Zero Divisor Graphs of Direct Systems of Abelian Groups

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Abstract. Let S be a commutative semigroup with zero ($0x = 0$ for all $x \in S$). In [2] the authors associated a graph to S. The vertices are the nonzero elements of S, with two vertices a, b joined by an edge in case $ab = 0$. We associate a commutative semigroup with zero to any direct system of abelian groups over a lower semilattice with least element. We use the associated graph to construct a number of classical graphs.

Definition 1. A commutative semigroup with zero is a set S with an associative binary operation such that

I. $x \cdot y \in S \ \forall x, y \in S$
II. $x \cdot y = y \cdot x \ \forall x, y \in S$
III. $\exists 0 \in S$ such that $x \cdot 0 = 0 \ \forall x \in S$.

Lemma 1. If S is a commutative semigroup with zero, the zero is unique.

Proof. Let S be a commutative semigroup with zero. Suppose there are two such zeros $a, b \in S$. Then $ax = a = xa$ and $bx = b = xb$, $\forall x \in S$ by definition of zero and by multiplicative commutativity in S.
Since

$$a = ax \ \forall x \in S, a = ab.$$

Furthermore,

$$xb = b \ \forall x \in S, ab = b.$$

Thus, $a = ab = b$. Hence, the zero is unique. □

Definition 2. A zero-divisor graph, $\Gamma(S)$, of a commutative semigroup, S, is a graph where the vertices of the graph are the nonzero zero-divisors of S, and two vertices x, y are connected by an edge in case $x \cdot y = 0$ in S. These graphs were explored in [2].

Definition 3. A poset with a least element is a nonempty set S together with a relation, $\leq$, on S x S which is reflexive, antisymmetric, transitive, and that also contains an element $\lambda \in S$, such that $\forall x \in S$, $\lambda \leq x$.

Example 1. An example of a poset with a least element is the set

$$P = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{a\}, \{b\}, \{a,b\}\}$$

where given $P_1, P_2 \in P$, $P_1 \leq P_2$ if $P_1 \subseteq P_2$.

Definition 4. A lower semilattice with least element is a poset with a unique least element such that any finite nonempty subset has a greatest lower bound.

Lemma 2. Let I be a lower semilattice with least element. If i, j, k are elements of I, then glb(i, glb(j, k)) = glb(j, glb(i, k)) = glb(k, glb(i, j)).

Proof. By definition of greatest lower bound, glb(i, glb(j, k)) $\leq$ glb(j, k) $\leq$ j. Similarly, glb(i, glb(j, k)) $\leq$ i and glb(i, glb(j, k)) $\leq$ glb(j,k) $\leq$ k, thus glb(i, glb(j, k)) $\leq$ glb(i,k). Hence, glb(i, glb(j,k)) $\leq$ glb(j, glb(i, k)).

Of course, it can be similarly shown that glb(j, glb(i, k)) $\leq$ glb(k, glb(i, j)), glb(k, glb(i, j)) $\leq$ glb(j, glb(i, k)), and glb(j, glb(i, k)) $\leq$ glb(i, glb(j, k)).

Thus, glb(i, glb(j, k)) $\leq$ glb(j, glb(i, k)) $\leq$ glb(k, glb(i, j)) and glb(k, glb(i, j)) $\leq$ glb(j, glb(i, k)) $\leq$ glb(i, glb(j, k)). Therefore, glb(i, glb(j, k)) = glb(j, glb(i, k)) = glb(k, glb(i, j)). $\square$

Definition 5. A morphism, $f : S \to T$, of lower semilattices, S and T, with least elements, λ_S and λ_T, respectively, satisfies the following property:

$$\forall s_1, s_2 \in S,\ s_1 \leq s_2 \Leftrightarrow f(s_1) \leq f(s_2)$$

Theorem 1. If $f : S \to T$, is a morphism of lower semilattices, S, T, with least elements, λ_S and λ_T, respectively, $f(S)$ is also a lower semilattice with least element $f(\lambda_S)$. More specifically, if $s_1, s_2 \in S$, then $f(glb(s_1, s_2)) = glb(f(s_1, s_2))$.

Proof. Take any two elements of $f(S)$, say $f(s_1)$ and $f(s_2)$. Since

$$glb(s_1, s_2) \leq s_1,$$

$$f(glb(s_1, s_2)) \leq f(s_1)$$

by the definition of the morphism of lower semilattices. Similarly, it can be argued that

$$glb(s_1, s_2) \leq s_2,$$

implies that

$$f(glb(s_1, s_2)) \leq f(s_2).$$

So $f(glb(s_1, s_2))$ is a lower bound of $f(s_1)$ and $f(s_2)$. We will now show that it is the *greatest* lower bound. Suppose that L_2 is another lower bound of $f(s_1)$ and $f(s_2)$. Since $S \to f(S)$ is onto, there exists an element in S, say L_1 such that $f(L_1) = L_2$. Since L_2 is a lower bound of $f(s_1)$ and $f(s_2)$ and $f(L_1) = L_2$, it follows that $f(L_1) \leq f(s_1)$ and $f(L_1) \leq f(s_2)$. Thus, by the definition of morphisms on lower semilattices, $L_1 \leq s_1$ and $L_1 \leq s_2$. Thus, L_1 is a lower bound of s_1 and s_2. By the definition of greatest lower bound, $L_1 \leq glb(s_1, s_2)$. By applying the defintion of morphisms of lower semilattices, we know that $f(L_1) \leq f(glb(s_1, s_2))$. Since $L_2 = f(L_1)$, it follows that $L_2 \leq f(glb(s_1, s_2))$. This implies that $f(glb(s_1, s_2))$ is the greatest lower bound of $f(s_1)$ and $f(s_2)$. Thus, given any two elements of $f(S)$, there will always exist a greatest lower bound.
We will now show that $f(S)$ has least element $f(\lambda_S)$. Let $L_4 \in f(S)$. Since $S \to f(S)$ is onto, there exists an element in S, say L_3 such that $f(L_3) = L_4$. By definition of least element, $\lambda_S \leq L_3$. By definition of morphisms of lower semilattices, $f(\lambda_S) \leq f(L_3) = L_4$. Since L_4 was chosen arbitrarily from $f(S)$, $f(\lambda_S)$ is the least element in $f(S)$.
Therefore, $f(S)$ is a lower semilattice with least element $f(\lambda_S)$. □

