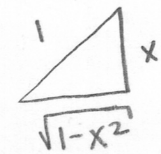
1. (12 total points) Evaluate the following integrals.

(a) (6 points) $\int \frac{1-x}{\sqrt{1-x^2}}\,dx$ Trig sub $x = \sin\theta$, $dx = \cos\theta\, d\theta$



$$\int \frac{(1-\sin\theta)\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta = \int 1-\sin\theta\, d\theta = \theta + \cos\theta + C$$

$$= \boxed{\arcsin x + \sqrt{1-x^2} + C}$$

(b) (6 points) $\int \frac{x^2-x+8}{x^3+4x}\,dx = \int \frac{x^2-x+8}{x(x^2+4)}\,dx$

$x^3 + 4x = x(x^2+4)$

$$\frac{A}{x} + \frac{Bx+C}{x^2+4} = \frac{x^2-x+8}{x(x^2+4)}$$

$A(x^2+4) + x(Bx+C) = x^2 - x + 8$

$x = 0$: $4A = 8$ $\boxed{A=2}$

$Ax^2 + Bx^2 = x^2 \Rightarrow 2 + B = 1$ $\boxed{B=-1}$

$Cx = -x \Rightarrow$ $\boxed{C=-1}$

$$\int \frac{2}{x} + \int \frac{-x-1}{x^2+4}\,dx = \int \frac{2}{x} - \int \frac{x}{x^2+4} - \int \frac{1}{x^2+4}$$

$u = x^2+4$
$du = 2x\,dx$

$$= \int \frac{2}{x} - \frac{1}{2}\int \frac{1}{u} - \int \frac{1}{x^2+4} = \boxed{2\ln|x| - \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C}$$

2. (12 total points) Evaluate the following integrals.

(a) (6 points) $\int x\,(e^x+\ln(x))\,dx$ $= \int xe^x + \int x\ln(x)\,dx$

$\int xe^x\,dx$ $\quad u=x \quad dv=e^x$ $\quad du=dx \quad v=e^x$ $\quad xe^x - \int e^x\,dx = xe^x - e^x + C$

$\int x\ln(x)\,dx$ $\quad u=\ln(x) \quad dv=x$ $\quad du=\frac{1}{x} \quad v=\frac{1}{2}x^2$ $\quad = \frac{1}{2}x^2\ln(x) - \frac{1}{2}\int x\,dx$

$= \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C$

$\therefore \int x(e^x+\ln(x))\,dx = \boxed{xe^x - e^x + \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C}$

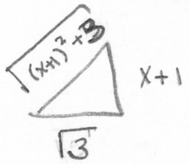
(b) (6 points) $\displaystyle\int_0^2 \frac{1}{\sqrt{x^2+2x+4}}\,dx$ Give your answer in exact form.

Complete the square

$x^2+2x+4 = (x+1)^2+3$

$\displaystyle\int \frac{1}{\sqrt{(x+1)^2+3}}$ $\quad x+1=\sqrt{3}\tan\theta$ $\quad dx=\sqrt{3}\sec^2\theta\,d\theta$

$\displaystyle = \int \frac{\sqrt{3}\sec^{\cancel{2}}\theta}{\sqrt{3}\cancel{\sec\theta}}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C$



$\displaystyle\int_0^2 \frac{1}{\sqrt{x^2+2x+4}}\,dx = \ln\left|\sqrt{(x+1)^2+3}\Big/\tfrac{1}{\sqrt{3}} + x+1\Big/\sqrt{3}\right|\Bigg|_0^2$

$= \ln\left|\sqrt{12}/\sqrt{3} + \frac{3}{\sqrt{3}}\right| - \ln\left|2/\sqrt{3} + 1/\sqrt{3}\right|$

$= \ln\left|\frac{2\sqrt{3}+3}{\sqrt{3}}\right| - \ln\left|3/\sqrt{3}\right|$

$\boxed{= \ln|2+\sqrt{3}| - \ln|\sqrt{3}|}$

3. (8 points) Consider the improper integral $\int_1^2 \frac{x}{\sqrt{x-1}}\,dx$.

