ur Name

Solutions

Your Signature

Student ID #

Quiz Section

Professor's Name

TA's Name

- This exam is closed book. You may use one 8.5″ × 11″ sheet of handwritten notes (both sides OK). Do not share notes. No photocopied materials are allowed.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- A scientific calculator is allowed, but graphing calculators are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- You may use any of the 20 integrals in the table on p. 484 of the text (p. 506 if you have the 5th edition of Stewart) without deriving them. **Show your work in evaluating any other integrals, even if they are on your note sheet**.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 10 pages, plus this cover sheet. Please make sure that your exam is complete.

Question	Points	Score
1	12	
2	12	
3	10	
4	8	
5	10	

Question	Points	Score
6	10	
7	10	
8	8	
9	10	
10	10	
Total	100	

1. (12 total points) Evaluate the following indefinite integrals.

(a) (6 points) $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\,dx$ (for $x \geq 3$)

① Use Trig substitution

Let $x = 3\sec\theta \Rightarrow dx = 3\tan\theta\sec\theta\, d\theta$

$$\therefore \int \frac{\sqrt{x^2-9}}{x} = \int \frac{3\tan\theta \cdot 3\tan\theta\sec\theta\, d\theta}{3\sec\theta}$$

$$= 3\int \tan^2\theta\, d\theta$$

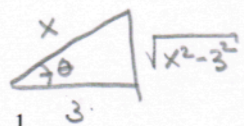
$$= 3\int \sec^2\theta - 1$$

$$= 3\int \sec^2\theta - \int 3$$

$$= 3\tan\theta - 3\theta + C$$

② Draw Triangle and sub in

$\frac{x}{3} = \sec\theta = \frac{h}{a}$



Also $\sec^{-1}\left(\frac{x}{3}\right) = \theta$.

$\therefore 3\tan\theta - 3\theta + C$

$$\boxed{= \frac{3\sqrt{x^2-9}}{3} - 3\sec^{-1}\left(\frac{x}{3}\right) + C}$$

(b) (6 points) $\displaystyle\int \frac{1}{(x+3)(\sqrt{x+4}-2)}\,dx$

① Try u-sub. Let $u = \sqrt{x+4} \Rightarrow u^2 = x+4$

$\Rightarrow 2u\,du = dx$

$$\therefore \int \frac{2u\,du}{(u^2-1)(u-2)} = \int \frac{2u\,du}{(u+1)(u-1)(u-2)}\,du$$

② By Partial Fractions,

$$\frac{2u}{(u+1)(u-1)(u-2)} = \frac{A}{u+1} + \frac{B}{u-1} + \frac{C}{u-2}$$

$\Rightarrow$
$$2u = A(u-1)(u-2) + B(u+1)(u-2) + C(u+1)(u-1)$$

Plug in $u=-1: -2 = A(-2)(-3) \Rightarrow A = -1/3$

$u = 1: 2 = B(2)(-1) \Rightarrow B = -1$

$u = 2: 4 = C(3)(1) \Rightarrow C = 4/3$

$$\therefore \int \frac{1}{(u+1)(u-1)(u-2)}\,du = \int \frac{-1/3}{u+1} + \int \frac{-1}{u-1} + \int \frac{4/3}{\ldots}$$

Hence,

$$= -\frac{1}{3}\ln|u+1| - \ln|u-1| + \frac{4}{3}\ln|u-2|$$

$$\boxed{= -\frac{1}{3}\ln\left|\sqrt{x+4}+1\right| - \ln\left|\sqrt{x+4}-1\right| + \frac{4}{3}\ln\left|\sqrt{x+4}-2\right| + C}$$

2. (12 total points) Evaluate the following definite integrals.

(a) (6 points) $\int_0^{\pi/2} (\sin^2\theta + \cos\theta)\sin^3\theta \, d\theta$ Give your answer in exact form.

