Your Name

Solutions

Your Signature

Student ID #

Quiz Section

Professor's Name

TA's Name

- This exam is closed book. You may use one 8.5″ × 11″ sheet of handwritten notes (both sides OK). Do not share notes. No photocopied materials are allowed.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- A scientific calculator is allowed, but graphing calculators are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- You may use any of the 20 integrals in the table on p. 484 of the text (p. 506 if you have the 5th edition of Stewart) without deriving them. **Show your work in evaluating any other integrals, even if they are on your note sheet**.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 10 pages, plus this cover sheet. Please make sure that your exam is complete.

Question	Points	Score
1	12	
2	12	
3	10	
4	10	
5	8	

Question	Points	Score
6	10	
7	8	
8	10	
9	10	
10	10	
Total	100	

1. (12 total points) Evaluate the following definite integrals.

(a) (6 points) $\int_0^{\pi/6} (\sin 2x)(\cos 2x)\, dx$

① Let $u=2x$ $du=2dx$ then

$$\int_0^{\pi/6} \sin(2x)\cos(2x)\,dx = \int_0^{\pi/3} \frac{1}{2}\sin(u)\cos(u)\,du$$

② Let $v=\sin(u)$ $dv=\cos(u)\,du$

$$\int_0^{\pi/3} \frac{1}{2}\sin(u)\cos(u) = \int_0^{\sqrt{3}/2} \frac{1}{2}v\,dv = \frac{1}{4}v^2\Big|_0^{\sqrt{3}/2}$$

$$= 1/4 \cdot 3/4 = \boxed{3/16}$$

(b) (6 points) $\int_2^3 \frac{8x^3}{2x^2 - x - 1}\, dx$

* Note since the highest power of the numerator > highest power of denominator

① Use long division:

$$\begin{array}{r} 4x+2 \\ 2x^2-x-1 \enclose{longdiv}{8x^3} \\ -\underline{8x^3-4x^2-4x} \\ 0+4x^2+4x \\ -\underline{4x^2-2x-2} \\ 6x+2 \end{array}$$

② Therefore, $\int_2^3 \frac{8x^3}{2x^2-x-1} = \int_2^3 4x+2+\frac{6x+2}{2x^2-x-1} = 2x^2+2x\Big|_2^3 + \int_2^3 \frac{6x+2}{2x^2-x-1}$

$$= 18+6-8-4+\underbrace{\int_2^3 \frac{6x+2}{2x^2-x-1}}_{ⓐ}$$

③ For ⓐ

$$\frac{6x+2}{(2x+1)(x-1)} = \frac{A}{(2x+1)} + \frac{B}{x-1} \Rightarrow A(x-1)+B(2x+1) = 6x+2$$

$x=1$ $B(3)=8$ $\boxed{B=8/3}$

$x=-1/2$ $A(-3/2) = -1$

$\boxed{A=2/3}$

④ Write Integral

$$2x^2+2x\Big|_2^3 + \int_2^3 \frac{2/3}{(2x+1)} + \int_2^3 \frac{8/3}{x-1} = 2x^2+2x\Big|_2^3 + \frac{2}{6}\ln|2x+1|\Big|_2^3 + 8/3\ln|x-1|\Big|_2^3$$

$\longrightarrow$

1b Continued

$$2x^2+2x\Big|_2^3 + \frac{1}{3}\ln|2x+1|\Big|_2^3 + \frac{8}{3}\ln|x-1|\Big|_2^3$$

$$= 18+6-8-4+\frac{1}{3}\ln|7|-\frac{1}{3}\ln|5|+\frac{8}{3}\ln|2|-8/3\ln|1|$$

$$\boxed{= 12+\frac{1}{3}\ln(7/5)+8/3\ln(2)}$$

2. (12 total points) Evaluate the following indefinite integrals.

