Problem 4.1

1) Since $\mathcal{F}[\text{sinc}(400t)] = \frac{1}{400}\Pi(\frac{f}{400})$, the bandwidth of the message signal is $W = 200$ and the resulting modulation index

$$\beta_f = \frac{k_f \max[|m(t)|]}{W} = \frac{k_f 10}{W} = 6 \Longrightarrow k_f = 120$$

Hence, the modulated signal is

$$\begin{aligned} u(t) &= A\cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau) \\ &= 100\cos(2\pi f_c t + +2\pi 1200 \int_{-\infty}^{t} \text{sinc}(400\tau)d\tau) \end{aligned}$$

2) The maximum frequency deviation of the modulated signal is

$$\Delta f_{\max} = \beta_f W = 6 \times 200 = 1200$$

3) Since the modulated signal is essentially a sinusoidal signal with amplitude $A = 100$, we have

$$P = \frac{A^2}{2} = 5000$$

4) Using Carson's rule, the effective bandwidth of the modulated signal can be approximated by

$$B_c = 2(\beta_f + 1)W = 2(6+1)200 = 2800 \text{ Hz}$$

Problem 4.2

1) The maximum phase deviation of the PM signal is

$$\Delta\phi_{\max} = k_p \max[|m(t)|] = k_p$$

The phase of the FM modulated signal is

$$\begin{aligned} \phi(t) &= 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau = 2\pi k_f \int_{0}^{t} m(\tau)d\tau \\ &= \begin{cases} 2\pi k_f \int_0^t \tau d\tau = \pi k_f t^2 & 0 \le t < 1 \\ \pi k_f + 2\pi k_f \int_1^t d\tau = \pi k_f + 2\pi k_f (t-1) & 1 \le t < 2 \\ \pi k_f + 2\pi k_f - 2\pi k_f \int_2^t d\tau = 3\pi k_f - 2\pi k_f (t-2) & 2 \le t < 3 \\ \pi k_f & 3 \le t \end{cases} \end{aligned}$$

The maximum value of $\phi(t)$ is achieved for $t = 2$ and is equal to $3\pi k_f$. Thus, the desired relation between k_p and k_f is

$$k_p = 3\pi k_f$$

2) The instantaneous frequency for the PM modulated signal is

$$f_i(t) = f_c + \frac{1}{2\pi}\frac{d}{dt}\phi(t) = f_c + \frac{1}{2\pi}k_p\frac{d}{dt}m(t)$$

For the $m(t)$ given in Fig. P-4.2, the maximum value of $\frac{d}{dt}m(t)$ is achieved for t in $[0,1]$ and it is equal to one. Hence,

$$\max(f_i(t)) = f_c + \frac{1}{2\pi}$$

For the FM signal $f_i(t) = f_c + k_f m(t)$. Thus, the maximum instantaneous frequency is

$$\max(f_i(t)) = f_c + k_f = f_c + 1$$

Problem 4.3

For an angle modulated signal we have $x(t) = A_c\cos(2\pi f_c t + \phi(t))$, therefore The lowpass equivalent of the signal is $x_l(t) = A_c e^{j\phi(t)}$ with Envelope A_c and phase $\pi(t)$ and in phase an quadrature components $A_c\cos(\phi(t))$ and $A_c\sin(\phi(t))$, respectively. Hence we have the following

$$\text{PM}\begin{cases} A_c & \text{envelope} \\ k_p m(t) & \text{phase} \\ A_c\cos\left(k_p m(t)\right) & \text{in-phase comp.} \\ A_c\sin\left(k_p m(t)\right) & \text{quadrature comp.} \end{cases} \qquad \text{FM}\begin{cases} A_c & \text{envelope} \\ 2\pi k_f\int_{-\infty}^{t} m(\tau)\,d\tau & \text{phase} \\ A_c\cos\left(2\pi k_f\int_{-\infty}^{t} m(\tau)\,d\tau\right) & \text{in-phase comp.} \\ A_c\sin\left(2\pi k_f\int_{-\infty}^{t} m(\tau)\,d\tau\right) & \text{quadrature comp.} \end{cases}$$

