Responses of LTI Systems

Def. $p \in \mathbb{C}$ is a pole of a transfer function $G(s)$ if $G(p) = \infty$.

Ex. $G(s) = \frac{(s+1)(s+2)}{(s+3)(s-4)}$

$\Rightarrow$ poles$(G(s)) = -3, 4$

$e^{j\theta} = \cos\theta + j\sin\theta$

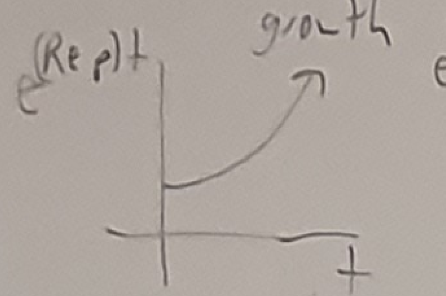
Ex. $H(s) = \frac{1}{s-p}$, $p \in \mathbb{C}$ $\Rightarrow$ pole$(H(s)) = p$

$\Rightarrow h(t) = e^{pt} = e^{(\mathrm{Re}\,p + j\,\mathrm{Im}\,p)t} = e^{(\mathrm{Re}\,p)t} e^{j(\mathrm{Im}\,p)t}$

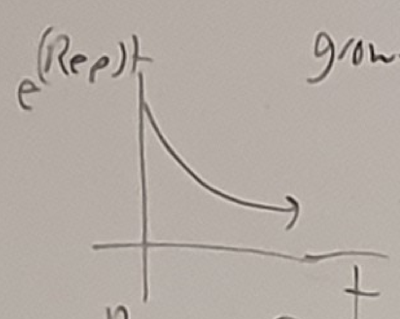
$\cos((\mathrm{Im}\,p)t) + j\sin((\mathrm{Im}\,p)t)$ //

$e^{(\mathrm{Re}\,p)t}$: growth/decay/constant

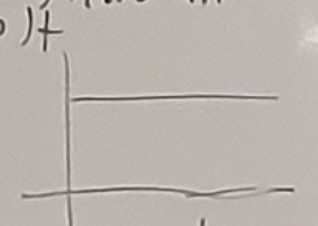
$e^{j(\mathrm{Im}\,p)t}$: sinusoidal oscillations with frequency $\mathrm{Im}(p)$



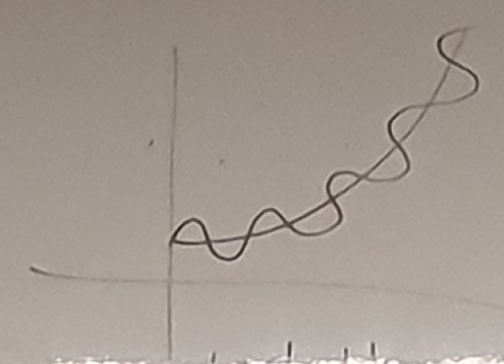
Re $p > 0$



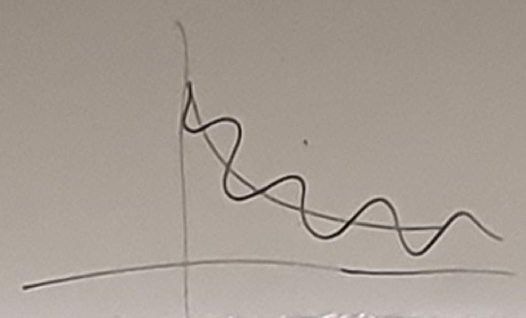
Re $p < 0$



Re $p = 0$



unstable system



stable systems

$\Rightarrow$ Re(p) determines the rate of growth or decay

Im(p) determines the frequency of sinusoidal oscillations

Ex. Re$(p) < 0 \Rightarrow$ stable system

Ex. $H(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2}$ for $p_1, p_2 \in \mathbb{C}$

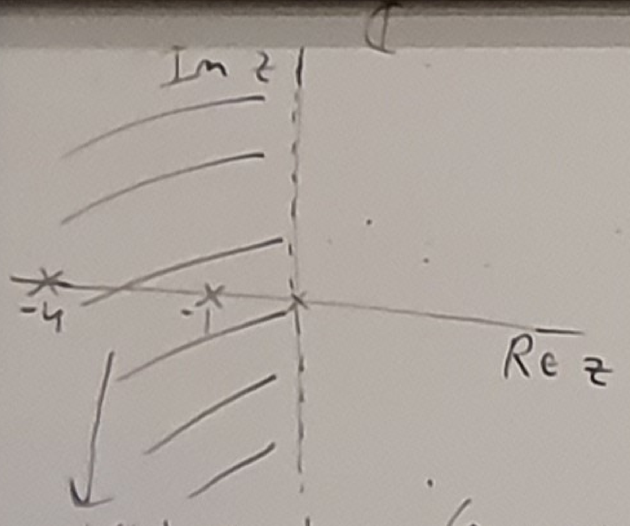
$\Rightarrow h(t) = c_1 e^{p_1 t} + c_2 e^{p_2 t}$

Cases:
- a. Re$(p_1) \not< 0$, Re$(p_2) \not< 0$
- b. Re$(p_1) < 0$, Re$(p_2) \not< 0$ — unstable
- c. Re$(p_1) \not< 0$, Re$(p_2) < 0$
- d. Re$(p_1) < 0$, Re$(p_2) < 0$ — stable

Def. A transfer function $H(s)$ is stable if all of its poles p satisfy Re$(p) < 0$.

