1. (12 points; 4pts each) Find the derivatives of the following functions. You do not have to simplify.

(a) $y = x^e + 5\,e^{x^2}$

$$\boxed{y' = e x^{e-1} + 5e^{x^2} \cdot 2x}$$

(b) $y = \sin\left(\sqrt{x \cos x}\right)$

$$\boxed{y' = \cos\left(\sqrt{x\cos x}\right) \cdot \left[\tfrac{1}{2}(x\cos x)^{-1/2} \cdot \left[\cos x - x\sin x\right]\right]}$$

(c) $y = x^{(2^x)}$

$$y = x^{(2^x)} = e^{\ln(x^{2^x})} = e^{2^x \ln(x)}$$

$$= e^{e^{x\ln(2)} \cdot \ln(x)}$$

Note: $2^x = e^{\ln(2^x)} = e^{x\ln(2)}$

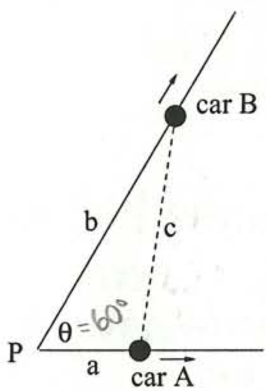
$$\boxed{y' = e^{e^{x\ln(2)} \ln(x)} \cdot \left[e^{x\ln(2)} \cdot \ln(2) \cdot \ln(x) + e^{x\ln(2)} \cdot \frac{1}{x}\right]}$$

2. (13 points) *In this problem you should use the law of cosines:*

$$a^2 + b^2 - 2ab\cos\theta = c^2,$$

where θ is an angle in the triangle, c is the opposite side, and a and b are the adjacent sides.

Two straight roads intersect at point P at a 60° angle. Car A is traveling away from P on one road, and car B on the other road. You're in car A, which has a device that can measure distance from car B and also the rate at which that distance is increasing. At a certain moment you have traveled 3 km away from P and are moving at 80 km/hr. At that time the device shows that your distance from car B is 7 km, and this distance is increasing at 100 km/hr.



What We Know

① $a = 3,\ da/dt = 80$ km/hr

② $c = 7,\ dc/dt = 100$ km/hr

③ $\theta = 60° = \pi/3$ ← This is NOT changing

At that instant find

(a) (5 pts) the distance car B has traveled from P;

Using Law of Cosines

$3^2 + b^2 - 2(3)b\cos(\pi/3) = 7^2$

$9 + b^2 - 6b(1/2) = 49$

$b^2 - 3b - 40 = 0$

$(b-8)(b+5) = 0$

$b = 8$ or -5

B/c $b \geq 0$, we ignore -5.

$\boxed{\text{Thus } b = 8\text{ km.}}$

(b) (8 pts) the speed at which car B is moving.

In this case, speed refers to db/dt

① Take derivative

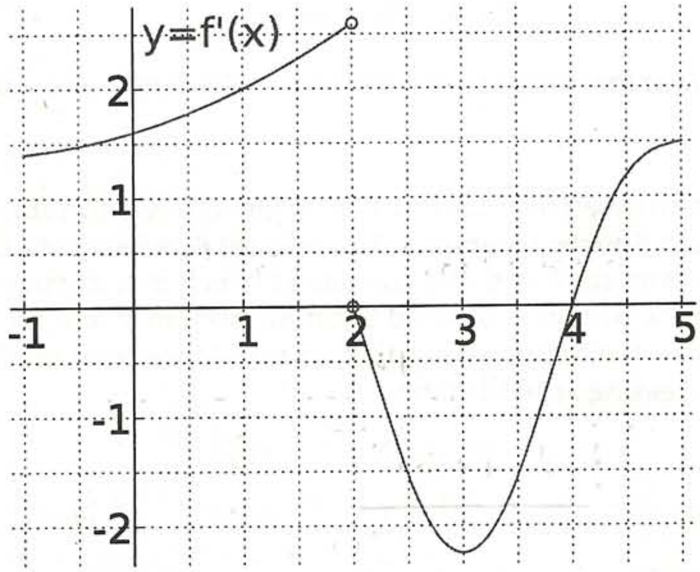
$2a\,da/dt + 2b\,db/dt - 2b\cos(\pi/3)\frac{da}{dt} - 2a\cos(\pi/3)\frac{db}{dt} = 2c\,dc/dt$

