Your Name

Solutions

Your Signature

Student ID #

Quiz Section

Professor's Name

TA's Name

- This exam is closed book. You may use one $8\frac{1}{2}$" $\times$ 11" sheet of handwritten notes (both sides). Do not share notes.
- Give your answers in exact form, except as noted in particular problems.
- Graphing calculators are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. You may use any of the 20 integrals on p. 484 of the text without deriving them. Show your work in evaluating any other integrals, even if they are on your note sheet.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 10 pages, plus this cover sheet. Please make sure that your exam is complete.

Question	Points	Score
1	12	
2	12	
3	12	
4	10	
5	8	
6	12	

Question	Points	Score
7	8	
8	10	
9	8	
10	8	
Total	100	

1. (12 total points) Evaluate the following integrals.

(a) (6 points) $\int \sec(x)\tan^3(x)\,dx$

Use u-substitution Let $u = \sec(x)$ $du = \tan x \sec x\,dx$

Then $\int \sec x \tan^3 x\,dx = \int \tan^2 x\,du = \int \sec^2 x - 1\,du$

$= \int u^2 - 1\,du = \frac{1}{3}u^3 - u + C$

$\boxed{= \frac{1}{3}\sec^3 x - \sec x + C}$

(b) (6 points) $\int \sin(\sqrt{x})\,dx$

1. Use u-substitution Let $u = \sqrt{x} \Rightarrow u^2 = x \Rightarrow 2u\,du = dx$

Thus $\int \sin(\sqrt{x})\,dx = \int \sin(u)\,2u\,du$

2. Use Integration By Parts

$w = 2u \qquad dv = \sin(u)\,du$

$dw = 2\,du \qquad v = -\cos(u)$

Hence,

$\int \sin(u)\,2u\,du = -2u\cos(u) + \int 2\cos(u)\,du$

$= -2u\cos(u) + 2\sin(u) + C$

$\boxed{= -2\sqrt{x}\cos(\sqrt{x}) + 2\sin(\sqrt{x}) + C}$

2. (12 total points) Evaluate the following integrals.

(a) (6 points) $\displaystyle\int \frac{1}{x(\sqrt{x+1}+2)}\,dx$

① Try u-sub. Let $u=\sqrt{x+1} \Rightarrow u^2 = x+1 \Rightarrow 2u\,du = dx$

Hence, $$\int \frac{2u\,du}{(u^2-1)[u+2]} = \int \frac{2u\,du}{(u-1)(u+1)(u+2)}$$

② Use Partial Fractions

$$\frac{2u}{(u-1)(u+1)(u+2)} = \frac{A}{u-1} + \frac{B}{u+1} + \frac{C}{u+2}$$

$$\Rightarrow \quad 2u = A(u+1)(u+2) + B(u-1)(u+2) + C(u+1)(u-1)$$

Plug in $u=1$: Then $2 = A(2)(3) \Rightarrow A = 1/3$

$u=-1$: Then $-2 = B(-2)(1) \Rightarrow B = 1$

$u=-2$: Then $-4 = C(-1)(-3) \Rightarrow C = -4/3$

$$\therefore \int \frac{2u\,du}{(u-1)(u+1)(u+2)} = \int \frac{1/3}{u-1} + \int \frac{1}{u+1} + \int \frac{-4/3}{u+2}$$

$$= \frac{1}{3}\ln|u-1| + \ln|u+1| - \frac{4}{3}\ln|u+2| + C$$

$$\boxed{= \frac{1}{3}\ln\left|\sqrt{x+1}-1\right| + \ln\left|\sqrt{x+1}+1\right| - \frac{4}{3}\ln\left|\sqrt{x+1}+2\right| + C}$$

(b) (6 points) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$

Try trig substitution

Let $x = 2\sin\theta$ $dx = 2\cos\theta\,d\theta$

Thus $$\int_0^1 \frac{x^2}{\sqrt{4-x^2}} = \int_0^{\pi/6} \frac{4\sin^2\theta \cdot 2\cos\theta}{2\cos\theta}\,d\theta$$

$$= \int_0^{\pi/6} 4\sin^2\theta\,d\theta$$

$$= \int_0^{\pi/6} 4\left(\tfrac{1}{2}\right)\left[1-\cos(2\theta)\right]d\theta$$

$$= 2\int_0^{\pi/6} 1 - 2\int_0^{\pi/6}\cos(2\theta)\,d\theta$$

$$= 2\theta\Big|_0^{\pi/6} - \sin(2\theta)\Big|_0^{\pi/6}$$

$$\boxed{= \pi/3 - \sqrt{3}/2}$$

3. (12 points) Decide if each of the following statements are TRUE or FALSE. **You will get 2 points for each correct answer, 1 point for each part left blank, and 0 points for each incorrect answer**. You do not need to justify your answers in this problem.

