ECE/CS 250: Computer Architecture

Combinational Logic: Boolean Algebra, Logic Gates

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Duke University

Slides are derived from work by
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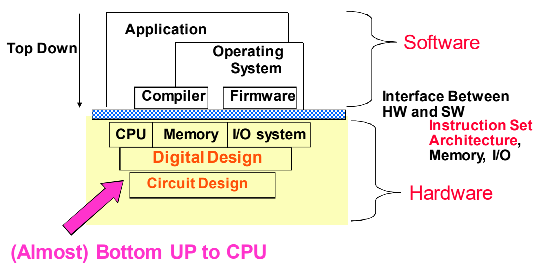
Reading

- Appendix B (parts 1,2,3,5,6,7,8,9,10)
- This material is covered in MUCH greater depth in ECE/CS 350 – please take ECE/CS 350 if you want to learn enough digital design to build your own processor

What We've Done, Where We're Going



Computer = Machine That Manipulates Bits

- Everything is in binary (bunches of 0s and 1s)
  - Instructions, numbers, memory locations, etc.
- Computer is a machine that operates on bits
  - Executing instructions → operating on bits
- Computers physically made of transistors
  - Electrically controlled switches
- We can use transistors to build logic
  - E.g., if this bit is a 0 and that bit is a 1, then set some other bit to be a 1
  - E.g., if the first 5 bits of the instruction are 10010 then set this other bit to 1 (to tell the adder to subtract instead of add)

How Many Transistors Are We Talking About?

Pentium III
- Processor Core 9.5 Million Transistors
- Total: 28 Million Transistors

Pentium 4
- Total: 42 Million Transistors

Core2 Duo (two processor cores)
- Total: 290 Million Transistors

Core2 Duo Extreme (4 processor cores, 8MB cache)
- Total: 590 Million Transistors

Core i7 with 6-cores
- Total: 2.27 Billion Transistors

How do they design such a thing? Carefully!

Abstraction!

- Use of abstraction (key to design of any large system)
  - Put a few (2-8) transistors into logic gate (OR, AND, XOR, …)
  - Combine gates into logical functions (add, select,….)
  - Combine adders, shifters, etc., together into modules
    - Units with well-defined interfaces for large tasks: e.g., decode
  - Combine a dozen of those into a core…
  - Stick 4 cores on a chip…

You are here:

- Use of abstraction (key to design of any large system)
  - Put a few (2-8) transistors into a logic gate
  - Combine gates into logical functions (add, select,….)
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    - Units with well-defined interfaces for large tasks: e.g., decode
  - Combine a dozen of those into a core…
  - Stick 4 cores on a chip…

Boolean Algebra

- **First step to logic: Boolean Algebra**
  - Manipulation of True / False (1/0)
  - After all: everything is just 1s and 0s

- **Given inputs (variables): A, B, X, P, Q…**
  - Compute outputs using logical operators, such as:
- **NOT:** !A **(=** ~A **=** $\overline{A}$**)**
- **AND:** A&B **(=** A·B **=** A*B = AB **=** A$\wedge$B) = A&&B in C/C++
- **OR:** A | B **(=** A+B **=** A $\vee$ B**) =** A || B in C/C++
- **XOR:** A ^ B (= A $\oplus$ B)
- NAND, NOR, XNOR, Etc.

Truth Tables

- Can represent as Truth Table: shows outputs for all inputs

a	NOT(a)
0	1
1	0

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a	b	AND(a,b)
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Truth Tables

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a	NOT(a)
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1	0

a	b	AND(a,b)
0	0	0
0	1	0
1	0	0
1	1	1

a	b	OR(a,b)
0	0	0
0	1	1
1	0	1
1	1	1

Truth Tables

- Can represent as truth table: shows outputs for all inputs

a	NOT(a)
0	1
1	0

a	b	AND(a,b)
0	0	0
0	1	0
1	0	0
1	1	1

a	b	OR(a,b)
0	0	0
0	1	1
1	0	1
1	1	1

a	b	XOR(a,b)
0	0	0
0	1	1
1	0	1
1	1	0

a	b	NAND(a,b)
0	0	1
0	1	1
1	0	1
1	1	0

a	b	NOR(a,b)
0	0	1
0	1	0
1	0	0
1	1	0

Any Inputs, Any Outputs

- Can have any # of inputs, any # of outputs
- Can have arbitrary functions:

a	b	c	f_1	f_2
0	0	0	0	1
0	0	1	1	1
0	1	0	1	0
0	1	1	0	0
1	0	0	1	0
1	0	1	1	1
1	1	0	0	1
1	1	1	1	1

