ECE/CS 250: Computer Architecture

Combinational Logic: Boolean Algebra, Logic Gates

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Slides are derived from work by
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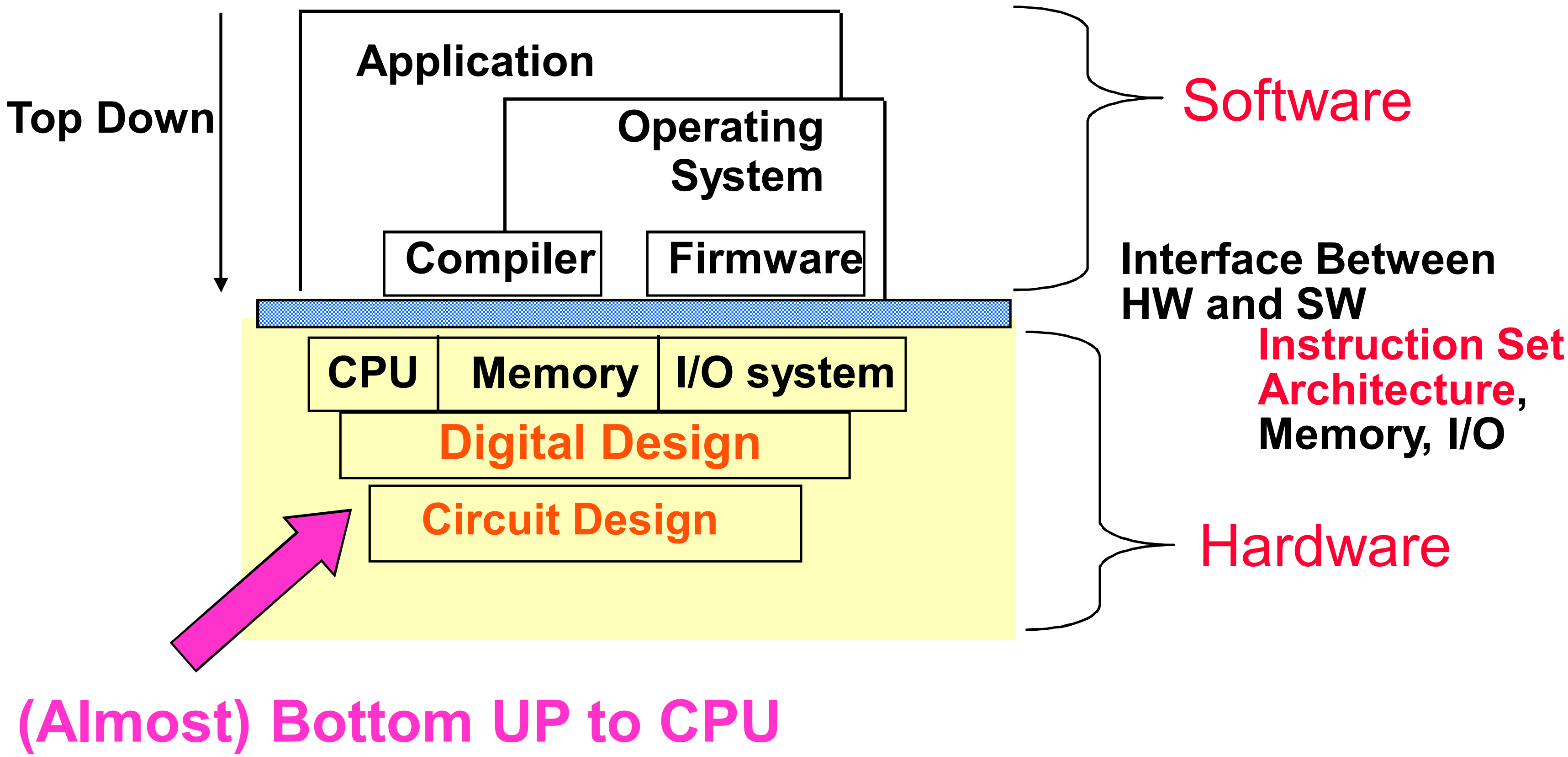
Reading

- **Appendix B (parts 1,2,3,5,6,7,8,9,10)**
- **This material is covered in MUCH greater depth in ECE/CS 350 – please take ECE/CS 350 if you want to learn enough digital design to build your own processor**

What We've Done, Where We're Going



Computer = Machine That Manipulates Bits

- Everything is in binary (bunches of 0s and 1s)
  - Instructions, numbers, memory locations, etc.
- Computer is a machine that operates on bits
  - Executing instructions → operating on bits

- Computers physically made of transistors
  - Electrically controlled switches
- We can use transistors to build logic
  - E.g., if this bit is a 0 and that bit is a 1, then set some other bit to be a 1
  - E.g., if the first 5 bits of the instruction are 10010 then set this other bit to 1 (to tell the adder to subtract instead of add)

How Many Transistors Are We Talking About?

Pentium III

- **Processor Core 9.5 Million Transistors**
- **Total: 28 Million Transistors**

Pentium 4

- **Total: 42 Million Transistors**

Core2 Duo (two processor cores)

- **Total: 290 Million Transistors**

Core2 Duo Extreme (4 processor cores, 8MB cache)

- **Total: 590 Million Transistors**

Core i7 with 6-cores

- **Total: 2.27 Billion Transistors**

How do they design such a thing? Carefully!

Abstraction!

- Use of abstraction (key to design of any large system)
  - Put a few (2-8) transistors into logic gate (OR, AND, XOR, …)
  - Combine gates into logical functions (add, select,….)
  - Combine adders, shifters, etc., together into modules
    Units with well-defined interfaces for large tasks: e.g., decode
  - Combine a dozen of those into a core…
  - Stick 4 cores on a chip…

You are here:

- Use of abstraction (key to design of any large system)
  - Put a few (2-8) transistors into a logic gate
  - Combine gates into logical functions (add, select,….)
  - Combine adders, muxes, etc together into modules
    - Units with well-defined interfaces for large tasks: e.g., decode
  - Combine a dozen of those into a core…
  - Stick 4 cores on a chip…

Boolean Algebra

- **First step to logic: Boolean Algebra**
  - **Manipulation of True / False (1/0)**
  - **After all: everything is just 1s and 0s**

- **Given inputs (variables): A, B, X, P, Q…**
  - **Compute outputs using logical operators, such as:**
- **NOT:** !A **(=** ~A **=** $\overline{A}$**)**
- **AND:** A&B **(=** A·B **=** A*B = AB **=** A∧B) = A&&B in C/C++
- **OR:** A | B **(=** A+B **=** A ∨ B**) =** A || B in C/C++
- **XOR:** A ^ B (= A ⊕ B)
- NAND, NOR, XNOR, Etc.

Truth Tables

- **Can represent as Truth Table: shows outputs for all inputs**

a	NOT(a)
0	1
1	0

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a	b	AND(a,b)
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0	1	0
1	0	0
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Truth Tables

- **Can represent as truth table: shows outputs for all inputs**

a	NOT(a)
0	1
1	0

a	b	AND(a,b)
0	0	0
0	1	0
1	0	0
1	1	1

a	b	OR(a,b)
0	0	0
0	1	1
1	0	1
1	1	1

Truth Tables

- Can represent as truth table: shows outputs for all inputs

a	NOT(a)
0	1
1	0

a	b	AND(a,b)
0	0	0
0	1	0
1	0	0
1	1	1

a	b	OR(a,b)
0	0	0
0	1	1
1	0	1
1	1	1

a	b	XOR(a,b)
0	0	0
0	1	1
1	0	1
1	1	0

a	b	NAND(a,b)
0	0	1
0	1	1
1	0	1
1	1	0

a	b	NOR(a,b)
0	0	1
0	1	0
1	0	0
1	1	0

Any Inputs, Any Outputs

- Can have any # of inputs, any # of outputs
- Can have arbitrary functions:

a	b	c	f_1	f_2
0	0	0	0	1
0	0	1	1	1
0	1	0	1	0
0	1	1	0	0
1	0	0	1	0
1	0	1	1	1
1	1	0	0	1
1	1	1	1	1

Let's Write a Truth Table for a Function…

- **Example:**

  (A & B) | !C

Start with Empty TT

Column Per Input

Column Per Output

A	B	C	Output

Let's write a Truth Table for a function…

- **Example:**
  - **(A & B) | !C**

Start with Empty TT

- **Column Per Input**
- **Column Per Output**

Fill in Inputs

- **Counting in Binary**

A	B	C	Output
0	0	0	

Let's write a Truth Table for a function…

- **Example:**

  (A & B) | !C

Start with Empty TT

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Counting in Binary

A	B	C	Output
0	0	0	
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A	B	C	Output
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

Let's write a Truth Table for a function…

- **Example:**

  (A & B) | !C

Start with Empty TT

Column Per Input

Column Per Output

Fill in Inputs

Counting in Binary

Compute Output

(0 & 0) | !0 = 0 | 1 = 1

A	B	C	Output
0	0	0	1
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

Let's write a Truth Table for a function…

- **Example:**

  (A & B) | !C

Start with Empty TT

  Column Per Input

  Column Per Output

Fill in Inputs

  Counting in Binary

Compute Output

  (0 & 0) | !1 = 0 | 0 = 0

A	B	C	Output
0	0	0	1
0	0	1	0
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

Let's write a Truth Table for a function…

- **Example:**
  (A & B) | !C

Start with Empty TT

Column Per Input

Column Per Output

Fill in Inputs

Counting in Binary

Compute Output

(0 & 1) | !0 = 0 | 1 = 1

A	B	C	Output
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

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0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

You try one…

- **Try one yourself (take 2 minutes):**
  - **(!A | B) & !C**

Suppose I turn it around…

- **Given a Truth Table, find the formula?**

Hmmm..

