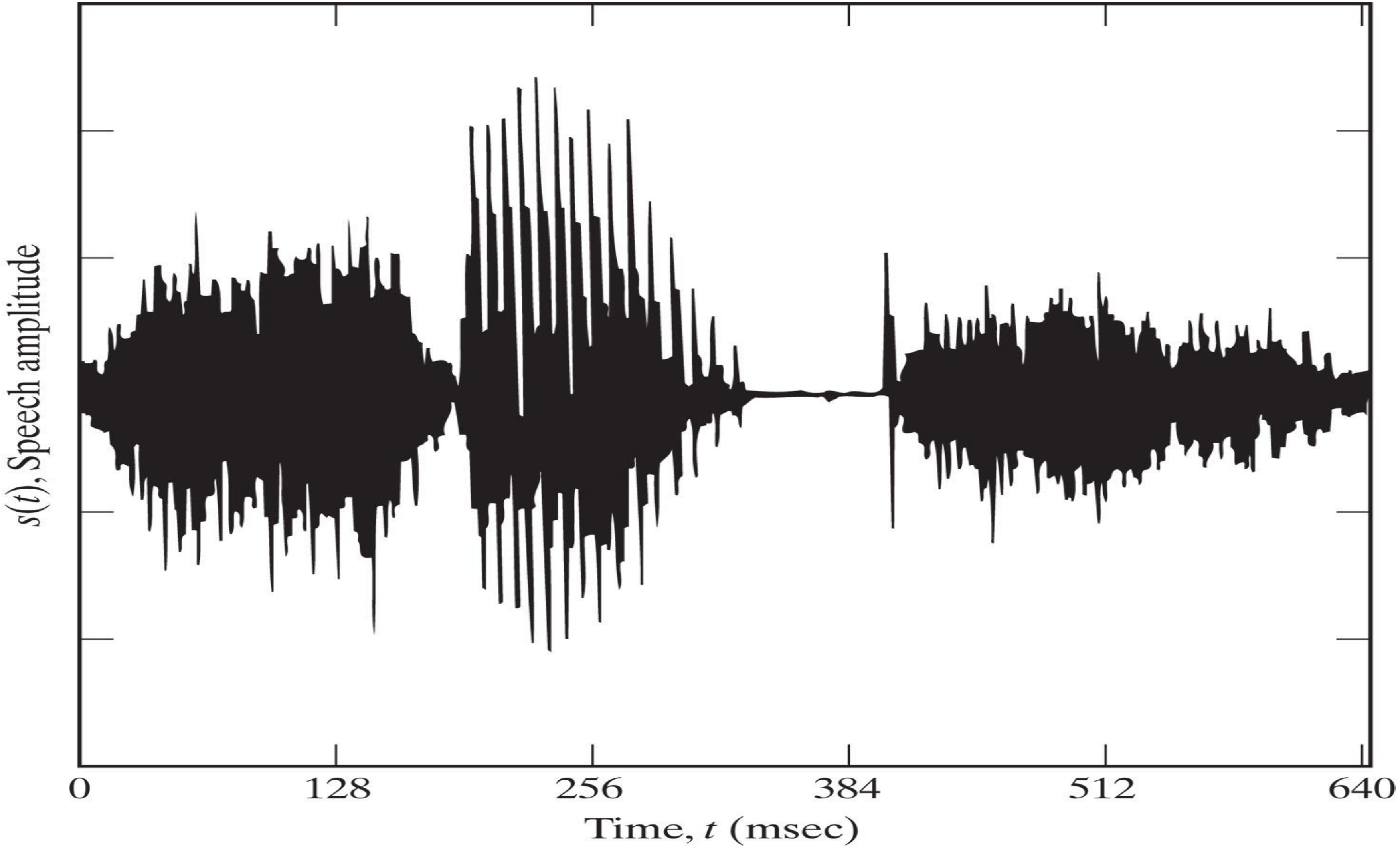
- Basic operations on signals include the following:
  - Time shifting $x(t-t_0)$
  - Time reversal $x(-t)$
  - Time scaling $x(at)$

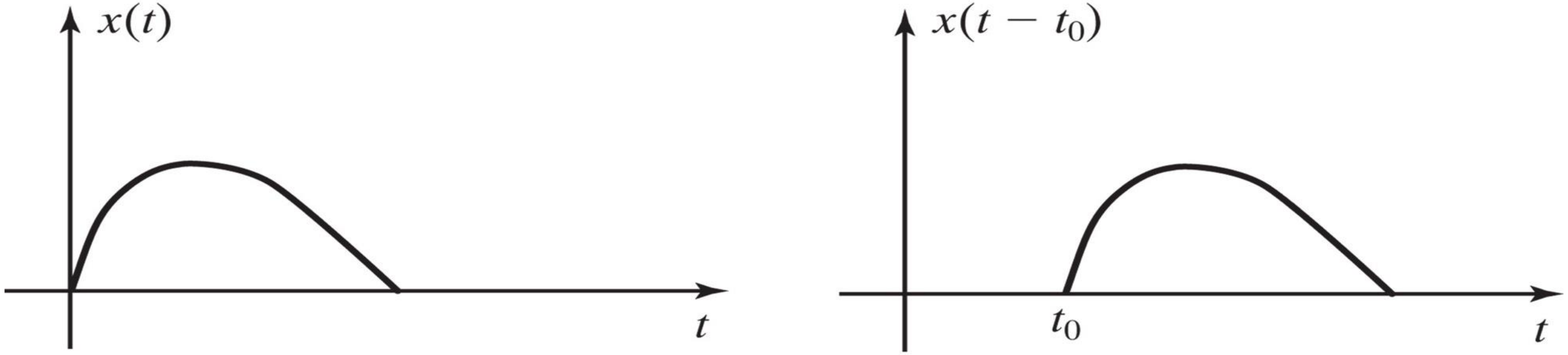
- Classification of signals:
  - Continuous-time vs discrete-time: $x(t)$ vs $x[n]$ (often discrete-time signals are obtained by sampling a continuous-time signal, i.e., $x[n]=x(nT_s)$ for a sampling interval T_s or sampling frequency $F_s=1/T_s$)
  - Real vs complex: for complex signals $x(t)=x_r(t)+j\,x_i(t)$ with $j=\sqrt{-1}$
  - Deterministic vs random signals (for the latter, at any given instant the signal is a random variable).
  - Periodic vs nonperiodic: Periodic signals satisfy $x(t)=x(t+T)\ \ \forall t$ for a given period T.
  - Causal vs noncausal: $x(t)$ is causal if $x(t)=0\ \ \forall t<0$
  - Even and odd symmetry: Even $x(t)=x(-t)\ \forall\ t$; Odd $x(t)=-x(-t)\ \forall\ t$. In general $x(t)=x_e(t)+\ x_o(t)$
  - Hermitian symmetry: for a complex signal, $x(t)$ is called Hermitian if its magnitude $|x(t)|=\sqrt{x_r(t)^2+x_i(t)^2}$ is even, and its phase $\angle x(t)=\arctan\frac{x_i(t)}{x_r(t)}$ is odd. In other words, $x(-t)=x(t)^*$
  - Energy and power-type signals:

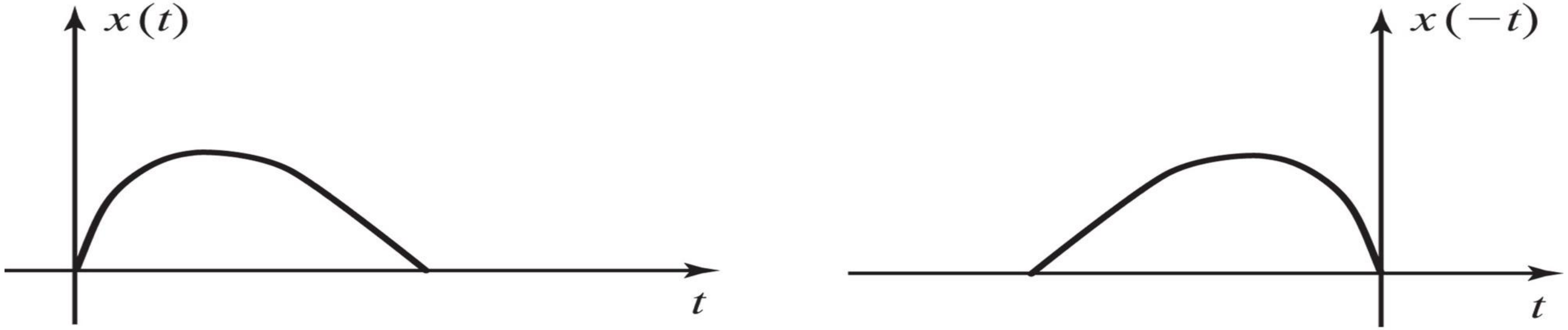
$$E_x=\int_{-\infty}^{\infty}|x(t)|^2dt,\ P_x=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2dt$$

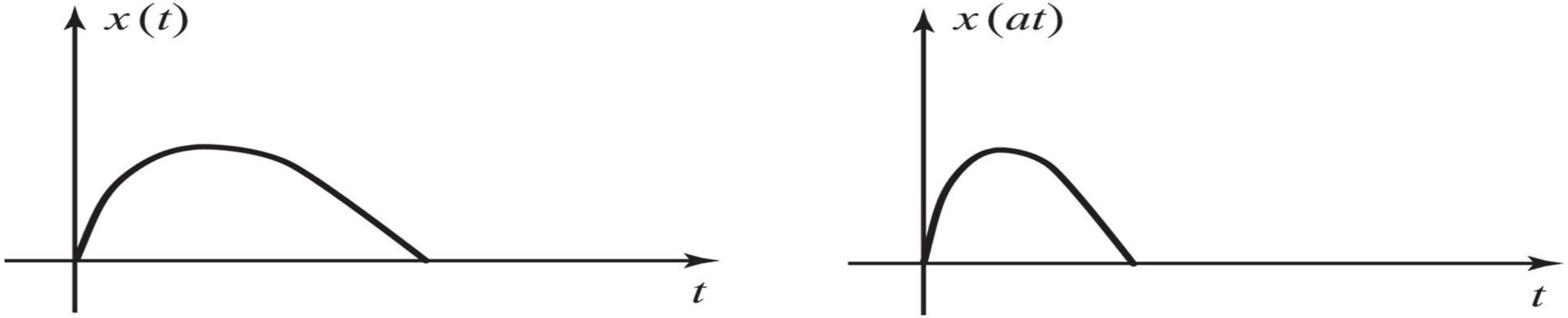
The signal $x(t)$ is called energy-type if E_x is finite. The signal is called power-type if $0<P_x<\infty$

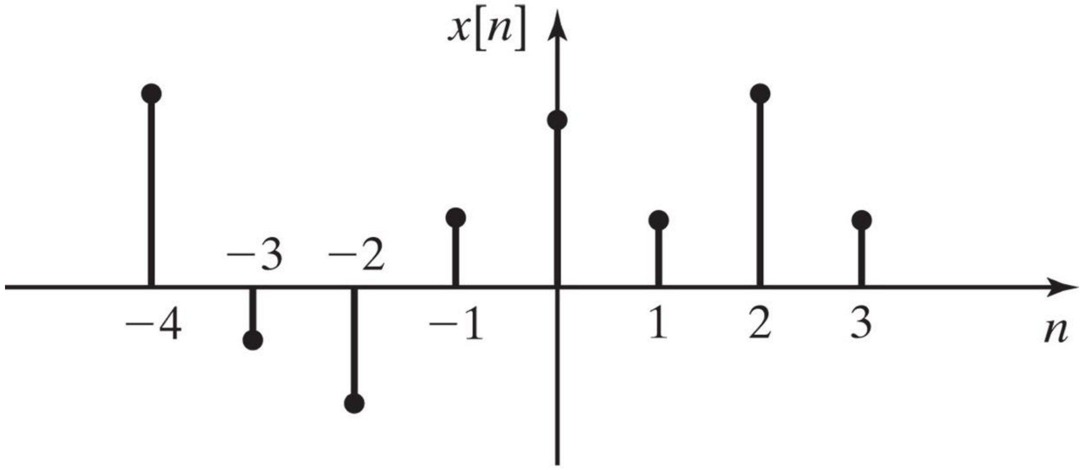
A sample speech waveform.

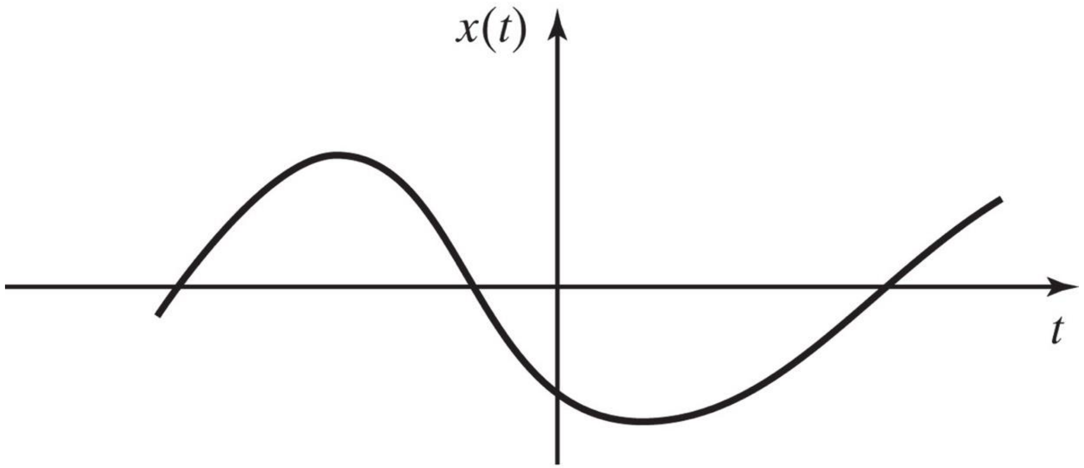



x(t)
t
x(t − t0)
t0
t


x(t)
t
x(−t)
t

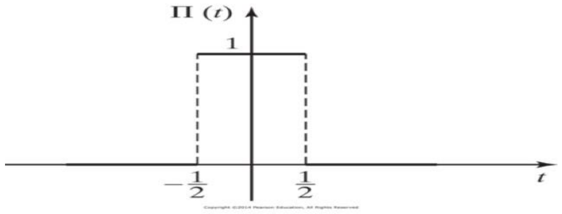

x(t)
t
x(at)
t


x[n]
−3
−2
−4
−1
1
2
3
n

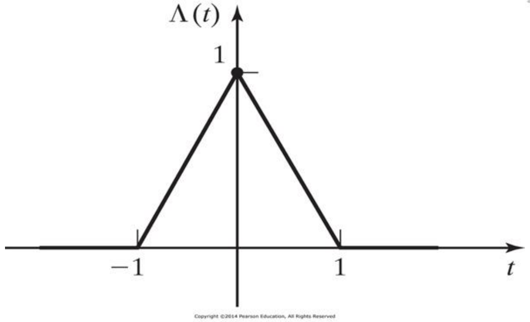

x(t)
t

Some Important Signals and Their Properties

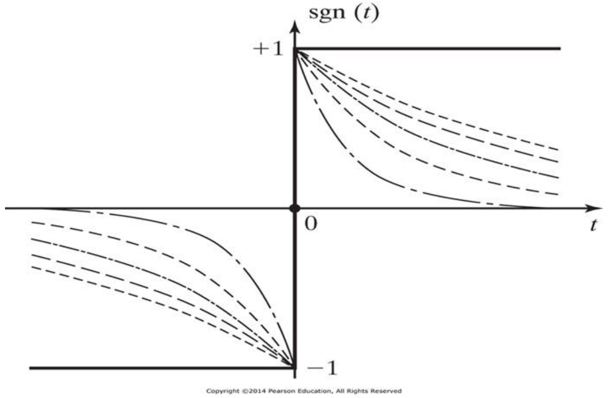
- The sinusoidal signal $x(t) = A\cos(2\pi f_0 t + \theta)$ and the complex exponential $x(t) = Ae^{j(2\pi f_0 t+\theta)}$
- The unit-step signal $u(t) = 1, t > 0$ and $u(t) = 0, t < 0$ $\left(u(0) = \frac{1}{2}\right)$
- The rectangular and triangular pulses: $\Pi(t) = \begin{cases} 1 & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$



$$\Lambda(t) = \begin{cases} 1+t & -1 \le t \le 0 \\ 1-t & 0 < t \le 1 \\ 0 & \text{otherwise} \end{cases}$$



