9. (10 points) A tank initially contains 1000 L of pure water. Brine that contains 0.07 kg of salt per liter of water enters the tank at a rate of 5 L/min. In addition, brine that contains 0.04 kg of salt per liter of water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 15 L/min. Find the amount of salt (in kg) in the tank as a function of time t in minutes.

Want to Use Rate in - Rate out

① Identify what you are trying to solve

$Q(t)$ = amt of salt (in kg) in the tank

② Set-Up the DE

$$\underbrace{\frac{dQ}{dt}}_{\text{rate salt is changing in the tank}} = \text{Rate in} - \text{Rate out}$$

(units kg/min)

ⓐ Rate in:

source 1 $\begin{cases} 0.07\ \text{kg/lit} = \text{concentration of salt coming in} \\ 5\ \text{L/min} = \text{rate sol. is coming in} \end{cases}$

source 2 $\begin{cases} 0.04\ \text{kg/lit} = \text{concentration of salt coming in} \\ 10\ \text{L/min} = \text{rate sol. is coming in} \end{cases}$

$$\text{Rate in} = \underbrace{0.07\ \text{kg/L} \cdot 5\ \text{L/min}}_{\text{source 1}} + \underbrace{0.04\ \tfrac{\text{kg}}{\text{L}} \times 10\ \text{L/min}}_{\text{source 2}}$$

$$= 0.35\ \text{kg/min} + 0.4\ \text{kg/min}$$

$$\boxed{= 0.75\ \text{kg/min}}$$

ⓑ Rate out:

solution comes out at 15L/min

What is its concentration of salt?

(i.e. How much of the 15L is salt)

$$\text{Concentration} = \frac{\text{amt of salt}}{\text{total Volume}} = \frac{Q(t)}{1000}\ \frac{\text{kg}}{\text{L}}$$

↑ initial Vol

$$\text{Rate out} = 15\ \text{L/min} \cdot \frac{Q}{1000} = \boxed{0.015\,Q}$$

$$\therefore \boxed{\frac{dQ}{dt} = 0.75 - 0.015Q}$$

③ Solve the DE

$$\int \frac{dQ}{0.75 - 0.015Q} = \int dt$$

$$\Rightarrow \frac{-1}{0.015} \ln|0.75 - 0.015Q| = t + C$$

$$\Rightarrow \ln|0.75 - 0.015Q| = -0.015t + C$$

$$\Rightarrow |0.75 - 0.015Q| = Ce^{-0.015t}$$

$$\Rightarrow 0.75 - Ce^{-0.015t} = 0.015Q$$

$$\Rightarrow \boxed{50 - Ce^{-0.015t} = Q}$$

$Q(0) = 0$ ← pure water

$$\Rightarrow 50 - C = 0 \Rightarrow \boxed{C = 50}$$

$$\Rightarrow \boxed{50 - 50e^{-0.015t} = Q(t)}$$

10. (12 total points) A lake has a volume of 1000 cubic meters and contains pure water. A farm near the lake begins using a pesticide. Runoff from the farmland into the lake is 10 cubic meters of water per day, with a concentration of 50 grams of pesticide per cubic meter of water.

The lake drains to the ocean at a rate of 10 cubic meters per day. You may assume that the pesticide mixes thoroughly with the water in the lake, and you should ignore other effects such as evaporation.

(a) (6 points) Let $y(t)$ denote the total amount of pesticide (in grams) in the lake after t days. Set up a differential equation for $y(t)$.

$$\frac{dy}{dt} = \text{Rate in} - \text{Rate out}$$

Rate in: Concentration * Rate of water coming in $= 50g/m^3 \cdot \frac{10\ m^3}{day} = \frac{500g}{day}$

Rate out: Concentration $= \frac{\text{amt of pesticides}}{\text{Total Volume}} = \frac{y}{1000} \leftarrow$ Observe Volume stays constant as 10 m^3/day goes in and out.

$\therefore$ Rate out $= \frac{10 \cdot y}{1000} = y/100$

$$\boxed{\frac{dy}{dt} = 500 - y/100}$$

(b) (6 points) Fish can survive a maximum concentration of 1 gram of pesticide per cubic meter of water. Solve the differential equation you found in part (a) and determine whether the fish will be alive after 10 days.

