Your Name

Your Signature

Student ID #

Quiz Section

Professor's Name

TA's Name

- This exam is closed book. You may use one $8.5'' \times 11''$ sheet of handwritten notes (both sides OK). Do not share notes. No photocopied materials are allowed.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- A scientific calculator is allowed, but graphing calculators are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- You may use any of the 20 integrals in the table on p. 484 of the text (p. 506 if you have the 5th edition of Stewart) without deriving them. **Show your work in evaluating any other integrals, even if they are on your note sheet**.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 10 pages, plus this cover sheet. Please make sure that your exam is complete.

Question	Points	Score
1	12	
2	12	
3	8	
4	10	
5	10	

Question	Points	Score
6	10	
7	8	
8	10	
9	12	
10	8	
Total	100	

1. (12 total points) Evaluate the following indefinite integrals.

(a) (6 points) $\displaystyle\int \frac{x^4}{\sqrt{2x^5 - x^{10}}}\,dx$

Let $u = x^5$ $du = 5x^4\,dx$ then

By completing the square

$$\int \frac{x^4}{\sqrt{2x^5 - x^{10}}}\,dx = \int \frac{1}{5}\frac{1}{\sqrt{2u-u^2}}\,du = \int \frac{1}{5}\frac{1}{\sqrt{1-(u-1)^2}}\,du$$

Let $v = u-1$ $dv = du$, then

$$= \int \frac{1}{5}\frac{1}{\sqrt{1-v^2}}\,dv$$ Let $v = \sin\theta$ $dv = \cos\theta\,d\theta$ $$= \int \frac{1}{5}\frac{\cos\theta\,d\theta}{\cos\theta\,d\theta}$$

$$= \int \frac{1}{5}\,d\theta = \frac{1}{5}\theta + C$$

Substituting back, $v = \sin\theta \Rightarrow \sin^{-1}(v) = \theta \Rightarrow \sin^{-1}(u-1) = \theta \Rightarrow \sin^{-1}(x^5-1) = \theta$. Hence

$$\boxed{\int \frac{x^4}{\sqrt{2x^5-x^{10}}}\,dx = \frac{1}{5}\sin^{-1}(x^5-1) + C}$$

(b) (6 points) $\displaystyle\int \frac{1}{x^3 - 4x^2 + 5x}\,dx$

$$\int \frac{1}{x^3-4x^2+5x}\,dx = \int \frac{1}{x(x^2-4x+5)}\,dx$$ Note that x^2-4x+5 is not reducible.

Use Partial Fractions,

$$\frac{1}{x(x^2-4x+5)} = \frac{A}{x} + \frac{Bx+C}{x^2-4x+5}$$

$$= \frac{A(x^2-4x+5) + Bx^2 + Cx}{x(x^2-4x+5)}$$

$\Rightarrow 1 = A5 \Rightarrow A = 1/5$
$0 = -4A + C \Rightarrow 4/5 = C$
$0 = A + B \Rightarrow B = -1/5$

$$\therefore \int \frac{1}{x(x^2-4x+5)} = \frac{1}{5}\int\frac{1}{x} + \int \frac{-1/5\,x + 4/5}{x^2-4x+5}$$

$$= \frac{1}{5}\ln|x| + \frac{1}{5}\int \frac{-x+4}{(x-2)^2+1}$$

$u = x-2$ $du = dx$

$v = u^2+1$ $dv = 2u\,du$

$$= \frac{1}{5}\ln|x| + \frac{1}{5}\int\frac{-u+2}{u^2+1}\,du = \frac{1}{5}\ln|x| + \frac{1}{5}\int\frac{2}{u^2+1} - \frac{1}{5}\int\frac{u}{u^2+1} = \boxed{\frac{1}{5}\ln|x| + \frac{2}{5}\tan^{-1}((x-2)) - \frac{1}{10}\ln|(x-2)^2+1| + C}$$

2. (12 total points) Evaluate the following definite integrals.

(a) (6 points) $\displaystyle\int_3^4 \frac{x^2}{(x-2)^4}\,dx$ Give your answer in exact form.

