1. (12 points; 4 pts each) Find the derivatives of the following functions. You do not have to simplify.

(a) $g(x) = \ln(1 + \cos(3x - 1))$

$$\boxed{g'(x) = \frac{1}{1+\cos(3x-1)} \cdot (-3\sin(3x-1))}$$

(b) $h(x) = \dfrac{5 + \tan^{-1} x}{e^x + 1}$

$$\boxed{h'(x) = \frac{\left(\frac{1}{1+x^2}\right)(e^x+1) - (e^x)(5+\tan^{-1}x)}{(e^x+1)^2}}$$

(c) $u(x) = (\ln x)^{(3^x)}$

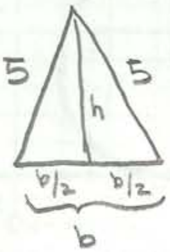
$u(x) = (\ln x)^{3^x} = e^{3^x \ln(\ln(x))} = e^{x\ln(3)\ln(\ln(x))}$

Note: $3^x = e^{\ln(3^x)} = e^{x\ln(3)}$

$$\boxed{u'(x) = e^{x\ln(3)\ln(\ln(x))} \cdot \left[\ln(3)\ln(\ln(x)) + x\ln(3)\left(\frac{1}{\ln(x)}\right)\cdot\frac{1}{x}\right]}$$

2. (12 points) An isosceles triangle has a base of length $b(t)$, which varies with time, and two equal sides of constant length 5 cm.

(a) (6 pts) Let $h(t)$ be the height of the triangle (the distance from the base to the vertex between the equal sides). Find a formula for $\frac{dh}{dt}$ in terms of b and $\frac{db}{dt}$.



$$5^2 = h^2 + (b/2)^2$$

$$0 = 2h\, dh/dt + 1/2\, b\, db/dt$$

$$\frac{-b\, db/dt}{4h} = dh/dt$$

$$h = \sqrt{25-(b/2)^2} \text{ thus}$$

$$\boxed{dh/dt = \frac{-b\, db/dt}{4\sqrt{25-(b/2)^2}}}$$

(b) (6 pts) Let $\theta(t)$ be the angle between the two equal sides. If $\frac{db}{dt} = 7$ cm/sec at the time when $\theta = \frac{\pi}{3}$, find $\frac{d\theta}{dt}$. $\boxed{\text{Goal: } d\theta/dt}$

Law of Cosines

$$5^2 + 5^2 - 2(5)(5)\cos\theta = b^2$$

$$50 - 50\cos\theta = b^2$$

$$50\sin\theta \frac{d\theta}{dt} = 2b\, db/dt$$

$$\frac{d\theta}{dt} = \frac{2b\, db/dt}{50\sin\theta} \longrightarrow \frac{d\theta}{dt} = \frac{2(5)\cdot 7}{50\sin(\pi/3)}$$

when $\theta = \pi/3$,

$$50 - 50\cos(\pi/3) = b^2$$

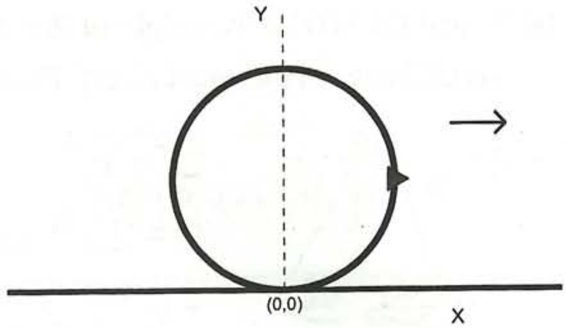
$$50 - 50(1/2) = b^2$$

$$25 = b^2 \Rightarrow b = 5$$

$$\boxed{\frac{d\theta}{dt} = \frac{70}{50(\sqrt{3}/2)} \text{ rad/sec}}$$

3. (12 points) A triangle is painted on the rim of a rolling bicycle wheel. The position of the triangle at time t is given by the parametric equations:

$$x(t) = t + \cos t$$
$$y(t) = 1 - \sin t$$



(a) (5 pts) Find the time t in the interval $[0, 2\pi]$ when the horizontal velocity of the triangle is a minimum. What is the minimum horizontal velocity? What are the coordinates $x(t)$ and $y(t)$ of the triangle at this time? Note trying to find min. of horizontal velocity