Definition 6. Let I be a lower semilattice with least element. Let $\{G_i\}_{i\in I}$ be a family of abelian groups, and suppose we have a family of group homomorphisms f_{ji}: $G_j \to G_i$ for all $i \leq j$ with the following properties:

I. f_{ii} is the identity in G_i
II. $f_{ki} = f_{ji} \circ f_{kj}$ for all $i \leq j \leq k$
Then the triple

$$(I, \{G_i\}_{i\in I}, \{f_{ji}\}_{\substack{i,j\in I \\ i\leq j}})$$

is called an direct system of abelian groups over a lower semilattice with least element.

Definition 7. The class of all direct systems of abelian groups over a lower semilattice with least element is a $\mathcal{DSAG}$. Given $\mathcal{F}$,$\mathcal{G} \in \mathcal{DSAG}$, define $hom(\mathcal{F}, \mathcal{G})$ to be the set whose elements are the pairs $(\theta, \{\gamma_{i,\theta(i)}\}_{i\in I})$ where:

I. $\theta : I \to J$ preserves order and greatest lower bound. Thus,

$$i_1 \leq i_2 \iff \theta(i_1) \leq \theta(i_2)$$

II. For each $i \in I$, $\gamma_{i,\theta(i)} : F_i \to G_{\theta(i)}$ is a group homomorphism such that whenever $i_1, i_2 \in I$ with $i_1 \geq i_2$ the following diagram commutes:

$$\begin{tikzcd}
F_{i_1} \arrow[r, "\gamma_{i_1,\theta(i_1)}"] \arrow[d, "f_{i_1,i_2}"'] & G_{\theta(i_1)} \arrow[d, "g_{\theta(i_1),\theta(i_2)}"] \\
F_{i_2} \arrow[r, "\gamma_{i_2,\theta(i_2)}"'] & G_{\theta(i_2)}
\end{tikzcd}$$

Diagram 1

Theorem 2. $\mathcal{DSAG}$ is a category.

Proof. Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, $\mathcal{M} \in \mathcal{DSAG}$ and

$$(\theta, \{\gamma_{i,\theta(i)}\}_{i\in I}) : \mathcal{F} \to \mathcal{G} \in hom(\mathcal{F}, \mathcal{G})$$

$$(\mu, \{\delta_{j,\mu(j)}\}_{j\in J}) : \mathcal{G} \to \mathcal{H} \in hom(\mathcal{G}, \mathcal{H})$$

$$(\nu, \{\epsilon_{k,\nu(k)}\}_{k\in K}) : \mathcal{H} \to \mathcal{M} \in hom(\mathcal{H}, \mathcal{M})$$

I. Associativity

$$\begin{aligned}
&(\nu, \{\epsilon_{k,\nu(k)}\}_{k\in K}) \circ ((\mu, \{\delta_{j,\mu(j)}\}_{j\in J} \circ (\theta, \{\gamma_{i,\theta(i)}\}_{i\in I})) \\
&\quad = (\nu, \{\epsilon_{k,\nu(k)}\}_{k\in K}) \circ (\mu \circ \theta, \{\delta_{\theta(i),\mu(\theta(i))} \circ \gamma_{i,\theta(i)}\}_{i\in I}) \\
&\quad = (\nu \circ (\mu \circ \theta), \{\epsilon_{\mu(\theta(i)),\nu(\mu(\theta(i)))} \circ (\delta_{\theta(i),\mu(\theta(i))} \circ \gamma_{i,\theta(i)})\}_{i\in I}) \\
&\quad = ((\nu \circ \mu) \circ \theta, \{(\epsilon_{\mu(\theta(i)),\nu(\mu(\theta(i)))} \circ \delta_{\theta(i),\mu(\theta(i))}) \circ \gamma_{i,\theta(i)}\}_{i\in I}) \\
&\quad = (\nu \circ \mu, \{\epsilon_{\mu(j),\nu(\mu(j))} \circ \delta_{j,\mu(j)}\}_{j\in J}) \circ (\theta, \{\gamma_{i,\theta(i)}\}_{i\in I}) \\
&\quad = ((\nu, \{\epsilon_{k,\nu(k)}\}_{k\in K}) \circ (\mu, \{\delta_{j,\mu(j)}\}_{j\in J})) \circ (\theta, \{\gamma_{i,\theta(i)}\}_{i\in I}).
\end{aligned}$$

II. Identity

For each object $\mathcal{G} \in \mathcal{DSAG}$ let $1_{\mathcal{G}} = (\iota, \{\tau_{j,\iota(j)}\}_{j\in J}) : \mathcal{G} \to \mathcal{G}$ where

$$\iota : J \to J, \iota(j) = j, \forall j \in J$$

and

$$\tau_{j,\iota(j)} : G_j \to G_j, \tau_{j,\iota(j)}(a) = a, \forall a \in G_j.$$

Thus, $(\iota, \{\tau_{j,\iota(j)}\}_{j\in J}) \circ (\theta, \{\gamma_{i,\theta(i)}\}_{i\in I}) = (\iota \circ \theta, \{\tau_{\theta(i),\iota(\theta(i))} \circ \gamma_{i,\theta(i)}\}_{i\in I})$

$$= (\theta, \{\gamma_{i,\theta(i)}\}_{i\in I}).$$

Likewise, $(\mu, \{\delta_{j,\mu(j)}\}_{j\in J}) \circ (\iota, \{\tau_{j,\iota(j)}\}_{j\in J}) = (\mu \circ \iota, \{\delta_{\iota(j),\mu(\iota(j))} \circ \tau_{j,\iota(j)}\}_{j\in J})$
$= (\mu, \{\delta_{j,\mu(j)}\}_{j\in J})$.