① u-substitution: Let $u = \cos\theta$ $du = -\sin\theta\, d\theta$

$$\Rightarrow \int_0^{\pi/2} (\sin^2\theta + \cos\theta)\sin^3\theta\, d\theta = \int_1^0 -(\sin^2\theta + u)\sin^2\theta\, d\theta$$

Since $\sin^2\theta = 1 - \cos^2\theta = 1 - u^2$, we have

$$\int_1^0 -(1-u^2+u)(1-u^2)\,du = \int_1^0 -1 + 2u^2 - u + u^3 - u^4\, du$$

$$= -u - \tfrac{1}{2}u^2 + 2/3u^3 + 1/4u^4 - 1/5u^5 \Big|_1^0$$

$$= 0 - \left(-1 - \tfrac{1}{2} + 2/3 + 1/4 - 1/5\right)$$

$$= \boxed{47/60}$$

(b) (6 points) $\int_0^2 t^3 e^{t^2-9}\, dt$ Give your answer in exact form.

① Let $u = t^2 - 9$ $du = 2t\,dt$

$$\int_0^2 t^3 e^{t^2-9} = \int_{-9}^{-5} \tfrac{1}{2}t^2 e^u\, du = \int_{-9}^{-5} \tfrac{1}{2}(u+9)e^u\, du$$

$$= \underbrace{\int_{-9}^{-5} \tfrac{1}{2}ue^u}_{①} + \underbrace{\int_{-9}^{-5} 9/2\, e^u\, du}_{②}$$

We know ② is

$$9/2\, e^u \Big|_{-9}^{-5} = 9/2\, e^{-5} - 9/2\, e^{-9}.$$

② To solve ①: Use int. by parts,

$w = 1/2u$ $dv = e^u$

$dw = 1/2du$ $v = e^u$

$$\int_{-9}^{-5} \tfrac{1}{2}ue^u = \tfrac{1}{2}ue^u\Big|_{-9}^{-5} - \int_{-9}^{-5}\tfrac{1}{2}e^u du = \tfrac{1}{2}(-5)e^{-5} - \tfrac{1}{2}(-9)e^{-9} - \tfrac{1}{2}e^{-5} + \tfrac{1}{2}e^{-9}$$

∴ ① + ② =

$$9/2\, e^{-5} - 9/2\, e^{-9} - \tfrac{5}{2}e^{-5} + \tfrac{9}{2}e^{-9} - \tfrac{1}{2}e^{-5} + \tfrac{1}{2}e^{-9}$$

$$\boxed{= 3/2\, e^{-5} + 1/2\, e^{-9}}$$

3. (10 points) Consider the improper integral $\displaystyle\int_1^\infty \frac{\tan^{-1}(x)}{x^2}\,dx$.

Evaluate the integral (if it converges) or explain carefully why it does not converge.

If it converges, give your answer in exact form.

$$\int_1^\infty \frac{\tan^{-1}x}{x^2}\,dx = \lim_{a\to\infty}\int_1^a \frac{\tan^{-1}x}{x^2}\,dx$$

* Use integration by parts:

$$u = \tan^{-1}x \qquad dv = \frac{1}{x^2}$$

$$du = \frac{1}{1+x^2}dx \qquad v = \frac{-1}{x}$$

$$\therefore \lim_{a\to\infty}\int_1^a \frac{\tan^{-1}x}{x^2}\,dx = \underbrace{\lim_{a\to\infty} -\frac{\tan^{-1}x}{x}\Big|_1^a}_{①} + \underbrace{\int_1^a \frac{1}{x(1+x^2)}dx}_{②}$$

For ①, recall $\lim_{x\to\infty}\tan^{-1}(x) = \pi/2$, therefore

$$\lim_{a\to\infty}\frac{\tan^{-1}(a)}{a} = \frac{\pi/2}{\infty} = 0 \quad \text{Thus} \quad ① = 0 + \frac{\tan^{-1}(1)}{1} = \pi/4$$

For ②, Use partial fractions

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$$

$$\Rightarrow 1 = A(1+x^2) + Bx^2 + Cx$$

$x=0 \quad 1 = A$

$C = 0$

$B = -A = -1$

$$\therefore ② = \lim_{a\to\infty}\int_1^a \frac{1}{x} + \lim_{a\to\infty}\int_1^a \frac{-x}{1+x^2} \qquad \begin{array}{l} u = 1+x^2 \\ du = 2x\,dx \end{array}$$

$$= \lim_{a\to\infty} \ln|x|\,\Big|_1^a + \lim_{a\to\infty} -\frac{1}{2}\int_2^a \frac{1}{u}$$

