1. Evaluate the following indefinite integrals.

(a) (5 points) $\int \sin(x)\sqrt{\cos(x)}\,dx$ $\quad u=\cos x \quad du=-\sin x\,dx$

$$\int \sin(x)\sqrt{\cos(x)}\,dx = -\int \sqrt{u}\,du = \frac{-2}{3}u^{3/2}+C$$

$$\boxed{= \frac{-2}{3}\left(\cos(x)\right)^{3/2}+C}$$

(b) (5 points) $\int \sqrt{3-2x-x^2}\,dx$

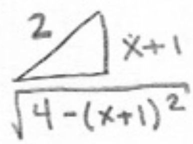
Complete the square $-(x^2+2x-3) = -\left((x+1)^2-4\right) = 4-(x+1)^2$

$$\Rightarrow \int \sqrt{3-2x-x^2}\,dx = \int \sqrt{4-(x+1)^2}\,dx \qquad \begin{array}{l} u=x+1 \\ du=dx \end{array}$$

$$= \int \sqrt{4-u^2}\,du \qquad \begin{array}{l} u=2\sin\theta \\ du=2\cos\theta\,d\theta \end{array}$$

$$= \int 2\cos\theta\sqrt{4-4\sin^2\theta}\,d\theta$$

$$= 4\int \cos^2\theta\,d\theta = 4\int \frac{1+\cos(2\theta)}{2}\,d\theta \qquad \frac{o}{h} = u/2 = \frac{x+1}{2} = \sin\theta$$



$$= 2\left(\theta+\tfrac{1}{2}\sin(2\theta)\right)+C$$

$$= 2\theta + 2\sin\theta\cos\theta + C$$

$$\boxed{= 2\sin^{-1}\left(\frac{x+1}{2}\right) + 2\left(\frac{x+1}{2}\right)\left(\frac{\sqrt{4-(x+1)^2}}{2}\right) + C}$$

2. Evaluate the following definite integrals.

(a) (5 points) $\int_0^{\pi} \sec\left(\frac{x}{3}\right) \tan^3\left(\frac{x}{3}\right) dx$ $u = x/3$ $du = \frac{1}{3} dx$

$= \int_0^{\pi/3} 3\sec(u)\tan^3(u)\, du$ $v = \sec(u)$ $dv = \sec(u)\tan(u)\, du$

$= \int_1^2 3\tan^2(u)\, dv = \int_1^2 3(\sec^2(u)+1)\, dv$

$= \int_1^2 3(v^2+1)\, dv = v^3 + 3v \Big|_1^2 = 8+6-1-3 = \boxed{10}$

(b) (5 points) $\int_{-1}^{2} \frac{x}{x^2+2x+10}\, dx$

Complete the square $\int_{-1}^{2} \frac{x}{(x+1)^2+9}\, dx$ $u = x+1$ $du = dx$

$= \int_0^3 \frac{u-1}{u^2+9}\, du$

$= \int_0^3 \frac{u}{u^2+9}\, du - \int_0^3 \frac{1}{u^2+9}\, du$

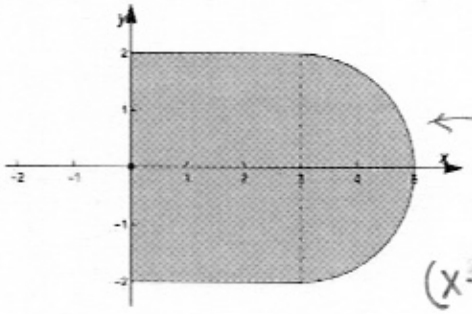
$v = u^2+9$
$dv = 2u\, du$

$= \int_9^{18} \frac{1}{2}\frac{1}{v}\, dv - \frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right)\Big|_0^3$

$= \frac{1}{2}\ln|v| \Big|_9^{18} - \frac{\pi}{12}$

$= \frac{1}{2}\ln\left|\frac{18}{9}\right| - \pi/12 = \boxed{\frac{1}{2}\ln(2) - \pi/12}$

3. (10 points) Consider the region in the xy-plane formed by a rectangle of height 4 and width 3 and a half-disk of radius 2 centered at $(3,0)$, as shown in the figure. Compute $\bar{x}$, the x-component of the centroid of the region.