Let's Write a Truth Table for a Function…

- Example:
  (A & B) | !C

Start with Empty TT
  Column Per Input
  Column Per Output

A	B	C	Output

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- Example:
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Fill in Inputs
  Counting in Binary

A	B	C	Output
0	0	0	

Let's write a Truth Table for a function…

- Example:
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0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

Let's write a Truth Table for a function…

- Example:
  (A & B) | !C

Start with Empty TT
  Column Per Input
  Column Per Output

Fill in Inputs
  Counting in Binary

Compute Output
  (0 & 0) | !0 = 0 | 1 = 1

A	B	C	Output
0	0	0	1
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

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0	0	1	0
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

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- **Example:**
  (A & B) | !C

Start with Empty TT
Column Per Input
Column Per Output

Fill in Inputs
Counting in Binary

Compute Output
(0 & 1) | !0 = 0 | 1 = 1

A	B	C	Output
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	
1	0	0	
1	0	1	
1	1	0	
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0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

You try one…

- **Try one yourself (take 2 minutes):**
  (!A | B) & !C

Suppose I turn it around…

- **Given a Truth Table, find the formula?**

Hmmm..

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Suppose I turn it around…

- **Given a Truth Table, find the formula?**

Hmmm …
Could write down every "true" case
Then OR together:

```
(!A & !B & !C)  |
(!A & !B & C)   |
(!A & B & !C)  |
(A & B &!C)  |
(A & B &C)
```

A	B	C	Output
0	0	0	1
0	**0**	**1**	**1**
0	**1**	**0**	**1**
0	1	1	0
1	0	0	0
1	0	1	0
1	**1**	**0**	**1**
1	**1**	**1**	**1**

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A	B	C	Output
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0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Suppose I turn it around…

- This approach is called "sum of products"
  - Works every time
  - Result is right…
  - But really ugly

```
(!A & !B & !C) |
(!A & !B & C)  |
(!A & B & !C) |
(A & B &!C) |
(A & B &C)
```

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
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```
(!A & !B & !C) |
(!A & !B & C)  |
(!A & B & !C) |
(A & B &!C) |
(A & B &C)
```

Could just be (A & B) here ?

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

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```
(!A & !B & !C) |
(!A & !B & C)  |
(!A & B & !C) |
(A&B)
```

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
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(!A & !B & !C) |
(!A & !B & C)  |
(!A & B & !C) |
(A&B)
```

Could just be (!A & !B) here

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Suppose I turn it around…

- This approach: "sum of products"
  - Works every time
  - Result is right…
  - But really ugly

```
(!A & !B) |
(!A & B & !C) |
(A&B)
```

Looks nicer…
Can we do better?

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Suppose I turn it around…

- This approach: "sum of products"
  - Works every time
  - Result is right…
  - But really ugly

```
(!A & !B) |
(!A & B & !C) |
(A&B)
```

This has a lot in common:
!A & (something)

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Suppose I turn it around…

- This approach: "sum of products"
  - Works every time
  - Result is right…
  - But really ugly

```
(!A & !(B & C)) |
(A & B)
```

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Just did some of these by intuition.. but

- Somewhat intuitive approach to simplifying
- This is math, so there are formal rules
  - Just like "regular" algebra

Boolean Function Simplification

- Boolean expressions can be simplified by using the following rules (bitwise logical):
  - `A & A = A` `A | A = A`
  - `A & 0 = 0` `A | 0 = A`
  - `A & 1 = A` `A | 1 = 1`
  - `A & !A = 0` `A | !A = 1`
  - `!!A = A`
  - & and | are both commutative and associative
  - & and | can be distributed: `A & (B | C) = (A & B) | (A & C)`
  - & and | can be subsumed: `A | (A & B) = A`

DeMorgan's Laws

- Two (less obvious) Laws of Boolean Algebra:
  - Let's push negations inside, flipping & and |

    `!(A & B) = (!A) | (!B)`

    `!(A | B) = (!A) & (!B)`

  - You should try this at home – build truth tables for both the left and right sides and see that they're the same

Using DeMorgan on Early Example

