Practice Midterm 2

Problem 1
A bag containing \$ 1 million dollars (in 100's) is 12 kg. It is lifted at a constant rate of 6 m/s. The money leaks out of the bag at a constant rate of 2 kg / s. Assuming gravity is $9.8 m/s^2$. Find the height, h, at which the work needed to lift the bag is 1,396.5 Joules. (use smaller of 2 values).

(1) Forces at Work here
a). The bag of money
b). Rope (ignore this as told nothing about it)

(2) Find Mass at height y
Initially bag's mass is 12 kg but loses \$ so need to figure out at height y how must mass lost.
Lifted at 6m/s and money leaks out at 2kg/s. Hence

$$\frac{2\,\text{kg}}{\text{sec}} \cdot \frac{\text{sec}}{6\,\text{m}} = \frac{1}{3}\,\text{kg}/\text{m}$$

losing in \$. Therefore

Mass at $y = 12 - \frac{1}{3}y$

(3) Find Force
Force = Mass * accel.
$= (12 - \frac{1}{3}y) \cdot 9.8$

(4) Find dist: dy

(5) Bounds (range of y)
$y = 0$ to h.

(6) Work Integral:

$$\int_0^h 9.8\left(12 - \tfrac{1}{3}y\right)dy = 1{,}396.5$$

$$9.8\left(12y - \tfrac{1}{6}y^2\right)\Big|_0^h = 1{,}396.5$$

$$9.8\left(12h - \tfrac{1}{6}h^2\right) = 1{,}396.5$$

$$12h - \tfrac{1}{6}h^2 = 142.5$$

$$-\tfrac{1}{6}h^2 + 12h - 142.5 = 0$$

$$h = \frac{-12 \pm \sqrt{12^2 - 4(-\frac{1}{6})(-142.5)}}{2(-\frac{1}{6})}$$

$$h = \frac{-12 \pm \sqrt{49}}{-\frac{1}{3}} = \frac{-12 \pm 7}{-\frac{1}{3}}$$

$$= 15 \text{ or } 57$$

Note that if $h = 57$, then you have negative mass in the bag $[12 - \frac{1}{3}(57) = -7]$ so this one isn't correct.

$\boxed{\text{Hence, } h = 15 \text{ meters}}$

Problem 2

1. $\int \frac{1}{x^{1/2}+x^{1/3}}\,dx$
2. $\int \frac{(x-2)}{(x^2-4)(x^2+2x+2)}\,dx$
3. $\int_0^{\sqrt{5}} (9-x^2)^{-3/2}\,dx$
4. $\int \frac{3\sec(x)\tan(x)}{(3\tan(x))^3}\,dx$
5. $\int_0^{\pi} x\sin^2(x)\cos(x)\,dx$

① $\int \frac{1}{x^{1/2}+x^{1/3}}\,dx$ $u = x^{1/6}$ $6u^5\,du = dx$
$u^6 = x$

$$= \int \frac{6u^5}{(x^{1/6})^3 + (x^{1/6})^2}\,du = \int \frac{6u^5}{u^3+u^2}\,du = \int \frac{6u^3}{u+1}\,du$$

Partial Fractions:

a). Long Division:

$$\begin{array}{r} 6u^2 - 6u + 6 \\ u+1 \overline{)\,6u^3 \qquad\qquad} \\ -\left(6u^3 + 6u^2\right) \\ \hline -6u^2 \qquad \\ -\left(-6u^2 - 6u\right) \\ \hline 6u \\ 6u + 6 \\ \hline -6 \end{array}$$

$$\Rightarrow \int \frac{6u^3}{u+1}\,du = \int 6u^2 - 6u + 6 - \frac{6}{u+1}$$

b). Don't Need forms b/c already know the integral:

$$\int 6u^2 - 6u + 6 - \frac{6}{u+1}\,du = 2u^3 - 3u^2 + 6u - 6\ln|u+1| + C$$

$$\boxed{= 2(x^{1/2}) - 3(x^{1/3}) + 6x^{1/6} - 6\ln|x^{1/6}+1| + C}$$

② $\int \frac{x-2}{(x^2-4)(x^2+2x+2)}\,dx = \int \frac{x-2}{(x-2)(x+2)(x^2+2x+2)} = \int \frac{1}{(x+2)(x^2+2x+2)}\,dx$

