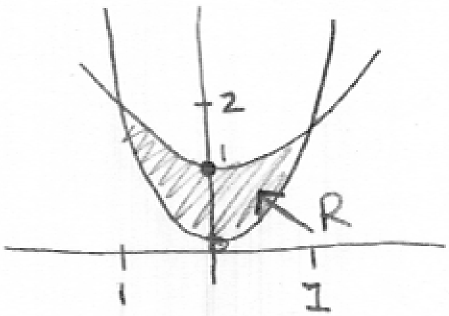
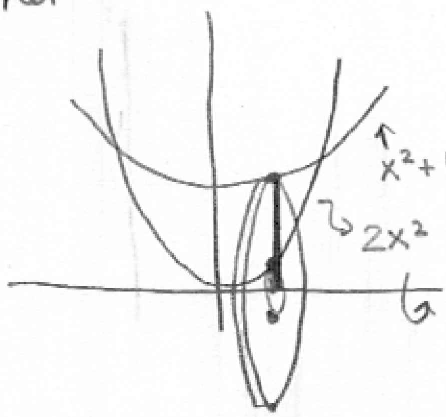
Worksheet 3 Solutions



Problem 1:

① Draw a typical washer

Note everything in terms of x



— Outer Radius

— Inner Radius

∴ outer radius $= x^2+1$

inner radius $= 2x^2$

② $\int_{-1}^{1} \pi\left[(\text{Outer Radius})^2 - (\text{Inner Radius})^2\right] dx$

$$= \int_{-1}^{1} \pi\left[(x^2+1)^2 - (2x^2)^2\right] dx$$

$$= \int_{-1}^{1} \pi\left[(x^4+2x^2+1-4x^4)\right] dx$$

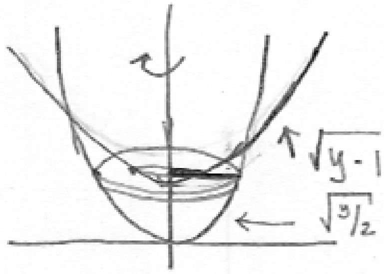
$$= \pi\left[\tfrac{1}{5}x^5 + \tfrac{2}{3}x^3 + x - \tfrac{4}{5}x^5\right]\Big|_{-1}^{1}$$

$$\boxed{= 32\pi/15}$$

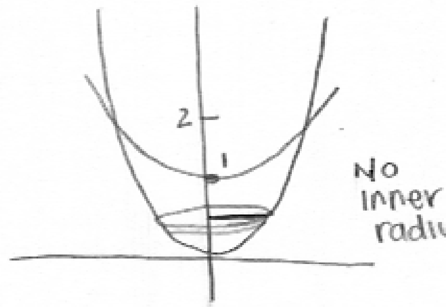
Problem 2:

① Draw a typical washer

Note need everything in terms of y



If $1 \le y \le 2$



if $0 \le y \le 1$

② Set up integral

$$\int_1^2 \pi \left[(\text{outer radius})^2 - (\text{inner radius})^2\right] dy$$

$$+ \int_0^1 \pi \left[(\text{outer radius})^2 - (\text{inner radius})^2\right] dy$$

$$= \int_1^2 \pi \left[\left(\sqrt{y/2}\right)^2 - \left(\sqrt{y-1}\right)^2 \right] dy + \int_0^1 \pi \left[\left(\sqrt{y/2}\right)^2 - 0^2 \right] dy$$

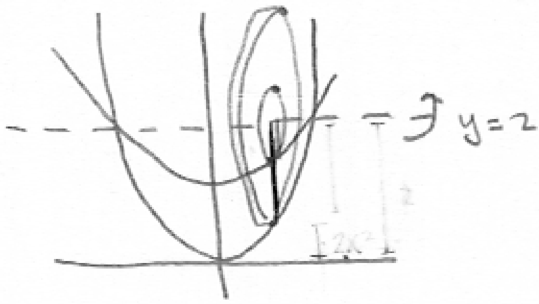
$$= \int_1^2 \pi \left[y/2 - y + 1 \right] dy + \int_0^1 \pi \left[y/2 \right] dy$$

$$= \pi \left[\frac{y^2}{4} - \frac{1}{2} y^2 + y \right] \Big|_1^2 + \pi \left[y^2/4 \right] \Big|_0^1$$

