1. (12 points; 4pts each) Find the derivatives of the following functions. You do not have to simplify.

(a) $f(x) = \sqrt{3+x} \cdot \sqrt[3]{5x^2 - 6}$

$$\boxed{f'(x) = \frac{1}{2}(3+x)^{-1/2}\sqrt[3]{5x^2-6} + \sqrt{3+x}\,\frac{1}{3}(5x^2-6)^{-2/3} \cdot 10x}$$

(b) $f(x) = \left(e^x - \frac{2}{4x^3}\right)^3$

$$\boxed{f'(x) = 3\left(e^x - \frac{2}{4x^3}\right)^2 \cdot \left[e^x + \frac{6}{4x^4}\right]}$$

(c) $f(x) = (\tan x)^{\ln x}$

$$f(x) = (\tan x)^{\ln x} = e^{\ln(\tan x^{\ln x})} = e^{\ln(x)\ln(\tan x)}$$

$$\boxed{f'(x) = e^{\ln(x)\ln(\tan x)}\left[\frac{\ln(\tan x)}{x} + \frac{\ln(x)\sec^2 x}{\tan x}\right]}$$

2. (12 points) Given the curve

$$x^{2/3} + y^{2/3} = 5$$

answer the following.

(a) Verify that the point $(8, 1)$ is on the curve and find the equation of the tangent line to the curve at this point.

① Take the derivative

$$\tfrac{2}{3}x^{-1/3} + \tfrac{2}{3}y^{-1/3} \cdot dy/dx = 0$$

$$dy/dx = \frac{-\sqrt[3]{y}}{\sqrt[3]{x}}$$

② Write tangent line

$$y = -\sqrt[3]{b}/\sqrt[3]{a}\,(x-a) + b$$

$$y = -\sqrt[3]{1}/\sqrt[3]{8}\,(x-8) + 1$$

$$\boxed{y = -1/2\,(x-8) + 1}$$

(b) Is the graph concave up or concave down at that point?

① Need to compute 2nd derivative

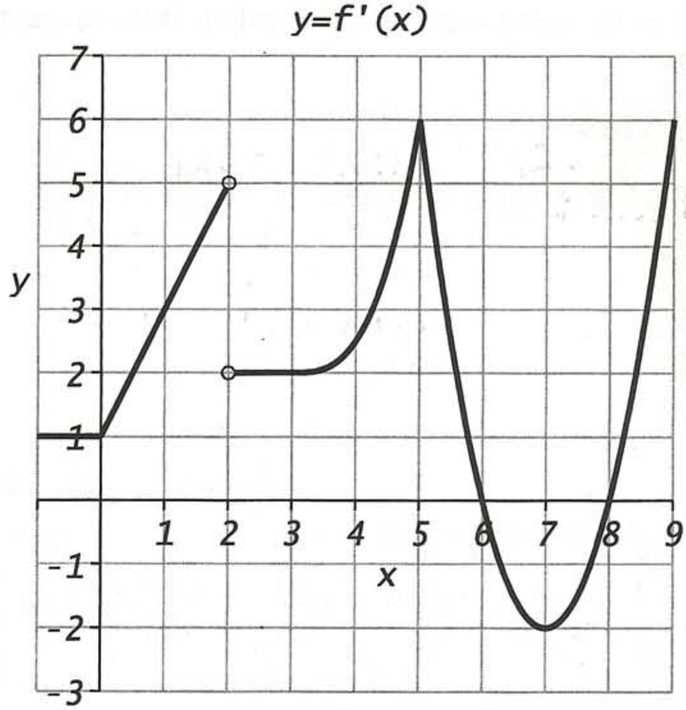
$$d^2y/dx^2 = \frac{-1/3\, y^{-2/3}\, dy/dx\, x^{1/3} + 1/3\, x^{-2/3}\, y^{1/3}}{(\sqrt[3]{x})^2}$$

$$d^2y/dx^2 = \frac{-1/3(-1/2)(1^{-2/3})\,8^{1/3} + \frac{1}{3}(8^{-2/3})\,1^{1/3}}{(\sqrt[3]{8})^2}$$

$\Rightarrow$ positive

$$\boxed{\text{Thus, concave up at } (8,1).}$$

3. (10 points) The following is a graph of $y = f'(x)$, the *derivative* of $f(x)$. Given that $f(0) = 0$, answer the following questions. You do not need to explain your answers. Each part is worth 1 point with no partial credit.