```
(!A & !(B & C)) |
(A & B)

=
(!A & (!B | !C)) |
(A & B)
```

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Simplification Example:

```
! (!A | !(A & (B | C)))
```


Simplification Example:

```
! (!A | !(A & (B | C)))
```
DeMorgan's
```
!!A & !! (A & (B | C))
```


Simplification Example:

```
! (!A | !(A & (B | C)))
```
DeMorgan's
```
!!A & !! (A & (B | C))
```
Double Negation Elimination
```
A & (A & (B | C))
```


Simplification Example:

```
! (!A | !(A & (B | C)))
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DeMorgan's
```
!!A & !! (A & (B | C))
```
Double Negation Elimination
```
A & (A & (B | C))
```
Associativity of &
```
(A & A) & (B | C)
```


Simplification Example:

```
! (!A | !(A & (B | C)))
```
DeMorgan's
```
!!A & !! (A & (B | C))
```
Double Negation Elimination
```
A & (A & (B | C))
```
Associativity of &
```
(A & A) & (B | C)
```
A & A = A
```
A & (B | C)
```

You try this:

Come up with a formula for this Truth Table

Simplify as much as possible

A	B	C	Output
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Something That Trips Up Students

!A&!B is not the same as !(AB)

Students tend to notice this when it's written this way but NOT when it's written with lines above the terms:

$\overline{A} \cdot \overline{B} \neq \overline{AB}$

"Not A and Not B" is not the same as "Not AB"

The former is true if (A,B) = (0,0).

The latter is true if (A,B) = (0,0) or (0,1) or (1,0).

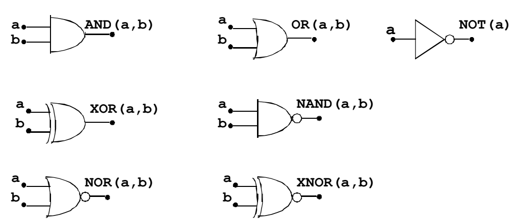
Applying the Theory

- Lots of good theory
- Can reason about complex Boolean expressions
- But why is this useful? (fun party trick)

Boolean Gates

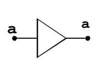
- Gates are electronic devices that implement simple Boolean functions (building blocks of hardware)

Examples



Guide to Remembering your Gates

- This one looks like it just points its input where to go
  - It just produces its input as its output
  - Called a buffer



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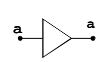
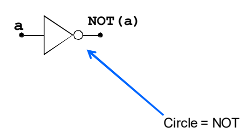
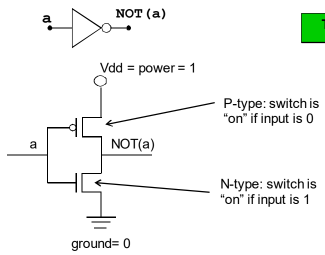


- A circle always means negate (invert)



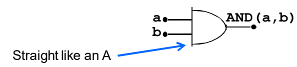
Brief Interlude: Building An Inverter

This is not on the test.