(a) (6 points) $\int x(7-x)^{2010}\,dx$

Use u-substitution, Let $u = 7-x$ $du = -dx$

$$\therefore \int x(7-x)^{2010}\,dx = -\int (7-u)\,u^{2010} = \int u^{2011} - 7u^{2010}$$

$$= \frac{u^{2012}}{2012} - \frac{7}{2011}u^{2011} + C$$

$$\boxed{= \frac{(7-x)^{2012}}{2012} - \frac{7}{2011}(7-x)^{2011} + C}$$

(b) (6 points) $\int x^3 \sin\left(x^2+1\right)\,dx$

① Let $u = x^2+1$ $du = 2x\,dx$

$$\int x^3 \sin(x^2+1)\,dx = \frac{1}{2}\int x^2 \sin(u)\,du = \frac{1}{2}\int (u-1)\sin(u)\,du$$

② Use Integration By parts,

$w = (u-1)\frac{1}{2}$ $dv = \sin(u)\,du$

$dw = \frac{1}{2}du$ $v = -\cos(u)$

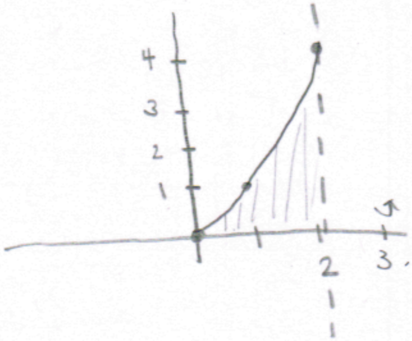
$$\therefore -\frac{1}{2}(u-1)\cos(u) + \int \frac{1}{2}\cos(u)\,du$$

$$= \frac{-1}{2}(u-1)\cos(u) + \frac{1}{2}\sin(u) + C$$

$$\boxed{= \frac{-1}{2}x^2\cos(x^2+1) + \frac{1}{2}\sin(x^2+1) + C}$$

3. (10 total points) Let $\mathscr{R}$ be the region bounded by the curve $y = x^4$, the line $x = 2$, and the x-axis.

(a) (2 points) Sketch the region $\mathscr{R}$.



(b) (4 points) The region $\mathscr{R}$ is rotated around the line $x = 3$ to form a solid. Set up an integral for the volume of this solid using <u>*CYLINDRICAL SHELLS*</u> and *EVALUATE THE INTEGRAL*.

① The height is x^4 and radius is $3-x$. Hence,

$$\int_0^2 x^4(3-x)\,2\pi\,dx = 2\pi\int_0^2 3x^4 - x^5\,dx$$

$$= 2\pi\left(\frac{3}{5}x^5 - \frac{1}{6}x^6\right)\Big|_0^2$$

$$= 2\pi\left(\frac{3}{5}(32) - \frac{1}{6}(64)\right)$$

$$= 2\pi\left(128/15\right)$$

$$\boxed{= \frac{256\pi}{15}}$$

(c) (4 points) Set up an integral for the volume of this solid using *WASHERS*.
DO NOT EVALUATE THE INTEGRAL.

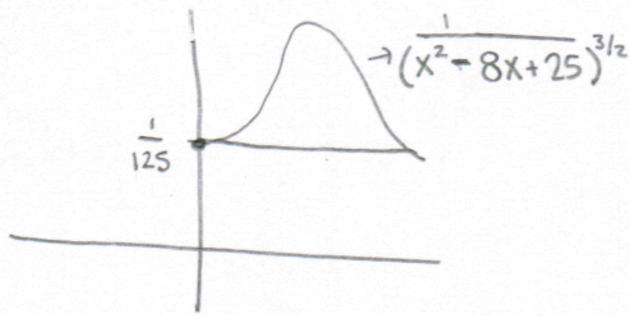
① Need to solve for x. $x = \pm\sqrt[4]{y}$. Since x always positive, chose the positive one. This is washers

Outer Radius $= 3 - \sqrt[4]{y}$, Inner Radius $= 3-2$

$$\boxed{\therefore\ \pi\int_0^4 (3-\sqrt[4]{y})^2 - (1)^2\,dy}$$

4. (10 points) Find the area of the region enclosed between the curve $y = \dfrac{1}{(x^2-8x+25)^{3/2}}$ and the line $y = \dfrac{1}{125}$. (Hint: To solve the equation to find the intersections, raise both sides to the 2/3 power.)