(a) $\lim_{h\to 0} \frac{f(5+h)-f(5)}{h} =$ $f'(5) = \boxed{6}$

(b) $\lim_{x\to 0} f(x) =$ $\boxed{0}$ [This is b/c $f(0)=0$ and $f'(x)$ con't at 0]

(c) Is the graph of $y = f(x)$ concave up or concave down at $x = 1$? Concave up b/c $f'(x)$ increasing

(d) $\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} =$ $\sim f'(2) = \boxed{2}$

(e) $\lim_{x\to 5} f'(x) =$ $\boxed{6}$

(f) $f''(1) =$ Slope at $f'(1) = \frac{5-1}{2-0} = \frac{4}{2} = \boxed{2}$

(g) $\lim_{x\to 0} \frac{f(x)}{x} =$ $\lim_{x\to 0} \frac{f(x)-0}{x-0} = f'(0) = \boxed{1}$

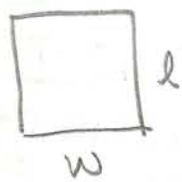
(h) What are the critical points of f in the domain $(-1, 9)$?

B/c $f'(x)$ undefined, 2,

(i) Is f increasing or decreasing at $x = 7$? Decreasing, b/c $f'(7)<0$

(j) Is f positive or negative at $x = -\frac{1}{2}$? Negative, b/c $f(0)=0$ and $f'(x)<0$ for $x \leq 0$.

4. (12 points) The length of a rectangle increases by 3 feet per minute while the width decreases by 2 feet per minute. When the length is 15 feet and the width is 8 feet, what is the rate at which the following changes. Make sure to state whether the rate is increasing or decreasing and include units.



$dl/dt = 3$ ft/min

$dw/dt = -2$ ft/min

Goal: Find the below when $l = 15$ft, $w = 8$ft.

(a) The area.

$A = l \cdot w$

$dA/dt = dl/dt \, w + dw/dt \, l$

$= 3(8) + (-2)(15)$

$\boxed{= -6 \text{ ft/min}}$

(b) The perimeter.

$P = 2l + 2w$

$dP/dt = 2 \, dl/dt + 2 \, dw/dt$

$= 2(3) + 2(-2)$

$= \boxed{2 \text{ ft/min}}$

(c) The length of the diagonal.

$d^2 = l^2 + w^2$

$d = \sqrt{l^2 + w^2}$

$$\frac{dd}{dt} = \frac{2l \, dl/dt + 2w \, dw/dt}{2\sqrt{l^2+w^2}} = \frac{2(15)(3) + 2(8)(-2)}{2\sqrt{15^2+8^2}} = \frac{58}{2(17)}$$

$\boxed{= 29/17 \text{ ft/min}}$

5. (12 points) A particle is moving in the plane and traces out a curve with parametric equations:

$$x(t) = \sqrt{3}\cos(\pi t) \qquad y(t) = \sin(\pi t),$$

with $\underline{0 \le t \le 3}$ seconds.

(a) Find the points on the curve where the slope is 1.

$$dy/dx = \frac{dy/dt}{dx/dt} = \frac{\cancel{\pi}\cos(\pi t)}{-\sqrt{3}\cancel{\pi}\sin(\pi t)} = \frac{1}{-\sqrt{3}}\cot(\pi t)$$

$$1 = -1/\sqrt{3}\cot(\pi t)$$

$$-\sqrt{3} = \cot(\pi t) \Rightarrow 11\pi/6 = \pi t, \quad \frac{5\pi}{6} = \pi t, \quad \frac{17\pi}{6} = \pi t$$

$$t = 11/6,\ 5/6,\ 17/6.$$

Hence, the points on the curve are

$t = 11/6$: $\sqrt{3}\cos(11\pi/6) = 3/2$ $\sin(11\pi/6) = -1/2$

$(3/2, -1/2)$

$t = 5/6$: $\sqrt{3}\cos(5\pi/6) = -3/2$ $\sin(5\pi/6) = 1/2$

$t = 17/6$: $\sqrt{3}\cos(17\pi/6) = -3/2$ $\sin(17\pi/6) = 1/2$

Thus, the points on the curve are $(3/2, -1/2)$ and $(-3/2, 1/2)$

(b) On the given time interval, find the **last** time when the slope is 1.