$(x-3)^2+y^2=4$

① Find Area

Area = Rectangle + semi-circle

$$= 4\cdot 3 + \frac{\pi(2^2)}{2} = 12 + 2\pi$$

② Find $\bar{x}$:

$$\bar{x} = \frac{1}{\text{Area}}\int_0^3 \underbrace{x(2 - -2)\,dx}_{\text{rectangle part}} + \frac{1}{\text{Area}}\int_3^5 2\sqrt{4-(x-3)^2}\,x$$

$u = x - 3$
$du = dx$

$$= \frac{1}{12+2\pi}\left(2x^2\Big|_0^3 + \int_0^2 2(u+3)\sqrt{4-u^2}\,du\right.$$

$$= \frac{1}{12+2\pi}\left(18 + \int_4^0 -\sqrt{v}\,dv\right.$$

$v = 4-u^2$
$dv = -2u\,du$

$$\left. + \int_0^2 6\sqrt{4-u^2}\,du\right)$$

$v = 2\sin\theta$
$dv = 2\cos\theta\,d\theta$

$$= \frac{1}{12+2\pi}\left(18 + \frac{-2}{3}v^{3/2}\Big|_4^0 + \int_0^{\pi/2} 24\cos^2\theta\,d\theta\right)$$

$$= \frac{1}{12+2\pi}\left(18 + \frac{16}{3} + 12\int_0^{\pi/2} 1+\cos(2\theta)\,d\theta\right)$$

$$= \boxed{\frac{1}{12+2\pi}\left(18 + \frac{16}{3} + 6\pi\right)}$$

4. (a) (5 points) Does the improper integral $\int_1^2 \frac{1}{\sqrt{x-1}}\,dx$ converge? If yes, to what?

$$\lim_{a\to 1^+} \int_a^2 \frac{1}{\sqrt{x-1}}\,dx = \lim_{a\to 1^+} 2\sqrt{x-1}\,\Big|_a^2 = \lim_{a\to 1^+} 2 - 2\sqrt{a-1}$$

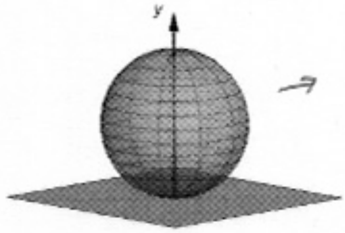
$= \boxed{2}$ Converges and goes to a

(b) (5 points) Does the improper integral $\int_2^\infty \frac{1}{\sqrt{x-1}}\,dx$ converge? If yes, to what?

$$\lim_{a\to \infty} \int_2^a \frac{1}{\sqrt{x-1}}\,dx = \lim_{a\to \infty} 2\sqrt{x-1}\,\Big|_2^a = \lim_{a\to \infty} 2\sqrt{a-1} - 2$$

$= \infty$ Does not converge

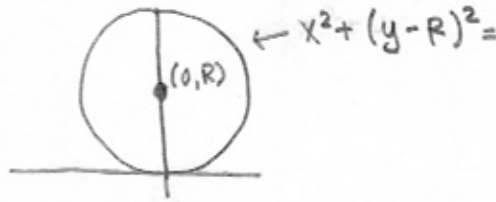
5. (10 points) A spherical tank of radius R meters is resting on the ground; so its center is located R meters above ground level (see figure). The tank is initially empty, and water is pumped from ground level into the tank until the tank is half full. Find the work (in Joules) required to do this.
Note: Water has a mass density of 1000 kilograms per cubic meter and the acceleration due to gravity is 9.8 meters/sec^2.



① Volume of a slice

$$\pi r^2\,dy = \pi\left[\sqrt{R^2-(y-R)^2}\right]^2 dy$$

$$= \pi\left(R^2-(y-R)^2\right)dy$$



② Mass of a slice

$$= 1000\pi\left(R^2-(y-R)^2\right)dy$$

③ Force of a slice

$$\text{mass} \times \text{accel} = 9800\pi\left(R^2-(y-R)^2\right)dy$$

④ Distance $= y$

⑤ Where you have slices of water is 0 to R ← bounds

$$\text{Work} = \int_0^R 9800\pi y\left(R^2-(y-R)^2\right)dy = 9800\pi\int_0^R y(2Ry-y^2)\,dy$$

$$(y-R)^2 = y^2-2Ry+R^2$$

$$= 9800\pi\left(\frac{2R}{3}y^3-\frac{1}{4}y^4\right)\Big|_0^R = 9800\pi\left(\frac{2}{3}R^4-\frac{1}{4}R^4\right)$$

$$\boxed{=\frac{9800\pi\cdot 5}{12}R^4}$$

6. Consider the curve $y = 2x^{5/2}$.

   (a) (4 points) Set up a definite integral for the arc length of this curve from $x = 1$ to $x = 4$.
   DO NOT EVALUATE THE INTEGRAL.