Separate the variables

$$\frac{dy}{500 - y/100} = dt$$

Integrate both sides

$$\int \frac{dy}{500 - y/100} = \int dt$$

$$-100 \ln|500 - y/100| = t + C$$

$$\ln|500 - y/100| = -t/100 + C$$

$$|500 - y/100| = Ce^{-t/100}$$

$$500 - y/100 = Ce^{-t/100}$$

$$\boxed{50{,}000 - Ce^{-t/100} = y(t)}$$

$y(0) = 0 \leftarrow$ pure water

$50000 = C$

$$\boxed{y(t) = 50{,}000 - 50{,}000e^{-t/100}}$$

$$y(10) = 50{,}000 - 50{,}000e^{-1/10} \approx 4{,}758.129$$

Concentration at $t=10$ $= \frac{y(10)}{1000} \approx 4.758 \frac{g}{m^3}$

As, $4.758 > 1 \Rightarrow$ $\boxed{\text{Fish dies}}$

10. (10 total points) You would like to be a multimillonaire in 30 years. You might win the lottery, or you can start investing. This problem is about investing.
Let $A(t)$ be the amount of money (in dollars) you have in your investment account at time t (in *years*). Let M be the amount (in dollars) that you deposit every *month*, so $12M$ is the amount (in dollars) that you deposit every *year*. The rate of change of the amount A in your account has two parts: the interest and your deposits. The part of the rate of change coming from the interest is proportional to the amount in your account, and the annual interest rate is 10%. Interest is compounded continuously, and assume that your deposits are also applied continuously.

(a) (4 points) Set up a differential equation for the rate of change of A with respect to time, taking into account both the interest you earn and the deposits you make. Be careful with units.

Interest earned $= .1A$

Deposit $= 12M$

$$\therefore \boxed{\frac{dA}{dt} = 12M + .1A}$$

(b) (4 points) You start with $A = 0$ at time $t = 0$. Solve the differential equation above. Your solution will involve M.

$$\int \frac{dA}{12M + .1A} = \int dt$$

$$10 \ln|12M + .1A| = t + C$$

$$\ln|12M + .1A| = t/10 + C$$

$$|12M + .1A| = Ce^{t/10}$$

$$12M + .1A = Ce^{t/10}$$

$$A(t) = Ce^{t/10} - 120M$$

$$0 = A(0) = Ce^{0} - 120M$$

$$C = 120M$$

$$\therefore \boxed{A(t) = 120Me^{t/10} - 120M}$$

(c) (2 points) Determine the value of M, the amount that you have to deposit every month, if you want to have 5 million dollars in your account after 30 years.

$$5{,}000{,}000 = 120Me^{30/10} - 120M = M(120e^{3} - 120)$$

$$\Rightarrow \boxed{M = \frac{5{,}000{,}000}{120e^{3} - 120} \approx \$2{,}183.15}$$

10. (12 total points) A tank contains 100 liters of water which has 2000 grams of salt dissolved in it. At noon, pure water begins to enter the tank at the rate of 10 liters per minute. The tank is kept thoroughly mixed, and the mixture leaves the tank at the rate of 15 liters per minute.

(a) (2 points) Express the volume of saltwater in the tank as a function of t, the number of minutes after noon.

Rate Vol. comes in is 10 lit/min

Rate Vol comes out is 15 lit/min

$\Rightarrow \frac{dV}{dt}$ = Rate in - Rate out

$= 10 - 15 = -5$ lit/min

$\therefore$ Volume $= V_0 - 5t$

$= \boxed{100 - 5t}$

(b) (4 points) Let $S(t)$ be the amount of salt in grams in the tank at time t. Set up a differential equation describing $\frac{dS}{dt}$.

- Rate in = 0 b/c pure water entering
  No salt
- Rate out
  - Concentration: $\frac{\text{Amt of salt at } t}{\text{Total Vol at } t} = \frac{S(t)}{100-5t}$
  - Rate of vol = 15 lit/min

Rate out $= 15\left(S(t)/100-5t\right)$

$\therefore \frac{dS}{dt} = 0 - \frac{15S}{100-5t}$

(c) (4 points) Solve the differential equation for $S(t)$.