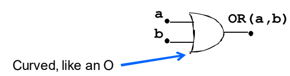


Guide to Remembering Your Gates

- AND Gates have a straight edge, like an A (in AND)

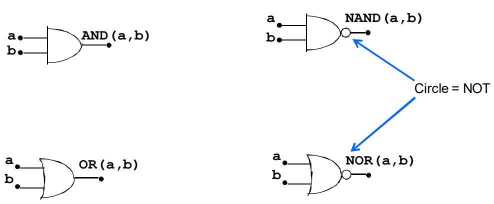


OR Gates have a curved edge, like an O (in OR)



Guide to Remembering Your Gates

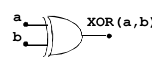
- If we stick a circle on them…



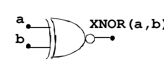
- We get NAND (NOT-AND) and NOR (NOT-OR)
  - NAND(a,b) = NOT(AND(a,b))

Guide to Remembering Your Gates

- XOR looks like OR (curved line)
  - But has two lines (like an X does)



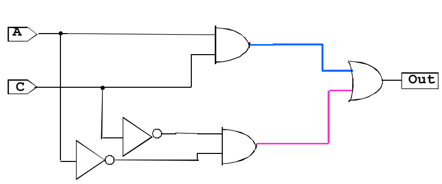
- Can put a dot for XNOR
  - XNOR is 1-bit "equals" by the way



Boolean Functions, Gates and Circuits

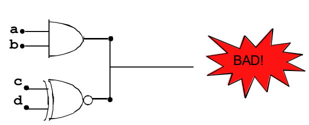
- Circuits are made from a network of gates.

`(!A & !C) | (A & C)`



A few more words about gates

- Gates have inputs and outputs
  - If you try to hook up two outputs, bad things happen (your processor catches fire)



  - If you don't hook up an input, it behaves kind of randomly (also not good, but not set-your-chip-on-fire bad)

Let's Make a Useful Circuit

- Pick between 2 inputs (called 2-to-1 MUX)
  - Short for multiplexor
- What might we do first?

Let's Make a Useful Circuit

- Pick between 2 inputs (called 2-to-1 MUX)
  - Short for multiplexor
- What might we do first?
  - Make a truth table?
    - S is selector:
      - S=0, pick A
      - S=1, pick B

A	B	S	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

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- Pick between 2 inputs (called 2-to-1 MUX)
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- Next: sum-of-products

```
(!A & B & S)  |
(A & !B & !S)  |
(A & B & !S )  |
(A & B & S)
```

A	B	S	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
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Let's Make a Useful Circuit

- Pick between 2 inputs (called 2-to-1 MUX)
  - Short for multiplexor
- What might we do first?
  - Make a truth table?
    - S is selector:
      - S=0, pick A
      - S=1, pick B
- Next: sum-of-products
- Simplify

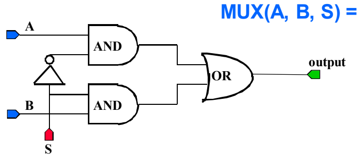
```
(A & !S)  |
(B &  S)
```

A	B	S	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

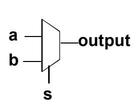
Circuit Example: 2x1 MUX

Draw it in gates:

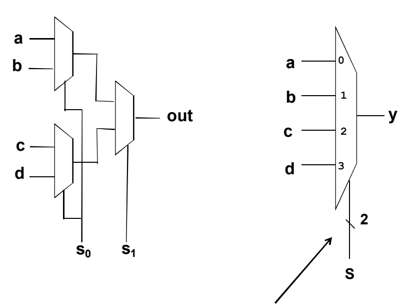
MUX(A, B, S) = (A & !S) | (B & S)



So common, we give it its own symbol:



Example 4x1 MUX

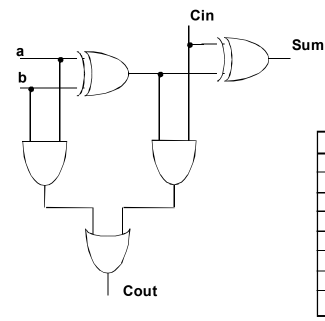


The / 2 on the wire means "2 bits"

Arithmetic and Logical Operations in ISA

- What operations are there?
- How do we implement them?
  - Consider a 1-bit Adder

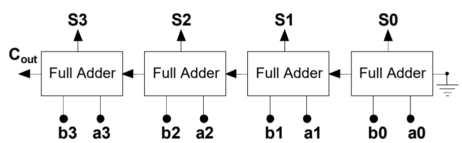
A 1-bit Full Adder




```
 01101100
 01101101
+00101100
 10011001
```

a	b	C_{in}	Sum	C_{out}
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Example: 4-bit adder