$\frac{dx}{dt}$ = horizontal velocity = $1 - \sin t$

$\frac{d^2x}{dt^2} = -\cos t = 0 \Rightarrow t = \pi/2$ or $3\pi/2$

$0<t<\pi/2$	$\pi/2$	$\frac{\pi}{2}<t<\frac{3\pi}{2}$	$3\pi/2$	$3\pi/2<t<$
$-$	0	$+$	0	$-$

Thus $t = \pi/2$ is a minimum and $(\pi/2, 0)$, the minmum velocity = 0

(b) (5 pts) Find the time t in the interval $[0, 2\pi]$ when the horizontal velocity of the triangle is a maximum. What is the maximum horizontal velocity? What are the coordinates $x(t)$ and $y(t)$ of the triangle at this time?

$dy/dt = -\cos t$ $\quad d^2y/dt^2 = \sin t = 0 \Rightarrow t = 0, \pi, 2\pi$

0	$0<t<\pi$	π	$\pi<t<2\pi$	2π
0	$+$	0	$-$	0

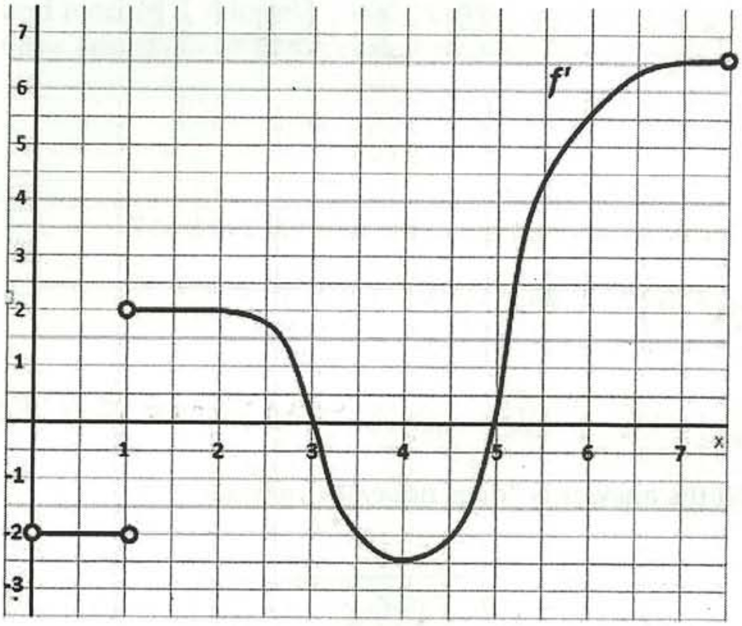
Thus $t = \pi$ is a maximum with maximur velocity 1, and position $(\pi - 1, 1)$

(c) (2 pts) Describe in words, the position of the triangle with respect to the wheel when its horizontal velocity is a maximum and when it is a minimum.

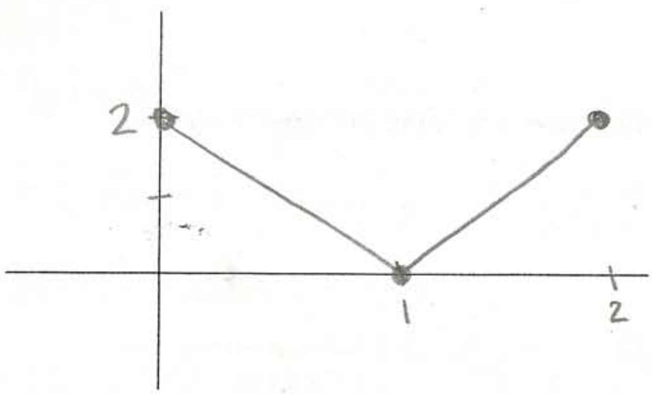
When horizontal velocity is maximize, It is at $t = 3\pi/2$. The triangle would be located at $(3\pi/2, 2)$ (i.e at the top of the rotation)

When vertical velocity is minimized $t = 0$ or 2π. The triangle would be located at $(1, 1) \rightarrow$ starting position and $(2\pi + 1, 1)$.