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Suppose I turn it around…

- **Given a Truth Table, find the formula?**

Hmmm …

Could write down every "true" case

Then OR together:

```
(!A & !B & !C) |
(!A & !B & C)  |
(!A & B & !C) |
(A & B &!C) |
(A & B &C)
```

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
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0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Suppose I turn it around…

- This approach is called "sum of products"
  - Works every time
  - Result is right…
  - But really ugly

```
(!A & !B & !C) |
(!A & !B & C)  |
(!A & B & !C)  |
(A & B &!C)  |
(A & B &C)
```

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
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(!A & !B & !C) |
(!A & !B & C)  |
(!A & B & !C) |
(A & B &!C) |
(A & B &C)
```

Could just be (A & B) here ?

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Suppose I turn it around…

- This approach: "sum of products"
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```
(!A & !B & !C)  |
(!A & !B & C)   |
(!A & B & !C)  |
(A&B)
```

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
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- This approach: “sum of products”
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```
(!A & !B & !C) |
(!A & !B & C)  |
(!A & B & !C) |
(A&B)
```

Could just be (!A & !B) here

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Suppose I turn it around…

- **This approach: "sum of products"**
  - Works every time
  - Result is right…
  - But really ugly

```
(!A & !B) |
(!A & B & !C) |
(A&B)
```

Looks nicer…

Can we do better?

A	B	C	Output
0	**0**	**0**	**1**
0	**0**	**1**	**1**
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Suppose I turn it around…

- This approach: “sum of products”
  - Works every time
  - Result is right…
  - But really ugly

```
(!A & !B) |
(!A & B & !C) |
(A&B)
```

This has a lot in common:

!A & (something)

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Suppose I turn it around…

- This approach: “sum of products”
  - Works every time
  - Result is right…
  - But really ugly

```
(!A & !(B & C)) |
(A & B)
```

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Just did some of these by intuition.. but

- Somewhat intuitive approach to simplifying
- This is math, so there are formal rules
  - Just like “regular” algebra

Boolean Function Simplification

- **Boolean expressions can be simplified by using the following rules (bitwise logical):**
  - `A & A = A` `A | A = A`
  - `A & 0 = 0` `A | 0 = A`
  - `A & 1 = A` `A | 1 = 1`
  - `A & !A = 0` `A | !A = 1`
  - `!!A = A`
  - **& and | are both commutative and associative**
  - **& and | can be distributed:** `A & (B | C) = (A & B) | (A & C)`
  - **& and | can be subsumed:** `A | (A & B) = A`

DeMorgan's Laws

- **Two (less obvious) Laws of Boolean Algebra:**
  - **Let's push negations inside, flipping & and |**

```
!(A & B) = (!A) | (!B)
```

```
!(A | B) = (!A) & (!B)
```

  - **You should try this at home – build truth tables for both the left and right sides and see that they're the same**

Using DeMorgan on Early Example

```
(!A & !(B & C)) |
(A & B)

=

(!A & (!B | !C)) |
(A & B)
```

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Simplification Example:

```
! (!A | !(A & (B | C)))
```

Simplification Example:

```
! (!A | !(A & (B | C)))
```

DeMorgan's

```
!!A & !! (A & (B | C))
```

Simplification Example:

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! (!A | !(A & (B | C)))
```

DeMorgan's

```
!!A & !! (A & (B | C))
```

Double Negation Elimination

```
A & (A & (B | C))
```

Simplification Example:

```
! (!A | !(A & (B | C)))
```

DeMorgan's

```
!!A & !! (A & (B | C))
```

Double Negation Elimination

```
A & (A & (B | C))
```

Associativity of &

```
(A & A) & (B | C)
```

Simplification Example:

`! (!A | !(A & (B | C)))`

DeMorgan's

`!!A & !! (A & (B | C))`

Double Negation Elimination

`A & (A & (B | C))`

Associativity of &

`(A & A) & (B | C)`

A & A = A

`A & (B | C)`

You try this:

Come up with a formula for this Truth Table

Simplify as much as possible

A	B	C	Output
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Something That Trips Up Students

!A&!B   is not the same as !(AB)

Students tend to notice this when it's written this way but NOT when it's written with lines above the terms:

$\overline{A} \cdot \overline{B} \neq \overline{AB}$

"Not A and Not B" is not the same as "Not AB"

The former is true if (A,B) = (0,0).

The latter is true if (A,B) = (0,0) or (0,1) or (1,0).

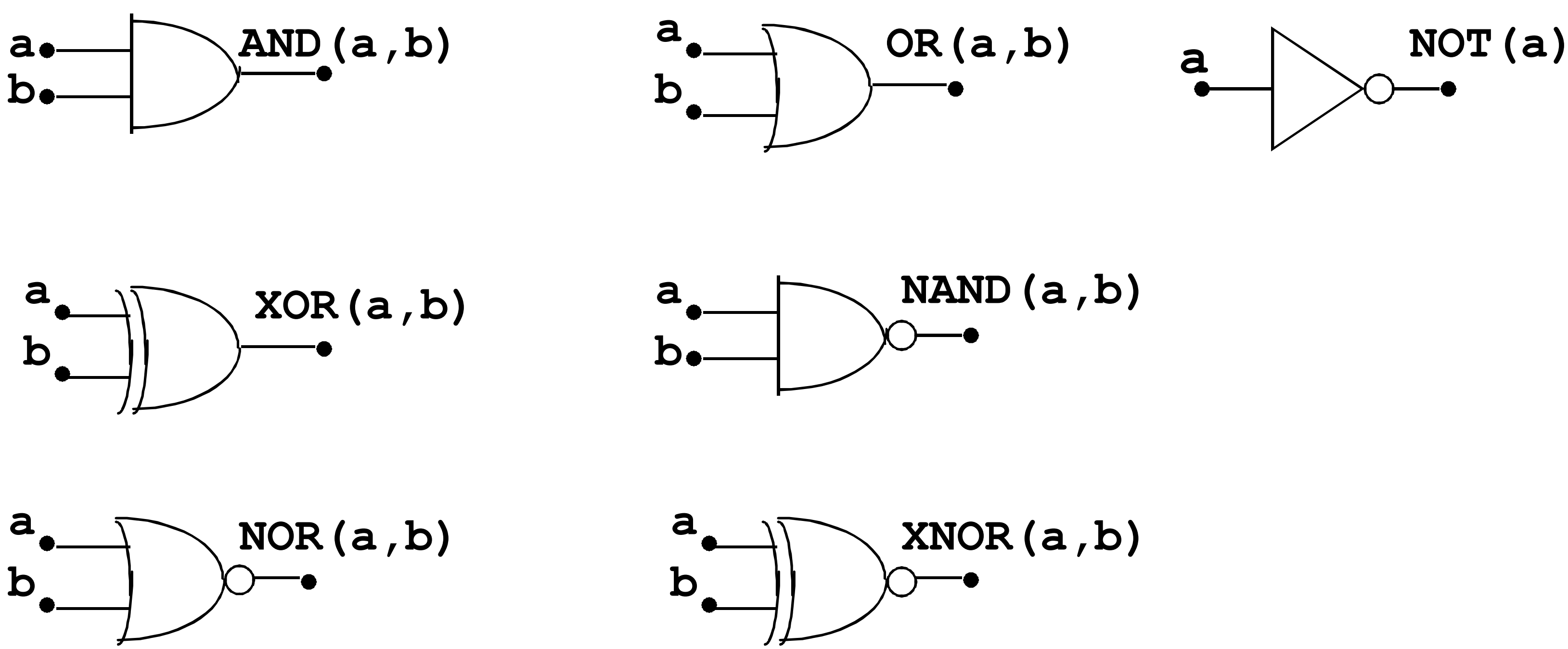
Applying the Theory

- **Lots of good theory**
- **Can reason about complex Boolean expressions**
- **But why is this useful?   (fun party trick)**

Boolean Gates

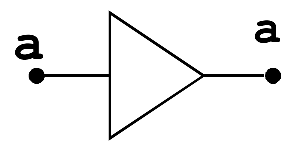
- **Gates are electronic devices that implement simple Boolean functions (building blocks of hardware)**

Examples



Guide to Remembering your Gates

- **This one looks like it just points its input where to go**
  - **It just produces its input as its output**
  - **Called a buffer**



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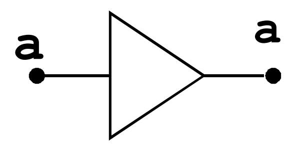
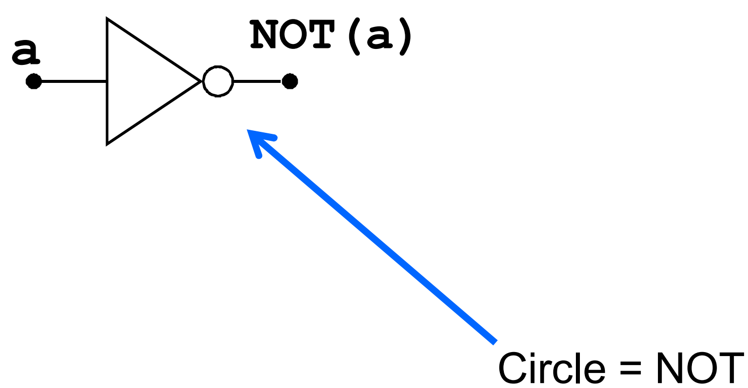
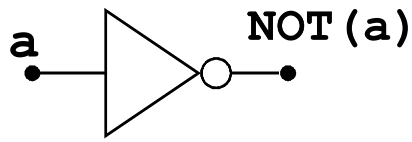


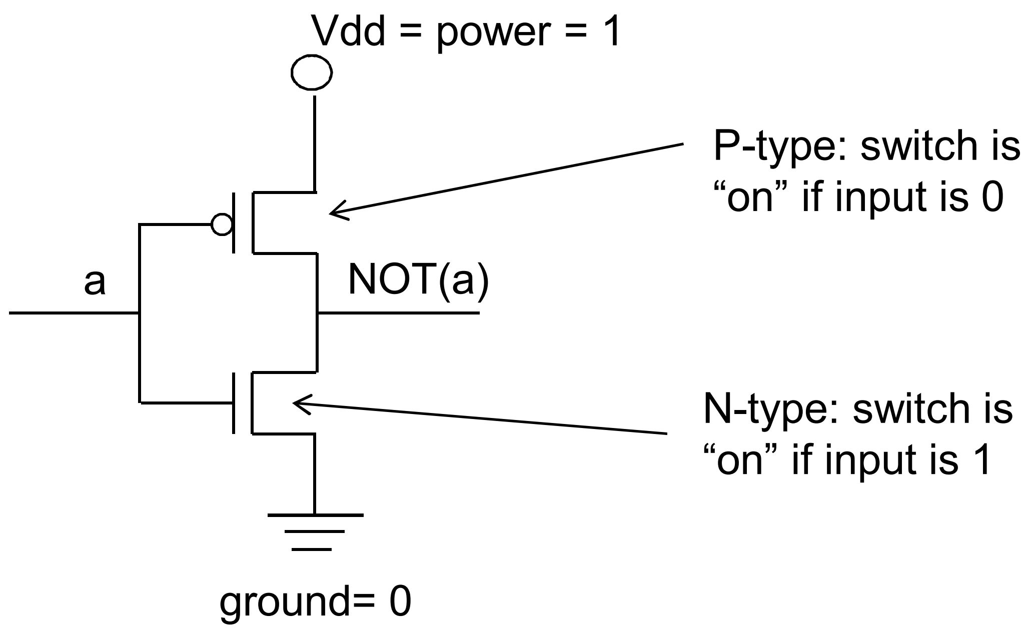
- **A circle always means negate (invert)**



Brief Interlude: Building An Inverter

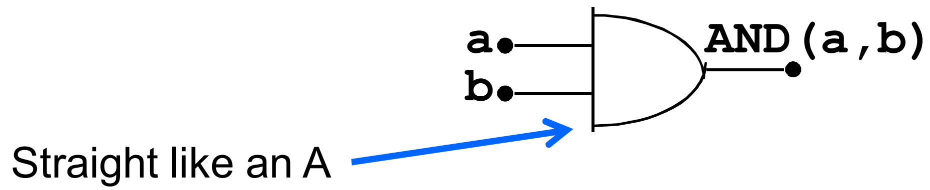
This is not on the test.