- The sinc signal: $\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$
- The sign or the Signum signal: $\mathrm{sgn}(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \\ 0 & t = 0 \end{cases}$ $(\mathrm{sgn}(t) = 2u(t) - 1)$

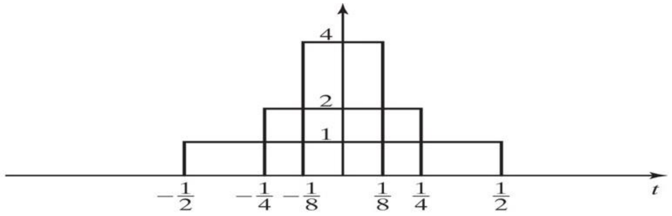


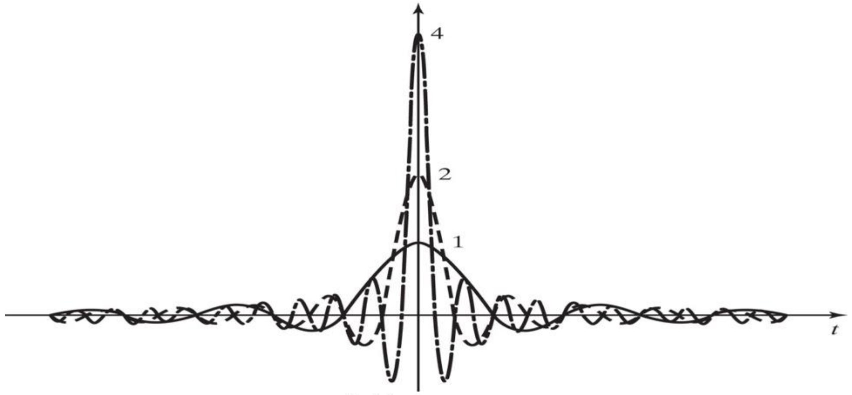
The Signum signal can be expressed as the limit of $x_n(t) = \begin{cases} e^{-\frac{t}{n}} & t > 0 \\ -e^{\frac{t}{n}} & t < 0 \\ 0 & t = 0 \end{cases}$

- The impulse or the Dirac Delta signal $\delta(t)$: a mathematical model to represent phenomena that have very high energy during a very small time duration (e.g., hammer blow, voltage or current spikes, …)

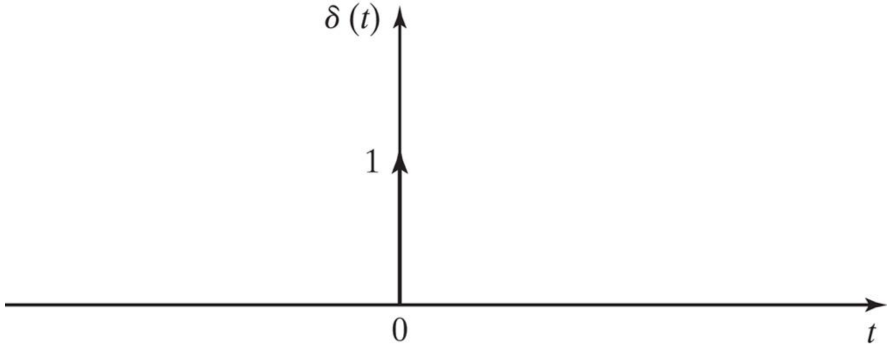
The Dirac Delta Impulse $\delta(t)$ Properties

- Mathematically, $\delta(t)$ is **not** a function, it is a Schwartz ***distribution*** or a ***functional***† that takes values on other signals or test functions
$$\int_{-\infty}^{\infty} \delta(t)\phi(t) = \phi(0) \text{ (sifting property)}$$
- Sometimes it is useful to think of $\delta(t)$ as a limit: $\delta(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Pi\left(\frac{t}{\epsilon}\right) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \text{sinc}\left(\frac{t}{\epsilon}\right)$





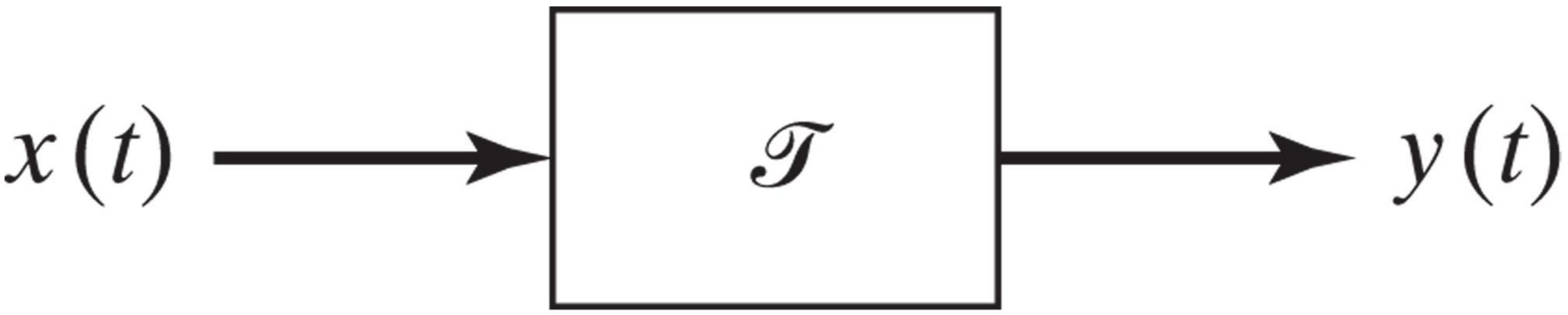
- $\delta(t) = 0\ \forall t \neq 0$, and $\int_{-\infty}^{\infty} \delta(t)dt = 1$
- $\delta(t - t_0)x(t) = x(t_0)\delta(t - t_0)$
- For any continuous signal at t_0, $\int_{-\infty}^{\infty} \delta(t - t_0)\,\phi(t)dt = \phi(t_0)$
- For any $a \neq 0,\ \delta(at) = \frac{1}{|a|}\delta(t)$
- $\delta(t) * x(t) = x(t)$
- $u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$ or $\delta(t) = \frac{d}{dt}u(t)$
- $u(t) * x(t) = \int_{-\infty}^{t} x(\tau)d\tau$



† Formally, a Schwartz distribution (or a functional) is a continuous linear map that associates a test function with a real-valued number.
For example,
$\delta : \phi(t) \mapsto \langle \delta | \phi \rangle = \phi(0)$
$u : \phi(t) \mapsto \langle u | \phi \rangle = \int_0^{\infty} \phi(t)dt$
Distributions are **not required and will not be covered** in this course, and they are not necessary for understanding the covered material. They are however necessary to make sense of some integrals and Fourier transforms that would otherwise diverge.
For those special cases, for the sake of brevity, while being rigorous, our approach will be to use the properties of the Fourier transform (which also apply to distributions) rather than calculating (divergent) integrals (cf. "Functional Analysis," W. Rudin, 1995).

Linear Time-Invariant (LTI) Systems

- A system is a function of signals: for an input signal *x(t)*, it associates a unique output signal *y(t)*
- The system is called linear if it satisfies the superposition property: the output to a linear combination of inputs is the same linear combination of the individual outputs.
- The system is time-invariant if whenever the input is shifted by a time t_0, the output is shifted by the same time t_0
- An LTI system is completely determined by its impulse response *h(t)* (its response to the Dirac delta). Its response to a given input *x(t)* is then determined by the convolution integral: $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$



2.2 Review of Fourier Series and Transform

A. **Fourier Series Representation for Periodic Signals**
B. **Fourier Transform**
C. **Properties of the Fourier Transform**
D. **Fourier Transform for Periodic Signals**

A. Fourier Series Representation for Periodic Signals

Definition: Let signal $x(t)$ be a periodic signal with fundamental period T (*i.e., the smallest number $T>0$ satisfying $x(t+T)=x(t)$ for all t*), satisfying the Dirichlet conditions. Then $x(t)$ can be expanded in terms of the complex exponential signals as

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{j\omega_0 nt}$$

$$X_n = \frac{1}{T}\int_T x(t) e^{-j\omega_0 nt} dt$$

- The coefficients X_n are called the *Fourier series coefficients* of the signal $x(t)$. These are, in general, complex numbers.

- $\omega_0 = 2\pi/T$ [rad/s] is called the *fundamental angular frequency*. The frequencies of the complex exponential signals are multiples of this frequency. The nth multiple of the fundamental frequency (for positive n) is called the nth *harmonic (or partials)*.

Why Fourier?

Very useful in studying LTI systems (and linear differential equations)

Complex exponentials are the eigenfunctions of LTI:

For continuous time LTI, one has

LTI

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = \left(\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau\right)e^{st}$$

$$= H(s)e^{st}$$

For discrete time LTI, one has

LTI

$$z^n \longrightarrow \boxed{h[n]} \longrightarrow \sum_{m=-\infty}^{\infty} h[m]z^{n-m} = \left(\sum_{m=-\infty}^{\infty} h[m]z^{-m}\right)z^n$$

$$= H(z)z^n$$

Alternative Representation for *Real* Periodic Signal: the Trigonometric Fourier Series

$$X_n = \frac{1}{T}\int_T x(t)\, e^{-j\omega_0 nt} dt = \underbrace{\frac{1}{T}\int_T x(t)\cos(\omega_0 nt) dt}_{A_n} - j\underbrace{\frac{1}{T}\int_T x(t)\sin(\omega_0 nt) dt}_{B_n}$$

$$X_n = A_n - jB_n \longrightarrow |X_n| = \sqrt{A_n^2 + B_n^2} \qquad \theta_n = -arctg(B_n / A_n)$$

$$\boxed{\begin{array}{l} x(t) = X_0 + 2\sum_{n=1}^{\infty} |X_n| \cos(\omega_0 nt + \theta_n) \\ \text{where} \qquad X_0 = \frac{1}{T}\int_T x(t) dt \end{array}}$$

Main steps in the proof :

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{j\omega_0 nt} = X_0 + \sum_{n=1}^{\infty}\left(X_n e^{j\omega_0 nt} + X_{-n} e^{-j\omega_0 nt}\right) = X_0 + 2\sum_{n=1}^{\infty} \mathrm{Re}\left(X_n e^{j\omega_0 nt}\right)$$

Fourier spectrum

- Time-domain representation

$$x(t) = \sum_n X_n e^{jn\omega_0 t} = \sum_n |X_n| e^{j(n\omega_0 t + \theta_n)}$$

$$\theta_n = \angle X_n$$

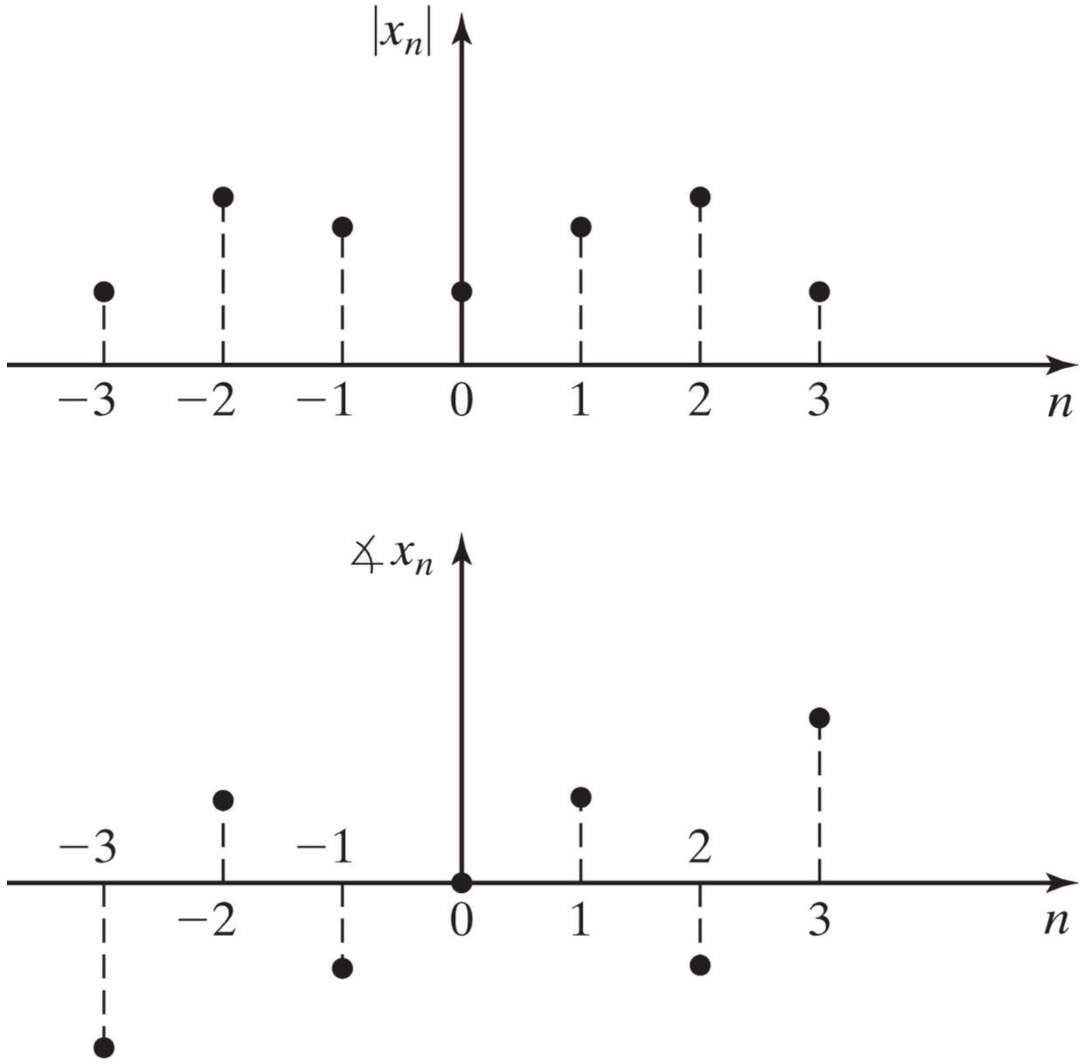
where $X_n = |X_n| e^{j\angle X_n}$.