② Plugging in everything

$2(3)(80) + 2(8)db/dt - 8(80) - 3db/dt = 2(7)(100)$

$13\,db/dt = 1560 \Rightarrow \boxed{db/dt = 120\text{ km/hr}}$

3. (15 points) Below is the graph of the DERIVATIVE of a continuous function $f(x)$.



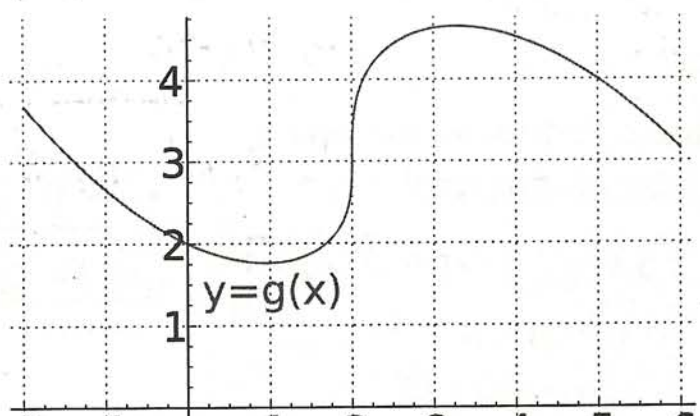
a. On what interval(s) is f increasing? $[-1,2) \cup (4,5)$ where $f'(x) > 0$

b. On what interval(s) is f concave down? $(2,3)$ where $f'(x)$ decreasing

c. Find all the value(s) of c such that $f(x)$ has a local maximum at c. DNE

$f'(x)=0$ and $f''(x) > 0$ which doesn't occur

Below is the graph of a continuous function $g(x)$.



d. Find all the value(s) of c such that $g(x)$ has a local minimum at c. $\boxed{1}$

e. Circle the x values that are among the critical numbers for $f(g(x))$ (where $f(x)$ is the function whose derivative is shown at the top of the page).

$x = -1$ $\qquad$ $x = 0$ $\qquad$ $x = 4$ $\qquad$ $x = 5$

*Critical Pts occur where $f(g(x))' = 0$ OR $f(g(x))'$ DNE

$f(g(x))' \Rightarrow f'(g(x))\,g'(x) = 0$ OR DNE

① $f'(g(-1))\,g'(-1) = f'(2.5)\,g'(-1) \neq 0$

② $f'(g(0))\,g'(0) = f'(2)(+) =$ DNE

③ $f'(g(4)) \cdot g'(4) = f'(4.5)(-) \neq 0$

④ $f'(g(5)) \cdot g'(5) = f'(4) \cdot - = 0$

4. (12 points)

Consider the curve defined by the equation

$$y^2 = 3x + 4\cos(xy).$$

(a) (4pts) Implicitly differentiate to find $\frac{dy}{dx}$.

$2y\,dy/dx = 3 - 4\sin(xy)\cdot[y + x\,dy/dx]$

$2y\,dy/dx + 4\sin(xy)x\,dy/dx = 3 - 4y\sin(xy)$

$$\boxed{dy/dx = \frac{3-4y\sin(xy)}{2y+4x\sin(xy)}}$$

(b) (4pts) Find the tangent line equation(s) at the y-intercept(s).

① Find the y-intercepts $\Rightarrow x=0$

$y^2 = 0 + 4\cos(0) = y^2 = 4 \Rightarrow y = 2$ OR -2

The points are $(0,2)$ and $(0,-2)$

② Find slopes at intercepts

$(0,-2)$: $dy/dx = \frac{3-4(-2)\sin(0)}{2(-2)+0} = -3/4$

$(0,2)$: $dy/dx = \frac{3-4(2)\sin(0)}{2(2)+0} = \frac{3}{4}$

③ The Equations are

$y = -3/4x - 2$

$y = 3/4x + 2$

(c) (4pts) Find the tangent line equation(s) at the x-intercept(s).