4. (14 points; 2 pts each part) The following is the graph of the **DERIVATIVE f′** of a function f (f is not shown). Answer the following questions; you do not need to explain your answers.



(a) Suppose $f(1) = 0$ and f is continuous for $x \geq 0$. Use the derivative graph above to sketch the graph of f over the interval $0 \leq x \leq 2$. Label the y-intercept.



$$f(x) = \begin{cases} -2(x-1) & x \leq 1 \\ 2(x-1) & x \geq 1 \end{cases}$$

(b) List the longest open interval over which f is decreasing (recall that the given graph is not f, but its derivative).

$\boxed{(3,5)}$

This is b/c $f'(x) < 0$ on $(3,5)$

4. continued

(c) List the longest interval over which f is concave up.

This is where $f'(x)$ is increasing $\boxed{(4,7)}$

(d) List x-coordinates of the local minimal points in the interval $2 < x < 7$.

3 local max b/c $f'(x)$ goes from $+ \mapsto -$

5 local min b/c $f'(x)$ goes from $- \mapsto +$

(e) Compute the following. If the answer is "does not exist", say so.

$$\lim_{h \to 0} \frac{f(0.5+h) - f(0.5)}{h} = f'(.5) = \boxed{-2}$$

(f) Compute the following. If the answer is "does not exist", say so.

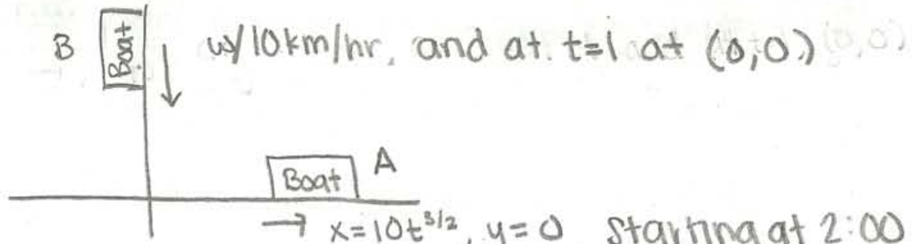
$$\lim_{x \to 1} f'(x) = \text{DNE}$$

$\lim_{x \to 1^+} f(x) = 2$ and $\lim_{x \to 1^-} f'(x) = -2$

(g) Compute the following. If the answer is "does not exist", say so.

$f''(4) = 0$ This is b/c $f''(4)$ = the slope of $f'(4)$.

5. (12 points) A boat leaves a dock at 2:00 PM and travels due east for an hour. At time t (in hours) after it leaves the dock, the boat is $10t^{3/2}$ km east of the dock. Another boat, coming from north of the dock, is heading due south at a speed of 10 km/h and reaches the dock at 3:00 PM. At what time were the two boats closest together? Give your answer to the closest minute. Justify your answer.



Let $t=0$ at 2:00

Boat B: $x=0,\ y=-10(t-1)$

Boat A: $x=10t^{3/2},\ y=0$ $\qquad t \geq 0$

$$\text{Distance}=\sqrt{(10t^{3/2}-0)^2+(0--10(t-1))^2}$$
$$=\sqrt{100t^3+(10(t-1))^2}$$

$$D' = \tfrac{1}{2}\left(100t^3+(10t-10)^2\right)^{-1/2}\cdot\left[300t^2+20(10t-10)\right]$$
$$0 = \tfrac{1}{2}\left(100t^3+(10t-10)^2\right)^{-1/2}\cdot\left[300t^2+20(10t-10)\right]$$
$$\Rightarrow 0 = 300t^2+200t-200$$

$$t=\frac{-20\pm\sqrt{20^2-4(30)(-20)}}{2(30)}=\frac{-20\pm\sqrt{2800}}{60}$$

Note that the only solution where t is positive is

$t=\dfrac{-20+\sqrt{2800}}{60}$ in hours $\quad t=-20+\sqrt{2800}$ mins ≈ 33 mins.

We need to check this is a min:

$0<t<.5485$	$.5485$	$.5485<t<1$
$-$	0	$+$

$\boxed{\text{Hence, } t=-20+\sqrt{2800}\approx 33\text{ min is minimum}}$

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6. (12 points; 4 pts each part) Compute the limit. If it is correct to say that the limit is ∞ or $-\infty$, then say so. If the limit does not exist, explain why. Justify your answers.