$\Rightarrow \int \frac{dS}{15S} = \int \frac{-dt}{100-5t}$

$\frac{1}{15}\ln|S| = 1/5 \ln|100-5t| + C$

$\ln|S| = 3\ln|100-5t| + C$

$= \ln|(100-5t)^3| + C$

↑ Never exceeds 100 so get rid of | |

$S = (100-5t)^3 C$

at $t=0$ $S(t) = 2000$

$S(0) = (100)^3 C = 2000$

$C = .002$

$\boxed{S(t) = .002(100-5t)^3}$

(d) (2 points) When will there be 1000 grams of salt in the tank?

Set $S(t) = 1000$ and solve for t

$1000 = .002(100-5t)^3$

$500{,}000 = (100-5t)$

$\sqrt[3]{500{,}000} = 100 - 5t$

$\frac{\sqrt[3]{500{,}000} - 100}{-5} = t \approx \boxed{4.125}$ hours after noon.

10. (10 total points) At time $t = 0$ a 5000 liter tank is full of pure water. Starting at that moment salt is added to it at the steady rate of 30 grams per hour. Assume that the salt is thoroughly mixed in the water. Meanwhile, pure water is entering the tank at 50 liters per hour and the salty water in the tank is leaving at the same rate. Let $y(t)$ be the amount of salt in grams in the water in the tank after t hours.

(a) (4 points) Find a differential equation for $y(t)$.

(b) (4 points) Solve for $y(t)$.

(c) (2 points) How much salt is in the tank in the limit as $t \to \infty$?

$\lim_{t\to\infty} 3000 - 3000e^{-t/100}$

$= \boxed{3000}$

$* \; e^{-t/100} \to 0$ as $t \to \infty$

① $Y(t)$ = amt of salt in grams in tank at time t (hours)

$dY/dt = g/h.$

② Solve Diff Q [Rate in - Rate out]

- Rate in [Amt of salt entering] = 30gr/hr
- Rate out
  - Concentration (% of salt per liter)

$$= \frac{\text{Amt of salt at } t}{\text{Total Water Volume at } t} = \frac{Y(t)}{5000}$$

Total Water Vol. at t

Rate in of Water = 50lit/hr

Rau' out = 50lit/hr

Total Rate inside Tank = 0

$\therefore$ Total Vol = 5000

- Rate sol. goes out = 50lit/hr

Rate out = $y/500\cancel{0} \cdot 5\cancel{0} = \frac{y(t)}{100}$ = Conce. * Rate sol goes out

$$\boxed{dY/dt = 30 - \frac{y(t)}{100} \quad \checkmark \text{ units}}$$

③ Solve Diff

$$\frac{dy}{30 - \frac{y(t)}{100}} = dt \qquad \int \quad -100 \ln\left|30 - \frac{y(t)}{100}\right| = t + C$$

$$\ln|30 - y(t)/100| = -t/100 + C$$

$$30 - \frac{y(t)}{100} = e^{-t/100 + C}$$

positive $= e^{-t/100} e^{C}$

$= Ce^{-t/100}$

$$30 - Ce^{-t/100} = \frac{y(t)}{100}$$

$$3000 - Ce^{-t/100}$$

④ Initial Cond. $y(0) = 0$

$C = 3000$

$$\boxed{y = 3000 - 3000e^{-t/100}}$$

10. (8 total points) Glucose is being fed intravenously to the bloodstream of a patient at 0.01 grams per minute. At the same time the patient's body converts the glucose and removes it from the bloodstream at a rate proportional to the amount of glucose present.

(a) (2 points) Let $g(t)$ be the amount of glucose in the bloodstream at time t in minutes. Let k be the proportionality constant mentioned above. Set up a differential equation for $g(t)$.

* Remember Rate in - Rate out

$$\boxed{\frac{dg}{dt} = .01 - kg}$$

Proportional means multiplied by some constant

(b) (4 points) Find the general solution to this differential equation. Show that the amount of glucose in the bloodstream always approaches $\frac{0.01}{k}$ as t becomes very large. (If it does not, you made a mistake somewhere. Go back and check your work.)

