Problem 2.5
Using the result of problem 2.4 we have:

1. The frequencies are 2000 and 5500, their ratio (and therefore the ratio of the periods) is rational, hence the sum is periodic.
2. The frequencies are 2000 and $\frac{5500}{\pi}$. Their ratio is not rational, hence the sum is not periodic.
3. The sum of two periodic discrete-time signal is periodic.

Problem 2.7
For the first two questions we will need the integral $I = \int e^{ax} \cos^2 x dx$.

$$\begin{aligned}
I &= \frac{1}{a}\int \cos^2 x \, de^{ax} = \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a}\int e^{ax}\sin 2x \, dx \\
&= \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a^2}\int \sin 2x \, de^{ax} \\
&= \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a^2}e^{ax}\sin 2x - \frac{2}{a^2}\int e^{ax}\cos 2x \, dx \\
&= \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a^2}e^{ax}\sin 2x - \frac{2}{a^2}\int e^{ax}(2\cos^2 x - 1) \, dx \\
&= \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a^2}e^{ax}\sin 2x - \frac{2}{a^2}\int e^{ax} \, dx - \frac{4}{a^2}I
\end{aligned}$$

Thus,

$$I = \frac{1}{4+a^2}\left[(a\cos^2 x + \sin 2x) + \frac{2}{a}\right]e^{ax}$$

2)

$$\begin{aligned}
E_x &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_2^2(t)dx = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-2t}\cos^2 t dt \\
&= \lim_{T\to\infty}\left[\int_{-\frac{T}{2}}^{0} e^{-2t}\cos^2 t dt + \int_{0}^{\frac{T}{2}} e^{-2t}\cos^2 t dt\right]
\end{aligned}$$

But,

$$\begin{aligned}
\lim_{T\to\infty}\int_{-\frac{T}{2}}^{0} e^{-2t}\cos^2 t dt &= \lim_{T\to\infty}\frac{1}{8}\left[(-2\cos^2 t + \sin 2t) - 1\right]e^{-2t}\Big|_{-\frac{T}{2}}^{0} \\
&= \lim_{T\to\infty}\frac{1}{8}\left[-3 + (2\cos^2\frac{T}{2} + 1 + \sin T)e^{T}\right] = \infty
\end{aligned}$$

since $2 + \cos\theta + \sin\theta > 0$. Thus, $E_x = \infty$ since as we have seen from the first question the second integral is bounded. Hence, the signal $x_2(t)$ is not an energy-type signal. To test if $x_2(t)$ is a power-type signal we find P_x.

$$P_x = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{0} e^{-2t}\cos^2 dt + \lim_{T\to\infty}\frac{1}{T}\int_{0}^{\frac{T}{2}} e^{-2t}\cos^2 dt$$

But $\lim_{T\to\infty} \frac{1}{T}\int_0^{\frac{T}{2}} e^{-2t}\cos^2 dt$ is zero and

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{0} e^{-2t}\cos^2 dt &= \lim_{T\to\infty}\frac{1}{8T}\left[2\cos^2\frac{T}{2}+1+\sin T\right]e^{T} \\ &> \lim_{T\to\infty}\frac{1}{T}e^{T} > \lim_{T\to\infty}\frac{1}{T}(1+T+T^2) > \lim_{T\to\infty} T = \infty\end{aligned}$$

Thus the signal $x_2(t)$ is not a power-type signal.

3)

$$\begin{aligned}E_x &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_3^2(t)dx = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} \mathrm{sgn}^2(t)dt = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt = \lim_{T\to\infty} T = \infty \\ P_x &= \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \mathrm{sgn}^2(t)dt = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt = \lim_{T\to\infty}\frac{1}{T}T = 1\end{aligned}$$

The signal $x_3(t)$ is of the power-type and the power content is 1.