$\Updownarrow$

A transfer function $H(s)$ is stable if all of its poles lie in the OLHP.



open left half plane (OLHP)

We'll mainly be interested in responses of stable systems.

- the time domain response of an LTI system $H(s)$ to an input $u(t)$:

$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}(H(s)U(s))$

Ex. $H(s) = \frac{(s+2)(s+3)}{(s+1)(s+4)}$, $U(s) = \frac{1}{s} = \mathbb{1}(s)$

poles$(H(s)) = -1, -4$ $\Rightarrow$ $H(s)$ is stable

$\Rightarrow Y(s) = H(s)U(s) = \frac{(s+2)(s+3)}{s(s+1)(s+4)} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{s+4}$

$\Rightarrow y(t) = \mathcal{L}^{-1}(Y(s)) = \mathbb{1}(t)\left(a + be^{-t} + ce^{-4t}\right)$

poles$(Y(s)) \subset$ poles$(H(s)) \cup$ poles$(U(s))$ $\Rightarrow$ the response $y(t)$ typically includes a term for each pole of $H(s)$ and $U(s)$

Typically:
- the terms in $Y(s) = H(s)U(s)$ that correspond to poles of $H(s)$ will have inverse transforms that decay to 0 with increasing time (because $H(s)$ is stable)
  - these terms represent the transient part of the response
- the terms that correspond to poles of $U(s)$ typically will give rise to the steady-state response

We will now study the responses of some simple systems
- standard 1st and 2nd order systems

Our main objective is to relate the positions of the poles in $\mathbb{C}$ to the forms of the corresponding terms of the transient response.
- This relationship is key to the analysis and design of controllers

Standard 1st-order system

$$H(s) = \frac{k}{s\tau+1}, \quad k, \tau > 0 \qquad \text{poles}(H(s)) = \frac{-1}{\tau}$$

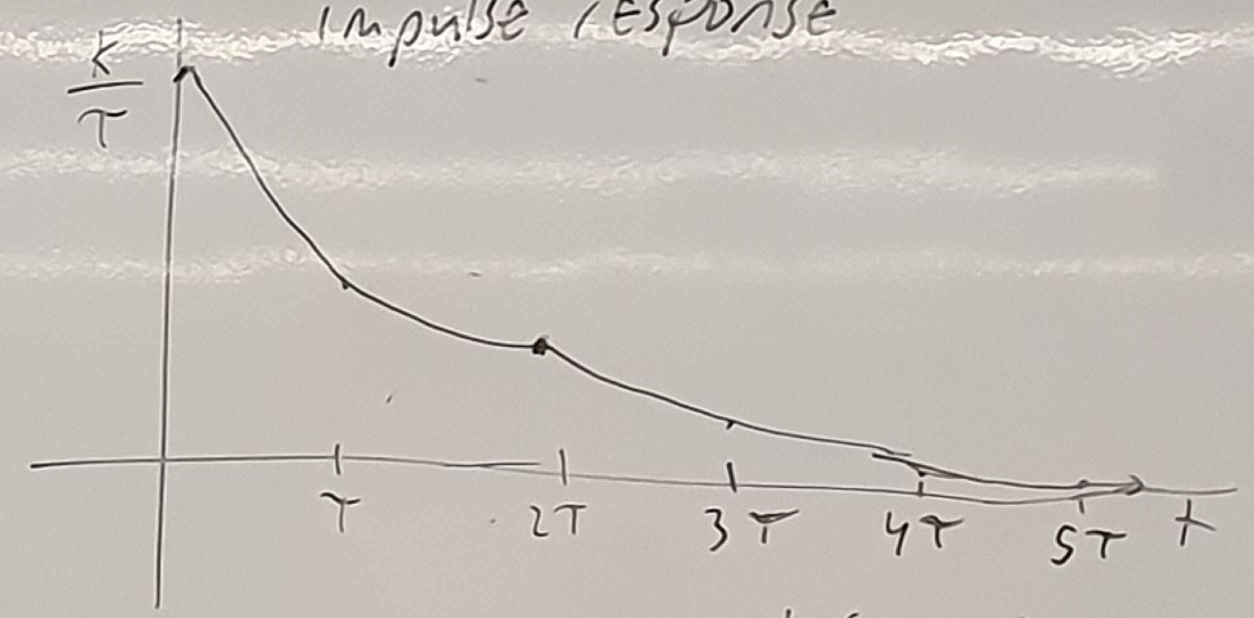
(e.x. RC circuit, RL circuit)

- impulse response:

$$y(t) = \mathcal{L}^{-1}(H(s)\cdot 1) = \frac{k}{\tau} e^{-\frac{t}{\tau}} \mathbb{1}(t)$$

$$\frac{k}{s\tau+1} = \frac{k}{\tau}\frac{1}{s+\frac{1}{\tau}}$$

impulse response



- step response:

$$y(t) = \mathcal{L}^{-1}\Big(H(s)\underbrace{\tfrac{1}{s}}_{\mathcal{L}(\mathbb{1}(t))}\Big)$$

$$Y(s) = H(s)\frac{1}{s} = \frac{k}{\tau}\frac{1}{s}\frac{1}{s+\frac{1}{\tau}} = \frac{k}{\cancel{\tau}}\left(\frac{\cancel{\tau}}{s} - \frac{\cancel{\tau}}{s+\frac{1}{\tau}}\right) = k\left(\frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}\right)$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}(Y(t)) = k\left(1 - e^{-\frac{t}{\tau}}\right)\mathbb{1}(t)$$