① Find the x-intercepts $\Rightarrow y=0$

$0^2 = 3x + 4\cos(0) \Rightarrow 0 = 3x + 4 \quad x = -4/3$

② Find slope at $(-4/3, 0)$

$dy/dx = \frac{3-0}{0+0} =$ undefined

③ When undefined slope, you have a vertical tangent line thus $x = -4/3$

5. (12 points)

Let $f(t)$ be defined by the formula

$$f(t) = \frac{t^2 e^{t-4}}{1+t}.$$

(a) Find the derivative of $f(t)$.

$$f'(t) = \frac{[2te^{t-4} + t^2e^{t-4}](1+t) - t^2e^{t-4}}{(1+t)^2}$$

$$= \boxed{\frac{e^{t-4}[(2t+t^2)(1+t) - t^2]}{(1+t)^2}}$$

(b) Find the tangent line to the graph of $f(t)$ at the point $(4, f(4))$.

① Find the slope.

$$f'(4) = \frac{e^0[(2(4)+4^2)(1+4) - 4^2]}{(1+4)^2} = \frac{24(5)-16}{25} = \frac{104}{25}$$

② Generalized Tangent Line

$y = 104/25(t-4) + f(4)$ $\quad f(4) = \frac{4^2e^0}{1+4} = \frac{16}{5}$

$$\boxed{y = \frac{104}{25}(t-4) + \frac{16}{5}}$$

(c) Use the tangent line approximation to find a number t for which $f(t) \approx 3$. Give your answer in decimal form to four digits after the decimal point.

We did the tangent line approx above and now just plug in 3 for "y".

① Tangent line approx: $y = 104/25(t-4) + 16/5$

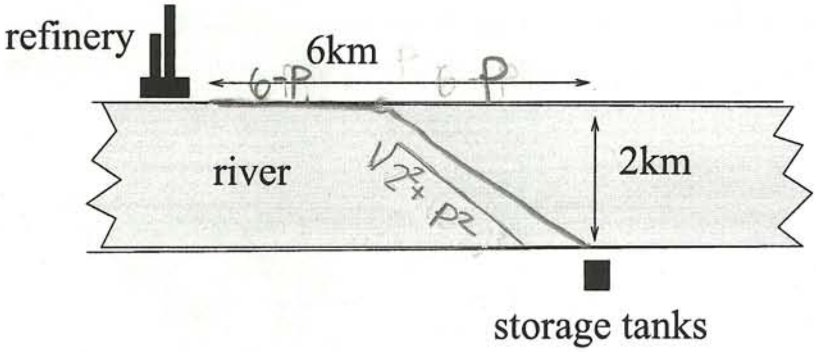
② Plug in given pt

$3 = 104/25(t-4) + 16/5$

$(3 - 16/5)(25/104) + 4 = t$

$$\boxed{t = 411/104 \approx 3.9519}$$

6. (12 points) An oil refinery is located on the north bank of a straight river that is 2 km wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 6 km east of the refinery. The cost of laying pipe is \$300,000/km over land to a point P on the north bank and \$500,000/km under the river to the tanks.



(a) To minimize the cost of the pipeline, where should P be located? Be sure to justify you have found the minimum.

① Cost of Pipe on land $= 300{,}000(6-P)$

Cost of Pipe in water $= 500{,}000\sqrt{2^2+P^2}$

Total Cost: $300{,}000(6-P) + 500000\sqrt{4+P^2}$

② Take derivative

$TC' = -300{,}000 + \frac{1}{2}500{,}000(4+P^2)^{-1/2}\cdot 2P$

$0 = -300{,}000 + 500{,}000P(4+P^2)^{-1/2}$

$0 = \dfrac{-300{,}000\sqrt{4+P^2} + 500{,}000P}{\sqrt{4+P^2}}$

$\Rightarrow 0 = -300{,}000\sqrt{4+P^2} + 500{,}000P$

$5/3\,P = \sqrt{4+P^2}$

$25/9\,P^2 = 4+P^2$

$16/9\,P^2 = 4 \Rightarrow P^2 = 36/16$

$P = \pm 6/4 = \pm 3/2$ can't be $P = -3/2$.