(a) The function $f(x) = \int_2^x |t|\,dt$ is differentiable. $\boxed{T}$

By Fundamental Thm of Calculus

(b) If two functions have the same derivative, they are the same function. $\boxed{F}$

Let $f(x) = x+2$ and $g(x) = x+1$. f' and g' are 1 but not the same function

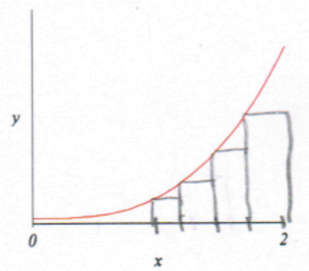
(c) $\frac{d}{dx}\left(\int_x^{x^2} \cos(1+t^3)\,dt\right) = \cos(1+x^6) - \cos(1+x^3)$. $\boxed{F}$

$= 2x\cos(1+x^6) - \cos(1+x^3)$

(d) A 10 foot rope weighing 4 pounds is hanging from the ceiling. The work done in pulling the whole rope up to the ceiling is less than 40 ft.lb. $\boxed{T}$

Look at the equation for work.

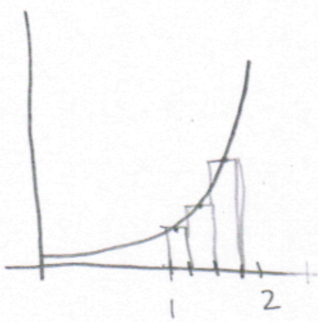
(e) Let $y = f(x)$ be the function whose graph is shown below. The estimate for $\int_1^2 f(x)\,dx$ using left endpoints with $n = 4$ is less than the actual value of this integral. $\boxed{T}$

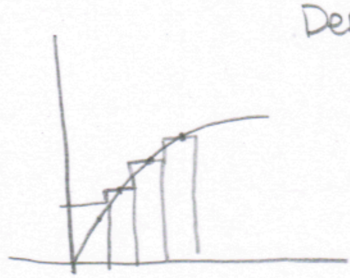


The Rectangles always below the actual area.

(f) Let $y = f(x)$ be the same function as in part (e). The Midpoint Rule estimate with $n = 4$ for $\int_1^2 f(x)\,dx$ is more than the actual value of this integral. $\boxed{F}$

Depends.





4. (10 points) Decide whether each of the following improper integrals converges or diverges. If an integral converges, find its value; if an integral diverges, explain why.

(a) $\displaystyle\int_2^\infty \frac{dx}{(x+2)(x-1)} = \lim_{a\to\infty}\int_2^a \frac{dx}{(x+2)(x-1)}$

By Partial Fractions,

$$\frac{1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$\Rightarrow 1 = A(x-1) + B(x+2)$$

Plug in $x=-2$: $1 = A(-3)$ $A = -1/3$

$x = 1$: $1 = B(3)$ $B = 1/3$

$$\text{so } \lim_{a\to\infty}\int_2^a \frac{dx}{(x+2)(x-1)} = \lim_{a\to\infty}\left[\int_2^a \frac{-\frac{1}{3}}{x+2} + \int_2^a \frac{\frac{1}{3}}{x-1}\right]$$

$$= \lim_{a\to\infty} -\frac{1}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| \Big|_2^a$$

$$= \lim_{a\to\infty} \frac{1}{3}\ln\left(\frac{x-1}{x+2}\right)\Big|_2^a$$

$$= \lim_{a\to\infty} \frac{1}{3}\ln\left(\frac{a-1}{a+2}\right) - \frac{1}{3}\ln\left(\frac{1}{4}\right)$$

$$\boxed{= -\frac{1}{3}\ln(1/4) = 1/3\ln(4)}$$

* Note $\displaystyle\lim_{a\to\infty}\frac{a-1}{a+2} = 1$ so

$$\lim_{a\to\infty}\ln\left(\frac{a-1}{a+2}\right) = \ln\left(\lim_{a\to\infty}\frac{a-1}{a+2}\right) = \ln(1) = 0$$