- Frequency-domain representation

$$\begin{cases} |X_n(\omega)| & \text{--- magnitude spectrum} \\ \angle X_n(\omega) & \text{--- phase spectrum} \end{cases}$$

Summary: The Fourier series is a means for expanding a periodic signal (or a "well-behaved" signal over a finite interval) in terms of orthogonal complex exponentials (coordinates transform in a vector space). The expansion in terms of complex exponentials is particularly useful when analyzing LTI systems and solving linear (partial) differential equations.

Example: The discrete spectrum of a theoretical signal *x(t)*.



Example: The discrete spectrum of audio signals
(cf. © “Tuning, Timbre, Spectrum, Scale,” W.A. Sethares, 2004)

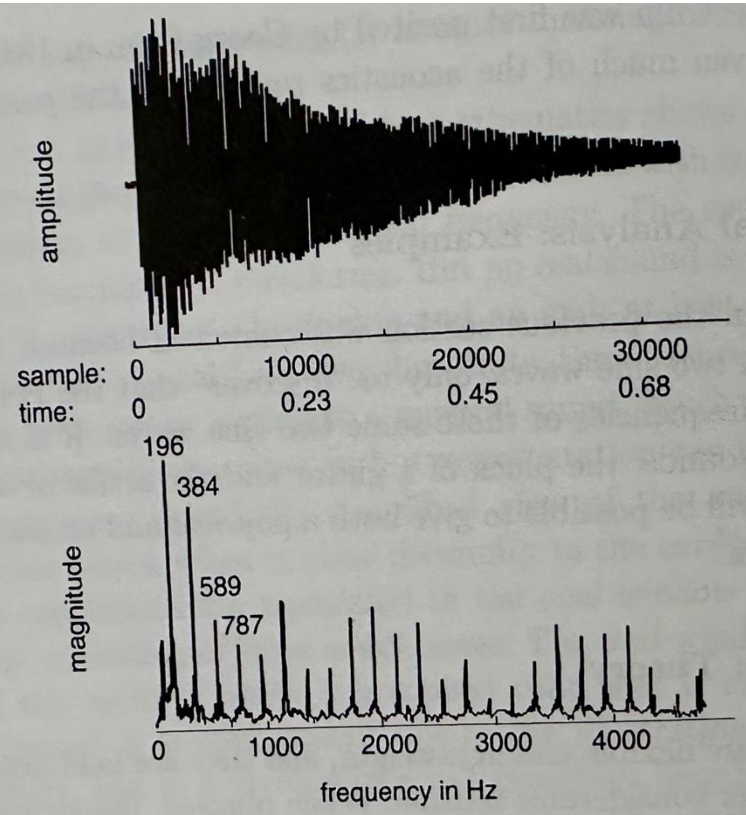


Fig. 2.5. Waveform of a guitar pluck and its spectrum. The top figure shows the first 3/4 second (32,000 samples) of the pluck of the G string of an acoustic guitar. The spectrum shows the fundamental at 196 Hz, and near integer harmonics at 384, 589, 787, ...

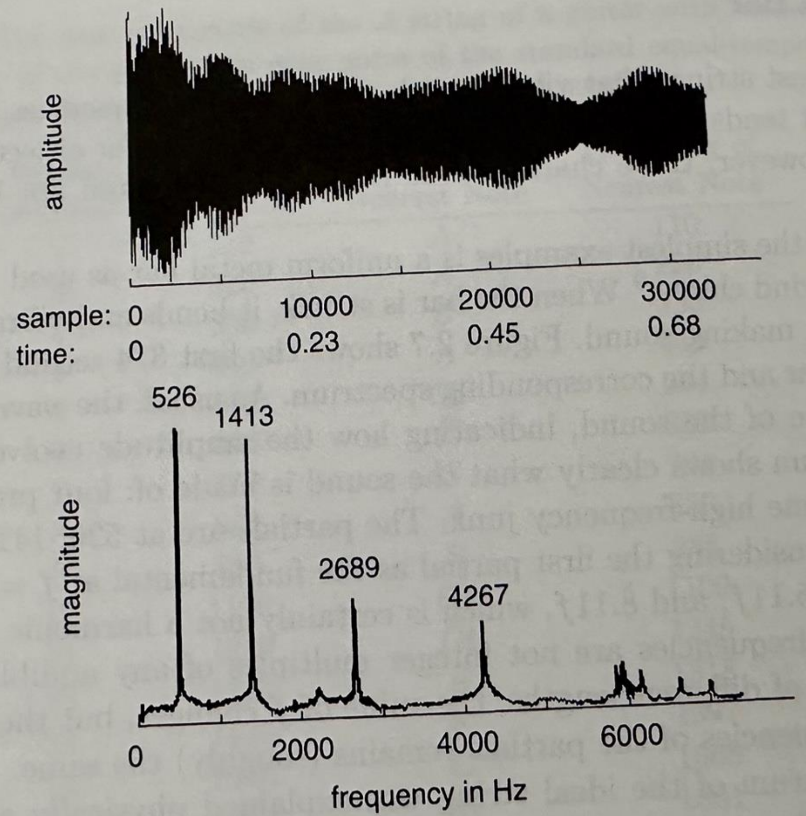
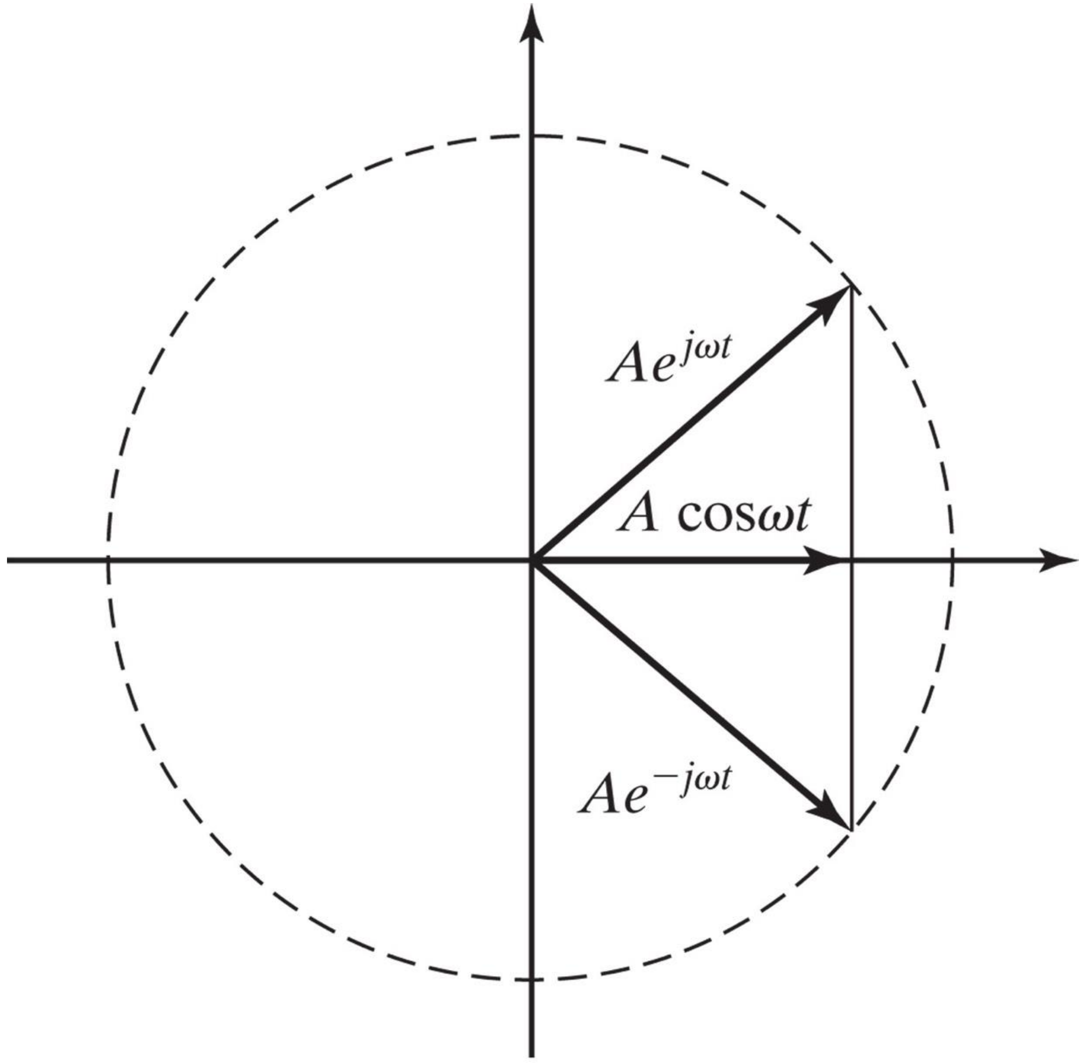
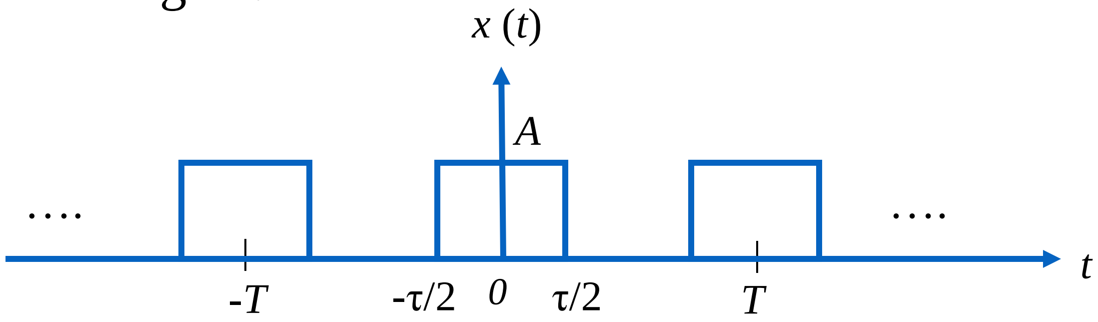


Fig. 2.7. Waveform of the strike of a metal bar and the corresponding spectrum. The top figure shows the first 3/4 second (32,000 samples) of the waveform in time. The spectrum shows four prominent partials.

A note on positive and negative frequencies: Phasors representing positive and negative frequencies.



Example: Periodic Gate Function. Determine the Fourier series expansion for this signal.

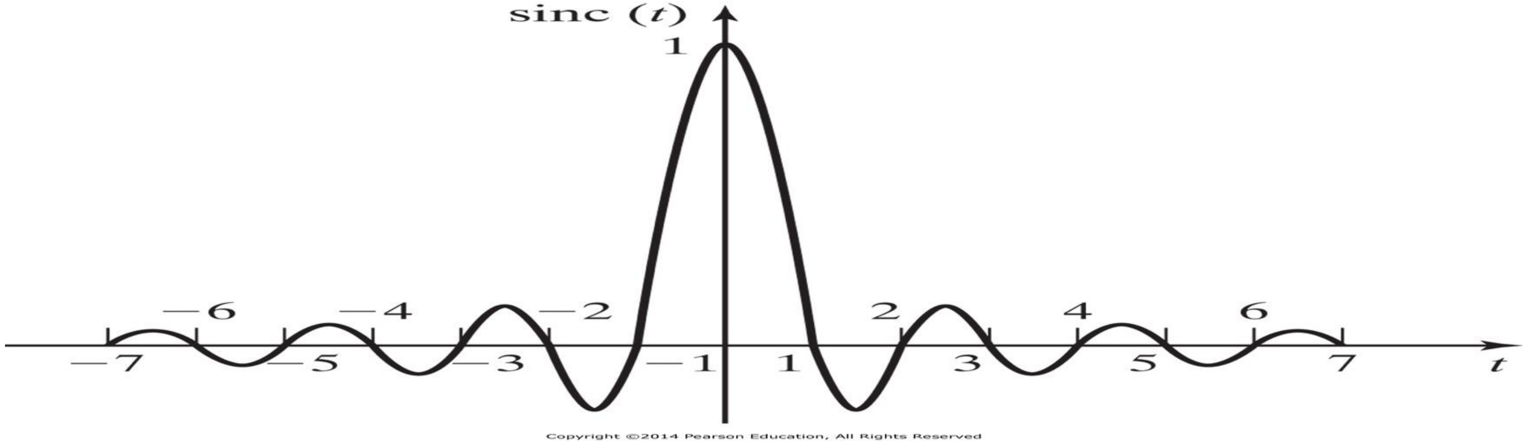


$$x(t) = A\sum_{n=-\infty}^{\infty}\Pi\left(\frac{t-nT}{\tau}\right)$$

Solution. Note that $x(t)$ is an even periodic function $x(t) = x(-t)$.

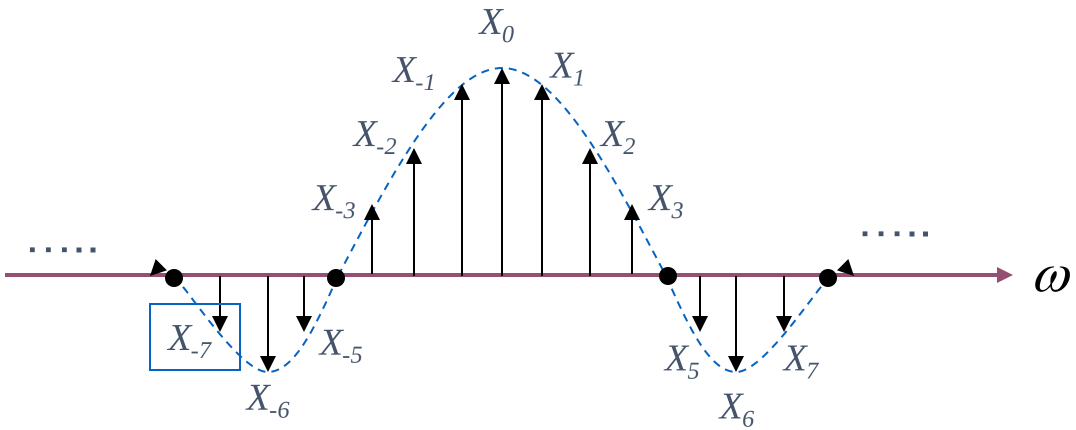
$$X_n = \frac{1}{T}\int_T x(t)\cos(\omega_0 nt)dt = \frac{A}{T}\int_{-\tau/2}^{\tau/2}\cos(\omega_0 nt)dt = \frac{A}{T}\frac{2}{\omega_0 n}\sin\left(\omega_0 n\frac{\tau}{2}\right) \quad \text{for } n \neq 0$$

$$X_0 = \frac{1}{T}\int_T x(t)dt = \frac{A}{T}\int_{-\tau/2}^{\tau/2}dt = \frac{A\tau}{T}$$