Let $u = x-2$ $du = dx$,

$$\int_3^4 \frac{x^2}{(x-2)^4}\,dx = \int_1^2 \frac{(u+2)^2}{u^4}\,du = \int_1^2 \frac{u^2+4u+4}{u^4}\,du$$

$$= \int_1^2 \frac{1}{u^2} + \int_1^2 \frac{4}{u^3} + \int_1^2 \frac{4}{u^4}$$

$$= -u^{-1}\Big|_1^2 - 2u^{-2}\Big|_1^2 - \frac{4}{3}u^{-3}\Big|_1^2$$

$$= \frac{-1}{2} + 1 - \frac{1}{2} + 2 - \frac{1}{6} + \frac{4}{3}$$

$$= \boxed{19/6}$$

(b) (6 points) $\displaystyle\int_0^{2\pi} |e^{-x}\sin x|\,dx$ Give your answer in exact form.

① Find where $e^{-x}\sin x = 0$ for $x \in [0, 2\pi]$. Since $e^{-x} \neq$ for all x, then $\sin x = 0$ if $x = \pi$ so $e^{-x}\sin x = 0$ at $0, 2\pi, \pi$.

② Need to figure out if $e^{-x}\sin x > 0$ or < 0 betwn $(0,\pi)$ and $(\pi, 2\pi)$. Pick a point in betwn and plug in

Let $x = \pi/2$, $e^{-\pi/2}\sin(\pi/2) > 0$

$x = 3\pi/2$, $e^{-3\pi/2}\sin(3\pi/2) < 0$.

$$\therefore \int_0^{2\pi} |e^{-x}\sin x|\,dx = \int_0^{\pi} e^{-x}\sin x + \int_\pi^{2\pi} -e^{-x}\sin x$$

Using integration by parts,

$u = \sin x$ $dv = e^{-x}$
$du = \cos x$ $v = -e^{-x}$

$$\int_0^\pi e^{-x}\sin x = -e^{-x}\sin x\Big|_0^\pi + \int_0^\pi e^{-x}\cos x\,dx$$
$$= \int_0^\pi e^{-x}\cos x\,dx$$

$\longrightarrow$ On next page

2b Continued

IBP

$$\int_0^{\pi} e^{-x}\sin x = \int_0^{\pi} e^{-x}\cos x\,dx$$

$u = \cos x \quad dv = e^{-x}$

$du = -\sin x\,dx \quad v = -e^{-x}$

$$= -e^{-x}\cos(x)\Big|_0^{\pi} - \int_0^{\pi} e^{-x}\sin(x)\,dx$$

$$\Rightarrow 2\int_0^{\pi} e^{-x}\sin(x) = -e^{-x}\cos(x)\Big|_0^{\pi}$$

$$\Rightarrow \int_0^{\pi} e^{-x}\sin(x) = \frac{1}{2}\left(-e^{-x}\cos(x)\right)\Big|_0^{\pi} = \frac{1}{2}\left[e^{-\pi}+1\right]$$

Similarly for

$$\int_{\pi}^{2\pi} e^{-x}\sin(x)\,dx = \frac{1}{2}\left(e^{-x}\cos(x)\right)\Big|_{\pi}^{2\pi}$$

$$= \frac{1}{2}\left(e^{-2\pi}+e^{-\pi}\right)$$

$$\therefore \int_0^{2\pi} |\sin(x)e^{-x}|\,dx = \frac{1}{2}\left[e^{-\pi}+1\right] + \frac{1}{2}\left[e^{-2\pi}+e^{-\pi}\right]$$

$$\boxed{= \frac{1}{2} + e^{-\pi} + \frac{1}{2}e^{-2\pi}}$$

3. (8 points) Find a continuous function $f(x)$ and a number $a > 0$ such that $16 + \int_a^x t^2 f(t)\,dt = x^4$. (*Hint: Differentiate both sides.*)

① Follow hint and differentiate both sides

$d/dx\left(16 + \int_a^x t^2 f(t)\,dt\right) = x^2 f(x)$ by FTC

$d/dx\, x^4 = 4x^3$

$\therefore\ x^2 f(x) = 4x^3 \Rightarrow \boxed{f(x) = 4x}$

② To find "a", plug f(x) back in. Thus

$16 + \int_a^x t^2 \cdot 4t = x^4$

$\Rightarrow 16 + \left(t^4 \Big|_a^x\right) = x^4$

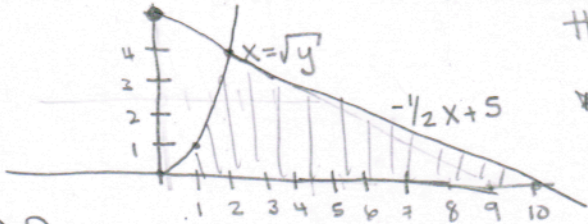
$\Rightarrow 16 + x^4 - a^4 = x^4$

$\Rightarrow 16 = a^4 \Rightarrow \boxed{a = 2}$

4. (10 total points) Let $\mathscr{R}$ be the region which is bounded on the left by the curve $x = \sqrt{y}$, bounded on the right by the line $y = -\frac{1}{2}x + 5$, and bounded below by the x-axis.