$$= \lim_{a\to\infty} \ln|x|\,\Big|_1^a - \frac{1}{2}\ln|1+x^2|\,\Big|_1^a$$

$$= \lim_{a\to\infty} \ln\left(\frac{x}{\sqrt{1+x^2}}\right)\Big|_1^a$$

*Note that $\displaystyle\lim_{x\to\infty}\frac{x}{\sqrt{1+x^2}} = 1$

$$\therefore \lim_{a\to\infty}\ln\left(\frac{a}{\sqrt{1+a^2}}\right) - \ln\left(\frac{1}{\sqrt{2}}\right)$$

$$= \ln(1) - \ln\left(\frac{1}{\sqrt{2}}\right)$$

$$= -\ln\left(\frac{1}{\sqrt{2}}\right)$$

$$= \tfrac{1}{2}\ln(2)$$

Putting ① and ② together, we have that

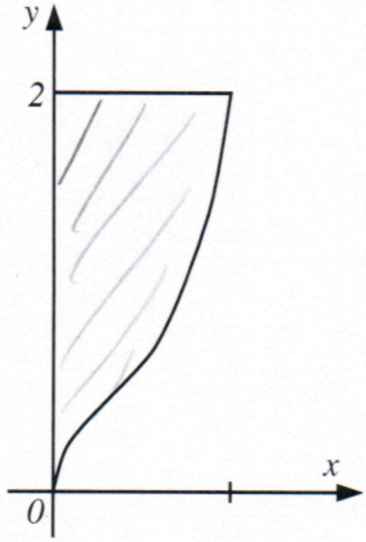
$$\boxed{\int_1^\infty \frac{\tan^{-1}x}{x^2} = \frac{\pi}{4} + \frac{1}{2}\ln(2)}$$

4. (8 points) Let $h(x) = \int_{\cos x}^{\ln x} \frac{e^{2t}}{\sqrt{t^2+1}}\,dt$. Find the derivative $h'(x)$.

Using Fundamental Thm of Calculus,

$$\boxed{h'(x) = \frac{\frac{1}{x} \cdot x^2 \cdot}{\sqrt{(\ln(x))^2+1}} + \frac{\sin(x)\, e^{2\cos(x)}}{\sqrt{\cos^2 x + 1}}}$$

5. (10 points) Find the area of the region in the first quadrant bounded on top by the line $y = 2$, on the left by the y-axis, and on the bottom by the curve $y = \sqrt{x/(1-x)}$. Give your answer in exact form.



① First find the bounds

$$2 = \sqrt{x/1-x}$$
$$4 = \frac{x}{1-x}$$
$$4 - 4x = x$$
$$4 = 5x$$
$$x = 4/5$$

② Set up integral

$$\int_0^{4/5} 2 - \sqrt{\frac{x}{1-x}}\, dx$$

$$= \int_0^{4/5} 2 - \int_0^{4/5} \frac{\sqrt{x}}{\sqrt{1-x}}$$

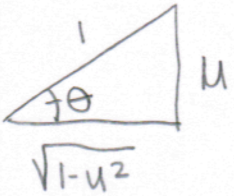
Let $u = \sqrt{1-x}$
$u^2 = 1-x$
$-2u\,du = -dx$

$$= 2x\Big|_0^{4/5} + \int_1^{\sqrt{1/5}} \frac{\sqrt{1-u^2}}{\cancel{u}} \cdot 2\cancel{u}\,du$$

$$= 8/5 + 2\int \cos^2\theta\, d\theta$$

$u = \sin\theta$
$du = \cos\theta\, d\theta$

$\sin\theta = o/h = u$



$$= 8/5 + \int \cancel{2}\left(\frac{1+\cos(2\theta)}{\cancel{2}}\right)$$

$$= 8/5 + \theta + \tfrac{1}{2}\sin(2\theta)$$

$$= 8/5 + \sin^{-1}(u) + \sin\theta\cos\theta$$

$$= 8/5 + \sin^{-1}(u) + u\sqrt{1-u^2}\Big|_1^{\sqrt{1/5}}$$

$$= 8/5 + \sin^{-1}(\sqrt{1/5}) + \sqrt{\tfrac{1}{5}}\sqrt{1 - 1/5} - \frac{\pi}{2}$$

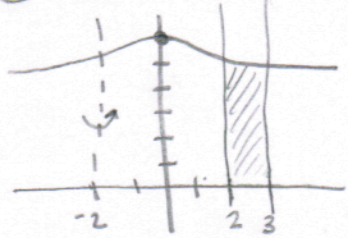
$$\boxed{\approx .89285}$$

6. (10 points) Let $\mathscr{R}$ be the region bounded on top by the curve

$$y = 5 + \frac{1}{1+x^2},$$

on the bottom by the x-axis, on the left by the line $x = 2$, and on the right by the line $x = 3$. Find the volume of the solid obtained by rotating $\mathscr{R}$ about the vertical line $x = -2$. Give your answer in exact form.