① Graph the function



② Find pts of intersection

$$\left(\frac{1}{125}\right)^{2/3} = \left(\frac{1}{(x^2-8x+25)^{3/2}}\right)^{2/3}$$

$$\Rightarrow (125)^{1/3} = \left[(x^2-8x+25)^{3/2}\right]^{1/3}$$

$$5 = [x^2-8x+25]^{1/2}$$

$$25 = x^2-8x+25$$

$$0 = x^2-8x$$

$$0 = x(x-8) \quad \boxed{x=0, x=8}$$

③ Make Chart

	$0 \le x \le 8$
Top	$1/(x^2-8x+25)^{3/2}$
Bottom	$1/125$

④ $$\int_0^8 \frac{1}{(\sqrt{x^2-8x+25})^3} - \frac{1}{125}\,dx$$

Trig

$$= \int_0^8 \frac{1}{(\sqrt{(x-4)^2+9})^3} - \frac{1}{125}$$

comp. the square

$u = x-4 \quad du = dx$

$$\int_{-4}^{4} \frac{1}{(\sqrt{u^2+9})^3} - \int_0^8 \frac{1}{125}$$

$u = 3\tan\theta$

$du = 3\sec^2\theta$

$$\Rightarrow \int \frac{3\sec^2\theta}{(\sqrt{9\tan^2\theta+9})^3}\,d\theta = \int \frac{3\sec^2\theta}{(\sqrt{9\sec^2\theta})^3}\,d\theta$$

Note
No bounds

$$= \int \frac{3\sec^2\theta}{(3\sec\theta)^3}\,d\theta$$

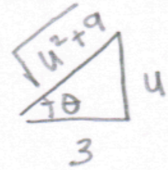
$$= \frac{1}{9}\int \frac{1}{\sec\theta}\,d\theta$$

$$= \frac{1}{9}\int \cos\theta\,d\theta$$

$$= \frac{1}{9}\sin\theta + C$$

By Triangle,

$u/3 = \tan\theta = \frac{o}{a}$



Hence,

$$\int_{-4}^{4} \frac{1}{(\sqrt{u^2+9})^3} - \int_0^8 \frac{1}{125}$$

$$= \frac{u}{9\sqrt{u^2+9}}\Bigg|_{-4}^{4} - \frac{1}{125} \times \Big|_0^8$$

$$= \frac{4}{9(5)} + \frac{4}{9(5)} - \frac{1}{125}(8)$$

$$\boxed{= \frac{128}{1125}}$$

5. (8 total points) Water is drawn from a well that is 35 meters deep using a leaky bucket that initially scoops up 20 kilograms of water from the bottom of the well. The mass of the bucket itself is 2 kilograms and the mass of the rope that is attached to the bucket is 0.2 kg/m. The rope is being pulled at a constant rate of 0.5 m/s. The bucket has a hole in it and water leaks from the bucket at a rate of 0.1 kg/s.

   (a) (3 points) Let y be the height (in meters) of the bucket above the bottom of the well. What is the mass of the water in the bucket when the bucket is y meters high?

   Rate that water leaks per meter $= .1\,\text{kg/s} \cdot \frac{1}{.5\,\text{m/s}} = .2\,\text{kg/m}$.

   $\therefore$ The mass of the water at height $(y) = 20\text{kg} - .2\frac{\text{kg}}{\text{m}}y$

   $\boxed{\text{Hence, mass of wat} = 20 - .2y}$

   (b) (5 points) The acceleration due to gravity is 9.8 m/s^2. Find the work done when the bucket is pulled from the bottom to the top of the well.

