gration Practice Problems

① $\int e^{-\sqrt{x}}\,dx$ $\quad u = -\sqrt{x}$

$du = \frac{-1}{2\sqrt{x}}\,dx = \frac{1}{2u}\,dx \Rightarrow 2u\,du = dx$

$= \int e^{u} \cdot 2u\,du$ Use IBP $\quad w = 2u \quad dv = e^{u}\,du$

$dw = 2\,du \quad v = e^{u}$

$= 2ue^{u} - \int 2e^{u}\,du$

$= 2ue^{u} - 2e^{u} + C$

$\boxed{= 2(-\sqrt{x})e^{-\sqrt{x}} - 2e^{-\sqrt{x}} + C}$

② $\int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx$ Get rid of $\cos x$ so set

$u = \sin x \quad du = \cos x\,dx$

$= \int \cos^2 x\,du$

$= \int 1 - \sin^2 x\,du$

$= \int 1 - u^2\,du = u - \frac{1}{3}u^3 + C = \boxed{\sin x - \frac{1}{3}(\sin^3 x) + C}$

③ $\int \frac{1}{\sqrt{x^2+4x+8}}\,dx$ Complete the square $\int \frac{1}{\sqrt{(x+2)^2+4}}\,dx$

Let $u = x+2 \quad du = dx$

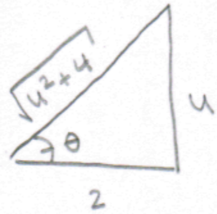
$= \int \frac{1}{\sqrt{u^2+4}}\,du$ Let $u = 2\tan\theta$ $\quad du = 2\sec^2\theta\,d\theta$ $\quad \int \frac{1}{\sqrt{4\tan^2\theta + 4}} \cdot 2\sec^2\theta\,d\theta$

$= \int \frac{1}{2\sqrt{\tan^2\theta+1}} \cdot 2\sec^2\theta\,d\theta = \int \frac{\sec^2\theta}{\sec\theta}\,d\theta = \int \sec\theta\,d\theta$

$= \ln|\tan\theta + \sec\theta| + C$

Draw the triangle:

$u = 2\tan\theta \Rightarrow u/2 = \tan\theta = {}^{o}/_{a}$



$$\therefore \ln|\tan\theta + \sec\theta| + C = \ln\left|\frac{u}{2} + \frac{\sqrt{u^2+4}}{2}\right| + C$$

$$\boxed{= \ln\left|\frac{x+2}{2} + \frac{\sqrt{(x+2)^2+4}}{2}\right| + C}$$

(4) $\displaystyle\int \frac{8x^3}{2x^2-x-1}\,dx$

(a) Do Long division

$$\begin{array}{r|l} & 4x^2+2 \\ \hline 2x^2-x-1 & 8x^3 \\ & -\;8x^3-4x^2-4x \\ \hline & 4x^2+4x \\ & -\;4x^2-2x-2 \\ \hline & 6x+2 \end{array}$$

Then $\displaystyle\frac{8x^3}{2x^2-x-1} = 4x+2 + \frac{6x+2}{2x^2-x-1}$

(b) Find the form

$$\frac{6x+2}{2x^2-x-1} = \frac{6x+2}{(2x+1)(x-1)} = \frac{A}{(2x+1)} + \frac{B}{(x-1)}$$

$$6x+2 = A(x-1) + B(2x+1)$$

Pick $x=1$: $6+2 = B(2+1) \Rightarrow \boxed{8/3 = B}$

Pick $x=-1/2$: $6(-1/2)+2 = A(-1/2-1) \Rightarrow \boxed{A = 2/3}$

(c) Integral

$$\int \frac{8x^3}{2x^2-x-1}\,dx = \int 4x+2 + \int \frac{8/3}{x-1} + \int \frac{2/3}{2x+1}\,dx$$

$$= 2x^2 + 2x + 8/3\ln|x-1| + \int \frac{2/3}{2u} \qquad u = 2x+1 \quad du = 2dx$$

$$\boxed{= 2x^2 + 2x + 8/3\ln|x-1| + \tfrac{1}{3}\ln|2x+1| + C}$$

5. $\displaystyle\int \frac{1}{(x+3)(\sqrt{x+4}-2)}\,dx$ $\quad u = \sqrt{x+4}$

$u^2 = x+4 \quad 2u\,du = dx$

$u^2 - 4 = x$

$$= \int \frac{2u}{(u^2-4+3)(u-2)}\,du = \int \frac{2u}{(u^2-1)(u-2)}\,du = \int \frac{2u}{(u+1)(u-1)(u-2)}\,du$$

