1. (10 total points) Evaluate the following definite integrals. Give your answers in exact form.

(a) (5 points) $\int_0^1 (x-3)\,e^{-2x}\,dx$

$u = x-3 \qquad dv = e^{-2x}$

$du = dx \qquad v = \frac{-1}{2}e^{-2x}$

$-\frac{1}{2}(x-3)e^{-2x}\Big|_0^1 + \frac{1}{2}\int_0^1 e^{-2x}\,dx$

$e^{-2} - \frac{3}{2} + \frac{-1}{4}e^{-2x}\Big|_0^1 = e^{-2} - \frac{3}{2} - \frac{1}{4}e^{-2} + \frac{1}{4}$

$= \boxed{\frac{3}{4}e^{-2} - \frac{5}{4}}$

(b) (5 points) $\int_0^2 \frac{1}{(x^2+4)^{3/2}}\,dx$ $\qquad x = 2\tan\theta \quad dx = 2\sec^2\theta\,d\theta$

$\int_0^{\pi/4} \frac{2\sec^2\theta}{(\sqrt{4\tan^2\theta+4})^3}\,d\theta = \int_0^{\pi/4} \frac{2\sec^2\theta}{8\sec^3\theta}\,d\theta$

$= \int_0^{\pi/4} \frac{1}{4}\cos\theta\,d\theta = \frac{1}{4}\sin\theta\Big|_0^{\pi/4} = \boxed{\frac{\sqrt{2}}{8}}$

2. (10 total points) Evaluate the following indefinite integrals.

(a) (5 points) $\displaystyle\int (\sin x + \cos^2 x)\sin x\,dx$

$$\int \sin^2 x + \int \cos^2 x \sin x \quad \to \begin{array}{l} u = \cos x \\ du = \sin x\,dx \end{array}$$

$$= \frac{1}{2}\int 1 - \cos(2x) - \int u^2\,du$$

$$= \boxed{\frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right) - \frac{1}{3}\cos^3 x + C}$$

(b) (5 points) $\displaystyle\int \frac{1}{(x+1)(\sqrt{x+5}+1)}\,dx$

$u = \sqrt{x+5} \quad u^2 = x+5$

$2u\,du = dx$

$$\int \frac{2u}{(u^2-4)(u+1)}\,du = \int \frac{2u}{(u-2)(u+2)(u+1)}\,du$$

$$\frac{A}{u-2} + \frac{B}{u+2} + \frac{C}{u+1} = \frac{2u}{(u-2)(u+2)(u+1)}$$

$$A(u+2)(u+1) + B(u-2)(u+1) + C(u-2)(u+2) = 2u$$

$u=2 \quad A(4)(3) = 4 \qquad A = 1/3$

$u=-2 \quad B(-4)(1) = -4 \qquad B = -1$

$u=-1 \quad C(-3)(-1) = -2 \qquad C = 2/3$

$$\int \frac{1/3}{u-2} + \int \frac{-1}{u+2} + \int \frac{2/3}{u+1} = \boxed{\frac{1}{3}\ln\left|\sqrt{x+5}-2\right| - \ln\left|\sqrt{x+5}+2\right| + \frac{2}{3}\ln\left|\sqrt{x+5}+1\right| + C}$$

3. (10 points) Let R be the region that is between the curve $y = \sqrt{x}e^{-x^2}$ and the x-axis, is bounded on the left by the line $x = 1$, and extends infinitely far out to the right. Let S be the solid obtained by rotating R around the x-axis.
Does S have finite volume? If so, find it, and give your answer in exact form.

Use Washers

$$\pi \int_1^\infty (\sqrt{x}e^{-x^2})^2\,dx = \lim_{b\to\infty} \int_1^b \pi(xe^{-2x^2})\,dx$$

