

9. (10 points) A tank initially contains 1000 L of pure water. Brine that contains 0.07 kg of salt per liter of water enters the tank at a rate of 5 L/min. In addition, brine that contains 0.04 kg of salt per liter of water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 15 L/min. Find the amount of salt (in kg) in the tank as a function of time t in minutes.

10. (12 total points) A lake has a volume of 1000 cubic meters and contains pure water. A farm near the lake begins using a pesticide. Runoff from the farmland into the lake is 10 cubic meters of water per day, with a concentration of 50 grams of pesticide per cubic meter of water.

The lake drains to the ocean at a rate of 10 cubic meters per day. You may assume that the pesticide mixes thoroughly with the water in the lake, and you should ignore other effects such as evaporation.

- (a) (6 points) Let $y(t)$ denote the total amount of pesticide (in grams) in the lake after t days. Set up a differential equation for $y(t)$.

- (b) (6 points) Fish can survive a maximum concentration of 1 gram of pesticide per cubic meter of water. Solve the differential equation you found in part (a) and determine whether the fish will be alive after 10 days.

10. (10 total points) You would like to be a multimillionaire in 30 years. You might win the lottery, or you can start investing. This problem is about investing.

Let $A(t)$ be the amount of money (in dollars) you have in your investment account at time t (in years). Let M be the amount (in dollars) that you deposit every *month*, so $12M$ is the amount (in dollars) that you deposit every *year*. The rate of change of the amount A in your account has two parts: the interest and your deposits. The part of the rate of change coming from the interest is proportional to the amount in your account, and the annual interest rate is 10%. Interest is compounded continuously, and assume that your deposits are also applied continuously.

- (a) (4 points) Set up a differential equation for the rate of change of A with respect to time, taking into account both the interest you earn and the deposits you make. Be careful with units.

- (b) (4 points) You start with $A = 0$ at time $t = 0$. Solve the differential equation above. Your solution will involve M .

- (c) (2 points) Determine the value of M , the amount that you have to deposit every month, if you want to have 5 million dollars in your account after 30 years.

10. (12 total points) A tank contains 100 liters of water which has 2000 grams of salt dissolved in it. At noon, pure water begins to enter the tank at the rate of 10 liters per minute. The tank is kept thoroughly mixed, and the mixture leaves the tank at the rate of 15 liters per minute.

(a) (2 points) Express the volume of saltwater in the tank as a function of t , the number of minutes after noon.

(b) (4 points) Let $S(t)$ be the amount of salt in grams in the tank at time t . Set up a differential equation describing $\frac{dS}{dt}$.

(c) (4 points) Solve the differential equation for $S(t)$.

(d) (2 points) When will there be 1000 grams of salt in the tank?

10. (10 total points) At time $t = 0$ a 5000 liter tank is full of pure water. Starting at that moment salt is added to it at the steady rate of 30 grams per hour. Assume that the salt is thoroughly mixed in the water. Meanwhile, pure water is entering the tank at 50 liters per hour and the salty water in the tank is leaving at the same rate. Let $y(t)$ be the amount of salt in grams in the water in the tank after t hours.

- (a) (4 points) Find a differential equation for $y(t)$.
- (b) (4 points) Solve for $y(t)$.
- (c) (2 points) How much salt is in the tank in the limit as $t \rightarrow \infty$?

10. (8 total points) Glucose is being fed intravenously to the bloodstream of a patient at 0.01 grams per minute. At the same time the patient's body converts the glucose and removes it from the bloodstream at a rate proportional to the amount of glucose present.
- (a) (2 points) Let $g(t)$ be the amount of glucose in the bloodstream at time t in minutes. Let k be the proportionality constant mentioned above. Set up a differential equation for $g(t)$.
- (b) (4 points) Find the general solution to this differential equation. Show that the amount of glucose in the bloodstream always approaches $\frac{0.01}{k}$ as t becomes very large. (If it does not, you made a mistake somewhere. Go back and check your work.)
- (c) (2 points) Suppose that there are 4.1 grams of glucose in the bloodstream at $t = 0$ and that as t becomes very large, the glucose level approaches 5.2 grams. How much glucose is in the blood one hour after starting? Give a decimal answer.

10. (10 points) When a cake is removed from an oven, the temperature of the cake is 210° F. The cake is left to cool at room temperature (70° F.), and after 30 minutes, the temperature of the cake is 140° F. According to Newton's law of cooling, the rate of change of temperature of a body is proportional to the temperature difference between the body and the environment. Set up and solve a differential equation to determine when the temperature of the cake will be 100° F.

11. (10 total points) The population P of Springfield is increasing steadily, for two reasons: (1) The birth rate exceeds the death rate, resulting in an increase of 2% per year (that is, a rate of increase equal to $.02P$ per year). (2) More people move to Springfield from elsewhere than move away, resulting in an additional increase of 1000 people per year.

(a) (2 points) Write down a formula for the total rate of increase of population, dP/dt .

(b) (6 points) At the beginning of 1995 (call this time $t = 0$), the population of Springfield was 25,000. Solve the differential equation you obtained in part (a) to find the population of Springfield t years later.

(c) (2 points) In what year will the population of Springfield be 60,000? (Your answer should look like “2009”, not like “14.217689...”.)

9. (12 total points) Your friend wins the lottery, and gives you P_0 dollars to help you pay your college expenses. The money is invested in a savings account that earns 10% annual interest, compounded continuously, and you withdraw the money continuously (a pretty good approximation to reality if you make regular frequent withdrawals) at a rate of \$3600 per year.

(a) (3 points) Set up a differential equation for the amount of money $P(t)$ (in dollars) left in the account at time t (in years).

(b) (3 points) Solve the differential equation to obtain a formula for $P(t)$. Your formula should involve P_0 .

(c) (3 points) If $P_0 = \$20,000$, then how much money is left in the account 4 years later?
Give your answer (in dollars) in decimal form, correct to the nearest cent (i.e., nearest \$0.01).

(d) (3 points) What should P_0 be so that after 4 years your account has exactly \$0 left over?
Give your answer (in dollars) in decimal form, correct to the nearest cent (i.e., nearest \$0.01).

11. (10 points) The swine flu epidemic has been modelled by the Gompertz function, which is a solution of the differential equation

$$\frac{dy}{dt} = 1.2 y (K - \ln(y)),$$

where $y(t)$ is the number of individuals (in thousands) in a large city that have been infected by time t , and K is a constant. Time t is measured in months, with $t = 0$ on July 9, 2009.

On July 9, 2009, 75 thousand individuals had been infected. One month later, 190 thousand individuals had been infected. Find

$$\lim_{t \rightarrow \infty} y(t),$$

which is the total number of individuals (in thousands) that will have been infected.

10. (10 total points) An object of mass m kg is dropped out of a airplane, and we assume that air resistance is proportional to the speed of the object. Let $s(t)$ be the distance dropped (in meters, positive pointing *down*) after t seconds, and let $v(t) = ds/dt$ be the velocity and $a(t) = dv/dt$ be the acceleration. The combined downward force on the object is

$$F = mg - kv,$$

where $g = 9.8$ meters/sec² is the acceleration due to gravity and k is a positive constant. By Newton's Second Law of motion,

$$F = ma = m \frac{dv}{dt}.$$

The mass of the object is $m = 10$ kg, and the constant k is $k = 2$.

- (a) (3 points) Set up a differential equation for the velocity $v(t)$.

- (b) (5 points) Solve the differential equation to obtain a formula for $v(t)$.

- (c) (2 points) What is the limiting velocity $\lim_{t \rightarrow \infty} v(t)$?