a. Do long division (None)

b. $$\frac{2u}{(u+1)(u-1)(u-2)} = \frac{A}{u+1} + \frac{B}{u-1} + \frac{C}{u-2}$$

$$\Rightarrow 2u = A(u-1)(u-2) + B(u+1)(u-2) + C(u-1)(u+1)$$

Let $u=1$: $2(1) = B(2)(-1) \Rightarrow B = -1$

$u=2$: $2(2) = C(1)(3) \Rightarrow C = 4/3$

$u=-1$: $2(-1) = A(-2)(-3) \Rightarrow A = -1/3$

c. $$\int \frac{2u}{(u+1)(u-1)(u-2)}\,du = \int \frac{-1/3}{u+1} + \int \frac{-1}{u-1} + \int \frac{4/3}{u-2}$$

$$= \frac{-1}{3}\ln|u+1| - \ln|u-1| + 4/3\ln|u-2| + C$$

$$\boxed{= \frac{-1}{3}\ln|\sqrt{x+4}+1| - \ln|\sqrt{x+4}-1| + \frac{4}{3}\ln|\sqrt{x+4}-2| + C}$$

6. $\displaystyle\int 2x\ln(x+5)\,dx$ $\quad u = x+5 \quad du = dx$

$\displaystyle\int 2(u-5)\ln(u)\,du$ $\quad$ use IBP $\quad w = \ln(u) \quad dv = 2(u-5)\,du$

$dw = 1/u\,du \quad v = u^2 - 10u$

$$= (u^2-10u)\ln(u) - \int \frac{u^2-10u}{u}\,du$$

$$= (u^2-10u)\ln(u) - \tfrac{1}{2}u^2 + 10u + C$$

$$\boxed{= \left([x+5]^2 - 10(x+5)\right)\ln(x+5) - \tfrac{1}{2}(x+5)^2 + 10(x+5) + C}$$

⑦ $\int \sec x \tan^3 x\,dx = \int \sec x \tan^2 x \tan x\,dx$ $u = \sec x$
$du = \sec x \tan x\,dx$

$= \int \tan^2 x\,du$

$= \int \sec^2 x - 1\,du = \int u^2 - 1\,du = \frac{1}{3}u^3 - u + C$

$\boxed{= \frac{1}{3}(\sec^3 x) - \sec x + C}$

⑧ $\int \frac{\sin(3t)\cos(3t)}{\cos^2(3t) - 3\cos(3t) + 2}\,dt$ $u = 3t$ $du = 3dt$

$\frac{1}{3}\int \frac{\sin(u)\cos(u)}{\cos^2(u) - 3\cos(u) + 2}\,du$ $v = \cos(u)$ $dv = -\sin(u)\,du$

$= \frac{-1}{3}\int \frac{v}{v^2 - 3v + 2}\,dv$ Solve w/ partial fractions

ⓐ No long division

$= \frac{-1}{3}\int \frac{v}{(v-1)(v-2)}\,dv$ ⓑ Find form

$\frac{\frac{-1}{3}v}{(v-1)(v-2)} = \frac{A}{v-1} + \frac{B}{v-2}$

$\Rightarrow -\frac{1}{3}v = A(v-2) + B(v-1)$

$v = 2:\ -\frac{2}{3} = B$

$v = 1:\ -\frac{1}{3} = A(-1) \Rightarrow A = \frac{1}{3}$

ⓒ Integrate $\int \frac{-\frac{1}{3}v}{(v-1)(v-2)}\,dv = \int \frac{\frac{1}{3}}{v-1} + \int \frac{-\frac{2}{3}}{v-2}$

$= \frac{1}{3}\ln|v-1| - \frac{2}{3}\ln|v-2| + C$

$\boxed{= \frac{1}{3}\ln|\cos(3t) - 1| - \frac{2}{3}\ln|\cos(3t) - 2| + C}$

⑨ $\int_1^2 \frac{x^3}{x^2 + x + \frac{1}{2}}\,dx$ ⓐ Long division

$$\begin{array}{r l}
 & x - 1 \\
x^2 + x - \frac{1}{2} & \overline{)\,x^3} \\
 & -(x^3 + x^2 - \frac{1}{2}x) \\
 & -x^2 + \frac{1}{2}x \\
 & -(-x^2 - x + \frac{1}{2}) \\
 & \frac{3}{2}x - \frac{1}{2}
\end{array}$$