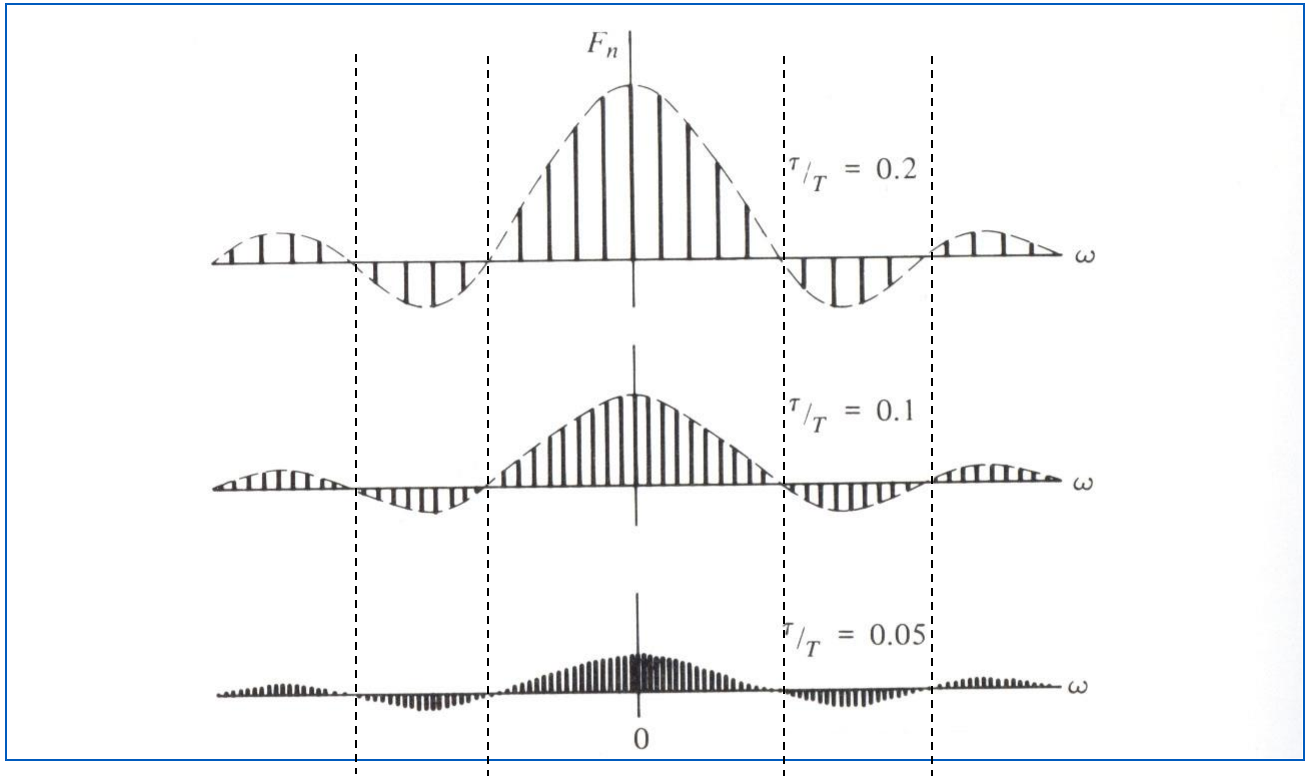


$$X_n = A\frac{\tau}{T}\,\mathrm{sinc}\left(\frac{n\tau}{T}\right)$$

Set $\quad T = 4\tau$



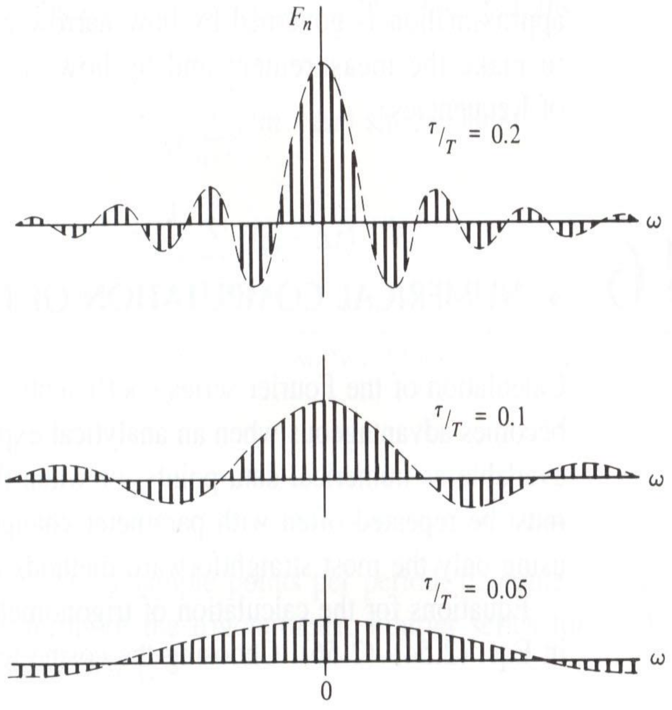
Spectrum for various values of τ/T, --- τ fixed

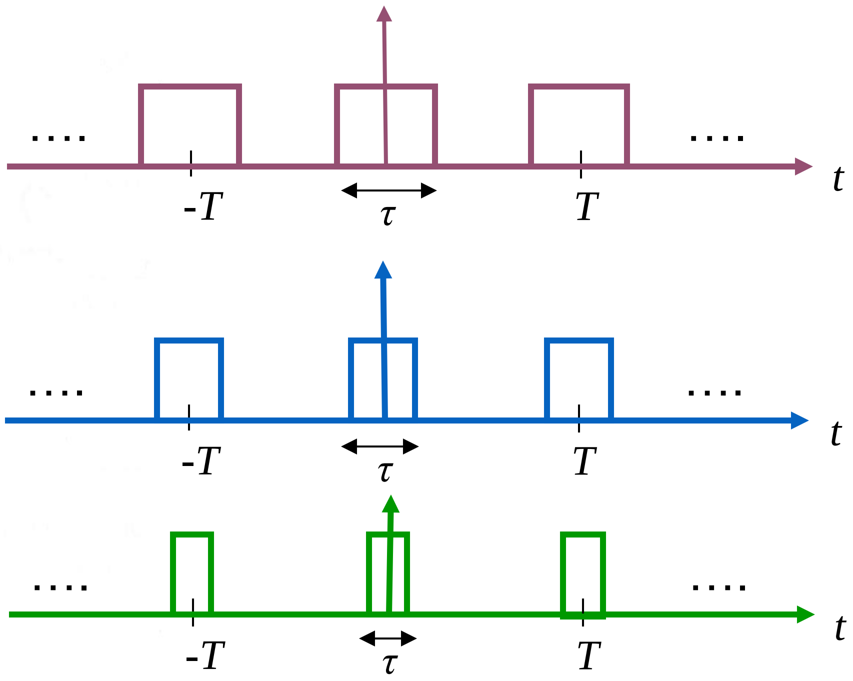


Observations:

- The amplitude decreases proportional to $1/T$.
- The spacing between lines decreases proportional to $1/T$.
- Zero crossings of the spectrum remain the same (not dependent on T).

Spectrum for various values of τ/T, --- T fixed





Observations:

- The amplitude decreases proportional to τ.

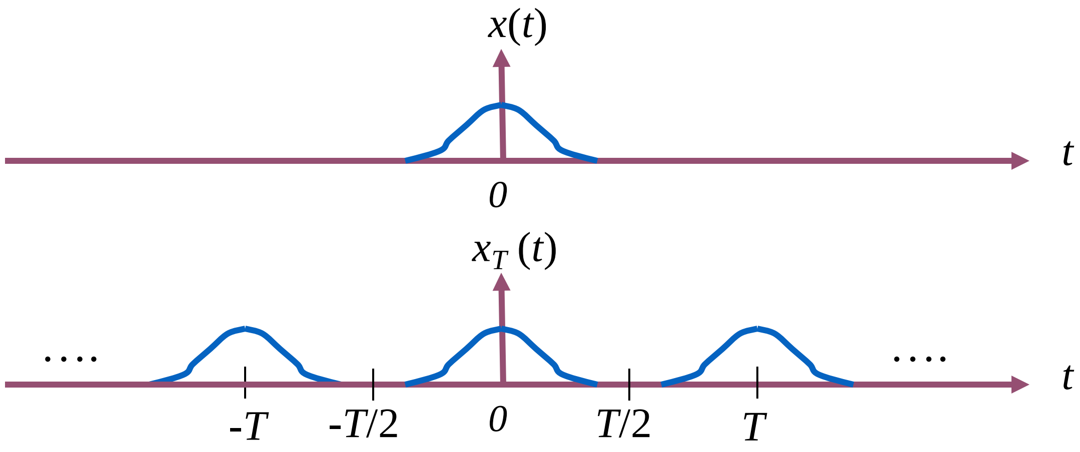
- The spectrum spreads as τ decreases. $\Rightarrow$ <u>There is an inverse relationship between pulse width in time (τ) and frequency *spread* of the spectrum.</u>

- <u>This is part of a more general phenomenon called the Heisenberg uncertainty principle.</u>

B. Fourier Transform

The Fourier series is a means for expanding a periodic signal in terms of complex exponentials. Now, consider an *aperiodic* function. How can we express it as a sum of exponential signals?

Construct a new periodic function $x_T(t)$ based on the original aperiodic signal $x(t)$.



$$\lim_{T\to\infty} x_T(t) = x(t)$$

$x_T(t)$ is a periodic signal and can be represented by Fourier series.

$$x_T(t) = \sum_{n=-\infty}^{+\infty} X_n e^{j\omega_0 nt} \quad \text{where} \quad X_n = \frac{1}{T}\int_T x_T(t) e^{-j\omega_0 nt} dt$$

$$\omega_n \overset{def}{=} n\omega_0$$

$$X(\omega_n) \overset{def}{=} TX_n$$

$$x_T(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{T} X(\omega_n) e^{j\omega_n t} \qquad X(\omega_n) = \int_T x_T(t) e^{-j\omega_n t} dt$$

Defining $\Delta\omega \overset{def}{=} 2\pi/T$ $\longrightarrow$ $x_T(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} X(\omega_n) e^{j\omega_n t}\Delta\omega$

As T becomes large,
$\Delta\omega$ becomes smaller.

$$\lim_{T\to\infty} x_T(t) = \lim_{T\to\infty} \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} X(\omega_n) e^{j\omega_n t}\Delta\omega$$

Riemann Integral

In the limiting case, the discrete lines in the spectrum merge and frequency spectrum becomes continuous.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

Fourier Transform $X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

Inverse Fourier Transform $x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$

$X(\omega)$ is also known as *spectral-density function* of $x(t)$.

$$x(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} X(\omega)$$

Fourier Transform Pair

Relationship between Fourier Series & Fourier Transform

$$X_n = \frac{1}{T} X(\omega)\Big|_{\omega \to n\omega_0}$$

$$X(\omega) = T X_n\Big|_{\omega_0 \to \frac{\omega}{n}}$$

Relationship of Fourier Transforms in ω and f

If the variable in the Fourier transform is chosen to be $f = \frac{\omega}{2\pi}$

(i.e., frequency in Hz or cycles per second) rather than ω, then we have

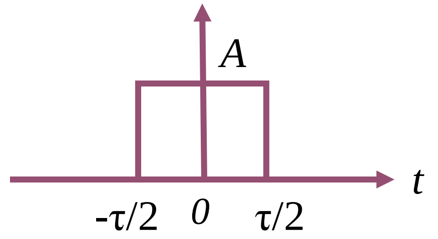
Fourier Transform

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

Inverse Fourier Transform

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

Example. Determine the Fourier transform of rectangular Gate function $x(t) = A\,\Pi\left(\frac{t}{\tau}\right)$:



<u>**Solution.**</u> Note that $x(t)$ is an even function. Thus

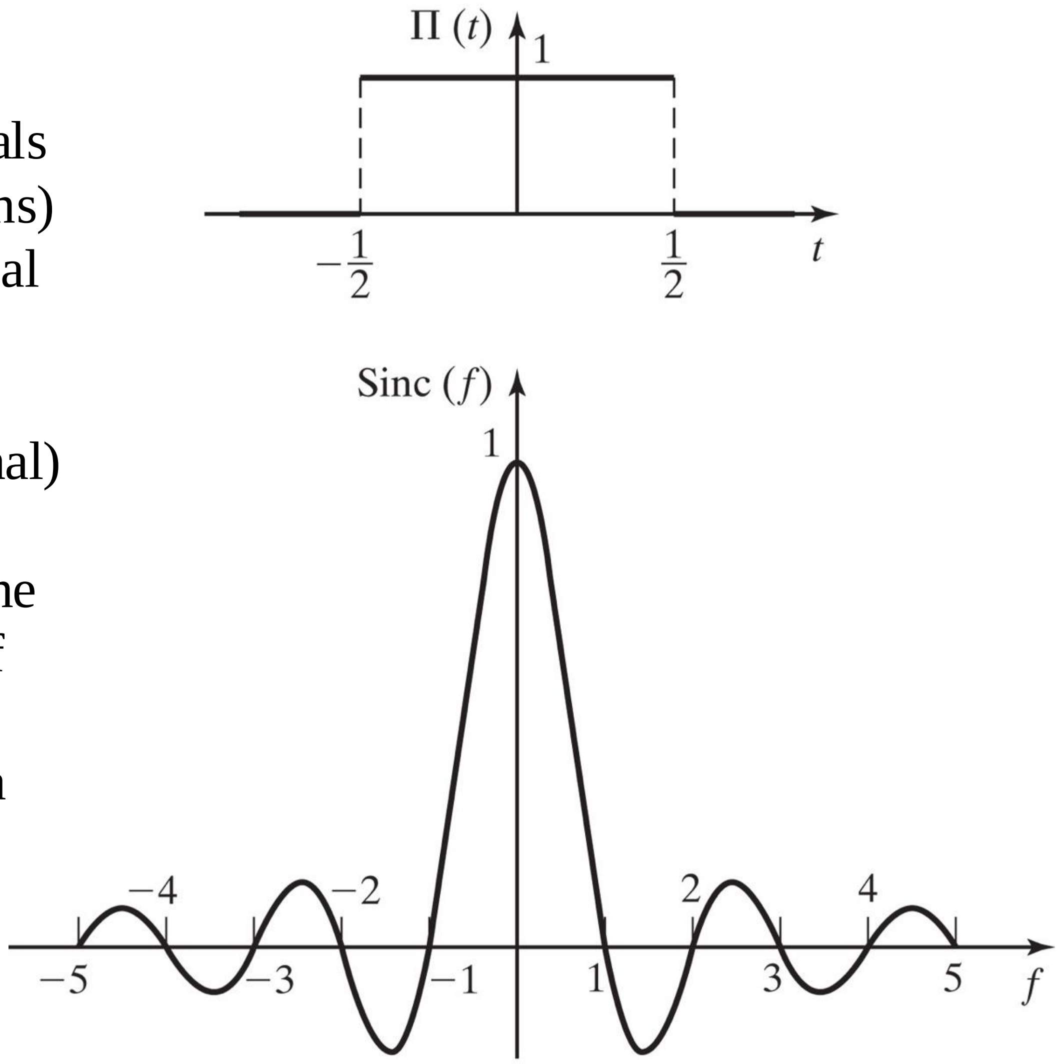
$$X(f) = \int_{-\infty}^{\infty} x(t)\cos(2\pi f t)\,dt = A\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \cos(2\pi f t)\,dt = \frac{2A}{2\pi f}\sin(\pi f \tau) = A\tau\,\mathrm{sinc}(f\tau)$$

Notice the Relation to the Fourier Series coefficients of a periodic gate function in the previous example, $X(f) = TX_n\big|_{\frac{1}{T}\to\frac{f}{n}} = TA\frac{\tau}{T}\,\mathrm{sinc}\left(\frac{n\tau}{T}\right)\big|_{\frac{1}{T}\to\frac{f}{n}}$.