(a) (5 points) Set up a definite integral (or integrals) *with respect to x* for the area of the region $\mathscr{R}$, and evaluate your integral(s). Give your answer in exact form.

① Draw a picture



*Read carefully: bounded below by the x-axis

* w/ respect to x means integral should have dx

② Set up integral. By the picture, we need 2 integrals: one at 0 to 2 and another at 2, 10.

$$\therefore \quad \int_0^2 x^2\,dx + \int_2^{10} -\tfrac{1}{2}x + 5\,dx$$

③ Now solve,

$\int_0^2 x^2 dx = \frac{1}{3}x^3 \Big|_0^2 = 8/3$ and $\int_2^{10} -\frac{1}{2}x + 5 dx = -\frac{1}{4}x^2 + 5x \Big|_2^{10} = 16$

$$\therefore \quad \int_0^2 x^2 dx + \int_2^{10} -\tfrac{1}{2}x + 5 dx = 8/3 + 16 = 56/3$$

(b) (5 points) Set up a definite integral (or integrals) *with respect to y* for the area of the region $\mathscr{R}$, and evaluate your integral(s). Give your answer in exact form.

* To solve using y, rotate your picture and pretend like x-axis
Thus you only need 1 integral.

$y = -\frac{1}{2}x + 5$
$y - 5 = -\frac{1}{2}x$
$10 - 2y = x$

$$\int_0^4 10 - 2y - \sqrt{y}\, dy$$

Now solve, $\int_0^4 10 - 2y - y^{1/2} dy = 10y - y^2 - \frac{2}{3}y^{3/2} \Big|_0^4$

$= 40 - 16 - 16/3$

$= \boxed{56/3}$

*Please note that your answers for a and b should agree.

5. (10 points) Let $\mathscr{E}$ be the region enclosed by the ellipse

$$(x-2)^2 + \frac{y^2}{4} = 1.$$

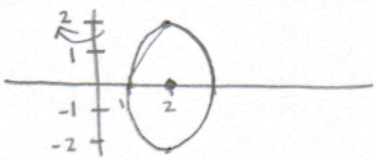
Find the volume of the (doughnut-shaped) solid obtained by rotating $\mathscr{E}$ about the y-axis. Give your answer in exact form.

① Draw a picture:

Recall that an equation of an ellipse is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$ where (h,k) is the center of the ellipse and a,b are the radii. "a" in the x direction, "b" in the y-direction.

Hence for us $(x-2)^2 + y^2/4 = 1$ is centered at $(2,0)$ w/ radii 1 and 2



② Should you use shells or washers/discs?

First question: Is it easier to solve for x or y? y in this case

Therefore b/c rotating about y-axis and easier to solve for y, use shells. Hence,

$$y^2/4 = 1-(x-2)^2 \Rightarrow y^2 = 4-4(x-2)^2$$

$$\Rightarrow y = \pm\sqrt{4(1-(x-2)^2)}$$

For simplicity, we will do the top half, Therefore, the height is $\sqrt{4(1-(x-2)^2)}$ with radius x. Hense,

$$\text{Volume of Ellipse} = 2\cdot 2\pi\int_1^3 x\sqrt{4(1-(x-2)^2)}\,dx$$

Use Trig substitution: Let $u = x-2$ $du = dx$

$$4\pi\int_1^3 2x\sqrt{1-(x-2)^2}\,dx = 8\pi\int_{-1}^{1}(u+2)\sqrt{1-u^2}\,du$$

Let $u = \sin\theta$, $du = \cos\theta\,d\theta$ $\quad 1 = \sin\theta \Rightarrow \theta = \pi/2 \quad -1 = \sin\theta \Rightarrow \theta = -\pi/2$

$$= 8\pi\int_{-\pi/2}^{\pi/2} \cos\theta(\sin\theta + 2)\cos\theta\,d\theta$$

$$= 8\pi\int_{-\pi/2}^{\pi/2} \cos^2\theta\sin\theta + 16\pi\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta$$

