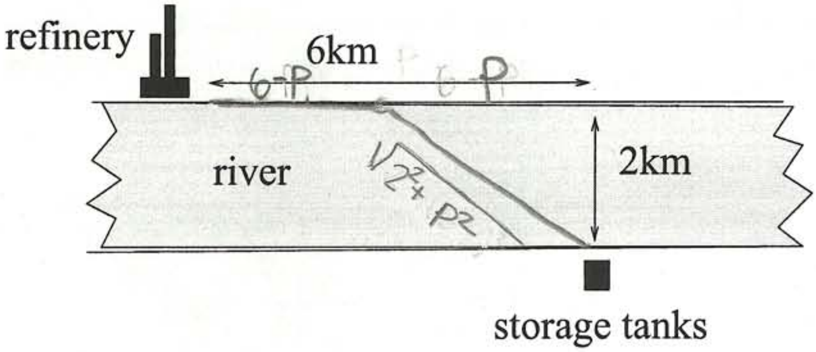
6. (12 points) An oil refinery is located on the north bank of a straight river that is 2 km wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 6 km east of the refinery. The cost of laying pipe is \$300,000/km over land to a point P on the north bank and \$500,000/km under the river to the tanks.



(a) To minimize the cost of the pipeline, where should P be located? Be sure to justify you have found the minimum.

① Cost of Pipe on land = $300{,}000(6-P)$

Cost of Pipe in water = $500{,}000\sqrt{2^2+P^2}$

Total Cost = $300{,}000(6-P) + 500000\sqrt{4+P^2}$

② Take derivative

$TC' = -300{,}000 + \frac{1}{2}500{,}000(4+P^2)^{-1/2}\cdot 2P$

$0 = -300{,}000 + 500{,}000P(4+P^2)^{-1/2}$

$0 = \dfrac{-300{,}000\sqrt{4+P^2} + 500{,}000P}{\sqrt{4+P^2}}$

$\Rightarrow 0 = -300{,}000\sqrt{4+P^2} + 500{,}000P$

$5/3\,P = \sqrt{4+P^2}$

$25/9\,P^2 = 4+P^2$

$16/9\,P^2 = 4 \Rightarrow P^2 = 36/16$

$P = \pm 6/4 = \pm 3/2$ can't be $P = -3/2$.

③ Check Endpts

$TC(3/2) = 300000(6-3/2) + 500000\sqrt{4+(\frac{3}{2})^2} = 2{,}600{,}000$

$TC(0) = 300{,}000(6) + 500{,}000(2)$
$= 2{,}800{,}000$

$TC(6) = 500000\sqrt{4+36}$
$= 3$ million + more

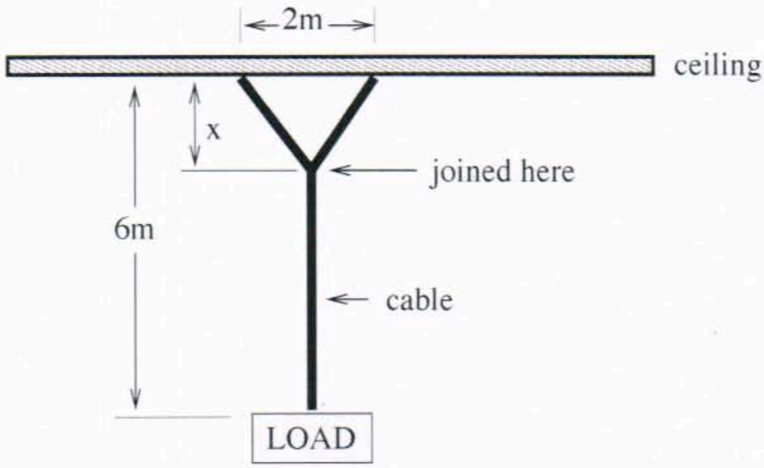
$\boxed{\text{Hence, } P = 3/2 \text{ is the minimum}}$

(b) What is the resultant minimum cost?

From above, the cost is

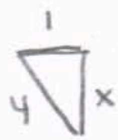
$TC(3/2) = 300{,}000(6-3/2) + 500000\sqrt{4+(3/2)^2} = \boxed{2{,}600{,}000}$

7. (12 points) A load must be suspended 6 meters below a high ceiling using cables attached to two supports that are 2 meters apart (see figure). How far below the ceiling (labeled x) should the cables be joined to minimize the total length of cable used? What is the minimum amount of cable needed? (Leave your answers in **exact** form and make sure to justify your minimum in some way.)



① Let $L(x)$ = length of the cable. Goal is to minimize $L(x)$

$$L(x) = 6 - x + 2y$$



$y = \sqrt{1+x^2}$

$$L(x) = \underbrace{6-x}_{\text{straight cable part}} + \underbrace{2\sqrt{1+x^2}}_{\text{the 2 triangle parts}}$$

$\boxed{x \in [0,6]}$

② $L'(x) = -1 + \dfrac{2x}{\sqrt{1+x^2}} \Rightarrow 0 = -1 + \dfrac{2x}{\sqrt{1+x^2}} \Rightarrow \dfrac{-\sqrt{1+x^2} + 2x}{\sqrt{1+x^2}} = 0$

$\Rightarrow 2x - \sqrt{1+x^2} = 0$

$2x = \sqrt{1+x^2}$

$4x^2 = 1 + x^2$

$3x^2 = 1$ $\boxed{x = \pm\sqrt{\tfrac{1}{3}}}$ Don't choose $-\sqrt{1/3}$ because not in domain.