$TC(0) = 300{,}000(6) + 500{,}000(2)$
$= 2{,}800{,}000$

$TC(6) = 500000\sqrt{4+36}$
$= 3$ million + more

$\boxed{\text{Hence, } P = 3/2 \text{ is the minimum}}$

③ Check Endpts

$TC(3/2) = 300000(6-3/2) + 500000\sqrt{4+(\frac{3}{2})^2} = 2{,}600{,}000$

(b) What is the resultant minimum cost?

From above, the cost is

$TC(3/2) = 300{,}000(6-3/2) + 500000\sqrt{4+(3/2)^2} = \boxed{2{,}600{,}000}$

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7. (12 points)

(a) (6pts) Compute the limit. If it is correct to say that the limit is ∞ or $-\infty$, then say so. If the limit does not exist, explain why.

$$\lim_{x\to 0}(e^x + x)^{1/x}$$

① Rewrite as

$$(e^x+x)^{1/x} = e^{\ln([e^x+x]^{1/x})} = e^{\frac{1}{x}\ln(e^x+x)}$$

② $\lim_{x\to 0}(e^x+x)^{1/x} = \lim_{x\to 0} e^{1/x\ln(e^x+x)} = e^{\lim_{x\to 0}\frac{1}{x}\ln(e^x+x)}$

* $\lim_{x\to 0}\frac{\ln(e^x+x)}{x} = \frac{0}{0}$ so L'Hopital Rule $= \lim_{x\to 0}\frac{\frac{e^x+1}{e^x+x}}{1} = \frac{1+1}{1+0} = 2$

Thus

$$\boxed{\lim_{x\to 0}(e^x+x)^{1/x} = e^2}$$

(b) (6pts) Find a value of c so that the following function is continuous everywhere.

$$f(x) = \begin{cases} \frac{4x-2\sin(2x)}{10x^3} & \text{if } x \neq 0 \\ c & \text{if } x = 0 \end{cases}$$

In order to be continuous

① $\lim_{x\to 0} f(x)$ exists ② $f(0) = \lim_{x\to 0} f(x) = c$

Hence,

$\lim_{x\to 0}\frac{4x-2\sin(2x)}{10x^3} = c$ to be continuous $= \frac{0}{0}$.

By L'Hopitall's Rule,

$\lim_{x\to 0}\frac{4-4\cos(2x)}{30x^2} = \frac{0}{0}$ By L'Hop $= \lim_{x\to 0}\frac{8\sin(2x)}{60x} = \frac{0}{0}$

By L'Hopital,

$$\lim_{x\to 0}\frac{16\cos(2x)}{60} = \boxed{\frac{16}{60} = c}$$

8. (12 points) A curve has parametric equations:

$$\begin{aligned} x &= 3t^2 + 2 \\ y &= 4t^3 + 2, \end{aligned}$$

with $t > 0$. Find the equation of the tangent line to the curve that passes through the point $(6, 2)$. We see "through" so (6,2) doesn't go through * the function

① Take derivative

$$dy/dx = \frac{dy/dt}{dx/dt} = \frac{12t^2}{6t} = 2t$$

② Generalized Tangent line

$$Y = 2t(x-6)+2$$

③ Plug in the $x, y = (3t^2+2,\ 4t^3+2)$ and solve for t

$$4t^3+2 = 2t(3t^2+2-6)+2$$

$$4t^3+2 = 6t^3-8t+2$$

$$0 = 2t^3-8t$$

$$0 = 2t(t^2-4)$$

$$0 = t \text{ OR } 0 = t^2-4 \Rightarrow t = \pm 2$$

B/c $t > 0$ by assumption, $t = 2$

④ Plug $t=2$ back into generalized tangent line

$$Y = 4(x-6)+2$$

$$Y = 4x-22$$