$\Downarrow$ $u = \cos\theta$

$du = -\sin\theta\,d\theta$

$$= 8\pi\int_0^0 u^2\,du + 16\pi\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta$$

Con't on Back $\Longrightarrow$

Problem 5 Autumn 2011

① Note that $\int_0^0 u^2 du = 0$. Hence only have

$$16\pi \int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta = 16\pi \int_{-\pi/2}^{\pi/2} \frac{1+\cos(2\theta)}{2} d\theta$$

$$= 8\pi \int_{-\pi/2}^{\pi/2} 1 + 8\pi \int_{-\pi/2}^{\pi/2} \cos(2\theta)$$

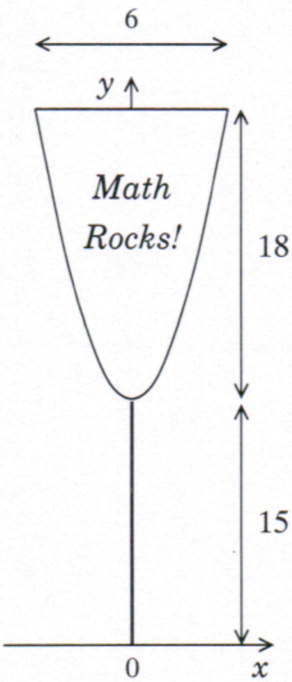
$u = 2\theta$
$du = 2d\theta$

$$8\pi\theta \Big|_{-\pi/2}^{\pi/2} + \underbrace{4\pi \sin(2\theta) \Big|_{-\pi/2}^{\pi/2}}_{=0}$$

$$= 8\pi(\pi/2) + 8\pi(\pi/2)$$

$$= \boxed{8\pi^2}$$

6. (10 points)



A flat math billboard is in the shape of (what else?) a parabola. Its top side is 6 feet wide and the billboard is 18 feet high, measured from the lowest to the highest point. It is mounted on a pole and the lowest point of the billboard is 15 feet above the ground.

Before it was mounted on the pole, the billboard was originally lying flat on the ground. The billboard weighs 3 pounds per square foot. Set up a definite integral for the work done in lifting this billboard up to where it now stands. Evaluate the integral and find the work done.

(*Hint: Slice the billboard in strips parallel to the straight edge.*)

① Volume (or in this case Area) of 1 slice

Recall that a parabola is given by $a(x-h)^2+k$ where h,k are the vertex. Therefore, (0,15) is our vertex so $y = ax^2+15$

To find a, note that (3,33) is a pt. Hence, $33 = a3^2 + 15 \Rightarrow a=2$. Thus $y = 2x^2+15$ or $\pm\sqrt{\frac{y-15}{2}} = x$.

$\boxed{\therefore \text{Area of 1 slice is } 2\sqrt{\frac{y-15}{2}}\,dy}$

② Force Since billboard weighs 3lbs per foot,

Force of 1 slice $= 6\sqrt{\frac{y-15}{2}}\,dy$

③ Distance 1 slice travels is y and the places where there is plank is 15 to 33

Hence

$$\text{Work} = \int_{15}^{33} 6y\sqrt{\frac{y-15}{2}}\,dy \qquad \text{Let } u = \frac{y-15}{2}, \quad du = 1/2\,dy$$

$$= 2\int_0^9 6(2u+15)\sqrt{u}\,du$$

$$= 2\int_0^9 12u^{3/2} + 90\sqrt{u}\,du$$

$$= 2\left(24/5\, u^{5/2}\Big|_0^9 + 60u^{3/2}\Big|_0^9\right)$$

$$= 2(1166.4 + 1620) = \boxed{5572.8 \text{ ft-lbs}}$$

7. (8 total points) Consider the curve $y = \sin x$.

   (a) (4 points) Set up a definite integral for the arc length of this curve for $0 \le x \le \pi/2$.
   DO NOT EVALUATE THE INTEGRAL.

The arc length $= \int_a^b \sqrt{1+(dy/dx)^2}\, dx$. Since $y = \sin x$, $\frac{dy}{dx} = \cos x$.