(b) $\displaystyle\int_1^\infty \frac{dx}{(x+2)(x-1)}$

$$\int_1^\infty \frac{dx}{(x+2)(x-1)} = \int_1^2 \frac{dx}{(x+2)(x-1)} + \int_2^\infty \frac{dx}{(x+2)(x-1)} = \lim_{a\to 1^+}\int_a^2 \frac{dx}{(x+2)(x-1)} + \lim_{b\to\infty}\int_2^b \frac{dx}{(x+2)(x-1)}$$

By Part a, the anti-derivative is

$$\lim_{a\to 1^+}\frac{1}{3}\ln\left(\frac{x-1}{x+2}\right)\Big|_a^2 + \lim_{b\to\infty}\frac{1}{3}\ln\left(\frac{x-1}{x+2}\right)\Big|_2^b$$

Note that as $a\to 1^+$, $\frac{x-1}{x+2}\to 0$. Thus $\displaystyle\lim_{a\to 1^+}\frac{1}{3}\ln\left(\frac{a-1}{a+2}\right) = \frac{1}{3}\ln(0) = -\infty$.

$\boxed{\therefore \text{ This diverges}}$

5. (8 total points) A bug is moving along the x-axis with velocity $v(t) = t^2 - t - 2$. At time $t = 0$, it is at the point $x = 1$.

   (a) (4 points) Find the net displacement of the bug over the time interval [0,3].

$$\text{Net displacement} = \int_0^3 t^2 - t - 2 \, dt$$
$$= \frac{1}{3}t^3 - \frac{1}{2}t^2 - 2t \Big|_0^3$$
$$= 9 - 9/2 - 6$$
$$\boxed{= -3/2}$$

   (b) (4 points) Find the total distance traveled by the bug over the time interval [0,3].

$$\text{Total Distance} = \int_0^3 |t^2 - t - 2| dt$$

① Find where $t^2 - t - 2 = 0$ for $t \in [0,3]$

$t^2 - t - 2 = (t-2)(t+1) = 0 \Rightarrow t = 2$ or -1

The only one in our time frame is 2.

② Pick points between $(0,2)$ and $(2,3)$ to find where positive OR negative

let $t = 1$: $1^2 - 1 - 2 < 0$

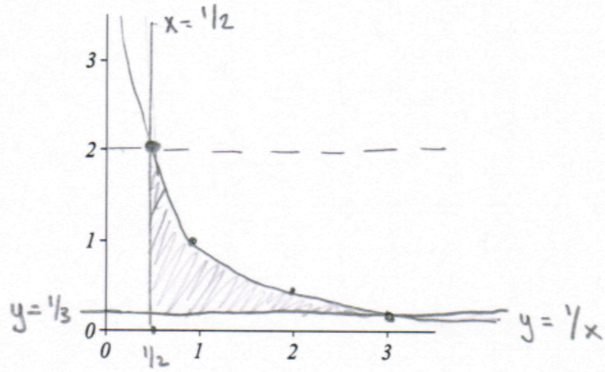
$t = 2.5$: $(2.5)^2 - 2.5 - 2 = 1.75 > 0$.

Thus

$$\int_0^3 |t^2 - t - 2| dt = \int_0^2 -(t^2 - t - 2) + \int_2^3 t^2 - t - 2$$
$$= -\frac{1}{3}t^3 + \frac{1}{2}t^2 + 2t \Big|_0^2 + 1/3\, t^3 - 1/2\, t^2 - 2t \Big|_2^3$$
$$= -\frac{8}{3} + 2 + 4 + 9 - 9/2 - 6 + 8/3 + 2 + 4 = \boxed{31/6}$$

6. (12 total points) The region R is bounded by the curve $y = 1/x$ and the lines $x = 1/2$ and $y = 1/3$.

(a) (2 points) Sketch the region R.



