3. (10 total points) The velocity function (in meters per second) for a particle moving along a line is given by $v(t) = t^2 - 5t + 6$.

(a) (5 points) Find the displacement of the particle during the time interval $0 \le t \le 4$.

$$\text{displacement} = \int_0^4 t^2 - 5t + 6\,dt = \frac{1}{3}t^3 - 5/2\,t^2 + 6t\Big|_0^4$$
$$= \frac{64}{3} - 40 + 24 = \boxed{16/3}$$

(b) (5 points) Find the total distance traveled by particle during the time interval $0 \le t \le 4$.

$$\int_0^4 |t^2 - 5t + 6|\,dt$$

$0 = t^2 - 5t + 6$
$0 = (t-3)(t-2)$
$t = 3, 2$

$0 \le t \le 2$	$2 \le t \le 3$	$3 \le t \le 4$
+	−	+

$$= \int_0^2 t^2 - 5t + 6 + \int_2^3 -t^2 + 5t - 6 + \int_3^4 t^2 - 5t + 6$$

$$= \frac{1}{3}t^3 - 5/2\,t^2 + 6t\Big|_0^2 - \frac{1}{3}t^3 + \frac{5}{2}t^2 - 6t\Big|_2^3 + \frac{1}{3}t^3 - \frac{5}{2}t^2 + 6t\Big|_3^4$$

$$= \frac{8}{3} - 10 + 12 - 9 + \frac{45}{2} - 18 + \frac{8}{3} - 10 + 12 + \frac{64}{3} - 40 + 24 - 9 + \frac{45}{2} - 18$$

$$= \boxed{17/3}$$

4. (10 total points)

Consider the shaded region in the figure, bounded (in clockwise order from the origin, as shown in the figure) by

the y-axis,

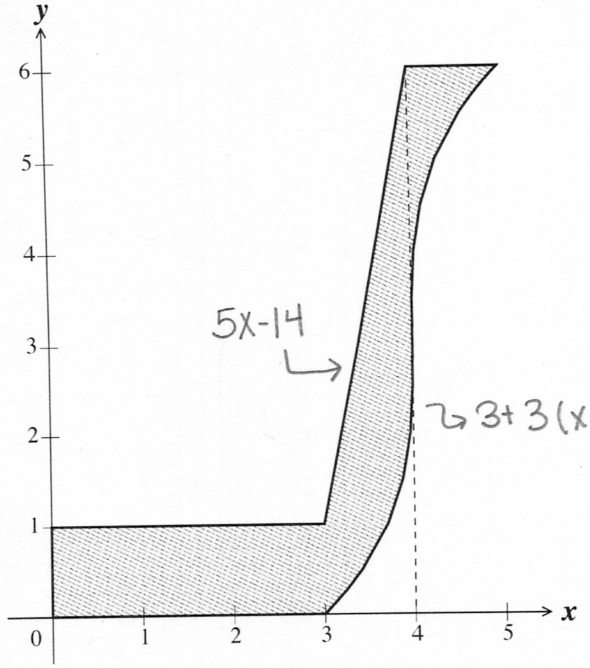
the line $y = 1$,

the line $y = 5x - 14$,

the line $y = 6$,

the curve $y = 3 + 3(x-4)^{1/3}$, and

the x-axis.



A Hallowe'en trick-or-treat bucket is formed by rotating this region around the y-axis.

(a) (5 points) Set up an integral (or a sum of integrals) using *SHELLS* which equals the volume of plastic needed to construct the bucket. DO NOT EVALUATE THE INTEGRAL(S). Shells

In terms of x

$$2\pi \int_0^3 1(x)\,dx + 2\pi \int_3^4 x\left[5x-14-(3+3(x-4)^{1/3})\right]dx$$

$$+\, 2\pi \int_4^5 x\left[6-(3+3(x-4)^{1/3})\right]dx$$

(b) (5 points) Set up an integral (or a sum of integrals) using *WASHERS* which equals the volume of plastic needed to construct the bucket. DO NOT EVALUATE THE INTEGRAL(S).

Washer – in terms of y

$$\pi \int_0^1 \left(4+(y/3-1)^3\right)^2 dy$$

$$+\, \pi \int_1^6 \left(4+(y/3-1)^3\right)^2 - \left[\frac{14+y}{5}\right]^2 dy$$

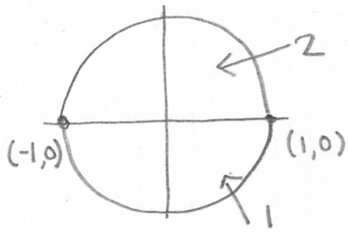
$y-3 = 3(x-4)^{1/3}$

$\left(\frac{y}{3}-1\right)^3 = x-4$

$x = 4 + \left(\frac{y}{3}-1\right)^3$

$X = 14+y/5$

5. (10 points) Find the coordinates of the center of mass of a circular plate of radius 1 with center at the origin (0,0) made with a material whose density is 2 on the upper semicircular region and 1 on the lower semicircular region.