Partial Fractions:

a). No long division

b). Find Forms:

$$\frac{1}{(x+2)(x^2+2x+2)} = \frac{A}{(x+2)} + \frac{Bx+C}{(x^2+2x+2)}$$

$$\Rightarrow \quad 1 = A(x^2+2x+2) + (Bx+C)(x+2)$$

on back ⇒

Continued Problem 2

$1 = A(x^2+2x+2) + (Bx+C)(x+2)$

Let $x=-2$: $1 = A(4-4+2)$

$1 = A(2) \Rightarrow \boxed{A = 1/2}$

To find B & C: Combine Like Terms

$1 + 0x + 0x^2 = 1/2 x^2 + x + 1 + Bx^2 + 2Bx + Cx + 2C$

$1 = 1 + 2C \quad \boxed{C=0}$

$0x^2 = 1/2 x^2 + Bx^2 \Rightarrow 0 = 1/2 + B \quad B = -1/2$

ⓒ Solve Integral:

$$\int \frac{1}{(x+2)(x^2+2x+2)} = \int \frac{1/2}{(x+2)} + \frac{-1/2 x + 0}{x^2+2x+2}$$

Complete Square:

$$\int \frac{-1/2 x}{x^2+2x+2} = \int \frac{-1/2 x}{(x+1)^2+1} \qquad u = x+1 \quad du = dx$$

$$= \int \frac{-1/2(u-1)}{u^2+1}$$

$$= \int \frac{-1/2 u}{u^2+1} + \int \frac{1/2}{u^2+1}$$

let
$v = u^2+1$
$dv = 2u\,du$

$$= \frac{-1}{4}\int \frac{1}{v}\,dv + \int \frac{1/2}{u^2+1}$$

$$= \frac{-1}{4}\ln|u^2+1| + 1/2 \tan^{-1}(u)$$

$$= -1/4 \ln|(x+1)^2+1| + 1/2 \tan^{-1}(x+1) + C$$

Thus,

$$\boxed{\int \frac{1/2}{(x+2)} + \frac{-1/2}{x^2+2x+2} = \frac{1}{2}\ln|x+2| - \frac{1}{4}\ln|(x+1)^2+1| + \frac{1}{2}\tan^{-1}(x+1) + C}$$

Problem 3

$$\int_0^{\sqrt{5}} (9-x^2)^{-3/2}$$

I'm going to do this w/o Δ-ing bounds:

$$\int (9-x^2)^{-3/2} = \int \frac{1}{(\sqrt{(9-x^2)})^3}\,dx$$

Use Trig:

$x = 3\sin\theta \quad dx = 3\cos\theta\,d\theta$

$$= \int \frac{1\,(3\cos\theta)}{(\sqrt{9-(3\sin\theta)^2})^3}\,d\theta$$

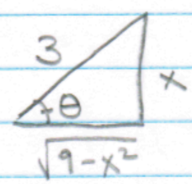
$$= \int \frac{3\cos\theta}{(\sqrt{9(1-\sin^2\theta)})^3}\,d\theta$$

$$= \int \frac{3\cos\theta}{(3\cos\theta)^3}\,d\theta = \int \frac{1}{9\cos^2\theta}\,d\theta$$

$$= \frac{1}{9}\int \sec^2\theta\,d\theta = \frac{1}{9}\tan\theta + C$$

Construct Triang

$x = 3\sin\theta \Rightarrow x/3 = \sin\theta = \frac{o}{h}$



Thus

$$\frac{1}{9}\tan\theta + C = \frac{1}{9}\left(\frac{x}{\sqrt{9-x^2}}\right) + C$$

$$\therefore \int_0^{\sqrt{5}} (9-x^2)^{-3/2} = \frac{1}{9}\left(\frac{x}{\sqrt{9-x^2}}\right)\Bigg|_0^{\sqrt{5}}$$

$$= \frac{1}{9}\left(\frac{\sqrt{5}}{\sqrt{9-5}}\right) = \frac{1}{9}\,\frac{\sqrt{5}}{2} = \boxed{\frac{\sqrt{5}}{18}}$$