Evaluate the integral (if it converges) or explain why it does not converge.

$$\int_1^2 \frac{x}{\sqrt{x-1}}\,dx \qquad u=\sqrt{x-1} \quad u^2=x-1$$
$$2u\,du=dx \quad u^2+1=x$$

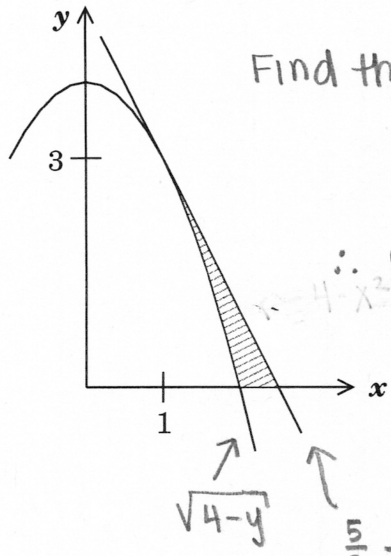
$$= \lim_{a\to 1^+} \int_a^2 \frac{x}{\sqrt{x-1}}\,dx = \lim_{a\to 1^+} \int_{\sqrt{a-1}}^{1} \frac{u^2+1}{\cancel{u}} \cdot 2\cancel{u}$$

$$= \lim_{a\to 1^+} \int_{\sqrt{a-1}}^{1} 2(u^2+1)\,du = \lim_{a\to 1^+} \frac{2}{3}u^3 + 2u \Big|_{\sqrt{a-1}}^{1}$$

$$= \lim_{a\to 1^+} \frac{2}{3} + 2 - \underbrace{\frac{2}{3}\left(\sqrt{a-1}\right)^3}_{\to 0} + \underbrace{2\sqrt{a-1}}_{\to 0}$$

$$= \boxed{8/3}$$

4. (10 points) Let R be the shaded region in the figure below, bounded by the x-axis, by the curve $y=4-x^2$, and by the line tangent to $y=4-x^2$ at the point $(1,3)$. Find the area of the region R.



Find the tangent line

$\frac{dy}{dx} = -2x \quad m = -2$

$\therefore y = -2(x-1)+3 \Rightarrow y = -2x+5$

Note if you solve in terms of x you must do 2 integrals

$$\text{Area: } \int_0^3 \frac{5}{2} - \frac{1}{2}y - \sqrt{4-y}\,dy = \int_0^3 \frac{5}{2} - \frac{1}{2}y\,dy - \int_0^3 \sqrt{4-y}\,dy$$

$u = 4-y$
$du = -dy$

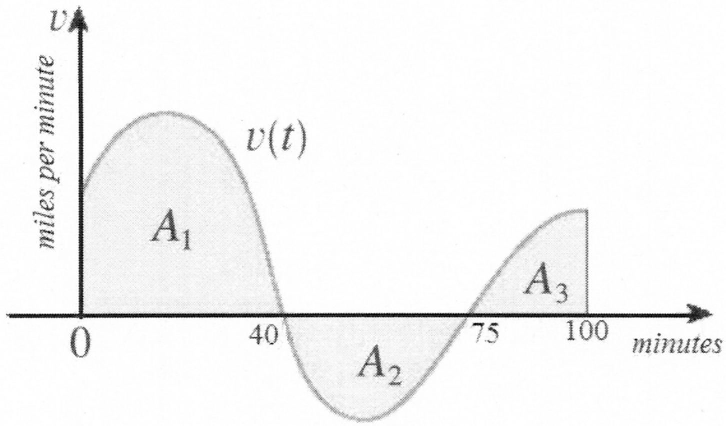
$$= \int_0^3 \frac{5}{2} - \frac{1}{2}y\,dy + \int_4^1 u^{1/2}\,du$$

$$= \frac{5}{2}y - \frac{1}{4}y^2 \Big|_0^3 + \frac{2}{3}u^{3/2}\Big|_4^1$$

$$= \frac{15}{2} - \frac{9}{4} + \frac{2}{3} - \frac{16}{3} = \boxed{\frac{7}{12}}$$

5. (10 total points) Bob the bicyclist is riding back and forth on a straight road. His velocity $v(t)$ (in miles per minute) is depicted in the graph below. Suppose the values of the areas labeled on the graph are:

$$A_1 = 16,\ A_2 = 6,\ \text{and}\ A_3 = 4.$$



Answer the following questions. Include units in your answers when appropriate.

(a) (2 points) What was the total distance traveled by Bob during the time interval $0 \le t \le 100$ minutes?

$A_1 + A_2 + A_3 = 16 + 6 + 4 = \boxed{26 \text{ miles}}$

(b) (2 points) How far from his initial position is Bob at the end of the 100 minutes?

$A_1 - A_2 + A_3 = 16 - 6 + 4 = \boxed{14 \text{ miles}}$

(c) (2 points) At what time, between $t = 0$ and $t = 100$ minutes, is Bob the farthest away from his starting position?

$\boxed{t = 40 \text{ minutes}}$

(d) (2 points) How many times during the time interval $0 \le t \le 100$ minutes is Bob exactly 11 miles away from his starting point?

one time btwn 0 to 40
" " 40 to 75
" " 75 to 100

$\boxed{3 \text{ times}}$

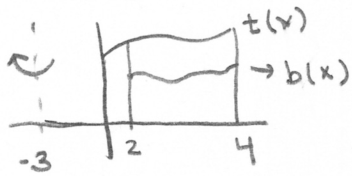
(e) (2 points) Compute the value of the definite integral $\int_{40}^{100} (20v(t) + 3)\, dt$.