Thus, $\mathcal{DSAG}$ is a category. $\square$

Lemma 3. Let $\mathcal{F}$,$\mathcal{G} \in \mathcal{DSAG}$ and

$$(\theta, {\gamma_{i,\theta(i)}}_{i\in I}) \in hom(\mathcal{F}, \mathcal{G})$$

The image of $\mathcal{F}$ under $(\theta, {\gamma_{i,\theta(i)}}_{i\in I})$ is in $\mathcal{DSAG}$.

Proof. The image of $\mathcal{F}$ under $(\theta, {\gamma_{i,\theta(i)}}_{i\in I})$ is

$$(\theta(I), \{\gamma_{i,\theta(i)}(F_i)\}_{i\in I}, \{g_{\theta(i_1),\theta(i_2)}\}_{\substack{i,j\in I\\ i\leq j}})$$

where

$$g_{\theta(i_1),\theta(i_2)} : \gamma_{i_1,\theta(i_1)}(F_{i_1}) \to \gamma_{i_2,\theta(i_2)}(F_{i_2})$$

If $a \in \gamma_{i_1,\theta(i_1)}(F_{i_1})$ then $\exists b \in F_{i_1}$ such that $a = \gamma_{i_2,\theta(i_2)}(b)$. By Diagram 1,

$$g_{\theta(i_1),\theta(i_2)}(a) = \gamma_{i_2,\theta(i_2)} \circ f_{i_1,i_2}(b)$$

Thus, $g_{\theta(i_1),\theta(i_2)}$ is a group homomorphism because it is the composition of group homomorphisms. By Theorem 1, $\theta(I)$ is a lower semilattice with least element. Further, $\{\gamma_{i,\theta(i)}(F_i)\}_{i\in I}$ is a family of groups indexed by $\theta(I)$ and

$$\{g_{\theta(i_1),\theta(i_2)}\}_{\substack{i,j\in I\\ i\leq j}}$$

is a family of group homomorphisms mapping this family of groups. Thus, the image of $\mathcal{F}$ under $(\theta, {\gamma_{i,\theta(i)}}_{i\in I})$ is in $\mathcal{DSAG}$. $\square$

Definition 8. If G is a group, let e_G denote the identity of G.

Example 2. In this example, G_i, G_j, and G_k make up the family of abelian groups where $i \leq j$ and k is not comparable to i and j. Although $f_{j\lambda}$ is not shown, $f_{j\lambda}$ is the composition of $f_{i\lambda}$ and f_{ji} .

$$\begin{tikzcd}
\mathbf{G}_j = \{e_{G_j}\} \arrow[d, dash, "f_{ji}"'] & & \\
\mathbf{G}_i = \{e_{G_i}\} \arrow[dr, dash, "f_{i\lambda}"'] & & \mathbf{G}_k = \{e_{G_k}\} \arrow[dl, dash, "f_{k\lambda}"] \\
& \mathbf{G}_\lambda = \{e_{G_\lambda}\} &
\end{tikzcd}$$

A direct system of abelian groups over a lower semilattice with least element

Now, let I be a lower semilattice with least element λ. Let

$$(I, \{G_i\}_{i\in I}, \{f_{ji}\}_{\substack{i,j\in I\\ i\leq j}})$$

be a direct system of abelian groups over I.
Let $\mathcal{G} = \{(g, i) : i \in I, g \in G_i\}$.
Define a binary operation, *, on $\mathcal{G}$ as follows:

$$(h, j) * (g, i) = \begin{cases} (f_{ji}(h) \cdot g, i) & \text{if } j \geq i, \\ (f_{ij}(g) \cdot h, j) & \text{if } j < i, \\ (f_{jk}(h) \cdot f_{ik}(g), k) & \text{otherwise} \end{cases}$$

where the greatest lower bound of j and i is k if j is not comparable to i.

Note that if $j = i$, $(f_{ji}(h) \cdot g, i) = (f_{ij}(g) \cdot h, j)$. Thus, we can also define

$$(h, j) * (g, i) = \begin{cases} (f_{ji}(h) \cdot g, i) & \text{if } j > i, \\ (f_{ij}(g) \cdot h, j) & \text{if } j \leq i, \\ (f_{jk}(h) \cdot f_{ik}(g), k) & \text{otherwise} \end{cases}$$

where the greatest lower bound of j and i is k if j is not comparable to i.

Theorem 3. $\mathcal{G}$ with * is a commutative semigroup with $0 = (e_{G_\lambda}, \lambda)$.

Proof. Let $\mathcal{G} = \{(g, i) : i \in I, g \in G_i\}$ with binary operation,*, as defined above.

I. Associativity
Let $i, j, k \in I$.
Suppose that the greatest lower bound of i and j is l, the greatest lower bound of i and k is m, the greatest lower bound of l and k is α, and the greatest lower bound of j and m is β.
Then $((h, j) * (g, i)) * (a, k)$

$$=(f_{jl}(h) \cdot f_{il}(g), l) * (a, k) \qquad \text{by definition of } *$$

$=(f_{l\alpha}(f_{jl}(h) \cdot f_{il}(g)) \cdot f_{k\alpha}(a), \alpha)$ by definition of $*$

$=((f_{l\alpha}(f_{jl}(h)) \cdot f_{l\alpha}(f_{il}(g))) \cdot f_{k\alpha}(a), \alpha)$ by definition of homomorphism

$=((f_{j\alpha}(h) \cdot f_{i\alpha}(g)) \cdot f_{k\alpha}(a), \alpha)$ by composition of homomorphism

$=(f_{j\alpha}(h) \cdot f_{i\alpha}(g) \cdot f_{k\alpha}(a), \alpha)$ the operation within G_α is associative

$=(f_{j\beta}(h) \cdot (f_{i\beta}(g) \cdot f_{k\beta}(a)), \beta)$ by Lemma 2, $\alpha = \beta$

$=(f_{j\beta}(h) \cdot (f_{m\beta}(f_{im}(g)) \cdot f_{m\beta}(f_{km}(a))), \beta)$ by composition of homomorphism

$=(f_{j\beta}(h) \cdot (f_{m\beta}(f_{im}(g) \cdot f_{km}(a))), \beta)$ by definition of homomorphism

$=(h, j) * (f_{im}(g) \cdot f_{km}(a), m)$ by definition of $*$

$=(h, j) * ((g, i) * (a, k))$ by definition of $*$

II. Commutativity

Let $(h, j), (g, i) \in \mathcal{G}$.