4)

First note that

$$\lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} A\cos(2\pi f t)dt = \sum_{k=-\infty}^{\infty} A\int_{k-\frac{1}{2f}}^{k+\frac{1}{2f}}\cos(2\pi f t)dt = 0$$

so that

$$\begin{aligned}\lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} A^2\cos^2(2\pi f t)dt &= \lim_{T\to\infty}\frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}}(A^2 + A^2\cos(2\pi 2 f t))dt \\ &= \lim_{T\to\infty}\frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 dt = \lim_{T\to\infty}\frac{1}{2}A^2 T = \infty\end{aligned}$$

$$\begin{aligned}E_x &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}}(A^2\cos^2(2\pi f_1 t) + B^2\cos^2(2\pi f_2 t) + 2AB\cos(2\pi f_1 t)\cos(2\pi f_2 t))dt \\ &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} A^2\cos^2(2\pi f_1 t)dt + \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} B^2\cos^2(2\pi f_2 t)dt + \\ &\quad AB\lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}}[\cos^2(2\pi(f_1+f_2) + \cos^2(2\pi(f_1-f_2)]dt \\ &= \infty + \infty + 0 = \infty\end{aligned}$$

Thus the signal is not of the energy-type. To test if the signal is of the power-type we consider two cases $f_1 = f_2$ and $f_1 \neq f_2$. In the first case

$$\begin{aligned}P_x &= \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}(A+B)^2\cos^2(2\pi f_1)dt \\ &= \lim_{T\to\infty}\frac{1}{2T}(A+B)^2\int_{-\frac{T}{2}}^{\frac{T}{2}} dt = \frac{1}{2}(A+B)^2\end{aligned}$$

If $f_1 \neq f_2$ then

$$
\begin{aligned}
P_x &= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} (A^2 \cos^2(2\pi f_1 t) + B^2 \cos^2(2\pi f_2 t) + 2AB \cos(2\pi f_1 t)\cos(2\pi f_2 t))dt \\
&= \lim_{T\to\infty} \frac{1}{T} \left[\frac{A^2 T}{2} + \frac{B^2 T}{2} \right] = \frac{A^2}{2} + \frac{B^2}{2}
\end{aligned}
$$

Thus the signal is of the power-type and if $f_1 = f_2$ the power content is $(A+B)^2/2$ whereas if $f_1 \neq f_2$ the power content is $\frac{1}{2}(A^2 + B^2)$

Problem 2.16

1) Nonlinear, since the response to $x(t) = 0$ is not $y(t) = 0$ (this is a necessary condition for linearity of a system, see also problem 2.21).

2) Nonlinear, if we multiply the input by constant -1, the output does not change. In a linear system the output should be scaled by -1.

4) Nonlinear, the output to $x(t) = 0$ is not zero.

5) Nonlinear. The system is not homogeneous for if $\alpha < 0$ and $x(t) > 0$ then $y(t) = T[\alpha x(t)] = 0$ whereas $z(t) = \alpha T[x(t)] = \alpha$.

6) Linear. For if $x(t) = \alpha x_1(t) + \beta x_2(t)$ then

$$
\begin{aligned}
T[\alpha x_1(t) + \beta x_2(t)] &= (\alpha x_1(t) + \beta x_2(t))e^{-t} \\
&= \alpha x_1(t)e^{-t} + \beta x_2(t)e^{-t} = \alpha T[x_1(t)] + \beta T[x_2(t)]
\end{aligned}
$$

7) Linear. For if $x(t) = \alpha x_1(t) + \beta x_2(t)$ then

$$\begin{aligned} T[\alpha x_1(t) + \beta x_2(t)] &= (\alpha x_1(t) + \beta x_2(t))u(t) \\ &= \alpha x_1(t)u(t) + \beta x_2(t)u(t) = \alpha T[x_1(t)] + \beta T[x_2(t)] \end{aligned}$$

8) Linear. We can write the output of this feedback system as

$$y(t) = x(t) + y(t-1) = \sum_{n=0}^{\infty} x(t-n)$$

Then for $x(t) = \alpha x_1(t) + \beta x_2(t)$

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} (\alpha x_1(t-n) + \beta x_2(t-n)) \\ &= \alpha \sum_{n=0}^{\infty} x_1(t-n) + \beta \sum_{n=0}^{\infty} x_2(t-n)) \\ &= \alpha y_1(t) + \beta y_2(t) \end{aligned}$$

