Name________________________________

**Math 125** **Second Midterm** **10:00 a.m., Feb. 28, 2013**

(80 minutes — 100 points)

**Please show all your work clearly. If you need more space, you can use the reverse side. A sheet of notes is permitted, but no calculator.**

1. Find the following indefinite integrals:

(a) (17 points) $\displaystyle\int t^2 e^{-t} dt.$

(b) (17 points) $\displaystyle\int \frac{\tan\theta \sec^2\theta \, d\theta}{\sec^2\theta - 8\sec\theta + 15}.$

(c) (17 points) $\displaystyle\int x^2 \operatorname{Arcsin}(x) dx.$

(a) IBP set $u = t^2$ $dv = e^{-t}dt$
$du = 2t$ $v = -e^{-t}$

$\int t^2 e^{-t} dt = -t^2 e^{-t} + \int 2te^{-t} dt$ apply IBP again

set $u = 2t$ $dv = e^{-t}dt$
$du = 2dt$ $v = -e^{-t}$

$\int t^2 e^{-t} dt = -t^2e^{-t} - 2te^{-t} + \int 2e^{-t} dt$

$\Rightarrow \int t^2 e^{-t} dt = \boxed{-t^2 e^{-t} - 2te^{-t} - 2e^{-t} + C}$

(b) $u = \sec\theta$ $du = \sec\theta\tan\theta\, d\theta$

$\int \frac{u}{u^2 - 8u + 15} du = \int \frac{u}{(u-5)(u-3)} du$

$\frac{u}{(u-5)(u-3)} = \frac{A}{u-5} + \frac{B}{u-3} \Rightarrow u = A(u-3) + B(u-5)$ $u = 3 \Rightarrow B = \frac{-3}{2}$
$u = 5 \Rightarrow A = 5/2$

$\int \frac{5/2}{u-5} + \int \frac{-3/2}{u-3} = \frac{5}{2}\ln|u-5| - \frac{3}{2}\ln|u-3| + C$

$\boxed{= \frac{5}{2}\ln|\sec\theta - 5| - \frac{3}{2}\ln|\sec\theta - 3| + C}$

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(c) $\int x^2 \arcsin(x)\,dx$

$u = \arcsin(x) \qquad dv = x^2$

$du = \frac{1}{\sqrt{1-x^2}}\,dx \qquad v = \frac{1}{3}x^3$

$= \frac{\arcsin(x)\cdot x^3}{3} - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx$

quick proof is to use
$u = 1 - x^2$
but will solve this w/ trig

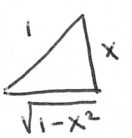
$x = \sin\theta$
$dx = \cos\theta\,d\theta$

$$-\frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \frac{-1}{3}\int \frac{\sin^3\theta}{\sqrt{1-\sin^2\theta}}\cdot \cos\theta\,d\theta$$

$$= \frac{-1}{3}\int \sin^3\theta\,d\theta = \frac{-1}{3}\int \sin\theta \sin^2\theta\,d\theta$$

$u = \cos\theta$
$du = -\sin\theta\,d\theta$

$$= \frac{1}{3}\int 1 - u^2\,du = \frac{1}{3}\left(u - \frac{1}{3}u^3\right) + C$$




$$= \frac{1}{3}\left(\cos\theta - \frac{1}{3}(\cos\theta)^3\right) + C$$

$$= \frac{1}{3}\left(\sqrt{1-x^2} - \frac{1}{3}\left(\sqrt{1-x^2}\right)^3\right) + C$$

$$\therefore \int x^2 \arcsin(x)\,dx = \boxed{\frac{x^3\arcsin(x)}{3} + \frac{1}{3}\left(\sqrt{1-x^2} - \frac{1}{3}\left(\sqrt{1-x^2}\right)^3\right) + C}$$

2. (a) (12 points) Evaluate the definite integral

$$\int_1^b \frac{dx}{(x^2+1)^{3/2}},$$

expressing your answer in terms of $b$; please simplify as much as possible. Then determine whether the improper integral as $b \to \infty$ converges or diverges; if it converges, find its value, writing your answer in exact form (simplified).