Case One: $j = i$

Then $(h, j) * (g, i) = (f_{ji}(h) \cdot g, i)$ by definition of *

$= (f_{ii}(h) \cdot g, i)$ by equality of i and j

$= (h \cdot g, i)$ by definition of inverse system

$= (h \cdot g, j)$ by equality of i and j

$= (g \cdot h, j)$ by commutative property

$= (f_{ii}(g) \cdot h, j)$ by definition of inverse system

$= (f_{ij}(g) \cdot h, j)$ by equality of i and j

$= (g, i) * (h, j)$ by definition of *

Case Two: $j > i$

Then $(h, j) * (g, i) = (f_{ji}(h) \cdot g, i) = (g, i) * (h, j)$.

Case Three: $j < i$
Then $(h, j) * (g, i) = (f_{ij}(g) \cdot h, j) = (g, i) * (h, j)$.

Case Four: Not comparable

Suppose the greatest lower bound of j and i is k.
Then $(h, j) * (g, i) = (f_{jk}(h) \cdot f_{ik}(g), (k))$

$$= (f_{ik}(g) \cdot f_{jk}(h), (k)) \qquad \text{by commutative property}$$

$$= (g, i) * (h, j).$$

III. Existence of $0 = (e_{G_\lambda}, \lambda)$
Let (h, i) be any arbitrary element from $\mathcal{G}$.
Since λ is the least element of the lower semilattice, $\lambda \leq i$.
Thus, $(h, i) * (e_{G_\lambda}, \lambda) = (f_{i\lambda}(h) \cdot e_{G_\lambda}, \lambda)$

$$= (e_{G_\lambda} \cdot e_{G_\lambda}, \lambda) \qquad \text{because } G_\lambda = \{e_{G_\lambda}\}$$

$$= (e_{G_\lambda}, \lambda)$$

□

Lemma 4. Suppose $\mathcal{G}$ with * is a commutative semigroup with zero derived from an direct system of abelian groups over a lower semilattice with least element as described in Theorem 1. If $s \in \mathcal{G}$ such that $s * s = 0$, then $s = 0$.

Proof. Let $s \in \mathcal{G}$ such that $s * s = 0$. By Theorem 1, $\mathcal{G} = \{(g, i) : i \in I, g \in G_i\}$. Since $s \in \mathcal{G}$, $s = (h, j)$ for some $h \in G_j$ where $G_j \in \{G_i\}_{i \in I}$ and $j \in I$.

$$\begin{aligned}
\text{Then } (e_{G_\lambda}, \lambda) &= 0 && \text{by Theorem 1} \\
&= s * s && \text{by assumption} \\
&= (h, j) * (h, j) \\
&= (f_{jj}(h) \cdot h, j) && \text{by definition of *} \\
&= (h \cdot h, j) && \text{by definition of direct system}
\end{aligned}$$

This implies that $j = \lambda$.
(†) So $(h, j) = (h, \lambda)$.

$$\begin{aligned}
\text{Now } (e_{G_\lambda}, \lambda) &= 0 && \text{by Theorem 1} \\
&= s * 0 && \text{by definition of zero}
\end{aligned}$$

$$
\begin{aligned}
&= (h, j) * (e_{G_\lambda}, \lambda) && \text{by Theorem 1} \\
&= (h, \lambda) * (e_{G_\lambda}, \lambda) && \text{by } (\dagger) \\
&= (f_{\lambda\lambda}(h) \cdot e_{G_\lambda}, \lambda) && \text{by definition of } * \\
&= (h \cdot e_{G_\lambda}, \lambda) && \text{by definition of direct system}
\end{aligned}
$$

($\dagger\dagger$) This implies that $e_{G_\lambda} = h \cdot e_{G_\lambda}$.

$$
\begin{aligned}
\text{Thus, } s &= (h, j) \\
&= (h, \lambda) && \text{by } (\dagger) \\
&= (h \cdot e_{G_\lambda}, \lambda) && \text{by definition of multiplicative identity} \\
&= (e_{G_\lambda}, \lambda) && \text{by } (\dagger\dagger) \\
&= 0 && \text{by Theorem 1} \qquad \square
\end{aligned}
$$

Theorem 4. The collection of commutative semigroups with 0 with commutative semigroup morphisms is a category.

Proof. See [2]. $\square$

Definition 9. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A covariant functor T from $\mathcal{C}$ to $\mathcal{D}$ is a pair of functions (T_1, T_2) where T_1 is an object function that assigns to each object C inside $\mathcal{C}$ another object D inside $\mathcal{D}$ where $D = T_1(C)$, and T_2 is a morphism function that assigns to each morphism $f : C \to C'$ of $\mathcal{C}$ another morphism $T_2(f) : T_1(C) \to T_1(C')$ of $\mathcal{D}$ such that the following is satisfied:

1. For every identity morphism 1_C of $\mathcal{C}$, $T_2(1_C) = 1_{T_1(C)}$

2. Let f and g be any two morphisms of $\mathcal{C}$ where $g \circ f$ is defined. Then $T_2(g \circ f) = T_2(g) \circ T_2(f)$.

Theorem 5. Consider the two categories: $\mathcal{DSAG}$ and the collection of commutative semigroups with 0. Let T be a pair of functions (T_1, T_2) with the following properties:

1. If $\mathcal{H} \in \mathcal{DSAG}$, then T_1 takes $\mathcal{H}$ and maps it to a commutative semigroup as follows:

$$T_1(\mathcal{H}) = T_1((I, \{H_i\}_{i \in I}, \{h_{ji}\}_{\substack{i,j \in I \\ i \leq j}})) = \{(h, i) : i \in I,\, h \in H_i\}$$

with the operation $*$ as defined in Theorem 3.