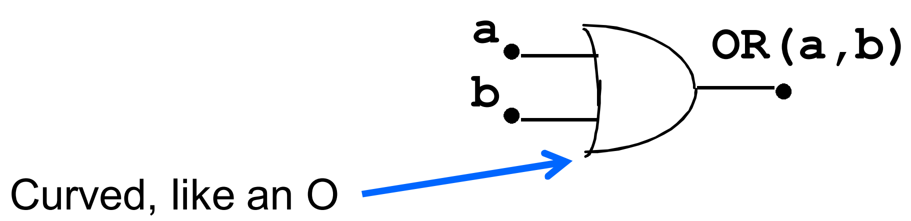


Guide to Remembering Your Gates

- **AND Gates have a straight edge, like an A (in AND)**

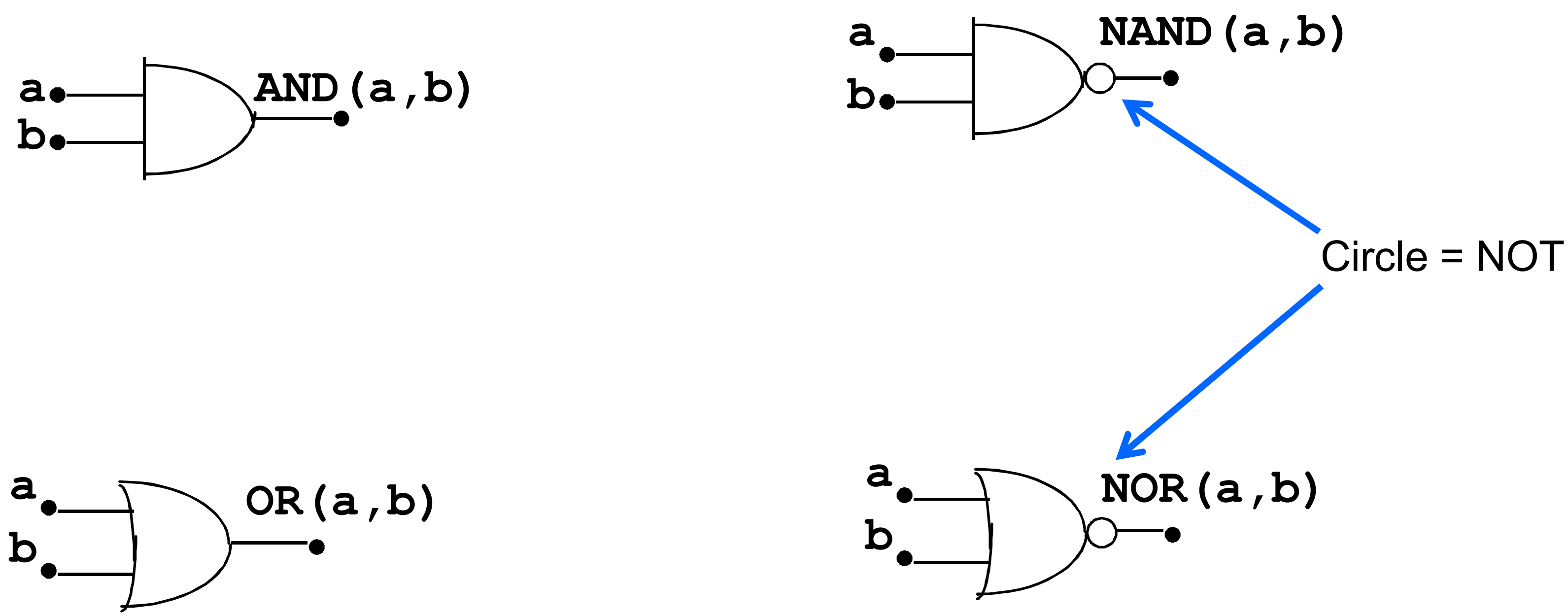


OR Gates have a curved edge, like an O (in OR)



Guide to Remembering Your Gates

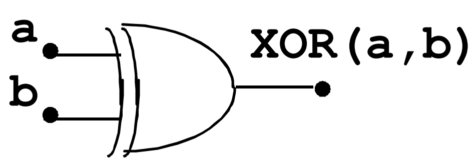
- If we stick a circle on them…



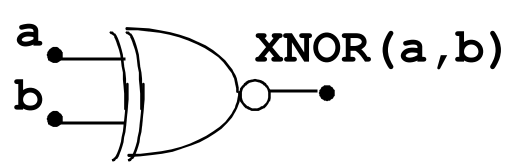
- We get NAND (NOT-AND) and NOR (NOT-OR)
  - NAND(a,b) = NOT(AND(a,b))

Guide to Remembering Your Gates

- XOR looks like OR (curved line)
  - But has two lines (like an X does)



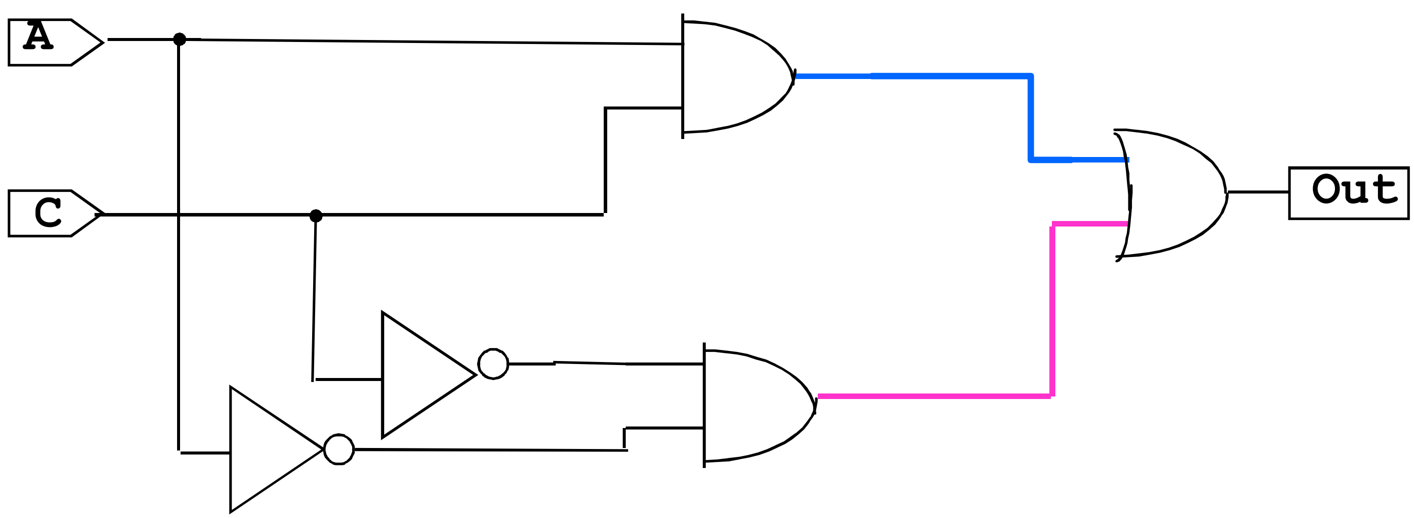
- Can put a dot for XNOR
  - XNOR is 1-bit "equals" by the way



Boolean Functions, Gates and Circuits

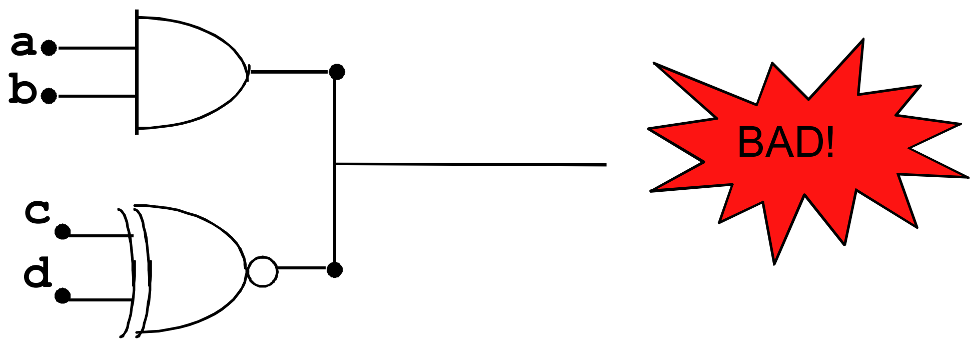
- **Circuits are made from a network of gates.**

`(!A & !C) | (A & C)`



A few more words about gates

- **Gates have inputs and outputs**
  - **If you try to hook up two outputs, bad things happen (your processor catches fire)**



  - **If you don't hook up an input, it behaves kind of randomly (also not good, but not set-your-chip-on-fire bad)**

Let's Make a Useful Circuit

- Pick between 2 inputs (called 2-to-1 MUX)
  - Short for multiplexor

- What might we do first?

Let's Make a Useful Circuit

- Pick between 2 inputs (called 2-to-1 MUX)
  - Short for multiplexor

- What might we do first?
  - Make a truth table?
    - S is selector:
      - S=0, pick A
      - S=1, pick B

A	B	S	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
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Let's Make a Useful Circuit

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  - Short for multiplexor
- What might we do first?
  - Make a truth table?
    - S is selector:
      - S=0, pick A
      - S=1, pick B
- Next: sum-of-products

```
(!A & B & S)  |
(A & !B & !S) |
(A & B & !S ) |
(A & B & S)
```

A	B	S	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

Let's Make a Useful Circuit

- Pick between 2 inputs (called 2-to-1 MUX)
  - Short for multiplexor
- What might we do first?
  - Make a truth table?
    - S is selector:
      - S=0, pick A
      - S=1, pick B
- Next: sum-of-products
- Simplify