Now $\dfrac{x^3}{x^2+x-1/2} = x-1+\dfrac{3/2x-1/2}{x^2+x-1/2}$

$$\int_1^2 x-1\,dx+\int_1^2 \frac{3/2x-1/2}{x^2+x-1/2} = \int_1^2 x-1\,dx+\int_1^2 \frac{3/2x-1/2}{(x+1/2)^2-3/4}\,dx$$

$u = x+1/2 \Rightarrow u-1/2=x$
$du=dx$

$$\int_{3/2}^{5/2} \frac{3/2u+5/4}{u^2-3/4}\,du = \frac{3}{2}\int_{3/2}^{5/2}\frac{u}{u^2-3/4}+\frac{5}{4}\int_{\frac{3}{2}}^{5/2}\frac{1}{u^2-3/4}$$

$v=u^2-3/4$
$dv=2u\,du$

$$=\frac{3}{4}\int_{\frac{3}{2}}^{11/2}\frac{1}{v}+\frac{5}{4}\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)\Big|_{3/2}^{5/2}$$

$$=\frac{3}{4}\ln|v|\,\Big|_{3/2}^{11/2}+5/4\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)\Big|_{3/2}^{5/2}$$

$= \ldots$

10 $\displaystyle\int x^3\sqrt{9+4x^2}\,dx$ $\quad x=\frac{3}{2}\tan\theta$
$dx = 3/2\sec^2\theta\,d\theta$

$$=\int \frac{27}{8}\tan^3\theta\sqrt{9+4(3/2\tan\theta)^2}\cdot\frac{3}{2}\sec^2\theta\,d\theta$$

$$=\int \frac{243}{16}\tan^3\theta\sqrt{1+\tan^2\theta}\cdot\sec^2\theta\,d\theta$$

$$=\int\frac{243}{16}\tan^3\theta\cdot\sec^3\theta\,d\theta = \frac{243}{16}\int\tan^2\theta\sec^2\theta\sec\theta\tan\theta$$

$u=\sec\theta$
$du=\sec\theta\tan\theta\,d\theta$

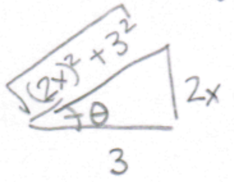
$$\frac{243}{16}\int\tan^2\theta\sec^2\theta\,du = \frac{243}{16}\int(\sec^2\theta-1)u^2\,du$$

$$=\frac{243}{16}\int(u^2-1)u^2\,du = \frac{243}{16}\int u^4-u^2\,du$$

$$=\frac{243}{16}\left(\frac{1}{5}u^5-\frac{1}{3}u^3\right)+C$$

$$=\frac{243}{16}\left(\frac{1}{5}\sec^5\theta-1/3\sec^3\theta\right)+C$$

Draw Triangle



$\frac{2x}{3} = \tan\theta = \frac{o}{a}$

Hence $\boxed{\frac{243}{16}\left(\frac{1}{5}\left(\frac{\sqrt{4x^2+9}}{9}\right)^5 - \frac{1}{3}\left(\frac{\sqrt{4x^2+9}}{9}\right)^3\right) + C}$

(11) $\int_1^3 \frac{1}{x^2+x^3}\,dx = \int_1^3 \frac{1}{x^2(1+x)}\,dx$

(a) Long division (No)

(b) Right Form

$$\frac{1}{x^2(1+x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1+x}$$

$$1 = Ax(1+x) + B(1+x) + C(x^2)$$

Pick $x=0$: $1 = B(1) \Rightarrow B=1$

$x=-1$: $1=C$

$x=1$: $1 = A(2)+2+1$ $A=-1$

(c) Integrate $\int_1^3 \frac{1}{x^2+x^3} = \int_1^3 \frac{-1}{x} + \int_1^3 \frac{1}{x^2} + \int_1^3 \frac{1}{1+x}$

$= -\ln|x|\Big|_1^3 + \left(\frac{-1}{x}\right)\Big|_1^3 + \ln|1+x|\Big|_1^3$

$\boxed{= -\ln|3| + \frac{-1}{3} + 1 + \ln(4) - \ln(2)}$

(12) $\int_0^{\pi/6} \sin(2x)\cos(2x)\,dx$ $u=2x$ $du=2dx$

$= \int_0^{\pi/3} \frac{1}{2}\sin(u)\cos(u)\,du$ $u=\sin x$ $du=\cos x\,dx$ $\int_0^{\sqrt{3}/2} \frac{1}{2}u\,du = \frac{1}{4}u^2\Big|_0^{\sqrt{3}/2}$

$= \boxed{\frac{1}{16}\cdot 3}$

(13) Same as #4.