③ Check endpoints to find minimum

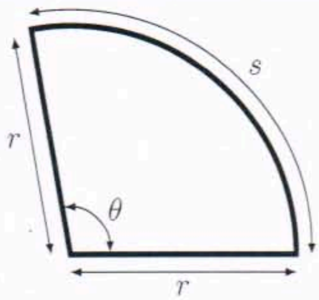
$x = \sqrt{1/3}$ $L(\sqrt{\tfrac{1}{3}}) = 6 - \sqrt{1/3} + 2\sqrt{1+\tfrac{1}{3}} \approx 7.73205$

$x = 0$ $L(0) = 6 - 0 + 2\sqrt{1+0^2} \approx 8$

$x = 6$ $L(6) = 6 - 6 + 2\sqrt{1+6^2} \approx 2\sqrt{37} \approx 12.1655$

Thus, the minimum is $x = \sqrt{1/3}$ w/ $L \approx 7.73205$

2. (12 points) You are a rancher. You wish to create an enclosure using fencing. The enclosure should have an area of 2000 square meters. The enclosure will have the shape of a circular sector as shown below.



For θ in radians, the area of such a sector is $\frac{1}{2}r^2\theta$ and the length of the curved part of the sector is $s = r\theta$.

Determine r and θ so that the amount of fencing needed is minimized.

Know: $2000 = \frac{1}{2}r^2\theta \Rightarrow \frac{4000}{r^2} = \theta$

Goal: Minimize

$$F = s + 2r = r\theta + 2r = \frac{r \cdot 4000}{r^2} + 2r = \frac{4000}{r} + 2r$$

$$F' = \frac{-4000}{r^2} + 2 = 0$$

$$\Rightarrow \frac{-4000 + 2r^2}{r^2} = 0 \Rightarrow -4000 + 2r^2 = 0$$

$$\Rightarrow 2r^2 = 4000$$

$$\Rightarrow r^2 = 2000$$

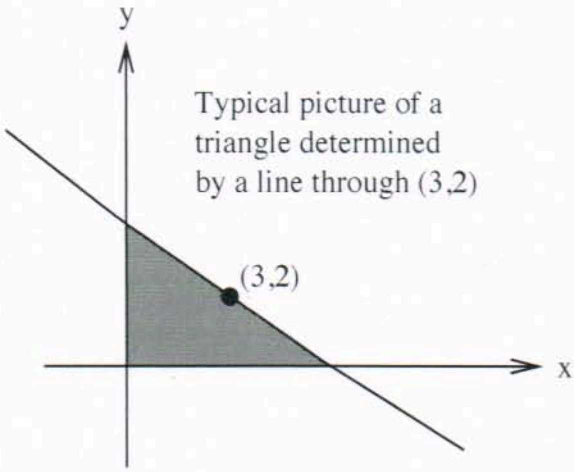
$$\Rightarrow r = \sqrt{2000}$$

If $r = \sqrt{2000}$, $\theta = \frac{4000}{\sqrt{2000}^2} = 2$.

Now this is a minimum b/c no boundary conditions.

Thus, the minimum fencing occurs when $\theta = 2$ and $r = \sqrt{2000}$

7. (12 points) Find the equation of the line passing through the point (3,2) which cuts off the triangle of least area from the first quadrant.



Pretend slope=m. then equation of the line is

$$y=m(x-3)+2$$

Since we need the Area of the triangle, we need to find the height and base, which correspond to the x and y intercept.

base= x-intercept is $0=m(x-3)+2$

$$\frac{3m-2}{m}=x=\text{base}$$

height= y-intercep is $y=m(0-3)+2$

$$2-3m=y=\text{height}$$

Therefore, Area of triangle= $\frac{1}{2}\left(\frac{3m-2}{m}\right)(2-3m)$

$$A(m)=6-\frac{9}{2}m-\frac{2}{m}$$

Differentiating $dA/dm=-9/2+\frac{2}{m^2}$

$$0=\frac{-9m^2+4}{2m^2} \Rightarrow m^2=4/9 \Rightarrow m=\pm 2/3$$

*Note that $m<0$ so $m=-2/3$.

We don't need to check b/c No bounds. Hence the optimal line is

$$\boxed{y=-2/3(x-3)+2}$$

8. (12 points) Nurl is designing a cylindrical container of volume 50π cubic centimeters. The top and bottom of the cylinder must be made of a material costing \$10 per square centimeter, while the rest of the container is made of a cheaper material that costs only \$3.20 per square centimeter. What is the surface area of the cheapest container Nurl can design?

$\text{Volume} = 50\pi = \pi r^2 h$

$TC = \$10 \cdot 2\pi r^2 + 3.20(2\pi r h)$

$50r^{-2} = h$. Plugging this in gives

$TC = 20\pi r^2 + 320\pi r^{-1}$

$TC' = 40\pi r - 320\pi r^{-2} = 0$

$\dfrac{40\pi r^3 - 320\pi}{r^2} = 0 \Rightarrow 40\pi r^3 - 320\pi = 0$

$40\pi r^3 = 320\pi$

$r^3 = 320/40 = 8$

$\boxed{r = 2}$

* We don't need to check the endpoints b/c neither r nor h could be 0.

The cheapest surface area is

$SA = 2\pi(2^2) + 2\pi(2)\,50\,2^{-2}$

$= 8\pi + 50\pi$

$\boxed{SA = 58\pi}$