① Forces at work
   ⓐ Rope
   ⓑ Bucket
   ⓒ Water

ⓐ 1). Find mass as a fun: of y
   Total rope mass is $.2\frac{\text{kg}}{\text{m}}(35) = 7\text{kg}$
   $\therefore$ mass $= 7 - .2y$
2). Find Force
   $F = m \cdot a = (7 - .2y)(9.8)$
3). dist = dy, bounds 0 to 35
4). Work of Rope
$$\int_0^{35} (7 - .2y)(9.8)\,dy$$
$$= 9.8(7y - .1y^2)\Big|_0^{35} = \boxed{1200.5}$$

ⓑ 1). Find mass of Bucket
   $m = 2$
2). Find Force
   $F = m \cdot a = 2(9.8)$
3). Dist = dy, bound = 0, 35
4). Work of Bucket
$$= \int_0^{35} 2(9.8)\,dy = 2(9.8)(35) = \boxed{686}$$

ⓒ 1). Find mass of Water
   $m = 20 - .2y$
2). Find Force
   $F = ma = (20 - .2y)(9.8)$
3). Bounds = 0 to 35, dist = dy
4). Work of Water
$$\int_0^{35} (20 - .2y)(9.8)\,dy = 9.8(20y - .1y^2)\Big|_0^{35} = \boxed{5659.5}$$

Hence,

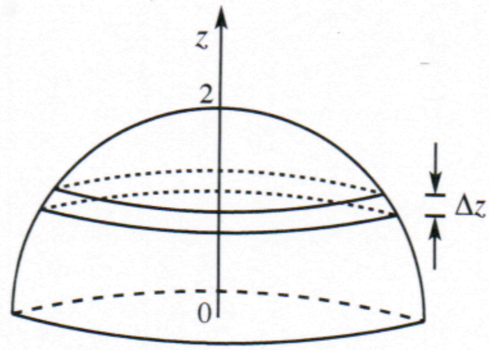
Total Work = Work Rope + Work Bucket + Work Water
$= 1200.5 + 686 + 5659.5$
$\boxed{= 7546}$

6. (10 total points)

A mound of snow is in the shape of a hemisphere of radius 2 feet. The snow is denser at the bottom than at the top because it compresses. Suppose the density of the snow is

$$\rho(z) = \frac{15}{z^2+9}$$

pounds per cubic foot at height z feet above the bottom of the mound.



In this problem, you will set up a definite integral for the weight of the mound of snow. Imagine dividing the mound into n thin horizontal slices of equal thickness Δz.

(a) (3 points) Each slice can be approximated by a disk. Find the (approximate) volume of a typical slice (the i^{th} slice) in terms of its height z_i above the bottom of the mound and Δz.

① Recall that the eq which draws a circle is $x^2+y^2=4$

Solving for x, we have $x = \pm\sqrt{4-y^2}$

$\therefore$ Volume of Slice $= \pi(4-y^2)\,dy$

$$\boxed{= \pi(4-z_i^2)\Delta z}$$

(b) (3 points) Find the (approximate) weight of this typical slice.

(Note weight = Force)

① Find mass = density × volume

$$\left(\frac{15}{z_i^2+9}\right) \times \pi(4-z_i^2)\Delta z$$

② Weight = Force

$$\boxed{= \left(\frac{15}{z_i^2+9}\right)\pi(4-z_i^2)\cdot 9.8\,\Delta z}$$

(c) (2 points) Write a sum which approximates the weight of the mound of snow.

Recall that $\Delta z = \frac{2-0}{n} = \frac{2}{n}$ and $z_i = a + \Delta z i = 0 + \frac{2}{n}i$

$$\boxed{\therefore \sum_{i=1}^{n} \frac{15}{(\frac{2}{n}i)^2+9} \cdot \pi\left(4-\left(\tfrac{2}{n}i\right)^2\right)\cdot 9.8 \cdot \frac{2}{n}}$$

(d) (2 points) For what definite integral is this a Riemann sum?

DO NOT EVALUATE THE INTEGRAL.

This is $\boxed{\displaystyle\int_0^2 \left(\frac{15}{z^2+9}\right)\pi(4-z^2)\cdot 9.8\,dz}$

7. (8 points) Consider the improper integral $\int_2^\infty \frac{\ln x}{x^4}\,dx$.