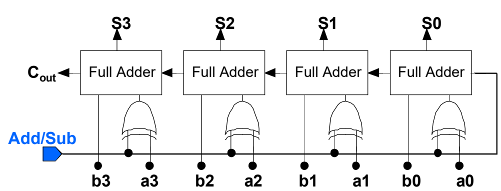


Subtraction

- How do we perform integer subtraction?
- What is the hardware?
  - Recall: hardware was why 2's complement was good idea
- Remember: Subtraction is just addition

  X – Y =
  X + (-Y) =
  X + (~Y +1)

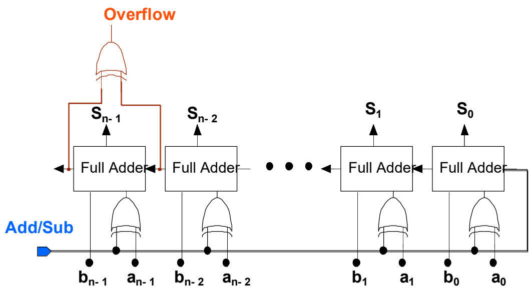
Example: Adder/Subtractor



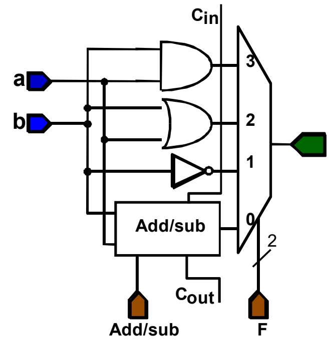
Overflow

- We can detect unsigned overflow by looking at CO
- How would we detect signed overflow?
  - If adding positive numbers and result "is" negative
  - If adding negative numbers and result "is" positive
  - At most significant bit of adder, check if CI != CO
  - Can check with XOR gate

Add/Subtract With Overflow Detection

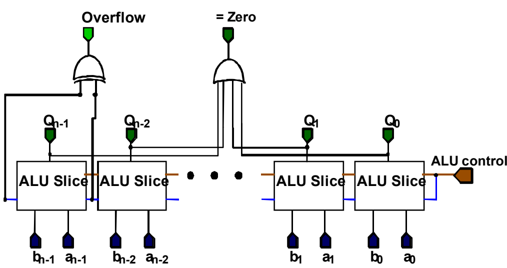


ALU Slice



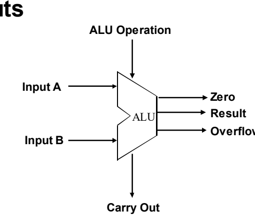
A	F	Q
0	0	a + b
1	0	a - b
-	1	NOT b
-	2	a OR b
-	3	a AND b

The ALU



Abstraction: The ALU

- General structure
- Two operand inputs
- Control inputs



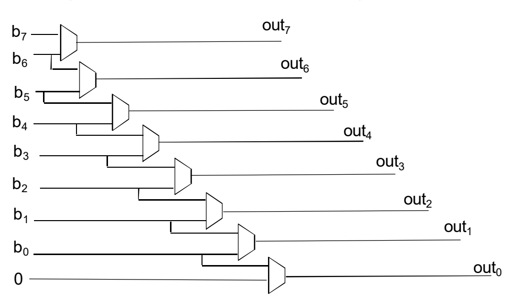
- We can build circuits for
  - Multiplication
  - Division
  - They are more complex (ECE/CS 350)

Another Operation We Might Want: Shift

- Remember the << and >> operations?
  - Shift left/shift right?
  - How would we implement these?
- Suppose you have an 8-bit number
  $b_7b_6b_5b_4b_3b_2b_1b_0$
- And you can shift it left by a 3-bit number
  $s_2s_1s_0$
- Option 1: Truth Table?
  - 2^{11} = 2048 rows? Yuck.