(b) (6 points) Find the volume of the solid obtained by rotating R about the y-axis.

Since given as $y = 1/x$ and rotating about the y-axis, use Shells

Height is $\frac{1}{x} - 1/3$ and radius is x.

Thus $\int_{1/2}^{3} 2\pi\left(\frac{1}{x} - \frac{1}{3}\right) x \, dx = \int_{1/2}^{3} 2\pi\left(1 - \frac{x}{3}\right) dx$

$= 2\pi\left(x - \frac{x^2}{6}\right)\Big|_{1/2}^{3}$

$= 2\pi\left(\frac{3}{2} - 11/24\right)$

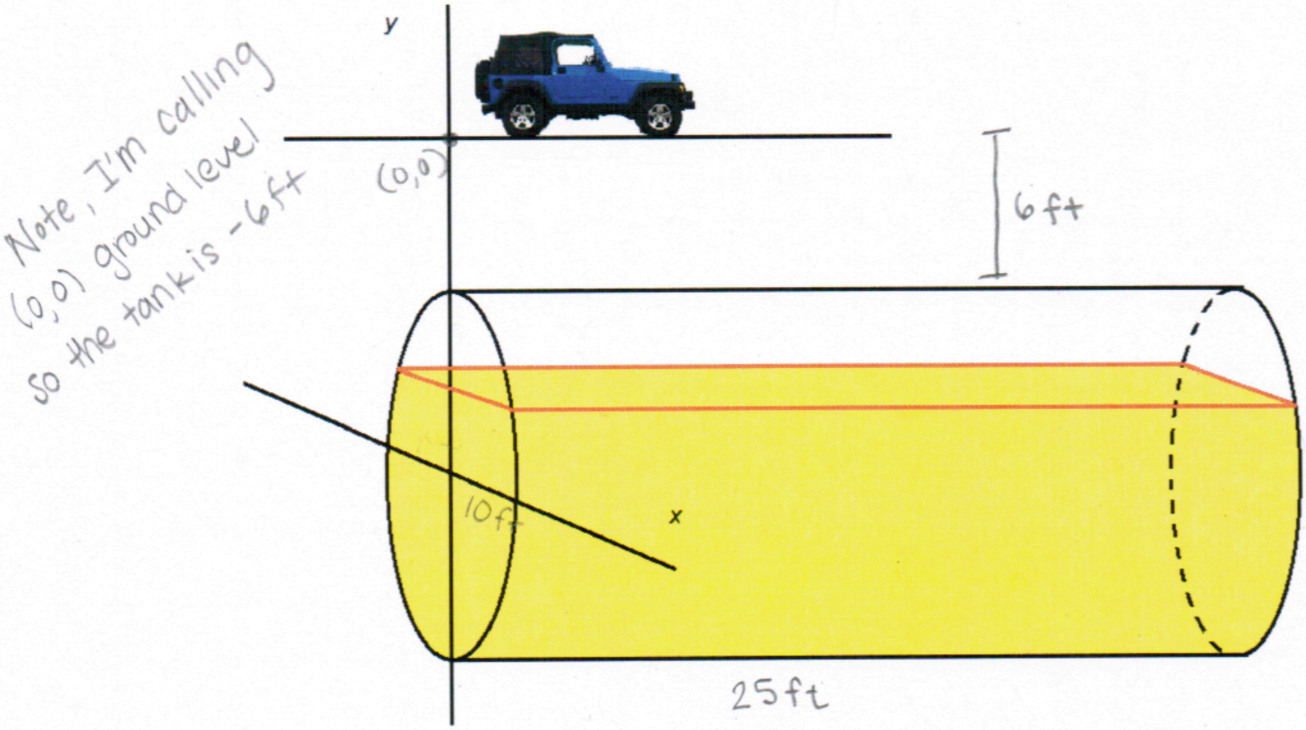
$= \boxed{\frac{25\pi}{12}}$

(c) (4 points) Express the volume of the solid obtained by rotating R about the line $y = 2$ as an integral, but **do not evaluate this integral.**

Since given as $y = 1/x$ and rotating horizontally, use washers/discs. This is washers. The outer radius is $2 - 1/3$ and inner radius is $2 - 1/x$. Thus

$$\boxed{\int_{1/2}^{3} (R^2 - r^2)\pi \, dx = \int_{1/2}^{3} \left[(5/3)^2 - \left(2 - \frac{1}{x}\right)^2\right]\pi \, dx}$$

7. (8 total points) Gasoline is stored in a cylindrical tank of radius 10 feet and length 25 feet lying under ground in a gas station. The tank is buried on its side with the highest part of the tank 6 feet below ground. The tank is initially full. Gasoline is now pumped into cars until the remaining gasoline is 12 feet measured at its deepest point. (Please note: The figure below is not drawn to scale.)