The last time as shown above is at $t = 17/6$

6. (10 points) Use linear approximation to approximate the value of

$$9^{1/3} - 2.$$

Show all of your work. You will receive **no credit** for simply evaluating this using your calculator.

* Note we don't know the function

① Find where the number is difficult to evaluate (i.e. $9^{1/3}$) and call it x. Thus,

$$f(x) = x^{1/3} - 2 \quad \text{approx. at } 9.$$

② Find a point that is good (i.e. easy to calculate) and close to 9

$$8^{1/3} - 2 = 0, \text{ so the pt } (8,0) \text{ good.}$$

③ Take derivative

$$f'(x) = \tfrac{1}{3}x^{-2/3}$$

④ Find generalized tangent line at (a,b) and plug in $(8,0)$

$$y = \tfrac{1}{3}a^{-2/3}(x-a) + b$$

$$y = \tfrac{1}{3}8^{-2/3}(x-8) + 0$$

$$y = 1/12\,(x-8)$$

⑤ Plug in point approximating

$$y = \tfrac{1}{12}(9-8) = \tfrac{1}{12}$$

$$\boxed{\text{Thus, } 9^{1/3} - 2 \approx 1/12}$$

Spring 2011 Final

7. (12 points) Compute the limit. If it is correct to say that the limit is ∞ or $-\infty$, then say so. If the limit does not exist, explain why.

(a) $\lim\limits_{x\to 0} \dfrac{\sin(3x)\sin(2x)}{x\sin(5x)} \cdot \dfrac{3}{3} = \lim\limits_{x\to 0} \dfrac{\sin(3x)}{3x} \cdot \dfrac{3\sin(2x)}{\sin(5x)} \cdot \dfrac{2x}{2x}$

$= \lim\limits_{x\to 0} \dfrac{\sin(3x)}{3x} \cdot \dfrac{\sin(2x)}{2x} \cdot \dfrac{6x}{\sin(5x)} \cdot \dfrac{5/6}{5/6}$

$= \lim\limits_{x\to 0} \dfrac{\sin(3x)}{3x} \cdot \dfrac{\sin(2x)}{2x} \cdot \dfrac{5x}{\sin(5x)} \cdot \dfrac{1}{(5/6)}$

$= \lim\limits_{x\to 0} \dfrac{\sin(3x)}{3x} \cdot \dfrac{\sin(2x)}{2x} \cdot \dfrac{1}{\frac{\sin(5x)}{5x}} \cdot \dfrac{6}{5}$ Note: $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

$= \boxed{6/5}$

(b) $\lim\limits_{x\to 2} \dfrac{e^{x^2} - e^4}{x-2}$

Looks like a derivative for e^{x^2}

Thus

$\lim\limits_{x\to 2} \dfrac{e^{x^2} - e^4}{x-2} = f'(2)$ $f'(x) = e^{x^2} \cdot 2x$

$\boxed{f'(2) = e^4 \cdot 8}$

(c) $\lim\limits_{t\to 0} \left(\dfrac{1}{t\sqrt{1+t}} - \dfrac{1}{t}\right) = \lim\limits_{t\to 0} \dfrac{1-\sqrt{1+t}}{t\sqrt{1+t}} \dfrac{(1+\sqrt{1+t})}{(1+\sqrt{1+t})} = \lim\limits_{t\to 0} \dfrac{1-1-t}{t\sqrt{1+t}\,(1+\sqrt{1+t})}$

$= \lim\limits_{t\to 0} \dfrac{-1}{\sqrt{1+t}\,(1+\sqrt{1+t})} = \dfrac{-1}{1(1+1)} = \boxed{\dfrac{-1}{2}}$

8. (8 points) For what values of a and b is the line $4x + y = b$ tangent to the parabola $y = ax^2$ when $x = 5$?

① Find the derivative

$$y' = 2ax$$

② Now we see that the slope of the line $4x + y = b$ is -4

Thus at $x = 5$

$$-4 = 2a(5) \Rightarrow a = -4/10 = -2/5$$

③ Now with $a = -2/5$, $y = -2/5 x^2$, at 5 $y = -2/5(5^2) = -10$.