9) Linear. Assuming that only a finite number of jumps occur in the interval $(-\infty, t]$ and that the magnitude of these jumps is finite so that the algebraic sum is well defined, we obtain

$$y(t) = T[\alpha x(t)] = \sum_{n=1}^{N} \alpha J_x(t_n) = \alpha \sum_{n=1}^{N} J_x(t_n) = \alpha T[x(t)]$$

where N is the number of jumps in $(-\infty, t]$ and $J_x(t_n)$ is the value of the jump at time instant t_n, that is

$$J_x(t_n) = \lim_{\epsilon \to 0} (x(t_n + \epsilon) - x(t_n - \epsilon))$$

For $x(t) = x_1(t) + x_2(t)$ we can assume that $x_1(t)$, $x_2(t)$ and $x(t)$ have the same number of jumps and at the same positions. This is true since we can always add new jumps of magnitude zero to the already existing ones. Then for each t_n, $J_x(t_n) = J_{x_1}(t_n) + J_{x_2}(t_n)$ and

$$y(t) = \sum_{n=1}^{N} J_x(t_n) = \sum_{n=1}^{N} J_{x_1}(t_n) + \sum_{n=1}^{N} J_{x_2}(t_n)$$

so that the system is additive.

Problem 2.39

1) Using Euler's relation we have

$$\begin{aligned} x_1(t) &= \cos(2\pi t) + \cos(4\pi t) \\ &= \frac{1}{2}\left(e^{i2\pi t} + e^{-j2\pi t} + e^{j4\pi t} + e^{-j4\pi t}\right) \end{aligned}$$

Therefore for $n = \pm 1, \pm 2$, $x_{1,n} = \frac{1}{2}$ and for all other values of n, $x_{1,n} = 0$.

2) Using Euler's relation we have

$$\begin{aligned} x_2(t) &= \cos(2\pi t) - \cos(4\pi t + \pi/3) \\ &= \frac{1}{2}\left(e^{i2\pi t} + e^{-j2\pi t} - e^{j(4\pi t+\pi/3)} - e^{-j(4\pi t+\pi/3)}\right) \\ &= \frac{1}{2}e^{i2\pi t} + \frac{1}{2}e^{-j2\pi t} + \frac{1}{2}e^{-j2\pi/3}e^{j4\pi t} + \frac{1}{2}e^{j2\pi/3}e^{-j4\pi t} \end{aligned}$$

from this we conclude that $x_{2,\pm 1} = \frac{1}{2}$ and $x_{2,2} = x_{2,-2}^* = \frac{1}{2}e^{-j2\pi/3}$, and for all other values of n, $x_{2,n} = 0$.

3) We have $x_3(t) = 2\cos(2\pi t) - \sin(4\pi t) = 2\cos(2\pi t) + \cos(4\pi t + \pi/2)$. Using Euler's relation as in parts 1 and 2 we see that $x_{3,\pm 1} = 1$ and $x_{3,2} = x_{3,-2}^* = j$, and for all other values of n, $x_{3,n} = 0$.

4) The signal $x_4(t)$ is periodic with period $T_0 = 2$. Thus

$$\begin{aligned} x_{4,n} &= \frac{1}{2}\int_{-1}^{1}\Lambda(t)e^{-j2\pi\frac{n}{2}t}dt = \frac{1}{2}\int_{-1}^{1}\Lambda(t)e^{-j\pi n t}dt \\ &= \frac{1}{2}\int_{-1}^{0}(t+1)e^{-j\pi n t}dt + \frac{1}{2}\int_{0}^{1}(-t+1)e^{-j\pi n t}dt \\ &= \frac{1}{2}\left(\frac{j}{\pi n}te^{-j\pi n t} + \frac{1}{\pi^2 n^2}e^{-j\pi n t}\right)\Bigg|_{-1}^{0} + \frac{j}{2\pi n}e^{-j\pi n t}\Bigg|_{-1}^{0} \\ &\quad -\frac{1}{2}\left(\frac{j}{\pi n}te^{-j\pi n t} + \frac{1}{\pi^2 n^2}e^{-j\pi n t}\right)\Bigg|_{0}^{1} + \frac{j}{2\pi n}e^{-j\pi n t}\Bigg|_{0}^{1} \\ &\quad \frac{1}{\pi^2 n^2} - \frac{1}{2\pi^2 n^2}(e^{j\pi n} + e^{-j\pi n}) = \frac{1}{\pi^2 n^2}(1 - \cos(\pi n)) \end{aligned}$$

