Week 2: Questions Taken From Past Midterms

(1) If $\int_2^4 (f(x)+g(x))\,dx = 11$, $\int_{-2}^3 f(x^2)x\,dx = 13$, and $\int_2^4 (3g(x)+2)\,dx = 7$,

$$\text{Find} \quad \int_2^9 f(x)\,dx.$$

Solution

(a) $\int_{-2}^3 xf(x^2)dx = 13$
Use u-substitution with $u = x^2$ and $\frac{du}{2x} = dx$. Hence,

$$\int_{-2}^3 xf(x^2)dx = \frac{1}{2}\int_4^9 f(u)du = 13$$
$$\int_4^9 f(u)du = 26$$

(b) $\int_2^4 (3g(x)+2)\,dx = 7$

$$\int_2^4 (3g(x)+2)dx = \int_2^4 3g(x)dx + \int_2^4 2dx = 7$$
$$\int_2^4 3g(x)dx + 2x|_2^4 = 7$$
$$\int_2^4 3g(x)dx + 2(4-2) = 7$$
$$\int_2^4 3g(x)dx = 3$$
$$\int_2^4 g(x)dx = 1$$

(c) $\int_2^4 (f(x)+g(x))dx = 11$

$$\int_2^4 (f(x)+g(x))dx = \int_2^4 f(x)dx + \int_2^4 g(x)dx = 11$$

Since above $\int_2^4 g(x)dx = 1$, then

$$\int_2^4 f(x)dx + \int_2^4 g(x)dx = 11$$
$$\int_2^4 f(x)dx + 1 = 11$$
$$\int_2^4 f(x)dx = 10$$

To Compute $\int_2^9 f(x)dx$.

$$\begin{aligned}\int_2^9 f(x)dx &= \int_2^4 f(x)dx + \int_4^9 f(x)dx \\ &= 10 + 26 \\ &= 36\end{aligned}$$

(2) Compute the following integrals

(a) $\int \frac{arctanx}{x^2+1}\, dx$

Solution
Use u-substitution with $u = tan^{-1}(x)$, $du(1+x^2) = dx$. Hence

$$\begin{aligned}\int \frac{tan^{-1}(x)}{1+x^2} dx &= \int u du \\ &= \frac{1}{2}u^2 + C \\ &= \frac{1}{2}(tan^{-1}(x))^2 + C\end{aligned}$$

(b) $\int x \sqrt[3]{x+2}\, dx$

Solution
Use u-substitution,

$$\begin{aligned}u &= \sqrt[3]{x+2} \\ u^3 &= x+2 \\ 3u^2 du &= dx\end{aligned}$$

So the integral we get is

$$\int x \sqrt[3]{x+2}\, dx = \int xu(3u^2)du$$

However we still have an x. Note that $u^3 = x+2$ so $u^3 - 2 = x$.

$$\begin{aligned}\int xu(3u^2)du &= \int (u^3-2)u(3u^2)du \\ &= \int 3u^6 - 6u^3 du \\ &= \frac{3}{7}u^7 - \frac{6}{4}u^4 + C \\ &= \frac{3}{7}(\sqrt[3]{x+2})^7 - \frac{6}{4}(\sqrt[3]{x+2})^4 + C\end{aligned}$$

(c) $\int_0^{\pi/2} \sin(2x)\, dx$
Solution
Use u-substitution $u = 2x$, $du = 2dx \implies \frac{du}{2} = dx$. Note that you should change your bounds $\frac{\pi}{2}$ and 0 when you switch to u.

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin(2x)dx &= \int_0^{\pi} \frac{\sin(u)}{2} du \\ &= \frac{-\cos(u)}{2}|_0^{\pi} \\ &= -\frac{cos(\pi)}{2} + \frac{\cos(0)}{2} \\ &= 1\end{aligned}$$

(d) $\int_{\pi/6}^{\pi/3} \frac{\sin(3x)}{2+cos(3x)}\, dx$

Solution

Let $u = 2 + \cos(3x)$ so $du = -\sin(3x) * 3dx \implies \frac{du}{-3\sin(x)} = dx$. Also $u(\frac{\pi}{3}) = 2 + \cos(\pi) = 1$ and $u(\frac{\pi}{6}) = 2$ Substituting this is in yields,

(e) $\int \frac{e^{2\sqrt{x}}}{\sqrt{x}}\, dx$

Solution

We see that what makes the integral "hard" to integrate is the $2\sqrt{x}$ in the exponent. Let's make that our u for u-substitution.