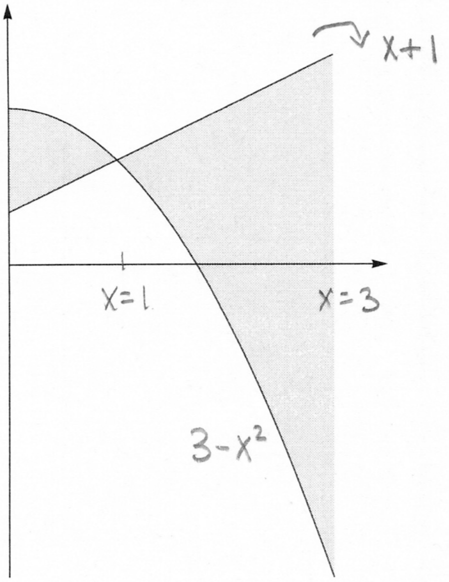
$u = -2x^2$
$du = -4x\,dx$

$$\lim_{b\to\infty} \int_{-2}^{-2b^2} \frac{-\pi}{4} e^u\,du$$

$$= \lim_{b\to\infty} \frac{-\pi}{4} e^u \Big|_{-2}^{-2b^2} = \lim_{b\to\infty} \underbrace{\frac{-\pi}{4} e^{-2b^2}}_{\to 0} + \frac{\pi}{4} e^{-2}$$

$$\boxed{= \frac{\pi}{4} e^{-2}}$$

4. (10 points) Find the total area (shaded in the figure below) that is between the curves $y=3-x^2$ and $y=x+1$ and between the lines $x=0$ and $x=3$.



Find the pts of intersection

$X+1=3-X^2$

$X^2+X-2=0$

$(X+2)(X-1)=0$

$X=-2 \quad X=1$

$$\int_0^1 3-X^2-X-1\,dX + \int_1^3 X+1-3+X^2\,dX$$

$$= \int_0^1 2-X^2-X\,dX + \int_1^3 X^2+X-2\,dX$$

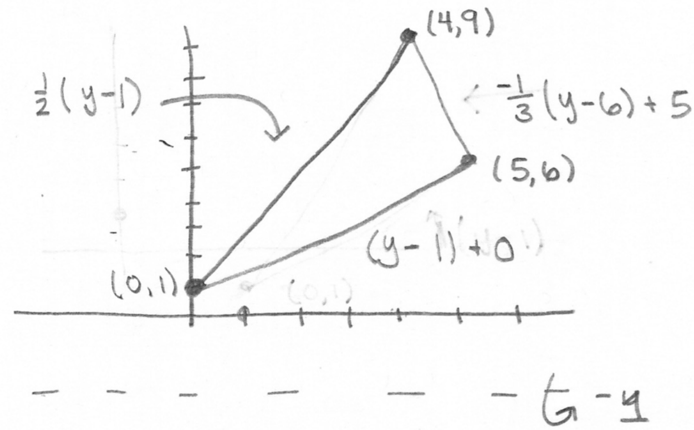
$$= 2X-\tfrac{1}{3}X^3-\tfrac{1}{2}X^2\Big|_0^1 + \tfrac{1}{3}X^3+\tfrac{1}{2}X^2-2X\Big|_1^3$$

$$= 2-\tfrac{1}{3}-\tfrac{1}{2}+9+9/2-6-\tfrac{1}{3}-\tfrac{1}{2}+2$$

$$= \boxed{59/6}$$

5. (10 points) Let T be the triangle with vertices $(0,1)$, $(4,9)$ and $(5,6)$. Using the method of *shells* (*not* washers), write the volume of the solid obtained by rotating T about the line $y=-1$ in terms of definite integrals. DO NOT EVALUATE THE INTEGRAL(S).

For shells, Need in y's



Volume

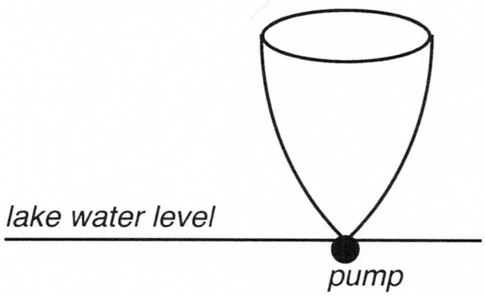
$$= 2\pi \int_1^6 \left(y-1-\tfrac{1}{2}(y-1)\right)(y+1)\,dy$$

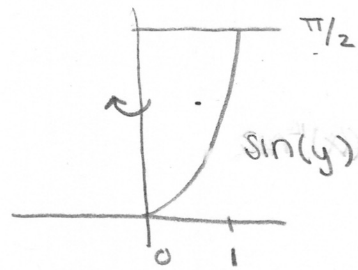
$$+ 2\pi \int_6^9 \left(-\tfrac{1}{3}(y-6)+5-\tfrac{1}{2}(y-1)\right)(y+1)\,dy$$

6. (10 points) A water container has the shape of a solid of revolution, obtained by rotating the region below the line $y = \pi/2$, above the curve $y = \sin^{-1} x$, and between the lines $x = 0$ and $x = 1$, about the y-axis. Dimensions on the x- and y-axes are in meters.