(a) (4 points) Express the volume of gasoline pumped as an integral, but **do not evaluate this integral.**

① Find eq. of circle: $x^2 + (y+16)^2 = 100$ [The tank in my coordinates is centered at $(0,-16)$]

$\therefore x = \pm\sqrt{100-(y+16)^2}$ [take the positive and double]

② Volume of single slice is: $2\left(\sqrt{100-(y+16)^2}\right) \cdot 25\, dy$

③ $\boxed{\text{Total Volume} = \int_{-14}^{-6} 2\left(\sqrt{100-(y+16)^2}\right) \cdot 25\, dy}$

(b) (4 points) Suppose that the filler cap of each car is 2 feet above the ground. Express the work done in pumping the gasoline as an integral, but **do not evaluate this integral.** The density of gasoline is 45 pounds per cubic foot.

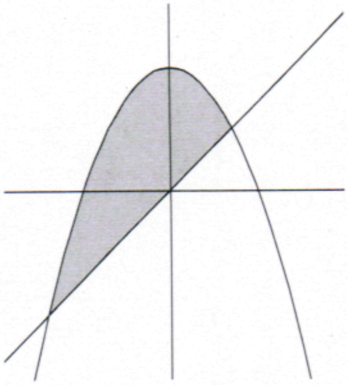
① Force: $2\left(\sqrt{100-(y+16)^2}\right) \cdot 25 \cdot 45\, dy$

② Distance travelled for 1 slice is $-y+2$

③ Work = Force * dist

$$\boxed{= \int_{-14}^{-6} 2\left(\sqrt{100-(y+16)^2}\right) \cdot 25 \cdot 45 \cdot (2-y)\, dy}$$

8. (10 points) Let R be the region bounded by the curve $y = 2 - x^2$ and the line $y = x$ as shown below. Find the center of mass of R.



① Find bounds

$x = 2 - x^2$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = -2$ or 1

② Find Area

$A = \int_{-2}^{1} 2 - x^2 - x \, dx$

$= 2x - \frac{1}{3}x^3 - \frac{1}{2}x^2 \Big|_{-2}^{1}$

$= 7/6 + 10/3$

$= 9/2$

③ Find $\bar{x}$

$\bar{x} = \frac{1}{A}\int_{-2}^{1} x(2 - x^2 - x)\, dx$

$= \frac{2}{9}\int_{-2}^{1} 2x - x^3 - x^2 \, dx$

$= \frac{2}{9}\left(x^2 - \frac{1}{4}x^4 - \frac{1}{3}x^3\right)\Big|_{-2}^{1}$

$= \frac{2}{9}\left(\frac{5}{12} - \frac{8}{3}\right)$

$\boxed{= -1/2}$

④ $\bar{y} = \frac{1}{2A}\int_{-2}^{1} f(x)^2 - g(x)^2 \, dx$

$= \frac{1}{9}\int_{-2}^{1} (2 - x^2)^2 - x^2 \, dx$

$= \frac{1}{9}\int_{-2}^{1} 4 - 4x^2 + x^4 - x^2$

$= \frac{1}{9}\int_{-2}^{1} 4 - 5x^2 + x^4$

$= \frac{1}{9}\left(4x - \frac{5}{3}x^3 + \frac{1}{5}x^5\right)\Big|_{-2}^{1}$

$= \frac{1}{9}\left(\frac{38}{15} + \frac{16}{15}\right)$

$\boxed{= \frac{2}{5}}$

9. (8 points) Find the solution of the differential equation

$$xy' + y = y^2$$

that satisfies $y(1) = -1$.