When $n = 0$ then

$$x_{4,0} = \frac{1}{2}\int_{-1}^{1}\Lambda(t)dt = \frac{1}{2}$$

Thus

$$x_4(t) = \frac{1}{2} + 2\sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2}(1 - \cos(\pi n))\cos(\pi n t)$$

5) The signal $x_5(t)$ is periodic with period $T_0 = 1$. For $n = 0$

$$x_{5,0} = \int_0^1 (-t+1)dt = (-\frac{1}{2}t^2 + t)\Big|_0^1 = \frac{1}{2}$$

For $n \neq 0$

$$\begin{aligned}
x_{5,n} &= \int_0^1 (-t+1)e^{-j2\pi n t}dt \\
&= -\left(\frac{j}{2\pi n}te^{-j2\pi n t} + \frac{1}{4\pi^2 n^2}e^{-j2\pi n t}\right)\Big|_0^1 + \frac{j}{2\pi n}e^{-j2\pi n t}\Big|_0^1 \\
&= -\frac{j}{2\pi n}
\end{aligned}$$

Thus,

$$x_5(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{\pi n}\sin 2\pi n t$$

6) The signal $x_6(t)$ is real even and periodic with period $T_0 = \frac{1}{2f_0}$. Hence, $x_{6,n} = a_{8,n}/2$ or

$$\begin{aligned}
x_{6,n} &= 2f_0 \int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0 t)\cos(2\pi n 2 f_0 t)dt \\
&= f_0 \int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1+2n)t)dt + f_0 \int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1-2n)t)dt \\
&= \frac{1}{2\pi(1+2n)}\sin(2\pi f_0(1+2n)t)\,|_{\frac{1}{4f_0}}^{\frac{1}{4f_0}} + \frac{1}{2\pi(1-2n)}\sin(2\pi f_0(1-2n)t)\,|_{\frac{1}{4f_0}}^{\frac{1}{4f_0}} \\
&= \frac{(-1)^n}{\pi}\left[\frac{1}{(1+2n)} + \frac{1}{(1-2n)}\right]
\end{aligned}$$

Problem 2.46

1) Using the Fourier transform pair

$$e^{-\alpha|t|} \xrightarrow{\mathcal{F}} \frac{2\alpha}{\alpha^2 + (2\pi f)^2} = \frac{2\alpha}{4\pi^2} \frac{1}{\frac{\alpha^2}{4\pi^2} + f^2}$$

and the duality property of the Fourier transform: $X(f) = \mathcal{F}[x(t)] \Rightarrow x(-f) = \mathcal{F}[X(t)]$ we obtain

$$\left(\frac{2\alpha}{4\pi^2}\right) \mathcal{F}\left[\frac{1}{\frac{\alpha^2}{4\pi^2} + t^2}\right] = e^{-\alpha|f|}$$

With $\alpha = 2\pi$ we get the desired result

$$\mathcal{F}\left[\frac{1}{1+t^2}\right] = \pi e^{-2\pi|f|}$$

2)

$$\begin{aligned}
\mathcal{F}[x(t)] &= \mathcal{F}[\Pi(t-3) + \Pi(t+3)] \\
&= \text{sinc}(f)e^{-j2\pi f3} + \text{sinc}(f)e^{j2\pi f3} \\
&= 2\text{sinc}(f)\cos(2\pi 3f)
\end{aligned}$$

3) $\mathcal{F}[\Pi(t/4)] = 4\,\text{sinc}(4f)$, hence $\mathcal{F}[4\Pi(t/4)] = 16\,\text{sinc}(4f)$. Using modulation property of FT we have $\mathcal{F}[4\Pi(t/4)\cos(2\pi f_0 t)] = 8\,\text{sinc}(4(f-f_0)) + 8\,\text{sinc}(4(f+f_0))$.