$$\int_1^4 \sqrt{1 + (5x^{3/2})^2}\, dx \qquad \frac{dy}{dx} = 5x^{3/2}$$

$$= \int_1^4 \sqrt{1 + 25x^3}\, dx$$

   (b) (6 points) Use the Trapezoid Rule with $n = 4$ subintervals to estimate the integral in part (a).
   Leave your answer in exact form.

$$1 \quad \frac{7}{4} \quad \frac{10}{4} \quad \frac{13}{4} \quad 4 \qquad \frac{4-1}{4} = \frac{3}{4}$$

$$\frac{b-a}{2(4)}\left[f(1) + 2f(2) + 2f(3) + f(4)\right]$$

$$= \frac{3}{8}\left[\sqrt{1+25} + 2\sqrt{1+25(\tfrac{7}{4})^3} + 2\sqrt{1+25(\tfrac{10}{4})^3} + 2\sqrt{1+25(\tfrac{13}{4})^3} + \sqrt{1+25(64)}\right]$$

7. Consider the region $\mathscr{R}$ above the x-axis and under the curve

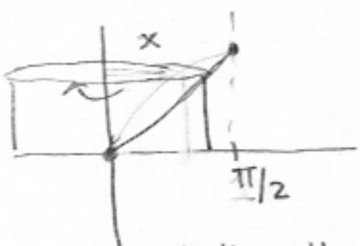
$$y = \sin^2 x \qquad \text{for} \qquad 0 \le x \le \pi/2.$$

(a) (4 points) Find the area of this region $\mathscr{R}$.

$$\int_0^{\pi/2} \sin^2 x \, dx = \int_0^{\pi/2} \frac{1-\cos(2x)}{2} \, dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin(2x) \Big|_0^{\pi/2} = \boxed{\frac{\pi}{4}}$$

(b) (6 points) The region $\mathscr{R}$ is rotated about the y-axis. Find the volume of the resulting solid.



Shell-method

$$2\pi \int_0^{\pi/2} \underbrace{\sin^2 x}_{\text{height}} \cdot \underbrace{x}_{\text{radius}} \, dx = \left(\int_0^{\pi/2} x \left(\frac{1-\cos(2x)}{2} \right) dx \right) 2\pi$$

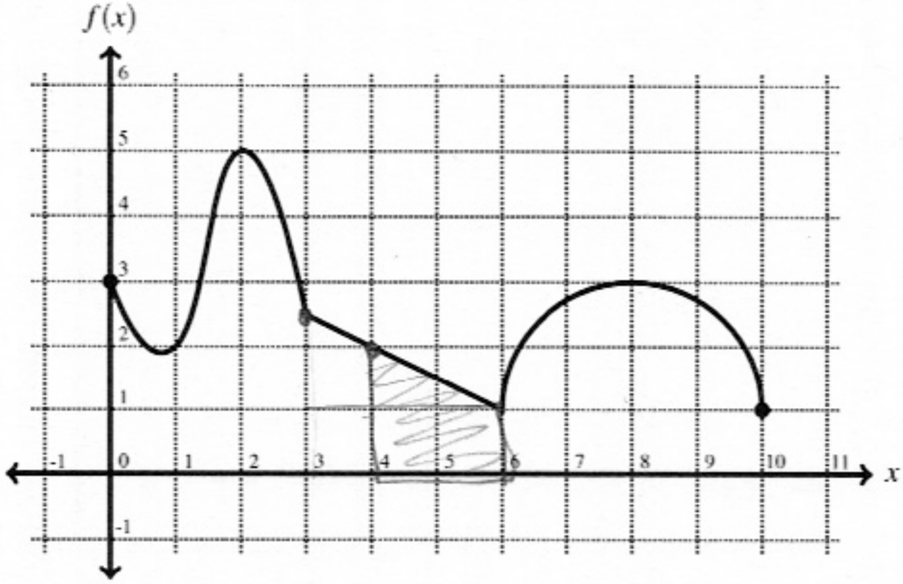
$$= \left[\int_0^{\pi/2} \frac{x}{2} \, dx - \frac{1}{2} \int_0^{\pi/2} x\cos(2x) \, dx \right] 2\pi \qquad \begin{matrix} u = x & dv = \cos(2x) \\ du = dx & v = \frac{\sin(2x)}{2} \end{matrix}$$

$$= \left[\frac{x^2}{4} \Big|_0^{\pi/2} - \frac{1}{2} \left(\frac{x\sin(2x)}{2} \Big|_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin 2x}{2} \, dx \right) \right] 2\pi$$

$$= \left[\frac{\pi^2}{16} + \frac{1}{4} \int_0^{\pi/2} \sin(2x) \, dx \right] 2\pi$$

$$2\pi \left(\frac{\pi^2}{16} - \frac{1}{8}\cos(2x) \Big|_0^{\pi/2} \right) = \left(\frac{\pi^2}{16} + \frac{1}{8} + \frac{1}{8} \right) 2\pi = \boxed{\frac{\pi^3}{8} + \frac{\pi}{2}}$$