Let $u = 2\sqrt{x}$ then $u^2 = 4x$ so $2udu = 4dx$. Substituting this in, we get:

$$\begin{aligned}\int \frac{e^{2\sqrt{x}}}{\sqrt{x}} dx &= \int \frac{e^u}{u}\frac{u}{2} du \\ &= \int \frac{e^u}{2} du \\ &= \frac{1}{2}e^u + C \\ &= \frac{1}{2}e^{2\sqrt{x}} + C\end{aligned}$$

(f) $\int (\sin x)^3\, dx$

Solution

Recall that

$$\sin(x)^2 = 1 - \cos(x)^2.$$

$$\begin{aligned}\int (\sin x)^3\, dx &= \int \sin(x)\sin(x)^2 dx \\ &= \int \sin(x)(1 - \cos(x)^2) dx \\ &= \int \sin(x) - \sin(x)\cos(x)^2 dx \\ &= \int \sin(x) dx - \int \sin(x)\cos(x)^2 dx\end{aligned}$$

We see that the first integral $\int \sin(x)dx = -cos(x) + C$. Now we turn our attention to the second integral. Use a u-substitution for the second integral with $u = \cos(x)$ and $du = -sin(x)dx$. Hence we get

$$\begin{aligned}\int \sin(x)\cos(x)^2 dx &= \int -u^2 du \\ &= \frac{-1}{3}u^3 + C \\ &= \frac{-1}{3}(\cos(x))^3 \text{ must convert back to } x \text{ because the integral has no bounds}\end{aligned}$$

Hence,

$$\int \sin(x)dx - \int \sin(x)\cos(x)^2 dx = -\cos(x) - \frac{-1}{3}(\cos(x))^3 + C$$

(3) Let $f(x) = \int_0^{\sqrt{2x+7}} \frac{dt}{t^4+9}$. Compute $f'(x)$.
Solution

a. Make it in the form $\int_c^{h(x)}$ where $h(x)$ is a function and c is a constant.

$$\int_0^{\sqrt{2x+7}} \frac{dt}{t^4+9}$$

is already in this form

b. Plug in $h(x)$ into the function you are integrating and get rid of integral then multiply by $h'(x)$

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{2x+7}^4 + 9} * \frac{d}{dx}\sqrt{2x+7} \\ &= \frac{1}{(\sqrt{2x+7})^4 + 9} * \frac{1}{\sqrt{2x+7}} \end{aligned}$$

(4) Pete is driving his car along a straight street. He starts at his work place and needs to deliver a packet to a customer. Not knowing the neighborhood too well he starts going in the wrong direction, but realizes his mistake soon. The velocity of his car is given by $v(t) = 90t^2 - 50t$ in mi/hour where t is measured in hours.

(a) Pete reaches his destination after one hour. How fair away does the customer live from Pete's work place?
Solution
Find the equations for $v(t)$ and $s(t)$

$$v(t) = 90t^2 - 50t$$
$$s(t) = 30t^3 - 25t^2$$

Plug in $t = 1$ to get the distance away

$$s(1) = 30(1^3) - 25(1^2) = 5 \text{ miles}$$

(b) Pete's car is quite friendly to the environment, it can drive 35 miles per gallon fuel. How much fuel did Pete use up for this journey?
Solution
Note that we need to find the **Total Distance** Pete traveled not just his position. Thus, we need $\int |v(t)|$. We see that

$$v(t) = 0$$

when $t = 0$ or $t = \frac{5}{9}$ and by testing the point $t = 1$ we know $v(t) > 0$. Hence it $v(t) < 0$ between $[0, \frac{5}{9}]$. This means we have the following integrals:

$$\begin{aligned}
\text{Total Distance } &= \int_0^{\frac{5}{9}} 50t - 90t^2 dt + \int_{\frac{5}{9}}^1 90t^2 - 50t dt \\
&= 25t^2 - 30t^3 \Big|_0^{\frac{5}{9}} + 30t^3 - 25t^2 \Big|_{\frac{5}{9}}^1 \\
&= \frac{625}{243} + \frac{1840}{243} \\
&= \frac{2,465}{243} \text{ miles}
\end{aligned}$$

To find the gallons he used, divide by the miles per gallon to get:

$$\frac{2,465}{243} \text{ miles} * \frac{1 \text{ gallon}}{35 \text{ miles}} = \frac{493}{170} \approx .2898 \text{ gallons}$$

(5) For $F(x) = \int_{\sqrt{x}}^1 \tan^{-1}(u)\, du$, find $F(1)$ and $F'(1)$.