2. If $\mathcal{G} \in \mathcal{DSAG}$ and $(\theta, \{\gamma_{i,\theta(i)}\}_{i \in I})$ is a morphism from $\mathcal{H}$ to $\mathcal{G}$, then

$$T_2((\theta, \{\gamma_{i,\theta(i)}\}_{i \in I})) : T_1((I, \{H_i\}_{i \in I}, \{h_{ji}\}_{\substack{i,j \in I \\ i \leq j}})) \to T_1((\theta(I), \{G_{\theta(i)}\}_{i \in I}, \{g_{\theta(j),\theta(i)}\}_{\substack{i,j \in I \\ i \leq j}}))$$

More specifically,

$$T_2((\theta, \{\gamma_{i,\theta(i)}\}_{i \in I}))(h, i) = (\gamma_{i,\theta(i)}(h), \theta(i))$$

Then $T = (T_1, T_2)$ is a functor. Consider the following diagram:

$$
\begin{tikzcd}
(I, \{H_i\}_{i \in I}, \{h_{ji}\}_{\substack{i,j \in I \\ i \le j}}) \arrow[rr, "{(\theta, \{\gamma_{i,\theta(i)}\}_{i \in I})}"] \arrow[d, "T_1"'] & {} \arrow[d, "T_2"] & (\theta(I), \{G_{\theta(i)}\}_{i \in I}, \{g_{\theta(j)\theta(i)}\}_{\substack{i,j \in I \\ i \le j}}) \arrow[d, "T_1"] \\
\{(h,i) : i \in I, h \in H_i\} \arrow[rr, "{(\theta, \{\gamma_{i,\theta(i)}\}_{i \in I})}"'] & {} & \{(\gamma_{i,\theta(i)}(h), \theta(i)) : i \in I, h \in H_i\}
\end{tikzcd}
$$

Proof. Let $\mathcal{H}$ be any element in $\mathcal{DSAG}$ and (h, i) be any element in $T_1(\mathcal{H})$. Suppose that $(\theta, \{\gamma_{i,\theta(i)}\}_{i \in I})$ is an identity morphism on $\mathcal{H}$. Then

$$T_2((\theta, \{\gamma_{i,\theta(i)}\}_{i \in I}))(h, i) = (\gamma_{i,\theta(i)}(h), \theta(i)) = (h, i)$$

Now suppose that $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are any elements in $\mathcal{DSAG}$, and let $(\theta, \{\gamma_{i,\theta(i)}\}_{i \in I})$ be any morphism from $\mathcal{F}$ to $\mathcal{G}$ and $(\mu, \{\delta_{j,\mu(j)}\}_{j \in J})$ be any morphism from $\mathcal{G}$ to $\mathcal{H}$. Also, let (a, i) be an arbitrary element in $T_1(\mathcal{F})$.

Now note that $T_2((\theta, \{\gamma_{i,\theta(i)}\}_{i \in I}) \circ (\mu, \{\delta_{j,\mu(j)}\}_{j \in J}))(a, i)$

$$= T_2(((\theta \circ \mu), (\{\gamma_{\mu(j),(\theta \circ \mu)(j)} \circ \delta_{j,\mu(j)}\}_{j \in J})))(a, i)$$

$$= ((\gamma_{\mu(j),(\theta \circ \mu)(j)} \circ \delta_{j,\mu(j)})(a), (\theta \circ \mu)(i))$$

$$= (\gamma_{i,\theta(i)}(a), \theta(i)) \circ (\delta_{j,\mu(j)}(a), \mu(i))$$

$$= T_2((\theta, \{\gamma_{i,\theta(i)}\}_{i \in I}))(a, i) \circ T_2((\mu, \{\delta_{j,\mu(j)}\}_{j \in J}))(a, i)$$

Next, we show that $T_2((\theta, \{\gamma_{i,\theta(i)}\}_{i \in I}))$ is a commutative semigroup homomorphism.

Let $i_1, i_2 \in I$ with greatest lower bound α. Let $h_1 \in H_{i_1}$ and $h_2 \in H_{i_2}$. Then $T_2((\theta, \{\gamma_{i,\theta(i)}\}_{i \in I}))((h_1, i_1) * (h_2, i_2))$

$= T_2((\theta, \{\gamma_{i,\theta(i)}\}_{i \in I}))(f_{i_1\alpha}(h_1) \cdot f_{i_2\alpha}(h_2), \alpha)$ by definition of $*$

$= (\gamma_{i,\theta(i)}(f_{i_1\alpha}(h_1) \cdot f_{i_2\alpha}(h_2)), \theta(\alpha))$ by definition of T_2

$= (\gamma_{i,\theta(i)}(f_{i_1\alpha}(h_1)) \cdot \gamma_{i,\theta(i)}(f_{i_2\alpha}(h_2)), \theta(\alpha))$ $\gamma_{i,\theta(i)}$ is a group homomorphism

$= (f_{\theta(i_1)\theta(\alpha)}(\gamma_{i,\theta(i)}(h_1)) \cdot f_{\theta(i_2)\theta(\alpha)}(\gamma_{i,\theta(i)}(h_2)), \theta(\alpha))$ by Theorem 1

$= (\gamma_{i,\theta(i)}(h_1), \theta(i_1)) * (\gamma_{i,\theta(i)}(h_2), \theta(i_2))$ by definition of $*$

□

By [2], a covariant functor, $Q = (Q_1, Q_2)$, exists from the category of commutative semigroups with 0 to the category of graphs and is defined as follows:

Let $\mathcal{G} = \{(g, i) : i \in I, g \in G_i\}$ be a commutative semigroup with 0.

1. The vertices of the graph $Q_1(\{(g, i) : i \in I, g \in G_i\})$ are the elements of $\mathcal{G}$.

2. If $(g, \alpha), (h, \beta)$ are distinct elements in $\mathcal{G}$, then there exists an edge between (g, α) and (h, β) precisely when α and β are both in the same lower semilattice and are comparable.