The container is placed above the surface of a lake, so that its lower end touches the water (see the figure below). The lower end contains a small opening through which a small but powerful pump can fill the container with water. Calculate the work needed to fill the empty container with water pumped from the lake up into the container. Give your answer in exact form.

Assume that the gravitational acceleration is 9.8 meters per second squared. The density of water is 1,000 kg per cubic meter.





① Volume of a slice

$\pi(\sin(y))^2\,dy$

② Force of a slice

$\pi(\sin(y)^2)(9.8)(1000)\,dy$

③ Distance = y

Note here water is pumped from the bottom up

④ Bounds 0 to $\pi/2$

$$W = \int_0^{\pi/2} \pi(9.8)(1000)(y\sin^2 y)\,dy$$

$u = y \quad dv = \frac{1}{2}(1-\cos(2y))\,dy$

$du = dy \quad v = \frac{1}{2}(y - \frac{1}{2}\sin(2y))$

$$9800\pi\left[\frac{y}{2}\left(y-\frac{1}{2}\sin(2y)\right)\Big|_0^{\pi/2} - \int_0^{\pi/2}\frac{1}{2}\left(y-\frac{1}{2}\sin(2y)\right)dy\right]$$

$$= 9800\pi\left[\frac{\pi^2}{8} - \left(\frac{1}{4}y^2 + \frac{1}{8}\cos(2y)\right)\Big|_0^{\pi/2}\right]$$

$$= 9800\pi\left[\pi^2/8 - \frac{\pi^2}{16} + \frac{1}{8} + \frac{1}{8}\right]$$

$$= \boxed{9800\pi\left[\pi^2/16 + 1/4\right]}$$

7. (10 total points)

(a) (4 points) Write a definite integral for the arclength L of the graph of $y = \sin(2x)$ from $x = 0$ to $x = \pi/3$.

$$L = \int_0^{\pi/3} \sqrt{1 + 4\cos^2(2x)}\, dx$$

(b) (6 points) Use the Trapezoid Rule with $n = 4$ subintervals to approximate the definite integral in part (a). Give your answer first in exact form, and then give a decimal approximation with at least five digits after the decimal point.

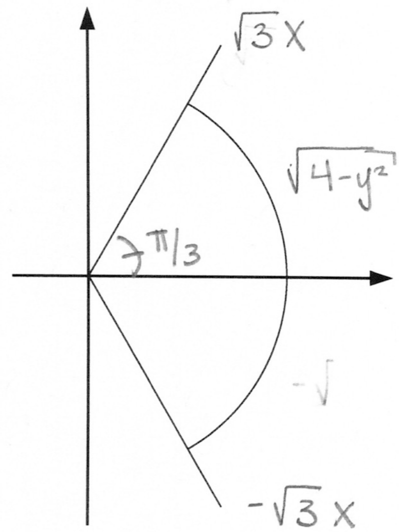
$$\Delta x = \frac{\pi/3 - 0}{4} = \frac{\pi}{12}$$

$$L \approx \frac{\pi}{24}\left[\sqrt{1+4\cos^2(0)} + 2\sqrt{1+4\cos^2(\pi/6)} + 2\sqrt{1+4\cos^2(\pi/3)} + 2\sqrt{1+4\cos^2(\pi/2)} + \sqrt{1+4\cos^2(2\pi/3)} \right]$$

$$= \boxed{1.633459}$$

8. (10 points)

Find the x- and y-coordinates of the center of mass of a uniform flat plate that is below the line $y = \sqrt{3}x$, above the line $y = -\sqrt{3}x$, and bounded on the right by the circle of radius 2 centered at the origin (see the figure).
Give your answer in exact form.



