9. (12 total points) A tank contains 100 liters of fresh water. Water containing s grams of salt per liter enters the tank at the rate of 5 liters/minute, and the well-mixed solution leaves at the same rate.

  (a) (6 points) Write down a differential equation for the amount of salt in the tank at time t. (This equation will contain s.)

  (b) (6 points) Suppose that after 10 minutes, the concentration of salt in the tank is 3 grams/liter. Find s.

10. (12 total points)   A curve has the property that the tangent line to the curve at each point (x,y) has y-intercept $(0,y/2)$, as shown in the picture below. In addition, the curve passes through the point $(3,1)$.

(a) (4 points) Derive a differential equation for the curve using its slope at the point (x,y).
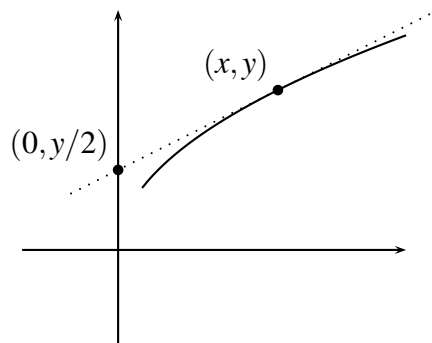
(b) (8 points) Solve the differential equation you obtained in part (a) to determine the equation of the curve.

10. (8 points) Find the solution of the differential equation

$$\frac{dy}{dx} = xe^{x+y}$$

which satisfies the initial condition $y(0) = 1$.

11. (10 points) The swine flu epidemic has been modelled by the Gompertz function, which is a solution of the differential equation

$$\frac{dy}{dt} = 1.2\, y\, (K - \ln(y)),$$

where $y(t)$ is the number of individuals (in thousands) in a large city that have been infected by time t, and K is a constant. Time t is measured in months, with $t = 0$ on July 9, 2009.

On July 9, 2009, 75 thousand individuals had been infected. One month later, 190 thousand individuals had been infected. Find

$$\lim_{t \to \infty} y(t),$$

which is the total number of individuals (in thousands) that will have been infected.

11. (8 total points) Suppose we have a colony of bacteria living in a Petri dish. Due to space limitations, there is a maximum number, k, of bacteria that can live in the dish. Let $P(t)$ be the population of the bacterial colony at time t. According to one model for population growth, the rate of growth of the population $\frac{dP}{dt}$ is proportional to the difference of the threshold population k and the present population; in other words

$$\frac{dP}{dt} = c(k - P). \tag{1}$$

The constant of proportionality c measures how quickly the bacteria multiply. For simplicity, we take $c = 1$.

(a) (6 points) Solve this differential equation for the unknown function $P(t)$.

(b) (2 points) If $k = 5,000,000$ and the initial population size is $P(0) = 1,000,000$, compute $\lim_{t \to \infty} P(t)$.

10. (8 points) Find the solution $y(x)$ for $x \geq 1$ of the initial value problem

$$\frac{y}{x^3}\frac{dy}{dx} = 4\ln(x) \ , \quad y(1) = 2.$$

10. (12 total points)   A 50-gallon tank initially contains 20 gallons of water in which 10 lbs of salt are dissolved. Pure water enters the tank at a rate of 4 gal/min. Simultaneously, a drain is open at the bottom of the tank, allowing the salt-water solution to leave the tank at a rate of 2 gal/min. The solution is kept thoroughly mixed.

  (a) (3 points) Find the volume $V(t)$ (in gallons) of the salt-water solution in the tank at time t (in minutes).

  (b) (3 points) Write a differential equation for the amount $y(t)$ (in lbs) of salt in the tank at time t.

  (c) (3 points) Solve this differential equation and use the initial amount of salt in the tank to find a formula for $y(t)$.

  (d) (3 points) What is the amount of salt in the tank at the moment that the tank becomes full?

9. (10 points) Find the solution of the differential equation that satisfies the given initial condition.

$$\frac{dy}{dx} = \frac{x\sin(x^2)}{y}, \qquad y(0) = -2.$$

11. (8 points) Find the function $y(x)$ which satisfies $\dfrac{dy}{dx} = \dfrac{x(y^2+1)}{\sqrt{x^2-1}}$ such that $y = 1$ when $x = \sqrt{2}$.

10. (10 points) At time $t = 0$, a tank contains 100 gallons of pure gasoline. A mixture whose volume is 30% ethanol and 70% gasoline is pumped into the tank at a rate of 2 gallons per minute. The solution is kept thoroughly mixed and drains from the tank at the same rate. Find a formula for the number of gallons of ethanol in the tank after t minutes.

8. (10 points) Find the solution of the differential equation that satisfies the given initial condition.

$$\frac{dP}{dt} = \sqrt{Pt}, \qquad P(4) = 1.$$

9. (12 total points)   In 2009 (which we take to be $t = 0$, with t in years) there are 5000 wolves in a big forest area. In the absence of hunting, the wolf population would increase at the rate of 1% per year. However, hunters are killing wolves at the steady rate of 100 wolves per year.

   (a) (4 points) Write a differential equation for $W(t)$.

   (b) (4 points) Solve this differential equation and use the initial number of wolves to find a formula for $W(t)$.

   (c) (4 points) In what year will the wolves in this forest area die out? Your answer should be some year in this century.