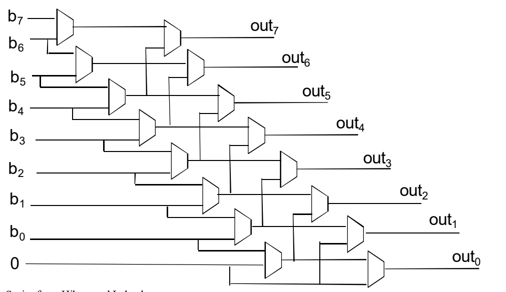
Let's simplify to 1-bit left-shift

- Simpler problem: 8-bit number left-shifted by 1 bit number (shift amount selects each mux)



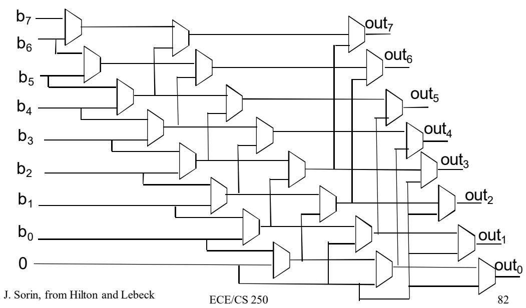
Let's simplify to 2-bit left shift

- Simpler problem: 8-bit number left-shifted by 2 bit number (i.e., can left-shift by 0, 1, 2, or 3)



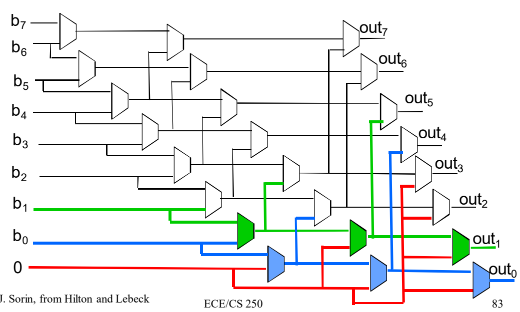
Now left-shifted by 3-bit number

- Full problem: 8-bit number left-shifted by 3 bit number (can shift by 0-7 bits)



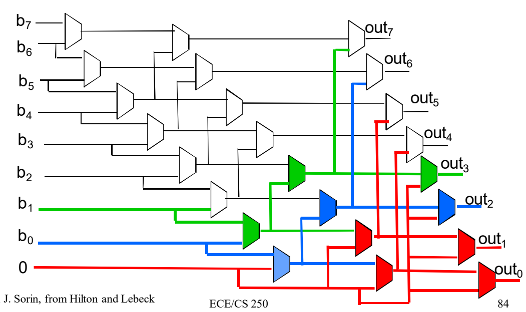
Now shifted by 3-bit number

- Shifter in action: shift by 000 (all muxes have S=0)



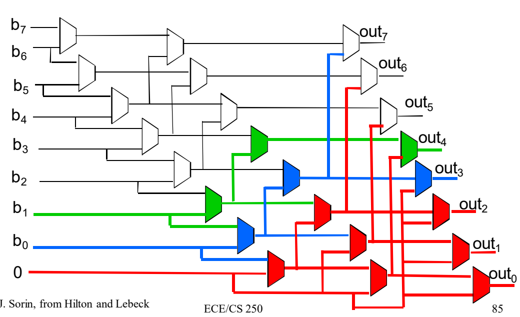
Now shifted by 3-bit number

- Shifter in action: shift by 010 (=2_{10})
  - From L to R: S = 0, 1, 0

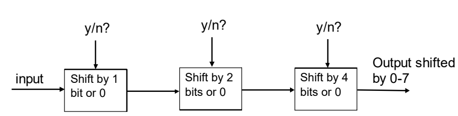


Now shifted by 3-bit number

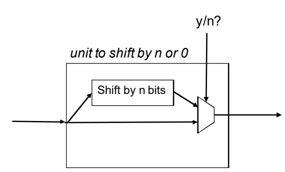
- Shifter in action: shift by 011
  - From L to R: S= 1, 1, 0 (reverse of shift amount)



Barrel Shifter



There are three control inputs (y/n). Any value from 0-7 can be achieved with them (000-111).



Summary

- Boolean Algebra & functions
- Logic gates (AND, OR, NOT, etc)
- Multiplexors
- Adder
- Arithmetic Logic Unit (ALU)