Problem 4

$$\int \frac{3\sec x \tan x}{(3+\tan(x))^3}\,dx = \int \frac{3\sec x \tan x}{27\tan^3 x}\,dx$$

$$= \frac{1}{9}\int \frac{\sec x}{\tan^2 x}\,dx$$

$$= \frac{1}{9}\int \frac{1}{\cos x}\cdot\frac{\cos^2 x}{\sin^2 x}\,dx$$

$$= \frac{1}{9}\int \frac{\cos x}{\sin x}\,dx \qquad \begin{aligned} u &= \sin x \\ du &= \cos x\,dx \end{aligned}$$

$$= \frac{1}{9}\int \frac{1}{u}\,du$$

$$= \frac{1}{9}\ln|u| + C = \boxed{\frac{1}{9}\ln|\sin x| + C}$$

PROBLEM 5

$$\int_0^{\pi} x\sin^2 x\cos x\,dx \qquad \begin{aligned} u &= x & dv &= \sin^2 x\cos x \\ du &= dx & v &= * \end{aligned}$$

$$* = \int \sin^2 x\cos x\,dx \qquad \begin{aligned} u &= \sin x \\ du &= \cos x\,dx \end{aligned}$$

$$= \int u^2\,du$$

$$= \frac{1}{3}u^3 = \frac{1}{3}(\sin^3 x)$$

$$\therefore \int_0^{\pi} x\sin^2 x\cos x\,dx = \frac{x}{3}\sin^3 x\Big|_0^{\pi} - \int_0^{\pi}\frac{1}{3}\sin^3 x\,dx$$

$$= 0 - \int_0^{\pi}\frac{1}{3}\sin^2 x\sin x\,dx \qquad \begin{aligned} u &= \cos x \\ du &= -\sin x\,dx \end{aligned}$$

$$= \int_1^{-1}\frac{1}{3}(1-u^2)\,du$$

Continued Problem 5

$$\int_1^{-1} \frac{1}{3}(1-u^2)\,du = \frac{1}{3}\left(u - \frac{1}{3}u^3\right)\Big|_1^{-1}$$

$$= \frac{1}{3}\left(-1+\frac{1}{3}\right) - \frac{1}{3}\left(1-\frac{1}{3}\right)$$

$$= \frac{-2}{9} - \frac{2}{9} = \boxed{\frac{-4}{9}}$$

Problem 3

1. Use the comparison test to determine if $\int_0^{1} \frac{1}{x^{1/2}+x^{1/3}}\,dx$ converges or diverges.

2. Evaluate

$$\int_1^{\infty} \frac{1+e^x}{e^x - e^{2x}}\,dx$$

① ⓐ. Find all relationships

on $[0,1]$ $x^{1/2} \le x^{1/2} + x^{1/3}$ b/c if a, b are positive # then $a+b \ge a$

or $a+b \ge b$.

also $x^{1/3} \le x^{1/2} + x^{1/3}$

$$\Rightarrow \frac{1}{x^{1/2}} \ge \frac{1}{x^{1/2}+x^{1/3}} \quad \text{or} \quad \frac{1}{x^{1/3}} \ge \frac{1}{x^{1/2}+x^{1/3}}$$

Let's use $x^{1/2}$ (but can do this w/ $x^{1/3}$)

ⓑ $\int_0^1 \frac{1}{x^{1/2}} \ge \int_0^1 \frac{1}{x^{1/2}+x^{1/3}}$ Now $\int_0^1 \frac{1}{x^{1/2}} = 2x^{1/2}\Big|_0^1 = 2.$