(a) $\lim_{x\to 1} \frac{x+7}{x^2-1} = \frac{8}{0}$ so know

$\lim_{x\to 1^+} \frac{x+7}{x^2-1} = \pm\infty$ and $\lim_{x\to 1^-} \frac{x+7}{x^2-1} = \pm\infty$

Now check signs

$\lim_{x\to 1^+} \frac{x+7}{x^2-1} = \frac{+}{+} = \infty$

$\lim_{x\to 1^-} \frac{x+7}{x^2-1} = \frac{+}{-} = -\infty$

Don't Match up

$\boxed{\text{Thus } \lim_{x\to 1} \frac{x+7}{x^2+1} = \text{DNE}}$

(b) $\lim_{x\to 0} \frac{(e^{3x}-1)}{\sin(x/4)}$

① Plug in and get $0/0 \to$ Step 2

② Step 2: (Rationalizing, etc don't seem like they work, but L'Hopital's does)

$\lim_{x\to 0} \frac{e^{3x}-1}{\sin(x/4)} = \lim_{x\to 0} \frac{3e^{3x}}{\frac{1}{4}\cos(\frac{x}{4})} = \frac{3}{1/4} = \boxed{12}$

(c) $\lim_{x\to\infty} \frac{\sqrt{1+5x^4}+\cos(x)}{3x^2+1}$

Since almost like Poly/Poly, try dividing by highest power

$\lim_{x\to\infty} \frac{\sqrt{1+5x^4}+\cos(x)}{3x^2+1} \frac{(\frac{1}{x^2})}{(1/x^2)} = \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^4}+5} + \frac{\cos x}{x^2}}{3+1/x^2}$

$= \frac{\sqrt{5} + \lim_{x\to\infty} \frac{\cos x}{x^2}}{3}$ ← Not $\frac{\sin x}{x}$ so maybe Squeeze Theorem

Squeeze Theorem

$-1 \le \cos x \le 1$

$\frac{-1}{x^2} \le \frac{\cos x}{x^2} \le \frac{1}{x^2}$

$\lim_{x\to\infty} \frac{-1}{x^2} \le \lim_{x\to\infty} \frac{\cos x}{x^2} \le \lim_{x\to\infty} \frac{1}{x^2}$

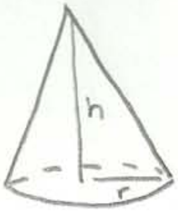
$\Rightarrow \quad 0 \le \lim_{x\to\infty} \frac{\cos x}{x^2} \le 0 \Rightarrow \lim_{x\to\infty} \frac{\cos x}{x^2} = 0$

Hence,

$\boxed{\lim_{x\to\infty} \frac{\sqrt{1+5x^4}+\cos x}{3x^2+1} = \frac{\sqrt{5}}{3}}$

8

7. (12 points) A right circular cone of height h and base radius r has total surface area consisting of its base area πr^2 plus its side area $\pi r\sqrt{r^2+h^2}$. Suppose you start out with a cone of height 8 cm and base radius 6 cm, and you want to change the dimensions in such a way that the total surface area remains the same. If you change the height to 8.04 cm, what is your new value for the base radius? Use implicit differentiation and the tangent line approximation. Please show your work clearly.



Surface Area $= \pi r^2 + \pi r\sqrt{r^2+h^2} = C \leftarrow$ constant

Use tangent line approx. where $r = x$ and $h = y$ variables

① Take derivative to find dh/dr

$$2\pi r + \pi\sqrt{r^2+h^2} + \pi r\left(\tfrac{1}{2}(r^2+h^2)^{-1/2}\right)\cdot\left(2r + 2h\tfrac{dh}{dr}\right) = 0$$

$$\frac{(-2\cancel{\pi} r - \cancel{\pi}\sqrt{r^2+h^2})\sqrt{r^2+h^2}}{\cancel{\pi} r} = r + h\,dh/dr$$