① Separate variables then solve

$$dg/dt = .01 - kg \Rightarrow \frac{dg}{.01-kg} = dt$$

$$\Rightarrow \int \frac{dg}{.01-kg} = \int dt$$

$$\Rightarrow \frac{-1}{k}\ln|.01-kg| = t + C$$

$$\ln|.01-kg| = -kt + C$$

$$.01 - kg = Ce^{-kt}$$

$$\boxed{\frac{.01 - Ce^{-kt}}{k} = g}$$

② Show that as $t \to \infty$ $g(t) \to .01/k$.

Note that $\lim_{t\to\infty} e^{-kt} \to 0$

$$\therefore \lim_{t\to\infty} \frac{.01 - Ce^{-kt}}{k} = \frac{.01}{k}$$

(c) (2 points) Suppose that there are 4.1 grams of glucose in the bloodstream at $t = 0$ and that as t becomes very large, the glucose level approaches 5.2 grams. How much glucose is in the blood one hour after starting? Give a decimal answer.

The two conditions we are given are

① $\lim_{t\to\infty} g(t) = 5.2$

② $g(0) = 4.1$

Start w/ the 1st condition: from part b)

$$\lim_{t\to\infty} g(t) = \frac{.01}{k} = 5.2 \Rightarrow k = \frac{1}{520}$$

Now we know k, we can solve for C

$$4.1 = \frac{.01 - Ce^{-0/520}}{1/520} = \frac{.01 - C}{1/520} \Rightarrow C = 11/5200$$

$$\therefore g(t) = \frac{.01 - \frac{11}{5200}e^{-t/520}}{(1/520)}$$

Hence at $t = 60$ (Note t = minutes)

$$g(60) = \frac{.01 - \frac{11}{5200}e^{-60/520}}{1/520}$$

$$\boxed{= 4.219874286\text{ g}}$$

10. (10 points) When a cake is removed from an oven, the temperature of the cake is 210° F. The cake is left to cool at room temperature (70° F.), and after 30 minutes, the temperature of the cake is 140° F. According to Newton's law of cooling, the rate of change of temperature of a body is proportional to the temperature difference between the body and the environment. Set up and solve a differential equation to determine when the temperature of the cake will be 100° F.

① $T(t)$ = the temperature of the cake in °F after t mins from removing from oven

② Set up diff

Newton's Law of cooling: rate of Δ of temp. is proportional to the temp. diff. btween cake & environment (room)

$$\frac{dT}{dt} = k\,(T(t) - T_e) = k(T-70)$$

k: portional; $T(t)$: current temp of cake; T_e: Temp of environ.

③ Solve differential

$$\int \frac{dT}{k(T-70)} = \int dt \Rightarrow \frac{1}{k}\ln|T-70| = t + C$$

$T>70$ b/c cake temp will never be below 70

drop absolute values

$$\ln(T-70) = kt + C \Rightarrow T-70 = Ce^{kt}$$

$$T = Ce^{kt} + 70$$

④ Initial Conditions

- $t=0$, Temp of cake is 210°F $\quad T(0) = 210$
- $t=30$, Temp is 140° $\quad T(30) = 140$

$$T(0) = Ce^{k(0)} + 70 = 210 \Rightarrow C + 70 = 210 \quad \boxed{C = 140}$$

$$T(30) = 140e^{k(30)} + 70 = 140 \Rightarrow e^{k(30)} = 1/2$$

$$\ln(1/2) = k(30) \Rightarrow \boxed{k = \frac{\ln(1/2)}{30} \approx -.0231049}$$

$\therefore$ $\boxed{T(t) = 140e^{\left(\frac{\ln(1/2)}{30}\right)t} + 70}$

Time when cake = 100°

$$100 = 140e^{(\ln(1/2)/30)t} + 70 \Rightarrow \frac{3}{14} = e^{\ln(1/2)/30\, t} \Rightarrow \ln\left(\frac{3}{14}\right) = \frac{\ln\left(\frac{1}{2}\right)}{30}t$$

$$\Rightarrow t = \ln(3/14)\cdot 30/\ln(1/2) \approx \boxed{66.67 \text{ mins}}$$

11. (10 total points) The population P of Springfield is increasing steadily, for two reasons: (1) The birth rate exceeds the death rate, resulting in an increase of 2% per year (that is, a rate of increase equal to $.02P$ per year). (2) More people move to Springfield from elsewhere than move away, resulting in an additional increase of 1000 people per year.