Now by comparison test,

$$\int_0^1 \frac{1}{x^{1/2}+x^{1/3}}\,dx \le \int_0^1 \frac{1}{x^{1/2}} = 2 \quad \text{so}$$

$$\boxed{\int_0^1 \frac{1}{x^{1/2}+x^{1/3}} \text{ converges}}$$

② a). Rewrite $\int_1^{\infty} \frac{1+e^x}{e^x - e^{2x}} = \lim_{b\to\infty} \int_1^b \frac{1+e^x}{e^x-e^{2x}}\,dx$

b). Solve integral: ~~Use IBP.~~

let $u = e^x$ $du = e^x dx \Rightarrow du/e^x = dx \Rightarrow du/u = dx$

$$\lim_{b\to\infty} \int_{e^1}^{e^b} \frac{1+u}{(u-u^2)u}\,du$$

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Problem 3 b)

$$\lim_{b\to\infty}\int_{e^1}^{e^b}\frac{1+u}{u^2(1-u)}\,du$$ Use Partial Fractions!

a). Long division: $\underline{\underline{\text{No}}}$

b). Find Form

$$\frac{1+u}{u^2(1-u)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{1-u}$$

$$\Rightarrow \quad 1+u = A(u)(1-u) + B(1-u) + C(u^2)$$

$u=1$: $1+1 = C \Rightarrow \boxed{C=2}$

$u=0$: $\boxed{1 = B}$

$u=-1$: $0 = -2A + 2 + 2 \Rightarrow \boxed{A=2}$

c). Solve Integral:

$$\lim_{b\to\infty}\int_{e^1}^{e^b}\frac{1+u}{u^2(1-u)} = \lim_{b\to\infty}\int_{e^1}^{e^b}\frac{2}{u} + \int_{e^1}^{e^b}\frac{1}{u^2} + \int_{e^1}^{e^b}\frac{2}{1-u}$$

$$= \lim_{b\to\infty} 2\ln|u| - \frac{1}{u} - 2\ln|1-u| \Big|_{e^1}^{e^b}$$

$$= \lim_{b\to\infty} \ln|u^2| - \frac{1}{u} - \ln|(1-u)^2| \Big|_{e^1}^{e^b}$$

$$= \lim_{b\to\infty} \ln\left|\frac{u^2}{u^2-2u+1}\right| - \frac{1}{u} \Big|_{e^1}^{e^b}$$

$$= \lim_{b\to\infty} \ln\left(\frac{u^2(1/u^2)}{(u^2-2u+1)\,1/u^2}\right) - \frac{1}{u} \Big|_{e^1}^{e^b}$$

$$= \lim_{b\to\infty} \ln\left(\frac{1}{1-2/u+1/u^2}\right) - \frac{1}{u} \Big|_{e^1}^{e^b}$$

$$= \lim_{b\to\infty} \ln\left(\frac{1}{1-2/e^b+1/e^{b^2}}\right) - \frac{1}{e^b} - \ln\left(\frac{1}{1-2/e+1/e^2}\right) + \frac{1}{e}$$

$$\boxed{= \ln(1) - 0 - \ln\left(\frac{1}{1-2/e+1/e^2}\right) + 1/e}$$ $\boxed{\text{Converges}}$

Problem 5

Use Simpson's Rule with $n = 6$ to approximate the integral

$$\int_2^5 \frac{1}{\ln(x)}\,dx$$

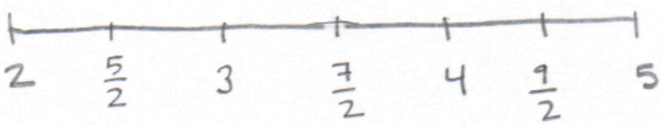
Maintain at least 4 digits of precision at all times.

Simpson's Rule

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\Big(f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + 2f(x_5) + 4f(x_6) + f(x_7)\Big)$$

① Find Δx and plug in pts

$$\Delta x = \frac{b-a}{n} = \frac{5-2}{6} = \frac{1}{2}$$



② $$\int_2^5 \frac{1}{\ln(x)}\,dx = \frac{1}{6}\Big(f(2) + 4f(5/2) + 2f(3) + 4f(\tfrac{7}{2}) + 2f(4) + 4f(\tfrac{9}{2}) + f(5)\Big)$$

$$\approx 1/6\,(15.545010)$$

$$\boxed{\approx 2.5908}$$