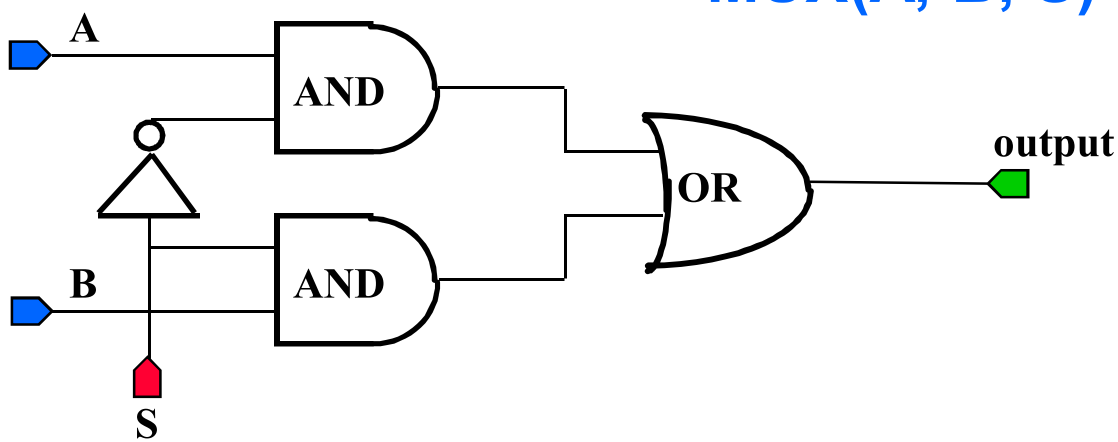
```
(A & !S) |
(B &  S)
```

A	B	S	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

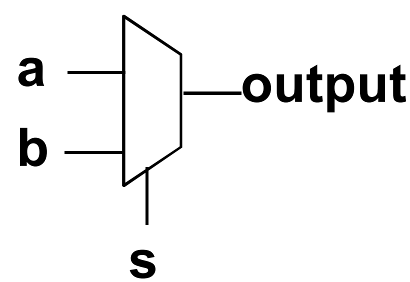
Circuit Example: 2x1 MUX

Draw it in gates:

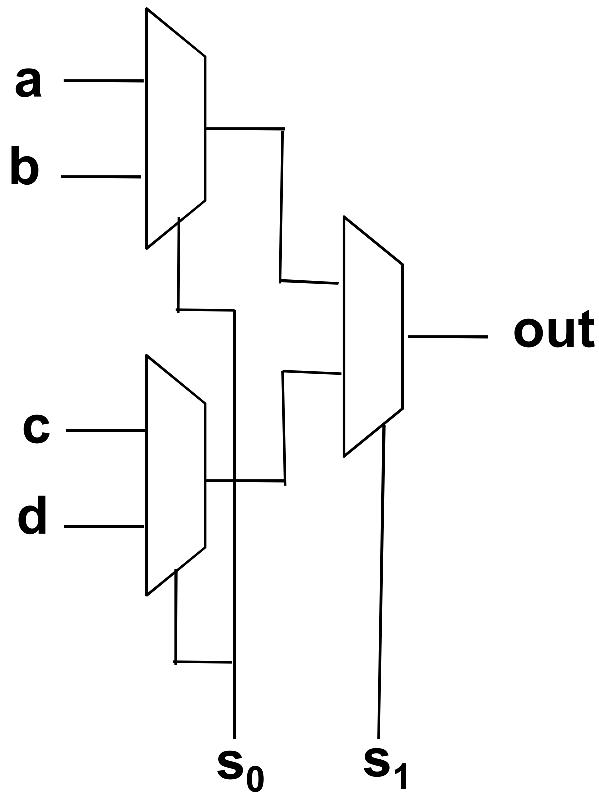
MUX(A, B, S) = (A & !S) | (B & S)

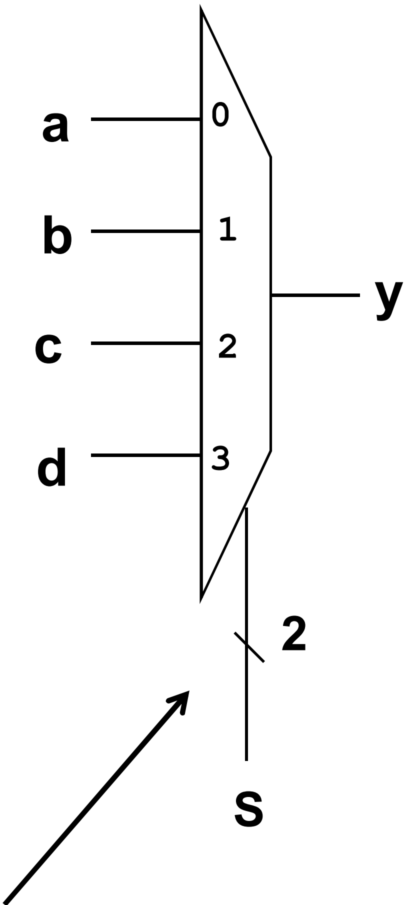


So common, we give it its own symbol:



Example 4x1 MUX



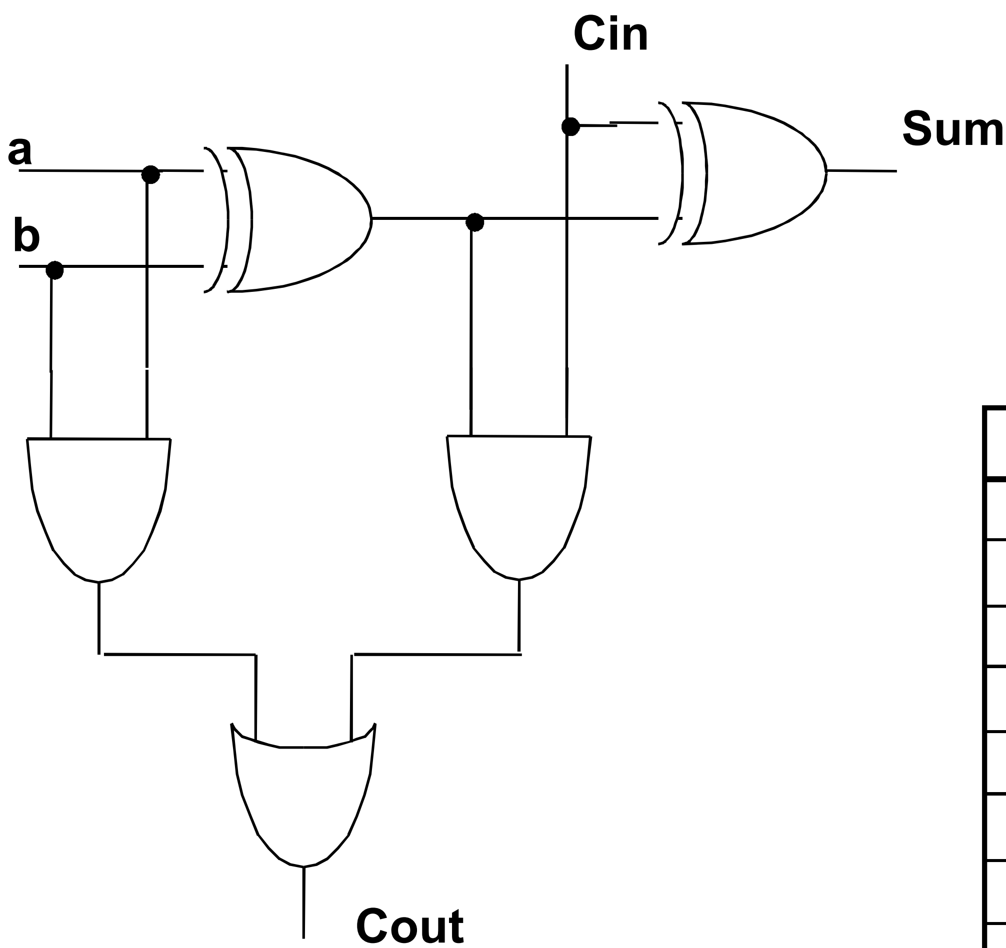


The / 2 on the wire means “2 bits”

Arithmetic and Logical Operations in ISA

- **What operations are there?**
- **How do we implement them?**
  - **Consider a 1-bit Adder**

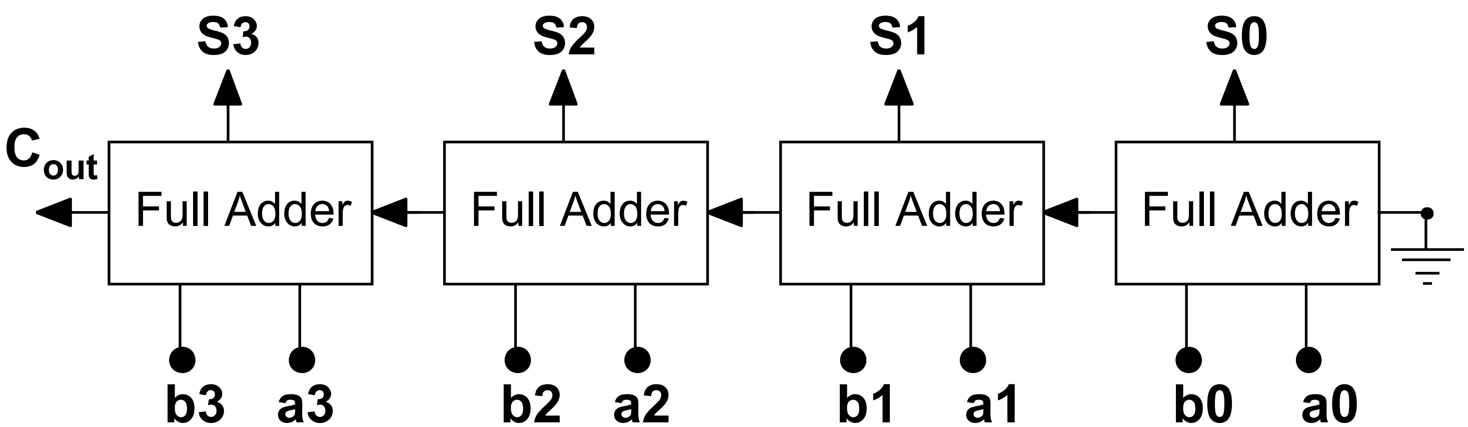
A 1-bit Full Adder




```
  01101100
  01101101
+00101100
  10011001
```

a	b	C_{in}	Sum	C_{out}
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Example: 4-bit adder



Subtraction

- **How do we perform integer subtraction?**
- **What is the hardware?**
  - **Recall: hardware was why 2's complement was good idea**