$\frac{1}{2}$ Area $\cdot 2 + \frac{1}{2}$ Area $\cdot 1 =$ mass

$$\frac{\pi}{2} \cdot 2 + \frac{\pi}{2} = \frac{3\pi}{2}$$

Area $= \pi r^2 = \pi$

$u = 1 - x^2 \quad du = -2x\,dx$

$$M_y = 2\int_{-1}^{1} x\sqrt{1-x^2}\,dx + \int_{-1}^{1} -x\sqrt{1-x^2}\,dx$$

$$= 2\int_{0}^{0} -\frac{1}{2}\sqrt{u}\,du + \int_{0}^{0} -\frac{1}{2}\sqrt{u}\,du = 0$$

$$M_x = 2\int_{-1}^{1} \frac{1}{2}(1-x^2)\,dx - \int_{-1}^{1} \frac{1}{2}(1-x^2)\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} (1-x^2)\,dx = \frac{1}{2}\left(x - \frac{1}{3}x^3\right)\Big|_{-1}^{1}$$

$$= \frac{1}{2}\left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) = \frac{1}{2}\left(\frac{2 \cdot 2}{3}\right) = \frac{2}{3}$$

$$\therefore \boxed{(\bar{x}, \bar{y}) = \left(0, \frac{4}{9\pi}\right)}$$

6. (10 points) Find the solution of the initial value problem

$$\frac{dy}{dx} = xe^{2x^2}(1+y^2), \quad y(0) = \sqrt{3}.$$

Give your answer in the form $y = f(x)$.

$$\int \frac{dy}{1+y^2} = \int xe^{2x^2}\,dx$$

$u = 2x^2$
$du = 4x\,dx$

$$\arctan(y) = \int xe^{2x^2}\,dx = \frac{1}{4}\int e^u\,du$$

$$= \frac{1}{4}e^{2x^2} + C$$

$$\arctan(\sqrt{3}) = \frac{1}{4} + C$$

$$\pi/3 = 1/4 + C$$

$$C = \pi/3 - 1/4$$

$$\therefore \arctan(y) = \frac{1}{4}e^{2x^2} + \pi/3 - 1/4$$

$$\boxed{y = \tan\left(\frac{1}{4}e^{2x^2} + \pi/3 - 1/4\right)}$$

7. (10 total points) The stale air in a crowded exam room initially contains 6.25 ft^3 of carbon dioxide (CO_2). An air conditioner is turned on at time $t = 0$ and blows fresher air into the room at a rate of 500 ft^3/min. The fresher air mixes with the stale air (assume it mixes instantaneously) and the well-mixed air leaves the room at the same rate of 500 ft^3/min. The incoming fresher air contains 0.01% CO_2 (by volume), and the air in the room has a total volume of 2500 ft^3. By their breathing, the people in the room generate an additional 0.08 ft^3 of CO_2 per minute (without changing the total volume of air in the room). Let $y(t)$ denote the amount (in ft^3) of CO_2 in the room, t minutes after the air conditioner is turned on.

(a) (4 points) Find a differential equation satisfied by $y(t)$. Simplify the differential equation, but wait until part (b) to solve it.

Rate in: Air from air conditioner $\frac{500\ ft^3}{min} \cdot 0.0001 = .05\ ft^3/min$

Air from breathing $0.08\ ft^3/min$

Rate out: Concentration = $y/2500$ Rate out = $500/2500 \cdot y$

$$\therefore \boxed{\frac{dy}{dt} = .05 + 0.08 - y/5 = 0.13 - y/5}$$

(b) (6 points) Now solve the differential equation from part (a), and solve for any constant(s) in your solution to find a formula for $y(t)$.

$$\int \frac{dy}{5.08 - y/5} = \int dt \Rightarrow -5\ln|0.13 - y/5| = t + C$$

$$0.13 - y/5 = Ce^{-t/5}$$

$$.65 - Ce^{-t/5} = y(t)$$

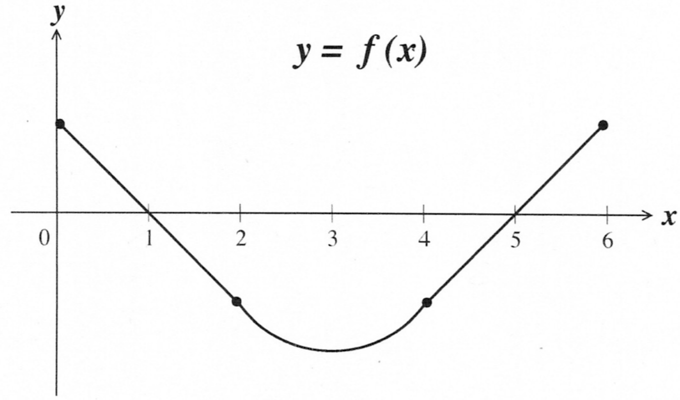
$$y(0) = 6.25$$

$$.65 - C = 6.25$$

$$C = -5.6$$

$$\boxed{\therefore y = .65 + 5.6e^{-t/5}}$$

8. (10 total points) The graph of $y = f(x)$ is given by



Let $A(x) = \int_0^x f(t)\,dt$ for $0 \leq x \leq 6$.

