Math 124 **Final Examination** **Winter 2011**

Print Your Name	Signature
Student ID Number	**Quiz Section**
Professor's Name	**TA's Name**

!!! READ...INSTRUCTIONS...READ !!!

1. Your exam contains 9 questions and 12 pages; PLEASE MAKE SURE YOU HAVE A COMPLETE EXAM.
2. The entire exam is worth 100 points. Point values for problems vary and these are clearly indicated. You have 2 hours and 50 minutes for this final exam.
3. Make sure to ALWAYS SHOW YOUR WORK; you will not receive any partial credit unless all work is clearly shown. If in doubt, ask for clarification.
4. There is plenty of space on the exam to do your work. If you need extra space, use the back pages of the exam and clearly indicate this.
5. You are allowed one 8.5×11 sheet of handwritten notes (both sides). Graphing calculators are NOT allowed; scientific calculators are allowed. Make sure your calculator is in radian mode.
6. Unless otherwise instructed, ALWAYS GIVE YOUR ANSWERS IN EXACT FORM. For example, $3\pi, \sqrt{2}, \ln(2)$ are in exact form; the corresponding approximations $9.424778, 1.4142, 0.693147$ are NOT in exact form.

Problem	Total Points	Score
1	16	
2	8	
3	6	
4	10	

Problem	Total Points	Score
5	10	
6	12	
7	12	
8	12	
9	14	
Total	100	

π

1. (16 points; 4pts each) Find the derivatives of the following functions. You do not have to simplify.

(a) $y = (1 + \cos^3 x)^{2/3}$

$$\boxed{y' = 2/3(1+\cos^3 x)^{-1/3} \cdot [-3\cos^2 x \cdot \sin x]}$$

(b) $y = \arctan(e^{\arctan x})$

$$\boxed{y' = \frac{1}{1+(e^{\arctan x})^2} \cdot \left[e^{\arctan x} \cdot \frac{1}{1+x^2}\right]}$$

1. continued

(c) $y = (\cos x)^{\sin x}$

① Rewrite the function

$$y = e^{\ln(\cos x^{\sin x})} = e^{\sin x \ln(\cos x)}$$

② Take derivative

$$\boxed{y' = e^{\sin x \ln(\cos x)} \cdot \left[\cos x \ln(\cos x) + \frac{1}{\cos x}(-\sin^2 x)\right]}$$

(d) $y = \dfrac{t}{(1+\sqrt{t})^{100}}$

$$\boxed{y' = \frac{(1+\sqrt{t})^{100} - 100t(1+\sqrt{t})^{99} \cdot \frac{1}{2} t^{-1/2}}{\left[(1+\sqrt{t})^{100}\right]^2}}$$

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2. (8 points)

Calculate the following limits. Make sure to justify all your steps.

(a) $\lim_{x\to 0} \frac{(\sin(x))^{12}}{x^{10}}$

$= \lim_{x\to 0} \underbrace{\frac{\sin x}{x} \cdot \frac{\sin x}{x} \cdot \ldots \cdot \frac{\sin x}{x}}_{10 \text{ times}} \cdot \underbrace{\sin x \cdot \sin x}_{2 \text{ times}}$

$= \lim_{x\to 0} \sin^2 x = \boxed{0}$

(b) $\lim_{x\to\infty} \left(\frac{x+1}{x-1}\right)^{x-1}$

① Rewrite as

$\left(\frac{x+1}{x-1}\right)^{x-1} = e^{(x-1)\ln\left(\frac{x+1}{x-1}\right)}$

② $\lim_{x\to\infty} e^{(x-1)\ln\left(\frac{x+1}{x-1}\right)} = e^{\lim_{x\to\infty}(x-1)\ln\left(\frac{x+1}{x-1}\right)}$

$\lim_{x\to\infty}(x-1)\ln\left(\frac{x+1}{x-1}\right) = \lim_{x\to\infty} \frac{\ln\left(\frac{x+1}{x-1}\right)}{\frac{1}{x-1}} = \frac{0}{0}$ By L'Hopital

$= \lim_{x\to\infty} \frac{\left(\frac{x-1}{x+1}\right)\cdot\left(\frac{x-1-x-1}{(x-1)^2}\right)}{\frac{-1}{(x-1)^2}} = \lim_{x\to\infty} 2\left(\frac{x-1}{x+1}\right) = 2\lim_{x\to\infty}\frac{x-1}{x+1}\frac{\left(\frac{1}{x}\right)}{(1/x)}$