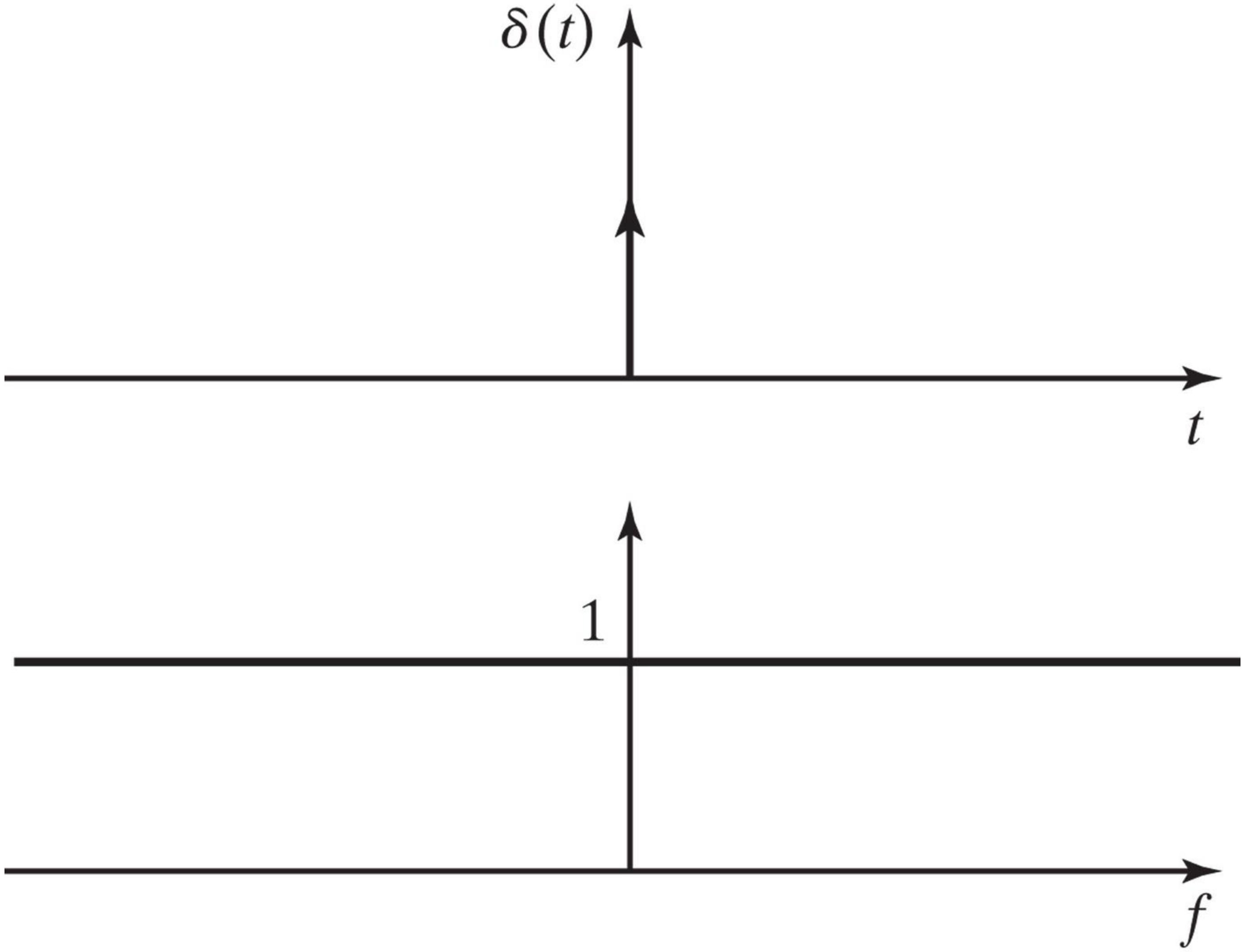
Observation: Compared to the spectrum of periodic signals, aperiodic signals have continuous spectrum.

In summary, for smooth enough signals (i.e., satisfying the Dirichlet conditions) Fourier analysis decomposes the signal into its frequency components which can be discrete multiples of the fundamental frequency (periodic signal) or continuous (nonperiodic signals) similarly to the prism decomposing the white light into the different colors of the rainbow. Fourier synthesis is a representation of periodic signals as a (discrete) sum of sines and cosines. For nonperiodic signals, this sum is continuum (i.e., integral).

$\Pi(t)$ and its Fourier transform



Impulse signal and its spectrum.

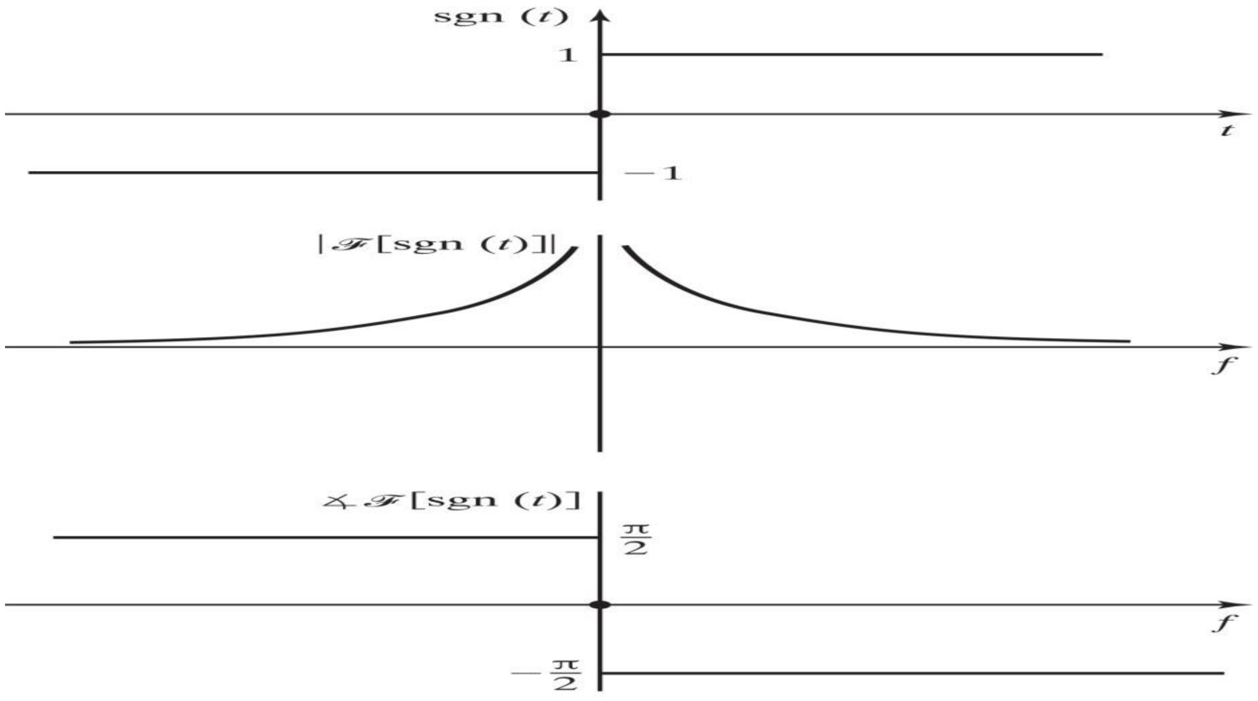


Example: determine the Fourier transform of the signum signal sgn(*t*)

Solution: as the signum signal doesn't satisfy the Dirichlet conditions, we consider its definition as a limit of exponentials that satisfy Dirichlet: $\mathrm{x_n}(t) = \begin{cases} e^{-\frac{t}{n}} & t > 0 \\ -e^{\frac{t}{n}} & t < 0 \\ 0 & t = 0 \end{cases}$

$$X_n(f) = \int_{-\infty}^{\infty} x_n(t) e^{-j2\pi \mathrm{ft}}\, dt = \int_{-\infty}^{0} -e^{\frac{t}{n}}\, e^{-j2\pi \mathrm{ft}}\, dt + \int_{0}^{\infty} e^{-\frac{t}{n}}\, e^{-j2\pi \mathrm{ft}}\, dt = \frac{-1}{\frac{1}{n}-2j\pi f} + \frac{1}{\frac{1}{n}+2j\pi f}$$

As *n* goes to ∞, $X(f) = \frac{1}{j\pi f}$ (to be revisited)



C. Properties of the Fourier Transform

- **Linearity (Superposition)**

$x_1(t) \leftrightarrow X_1(f), x_2(\mathrm{t}) \leftrightarrow X_2(f) \Rightarrow a_1x_1(t) + a_2x_2(t) \leftrightarrow a_1X_1(f) + a_2X_2(f)$

a_1, a_2: arbitrary constants

- **Complex Conjugate**

$x(t) \leftrightarrow X(f)$
$x^*(t) \leftrightarrow X^*(-f)$

$x^*(t)$: Complex conjugate of $x(t)$

- **Duality**

$x(t) \leftrightarrow X(f) = \mathcal{F}\{x(t)\}$
$X(t) \leftrightarrow \mathcal{F}\{X(t)\} = x(-f)$

Example: Find the Fourier transform of $x(t) = 1$ and $y(t) = \text{sinc}(t)$?

Solution: by the duality property

$$X(f) = \mathcal{F}\{1\} = \delta(-f) = \delta(f)$$
$$Y(f) = \mathcal{F}\{\text{sinc}(t)\} = \Pi(-f) = \Pi(f)$$

Example: Find the Fourier transform of the unit step signal?

Solution: $u(t) = \frac{1}{2} + \frac{1}{2}\text{sgn}(t) \Rightarrow U(f) = \mathcal{F}\{u(t)\} = \frac{1}{2}\mathcal{F}\{1\} + \frac{1}{2}\mathcal{F}\{\text{sgn}(t)\} = \frac{1}{2}\delta(f) + \frac{1}{2j\pi f}$

Example: Find the Fourier transform $x(t) = \frac{1}{t}$?

C. Properties of the Fourier Transform (continued)

- Time scaling: $x(t) \leftrightarrow X(f)$

For any $a \neq 0,\ x(a\,t) \leftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$

- Time shift: $x(t) \leftrightarrow X(f)$

  $x(t - t_0) \leftrightarrow X(f)e^{-j2\pi f t_0}$

- Frequency shift (modulation property): $x(t) \leftrightarrow X(f)$

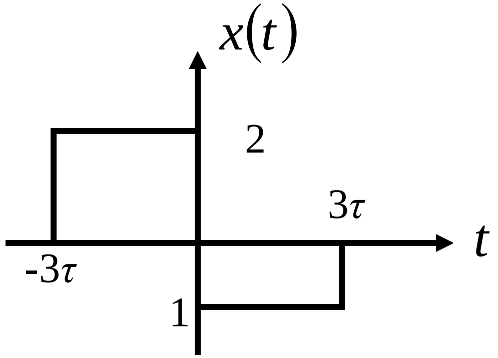
  $e^{j2\pi f_0 t}x(t) \leftrightarrow X(f - f_0)$

- Convolution: $\mathcal{F}\{x(t) * h(t)\} = X(f)H(f)$

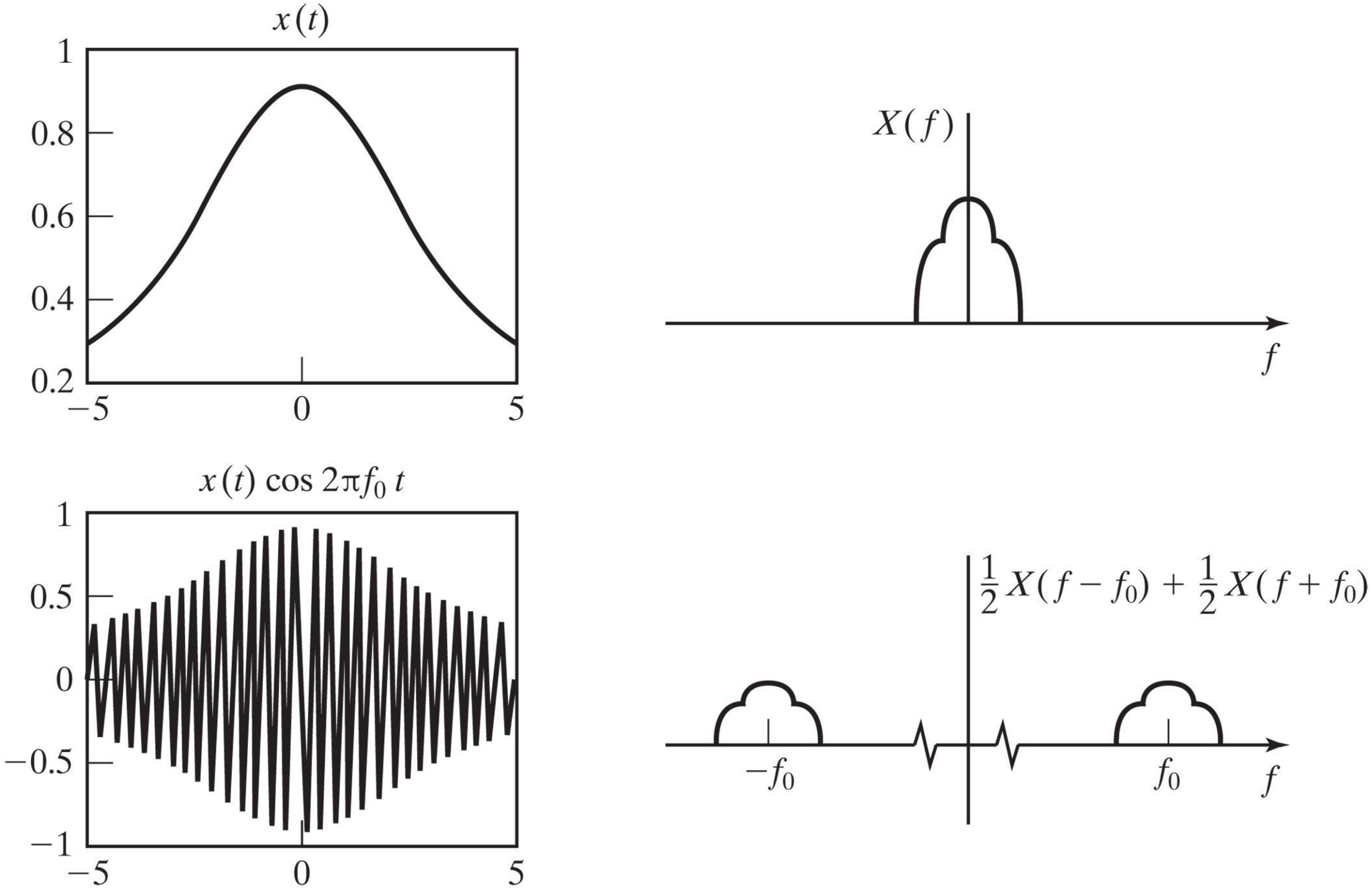
- Multiplication: $\mathcal{F}\{x(t)y(t)\} = X(f) * Y(f)$

Example: Find the Fourier transforms of $x(t)$ and

$y(t) = A\cos(2\pi f_c t)\Pi\left(\frac{t}{\tau}\right)$



Effect of modulation in both the time and frequency domain.