14) $\int \frac{\sin x}{\cos^2 x - 5\cos x + 6}\,dx$ $\quad u = \cos x$, $du = -\sin x\,dx$

$= \int \frac{-1}{u^2 - 5u + 6}\,du = \int \frac{-1}{(u-3)(u-2)}\,du$

a) No long division

b) Find Form

$\frac{-1}{(u-3)(u-2)} = \frac{A}{u-3} + \frac{B}{u-2} \Rightarrow -1 = A(u-2) + B(u-3)$

$u=2: -1 = B(-1) \quad B = 1$

$u=3: -1 = A(1) \quad A = -1$

c) $\int \frac{-1}{(u-3)(u-2)} = \int \frac{-1}{u-3} + \int \frac{1}{u-2} = -\ln|u-3| + \ln|u-2| + C$

$\boxed{= -\ln|\cos x - 3| + \ln|\cos x - 2| + C}$

15) $\int \frac{t^3}{\sqrt{t^2+4}}$ $\quad t = 2\tan\theta$, $dt = 2\sec^2\theta\,d\theta$ $\quad \int \frac{16\tan^3\theta\sec^2\theta\,d\theta}{\sqrt{4\tan^2\theta + 4}}$

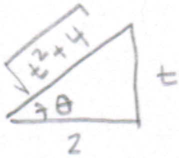
$= \int \frac{16\tan^3\theta\sec^2\theta}{2\sec\theta}\,d\theta = 8\int \tan^2\theta\tan\theta\sec\theta\,d\theta$ $\quad u = \sec\theta$, $du = \sec\theta\tan\theta$

$= 8\int \tan^2\theta\,du = 8\int \sec^2\theta - 1\,du = 8\int u^2 - 1\,du$

$= 8(1/3\,u^3 - u) + C = 8(\frac{1}{3}\sec^3\theta - \sec\theta) + C$

Construct Triangle

$\frac{t}{2} = \tan\theta = \frac{o}{a}$



$\therefore \boxed{8\left(\frac{1}{3}\left(\frac{\sqrt{t^2+4}}{2}\right)^3 - \frac{\sqrt{t^2+4}}{2}\right) + C}$

16) $\int_{\pi/4}^{\pi/3} \frac{\ln(\tan\theta)}{\sin\theta\cos\theta}\,d\theta$ $\quad u = \ln(\tan\theta)$

$du = \frac{1}{\tan\theta}\cdot\sec^2\theta\,d\theta = \frac{1}{\cos^2\theta}\cdot\frac{\cos\theta}{\sin\theta}\,d\theta = \frac{1}{\cos\theta\sin\theta}\,d\theta$

$= \int_0^{\ln(3/\sqrt{2})} u\,du = 1/2\,u^2 \Big|_0^{\ln(3/\sqrt{3})} = \boxed{\frac{1}{2}\left(\ln(3/\sqrt{3})\right)^2}$

(17) $\int x^3\sin(x^2+1)\,dx \quad u=x^2+1 \quad du=2x\,dx$

$$=\frac{1}{2}\int x^2\sin(u)\,du=\frac{1}{2}\int(u-1)\sin(u)\,du=\int\frac{1}{2}u\sin(u)-\int\frac{1}{2}\sin(u)$$

$w=1/2u \quad dv=\sin(u)\,du$
$dw=1/2\,du \quad v=-\cos(u)$

$$=\frac{-1}{2}u\cos(u)+\int 1/2\cos(u)\,du-\int 1/2\sin(u)\,du$$

$$=\frac{-1}{2}u\cos(u)+1/2\sin(u)+1/2\cos(u)+C$$

$$\boxed{=-1/2(x^2+1)\cos(x^2+1)+1/2\sin(x^2+1)+1/2\cos(x^2+1)+C}$$

(18) $\int\ln(1+\sqrt{x})\,dx \quad u=1+\sqrt{x}$
$du=1/2\sqrt{x}\,dx \Rightarrow du=\frac{1}{2(u-1)}dx$

$=\int 2(u-1)\ln(u)\,du \quad w=\ln(u) \quad dv=2(u-1)$
$dw=1/u\,du \quad v=u^2-2u\,du$

$$=(u^2-2u)\ln(u)-\int u-2\,du=(u^2-2u)\ln(u)-\frac{1}{2}u^2+2u+C$$

$$\boxed{\begin{aligned}&=\left((1+\sqrt{x})^2-2(1+\sqrt{x})\right)\ln(1+\sqrt{x})\\&-\frac{1}{2}(1+\sqrt{x})^2+2(1+\sqrt{x})+C\end{aligned}}$$