(b) (12 points) Evaluate the definite integral

$$\int_{4+\delta}^{20} \frac{dx}{(x-4)^{5/4}},$$

expressing your answer in terms of $\delta$; please write your answer in simplified form. Then determine whether the improper integral as $\delta \to 0$ converges or diverges; if it converges, find its value in exact form.

(a) $\int_1^b \frac{dx}{(\sqrt{1+x^2})^3}$ $\quad x = \tan\theta$, $dx = \sec^2\theta \, d\theta$ $\quad \int_{\pi/4}^{\arctan(b)} \frac{\sec^2\theta}{(\sqrt{1+\tan^2\theta})^3} d\theta = \int_{\pi/4}^{\arctan(b)} \frac{1}{\sec\theta} d\theta$

$= \int_{\pi/4}^{\arctan(b)} \cos\theta \, d\theta = \sin\theta \Big|_{\pi/4}^{\arctan(b)} = \boxed{\sin(\arctan(b)) - \frac{\sqrt{2}}{2}}$

$\lim_{b\to\infty} \int_1^b \frac{dx}{(\sqrt{1+x^2})^3} = \lim_{b\to\infty} \sin(\arctan(b)) - \frac{\sqrt{2}}{2} = \sin(\frac{\pi}{2}) - \frac{\sqrt{2}}{2} = \boxed{1 - \sqrt{2}/2}$

(b) $\int_{4+\delta}^{20} \frac{dx}{(x-4)^{5/4}}$ $\quad u = x-4$, $du = dx$ $\quad = \int_\delta^{16} u^{-5/4} du = -4u^{-1/4} \Big|_\delta^{16}$

$= -4\left(\frac{1}{(16)^{1/4}}\right) + 4\left(\frac{1}{\delta^{1/4}}\right) = \boxed{\frac{4}{\delta^{1/4}} - 2}$

$\lim_{\delta\to 0} \int_{4+\delta}^{20} \frac{dx}{(x-4)^{5/4}} = \lim_{\delta\to 0} \frac{4}{\delta^{1/4}} - 2 = \infty$ $\quad \boxed{\text{diverges}}$

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3. (25 points) You want to find how far an object travels between $t = 0$ and $t = 30$ sec, starting at $t = 0$. You take readings of the horizontal and vertical velocities at 5-sec intervals. Let $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ denote the seven horizontal velocity readings, and let $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6$ denote the seven vertical velocity readings taken during the 30-second time period. Using Simpson's rule, write an expression in terms of the $\alpha$'s and $\beta$'s for the length of the object's trajectory between $t = 0$ and $t = 30$.

$$\text{arclength} = \int_0^{30} \sqrt{(\underset{\alpha}{\underset{\uparrow}{x'(t)}})^2 + (\underset{\beta}{\underset{\uparrow}{y'(t)}})^2}\, dt$$

Note $\alpha_0 = x'(0)$, $\alpha_1 = x'(5)$, $\alpha_2 = x'(10)$, $\alpha_3 = x'(15)$
$\alpha_4 = x'(20)$ $\alpha_5 = x'(25)$, $\alpha_6 = x'(30)$

$\beta_0 = y'(0)$, $\beta_1 = y'(5)$, $\beta_2 = y'(10)$, $\beta_3 = y'(15)$,
$\beta_4 = y'(20)$, $\beta_5 = y'(25)$, $\beta_6 = y'(30)$

$\Delta t = 5$

$$\boxed{\text{arclength} \approx \frac{5}{3}\left(\sqrt{\alpha_0^2+\beta_0^2} + 4\sqrt{\alpha_1^2+\beta_1^2} + 2\sqrt{\alpha_2^2+\beta_2^2} + 4\sqrt{\alpha_3^2+\beta_3^2} + 2\sqrt{\alpha_4^2+\beta_4^2} + 4\sqrt{\alpha_5^2+\beta_5^2} + \sqrt{\alpha_6^2+\beta_6^2}\right)}$$