C. Properties of the Fourier Transform (continued)

- Differentiation: $x(t) \leftrightarrow X(f)$

  $$\frac{\mathrm{d}}{\mathrm{dt}} x(t) \leftrightarrow 2j\pi f\; X(f)$$

- Integration: $x(t) \leftrightarrow X(f)$

  $$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{2} X(0)\delta(f) + \frac{1}{j2\pi f} X(f)$$

To prove the last property, utilize $\int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t)$

- Parseval's Relation: $x(t) \leftrightarrow X(f), \;\; y(t) \leftrightarrow Y(f)$

  $$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$$

Example: Using Parseval's theorem find the values of the integrals

$\int_{-\infty}^{\infty} \text{sinc}^4(t)dt$ and $\int_{-\infty}^{\infty} \text{sinc}^3(t)dt$

D. Fourier Transform of Periodic Signals

We have already seen that a periodic signal can be represented by its exponential Fourier Series, i.e.,

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{j\omega_0 nt} \qquad \omega_0 = 2\pi/T$$

Taking the Fourier transform,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} X_n e^{j\omega_0 nt}\right\} = \sum_{n=-\infty}^{\infty} X_n \mathcal{F}\{e^{j\omega_0 nt}\} = 2\pi \sum_{n=-\infty}^{\infty} X_n \delta(\omega - n\omega_0)$$

The Fourier transform of a periodic signal consists of a set of impulses located at the multiples of the fundamental frequency. The weight of each impulse is 2π times the value of its corresponding coefficient in the Fourier series.

In the Hz frequency representation $X(f) = \sum_{n=-\infty}^{\infty} X_n \delta\left(f - \frac{n}{T}\right)$

Example: Find the Fourier transform of the impulse train $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$

Solution: By linearity, $X(f) = \mathcal{F}\{\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\} = \sum_{n=-\infty}^{\infty} e^{-j2\pi f n T_s} = \sum_{n=-\infty}^{\infty} e^{j2\pi f n T_s}$

On the other hand, $x(t)$ is periodic of period T_s. Therefore, its Fourier series representation coefficients are
$X_n = \frac{1}{T_s}\int_0^{T_s} x(t) e^{-\frac{j2\pi n}{T_s}t}\, dt = \frac{1}{T_s}$ ➔ $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} e^{\frac{j2\pi n}{T_s}t}$.
Substituting f for t and $1/T_s$ for T_s in the relation above yields:

$$X(f) = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right\} = \sum_{n=-\infty}^{\infty} e^{j2\pi f n T_s} = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_s}\right)$$

In particular, when $T_s = 1$:

$$\mathcal{F}\left\{\sum_{n=-\infty}^{\infty} \delta(t - n)\right\} = \sum_{n=-\infty}^{\infty} \delta(f - n)$$

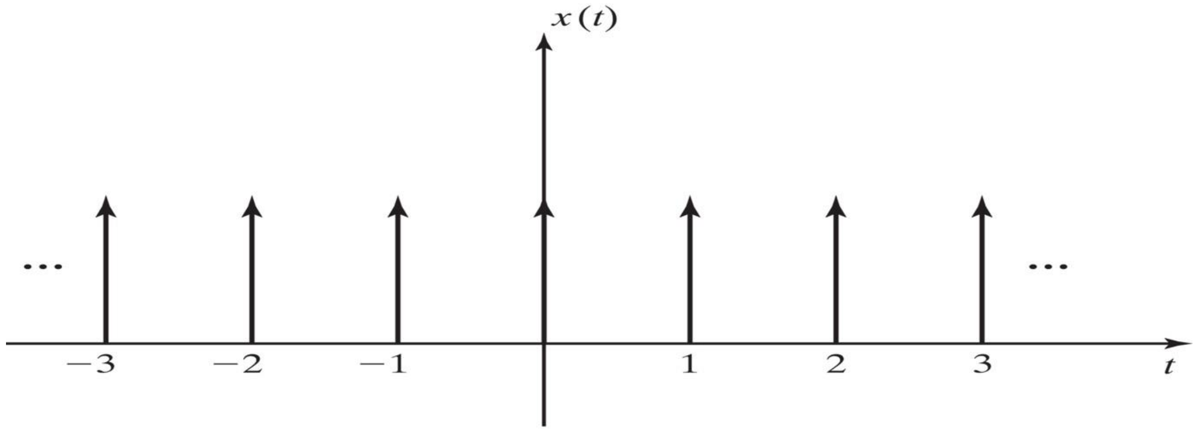


TABLE 2.1 TABLE OF FOURIER-TRANSFORM PAIRS

Time Domain	Frequency Domain
$\delta(t)$	1
1	$\delta(f)$
$\delta(t-t_0)$	$e^{-j2\pi f t_0}$
$e^{j2\pi f_0 t}$	$\delta(f-f_0)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}\delta(f-f_0)+\frac{1}{2}\delta(f+f_0)$
$\sin(2\pi f_0 t)$	$-\frac{1}{2j}\delta(f+f_0)+\frac{1}{2j}\delta(f-f_0)$
$\Pi(t)$	$\mathrm{sinc}(f)$
$\mathrm{sinc}(t)$	$\Pi(f)$
$\Lambda(t)$	$\mathrm{sinc}^2(f)$
$\mathrm{sinc}^2(t)$	$\Lambda(f)$

TABLE 2.1 TABLE OF FOURIER-TRANSFORM PAIRS

Time Domain	Frequency Domain
$e^{-\alpha t}u_{-1}(t), \alpha > 0$	$\frac{1}{\alpha + j2\pi f}$
$te^{-\alpha t}u_{-1}(t), \alpha > 0$	$\frac{1}{(\alpha + j2\pi f)^2}$
$e^{-\alpha\|t\|}$	$\frac{2\alpha}{\alpha^2 + (2\pi f)^2}$
$e^{-\pi t^2}$	$e^{-\pi f^2}$
$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$
$u_{-1}(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
$\delta'(t)$	$j2\pi f$
$\delta^{(n)}(t)$	$(j2\pi f)^n$
$\frac{1}{t}$	$-j\pi\mathrm{sgn}(f)$
$\sum_{n=-\infty}^{n=+\infty}\delta(t-nT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{n=+\infty}\delta\left(f-\frac{n}{T_0}\right)$

TABLE 2.2 TABLE OF FOURIER-TRANSFORM PROPERTIES

Signal	Fourier Transform
$\alpha x_1(t) + \beta x_2(t)$	$\alpha X_1(f) + \beta X_2(f)$
$X(t)$	$x(-f)$
$x(at)$	$\frac{1}{\lvert a \rvert} X\left(\frac{f}{a}\right)$
$x(t - t_0)$	$e^{-j2\pi f t_0} X(f)$
$e^{j2\pi f_0 t} x(t)$	$X(f - f_0)$
$x(t) \star y(t)$	$X(f)Y(f)$
$x(t)y(t)$	$X(f) \star Y(f)$
$\frac{d}{dt} x(t)$	$j2\pi f X(f)$
$\frac{d^n}{dt^n} x(t)$	$(j2\pi f)^n X(f)$
$t x(t)$	$\left(\frac{j}{2\pi}\right) \frac{d}{df} X(f)$
$t^n x(t)$	$\left(\frac{j}{2\pi}\right)^n \frac{d^n}{df^n} X(f)$
$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\frac{X(f)}{j2\pi f} + \frac{1}{2} X(0)\delta(f)$

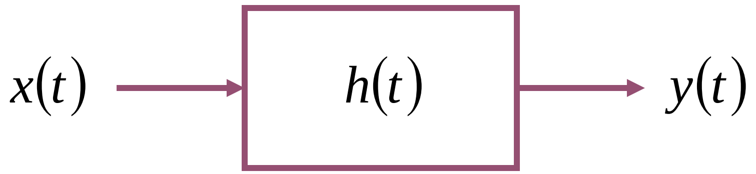
2.3 Fourier Analysis of Linear Time Invariant (LTI) Systems

A. **Transmission over LTI Systems**
B. **System Bandwidth**
C. **Distortionless Transmission**
D. **Energy/Power Transmission Over LTI Systems**

A. Transmission Over LTI Systems

LTI systems provide good and accurate models for a large class of communication systems. Some basic components of transmitters and receivers, such as filters, *amplifiers* and equalizers can be modeled as LTI systems.

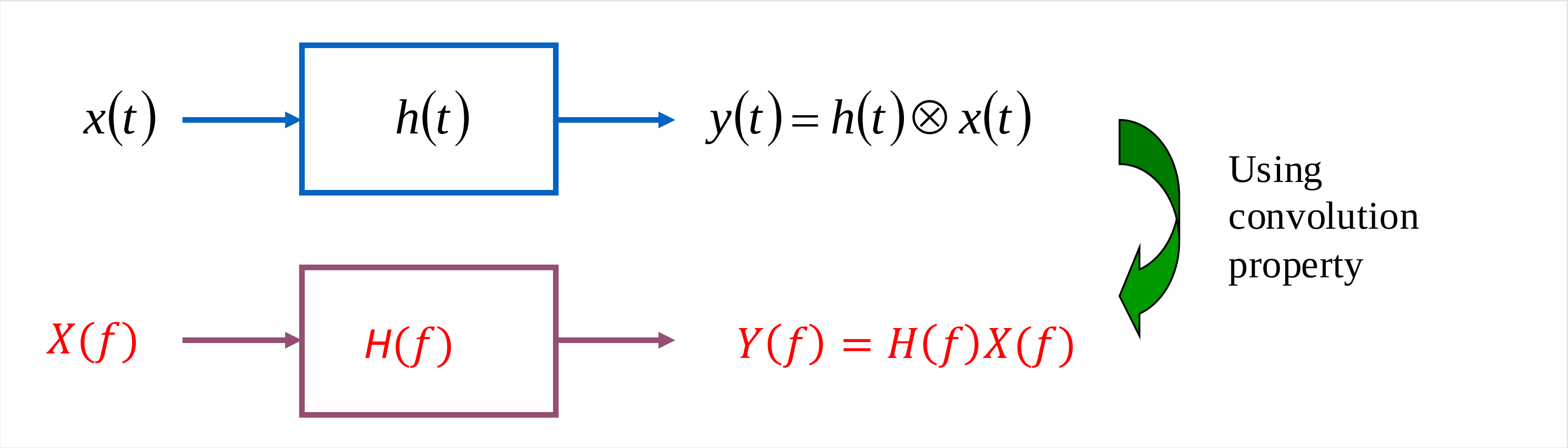
The *impulse response* $h(t)$ of a system is the system's response to a unit impulse. In other words, $h(t) = y(t)$ when $x(t) = \delta(t)$.



$$y(t) = h(t) \otimes x(t)$$

$$= \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$$

The *frequency response of the system* is the Fourier Transform of the impulse response: $H(f) = \mathcal{F}\{h(t)\}$



In time domain, we have the convolution integral. On the other hand, in the frequency domain, the input-output relation is much simpler, just a multiplication.

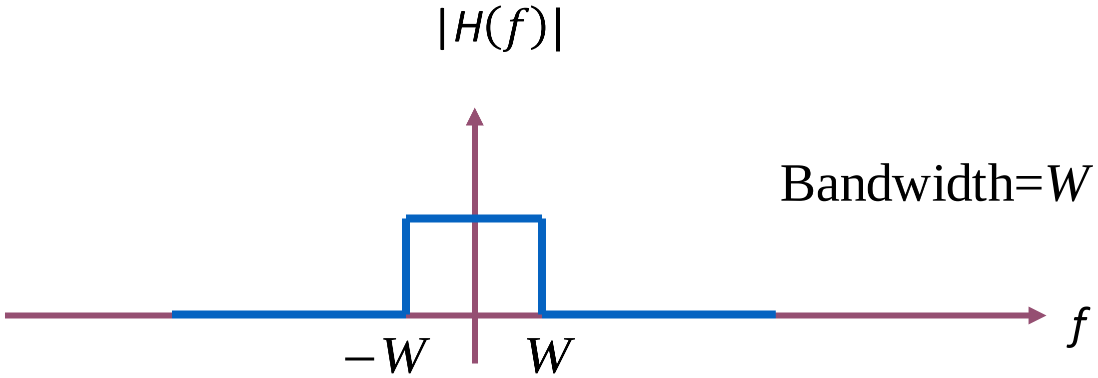
An LTI acts as a filter on the various frequency components of the input signal. It might affect both its amplitude and phase.

$$|Y(f)|e^{j\theta_y(f)} = |H(f)|e^{j\theta_y(f)}|X(f)|e^{j\theta_x(f)}$$

$$|Y(f)| = |H(f)| \cdot |X(f)|$$

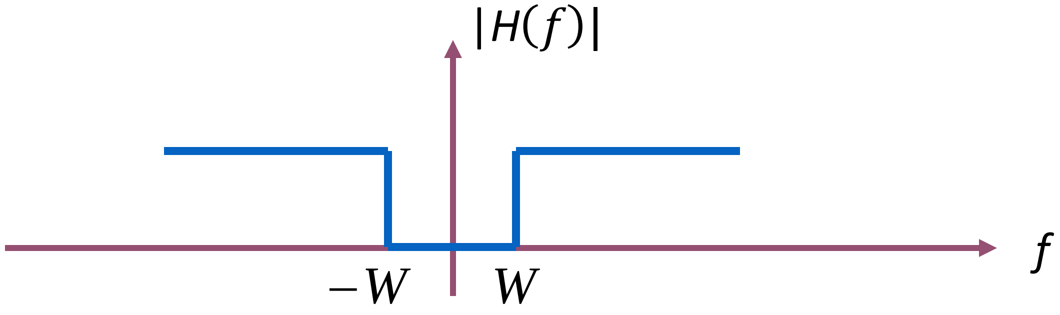
$$\theta_y(f) = \theta_h(f) + \theta_x(f)$$

Ideal Low Pass Filter (LPF)

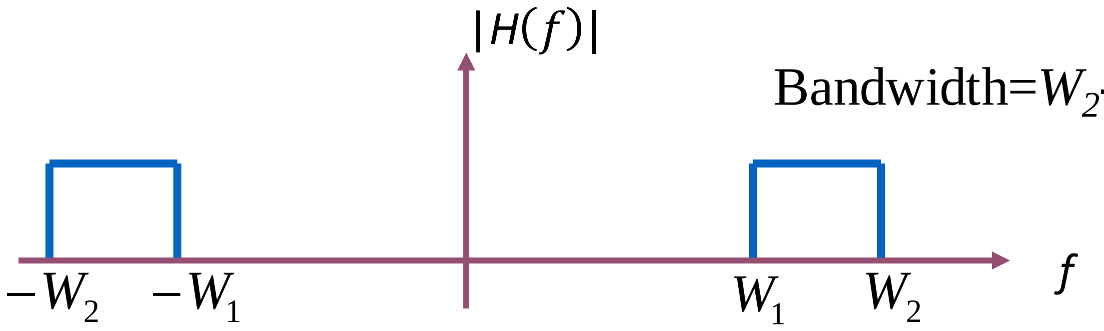


Bandwidth=W Hz

Ideal High Pass Filter (HPF)



Ideal Band Pass Filter (BPF)

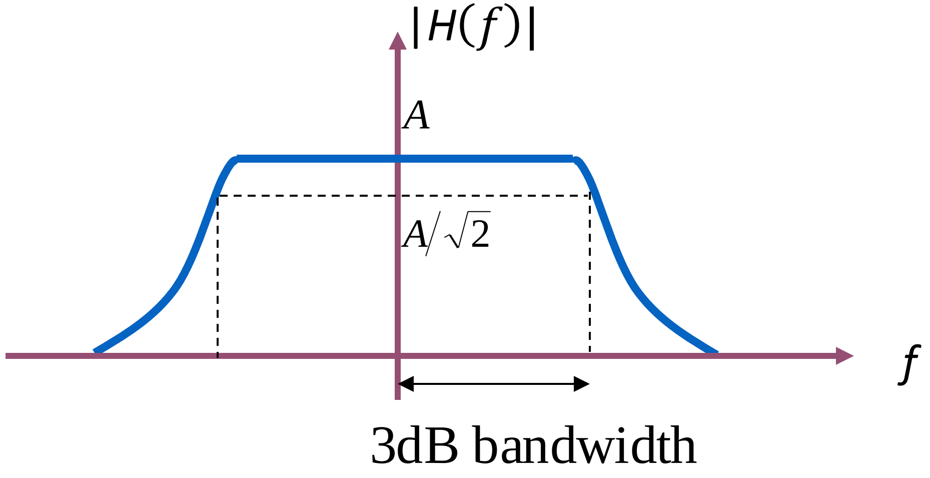


Bandwidth=W_2-W_1

B. Bandwidth of a System

In practice, filters do not have sharp transitions as ideal ones. The bandwidth is then defined as " the interval of positive frequencies over which the magnitude of transfer function remains within a given numerical factor ". Although different criterion might be used, a widely used definition is *3dB bandwidth (or half-power bandwidth).*

According to this criteria, the bandwidth is defined as the band of frequencies at which the magnitude of the transfer function is at least $1/\sqrt{2}$ of its max. value. It is called as *3dB bandwith*, because reducing the amplitude by a factor of $\sqrt{2}$ (i.e., reduce power by 2) corresponds to a decrease of 3dB on logarithmic scale.