Problem 4.4

1) Since an angle modulated signal is essentially a sinusoidal signal with constant amplitude, we have

$$P = \frac{A_c^2}{2} \Longrightarrow P = \frac{100^2}{2} = 5000$$

The same result is obtained if we use the expansion

$$u(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta)\cos(2\pi(f_c + nf_m)t)$$

along with the identity

$$J_0^2(\beta) + 2\sum_{n=1}^{\infty} J_n^2(\beta) = 1$$

2) The maximum phase deviation is

$$\Delta\phi_{\max} = \max|4\sin(2000\pi t)| = 4$$

3) The instantaneous frequency is

$$\begin{aligned} f_i &= f_c + \frac{1}{2\pi}\frac{d}{dt}\phi(t) \\ &= f_c + \frac{4}{2\pi}\cos(2000\pi t)2000\pi = f_c + 4000\cos(2000\pi t) \end{aligned}$$

Hence, the maximum frequency deviation is

$$\Delta f_{\max} = \max|f_i - f_c| = 4000$$

4) The angle modulated signal can be interpreted both as a PM and an FM signal. It is a PM signal with phase deviation constant $k_p = 4$ and message signal $m(t) = \sin(2000\pi t)$ and it is an FM signal with frequency deviation constant $k_f = 4000$ and message signal $m(t) = \cos(2000\pi t)$.

Problem 4.6

1) If the output of the narrowband FM modulator is,

$$u(t) = A\cos(2\pi f_0 t + \phi(t))$$

then the output of the upper frequency multiplier ($\times n_1$) is

$$u_1(t) = A\cos(2\pi n_1 f_0 t + n_1\phi(t))$$

After mixing with the output of the second frequency multiplier $u_2(t) = A\cos(2\pi n_2 f_0 t)$ we obtain the signal

$$\begin{aligned} y(t) &= A^2\cos(2\pi n_1 f_0 t + n_1\phi(t))\cos(2\pi n_2 f_0 t) \\ &= \frac{A^2}{2}(\cos(2\pi(n_1+n_2)f_0 + n_1\phi(t)) + \cos(2\pi(n_1-n_2)f_0 + n_1\phi(t))) \end{aligned}$$

The bandwidth of the signal is $W = 15$ KHz, so the maximum frequency deviation is $\Delta f = \beta_f W = 0.1 \times 15 = 1.5$ KHz. In order to achieve a frequency deviation of $f = 75$ KHz at the output of the wideband modulator, the frequency multiplier n_1 should be equal to

$$n_1 = \frac{f}{\Delta f} = \frac{75}{1.5} = 50$$

Using an up-converter the frequency modulated signal is given by

$$y(t) = \frac{A^2}{2}\cos(2\pi(n_1+n_2)f_0 + n_1\phi(t))$$

Since the carrier frequency $f_c = (n_1+n_2)f_0$ is 104 MHz, n_2 should be such that

$$(n_1+n_2)100 = 104 \times 10^3 \Longrightarrow n_1+n_2 = 1040 \text{ or } n_2 = 990$$

2) The maximum allowable drift (d_f) of the 100 kHz oscillator should be such that

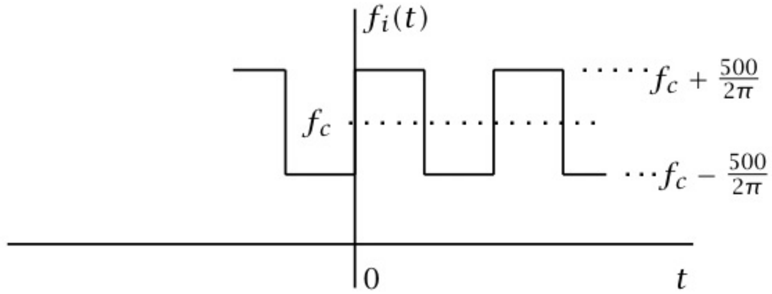
$$(n_1+n_2)d_f = 2 \Longrightarrow d_f = \frac{2}{1040} = .0019 \text{ Hz}$$