$= 2\lim_{x\to\infty}\frac{1-1/x}{1+1/x} = 2(1) = 2$

$\boxed{\text{Thus } \lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^{x-1} = e^2}$

3. (6 points) For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 6x, & \text{if } x < 5 \\ x^3 - cx, & \text{if } x \geq 5 \end{cases}$$

The function is con't if

$$\lim_{x \to 5^-} cx^2 + 6x = \lim_{x \to 5^+} x^3 - cx$$

$$c(25) + 6(5) = 5^3 - c(5)$$

$$25c + 30 = 125 - 5c$$

$$30c = 95$$

$$\boxed{c = 16/5 = 3.2}$$

4. (10 points) Consider the curve defined by the parametric equations

$$x = \frac{1}{3}t^3 - \ln t, \quad y = \frac{81}{2}t^2 + \frac{8}{t^2} + 3,$$

where $t > 0$.

(a) Find all the horizontal tangent lines to the curve.

Horizontal Tangent line where $dy/dx = \frac{dy/dt}{dx/dt} = 0$

$\Rightarrow dy/dt = 0$ and $dx/dt \neq 0$.

$dy/dt = 81t - 16t^{-3} = 0$

$\frac{81t^4 - 16}{t^3} = 0 \Rightarrow 81t^4 - 16 = 0$

$t^4 = 16/81 \Rightarrow \boxed{t = \pm 2/3}$

At $t = 2/3$, $x = 1/3(\frac{2}{3})^3 - \ln(2/3)$ $y = 81/2(2/3)^2 + 8/(2/3)^2 + 3 = 39$

Thus, the tangent line is $\boxed{y = 39}$

(b) Find all the vertical tangent lines to the curve.

Now you want to find where vertical $\Rightarrow dy/dx =$ undefined

Since $dy/dx = \frac{dy/dt}{dx/dt}$, then vertical $\Rightarrow dx/dt = 0$.

Hence,

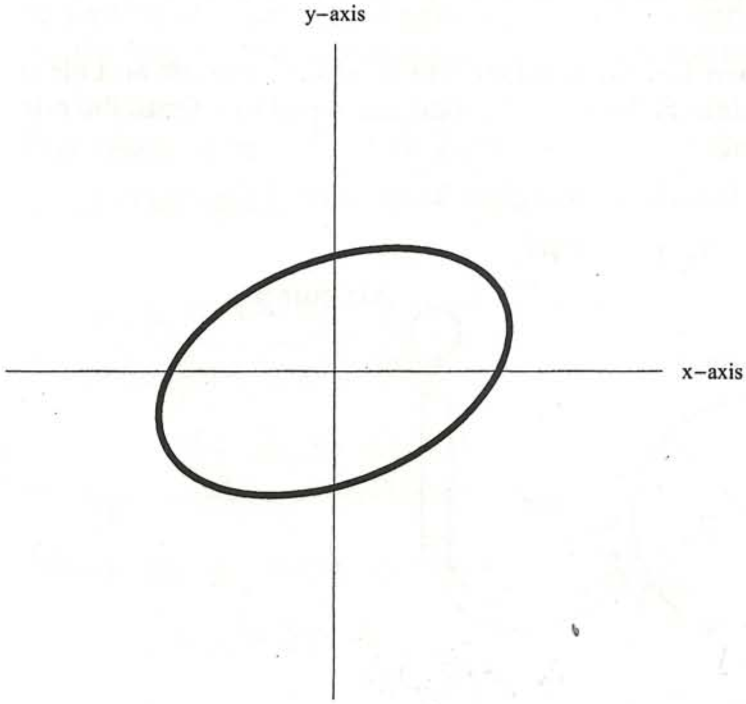
$dx/dt = t^2 - 1/t = 0$

$\Rightarrow \frac{t^3 - 1}{t} = 0 \Rightarrow t^3 - 1 = 0 \Rightarrow t = 1$

$x(1) = 1/3 - 0$; $y(1) = 81/2 + 8 + 3$

$\boxed{\text{Thus, the vertical tangent line is: } x = 1/3}$

5. (10 points) The graph of the equation $x^2 - xy + 2y^2 = 4$ is a tilted ellipse, as pictured below.



(a) Find a formula for the implicit derivative $\frac{dy}{dx}$.

$2x - y - x\,dy/dx + 4y\,dy/dx = 0$

$(4y - x)\,dy/dx = y - 2x$

$\boxed{dy/dx = \dfrac{y-2x}{4y-x}}$

(b) Find the coordinates of a point on the ellipse where the tangent line is parallel to the line with equation $y = x + 4$. (Note: there are two correct answers; either will be accepted.) Give your answer in exact form.