(a) (2 points) Write down a formula for the total rate of increase of population, dP/dt.

• Rate in (# of people ent. per year)

- .02 (P(t)) → # of people born each year
- 1000 people/year move there

$$\boxed{dP/dt = .02P + 1000}$$

① Solve diff Eq

(b) (6 points) At the beginning of 1995 (call this time $t = 0$), the population of Springfield was 25,000. Solve the differential equation you obtained in part (a) to find the population of Springfield t years later.

$$\frac{dP}{.02P+1000} = dt \Rightarrow \int \frac{dP}{.02P+1000} = \int dt$$

$$\frac{1}{.02} \ln|.02P+1000| = t + C$$

$P \geq 0$ so No need for absol. val.

$$= 50\ln(.02P+1000) = t + C$$

$$\ln(.02P+1000) = \frac{1}{50}t + C$$

$$.02P + 1000 = e^{\frac{1}{50}t + C} = e^C e^{50t} = Ce^{50t}$$

$$P = \frac{Ce^{1/50t} - 1000}{.02}$$

$$P = Ce^{\frac{1}{50}t} - 50{,}000$$

② Initial Conditions $P(0) = 25{,}000$

$$25000 = Ce^{\frac{1}{50}(0)} - 50{,}000$$

$$75000 = C$$

$$\boxed{\therefore P(t) = 75{,}000e^{1/50t} - 50{,}000}$$

(c) (2 points) In what year will the population of Springfield be 60,000? (Your answer should look like "2009", not like "14.217689...".)

Set $P(t) = 60{,}000$ solve for t

$$60000 = 75{,}000e^{1/50t} - 50{,}000$$

$$110{,}000 = 75{,}000e^{1/50t}$$

$$\frac{22}{15} = e^{1/50t}$$

$$\ln(22/15) = 1/50\, t$$

$$50\ln(22/15) = t$$

$t \approx 19.1496$ years after 1995

$$\therefore t = 1995 + 19.1496$$

$$\boxed{\approx 2014}$$

9. (12 total points) Your friend wins the lottery, and gives you P_0 dollars to help you pay your college expenses. The money is invested in a savings account that earns 10% annual interest, compounded continuously, and you withdraw the money continuously (a pretty good approximation to reality if you make regular frequent withdrawals) at a rate of \$3600 per year.

   (a) (3 points) Set up a differential equation for the amount of money $P(t)$ (in dollars) left in the account at time t (in years).

   Recall, Rate in - Rate out

   $$\boxed{\frac{dP}{dt} = .1P(t) - 3{,}600}$$

   (b) (3 points) Solve the differential equation to obtain a formula for $P(t)$. Your formula should involve P_0.

   ① Separate and solve

   $$\frac{dP}{\frac{P}{10} - 3{,}600} = dt$$

   $$\Rightarrow \int \frac{dP}{P/10 - 3600} = \int dt$$

   $$\Rightarrow 10 \ln|P/10 - 3600| = t + C$$

   $$\Rightarrow \ln|P/10 - 3600| = \frac{t}{10} + C$$

   $$P/10 - 3600 = Ce^{t/10}$$

   $$\Rightarrow P(t) = Ce^{t/10} + 36{,}000$$

   ② Initial conditions: $P(0) = P_0$

   $$\therefore P_0 = Ce^0 + 36000$$

   $$\Rightarrow C = P_0 - 36000$$

   $$\therefore \boxed{P(t) = (P_0 - 36{,}000)e^{t/10} + 36{,}000}$$

   (c) (3 points) If $P_0 = \$20{,}000$, then how much money is left in the account 4 years later? Give your answer (in dollars) in decimal form, correct to the nearest cent (i.e., nearest \$0.01).

   $$P(t) = -16{,}000e^{t/10} + 36{,}000$$

   $$P(4) = -16{,}000e^{4/10} + 36{,}000 \boxed{= \$12{,}130.80}$$

   (d) (3 points) What should P_0 be so that after 4 years your account has exactly \$0 left over? Give your answer (in dollars) in decimal form, correct to the nearest cent (i.e., nearest \$0.01).

