9. (12 total points) A tank contains 100 liters of fresh water. Water containing s grams of salt per liter enters the tank at the rate of 5 liters/minute, and the well-mixed solution leaves at the same rate.

   (a) (6 points) Write down a differential equation for the amount of salt in the tank at time t. (This equation will contain s.)

Let y = amt of salt in water

dy/dt = Rate in - Rate out

$$= \frac{s\ g}{lit.} \cdot \frac{5\ lit}{min} - \frac{y}{100} \cdot 5\ lit/min$$

b/c fresh water (no salt)

$$\boxed{\text{Thus } \frac{dy}{dt} = 5s - \frac{1}{20}y}$$

   (b) (6 points) Suppose that after 10 minutes, the concentration of salt in the tank is 3 grams/liter. Find s.

① Solve diff Q

$$\frac{dy}{5s - \frac{1}{20}y} = dt \Rightarrow \int \frac{dy}{5s - 1/20y} = \int dt \Rightarrow -20\ln|5s - \tfrac{1}{20}y| = t + C$$

$$\Rightarrow \ln|5s - \tfrac{1}{20}y| = -1/20t + C$$

$$|5s - 1/20y| = Ce^{-1/20t} \Rightarrow Ce$$

At $t=0$, 100 lit. of fresh water so NO salt, thus $y=0$

$$|5s - 1/20(0)| = Ce^0 \Rightarrow 5s = C.$$

Thus, $5s - 1/20y = 5se^{-1/20t}$

$$20(5s - 5se^{-1/20t}) = y$$

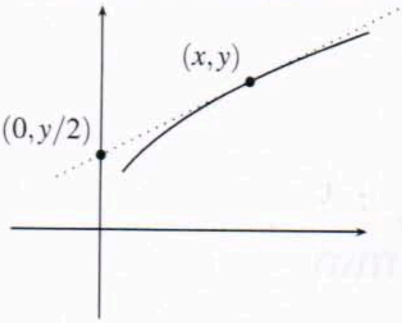
At $t=10$, $y = 3 \cdot 100$

$$20(5s - 5se^{-1/2}) = 300$$

$$5s(1 - e^{-1/2}) = 300/20 \Rightarrow \boxed{s = \frac{300}{20 \cdot 5(1 - e^{-1/2})} = 7.62448}$$

10. (12 total points) A curve has the property that the tangent line to the curve at each point (x,y) has y-intercept $(0,y/2)$, as shown in the picture below. In addition, the curve passes through the point $(3,1)$.

(a) (4 points) Derive a differential equation for the curve using its slope at the point (x,y).



$$\text{Slope} = \frac{y - y/2}{x - 0} = \frac{y/2}{x} = \frac{y}{2x}$$

$$\boxed{\Rightarrow \frac{dy}{dx} = \frac{y}{2x}}$$

(b) (8 points) Solve the differential equation you obtained in part (a) to determine the equation of the curve.

① $\frac{dy}{dx} = \frac{y}{2x} \Rightarrow \frac{dy}{y} = \frac{dx}{2x} \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{2x}$

$\Rightarrow \ln|y| = \frac{1}{2}\ln|x| + C$

$\Rightarrow y = Ce^{1/2 \ln(x)}$

$\Rightarrow y = Ce^{\ln(x^{1/2})}$

$= Cx^{1/2}$

② Find C, know $x=3$, $y=1$

$1 = C(3^{1/2}) \Rightarrow C = 1/\sqrt{3}$. Thus

$$\boxed{y = \frac{1}{\sqrt{3}} x^{1/2}}$$

10. (8 points) Find the solution of the differential equation

$$\frac{dy}{dx} = xe^{x+y}$$

which satisfies the initial condition $y(0) = 1$.

$$\frac{dy}{dx} = xe^x e^y \Rightarrow \frac{dy}{e^y} = xe^x dx$$

$$\int e^{-y} dy = \int xe^x dx$$

IBP $u = x$ $dv = e^x$
$du = dx$ $v = e^x$

$$-e^{-y} = xe^x - \int e^x dx$$

$$-e^{-y} = xe^x - e^x + C$$

$$\ln(e^{-y}) = \ln(e^x - xe^x + C)$$

$$\Rightarrow -y = \ln(e^x - xe^x + C)$$

$$y = -\ln(e^x - xe^x + C)$$

Use initial condition, $y(0) = 1$

$$1 = -\ln(e^0 - 0e^0 + C)$$

$$1 = -\ln(1 + C) \Rightarrow -1 = \ln(1+C) \Rightarrow e^{-1} = 1 + C \Rightarrow C = e^{-1} - 1$$

$$\boxed{\therefore\ y = -\ln(e^x - xe^x + e^{-1} - 1)}$$

11. (10 points) The swine flu epidemic has been modelled by the Gompertz function, which is a solution of the differential equation

$$\frac{dy}{dt} = 1.2\,y\,(K - \ln(y)),$$

where $y(t)$ is the number of individuals (in thousands) in a large city that have been infected by time t, and K is a constant. Time t is measured in months, with $t = 0$ on July 9, 2009.