Solution

a. Find $F(1)$

$$\begin{aligned}
F(1) &= \int_{\sqrt{1}}^1 \tan^{-1}(u)\, du \\
&= 0
\end{aligned}$$

This is because an integral from $[a, a]$ where a is the same number, means you are taking the area from a to a which doesn't have a length. Thus it is zero.

b. Find $F'(1)$.

Since it is finding a derivative, apply the fundamental theorem of calculus. Recall the steps for doing this,

i. Set it up so that $\int_c^{f(x)} g(t)\, dt$. Get the function in the bound to be the top bound.

$$\int_{\sqrt{x}}^1 \tan^{-1}(u) du = -\int_1^{\sqrt{x}} \tan^{-1}(u) du$$

ii. Plug in $f(x)$ into the function for u, getting rid of the integral sign, and then times by the derivative

$$\begin{aligned}
F'(u) &= \tan^{-1}(\sqrt{x}) * \frac{d}{dx}\sqrt{x} \\
&= \tan^{-1}(\sqrt{x}) * \frac{1}{2}\frac{1}{\sqrt{x}}
\end{aligned}$$

iii. Now plug in $F'(1)$ to get

$$F'(1) = \tan^{-1}(\sqrt{1}) * \frac{1}{2}\frac{1}{\sqrt{1}} = \frac{\pi}{8}$$

Additional Practice Questions From Another Calculus Book

(1) Evaluate the integral:

(a) $\int_0^5 |x^2 - 4x + 3|\, dx$

Solution

Note that

$$|x^2 - 4x + 3| = \begin{cases} x^2 - 4x + 3 & \text{if } x \leq 2 \text{ and } x \geq 3 \\ -x^2 + 4x - 3 & \text{if } 2 \leq x \leq 3 \end{cases}$$

Hence we get,

$$\int_0^2 x^2 - 4x + 3dx + \int_2^3 -x^2 + 4x - 3dx + \int_3^5 x^2 - 4x + 3dx$$

$$\frac{1}{3}x^3 - 2x^2 + 3x\Big|_0^2 - \frac{1}{3}x^3 + 2x^2 - 3x\Big|_2^3 + \frac{1}{3}x^3 - 2x^2 + 3x\Big|_3^5$$

$$\frac{2}{3} + \frac{2}{3} + \frac{20}{3} = 8$$

(b) $\int_0^\pi |\cos(x)|\, dx$

Solution

$$\int_0^\pi |\cos(x)|\, dx = \int_0^{\frac{\pi}{2}} \cos(x)dx + \int_{\frac{\pi}{2}}^{\pi} -\cos(x)dx$$

$$= \sin(x)\Big|_0^{\frac{\pi}{2}} - \sin(x)\Big|_{\frac{\pi}{2}}^{\pi}$$

$$= 1 + 1 \qquad = 2$$

(2) Calculate the Derivative

(a) $\frac{d}{du}\int_{-u}^{3u} \sqrt{x^2+1}\, dx$

Solution

i. Get in the form $\int_c^{f(x)}$

$$\int_{-u}^{3u} \sqrt{x^2+1}\, dx = \int_0^{3u} \sqrt{x^2+1}\, dx - \int_0^{-u} \sqrt{x^2+1}\, dx$$

ii. Plug in $f(x)$ and multiply by the derivative

$$\frac{d}{du} = \sqrt{(3u)^2+1} * 3 - (-1)\sqrt{(-u)^2+1}$$

(b) $\frac{d}{dx}\int_{x^2}^{x^4}\sqrt{t}\,dt$
Solution

i. Get in the form $\int_c^{f(x)}$

$$\int_{x^2}^{x^4}\sqrt{t}\,dt = \int_0^{x^4}\sqrt{t}\,dt - \int_0^{x^2}\sqrt{t}\,dt$$

ii. Plug in and multiply by the derivative

$$\boxed{\sqrt{x^4} * 4x^3 - \sqrt{x^2} * 2x}$$

(3) Let $f(x) = x^2 - 5x - 6$ and $F(x) = \int_0^x f(t)\,dt$

(a) Find the critical points of $F(x)$ and determine whether they are local minima or local maxima.

(b) Find the points of inflection of $F(x)$ and determine whether the concavity changes from up to down or down to up.