Evaluate the integral (if it converges) or explain carefully why it does not converge.

$$\int_2^\infty \frac{\ln(x)}{x^4}\,dx = \lim_{a\to\infty}\int_2^a \frac{\ln(x)}{x^4}$$

Using integration by parts,

$u=\ln(x)$ $\qquad dv = x^{-4}\,dx$

$du = \frac{1}{x}\,dx$ $\qquad v = -\frac{1}{3}\left(\frac{1}{x^3}\right)$

$$= \lim_{a\to\infty} -\frac{1}{3}\,\frac{\ln(x)}{x^3}\Big|_2^a + \frac{1}{3}\int_2^a \frac{1}{x^4}\,dx$$

$$= \lim_{a\to\infty} \frac{-\ln(x)}{3x^3}\Big|_2^a - \frac{1}{3}\left(\frac{1}{3}\right)\left(\frac{1}{x^3}\right)\Big|_2^a$$

$$= \lim_{a\to\infty} \frac{-\ln(a)}{3a^3} + \frac{\ln(2)}{3(2^3)} - \frac{1}{9}\left(\frac{1}{a^3}\right) + \frac{1}{72}$$ b/c got $\frac{\infty}{\infty}$

L'Hop

$$\lim_{a\to\infty} \frac{-\frac{1}{a}}{9a^2} + \frac{\ln(2)}{24} - \frac{1}{9}\left(\frac{1}{a^3}\right) + \frac{1}{72}$$

$$= \lim_{a\to\infty} \frac{-1}{9a^3} + \frac{\ln(2)}{24} - \frac{1}{9a^3} + \frac{1}{72}$$

$$\boxed{= \frac{\ln(2)}{24} + \frac{1}{72}}$$

8. (10 points) Find the solution of the differential equation $\dfrac{dL}{dt} = 3L^2\sqrt{100-t^2}$ that satisfies the initial condition $L(0) = -\dfrac{1}{2}$.

① $\dfrac{dL}{dt} = 3L^2\sqrt{100-t^2} \Rightarrow \dfrac{dL}{3L^2} = \sqrt{100-t^2}\,dt$

$\Rightarrow \displaystyle\int \frac{dL}{3L^2} = \int \sqrt{100-t^2}\,dt$

$\hookrightarrow t = 10\sin\theta$

$dt = 10\cos\theta\, d\theta$

$$\frac{-1}{3}\frac{1}{L} = \int \sqrt{100(1-\sin^2\theta)}\cdot 10\cos\theta\, d\theta$$

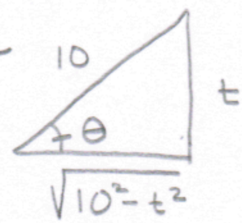
$$= \int 10\cos\theta \cdot 10\cos\theta\, d\theta$$

$$= \int 100\cos^2\theta\, d\theta = \int \frac{100}{2}(1+\cos(2\theta))\,d\theta$$

$$= 50\left[\theta + \tfrac{1}{2}\sin(2\theta)\right] + C$$

$$= 50\left[\sin^{-1}\left(\tfrac{t}{10}\right) + \tfrac{1}{2}\cdot 2\sin\theta\cos\theta\right] + C$$

$$= 50\left[\sin^{-1}\left(\tfrac{t}{10}\right) + \frac{t}{10}\cdot\frac{\sqrt{100-t^2}}{10}\right] + C$$



$$\frac{1}{L} = -150\left[\sin^{-1}\left(\tfrac{t}{10}\right) + \frac{t\sqrt{100-t^2}}{100}\right] + C$$

$$L = \frac{1}{-150(\sin^{-1}(t/10)) - 3/2\, t\sqrt{100-t^2} + C}$$

② $L(0) = -1/2$

$$\frac{-1}{2} = \frac{1}{-150(\sin^{-1}(0)) - 3/2(0)\sqrt{100-0} + C} \Rightarrow \frac{-1}{2} = \frac{1}{C} \Rightarrow C = -2$$

$$\boxed{\therefore\ L = \frac{1}{-150\sin^{-1}(t/10) - \frac{3}{2}t\sqrt{100-t^2} - 2}}$$