4)

$$\mathcal{F}[t\text{sinc}(t)] = \frac{1}{\pi}\mathcal{F}[\sin(\pi t)] = \frac{j}{2\pi}\left[\delta(f+\frac{1}{2}) - \delta(f-\frac{1}{2})\right]$$

The same result is obtain if we recognize that multiplication by t results in differentiation in the frequency domain. Thus

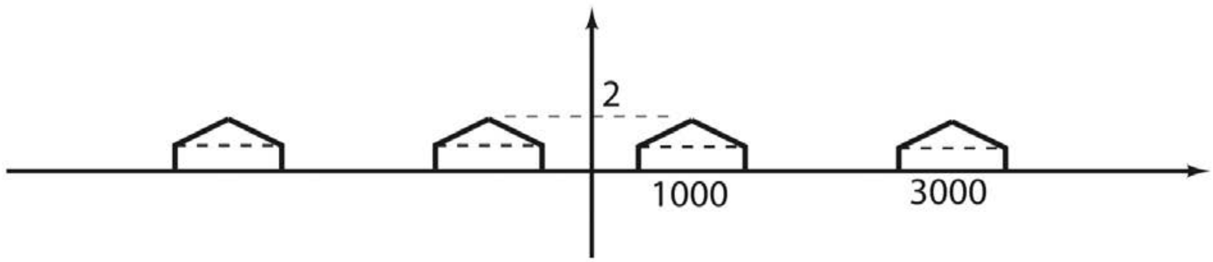
$$\mathcal{F}[t\text{sinc}] = \frac{j}{2\pi}\frac{d}{df}\Pi(f) = \frac{j}{2\pi}\left[\delta(f+\frac{1}{2}) - \delta(f-\frac{1}{2})\right]$$

5)

$$\begin{aligned}
\mathcal{F}[t\cos(2\pi f_0 t)] &= \frac{j}{2\pi}\frac{d}{df}\left(\frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)\right) \\
&= \frac{j}{4\pi}\left(\delta'(f-f_0) + \delta'(f+f_0)\right)
\end{aligned}$$

Problem 2.47

$x_1(t) = -x(t) + x(t)\cos(2000\pi t) + x(t)\,(1+\cos(6000\pi t))$ or $x_1(t) = x(t)\cos(2000\pi t) + x(t)\cos(6000\pi t)$. Using modulation property, we have $X_1(f) = \frac{1}{2}X(f-1000) + \frac{1}{2}X(f+1000) + \frac{1}{2}X(f-3000) + \frac{1}{2}X(f+3000)$. The plot is given below:



Problem 2.50

(Convolution theorem:)

$$\mathcal{F}[x(t) \star y(t)] = \mathcal{F}[x(t)]\mathcal{F}[y(t)] = X(f)Y(f)$$

Thus

$$\begin{aligned}
\text{sinc}(t) \star \text{sinc}(t) &= \mathcal{F}^{-1}[\mathcal{F}[\text{sinc}(t) \star \text{sinc}(t)]] \\
&= \mathcal{F}^{-1}[\mathcal{F}[\text{sinc}(t)] \cdot \mathcal{F}[\text{sinc}(t)]] \\
&= \mathcal{F}^{-1}[\Pi(f)\Pi(f)] = \mathcal{F}^{-1}[\Pi(f)] \\
&= \text{sinc}(t)
\end{aligned}$$

Problem 2.51

$$\begin{aligned}
\mathcal{F}[x(t)y(t)] &= \int_{-\infty}^{\infty} x(t)y(t)e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} X(\theta)e^{j2\pi\theta t}d\theta\right) y(t)e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} X(\theta)\left(\int_{-\infty}^{\infty} y(t)e^{-j2\pi(f-\theta)t}dt\right)d\theta \\
&= \int_{-\infty}^{\infty} X(\theta)Y(f-\theta)d\theta = X(f) \star Y(f)
\end{aligned}$$