$$dh/dr = \frac{(-2r - \sqrt{r^2+h^2})(\sqrt{r^2+h^2})}{hr} - \frac{r}{h}$$

At $h=8$, $r=6$

$$\frac{dh}{dr} = -127/9$$

② The Tangent Line is

$$h = -127/9\,(r-6) + 8$$

$$8.04 = -127/9\,(r-6) + 8$$

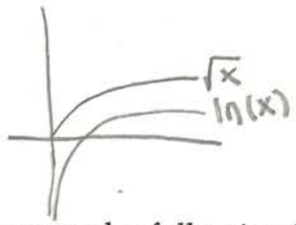
$$\boxed{r \approx 5.99717}$$

8. (14 points) Let $y = f(x) = \ln(x) - \sqrt{x}$ for $x > 0$.

Autum 2011

(a) Compute the following limit. If it is correct to say that the limit is ∞ or $-\infty$, then say so. If the limit does not exist, explain why. Justify your answers.

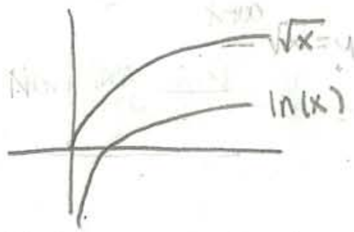
$\lim_{x\to 0^+}$ f(x).



$\lim_{x\to 0^+} \ln(x) - \sqrt{x} = \lim_{x\to 0^+} \ln(x) = -\infty$

(b) Compute the following limit. If it is correct to say that the limit is ∞ or $-\infty$, then say so. If the limit does not exist, explain why. Justify your answers.

$\lim_{x\to+\infty}$ f(x).



As shown by the picture, we know that $\sqrt{x} > \ln(x)$ and growing.

$\boxed{\text{Thus } \lim_{x\to\infty} \ln(x) - \sqrt{x} = -\infty}$

(c) Find any vertical or horizontal asymptote(s).

Vertical is where the function is undefined → DNE (our domain is only $x > 0$).

Horizontal is $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ which DNE b/c above the limit goes off to $-\infty$.

(d) Compute $f'(x)$ and $f''(x)$. Label and box each derivative.

$$\boxed{\begin{aligned} f'(x) &= \frac{1}{x} - \frac{1}{2}\frac{1}{\sqrt{x}} \\ f''(x) &= \frac{-1}{x^2} + \frac{1}{4}\frac{1}{x^{3/2}} \end{aligned}}$$

8. continued

(e) Find the interval(s) where f(x) is increasing, the interval(s) where f(x) is decreasing, and any local maximum or local minimum point(s) (both coordinates).

① Take Derivative and identify Places where undefined or $f'(x) = 0$

undefined at 0 but $x > 0$, so don't count it

$$\frac{1}{x} - \frac{1}{2\sqrt{x}} = 0 \Rightarrow \frac{2-\sqrt{x}}{2x} = 0 \Rightarrow 2-\sqrt{x} = 0 \quad \boxed{x=4}$$

② Make chart and plug in pts

	$0<x<4$	4	$4<x<\infty$
$f'(x)$	+	0	−

local max at $(4, \ln(4)-2)$
increasing on $(0,4)$
decreasing on $(4,\infty)$

(f) Find the interval(s) where f(x) is concave upward, the interval(s) where f(x) is concave downward, and any point(s) of inflection (both coordinates).

① Take 2nd derivative and identify places where undefined or $f''(x) = 0$

Undefined at $x=0$, but $x > 0$ by assumption

$$\frac{-1}{x^2} + \frac{1}{4}\frac{1}{(\sqrt{x})^3} = \frac{-1}{(\sqrt{x})^4} + \frac{1}{4}\frac{1}{(\sqrt{x})^3} = \frac{-4+\sqrt{x}}{4(\sqrt{x})^4} = 0 \Rightarrow -4+\sqrt{x} = 0$$

$$\boxed{x=16}$$

② Make chart

	$0<x<16$	16	$16<x<\infty$
$f'(x)$	−	0	+

Inflection Pt at $(16, \ln(16)-4)$
concave up on $(16,\infty)$
concave down on $(0,16)$

(g) Sketch the graph of $y = f(x)$, showing clearly all points you found in parts (e) and (f).