(4) Calculate the sums:

(a) $\sum_{j=3}^{4}\sin\left(j\frac{\pi}{2}\right)$

Solution

$$\boxed{\sum_{j=3}^{4}\sin\left(j\frac{\pi}{2}\right) = \sin\left(3 * \frac{\pi}{2}\right) + \sin\left(4 * \frac{\pi}{2}\right)}$$

(b) $\sum_{j=0}^{2} 3^{j-1}$
Solution

$$\boxed{\sum_{j=0}^{2} 3^{j-1} = 3^{0-1} + 3^{1-1} + 3^{2-1}}$$

(c) $\sum_{k=3}^{5}\frac{1}{k-1}$
Solution

$$\boxed{\sum_{k=3}^{5}\frac{1}{k-1} = \frac{1}{3-1} + \frac{1}{4-1} + \frac{1}{5-1}}$$

(5) Describe the area represented by the limits

(a) $\lim_{N\to\infty}\frac{5}{N}\sum_{j=1}^{N} e^{-2+\frac{5j}{N}}$
Solution
Recall that

$$\lim_{N\to\infty}\sum_{j=1}^{N} f(a + i\Delta x)\Delta x$$

We need to find $f(x)$ and $[a, b]$

i. Guess the $f(x)$ and Δx. This point you just have to have a feel for what is the function and what is Δx. In this case,

$$f(x) = e^x, \text{ and } \Delta x = \frac{5}{N}.$$

ii. Find a by seeing that it is $f(a + i\Delta x)$.

$$\begin{aligned} a + i\Delta x &= -2 + \frac{5j}{N} \\ a + j\frac{5}{N} &= -2 + \frac{5j}{N} \\ a &= -2 \end{aligned}$$

iii. Now find b, by noting that $\Delta x = \frac{b-a}{n}$. Hence we get,

$$\begin{aligned} \frac{b-a}{n} &= \frac{5}{n} \\ \frac{b+2}{n} &= \frac{5}{n} \\ b+2 &= 5 \\ b &= 3 \end{aligned}$$

$$\boxed{\lim_{N\to\infty} \frac{5}{N} \sum_{j=1}^{N} e^{-2+\frac{5j}{N}} = \int_{-2}^{3} e^x dx}$$

(b) $\lim_{N\to\infty} \frac{3}{N} \sum_{j=1}^{N} \left(2 + \frac{3j}{N}\right)^4$

Solution

Like the above problem.

i. Guess that $f(x) = x^4$ and $\Delta x = \frac{3}{N}$

ii. Use $a + i\Delta x$ to find a

$$\begin{aligned} a + j\Delta x &= 2 + \frac{3j}{N} \\ a + j\frac{3}{N} &= 2 + \frac{3j}{N} \\ a &= 2 \end{aligned}$$

iii. Find b by using $\Delta x = \frac{b-a}{n}$.

$$\begin{aligned} \frac{b-a}{N} &= \frac{3}{N} \\ \frac{b-2}{N} &= \frac{3}{N} \\ b-2 &= 3 \\ b &= 5 \end{aligned}$$

$$\boxed{\lim_{N\to\infty} \frac{3}{N} \sum_{j=1}^{N} \left(2 + \frac{3j}{N}\right)^4 = \int_{2}^{5} x^4 dx}$$

(6) Calculate the limit for the given function and interval.

(a) $f(x) = 9x$, $[0, 2]$

Solution

Use the fact that

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(a + i\Delta x)\Delta x.$$

We need to find Δx and $a + i\Delta x$.

i. Find Δx

$$\begin{aligned} \Delta x &= \frac{b-a}{n} \\ &= \frac{2-0}{n} \\ &= \frac{2}{n} \end{aligned}$$

ii. Find $a + i\Delta x$.

$$a + i\Delta x = 0 + \frac{2i}{n}$$

iii. Plug in $a + i\Delta x$ into $f(x)$ hence getting

$$\boxed{\lim_{n\to\infty} \sum_{i=1}^{n} 9\left(\frac{2i}{n}\right)\frac{2}{n}}$$

(b) $f(x) = \frac{1}{2}x + 2$, $[1, 4]$

Solution

Use the fact that

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(a + i\Delta x)\Delta x.$$

We need to find Δx and $a + i\Delta x$.

i. Find Δx

$$\begin{aligned} \Delta x &= \frac{b-a}{n} \\ &= \frac{4-1}{n} \\ &= \frac{3}{n} \end{aligned}$$

ii. Find $a + i\Delta x$

$$a + i\Delta x = 1 + \frac{3i}{n}$$

iii. Plug in $a + i\Delta x$ into $f(x)$ hence getting

$$\boxed{\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{2}\left(1 + \frac{3i}{n}\right) + 2\right)\frac{3}{n}}$$