(1) Find the function $F(x)$ such that $F'(x) = x\sqrt{3x+1}$ and $F(0) = 0$.

Solution
Recall that,

$$F(x) = \int x\sqrt{3x+1}dx.$$

Using u-substitution,

$$\begin{aligned}&\text{Let } u = \sqrt{3x+1}\\ &u^2 = 3x+1\\ &2udu = 3dx\\ &\tfrac{2u}{3}du = dx\end{aligned}$$

Substituting back into $F'(x)$

$$\int x\sqrt{3x+1}dx = \int x * u\frac{2u}{3}du.$$

But there is a problem, we still have an x. However, $u^2 = 3x+1$ so $x = \frac{u^2-1}{3}$. Replacing this,

$$\begin{aligned}\int x\sqrt{3x+1}dx &= \int \frac{u^2-1}{3} * u * \frac{2u}{3}du\\ &= \int \frac{2u^4-2u^2}{9}du\\ &= \frac{2u^5}{45} - \frac{2u^3}{27} + C\end{aligned}$$

Replacing back $u = \sqrt{3x+1}$ gives,

$$F(x) = \frac{2\sqrt[5]{3x+1}}{45} - \frac{2\sqrt[3]{3x+1}}{27} + C.$$

Using $F(0) = 0$,

$$\begin{aligned}\frac{2\sqrt[5]{1}}{45} - \frac{2\sqrt[3]{1}}{27} + C &= 0\\ C &= \frac{4}{135}\end{aligned}$$

Thus,

$$\boxed{F(x) = \frac{2\sqrt[5]{3x+1}}{45} - \frac{2\sqrt[3]{3x+1}}{27} + \frac{4}{135}}$$

(2) Express the limit as a definite integral then evaluate:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\pi}{4n}\tan\frac{k\pi}{4n}$$

Solution
Recall that

$$\sum_{i=1}^{n} f(a+i\Delta x)\Delta x.$$

We need to find $[a,b]$ and $f(x)$.

We guess that $f(x) = \tan(x)$ and $\Delta x = \frac{\pi}{4n}$. Thus,

$$\begin{aligned}\frac{k\pi}{4n} &= a + i\Delta x\\ &= a + \frac{k\pi}{4n}\\ 0 &= a\end{aligned}$$

Now that we know a, let's find b.

$$\begin{aligned}\Delta x &= \frac{b-a}{n}\\ \frac{\pi}{4n} &= \frac{b-0}{n}\\ \frac{\pi}{4} &= b \text{ the } n \text{ cancels out}\end{aligned}$$

Therefore, $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\pi}{4n}\tan\frac{k\pi}{4n} = \int_0^1 \tan(x)dx$

(3) An object is tossed into the air at time $t = 0$. At time $t = 1$ seconds the object has reached height 14.7 m. Assuming that there is no air resistance, answer the following questions (recall that the gravitational constant $9.8m/s^2$; give your answers in decimals.

(a) What is the maximal height that the object reaches?

(b) What is the **total distance** that the object flies from time 0 until time $t = 3$?

Solution Find $a(t), v(t)$, and $s(t)$.

$$\begin{aligned}a(t) &= -9.8\\ v(t) &= -9.8t + C\\ s(t) &= -4.9t^2 + Ct + D\end{aligned}$$

By the statement of the problem, $s(1) = 14.7$, $s(0) = 0$ (it starts at height 0). Therefore,

$$s(0) = D = 0.$$

To find C, we will use $s(1) = 14.7$.

$$\begin{aligned}s(1) = -4.9(1^2) + C &= 14.7\\ C &= 19.6\end{aligned}$$

Hence

$$\begin{aligned}a(t) &= -9.8\\ v(t) &= -9.8t + 19.6\\ s(t) &= -4.9t^2 + 19.6t\end{aligned}$$

(a) The maximum height.

The maximum height occurs when $v(t) = 0$, so

$$\begin{aligned}v(t) = 0 &= -9.8t + 19.6\\ 2 &= t\end{aligned}$$

Thus when $t = 2$, it reaches the maximum height. Therefore the maximum height occurs,

$$\begin{aligned}s(2) &= -4.9(2^2) + 19.6(2)\\ &= 19.6\end{aligned}$$

The maximum height occurs at 19.6 meters.

(b) Total Distance

$$\text{Total Distance } = \int_{|} v(t)|dt.$$

We know that the height is increasing and positive between $[0, 2]$ then decreases so

$$\text{Total Distance } = \int_0^2 -9.8t + 19.6dt + \int_2^3 -19.6 + 9.8tdt.$$

Hence,

$$-4.9t^2 + 19.6t|_0^2 = 19.6$$
$$4.9t^2 - 19.6t|_2^3 = 4.9$$

The total distance is $19.6 + 4.9 = 24.5$ meters.

(4) Use the midpoint rule with $n = 3$ subdivisions to find the approximate value of $\int_0^6 \frac{x^2+5}{x^3+1}dx$. Give your answer to two decimal places.

Solution
Remember that

$$\int_a^b f(x)dx = \Delta x(f(x_1^*) + f(x_2^*) + f(x_3^*) + \ldots).$$

$$\Delta x = \frac{b-a}{n} = \frac{6-0}{3} = 2$$

The midpoints occur at $1, 3, 5$.

$$\begin{aligned} 2(f(1) + f(3_+ f(5)) &= 2\left(\frac{1^2+5}{1^3+1} + \frac{3^2+5}{3^3+1} + \frac{5^2+5}{5^3+1}\right) \\ &= 2(3 + \frac{1}{2} + \frac{5}{21}) \\ &= \frac{157}{21} \\ &\approx 7.47619 \end{aligned}$$