By symmetry, $\boxed{\bar{y} = 0}$

$\tan\theta = \sqrt{3} \Rightarrow \theta = \pi/3$

Area $= \frac{2\pi/3}{2\pi}$ of a circle $= \frac{1}{3}\pi(2^2) = \boxed{\frac{4\pi}{3}}$

$\sqrt{4-y^2} = y/\sqrt{3}$

$4 - y^2 = y^2/3$

$4 = \frac{4}{3}y^2$

$y = \pm\sqrt{3}$

$\frac{1}{2}\int_0^{\sqrt{3}} (4-y^2) - y^2/3 \, dy + \frac{1}{2}\int_{-\sqrt{3}}^0 (4-y^2) - y^2/3$

$= \frac{1}{2}\int_{-\sqrt{3}}^{\sqrt{3}} 4 - 4/3 y^2 \, dy$

$= \frac{1}{2}\left(4y - 4/9 y^3\right)\Big|_{-\sqrt{3}}^{\sqrt{3}}$

$= \frac{1}{2}\left(4\sqrt{3} - \frac{4}{9}3\sqrt{3} + 4\sqrt{3} - \frac{4}{9}3\sqrt{3}\right) = 2\left(2\sqrt{3} - \frac{2}{3}\sqrt{3}\right)$

$= 8/3\sqrt{3}$

$\therefore \bar{x} = \frac{8/3\sqrt{3}}{4\pi/3} = \boxed{\frac{2\sqrt{3}}{\pi}}$

$\boxed{(\bar{x}, \bar{y}) = \left(\frac{2\sqrt{3}}{\pi}, 0\right)}$

9. (10 points) Find the solution of the initial value problem

$$\frac{dy}{dx} = \frac{(x+3)(y+2)}{x^2+9}, \quad y(0) = 10.$$

Give your answer in the form $y = f(x)$.

$$\int \frac{dy}{y+2} = \int \frac{(x+3)}{x^2+9}\,dx$$

$$\int \frac{dy}{y+2} = \int \frac{x}{x^2+9}\,dx + 3\int \frac{1}{x^2+9}\,dx$$

$u = x^2+9$
$du = 2x\,dx$

$$\ln|y+2| = \tfrac{1}{2}\ln|x^2+9| + \arctan(x/3) + C$$

$$y+2 = C\sqrt{x^2+9}\,e^{\arctan(x/3)}$$

$$12 = C\sqrt{9}\,e^0 ; \quad 12 = 3C \Rightarrow C = 4$$

$$y+2 = 4\sqrt{x^2+9}\,e^{\arctan(x/3)}$$

$$\boxed{y = 4\sqrt{x^2+9}\,e^{\arctan(x/3)} - 2}$$

10. (10 points) A large vat initially contains 10 liters of pure water. A solution of salt in water is pumped into the vat at a rate of 2 liters per hour. The incoming solution contains b grams of salt per liter. The solution in the vat is kept thoroughly mixed and is drained from the vat at a rate of 2 liters per hour. After 8 hours, the concentration of salt in the vat is 3 grams per liter.

What is the concentration b of salt in the incoming mixture?

$Q(t)$ = amt of salt in the vat at time t

$\frac{dQ}{dt}$ = Rate in − Rate out

Rate in: concentration * Rate in of vol. $= b \cdot 2 \; g/hr$

Rate out: Concentration $= \frac{Q}{10} \Rightarrow$ Rate out $= 2 \cdot Q/10$

$\frac{dQ}{dt}$ = Rate in − Rate out $= 2b - Q/5$

$\int \frac{dQ}{2b - Q/5} = \int dt \Rightarrow -5\ln|2b - Q/5| = t + C$

$\cancel{\ln} 2b - Q/5 = Ce^{-t/5} \Rightarrow 10b - Ce^{-t/5} = Q(t)$

$Q(0) = 0 \quad 10b - C = 0 \Rightarrow C = 10b$

$Q(t) = 10b - 10be^{-t/5}$

Concentration at $t=8$ $= \frac{Q(8)}{10} = \frac{1\cancel{0}b - 10be^{-8/5}}{1\cancel{0}} = 3$

$$\boxed{b = \frac{3}{1 - e^{-8/5}} \approx 3.7589}$$