(19) $\int\frac{\sqrt{y-4}}{y}\,dy \quad u=\sqrt{y-4} \quad u^2=y-4$
$2u\,du=dy$

$=\int\frac{u(2u)}{u^2+4}\,du$ By Partial Fractions

(a) Long division

$$\begin{array}{r} 2 \\ u^2+4\ \overline{)\,2u^2} \\ -2u^2+8 \\ \hline -8 \end{array}$$

$$\frac{2u^2}{u^2+4}=2-\frac{8}{u^2+4}$$

(b) Integrate $\int\frac{2u^2}{u^2+4}\,du=\int 2-\int\frac{8}{u^2+4}\,du=2u-4\tan^{-1}\left(\frac{u}{2}\right)+C$

$$\boxed{=2\sqrt{y-4}-4\tan^{-1}\left(\frac{\sqrt{y-4}}{2}\right)+C}$$

20 $\int \frac{1}{x(\sqrt{x+1}+2)}\,dx$ $\quad u=\sqrt{x+1}$ $\quad u^2=x+1 \quad 2u\,du=dx$

$$= \int \frac{2u}{(u^2-1)(u+2)}\,du = \int \frac{2u}{(u-1)(u+1)(u+2)}\,du$$

a) No long division

b) Forms $\frac{2u}{(u-1)(u+1)(u+2)} = \frac{A}{u-1} + \frac{B}{u+1} + \frac{C}{u+2}$

$$\Rightarrow 2u = A(u+1)(u+2) + B(u-1)(u+2) + C(u-1)(u+1)$$

Pick $u=1$: $2 = A(2)(3) \Rightarrow A = 1/3$

$u=-1$: $-2 = B(-2)(1) \Rightarrow B = 1$

$u=-2$: $-4 = C(-3)(-1) \Rightarrow C = -4/3$

c) $\int \frac{2u}{(u^2-1)(u+2)} = \int \frac{1/3}{u-1} + \int \frac{1}{u+1} + \int \frac{-4/3}{u+2} \Rightarrow \frac{1}{3}\ln|u-1| + \ln|u+1| - \frac{4}{3}\ln|u+2|$

$$\boxed{= \frac{1}{3}\ln\left|\sqrt{x+1}-1\right| + \ln\left|\sqrt{x+1}+1\right| - \frac{4}{3}\ln\left|\sqrt{x+1}+2\right| + C}$$

21 $\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$ $\quad x = 2\sin\theta$ $\quad dx = 2\cos\theta\,d\theta$ $\quad \int_0^{\pi/6} \frac{4\sin^2\theta \cdot 2\cos\theta}{\sqrt{4-4\sin^2\theta}}\,d\theta$

$$= \int_0^{\pi/6} 4\sin^2\theta\,d\theta = \int_0^{\pi/6} 4\left(\frac{1+\cos(2\theta)}{2}\right)d\theta$$

$$= 2\theta + \sin(2\theta)\Big|_0^{\pi/6} = \boxed{\frac{\pi}{3} + \frac{\sqrt{3}}{2}}$$

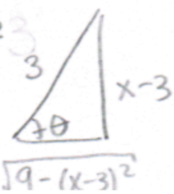
22 $\int \frac{x}{\sqrt{-x^2+6x}}\,dx$ Complete Square $\quad -(x^2-6x) = -([x-3]^2-9) = 9-(x-3)^2$

$$= \int \frac{x}{\sqrt{9-(x-3)^2}} \quad u = x-3 \quad du = dx \quad \int \frac{u+3}{\sqrt{9-u^2}} \quad u = 3\sin\theta \quad du = 3\cos\theta\,d\theta$$

$$= \int \frac{(3\sin\theta+3)(3\cos\theta)}{\sqrt{9-9\sin^2\theta}}\,d\theta = \int 3\sin\theta + 3\,d\theta = -3\cos\theta + 3\theta + C$$

Triangle

$\frac{u}{3} = \sin\theta = \frac{o}{h} = \frac{x-3}{3}$



$$\boxed{\therefore -3\left(\frac{\sqrt{9-(x-3)^2}}{3}\right) + 3\sin^{-1}\left(\frac{x-3}{3}\right) + C}$$