① Draw Picture



② Shells OR discs/washers?

Since $y = 5 + \frac{1}{1+x^2}$ and rotating about vertical line use shells.

③ Height $= 5 + \frac{1}{1+x^2}$, Radius $2+x$

$\therefore \int_2^3 2\pi(2+x)\left(5+\frac{1}{1+x^2}\right)dx$

$= \int_2^3 20\pi + \int_2^3 10\pi x + \int_2^3 \frac{4\pi}{1+x^2} + \int_2^3 \frac{2\pi x}{1+x^2}$ $\quad u = 1+x^2$, $du = 2x\,dx$

$= 20\pi x\Big|_2^3 + 5\pi x^2\Big|_2^3 + 4\pi\tan^{-1}(x)\Big|_2^3 + \int_5^{10}\frac{\pi}{u}du$

$= 20\pi + 25\pi + 4\pi\tan^{-1}(3) - 4\pi\tan^{-1}(2) + \pi\ln|u|\Big|_5^{10}$

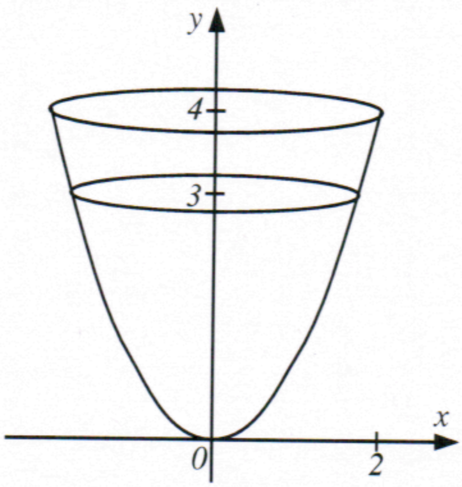
$\boxed{= 45\pi + 4\pi(\tan^{-1}(3) - \tan^{-1}(2)) + \pi\ln(10/5)}$

7. (10 points) The portion of the graph $y = x^2$ between $x = 0$ and $x = 2$ is rotated around the y-axis to form a container. The container is partially filled with water, up to the level $y = 3$.

Find the work required to pump all of the water out over the top of the side of the container.

Give your answer (in joules = $\text{kg} \cdot \text{m/s}^2 \cdot \text{m}$) in exact form.

(Distance is measured in meters, the density of water is 1000 kg/m^3, and use 9.8 m/s^2 for the acceleration g due to gravity.)



① Volume of one slice

$\pi r^2 dy = \pi y\, dy \quad y = x^2 \Rightarrow \sqrt{y} = x = \text{radius}$

② Force:

Mass $= 1000 \cdot \pi y\, dy$

Force $= 9.8 \times 1000 \cdot \pi y\, dy$

③ Distance travelled

$4 - y$

$\therefore \int_0^3 9.8(1000)\pi y(4-y)\, dy$

$= \int_0^3 9800\pi(4y - y^2)\, dy$

$= 19{,}600\pi y^2 - \frac{9800\pi}{3} y^3 \Big|_0^3$

$\boxed{= 88{,}200\pi \text{ joules}}$

8. (8 total points) Consider the curve $y = x + \frac{2}{3}x^{3/2}$.

(a) (4 points) Set up a definite integral for the arc length of this curve for $1 \le x \le 3$.
DO NOT EVALUATE THE INTEGRAL.

Recall arc length $= \int_a^b \sqrt{1+(dy/dx)^2}\, dx$

$\frac{dy}{dx} = 1 + \sqrt{x}$

$$\boxed{\therefore \text{ arc length} = \int_1^3 \sqrt{1+(1+\sqrt{x})^2}\, dx}$$

(b) (4 points) Use Simpson's rule with $n = 4$ subintervals to estimate the integral in part (a). Give your answer in decimal form, correct to at least the third digit after the decimal point.