$20\int_{40}^{100} v(t) + 3t\Big|_{40}^{100}$

$= 20[-6+4] + 3[100-40] = -40 + 180 = \boxed{140 \text{ miles}}$

6. (10 total points) Let $\mathscr{R}$ be the region between the lines $x = 2$ and $x = 4$ that is bounded on the top by the curve $y = t(x)$ and bounded on the bottom by the curve $y = b(x)$, where $t(x)$ and $b(x)$ are continuous functions satisfying $t(x) > b(x)$.

(a) (6 points) Set up a definite integral *with respect to x* for the volume of the solid obtained by rotating the region $\mathscr{R}$ about the vertical line $x = -3$. Your answer should be a definite integral that involves $t(x)$ and $b(x)$.



Want to use Shells

$$\boxed{2\pi \int_2^4 [t(x) - b(x)][x+3]\,dx}$$

(b) (4 points) Suppose we have the following table of values for $t(x)$ and $b(x)$.

x	$b(x)$	$t(x)$
0	0.0	12.0
0.5	6.4	12.1
1	8.3	12.5
1.5	9.6	13.1
2	10.5	14.0
2.5	11.2	15.1
3	11.8	16.5
3.5	12.3	18.1
4	12.7	20.0
4.5	13.1	22.1
5	13.4	24.5

Use the relevant data from this table and the Trapezoid Rule with $n = 4$ subintervals to approximate the value of the volume of the solid in part (a). Give your answer in decimal form, correct to at least the first digit after the decimal point.

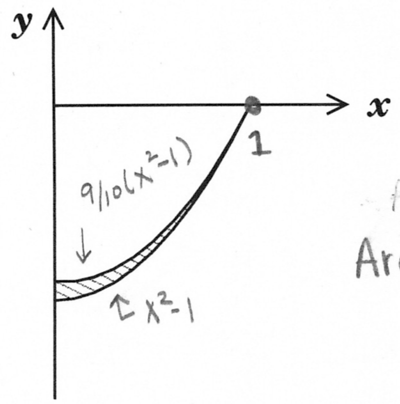
$$\Delta x = \frac{4-2}{4} = \frac{1}{2}$$

$$\frac{2\pi}{4}\Big[(2+3)[t(2)-b(2)] + 2(2.5+3)[t(2.5)-b(2.5)] + 2(3+3)[t(3)-b(3)] + 2(3.5+3)[t(3.5)-b(3.5)] + (4+3)[t(4)-b(4)]\Big]$$

$$= \frac{\pi}{2}\Big[5(14-10.5) + 2(5.5)(15.1-11.2) + 2(6)(16.5-11.8) + 2(6.5)(18.1-12.3) + 7(20-12.7)\Big]$$

$$\boxed{\approx 382.17}$$

7. (10 points) Find the center of mass of the region in the fourth quadrant bounded above by the curve $y = \frac{9}{10}(x^2 - 1)$, bounded below by the curve $y = x^2 - 1$, and bounded on the left by the y-axis.



$$\frac{9}{10}(x^2-1) = x^2-1$$

$$\frac{-1}{10}x^2 + 1/10 = 0 \quad x^2 = 1 \quad x = \pm 1$$

$$\text{Area} = \int_0^1 \frac{-1}{10}x^2 + \frac{1}{10}\,dx$$

$$= \frac{-1}{30}x^3 + \frac{1}{10}x \Big|_0^1 = \frac{-1}{30} + \frac{1}{10} = \frac{2}{30} = \boxed{\frac{1}{15}}$$

$$M_y = \int_0^1 x\left(\frac{-1}{10}x^2 + \frac{1}{10}\right)dx = \int_0^1 \frac{-1}{10}x^3 + \frac{1}{10}x\,dx$$

$$= \frac{-1}{40}x^4 + \frac{1}{20}x^2 \Big|_0^1$$

$$= \frac{-1}{40} + \frac{1}{20} = \boxed{\frac{1}{40}}$$

$$M_x = \frac{1}{2}\int_0^1 \left(\frac{9}{10}(x^2-1)\right)^2 - (x^2-1)^2\,dx = \frac{1}{2}\int_0^1 \left(\frac{81}{100} - 1\right)(x^2-1)^2\,dx$$

$$= \frac{-19}{200}\int_0^1 x^4 - 2x^2 + 1\,dx = \frac{-19}{200}\left(\frac{1}{5}x^5 - \frac{2}{3}x^3 + x\right)\Big|_0^1$$