$$= \frac{\pi}{4} + \frac{\pi}{4} = \boxed{\frac{\pi}{2}}$$

Problem 3

① Draw a typical washer



Outer radius $= 2 - 2x^2$

Inner radius $= 2 - (x^2+1)$

②

$$\int_{-1}^{1} \pi\left[(2-2x^2)^2 - (2-(x^2+1))^2\right] dx$$

$$= \int_{-1}^{1} \pi\left[4 - 8x^2 + 4x^4 - (x^4 - 2x^2 + 1)\right] dx$$

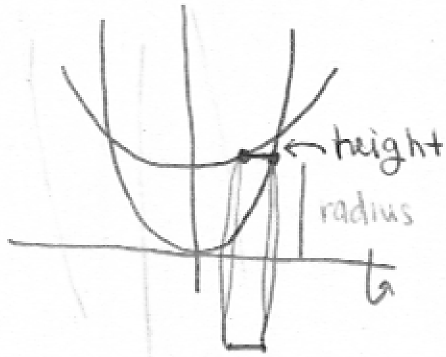
$$= \pi\left[3x - \frac{6}{3}x^3 + \frac{3}{5}x^5\right]\Big|_{-1}^{1}$$

$$= \boxed{\frac{\pi \cdot 16}{5}}$$

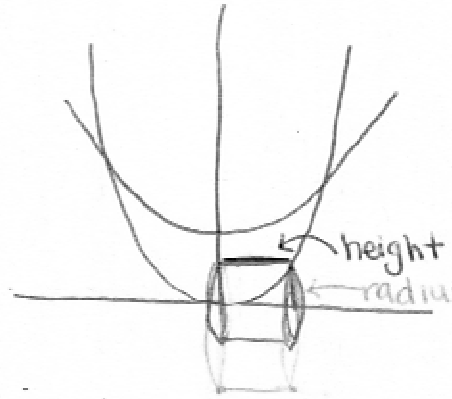
Problem 4

* Need everything in terms of y

Observe, I am only going to find the area of ½ of the region. By symmetry, I can multiply by 2 to get the full region



if $1 \le y \le 2$



if $0 \le y \le 1$

Height $= \sqrt{\frac{y}{2}} - \sqrt{y-1}$
radius $= y$

Height $= \sqrt{\frac{y}{2}}$
radius $= y$

$$\frac{1}{2}\text{ Volume} = 2\pi \int \text{height} * \text{radius} = 2\pi \left(\int_1^2 \left(\sqrt{\tfrac{y}{2}} - \sqrt{y-1}\right) y\, dy + \int_0^1 \left(\sqrt{y/2}\right) y\, dy \right)$$

$$= 2\pi \left[\int_1^2 \frac{y^{3/2}}{\sqrt{2}}\, dy - \int_1^2 \sqrt{y-1} \cdot y\, dy + \int_0^1 \frac{y^{3/2}}{\sqrt{2}}\, dy \right]$$

$u = y-1$
$du = dy$

$$= 2\pi \left[\int_1^2 \frac{y^{3/2}}{\sqrt{2}}\, dy - \int_0^1 (u+1)\sqrt{u}\, du + \int_0^1 y^{3/2}/\sqrt{2}\, dy \right]$$

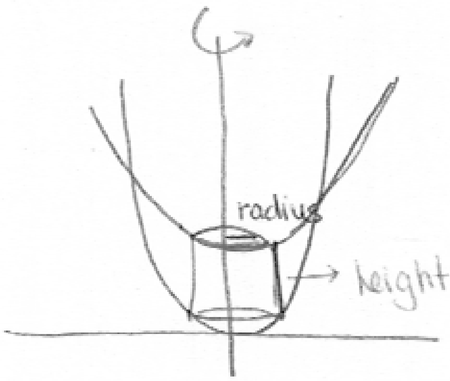
$$= 2\pi \left[\frac{2}{5\sqrt{2}} y^{5/2} \Big|_1^2 - \left(\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right) \Big|_0^1 + \frac{2}{5\sqrt{2}} y^{5/2} \Big|_0^1 \right]$$

$$= 2\pi \left[\frac{2}{5}(\sqrt{2})^4 - \left(\frac{2}{5} + \frac{2}{3}\right) \right] = 16\pi/15$$

Since calculating only ½ vol, then total vol. is

$$2 \cdot \frac{16\pi}{15} = \boxed{\frac{32\pi}{15}}$$

Problem 5



Everything in terms of x

$$\text{Height} = x^2+1-2x^2$$

$$\text{Radius} = x$$

$$\therefore \text{Volume} = 2\pi \int_0^1 \overbrace{(x^2+1-2x^2)}^{\text{Height}} \overset{\text{radius}}{x}\, dx$$