$\therefore$

arc length $= \int_0^{\pi/2} \sqrt{1+\cos^2 x}\, dx$

   (b) (4 points) Use Simpson's rule with $n = 4$ subintervals to estimate the integral in part (a). Give your answer in decimal form, correct to at least the third digit after the decimal point.

① Find plug in points

$\frac{\pi/2 - 0}{4} = \frac{\pi}{8} = \Delta x$

$0 \quad \frac{\pi}{8} \quad \frac{\pi}{4} \quad \frac{3\pi}{8} \quad \frac{\pi}{2}$

② Simpson's Rule $= \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$

$= \frac{\pi/8}{3}\left[f(0) + 4f(\pi/8) + 2f(\pi/4) + 4f(\frac{3\pi}{8}) + f(\pi/2)\right]$ $\quad f(x) = \sqrt{1+\cos^2 x}$

$= \frac{\pi}{24}\left[\sqrt{2} + 4\sqrt{1+\cos^2(\frac{\pi}{8})} + 2\sqrt{1+\cos^2(\frac{\pi}{4})} + 4\sqrt{1+\cos^2(3\pi/8)} + \sqrt{1+\cos^2(\pi/2)}\right)$

$= \frac{\pi}{24}(14.59240389) = \boxed{1.910141203}$

8. (10 points) Find the solution of the differential equation

$$x'(t) = t + tx - 3 - 3x$$

that satisfies the initial condition $x(0) = -5$.

① Get all the x's on one side and t's on the other without adding and substracting

$$\frac{dx}{dt} = t + tx - 3 - 3x$$

$$\frac{dx}{dt} = (t-3)(x+1)$$

$$\Rightarrow \frac{dx}{x+1} = (t-3)\,dt$$

$$\Rightarrow \int \frac{dx}{x+1} = \int t-3\,dt \Rightarrow \ln|x+1| = \tfrac{1}{2}t^2 - 3t + C$$

$$|x+1| = Ce^{1/2t^2}e^{-3t}$$

$$x+1 = \pm Ce^{1/2t^2}e^{-3t}$$

$$x = \pm Ce^{1/2t^2}e^{-3t} - 1$$

② Plug in initial condition.

$$-5 = \pm Ce^0e^0 - 1 \Rightarrow -4 = \pm C \Rightarrow C = -4.$$

Hence the solution is

$$\boxed{x(t) = -4e^{1/2t^2}e^{-3t} - 1}$$

9. (12 total points)

(a) (8 points) The volume of a lake is 10^5 m^3. A stream brings water into the lake at the rate of 0.2×10^5 m^3 per day, and another stream flows out of the lake at the same rate. A plant started releasing a harmful liquid chemical into the incoming stream, so that 0.1% (by volume) of the inflow now consists of the chemical. Assume that the contaminated water flowing into the lake mixes instantaneously with the water already in the lake. Find the volume of the chemical present in the lake 5 days after the start of the contamination. Give your answer in m^3 in decimal form, correct to at least the second digit after the decimal point.

① Set up differential equation: (Rate in − Rate out) and solve

Let y = the volume of chemical in the lake

$$dy/dt = [.2\times10^5](.001) - [.2\times10^5]\frac{y}{10^5} \leftarrow \text{volume in lake is constant}$$

$$\Rightarrow \frac{dy}{20-.2y} = dt \Rightarrow \int\frac{dy}{20-.2y} = \int dt$$

$$\Rightarrow -5\ln|20-.2y| = t + C$$

$$\Rightarrow |20-.2y| = Ce^{-1/5t}$$

$$\Rightarrow 20 \pm Ce^{-1/5t} = .2y$$

$$\Rightarrow 100 \pm Ce^{-1/5t} = y$$

② Solve initial value. Note at $t=0$, $y=0$ [No chemical in lake] Hence,

$C=100$ so $100-100e^{-1/5t} = y$

③ Plug in $t=5$: $\boxed{100-100e^{-1} = y \approx 63.212\,\text{m}^3}$

(b) (4 points) Ten liters of water per day pass through the body of a fish that lives in the lake. All the chemical present in the water that passes through the body of the fish is retained by the body. Compute how much of the chemical (by volume) has accumulated in the body of the fish 5 days after the start of the contamination. Give your answer in liters in decimal form, correct to at least the fourth digit after the decimal point.