$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$

Simpson's Rule is

$\frac{1/2}{3}\left[f(1) + 4f(\tfrac{3}{2}) + 2f(2) + 4f(\tfrac{5}{2}) + f(3)\right]$

$= \frac{1}{6}\left[\sqrt{5} + 4\sqrt{1+(1+\sqrt{\tfrac{3}{2}})^2} + 2\sqrt{1+(1+\sqrt{2})^2} + 4\sqrt{1+(1+\sqrt{\tfrac{5}{2}})^2} + \sqrt{1+(1+\sqrt{3})^2}\right)$

$\boxed{= 5.200}$

9. (10 points) Find the solution of the differential equation

$$\frac{dy}{dx} = \frac{2\sqrt{4-y^2}}{ye^x}$$

that satisfies the condition $y(0) = \sqrt{3}$. Solve for y, giving your answer in the form $y = f(x)$.

① Separate variables and solve

$$\frac{y\,dy}{2\sqrt{4-y^2}} = e^{-x}dx \Rightarrow \int \frac{y\,dy}{2\sqrt{4-y^2}} = \int e^{-x}dx$$

$u = 4-y^2 \quad du = -2y\,dy$

$$\Rightarrow \int -\frac{1}{4}u^{-1/2} = -e^{-x} + C$$

$$\Rightarrow -\frac{1}{2}u^{1/2} = -e^{-x} + C$$

$$\Rightarrow -\frac{1}{2}\sqrt{4-y^2} = -e^{-x} + C$$

$$\Rightarrow \sqrt{4-y^2} = 2e^{-x} + C$$

$$\Rightarrow 4 - y^2 = (2e^{-x} + C)^2$$

$$\Rightarrow 4 - (2e^{-x}+C)^2 = y^2$$

$$\Rightarrow \sqrt{4-(2e^{-x}+C)^2} = y$$

② Plug in initial conditions:

$$\sqrt{3} = \sqrt{4-(2e^0+C)^2}$$

$$\Rightarrow 3 = 4-(2+C)^2$$

$$1 = (2+C)^2$$

$$C = -1$$

$$\therefore y = \sqrt{4-(2e^{-x}-1)^2}$$

$$= \sqrt{4-4e^{-2x}+4e^{-x}-1}$$

$$\boxed{= \sqrt{3-4e^{-2x}+4e^{-x}}}$$

10. (10 total points) An object of mass m kg is dropped out of a airplane, and we assume that air resistance is proportional to the speed of the object. Let $s(t)$ be the distance dropped (in meters, positive pointing *down*) after t seconds, and let $v(t) = ds/dt$ be the velocity and $a(t) = dv/dt$ be the acceleration. The combined downward force on the object is

$$F = mg - kv,$$

where $g = 9.8$ meters/sec^2 is the acceleration due to gravity and k is a positive constant. By Newton's Second Law of motion,

$$F = ma = m\frac{dv}{dt}.$$

The mass of the object is $m = 10$ kg, and the constant k is $k = 2$.

(a) (3 points) Set up a differential equation for the velocity $v(t)$.

By above, $mg - kv = m\,dv/dt$

$\Rightarrow 10 \cdot 9.8 - 2v = 10\,dv/dt$

$\boxed{\Rightarrow 9.8 - v/5 = dv/dt}$

(b) (5 points) Solve the differential equation to obtain a formula for $v(t)$.

① Separate the variables and solve

$$dt = \frac{dv}{9.8 - 1/5\,v} \Rightarrow \int dt = \int \frac{dv}{9.8 - \frac{1}{5}v}$$

$$t + C = -5\ln|9.8 - 1/5\,v|$$

$$\frac{-t}{5} + C = \ln|9.8 - 1/5\,v|$$

$$Ce^{-t/5} = 9.8 - 1/5\,v$$

$$1/5\,v = 9.8 - Ce^{-t/5}$$

$$v = 49 - Ce^{-t/5}$$

② Solve initial value (Since dropped $v = 0$ at $t = 0$)

$0 = 49 - C \Rightarrow C = 49$ $\boxed{v = 49 - 49e^{-t/5}}$

(c) (2 points) What is the limiting velocity $\lim_{t\to\infty} v(t)$?

Since $\lim_{t\to\infty} e^{-t/5} = 0$, then $\lim_{t\to\infty} 49 - 49e^{-t/5} = \boxed{49}$