Since the line $y=x+4$ is parallel, then the slopes must be the same. Hence,

$1 = y-2x/4y-x \Rightarrow 4y-x = y-2x \Rightarrow 3y = -x \Rightarrow y = -\frac{1}{3}x$

Plugging into the ellipse equation, we get

$x^2 + 1/3\,x^2 + 2/9\,x^2 = 4$

$14/9\,x^2 = 4 \Rightarrow x^2 = 36/14$

$x = \pm\, 6/\sqrt{14}$

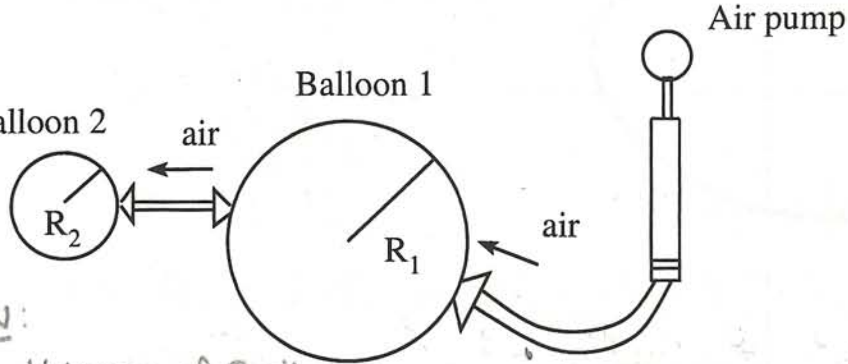
$\boxed{\text{Thus, the solutions are } \left(6/\sqrt{14}, \frac{-6}{3\sqrt{14}}\right) \text{ and } \left(-6/\sqrt{14}, \frac{6}{3\sqrt{14}}\right)}$

6. (12 points) Balloon 1 is linked by a large tube to an air pump and by a smaller tube to Balloon 2 (see picture). The radius of Balloon 1 is R_1 and the radius of Balloon 2 is R_2.

Air is being pumped in Balloon 1 at the constant rate of $101\text{cm}^3/\text{minute}$ and air is leaking out of Balloon 1 (and into Balloon 2) at a total rate equal to π times the rate of change of R_1, in $\text{cm}^3/\text{minute}$.

At time t_0 measurements say that $R_1 = 5$ and $R_2 = 2$.

Calculate the rate of change of R_2 at that time.



KNOW:

V_1 = Volume of Balloon 1

V_2 = Volume of Balloon 2

$\text{Volume}_1 = \pi R_1^2$

$\text{Volume}_2 = \pi R_2^2$

$\frac{dV_2}{dt} = \pi \frac{dR_1}{dt}$, $\frac{dV_1}{dt}$ = Rate in - Rate out = $\frac{101\text{cm}^3}{\text{min}} - \pi \frac{dR_1}{dt}$

$dV_1/dt = 2\pi R_1 \, dR_1/dt = 101 - \pi \, dR_1/dt$

$2\pi(5) \, dR_1/dt = 101 - \pi \, dR_1/dt$

$11\pi \, dR_1/dt = 101 \Rightarrow \boxed{dR_1/dt = \frac{101}{11\pi}}$

$\frac{dV_2}{dt} = 2\pi R_2 \, dR_2/dt = \pi \, dR_1/dt$

$2\pi(2) \, dR_2/dt = \pi (101/11\pi)$

$\boxed{dR_2/dt = \frac{101}{44\pi}}$

7. (12 points) A particle is traveling along a curve with parametric equations $x = x(t), y = y(t)$. The implicit equation of the curve is $y^2 = x^3 + 3x$. At time $t = 0$, the particle is located at the point $(1, -2)$ and its vertical velocity $\frac{dy}{dt}$ is 2 units/sec. Use the tangent line approximation to estimate the location of the particle at time $t = 0.1$.

* The idea is to write a tangent line approx. for the x and y coordinates separately

At time $t=0$, $y=-2$ and $dy/dt=2$. Hence

The tangent line is then
$$y = 2(t-0) - 2$$

Now for x, we need to find dx/dt.

$$y^2 = x^3+3x \Rightarrow 2y\,dy/dt = 3x^2\,dx/dt + 3\,dx/dt$$
$$2(-2)(2) = 3(1^2)\,dx/dt + 3\,dx/dt$$
$$-8 = 6\,dx/dt$$
$$-8/6 = dx/dt$$

At $t=0$, $x=1$, and $dx/dt = -8/6$.