   Now

   $$0 = (P_0 - 36{,}000)e^{4/10} + 36{,}000$$

   $$\Rightarrow \frac{-36{,}000}{e^{4/10}} = P_0 - 36000$$

   $$\Rightarrow \frac{-36{,}000}{e^{4/10}} + 36{,}000 = P_0 = \boxed{\$11{,}868.48}$$

11. (10 points) The swine flu epidemic has been modelled by the Gompertz function, which is a solution of the differential equation

$$\frac{dy}{dt} = 1.2\,y\,(K - \ln(y)),$$

where $y(t)$ is the number of individuals (in thousands) in a large city that have been infected by time t, and K is a constant. Time t is measured in months, with $t = 0$ on July 9, 2009.

On July 9, 2009, 75 thousand individuals had been infected. One month later, 190 thousand individuals had been infected. Find

$$\lim_{t\to\infty} y(t),$$

which is the total number of individuals (in thousands) that will have been infected.

* Note t is in mnths. Thus $t=1 \Rightarrow y = 190$

y is in 000's

① Solve diff Q

$$\frac{dy}{1.2y(k-\ln(y))} = dt \Rightarrow \int \frac{dy}{1.2y(k-\ln(y))} = \int dt$$

$u = \ln(y)$

$du = 1/y\,dy$

$t + C$

$$\int \frac{du}{1.2(k-u)}$$

$$\Rightarrow -\frac{1}{1.2}\ln|k-u| = t + C \Rightarrow -\frac{1}{1.2}\ln|k-\ln(y)| = t + C$$

Thus $\ln|k-\ln(y)| = -1.2t + C \Rightarrow k - \ln(y) = e^{-1.2t}\cdot C$

$$\Rightarrow k - Ce^{-1.2t} = \ln(y)$$

② At $t=0$, $y=75$ and $t=1$, 190

$k - Ce^{0} = \ln(75) \Rightarrow \boxed{k = C + \ln(75)}$

$C + \ln(75) - Ce^{-1.2} = \ln(190)$

$C(1-e^{-1.2}) = \ln(190) - \ln(75) \Rightarrow \boxed{C = \frac{\ln(190/75)}{(1-e^{-1.2})}}$

③ $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} e^{k} e^{-Ce^{-1.2t}} = e^{k}$

$= e^{C+\ln(75)} = \boxed{283.6286}$

10. (10 total points) An object of mass m kg is dropped out of a airplane, and we assume that air resistance is proportional to the speed of the object. Let $s(t)$ be the distance dropped (in meters, positive pointing *down*) after t seconds, and let $v(t) = ds/dt$ be the velocity and $a(t) = dv/dt$ be the acceleration. The combined downward force on the object is

$$F = mg - kv,$$

where $g = 9.8$ meters/sec^2 is the acceleration due to gravity and k is a positive constant. By Newton's Second Law of motion,

$$F = ma = m\frac{dv}{dt}.$$

The mass of the object is $m = 10$ kg, and the constant k is $k = 2$.

(a) (3 points) Set up a differential equation for the velocity $v(t)$.

By above, $mg - kv = m\,dv/dt$

$\Rightarrow 10 \cdot 9.8 - 2v = 10\,dv/dt$

$\boxed{\Rightarrow 9.8 - v/5 = dv/dt}$

(b) (5 points) Solve the differential equation to obtain a formula for $v(t)$.

① Separate the variables and solve

$dt = \dfrac{dv}{9.8 - 1/5\,v} \Rightarrow \int dt = \int \dfrac{dv}{9.8 - \frac{1}{5}v}$

$t + C = -5\ln|9.8 - 1/5\,v|$

$\dfrac{-t}{5} + C = \ln|9.8 - 1/5\,v|$

$Ce^{-t/5} = 9.8 - 1/5\,v$

$1/5\,v = 9.8 - Ce^{-t/5}$

$v = 49 - Ce^{-t/5}$

② Solve initial value (Since dropped $v=0$ at $t=0$)

$0 = 49 - C \Rightarrow C = 49$ $\boxed{v = 49 - 49e^{-t/5}}$

(c) (2 points) What is the limiting velocity $\lim_{t\to\infty} v(t)$?

Since $\lim_{t\to\infty} e^{-t/5} = 0$, then $\lim_{t\to\infty} 49 - 49e^{-t/5} = \boxed{49}$