3. Let $\mathcal{H} = \{(h, j) : j \in J, h \in H_j\}$ be another commutative semigroup with 0 element and let σ be a morphism from $\mathcal{G}$ to $\mathcal{H}$. Let (g, i) be any element in $\mathcal{G}$. Define $Q_2(\sigma)((g, i)) = \sigma((g, i))$.

Since the composition of functors is a functor, the composition of Q and T as described in Theorem 5 is a functor from $\mathcal{DSAG}$ to the category of graphs. More specifically, $(Q \circ T) = ((Q \circ T)_1, (Q \circ T)_2)$ is defined as follows:

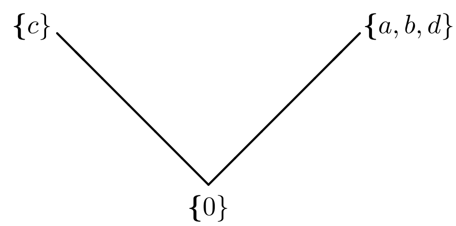
Let $\mathcal{H}, \mathcal{M}$ be any elements in $\mathcal{DSAG}$, $(\theta, \{\gamma_{i,\theta(i)}\}_{i \in I})$ be any morphism from $\mathcal{H}$ to $\mathcal{M}$, and (h, i) be any element in $T_1(\mathcal{H})$. Then

$$\begin{aligned}(Q \circ T)_1(\mathcal{H}) &= Q_1(T_1(\mathcal{H})) \\ &= Q_1(T_1((I, \{H_i\}_{i \in I}, \{h_{ji}\}_{\substack{i,j \in I \\ i \leq j}}))) \\ &= Q_1(\{(h, i) : i \in I,\ h \in H_i\})\end{aligned}$$

which is a graph whose vertices and edges can be found by using the rules as outlined above. Also,

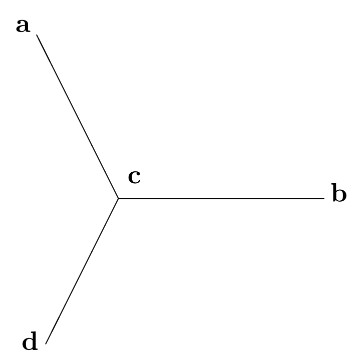
$$\begin{aligned}(Q \circ T)_2((\theta, \{\gamma_{i,\theta(i)}\}_{i \in I}))(h, i) &= Q_2(T_2((\theta, \{\gamma_{i,\theta(i)}\}_{i \in I})(h, i)) \\ &= Q_2((\gamma_{i,\theta(i)}(h), \theta(i))) \\ &= (\gamma_{i,\theta(i)}(h), \theta(i))\end{aligned}$$

Example 3. Consider the following direct system of abelian groups.



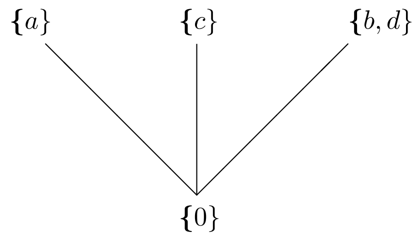
Note that $ac = bc = dc = 0$, but $ab, ad, bd \neq 0$.

The corresponding zero divisor graph is



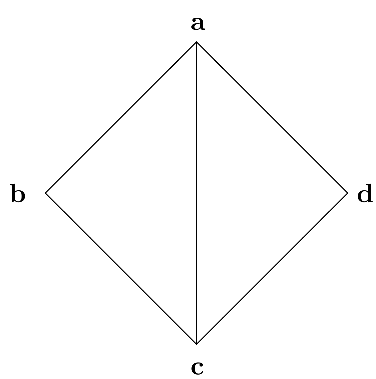
This is known as the Star Graph, S_3.

Example 4. Consider the following direct system of abelian groups.



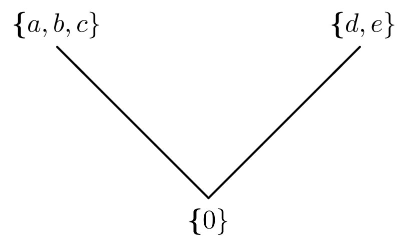
Note that $ab = ad = ac = bc = dc = 0$, but $bd \neq 0$.

The corresponding zero divisor graph is



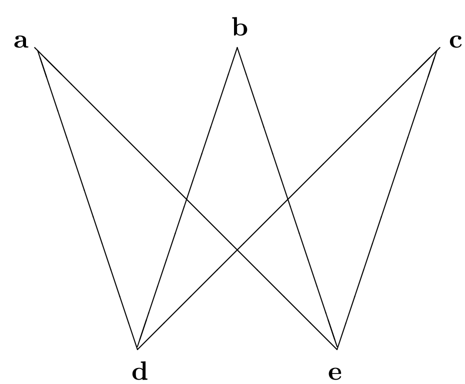
This is known as the Diamond Graph.

Example 5. Consider the following direct system of abelian groups.



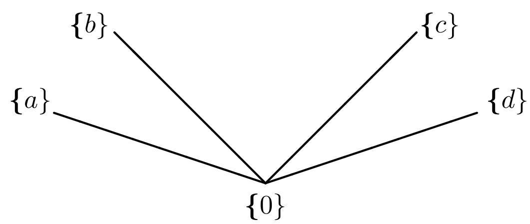
Note that $ad = ae = bd = be = cd = ce = 0$, but $ab, ac, bc, de \neq 0$.

The corresponding zero divisor graph is



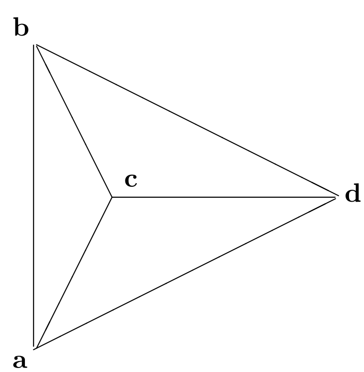
This is known as the Complete Bitartite Graph, $K_{2,3}$.