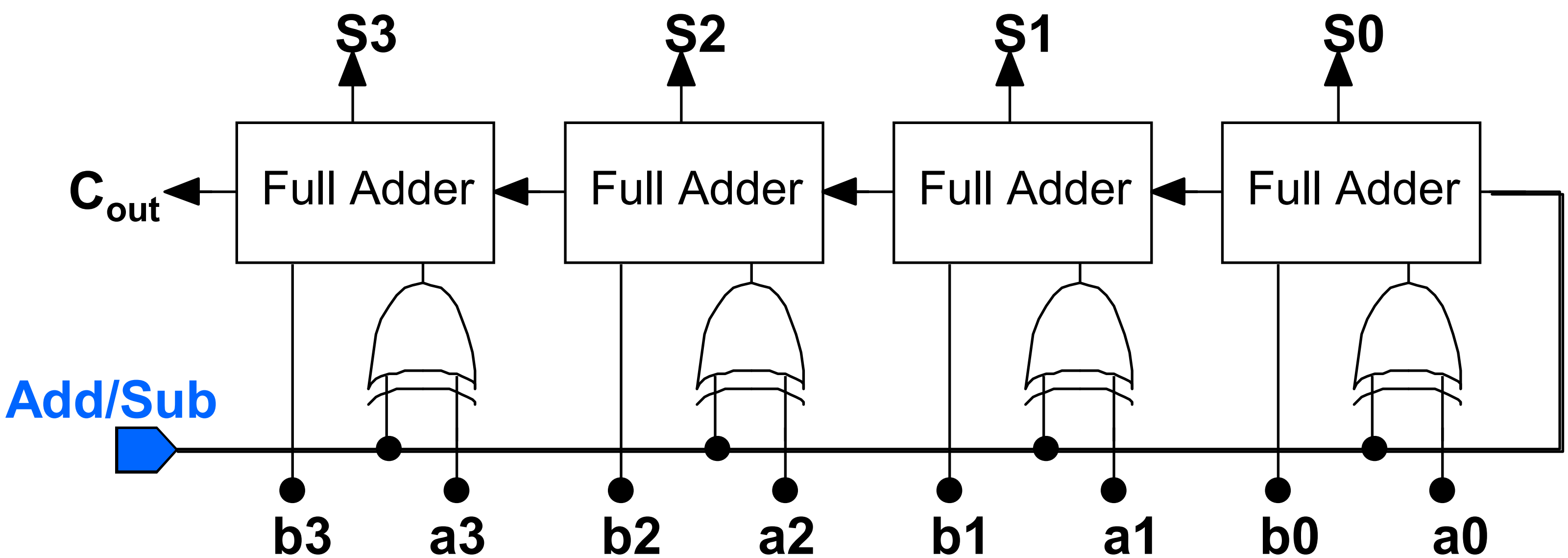
- **Remember: Subtraction is just addition**

  X – Y =

  X + (-Y) =

  X + (~Y +1)

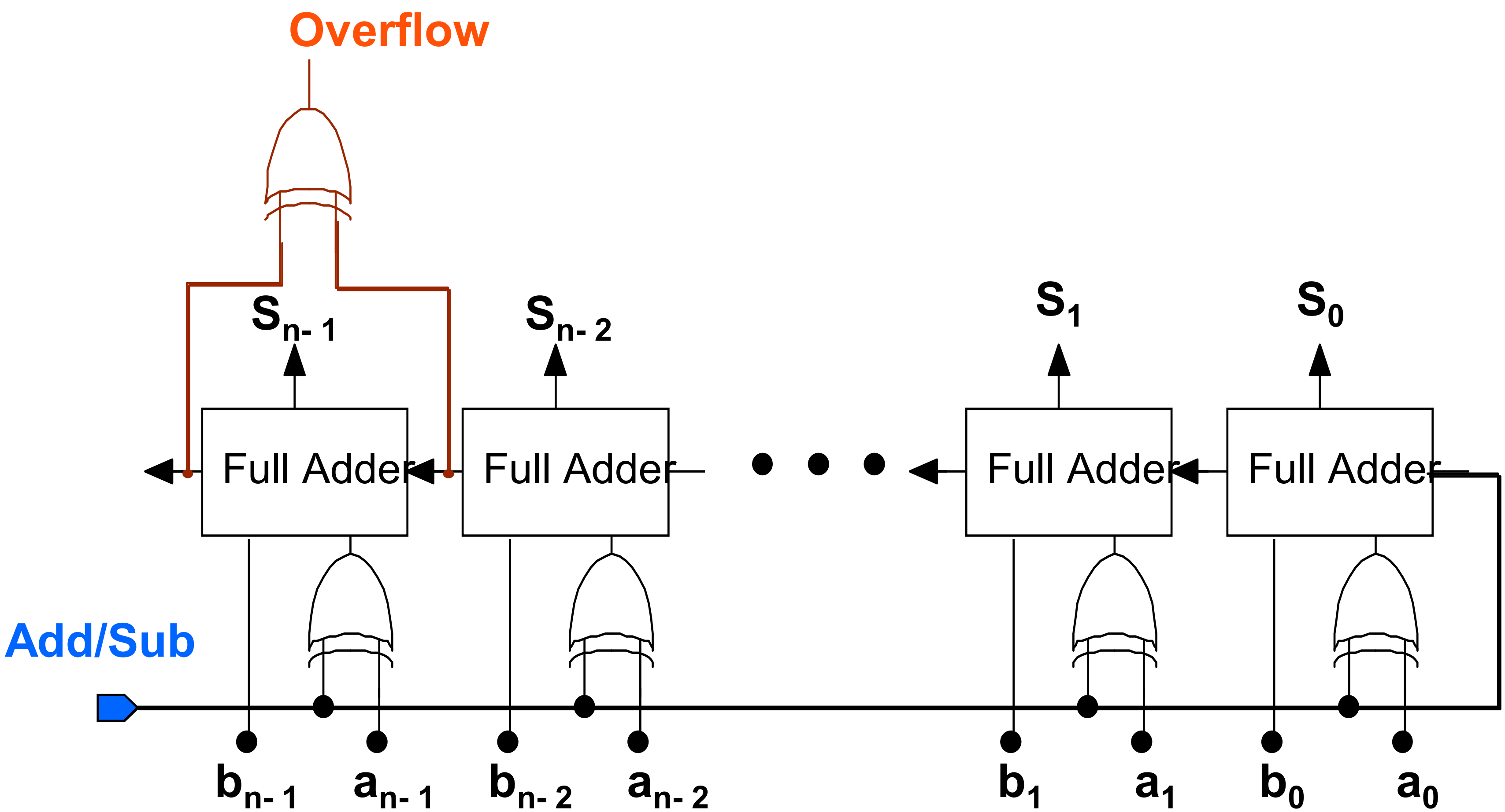
Example: Adder/Subtractor



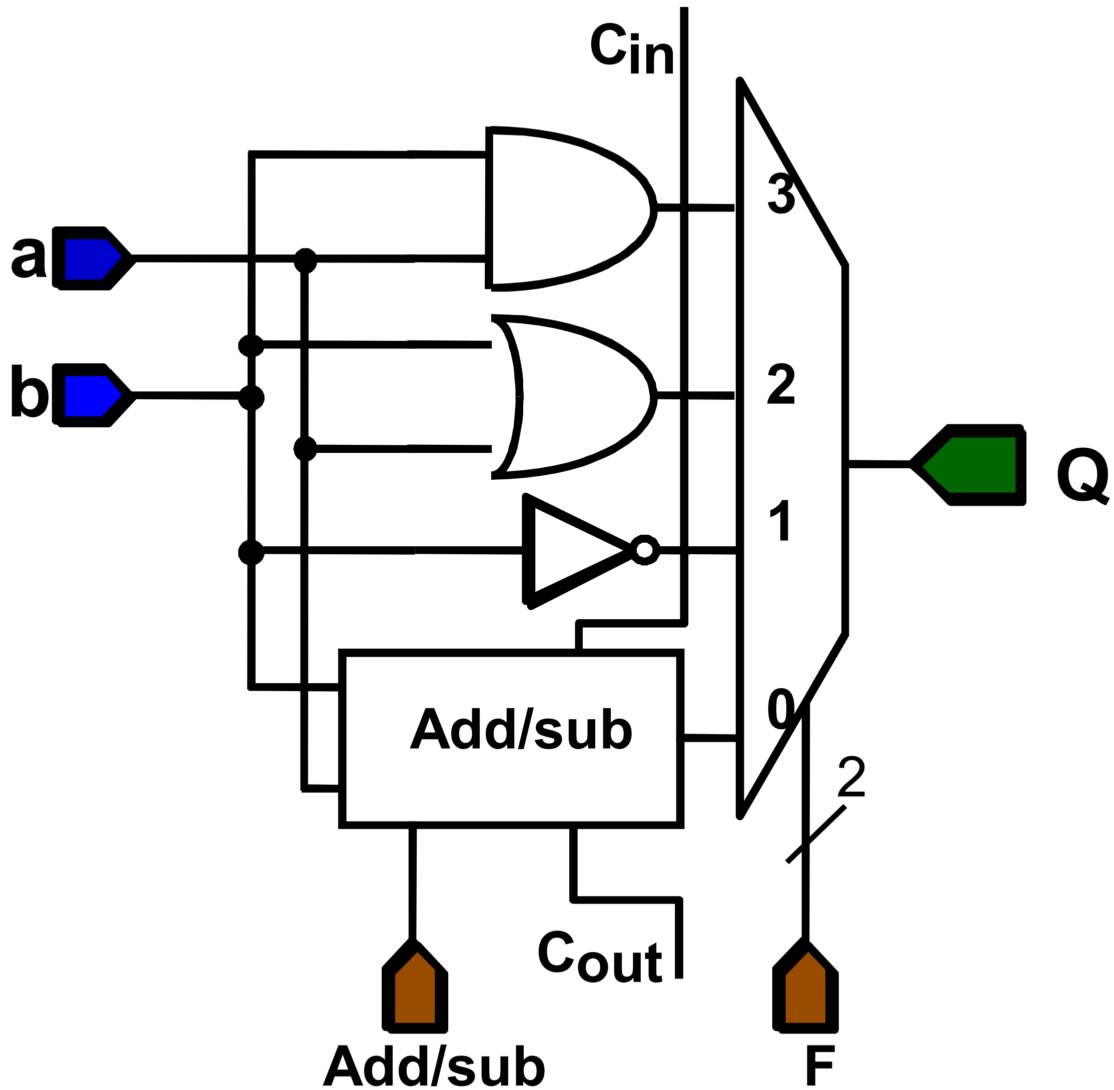
Overflow

- We can detect unsigned overflow by looking at CO
- How would we detect signed overflow?
  - If adding positive numbers and result “is” negative
  - If adding negative numbers and result “is” positive
  - At most significant bit of adder, check if CI != CO
  - Can check with XOR gate