C. Distortionless Transmission

For ideal distortionless transmission, the output signal of an LTI should be

$$y(t) = kx(t-\tau)$$

where k and τ are constant. Taking the Fourier transform of the above identity, we have

$$Y(f) = k\, X(f)e^{-j2\pi f\tau}$$

Thus, the required system transfer function for distortionless transmission is given by

$$H(f) = k\, e^{-j2\pi f\tau}$$

In other words, to achieve ideal distortionless transmission, the overall system response must have a constant magnitude response and its phase shift must be linear with frequency.

D. Transmission of Energy/Power Over LTI

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f) = X(f)H(f)$$

- Recall the definitions of energy and power-type signals:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt,$$

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

The signal $x(t)$ is called energy-type if E_x is finite. The signal is called power-type if $0 < P_x < \infty$

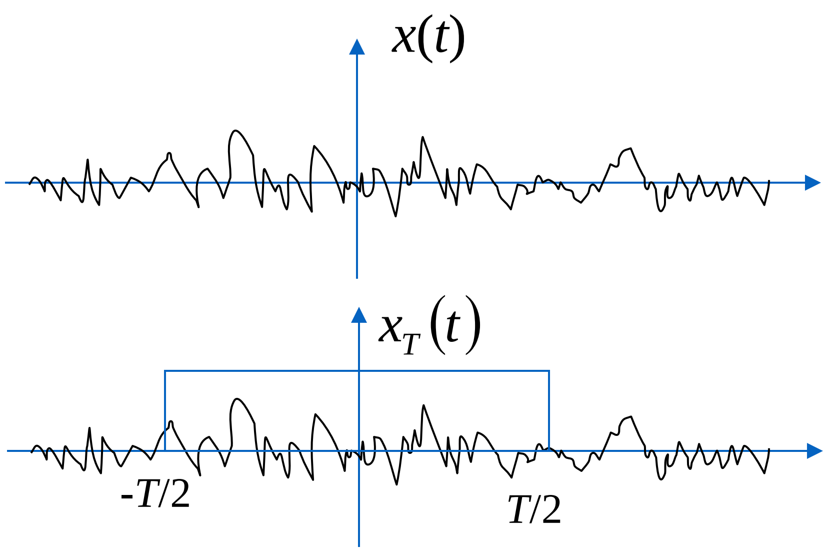
- **Parseval's theorem** #: $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df,$ $|X(f)|^2$ is defined as the energy spectral density

- Similarly we can define the power spectral density (PSD) as $S_x(f)$ such that $P_x = \int_{-\infty}^{\infty} S_x(f) df$

- Question: What is $S_x(f)$?

#The general form of Parseval's theorem is: $\int_{-\infty}^{\infty} x(t) y^*(t) dt = \int_{-\infty}^{\infty} X(f) Y^*(f) df$

Derivation of $S_x(f)$



$x(t)$: power signal

$x_T(t)$: truncated to $-T/2$ and $T/2$

$$x_T(t) = x(t)\Pi\left(\frac{t}{T}\right)$$

We now compute the energy content of $x_T(t)$, then divide by T and take the limit:

$$P_x = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2 dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}|x_T(t)|^2 dt = \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}|X_T(f)|^2 df = \int_{-\infty}^{\infty}\lim_{T\to\infty}\frac{1}{T}|X_T(f)|^2 df = \int_{-\infty}^{\infty}S_x(f)df$$

The last relationship should hold over each frequency increment ➔ $S_x(f) = \lim_{T\to\infty}\frac{1}{T}|X_T(f)|^2$

Power Spectral Density for Periodic Signals

The previous discussion on PSD holds for any general power signal.
Now, we assume that $x(t)$ is a periodic power signal.

For a periodic signal, each period contains a replica of the function and the limiting operation can be omitted as long as T is taken as the period.

$$P_x = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)x^*(t)dt$$

Express in terms of Fourier series

$$= \frac{1}{T}\int_{-T/2}^{T/2} \left(\sum_{m=-\infty}^{\infty} X_m e^{jm\omega_0 t}\right)\left(\sum_{n=-\infty}^{\infty} X_n^* e^{-jn\omega_0 t}\right)dt$$

Changing the order of summation and integration

$$= \sum_{m=-\infty}^{\infty} X_m \sum_{n=-\infty}^{\infty} X_n^* \frac{1}{T}\int_{-T/2}^{+T/2} e^{j(m-n)\omega_0 t}dt$$

$$\frac{1}{T}\int_{-T/2}^{+T/2} e^{j(m-n)\omega_0 t}dt = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

$$= \sum_{n=-\infty}^{\infty} X_n X_n^* = \sum_{n=-\infty}^{\infty} |X_n|^2$$

Fourier Transform

$$X(f) = \sum_{n=-\infty}^{\infty} X_n \delta\left(f - \frac{n}{T}\right)$$

Power Spectral Density

$$S_x(f) = \sum_{n=-\infty}^{\infty} \left|X_n\right|^2 \delta\left(f - \frac{n}{T}\right)$$

PSD of periodic signals have discrete components. It consists of a series of impulse functions with weights corresponding to the magnitude squared of respective Fourier series coefficients. The average power is given by

$$P_x = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \int_{-\infty}^{\infty} S_x(f) df = \sum_{n=-\infty}^{\infty} \left|X_n\right|^2$$

Example. Find the power spectral density of the periodic signal $x(t) = A\cos(2\pi f_0 t + \theta)$

Transmission of PSD over LTI

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f) = X(f)H(f)$$

Energy spectral density of the output signal

$$|Y(f)|^2 = |X(f)|^2 |H(f)|^2$$

Energy of the output signal

$$E_y = \int_{-\infty}^{\infty} |X(f)|^2 |H(f)|^2 df$$

Power spectral density of the output signal

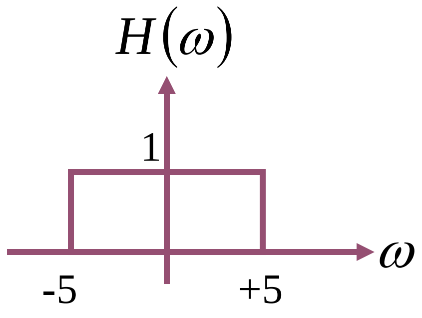
$$S_y(f) = S_x(f)|H(f)|^2$$

Power of the output signal

$$P_y = \int_{-\infty}^{\infty} S_x(f)|H(f)|^2 \, df$$

Example. A voltage signal is described by $x(t) = e^{-5t}u(t)$.
It is applied to the input of an ideal low-pass filter. The gain of the filter is unity, the bandwidth is 5 rad/sec and the resistance levels are 50 Ω.
Calculate the energy of the input signal and of the output signal.

Solution.



$$H(\omega) = \begin{cases} 1, & |\omega| < 5 \\ 0, & |\omega| > 5 \end{cases} \qquad |X(\omega)|^2 = \frac{1}{\omega^2 + 25}$$

$$E_x = \frac{1}{R}\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{50}\int_{0}^{\infty} e^{-10t} dt = 2 \cdot 10^{-3} \text{ [joule]}$$

$$E_y = \frac{1}{2\pi R}\int_{-\infty}^{\infty} |H(\omega)|^2 |X(\omega)|^2 d\omega = \frac{1}{2\pi 50}\int_{-5}^{+5} \frac{d\omega}{\omega^2 + 25} = 1 \cdot 10^{-3} \text{ [joule]}$$

2.4 Correlation and Spectral Density

A. Autocorrelation Functions
B. Properties of Autocorrelation Functions
C. Cross-correlation Functions
D. Remark on Correlation of Random Signals

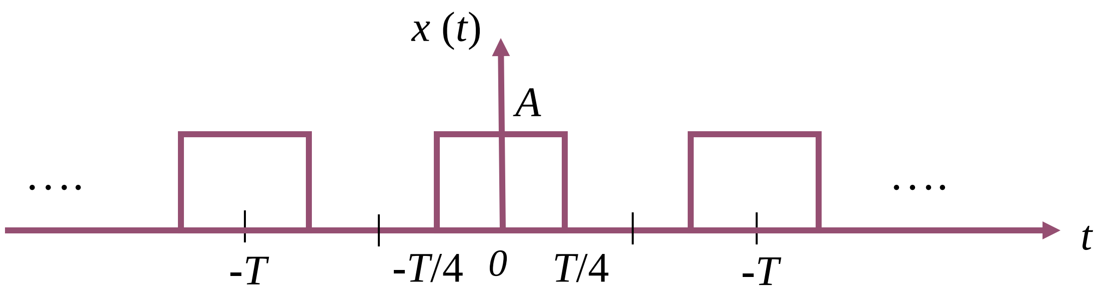
A. Autocorrelation Function

$$S_x(f) = \lim_{T\to\infty} \frac{1}{T} |X_T(f)|^2$$

Taking the inverse Fourier transform of PSD, after some manipulations, one can show that (Khinchin-Wiener theorem):

$$\mathcal{F}^{-1}\{S_x(f)\} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^*(t)x(t+\tau)dt = R_x(\tau)$$

Example 6 . Determine the autocorrelation of the following signal.



Solution.

$$R_x(\tau)=\frac{1}{T}\int_{-T/2}^{T/2} x^*(t)x(t+\tau)dt$$

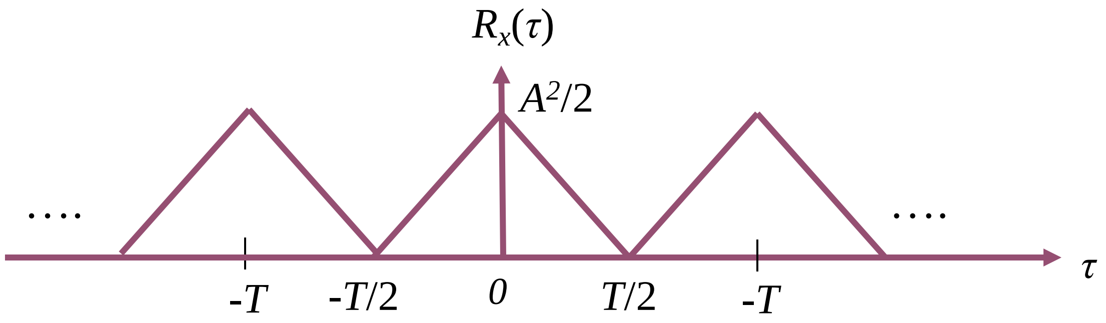
Since $x(t)$ is periodic, the limiting operation can be omitted. This is the convolution of the retangular function with itself. Thus we have

For the case: $-T/2<\tau<0$

$$R_x(\tau)=\frac{1}{T}\int_{-T/4}^{T/4+\tau} A^2dt = A^2\left(\frac{1}{2}+\frac{\tau}{T}\right)$$

For the case: $0<\tau<T/2$

$$R_x(\tau)=\frac{1}{T}\int_{\tau-T/4}^{T/4} A^2dt = A^2\left(\frac{1}{2}-\frac{\tau}{T}\right)$$



Autocorrelation function is widely used in signal analysis. It is especially useful for the detection or recognition of signals that are masked by additive noise.

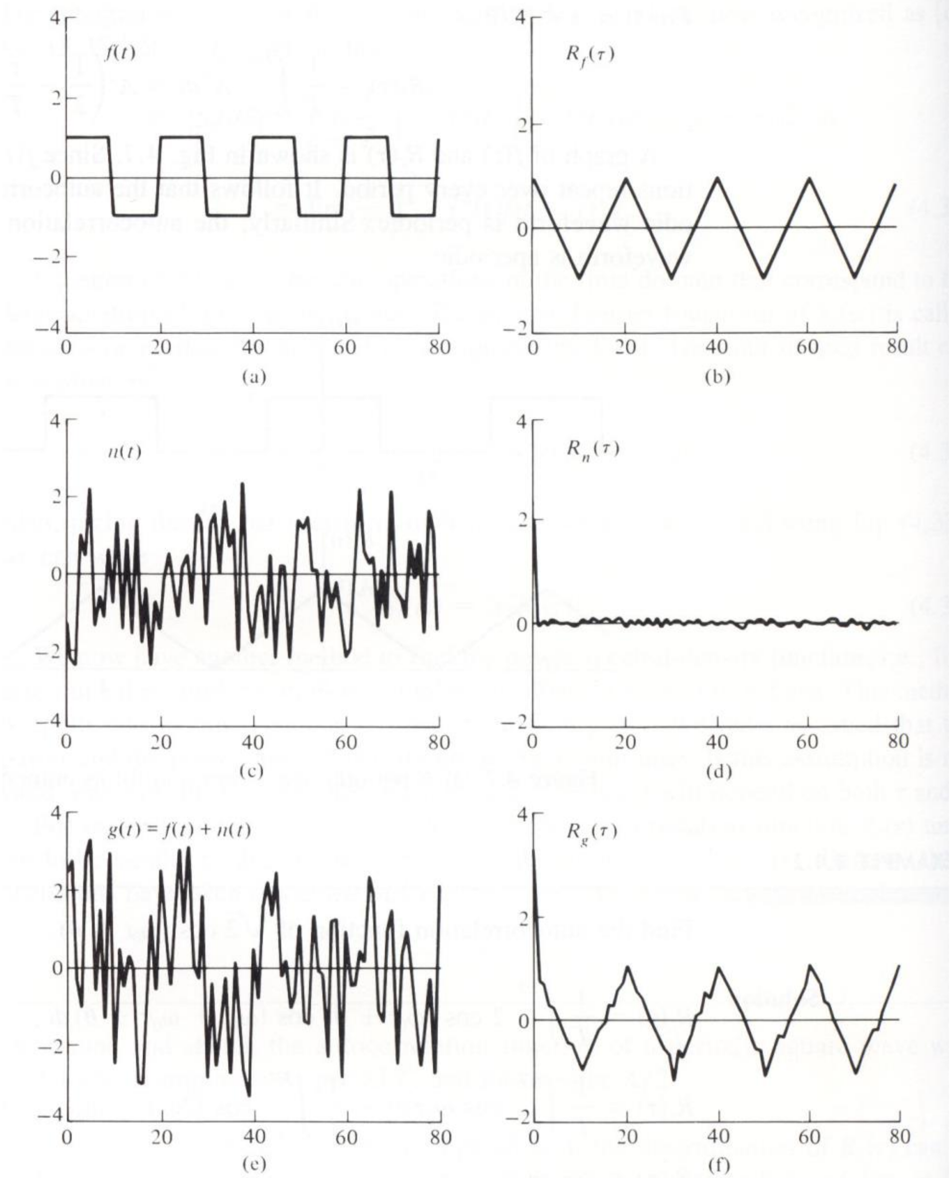
a) Original signal

c) Noise

e) Original signal + noise

b), d) and f) : corresponding autocorrelation functions

The autocorrelation function of the original signal is still recognizable in the noisy case!