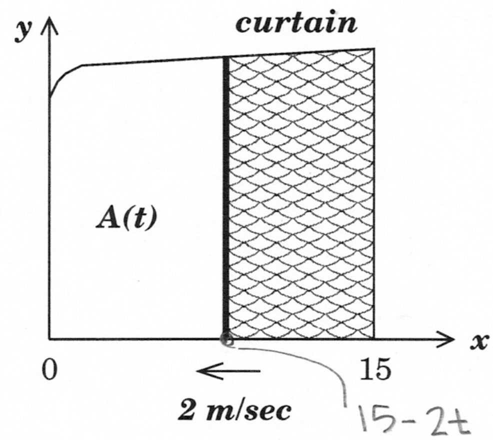
$$= \frac{-19}{200}\left(\frac{1}{5} - \frac{2}{3} + 1\right) = \boxed{\frac{-19}{375}}$$

$$\boxed{(\bar{x}, \bar{y}) = \left(\frac{15}{40}, \frac{-19}{375}\cdot 15\right) = \left(\frac{3}{8}, -19/25\right)}$$

8. (8 total points) A stage opening is bounded by the x-axis, the y-axis, the line $x = 15$, and the curve

$$y = \sqrt{10 + x^{1/3}}.$$

The units on the x and y axes are meters. Initially, the stage curtain is completely open. At time $t = 0$, a vertical pole pulling the curtain starts on the right side of the stage opening ($x = 15$) and moves to the left at a constant speed of 2 m/sec. Let $A(t)$ be the area that is not yet covered by the curtain at time t seconds (the enclosed white area in the figure below).



(a) (4 points) Express $A(t)$ as a definite integral.

$$A(t) = \int_0^{15-2t} \sqrt{10 + x^{1/3}}\, dx$$

(b) (4 points) Find $\frac{dA}{dt}$ when $t = 3.5$ sec. Give your answer in exact form and include correct units. The FTC

$$\frac{dA}{dt} = \sqrt{10 + (15-2t)^{1/3}} \cdot (-2)$$

$$A'(3.5) = \sqrt{10 + (15-7)^{1/3}}\ (-2)$$

$$= \boxed{-2\sqrt{12}\ m^2/sec}$$

9. (10 points) Find the solution of the differential equation

$$\frac{dy}{dt} = \sqrt{5t}\,(1+y^2)$$

that satisfies the initial condition $y(0) = 1$. Solve for y, giving your answer in the form $y = f(t)$.

1). Separate the variables

$$\frac{dy}{1+y^2} = \sqrt{5t}\,dt$$

2). Integrate both sides

$$\int \frac{dy}{1+y^2} = \int \sqrt{5}\,t^{1/2}\,dt$$

$$\arctan(y) = \frac{2}{3}\sqrt{5}\,t^{3/2} + C$$

3). Solve for C

$$\arctan(1) = \frac{2}{3}\sqrt{5}\cdot 0^{3/2} + C$$

$$\frac{\pi}{4} = C$$

4). Solve for y

$$\arctan(y) = \frac{2}{3}\sqrt{5}\,t^{3/2} + \pi/4$$

$$\boxed{y = \tan\left(\frac{2}{3}\sqrt{5}\,t^{3/2} + \pi/4\right)}$$

10. (10 points) Let P denote the total weight (in kilograms) of water lilies on the surface of a lake. The rate of increase of P depends on the amount of sunlight and is given by the equation

$$\frac{dP}{dt} = s(t)P.$$

Here, t denotes the time (in hours) after midnight, and

$$s(t) = 0.03\cos^2\left(\frac{\pi}{12}t\right) \quad (\text{for } 6 \le t \le 18)$$

is a function that depends on the amount of sunlight at time t.

One day, at 6:00am, there were 100 kilograms of lilies on the lake. How many kilograms of lilies were on the lake at 6:00pm that day? Give your answer in decimal form, correct to at least the second digit after the decimal point.

1) Separate the variables

$$\frac{dP}{P} = s(t)dt = 0.03\cos^2\left(\frac{\pi}{12}t\right)dt$$

2) $$\int \frac{dP}{P} = \int 0.03\cos^2\left(\frac{\pi}{12}t\right)dt = \frac{0.03}{2}\int 1+\cos(\pi/6\, t)\,dt$$

$$\ln|P| = 0.015\left(t + 6/\pi \sin(\pi/6\, t)\right) + C$$

$$P = Ce^{0.015(t + 6/\pi \sin(\pi/6\, t))}$$

$P(6) = 100$ $\quad 100 = Ce^{0.015(6 + 6/\pi \sin(\pi))}$

$$C = \frac{100}{e^{0.015(6)}}$$

$$P(12) = 100e^{-0.015(6)}e^{0.015(12)} = \boxed{109.417 \text{ lilies}}$$