The tangent line is then
$$x = -8/6(t-0)+1$$

To Estimate: Plug in $t=.1$

$$y = 2(.1-0)-2 = -1.8$$

To $x = -8/6(.1-0)+1 = 13/15$

Thus, the location is $(13/15, -1.8)$

8. (12 points) Nurl is designing a cylindrical container of volume 50π cubic centimeters. The top and bottom of the cylinder must be made of a material costing \$10 per square centimeter, while the rest of the container is made of a cheaper material that costs only \$3.20 per square centimeter. What is the surface area of the cheapest container Nurl can design?

Volume $= 50\pi = \pi r^2 h$

$TC = \$10 \cdot 2\pi r^2 + 3.20(2\pi r h)$

$50r^{-2} = h$. Plugging this in gives

$TC = 20\pi r^2 + 320\pi r^{-1}$

$TC' = 40\pi r - 320\pi r^{-2} = 0$

$$\frac{40\pi r^3 - 320\pi}{r^2} = 0 \Rightarrow 40\pi r^3 - 320\pi = 0$$

$$40\pi r^3 = 320\pi$$

$$r^3 = 320/40 = 8$$

$$\boxed{r = 2}$$

* We don't need to check the endpoints b/c neither r nor h could be 0.

The cheapest surface area is

$SA = 2\pi(2^2) + 2\pi(2)\,50\,(2^{-2})$

$= 8\pi + 50\pi$

$\boxed{SA = 58\pi}$

9. (14 points) Let $f(x) = e^{\frac{1}{x-2}}$.

(a) Find the largest possible domain for the function.

Since $x=2$, causes $e^{1/x-2}$ to be undefined,

$\boxed{\text{Domain is } (-\infty,2)\cup(2,\infty)}$

(b) Find $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$.

$$\lim_{x\to 2^-} e^{\frac{1}{x-2}} = e^{\lim_{x\to 2^-}\frac{1}{x-2}} = e^{-\infty} = \boxed{0}$$

$$\lim_{x\to 2^+} e^{\frac{1}{x-2}} = e^{\lim_{x\to 2^+}\frac{1}{x-2}} = e^{\infty} = \boxed{\infty}$$

(c) Find all asymptotes for f (either vertical or horizontal).

Horizontal Asymptotes: $\lim_{x\to\infty} e^{1/x-2} = e^0 = 1$

$\lim_{x\to-\infty} e^{1/x-2} = e^0 = \boxed{1}$

$\boxed{\text{The horizontal asympt is } y=1}$

Vertical Asymptotes: (where the funct. undefined)

$\boxed{\text{This occurs when } x=2.}$

(d) Calculate the intervals where f is increasing or decreasing.

① Find derivative

undefined at $x=2$

$$f'(x) = e^{\frac{1}{x-2}} \cdot \left(\frac{-1}{(x-2)^2}\right) = \frac{-e^{\frac{1}{x-2}}}{(x-2)^2}$$

$$0 = \frac{-e^{\frac{1}{x-2}}}{(x-2)^2} \Rightarrow 0 = e^{1/x-2}$$, but this never occurs

② Make Chart

$-\infty<x<2$	2	$2<x<\infty$
$-$	DNE	$-$

$$f'(0) = \frac{-e^{\frac{1}{-2}}}{4} < 0$$

$$f'(3) = \frac{-e^1}{4} < 0$$

$\boxed{\text{The function is increasing nowhere; decreasing on } (-\infty,2)\cup(2,\infty)}$

9. continued

(e) Find all local extrema (if any) for f.

There are <u>NO</u> local extrema. View chart prior.

(f) Calculate the intervals where f is concave up or concave down.

① Take derivative: $f''(x) = e^{\frac{1}{x-2}}\left(\frac{-1}{x-2}\right)\left(\frac{-1}{(x-2)^2}\right) + e^{\frac{1}{x-2}} 2\cdot(x-2)^{-3}$

$= \frac{e^{\frac{1}{x-2}}}{(x-2)^3}\cdot 3 = 0$ undefined at 2 and doesn't equal 0.

② Make chart

$-\infty < x < 2$	2	$2 < x < \infty$
$-$	DNE	$+$

$f''(1) = \frac{e^{-1}}{-1}\cdot 3 < 0$

$f''(3) = \frac{e^{1}}{1}\cdot 3 > 0$

Concave up on $(2, \infty)$
Concave down on $(-\infty, 2)$

(g) Based on all of the above, sketch the graph of f, labeling all extrema and indicating any asymptotes.