8. The graph of $f(x)$ is shown below. Use it to answer the following questions.



(a) (4 points) Compute the average value of $f(x)$ on the interval $[4, 10]$.

$$\text{avg } f = \frac{1}{10-4}\int_4^{10} f(x) = \frac{1}{6}\left[\text{Area of triangle + area of semi}\right.$$

$$= \frac{1}{6}\left[\frac{1\cdot 2}{2} + 2 + \frac{\pi(2^2)}{2} + 4\right]$$

$$= \frac{1}{6}\left[7 + 2\pi\right]$$

(b) (6 points) Let $g(x) = \int_{x^2}^{7} f(t)\,dt$. Calculate $g''(2)$.

$$g(x) = -\int_7^{x^2} f(t)\,dt$$

$$g'(x) = -f(x^2)\cdot 2x \leftarrow \text{FTC}$$

$$g''(x) = -2f(x^2) - 2x\,f'(x^2)\cdot 2x$$

$$= -2f(x^2) - 4x^2 f'(x^2)$$

$$f'(4) = \frac{1-\frac{5}{2}}{6-3} = \frac{-3}{2(3)} =$$

$$g''(2) = -2f(4) - 16f'(4)$$

$$= -2(2) - 16\left(\frac{-1}{2}\right)$$

$$= -4 + 8 = \boxed{4}$$

9. (10 points) Find the solution of the initial value problem

$$\frac{dy}{dx} = \frac{x\cos(x^2)}{y}, \quad y(0) = -3.$$

Give your answer in the form $y = f(x)$.

$y\,dy = x\cos(x^2)\,dx$

Integrate both sides

$\frac{1}{2}y^2 = \int x\cos(x^2)\,dx \quad u = x^2, \; du = 2x\,dx$

$= \frac{1}{2}\int \cos(u)\,du$

$= \frac{1}{2}\sin(x^2) + C$

$y^2 = \sin(x^2) + C$

$y = \pm\sqrt{\sin(x^2) + C}$

$-3 = \pm\sqrt{0 + C}$

$\Rightarrow 3 = \sqrt{C} \Rightarrow C = 9$

$\boxed{y = -\sqrt{\sin(x^2) + 9}}$

10. (10 points) Among the 30 thousand students of the University of Washington, the rate of the spread of the app *Sbreak* is proportional to the product of the number of students P (in thousands) who have the app on their phones and the number of students $30-P$ who do not (again in thousands). That is,

$$\frac{dP}{dt} = kP(30-P)$$

where k is a positive proportionality constant, P is in thousand of students and t is in hours.

Initially, one thousand students have the app on their phones. In 4 hours, 5 thousand students have the app on their phones. According to this model, when will half of the students have the app on their phones?

$$\int \frac{dP}{P(30-P)} = \int k\,dt$$

$$\Rightarrow \frac{A}{P} + \frac{B}{30-P} = \frac{1}{P(30-P)} \qquad (30-P)A + BP = 1 \qquad B = \frac{1}{30} \quad A = \frac{1}{30}$$

$$\Rightarrow \int \frac{\frac{1}{30}}{30-P} + \frac{\frac{1}{30}}{P}\,dP = kt + C$$

$$\Rightarrow -\frac{1}{30}\ln|30-P| + \frac{1}{30}|P| = kt + C$$

$$\Rightarrow \frac{1}{30}\ln\left|\frac{P}{30-P}\right| = kt + C \qquad P(0) = 1$$

$$\ln\left|\frac{P}{30-P}\right| = 30kt + C$$

$$\frac{P}{30-P} = Ce^{30kt} \qquad P(0) = 1$$

$$C = \frac{1}{29}$$

$$\Rightarrow \frac{P}{30-P} = \frac{1}{29}e^{30kt} \qquad P(4) = 5$$

$$\Rightarrow \frac{5}{25} = \frac{1}{29}e^{30(4)k} \Rightarrow \frac{\ln\left|\frac{29}{5}\right|}{120} = k$$

Want to know when $P = 15$

$$1 = \frac{1}{29}e^{\ln\left(\frac{29}{5}\right)\frac{1}{4}t}$$

$$\frac{4\ln(29)}{\ln\left(\frac{29}{5}\right)} = t \approx 7.66.$$