(a) (2 points) On what interval(s) in x is $A(x)$ positive?

[0, 2]

(b) (2 points) On what interval(s) in x is $A(x)$ increasing?

[0,1] and [5,6]

(c) (2 points) On what interval(s) in x is the graph of $y = A(x)$ concave up?

[3, 6]

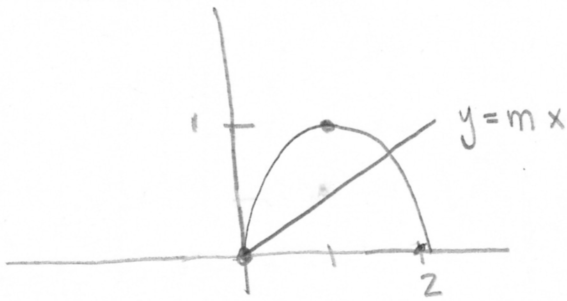
(d) (2 points) At what value(s) of x does $A(x)$ have an absolute maximum?

X = 1

(e) (2 points) On what interval(s) in x is the graph of $y = (A(x))^2$ increasing?

[0,1] and [2, 5]

9. (10 points) Find the slope m of the line $y = mx$ through the origin that divides the region bounded between the parabola $y = 2x - x^2$ and the x-axis into two regions with equal area.



Total Area $= \int_0^2 2x - x^2 \, dx = x^2 - \frac{1}{3}x^3 \Big|_0^2 = 4 - 8/3 = 4/3$

$mx = 2x - x^2 \qquad x^2 + (m-2)x = 0 \qquad x = 0$ and $x = 2-m$

$\therefore \frac{1}{2}$ Area $= \frac{4}{6} = \int_0^{2-m} 2x - x^2 - mx \, dx$

$= x^2 - \frac{1}{3}x^3 - \frac{m}{2}x^2 \Big|_0^{2-m}$

$= (1 - \frac{m}{2})[2-m]^2 - \frac{1}{3}(2-m)^3$

$(2-m)(2-m) = 4 - 4m + m^2$

$(2-m)(2-m)(2-m)$

$= (4 - 4m + m^2)(2-m)$

$= 8 - 12m + 6m^2 - m^3$

$= (1 - \frac{m}{2})[4 - 4m + m^2] - \frac{1}{3}[8 - 12m + 6m^2 - m^3]$

$= 4 - 2m + 3m^2 - \frac{1}{2}m^3 - \frac{8}{3} + 4m - 2m^2 + \frac{1}{3}m^3$

$= \frac{-1}{6}m^3 + m^2 - 2m + 4/3$

Solving this gives $m = 2 - 4^{1/3}$

10. (10 total points)

(a) (5 points) In this part, treat b and c as constants. Evaluate the definite integral

$$\int_0^b \left(\frac{2x}{x^2+1} - \frac{c}{5x+1}\right) dx.$$

Your answer will involve both b and c.

$$\int_0^b \frac{2x}{x^2+1}\,dx - \int_0^b \frac{c}{5x+1}\,dx$$

$u = x^2+1$

$du = 2x\,dx$

$$\int_1^{b^2+1} \frac{1}{u} - \int_0^b \frac{c}{5x+1} = \ln|b^2+1| - \frac{c}{5}\ln|5x+1|\Big|_0^b$$

$$= \boxed{\ln|b^2+1| - \frac{c}{5}\ln|5b+1|}$$

(b) (5 points) Use your answer to part (a) to find the value of the constant c for which the improper integral

$$\int_0^\infty \left(\frac{2x}{x^2+1} - \frac{c}{5x+1}\right) dx$$

converges, and evaluate the improper integral for this value of c.

$$\lim_{b\to\infty} \int_0^b \left(\frac{2x}{x^2+1} - \frac{c}{5x+1}\right) = \lim_{b\to\infty} \ln|b^2+1| - \frac{c}{5}\ln|5b+1|$$

$$= \lim_{b\to\infty} \ln\left|\frac{b^2+1}{(5b+1)^{c/5}}\right| \quad \text{If } \boxed{c = 10}$$

$$= \lim_{b\to\infty} \ln\left|\frac{b^2+1}{(5b+10)^2}\right| = \boxed{\ln\left(\frac{1}{25}\right)}$$