Determination of the autocorrelation of the output of LTI

$$x(t) \longrightarrow \boxed{\quad h(t) \quad} \longrightarrow y(t) = h(t) \otimes x(t) = \int_{-\infty}^{\infty} h(u)x(t-u)du$$

$$R_y(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} y(t)y^*(t-\tau)dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[\int_{-\infty}^{\infty} h(u)x(t-u)du\right]\left[\int_{-\infty}^{\infty} h^*(v)x^*(t-\tau-v)dv\right]dt$$

Variable change $s=t-u$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h^*(v)\left[\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(s)x^*(s+u-\tau-v)ds\right]dudv$$

Definition of R_x

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h^*(v)R_x(\tau+v-u)dudv$$

Definition of convolution

$$= \int_{-\infty}^{\infty} \left[R_x(\tau+v) \otimes h(\tau+v)\right]h^*(v)dv$$

$$R_y(\tau) = R_x(\tau) \otimes h(\tau) \otimes h^*(-\tau)$$

Time domain

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = h(t) \otimes x(t)$$

$$R_y(\tau) = R_x(\tau) \otimes h(\tau) \otimes h^*(-\tau)$$

Frequency domain

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f) = X(f)H(f)$$

$$S_y(f) = S_x(f)|H(f)|^2$$

B. Properties of Autocorrelation Functions

- **Symmetry**

$$R_x^*(\tau) = R_x(-\tau)$$

- **Mean-Square Value**

$$R_x(\tau)\big|_{\tau=0} = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)x^*(t)dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = P_x$$

- **Periodicity**

If $x(t+T) = x(t)$, i.e. $x(t)$ periodic, $R_x(\tau+T) = R_x(\tau)$

- **Maximum Value**

The autocorrelation function is bounded by its mean square value:

$$|R_x(\tau)| \le R_x(0)$$

C. Crosscorrelation Function

Definition. Crosscorrelation between two signal $x(t)$ and $y(t)$ is defined as

$$R_{xy}(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}x^*(t)y(t+\tau)dt$$

$R_{xy}(\tau)=0 \Rightarrow$ $x(t)$ and $y(t)$ are uncorrelated.

The crosscorrelation between two signals measures the *similarity* between them.

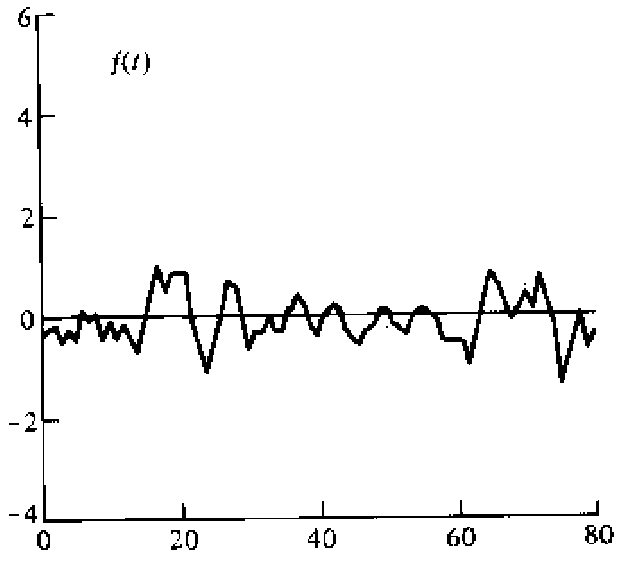
Example. Let $z(t)=x(t)+y(t)$ Determine $R_z(\tau)$

Solution.

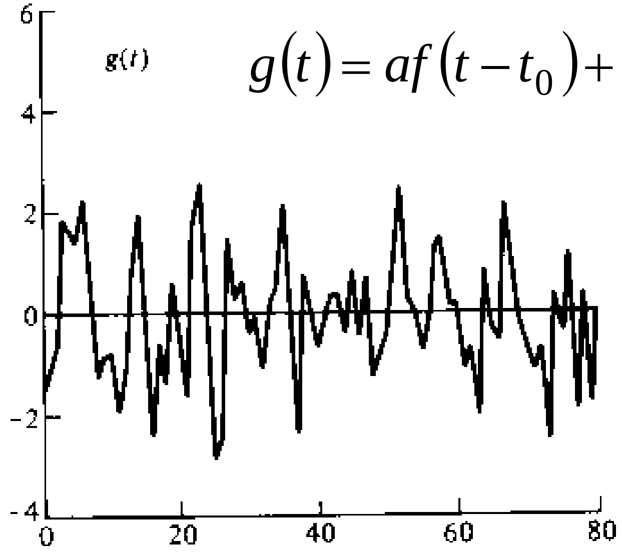
$$R_z(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\left[x^*(t)+y^*(t)\right]\left[x\ (t+\tau)+y(t+\tau)\right]dt$$

$$=R_x(\tau)+R_y(\tau)+R_{xy}(\tau)+R_{yx}(\tau)$$

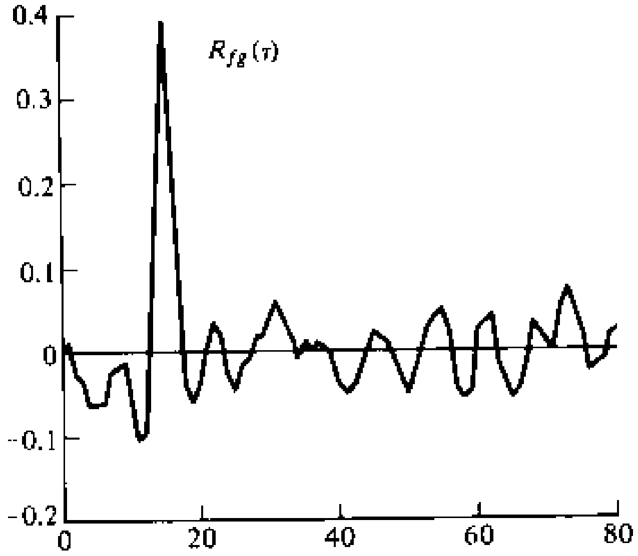
$$g(t) = af(t - t_0) + n(t)$$



(a) Random signal



(b) Random signal + noise



(c) Crosscorrelation

Cross-correlation can be used to find the time delay.

Typically used in synchronization for communication systems.

2.5 Noise

A. **Noise Sources in Communication**
B. **Time-averaged Noise Representation**
C. **Band-limited Additive White Noise**
D. **Thermal Noise**

A. Noise Sources in Communications

Noise consists of any unwanted signals that tend to disturb the transmission and processing of desired signals in communication systems. Noise may be random or deterministic.

In communications, the source of noise can be:

external --- atmospheric noise, man made noise, etc.

internal --- thermal noise

Thermal noise is produced as a result of the random motion of the thermally excited free electrons in a conducting medium, such as a resistor.

B. Time Averaged Noise Representation

In the case of random noise, time-average representations are useful.

Mean (DC) value:

$$\overline{n(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} n(t)\,dt$$

Mean-square value:
(=Power)

$$P_n = \overline{n^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |n(t)|^2\,dt$$

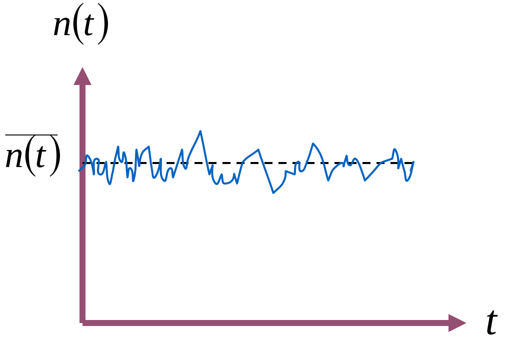
$\sqrt{\overline{n^2(t)}}$: root-mean-square (rms)

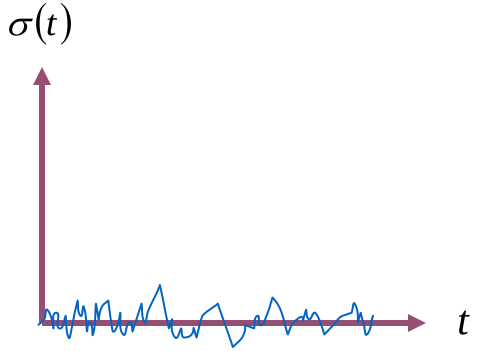
AC component:

$$\sigma(t) \overset{def}{=} n(t) - \overline{n(t)}$$

Signal-to-noise ratio (SNR) is widely used as a performance measure.

$$\frac{S}{N} = \frac{\overline{s^2(t)}}{\overline{n^2(t)}} = \frac{\text{signal power}}{\text{noise power}} \qquad \left[\frac{S}{N}\right]_{dB} = 10\log_{10}\left(\frac{\overline{s^2(t)}}{\overline{n^2(t)}}\right)$$





$\overline{n(t)}$: constant

$\overline{\sigma(t)} = 0$

$\sigma(t) = n(t) - \overline{n(t)}$

What are the DC power and AC power of noise $n(t)$?

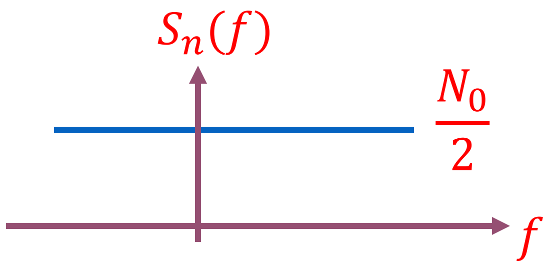
$$\overline{n^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left|\overline{n(t)} + \sigma(t)\right|^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left|\overline{n(t)}\right|^2 dt + \underbrace{\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \overline{n(t)}\sigma^*(t)dt}_{=0} + \underbrace{\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[\overline{n(t)}\right]^* \sigma(t)dt}_{=0} + \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left|\sigma(t)\right|^2 dt$$

$$= \underbrace{\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left|\overline{n(t)}\right|^2 dt}_{\text{DC power}} + \underbrace{\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left|\sigma(t)\right|^2 dt}_{\text{AC power}}$$

C. Band-Limited White Noise

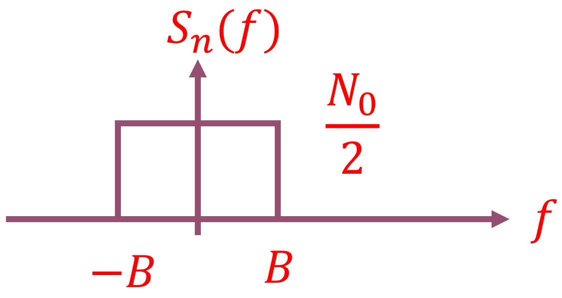
A flat power spectrum contains all frequency components with equal power weighting and is designated as *white*, in an analogy to white light.



$$P_n = \overline{n^2(t)} = \int_{-\infty}^{\infty} S_n(f)df = \infty$$

Infinite power!

In general, the bandwidth of the receiver (i.e., front end filter) is narrower than the bandwidth limitations of the noise process. If noise has a flat PSD extending beyond the bandwidth of a given system, the noise appears to the system as if it were band-limited and white.



$$P_n = \overline{n^2(t)} = \int_{-B}^{B} \frac{N_0}{2} df = N_0 B$$

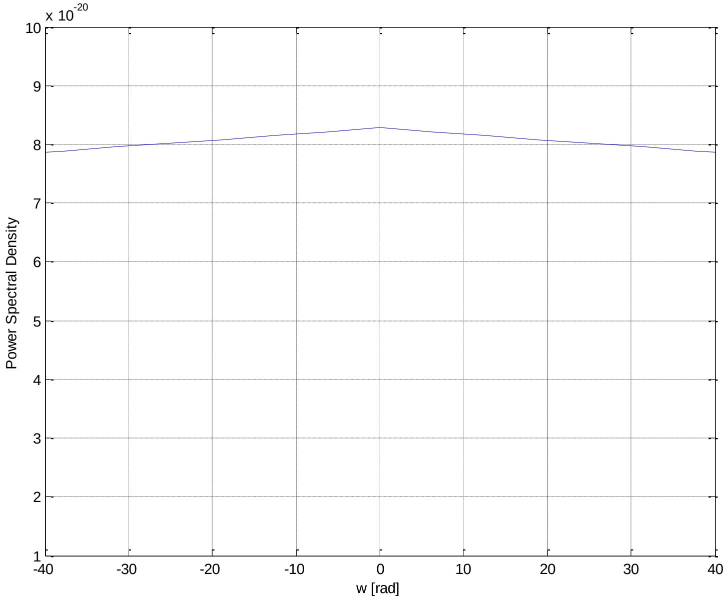
D. Thermal Noise

$$S_n(\omega) = \frac{h|\omega|}{\pi\left[\exp\left(\frac{h|\omega|}{2\pi kT}\right) - 1\right]}$$

T: temperature [K]

k: Boltzmann's constant (=1.38×10^{-23} joule/K)

h: Planck's constant (=6.625×10^{-34} joule-sec)



- Achieves its maximum at $\omega = 0$. The value of this maximum is 2*kT*.
- $\omega \to \infty \quad PSD \to 0$ The rate of convergence is very slow.

$$S_n(\omega) \cong 2kT \qquad |\omega| << 2\pi kT/h$$

- $kT/h \approx 6 \cdot 10^{12}$ Hz Beyond the operating frequency of conventional communication systems!

Conclusion: Thermal noise can be assumed white for all practical purposes!

Important Remark

The auto/cross correlation of a random signal, therefore, power spectral density, when satisfies the condition of an *ergodic random process* (i.e., its time averages are equal to its ensemble averages), are the same as the ones we defined for a deterministic signal. So, in the content of ECE 318, the results about deterministic signals can be applied to random signals when we discuss the properties related to their correlation, such as power spectral density.

Summary on Time Average Operators

$$< x(t) >= \overline{x(t)} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)dt$$

Bar indicates time averaging.

where T is the period if $x(t)$ is periodic with the period T. In this case, the limitation operator can be omitted.

1. DC value of $x(t)$: $x_{dc} =< x(t) >$

2. Average power of $x(t)$: $P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \int_{-\infty}^{\infty} S_x(f) df = R_x(0)$

3. Root mean square (rms) value and average (normalized) power:

$$x_{rms} = \sqrt{< x^2(t) >} = \sqrt{P_x}$$