Problem 4.8

1) The instantaneous frequency is given by

$$f_i(t) = f_c + \frac{1}{2\pi}\frac{d}{dt}\phi(t) = f_c + \frac{1}{2\pi}100m(t)$$

A plot of $f_i(t)$ is given in the next figure



2) The peak frequency deviation is given by

$$\Delta f_{\max} = k_f \max[|m(t)|] = \frac{100}{2\pi}5 = \frac{250}{\pi}$$

Problem 4.9

1) The modulation index is

$$\beta = \frac{k_f \max[|m(t)|]}{f_m} = \frac{\Delta f_{\max}}{f_m} = \frac{20 \times 10^3}{10^4} = 2$$

The modulated signal $u(t)$ has the form

$$\begin{aligned} u(t) &= \sum_{n=-\infty}^{\infty} A_c J_n(\beta)\cos(2\pi(f_c + nf_m)t + \phi_n) \\ &= \sum_{n=-\infty}^{\infty} 100 J_n(2)\cos(2\pi(10^8 + n10^4)t + \phi_n) \end{aligned}$$

The power of the unmodulated carrier signal is $P = \frac{100^2}{2} = 5000$. The power in the frequency component $f = f_c + k10^4$ is

$$P_{f_c+kf_m} = \frac{100^2 J_k^2(2)}{2}$$

The next table shows the values of $J_k(2)$, the frequency $f_c + kf_m$, the amplitude $100J_k(2)$ and the power $P_{f_c+kf_m}$ for various values of k.

Index k	$J_k(2)$	Frequency Hz	Amplitude $100J_k(2)$	Power $P_{f_c+kf_m}$
0	.2239	10^8	22.39	250.63
1	.5767	$10^8 + 10^4$	57.67	1663.1
2	.3528	$10^8 + 2 \times 10^4$	35.28	622.46
3	.1289	$10^8 + 3 \times 10^4$	12.89	83.13
4	.0340	$10^8 + 4 \times 10^4$	3.40	5.7785

As it is observed from the table the signal components that have a power level greater than 500 (= 10% of the power of the unmodulated signal) are those with frequencies $10^8 + 10^4$ and $10^8 + 2 \times 10^4$. Since $J_n^2(\beta) = J_{-n}^2(\beta)$ it is conceivable that the signal components with frequency $10^8 - 10^4$ and $10^8 - 2 \times 10^4$ will satisfy the condition of minimum power level. Hence, there are four signal components that have a power of at least 10% of the power of the unmodulated signal. The components with frequencies $10^8 + 10^4$, $10^8 - 10^4$ have an amplitude equal to 57.67, whereas the signal components with frequencies $10^8 + 2 \times 10^4$, $10^8 - 2 \times 10^4$ have an amplitude equal to 35.28.

2) Using Carson's rule, the approximate bandwidth of the FM signal is

$$B_c = 2(\beta + 1)f_m = 2(2 + 1)10^4 = 6 \times 10^4 \text{ Hz}$$

Problem 4.11

1) The PM modulated signal is

$$\begin{aligned} u(t) &= 100\cos(2\pi f_c t + \frac{\pi}{2}\cos(2\pi 1000t)) \\ &= \sum_{n=-\infty}^{\infty} 100 J_n\left(\frac{\pi}{2}\right)\cos(2\pi(10^8 + n10^3)t) \end{aligned}$$

The next table tabulates $J_n(\beta)$ for $\beta = \frac{\pi}{2}$ and $n = 0, \ldots, 4$.