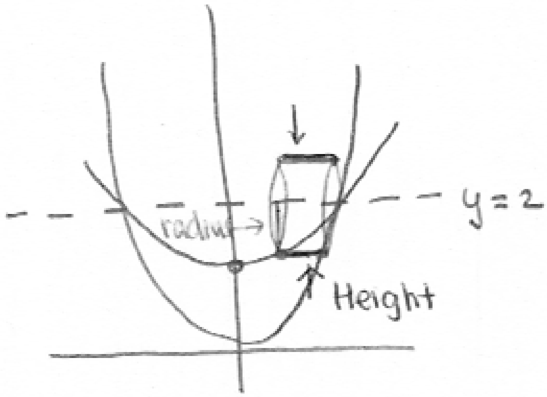
$$= 2\pi \left(-\tfrac{1}{4}x^4 + \tfrac{1}{2}x^2\right)\Big|_0^1$$

$$= 2\pi\left(\tfrac{-1}{4}+\tfrac{1}{2}\right) = \boxed{\tfrac{\pi}{2}}$$

Problem 6

Everything in terms of y

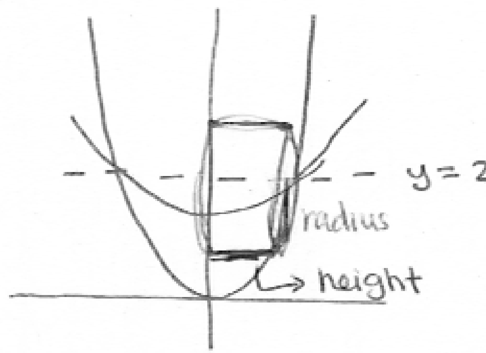
Computing $\frac{1}{2}$ the Volume



if $1 \le y \le 2$

Height $= \sqrt{y/2} - \sqrt{y-1}$

radius $= 2 - y$



If $0 \le y \le 1$

Height $= \sqrt{y/2}$

radius $= 2 - y$

$$\frac{1}{2}\text{Volume} = 2\pi\left(\int_1^2 \left(\sqrt{\tfrac{y}{2}} - \sqrt{y-1}\right)(2-y)\,dy + \int_0^1 \left(\sqrt{\tfrac{y}{2}}\right)(2-y)\,dy\right)$$

$$= 2\pi\left[\int_1^2 \frac{2}{\sqrt{2}} y^{1/2} - \frac{y^{3/2}}{\sqrt{2}}\,dy - \int_1^2 \sqrt{y-1}\,(2-y)\,dy + \int_0^1 \frac{2y^{1/2}}{\sqrt{2}} - \frac{y^{3/2}}{\sqrt{2}}\,dy\right]$$

$\rightarrow u = y - 1$

$du = dy$

$u + 1 = y$

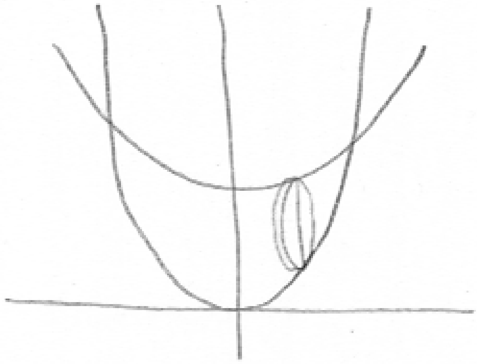
$$= 2\pi\left[\frac{4}{3\sqrt{2}} y^{3/2} - \frac{2}{5\sqrt{2}} y^{5/2}\Big|_0^2 - \int_0^1 \sqrt{u}\,(1-u)\,du\right.$$

$$= 2\pi\left(\left[\frac{8}{3} - \frac{8}{5}\right] - \left(\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\Big|_0^1\right)\right)$$

$$= 2\pi\left(\frac{8}{3} - \frac{8}{5} - \frac{2}{3} + \frac{2}{5}\right) = \frac{8\pi}{5}$$

$\therefore$ Total Volume $= \dfrac{2 \cdot 8\pi}{5} = \boxed{\dfrac{16\pi}{5}}$

Problem 7



diameter $= x^2+1-2x^2$

$$\int_{-1}^{1} \pi r^2\,dx = \int_{-1}^{1} \pi \left(\frac{x^2+1-2x^2}{2}\right)^2 dx$$

$$= \int_{-1}^{1} \pi \left(\frac{1-2x^2+x^4}{4}\right) dx$$

$$= \frac{\pi}{4}\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]\Big|_{-1}^{1}$$

$$= \frac{\pi}{4}\left[1-\frac{2}{3}+\frac{1}{5}-\left(-1+\frac{2}{3}-\frac{1}{5}\right)\right] = \boxed{\frac{4\pi}{15}}$$