Add/Subtract With Overflow Detection

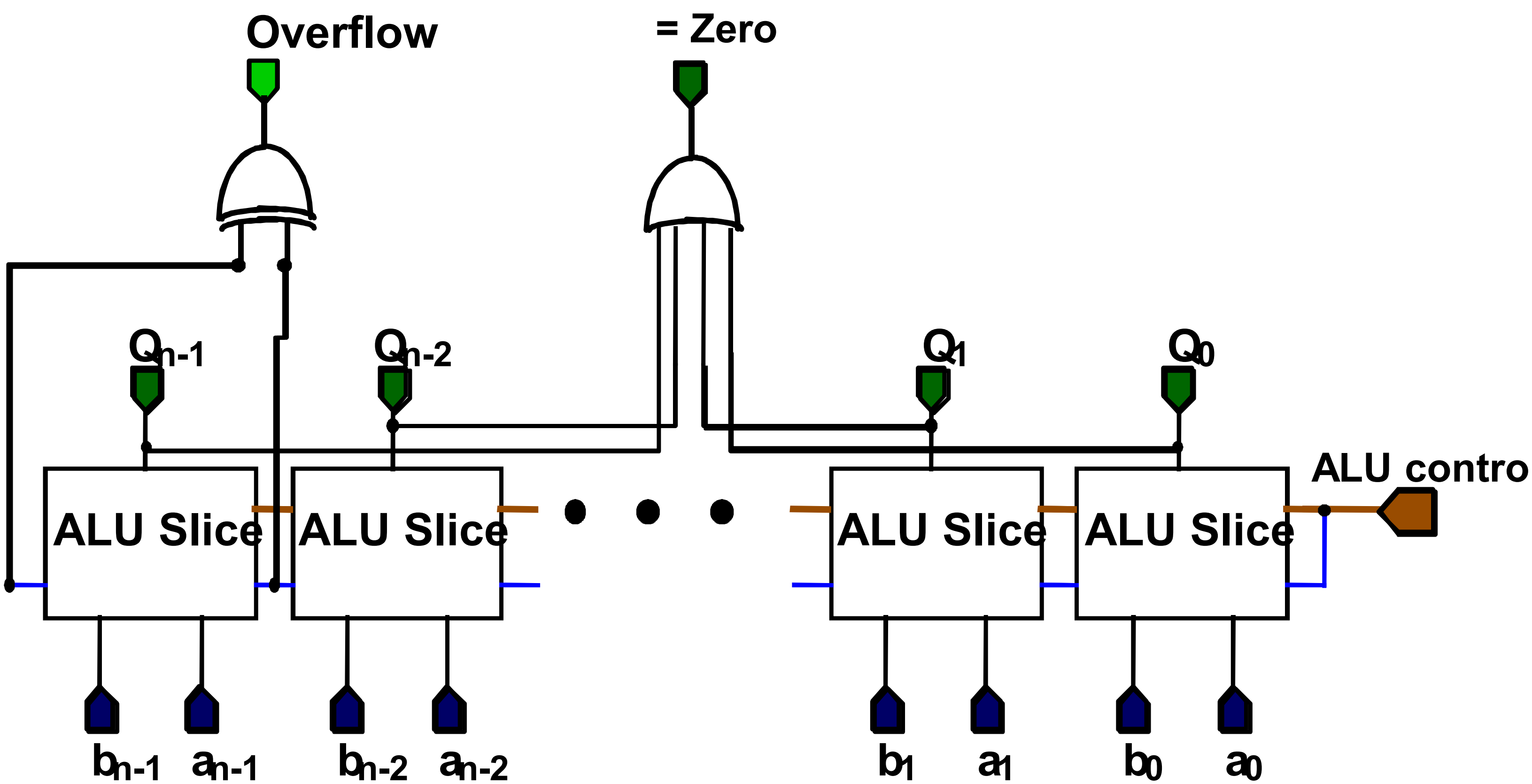


ALU Slice



A	F	Q
0	0	a + b
1	0	a - b
-	1	NOT b
-	2	a OR b
-	3	a AND b

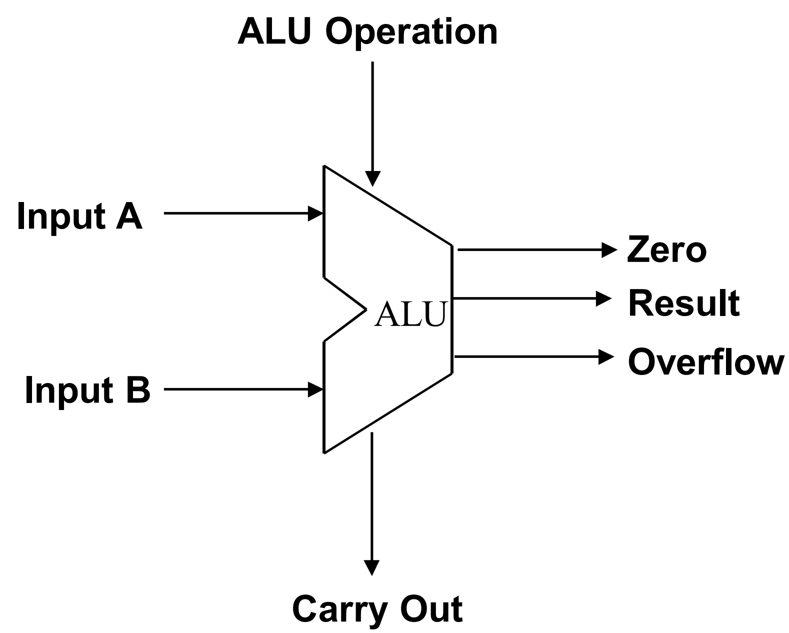
The ALU



Overflow
= Zero
Q_{n-1}
Q_{n-2}
Q_1
Q_0
ALU control
ALU Slice
ALU Slice
ALU Slice
ALU Slice
b_{n-1}
a_{n-1}
b_{n-2}
a_{n-2}
b_1
a_1
b_0
a_0

Abstraction: The ALU

- General structure
- Two operand inputs
- Control inputs



- We can build circuits for
  - Multiplication
  - Division
  - They are more complex (ECE/CS 350)

Another Operation We Might Want: Shift

- Remember the << and >> operations?
  - Shift left/shift right?
  - How would we implement these?
- Suppose you have an 8-bit number

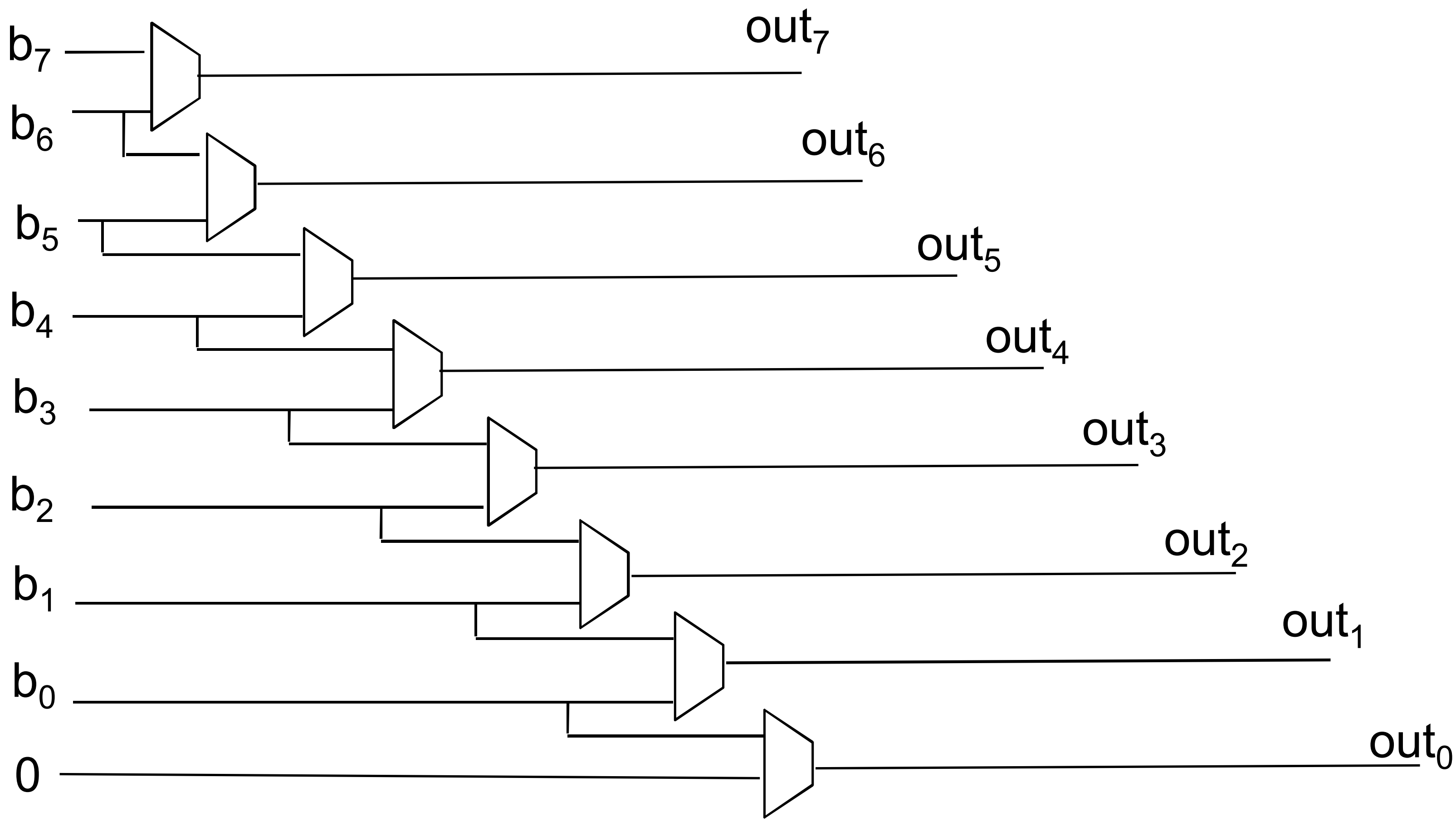
  $b_7b_6b_5b_4b_3b_2b_1b_0$
- And you can shift it left by a 3-bit number

  $s_2s_1s_0$

- Option 1: Truth Table?
  - 2^{11} = 2048 rows? Yuck.