n	0	1	2	3	4
$J_n(\beta)$	.4720	.5668	.2497	.0690	.0140

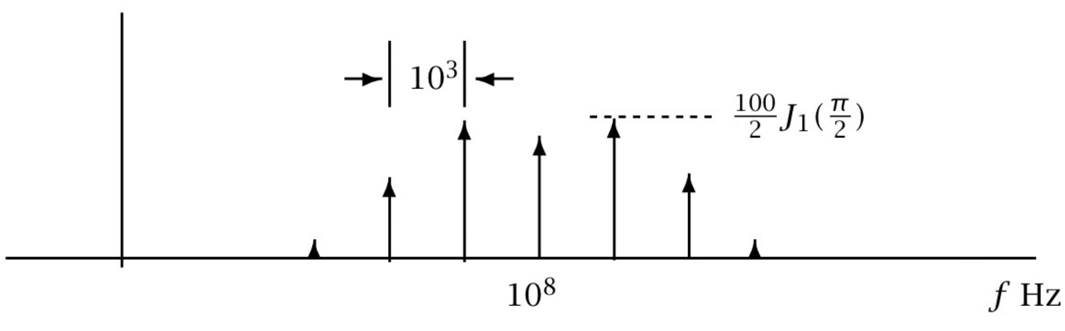
The total power of the modulated signal is $P_{\text{tot}} = \frac{100^2}{2} = 5000$. To find the effective bandwidth of the signal we calculate the index k such that

$$\sum_{n=-k}^{k} \frac{100^2}{2} J_n^2\left(\frac{\pi}{2}\right) \geq 0.99 \times 5000 \Longrightarrow \sum_{n=-k}^{k} J_n^2\left(\frac{\pi}{2}\right) \geq 0.99$$

By trial end error we find that the smallest index k is 2. Hence the effective bandwidth is

$$B_{\text{eff}} = 4 \times 10^3 = 4000$$

In the the next figure we sketch the magnitude spectrum for the positive frequencies.



2) Using Carson's rule, the approximate bandwidth of the PM signal is

$$B_{\text{PM}} = 2(\beta_p + 1)f_m = 2(\frac{\pi}{2} + 1)1000 = 5141.6$$

As it is observed, Carson's rule overestimates the effective bandwidth allowing in this way some margin for the missing harmonics.

Problem 4.13

1) If the signal $m(t) = m_1(t) + m_2(t)$ DSB modulates the carrier $A_c \cos(2\pi f_c t)$ the result is the signal

$$\begin{aligned} u(t) &= A_c m(t) \cos(2\pi f_c t) \\ &= A_c (m_1(t) + m_2(t)) \cos(2\pi f_c t) \\ &= A_c m_1(t) \cos(2\pi f_c t) + A_c m_2(t) \cos(2\pi f_c t) \\ &= u_1(t) + u_2(t) \end{aligned}$$

where $u_1(t)$ and $u_2(t)$ are the DSB modulated signals corresponding to the message signals $m_1(t)$ and $m_2(t)$. Hence, AM modulation satisfies the superposition principle.

2) If $m(t)$ frequency modulates a carrier $A_c \cos(2\pi f_c t)$ the result is

$$\begin{aligned} u(t) &= A_c \cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{\infty} (m_1(\tau) + m_2(\tau)) d\tau) \\ &\neq A_c \cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{\infty} m_1(\tau) d\tau) \\ &\quad + A_c \cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{\infty} m_2(\tau) d\tau) \\ &= u_1(t) + u_2(t) \end{aligned}$$

where the inequality follows from the nonlinearity of the cosine function. Hence, angle modulation is not a linear modulation method.

Problem 4.17

Using Carson's rule we obtain

$$B_c = 2(\beta + 1)W = 2(\frac{k_f \max[|m(t)|]}{W} + 1)W = \begin{cases} 20020 & k_f = 10 \\ 20200 & k_f = 100 \\ 22000 & k_f = 1000 \end{cases}$$