$xy' + y = y^2 \Rightarrow x\,dy/dx + y = y^2$

① Separate variables w/o adding or subtracting

$$x\,dy/dx = y^2 - y$$

$$\Rightarrow \frac{dy}{y^2 - y} = \frac{dx}{x}$$

$$\Rightarrow \underbrace{\int \frac{dy}{y(y-1)}}_{①} = \underbrace{\int \frac{dx}{x}}_{②}$$

Find ①. Use Partial Fractions

$$\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1}$$

$$\Rightarrow 1 = A(y-1) + By$$

Plug in $y = 0$, $1 = A(-1) \Rightarrow A = -1$

$y = 1$, $1 = B$

$$\therefore \int \frac{dy}{y(y-1)} = \int \frac{-1}{y} + \int \frac{1}{y-1} = -\ln|y| + \ln|y-1| = \ln\left|\frac{y-1}{y}\right|$$

Find ②, $\int \frac{dx}{x} = \ln|x| + C$

$$\therefore \ln\left|\frac{y-1}{y}\right| = \ln|x| + C \Rightarrow \left|\frac{y-1}{y}\right| = C|x| \Rightarrow \frac{y-1}{y} = Cx$$

$$\Rightarrow y - Cxy = 1$$

$$\Rightarrow y(1 - Cx) = 1$$

$$\Rightarrow y = \frac{1}{1 - Cx}$$

② Solve initial Value

$$y = \frac{1}{1 - Cx}$$

$$\Rightarrow -1 = \frac{1}{1 - C}$$

$$\Rightarrow -1 + C = 1$$

$$\Rightarrow C = 2$$

$$\boxed{\therefore\; y = \frac{1}{1 - 2x}}$$

0. (8 total points) Glucose is being fed intravenously to the bloodstream of a patient at 0.01 grams per minute. At the same time the patient's body converts the glucose and removes it from the bloodstream at a rate proportional to the amount of glucose present.

(a) (2 points) Let $g(t)$ be the amount of glucose in the bloodstream at time t in minutes. Let k be the proportionality constant mentioned above. Set up a differential equation for $g(t)$.

* Remember Rate in - Rate out

$$\boxed{\frac{dg}{dt} = .01 - kg}$$

Proportional means multiplied by some constant

(b) (4 points) Find the general solution to this differential equation. Show that the amount of glucose in the bloodstream always approaches $\frac{0.01}{k}$ as t becomes very large. (If it does not, you made a mistake somewhere. Go back and check your work.)

① Separate variables then solve

$$dg/dt = .01 - kg \Rightarrow \frac{dg}{.01-kg} = dt$$

$$\Rightarrow \int \frac{dg}{.01-kg} = \int dt$$

$$\Rightarrow \frac{-1}{k} \ln|.01-kg| = t + C$$

$$\ln|.01-kg| = -kt + C$$

$$.01 - kg = Ce^{-kt}$$

$$\boxed{\frac{.01 - Ce^{-kt}}{k} = g}$$

② Show that as $t \to \infty$ $g(t) \to .01/k$.

Note that $\lim_{t\to\infty} e^{-kt} \to 0$

$$\therefore \lim_{t\to\infty} \frac{.01 - Ce^{-kt}}{k} = \frac{.01}{k}.$$

(c) (2 points) Suppose that there are 4.1 grams of glucose in the bloodstream at $t = 0$ and that as t becomes very large, the glucose level approaches 5.2 grams. How much glucose is in the blood one hour after starting? Give a decimal answer.

The two conditions we are given are

① $\lim_{t\to\infty} g(t) = 5.2$

② $g(0) = 4.1$

Start w/ the 1st condition: from part b)

$$\lim_{t\to\infty} g(t) = \frac{.01}{k} = 5.2 \Rightarrow k = \frac{1}{520}$$

Now we know k, we can solve for C

$$4.1 = \frac{.01 - Ce^{-0/520}}{1/520} = \frac{.01 - C}{1/520} \Rightarrow C = 11/5200$$

$$\therefore g(t) = \frac{.01 - \frac{11}{5200} e^{-t/520}}{(1/520)}$$

Hence at $t = 60$ (Note t = minutes)

$$g(60) = \frac{.01 - \frac{11}{5200} e^{-60/520}}{1/520}$$

$$\boxed{= 4.219874286\ g}$$