① Set up differential equation and solve

Let f = volume of chemical in fish

$$\frac{df}{dt} = \frac{y}{10^5}\cdot 10 - 0 \qquad df = \frac{100-100e^{-.2t}}{10000}\,dt$$

$$f = \int .01 - .01e^{-.2t}\,dt$$

$$f = .01t + .05e^{-.2t} + C$$

② Solve the initial value. At $t=0$, $f=0$, hence

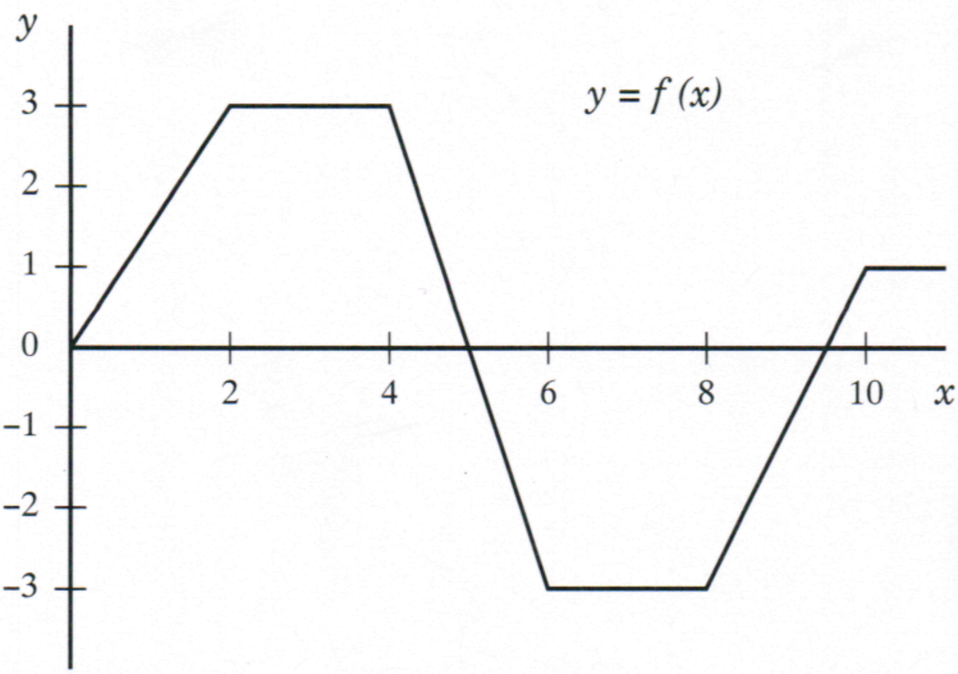
$$0 = 0 + .05 + C \Rightarrow C = -.05$$

$$\Rightarrow f = .01t + .05e^{-.2t} - .05$$

③ Plug in $t=5$,

$$f = .01(5) + .05e^{-.2(5)} - .05 \;\boxed{\approx .0183939 \text{ liters}}$$

10. (8 total points) Let f be the function whose graph is given below, and let $g(x) = \int_4^x f(t)\,dt$.



Evaluate the following:

(a) (2 points) $g(0) =$

By definition, $g(0) = \int_4^0 f(t)\,dt = -\int_0^4 f(t)\,dt = -\left(\frac{3\cdot 2}{2} + 2\cdot 3\right)$

$\boxed{= -9}$

(b) (2 points) $g'(2) =$

By fund. thm of Calculus, $g'(x) = f(x)$ $\therefore g'(2) = f(2)$

Looking at the graph (NOT area), $f(2) = \boxed{3}$

(c) (2 points) $g''(9) =$

By part (b), $g'(x) = f(x)$ so $g''(x) = f'(x)$. Recall that the derivative of the function is the slope of the tangent line. Therefore $g''(9) = f'(9)$ is $\frac{1 - -3}{10-8} = \frac{4}{2} = \boxed{2}$

(d) (2 points) $\int_0^2 t\,f(t^2)\,dt =$

Let $u = t^2$ $du = 2t\,dt \Rightarrow du/2t = dt$

$\int_0^2 t f(t^2)\,dt = \int_0^4 \frac{1}{2} f(u) = \frac{1}{2}\int_0^4 f(u) = -\frac{1}{2} g(0)$

$\boxed{= 9/2 \text{ by part a}}$