Example 6. Consider the following direct system of abelian groups.



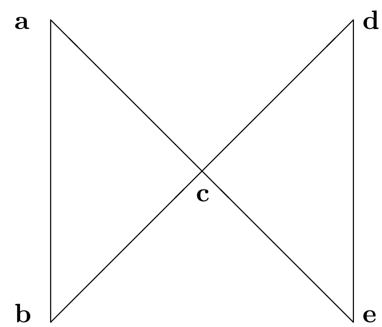
Note that $ab = ac = ad = bc = bd = dc = 0$.

The corresponding zero divisor graph is



This is known as the Wheel Graph.

Example 7. Consider the following Butterfly Graph.



According to this graph, $ab = ac = bc = dc = de = ec = 0$, but $ad, ae, bd, be \neq 0$.

It is not possible to construct an direct system of abelian groups over a lower semilattice with a least element from the Butterfly Graph.

To see this, let S be a commutative semigroup with zero derived from an direct system of abelian groups over a lower semilattice with least element. Suppose that it is possible to construct a zero divisor graph from this direct system and that the zero divisor graph is the Butterfly Graph above.

Then $(bd)c = b(dc)$	by associativity
$= b(0)$	by definition of zero divisor graph
$= 0$	by definition of zero

(† † †) So $(bd)c = 0$.

But $c, bd \neq 0$ by the definition of zero divisor graph, $c \in S$, and $bd \in S$ by multiplicative closure, so bd is, by definition, a zero divisor of S. So bd must be one of the vertices of the Butterfly Graph above.

Case One: $bd = a$

Then $0 = b(0)$	by definition of zero
$= b(de)$	by definition of zero divisor graph
$= (bd)e$	by associativity

$= ae$ by assumption

$\neq 0$ by definition of zero divisor graph

Case Two: $bd = b$
Then $0 = b(0) = b(de) = (bd)e = be \neq 0$.

Case Three: $bd = d$
Then $0 = (0)d = (ab)d = a(bd) = ad \neq 0$.

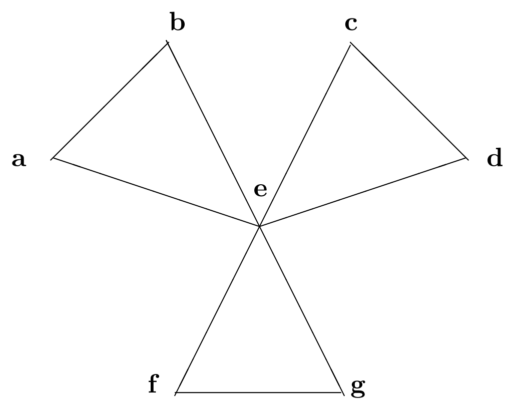
Case Four: $bd = e$
Then $0 = (0)d = (ab)d = a(bd) = ae \neq 0$.

So $bd = c$ since c is the last possible vertex.
Then $0 = (bd)c$ by $(\dagger\dagger\dagger)$

$= cc$

Since $cc = 0$ and S is a commutative semigroup with zero derived from an direct system of abelian groups over a lower semilattice with least element, $c = 0$ by Lemma 2. But $c \neq 0$ by definition of zero divisor graph. This is a contradiction. Thus, there is no corresponding direct system of abelian groups over a lower semilattice with a least element for the Butterfly Graph.

Example 8. Consider the following Friendship Graph, F_3.



According to this graph, $ab = ae = be = cd = ce = de = fg = fe = ge = 0$, but $ac, ad, af, ag, bc, bd, bf, bg, cf, cg, df, dg \neq 0$.

However, it is not possible to construct an direct system of abelian groups over a lower semilattice with a least element from the Friendship Graph, F_3. The proof of this is similar to the proof in the last example.

Definition 10. Let S be a commutative semigroup with zero. Let $I \subseteq S$ and $0 \in I$. Then I is an ideal of S if whenever $s \in S$ and $a \in I$, $as \in I$.

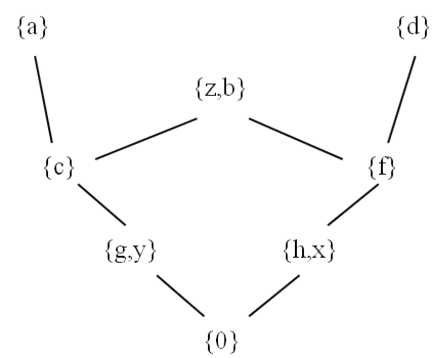
Following Rosen in [3], informally define a path to be a sequence of edges that begins at a vertex of a graph and travels along edges of the graph, always connecting pairs of adjacent vertices. The length of a path is the number of edges in the path. A simple cycle is a path with no repeated edges such that the initial and terminal vertex are the same. See page 567-568 of [3] for more formal definitions.

Theorem 6. If $\{0, x\}$ is not an ideal in S for any nonzero $x \in S$ and the cardinality of the set containing 0 and the nonzero zero divisors of S is greater than or equal to 3, then every pair of vertices in the zero divisor graph corresponding to S is contained in a simple cycle of length ≤ 6.

Note that when the authors in [2] refer to cycle, they mean simple cycle.

Proof. See [2]. □

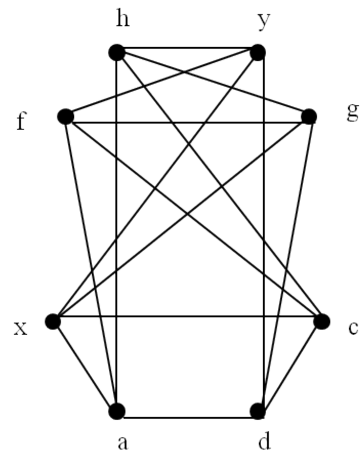
The following semilattice S of abelian groups satisfies Theorem 2, and the vertices a, d are contained in a simple cycle of length 6 but not any simple cycle of shorter length. Without loss of generality, assume that g is the identity of $\{g, y\}$ and h is the identity of $\{h, x\}$.