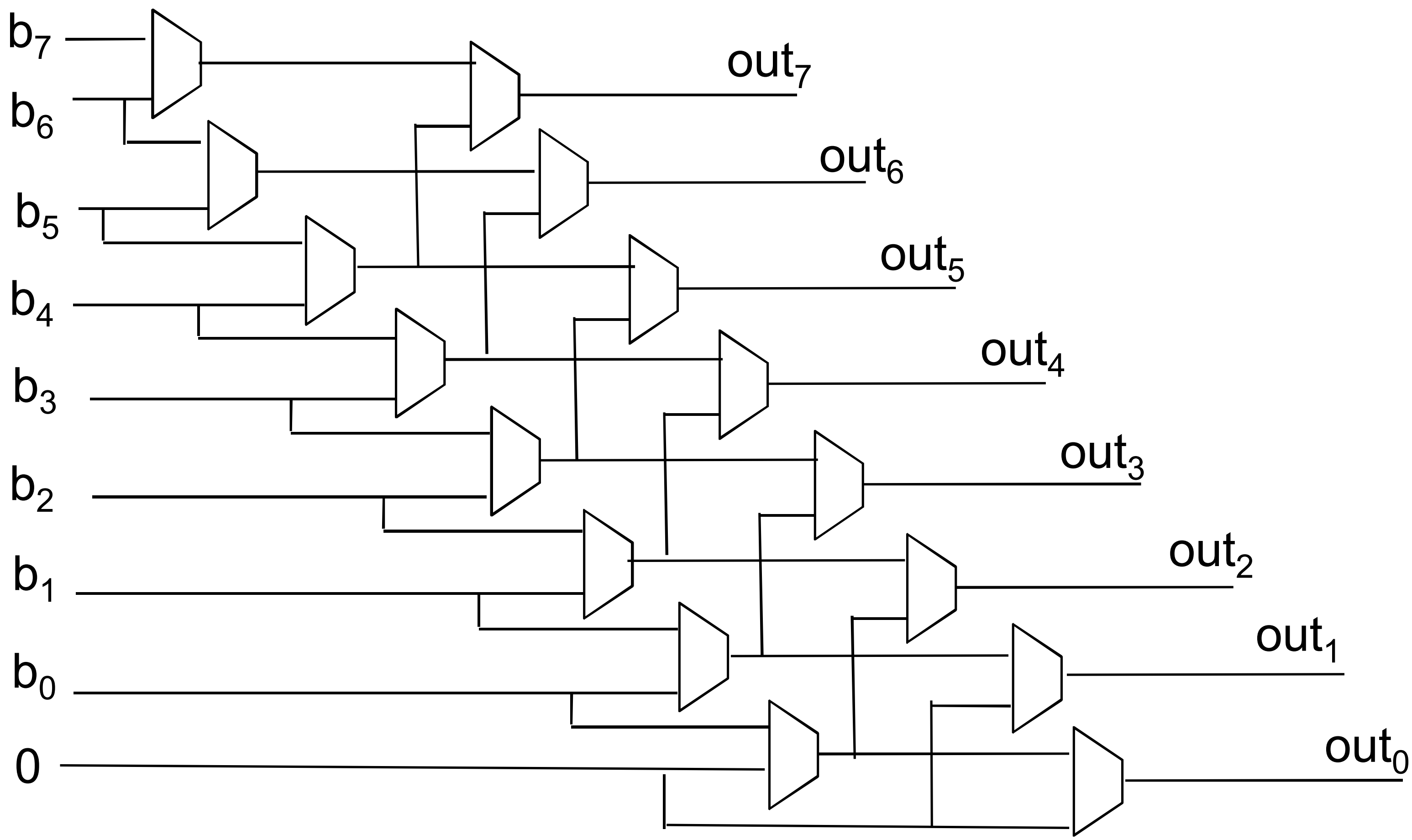
Let's simplify to 1-bit left-shift

- **Simpler problem: 8-bit number left-shifted by 1 bit number (shift amount selects each mux)**



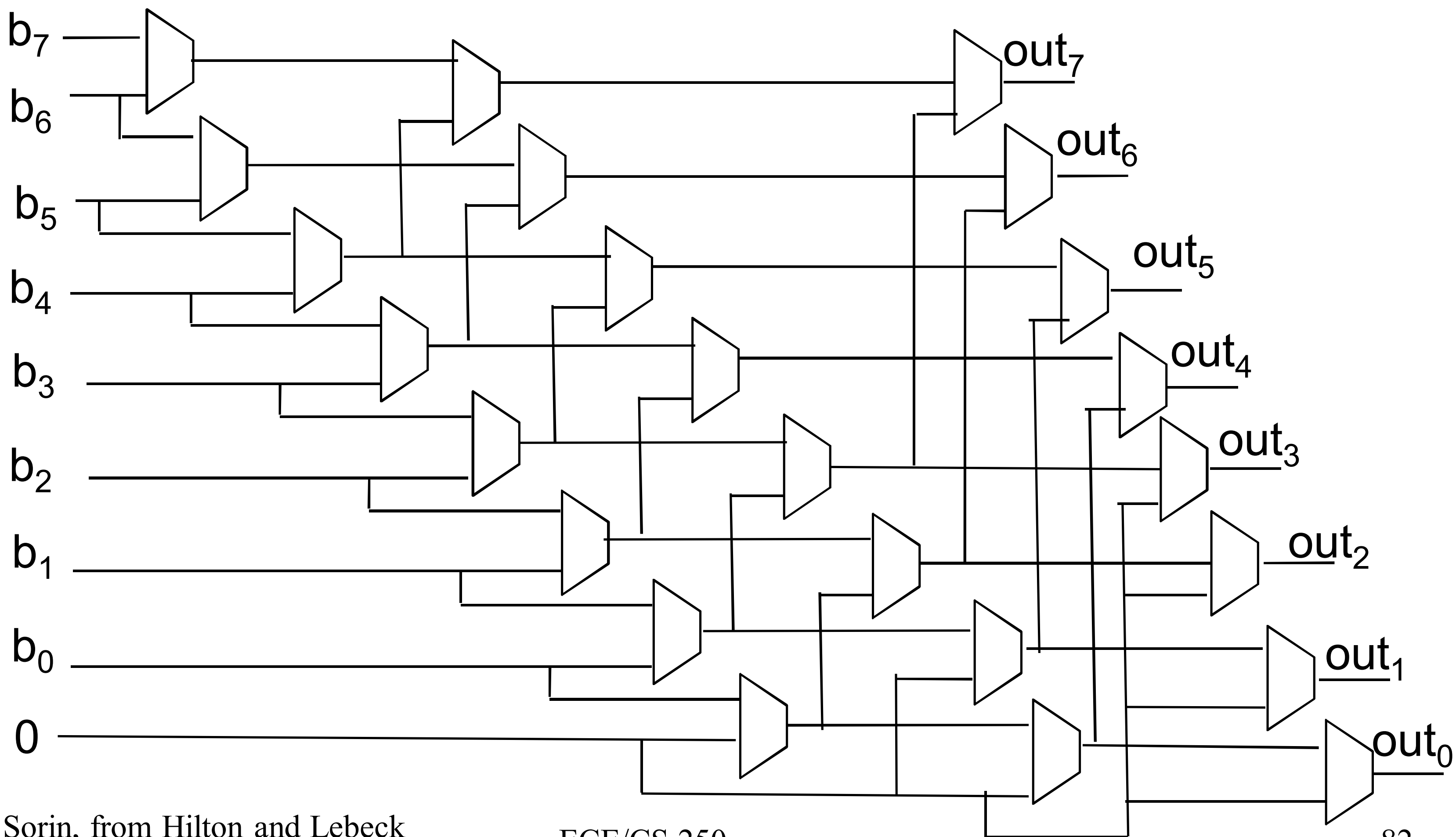
Let’s simplify to 2-bit left shift

- Simpler problem: 8-bit number left-shifted by 2 bit number (i.e., can left-shift by 0, 1, 2, or 3)



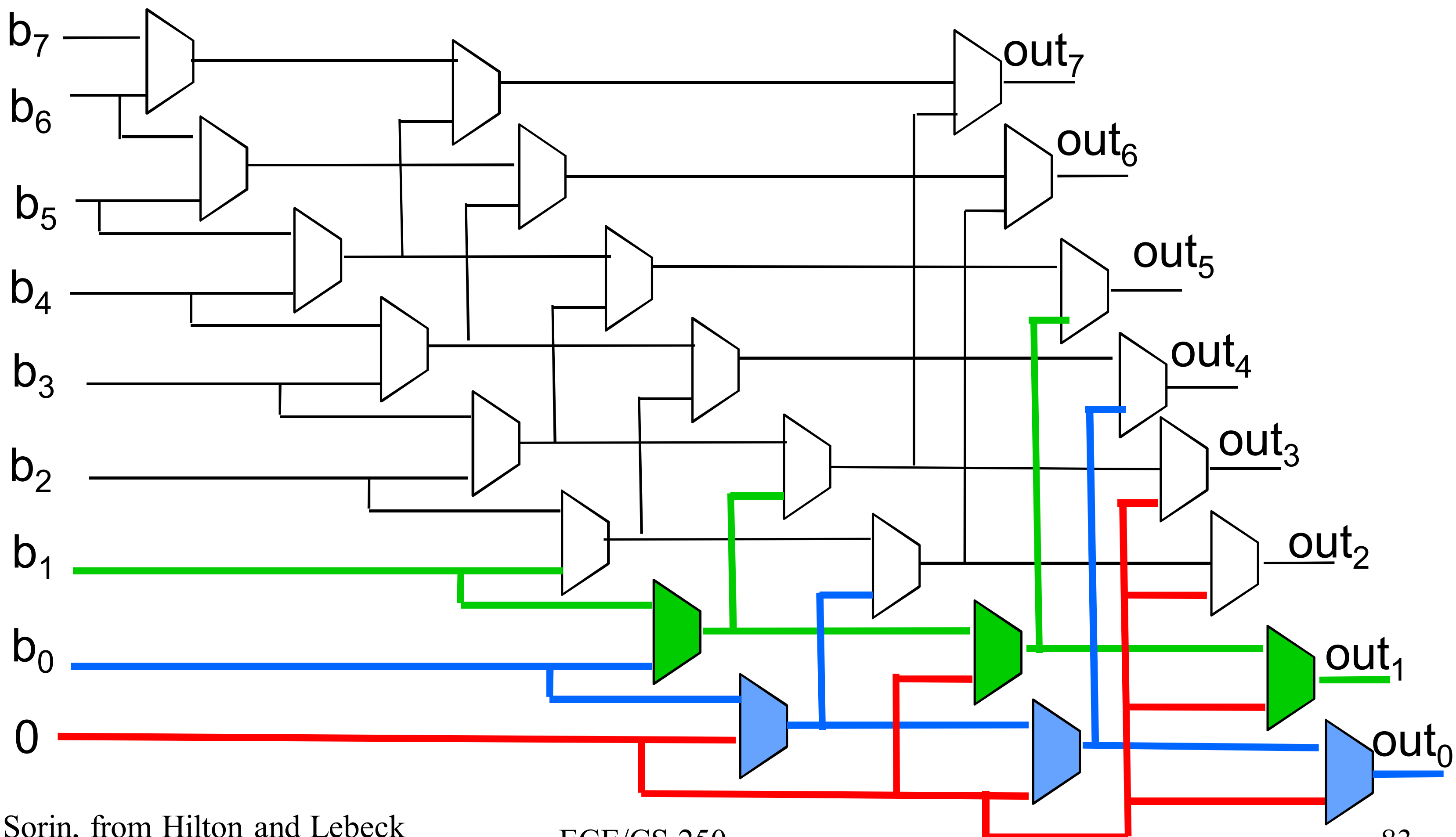
Now left-shifted by 3-bit number

- **Full problem: 8-bit number left-shifted by 3 bit number (can shift by 0-7 bits)**