Thus, the tangent line at $x=5$ is

$$y = 4(x-5) - 10$$

$$y = 4x - 20 - 10$$

$$y = 4x - 30$$

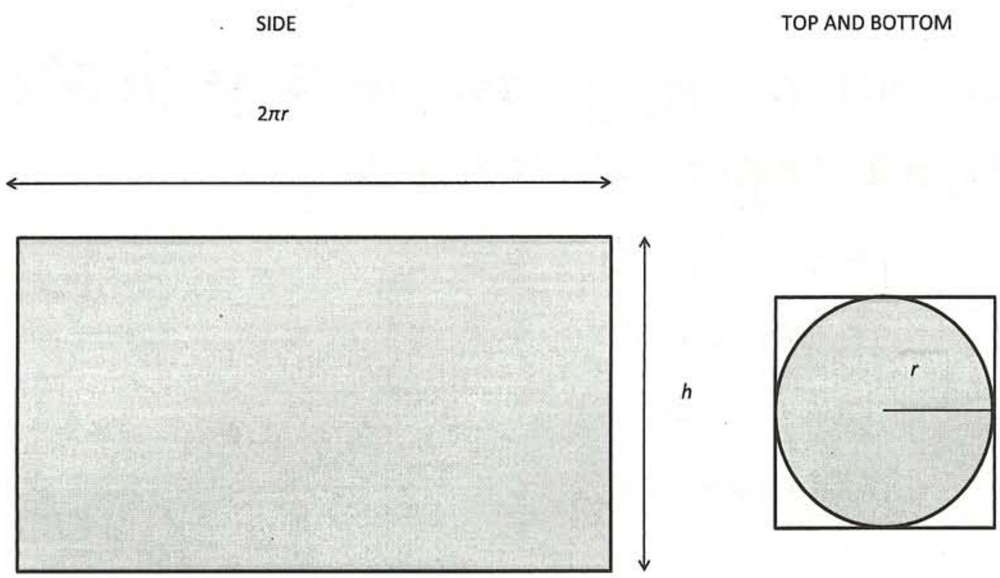
Hence, $b = -30$.

We get that

$$\boxed{a = -2/5, \quad b = -30}$$

9. (12 points) A cylindrical can of volume 250 cubic centimeters is to be made from aluminum. The side is made from a thin sheet which costs 0.2 cents per square centimeter. The top and bottom of the can is made from a thicker sheet which costs 0.4 cents per square centimeter. Moreover, since the top and the bottom are circles, they have to be cut from square pieces. The wasted area between the circle and the square can be sold back to the aluminum supplier at a price of 0.1 cent per square centimeter to be recycled. What are the radius r and height h of the minimal cost can?

Give your answer in exact form and as a decimal approximation. Note: Given a cylinder of radius r and height h, the volume is $V = \pi r^2 h$.



- $250 = \pi r^2 h \Rightarrow \dfrac{250}{\pi r^2} = h$
- $TC = \underbrace{.2h \cdot 2\pi r}_{\text{side}} + \underbrace{2 \cdot .4(2r)^2}_{2\cdot\text{bottom}} - \underbrace{2 \cdot .1[(2r)^2 - \pi r^2]}_{\text{savings from the cutout}}$

$TC = .4\pi h r + 3.2r^2 - .8r^2 + .2\pi r^2$

$TC = .4 \cdot 250 r^{-1} + 3.2r^2 - .8r^2 + .2\pi r^2$

$TC = 100r^{-1} + 2.4r^2 + .2\pi r^2$

Take derivative: $TC' = -100r^{-2} + 4.8r + .4\pi r$

$0 = \dfrac{4.8r^3 + .4\pi r^3 - 100}{r^2} \Rightarrow 0 = 4.8r^3 + .4\pi r^3 - 100$

$100 = (4.8 + .4\pi)r^3$

$\boxed{\sqrt[3]{\dfrac{100}{4.8 + .4\pi}} = r \approx 2.54638}$

another page

Use this page if you need more space on #9.

Now for h,

$$h = \frac{250}{\pi\left(\frac{100}{4.8+.4\pi}\right)^{2/3}} \approx 12.2728$$

* Note we don't have to check extremes like $h=0$ or $r=0$ b/c then Volu $\neq 250$.