Semilattice S of abelian groups from [2]

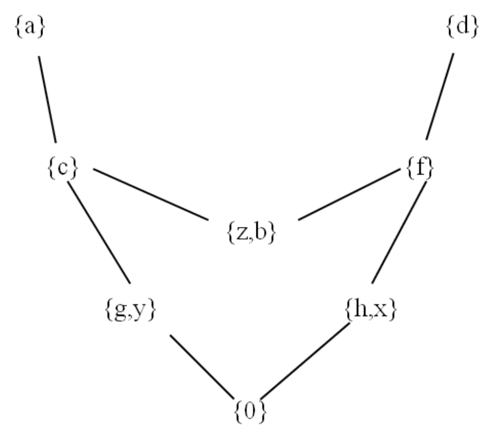
But, if a zero divisor graph were to be constructed from the given example, a, d would be contained in a simple cycle of length 4 ($a \to h \to y \to d \to a$) which contradicts with the example's claim.

Here is the zero divisor graph corresponding to the semilattice S.



Zero Divisor Graph corresponding to [2]

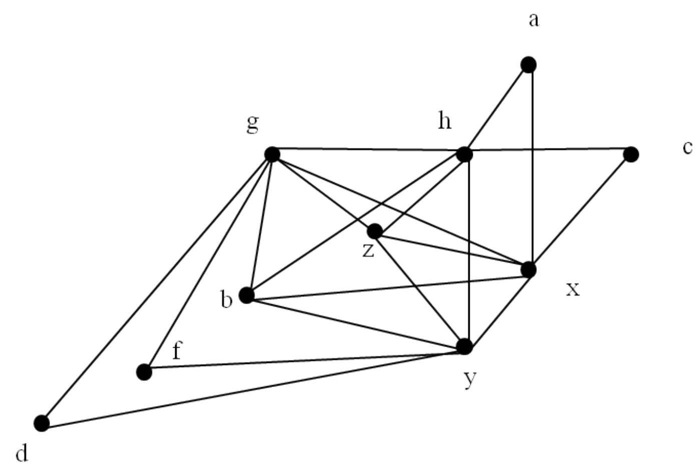
In the following example, the vertices a, d are contained in a simple cycle of length 6 but not any simple cycle of shorter length. Assume, once again, that g is the identity of $\{g, y\}$ and h is the identity of $\{h, x\}$. Also assume z is the identity of $\{z, b\}$.



Correction of semilattice S

One can check that $\{0, u\}$ is not an ideal for any nonzero $u \in S$ and the cardinality of the set containing 0 and the nonzero zero divisors of S is greater than or equal to 3.

The corresponding zero divisor graph for this direct system is the following:



Zero Divisor Graph corresponding to the correction of semilattice S

In this semilattice, a and d can be contained in a simple cycle of length 6:

$$(a \to x \to y \to d \to g \to h \to a)$$

but cannot be contained in a simple cycle of length less than 6.
To see this, suppose a, d are contained in a simple cycle of length less than 6. A simple cycle of length 2 or 1 is impossible since there does not exist an edge between a and d. Thus, the simple cycle must be of length 5, 4, or 3.

If a, d are contained in a simple cycle of length less than 6, then either the path from a to d or the path from d to a must have length 1 or 2.

Simple Cycle of Length 5

Length of Path from a to d	Length of Path from d to a
2	3
1	4

Simple Cycle of Length 4

Length of Path from a to d	Length of Path from d to a
2	2
1	3

Simple Cycle of Length 3

Length of Path from a to d	Length of Path from d to a
1	2

According to [3], the adjacency matrix for the zero divisor graph corresponding to the correction is

$$
A \quad = \quad \begin{array}{cc}
 & \begin{array}{cccccccccc} a & b & c & d & f & g & h & x & y & z \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \\ f \\ g \\ h \\ x \\ y \\ z \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}
\end{array}
$$

Adjacency Matrix for the Zero Divisor Graph

The number of paths of length 1 from a to d is the $a_{1,4}$ entry of A. Since this entry is 0, there exists no path of length 1 from a to d. A^2 gives the number of different paths of length 2 from one vertex to another. See [3] for details.

$$A^2 = \begin{array}{cc} & \begin{array}{cccccccccc} a & b & c & d & f & g & h & x & y & z \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ f \\ g \\ h \\ x \\ y \\ z \end{array} & \begin{pmatrix} 2 & 2 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 2 \\ 2 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 4 \\ 2 & 2 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 & 2 & 0 & 2 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 & 2 & 0 & 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 & 0 & 6 & 2 & 2 & 6 & 2 \\ 0 & 2 & 0 & 2 & 2 & 2 & 6 & 6 & 2 & 2 \\ 0 & 2 & 0 & 2 & 2 & 2 & 6 & 6 & 2 & 2 \\ 2 & 2 & 2 & 0 & 0 & 6 & 2 & 2 & 6 & 2 \\ 2 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 4 \end{pmatrix} \end{array}$$

Square of the Adjacency Matrix

Hence, the number of paths of length 2 from a to d is the $a_{1,4}$ entry of A^2. Since this entry is 0, there exists no path of length 2 from a to d. Although the adjacency matrix accounts for directed or undirected edges with multiple edges and loops allowed according to [3], if the entry for the a_{ij} entry is 0, there would exist no path from the vertex corresponding to i to the vertex corresponding to j regardless. Since there does not exist a path of length 1 or 2 from a to d and vice versa, a and d must be contained in a simple cycle of length 6 or greater.

References

[1] T.W. Hungerford, *Algebra*, Springer-Verlag, v.73, Graduate Texts in Mathematics, New York, (1974), 13.

[2] F.R. DeMeyer , T. McKenzie, K. Schneider, *The zero-divisor graph of a commutative semigroup*, Semigroup Forum, **65** (2002), 206–214 .

[3] K.H. Rosen, *Discrete Mathematics and Its Applications Fifth Edition*, McGraw-Hill, Boston, (2003), 574.