9. (10 points) A tank initially contains 1000 L of pure water. Brine that contains 0.07 kg of salt per liter of water enters the tank at a rate of 5 L/min. In addition, brine that contains 0.04 kg of salt per liter of water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 15 L/min. Find the amount of salt (in kg) in the tank as a function of time t in minutes.

let $s(t)$ = amt of salt in water

① Set up Diff Q

$$\frac{ds}{dt} = \text{Rate in} - \text{Rate out} = \frac{.07\text{ kg}}{L} \cdot \frac{5L}{\min} + \frac{.04\text{kg}}{L} \cdot \frac{10L}{\min} - \frac{s}{1000} \cdot \frac{15L}{\min}$$

$$= .35 + .4 - \frac{15s}{1000} = .75 - \frac{15}{1000}s$$

② Solve:

$$\frac{ds}{.75 - \frac{15}{1000}s} = dt \Rightarrow \int \frac{ds}{.75 - 15/1000 s} = \int dt$$

$$\Rightarrow \frac{-1000}{15} \ln|.75 - 15/1000 s| = t + C$$

$$\Rightarrow \ln|.75 - \frac{15}{1000}s| = \frac{-15}{1000}t + C$$

$$\Rightarrow .75 - \frac{15}{1000}s = Ce^{-15/1000 t}$$

$$\Rightarrow .75 - Ce^{-15/1000 t} = -15/1000 s$$

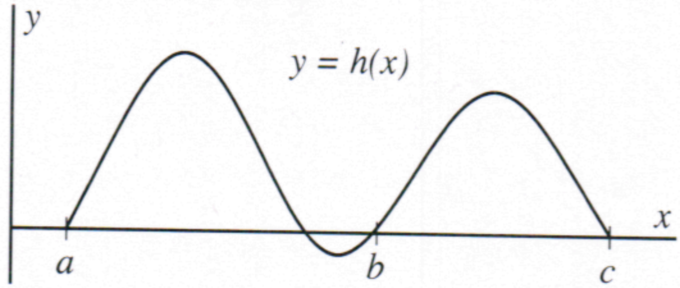
$$Ce^{-15/1000 t} - 50 = s$$

③ At $t = 0$, $s(t) = 0$

$$Ce^0 - 50 = 0 . C = 50$$

$$\boxed{\therefore 50e^{-15/1000 t} - 50 = s(t)}$$

10. (10 total points) Consider the graph of $y = h(x)$:



(a) (2 points) Is $\int_a^b h(x)\,dx$ positive or negative?

Circle one: Positive Negative

Since $\int_a^b h(x)dx$ = Area under the curve from a to b.

(b) (2 points) Is $\int_c^a h(x)\,dx$ positive or negative?

Since $\int_c^a h(x)dx = -\int_a^c h(x)dx$ and the area under the curve from a to c is positive.

Circle one: Positive Negative

(c) (2 points) Let $f(x)$ be a function whose derivative $f'(x)$ is defined and continuous for all real numbers x. Is it always true that for any real number a,

$$f(a) = f(0) + \int_0^a f'(x)\,dx?$$

Circle one: Always true Might be false

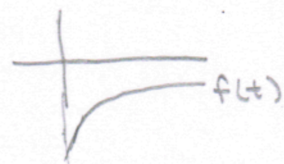
This is b/c of FTC.

(d) (2 points) Let $f(x)$ be a function that is continuous and increasing for all real numbers x, and let

$$g(x) = \int_0^x f(t)dt.$$

Is it always true that $g(x)$ is increasing for all real numbers x?

Consider



Circle one: Always true Might be false

(e) (2 points) Let a be a real number. Is it always true that

$$\int_0^a \sin x\,dx \le \int_0^a |\sin x|\,dx?$$

Circle one: Always true Might be false

If a is positive, then always true

If a is negative, then could be false.