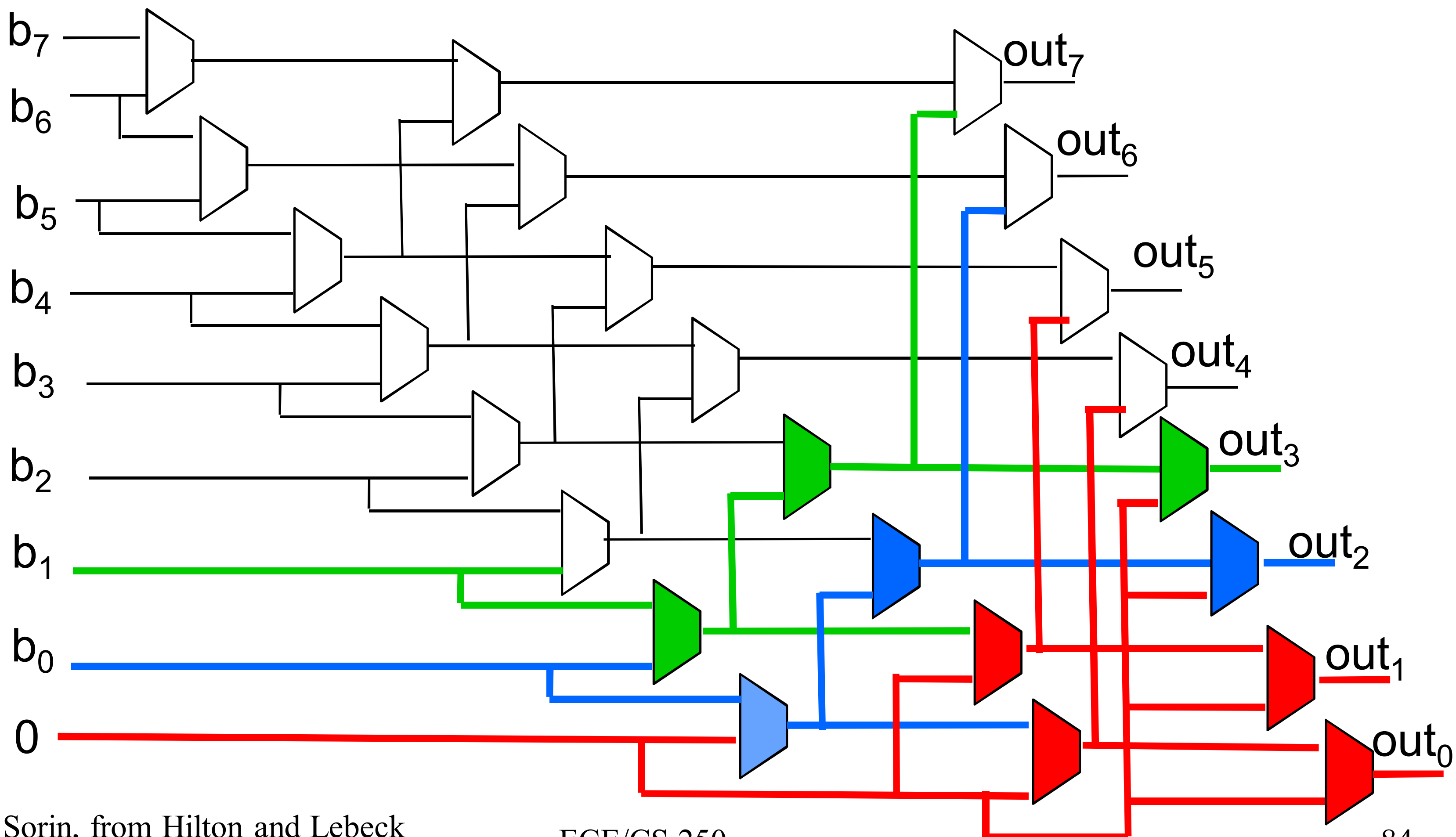
Now shifted by 3-bit number

- **Shifter in action: shift by 000 (all muxes have S=0)**



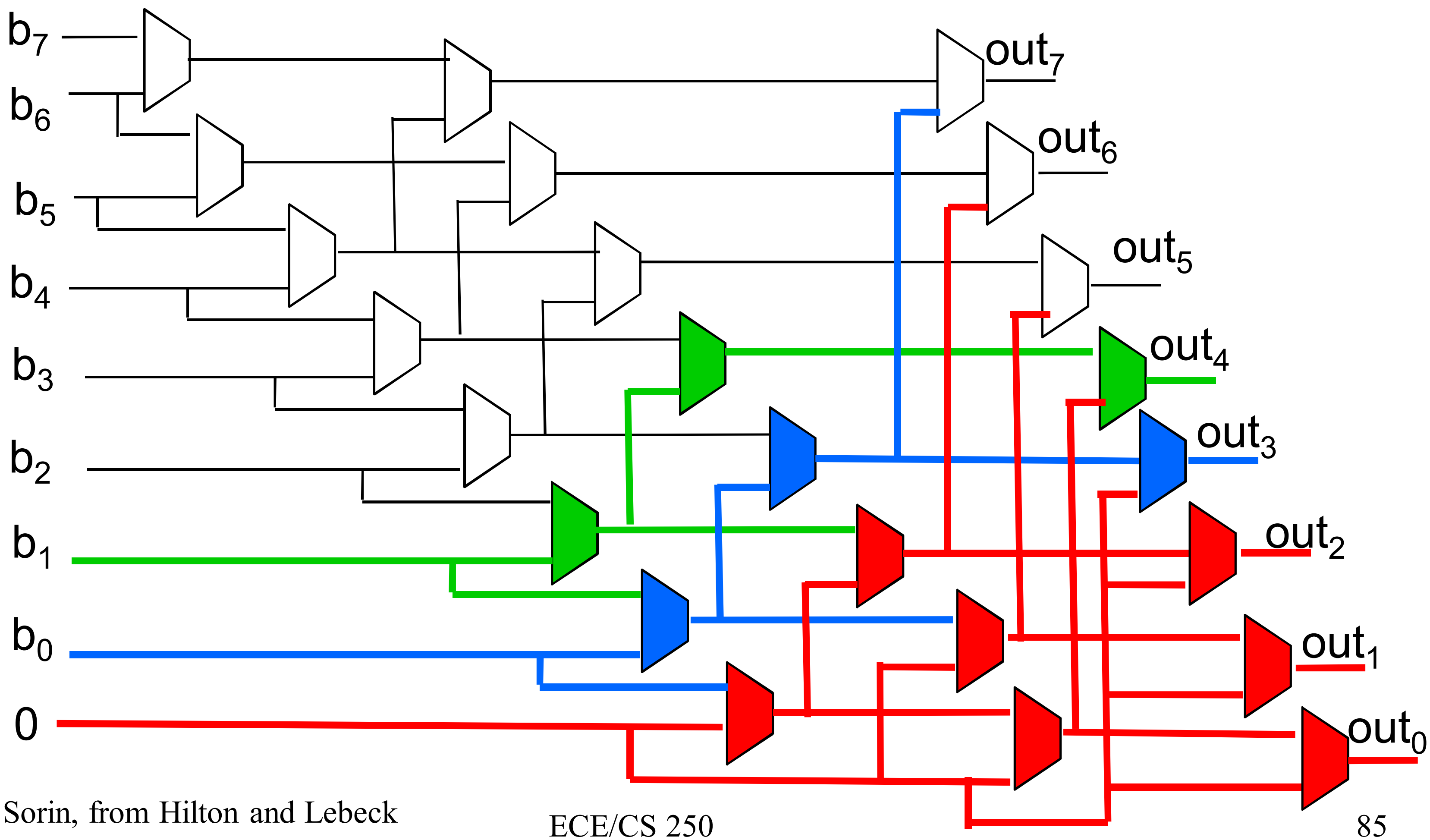
Now shifted by 3-bit number

- **Shifter in action: shift by 010 (=2_{10})**
  - **From L to R: S = 0, 1, 0**

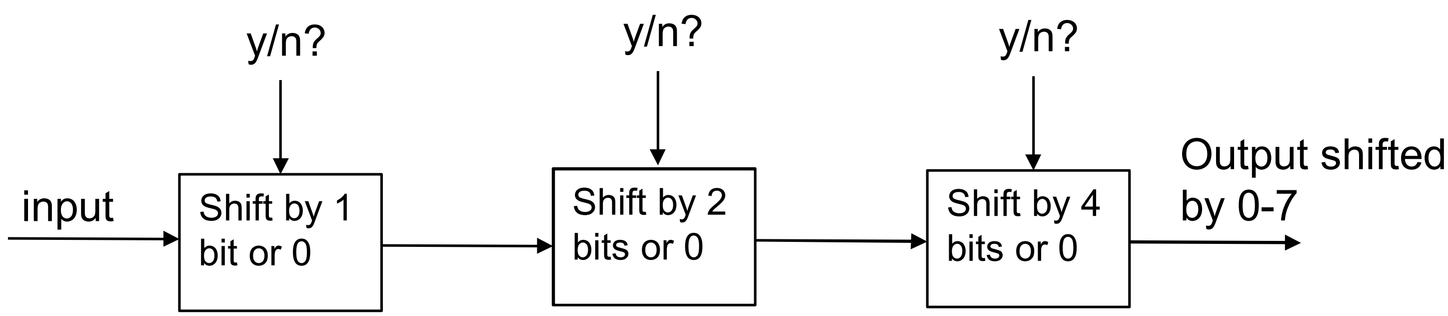


Now shifted by 3-bit number

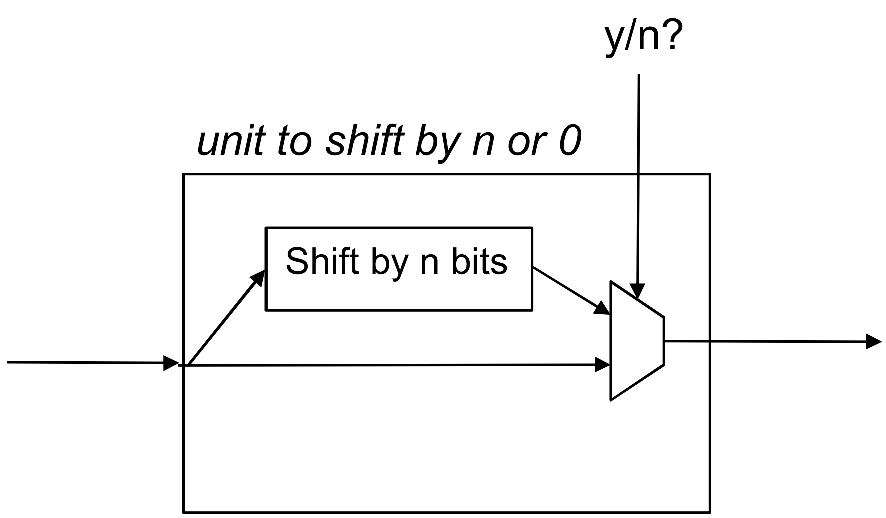
- **Shifter in action: shift by 011**
  - **From L to R: S= 1, 1, 0 (reverse of shift amount)**



Barrel Shifter



There are three control inputs (y/n). Any value from 0-7 can be achieved with them (000-111).



Summary

- Boolean Algebra & functions
- Logic gates (AND, OR, NOT, etc)
- Multiplexors
- Adder
- Arithmetic Logic Unit (ALU)