On July 9, 2009, 75 thousand individuals had been infected. One month later, 190 thousand individuals had been infected. Find

$$\lim_{t\to\infty} y(t),$$

which is the total number of individuals (in thousands) that will have been infected.

* Note t is in mnths. Thus $t=1 \Rightarrow y = 190$

y is in 000's

① Solve diff Q

$$\frac{dy}{1.2y(K-\ln(y))} = dt \Rightarrow \int \frac{dy}{1.2y(K-\ln(y))} = \int dt$$

$u = \ln(y)$

$du = 1/y\,dy$

$t + C$

$$\int \frac{du}{1.2(K-u)}$$

$$\Rightarrow -\frac{1}{1.2}\ln|K-u| = t + C \Rightarrow \frac{-1}{1.2}\ln|K-\ln(y)| = t + C$$

Thus $\ln|K-\ln(y)| = -1.2t + C \Rightarrow K - \ln(y) = e^{-1.2t}\cdot C$

$$\Rightarrow K - Ce^{-1.2t} = \ln(y)$$

② At $t=0$, $y=75$ and $t=1$, 190

$K - Ce^{0} = \ln(75) \Rightarrow \boxed{K = C + \ln(75)}$

$C + \ln(75) - Ce^{-1.2} = \ln(190)$

$C(1-e^{-1.2}) = \ln(190) - \ln(75) \Rightarrow \boxed{C = \frac{\ln(190/75)}{(1-e^{-1.2})}}$

③ $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} e^{K} e^{-Ce^{-1.2t}} = e^{K}$

$= e^{C+\ln(75)} = \boxed{283.6286}$

11. (8 total points) Suppose we have a colony of bacteria living in a Petri dish. Due to space limitations, there is a maximum number, k, of bacteria that can live in the dish. Let $P(t)$ be the population of the bacterial colony at time t. According to one model for population growth, the rate of growth of the population $\frac{dP}{dt}$ is proportional to the difference of the threshold population k and the present population; in other words

$$\frac{dP}{dt} = c(k-P). \tag{1}$$

The constant of proportionality c measures how quickly the bacteria multiply. For simplicity, we take $c = 1$.

(a) (6 points) Solve this differential equation for the unknown function $P(t)$.

$$\frac{dP}{dt} = (k-P) \Rightarrow \frac{dP}{k-P} = dt \Rightarrow \int \frac{dP}{k-P} = \int dt$$

$$\Rightarrow -\ln|k-P| = t + C$$

$$\Rightarrow |k-P| = Ce^{-t} \Rightarrow \boxed{k - Ce^{-t} = P}$$

(b) (2 points) If $k = 5{,}000{,}000$ and the initial population size is $P(0) = 1{,}000{,}000$, compute $\lim_{t\to\infty} P(t)$.

$P(t) = 5{,}000{,}000 - Ce^{-t}$ $\quad P(0) = 5{,}000{,}000 - C = 1{,}000{,}000$

$\Rightarrow C = 4{,}000{,}000$

Thus

$$\lim_{t\to\infty} 5000{,}000 - 4{,}000{,}000\, e^{-t} = \boxed{5000{,}000}$$

10. (8 points) Find the solution $y(x)$ for $x \geq 1$ of the initial value problem

$$\frac{y}{x^3}\frac{dy}{dx} = 4\ln(x), \quad y(1) = 2.$$

(1)

$$\frac{y\,dy}{x^3dx} = 4\ln(x) \Rightarrow y\,dy = 4x^3\ln(x)\,dx$$

$$\int y\,dy = \int 4x^3\ln(x)\,dx \quad \text{IBP} \longrightarrow \quad u=\ln(x) \quad dv = x^3$$

$$du = 1/x \quad v = 1/4x^4$$

$$\frac{1}{2}y^2 = \left(\frac{1}{4}x^4\ln(x) - \int\frac{1}{4}x^3dx\right)4$$

$$1/2y^2 = (1/4x^4\ln(x) - 1/16x^4 + C)4$$

$$y^2 = 2x^4\ln(x) - 1/2x^4 + C$$

$$y = \sqrt{2x^4\ln(x) - \frac{1}{2}x^4 + C}$$

(2) Solve $y(1) = 2$

$$2 = \sqrt{2(1^4)\ln(1) - 1/2(1^4) + C} \Rightarrow 2 = \sqrt{C - 1/2}$$

$$\Rightarrow 4 = C - 1/2 \Rightarrow \boxed{C = 9/2}$$

$$\boxed{\therefore\ y = \sqrt{2x^4\ln(x) - 1/2x^8 + 9/2}}$$

10. (12 total points) A 50-gallon tank initially contains 20 gallons of water in which 10 lbs of salt are dissolved. Pure water enters the tank at a rate of 4 gal/min. Simultaneously, a drain is open at the bottom of the tank, allowing the salt-water solution to leave the tank at a rate of 2 gal/min. The solution is kept thoroughly mixed.

   (a) (3 points) Find the volume $V(t)$ (in gallons) of the salt-water solution in the tank at time t (in minutes).

$$V(t) = V_0 + (\text{Rate in} - \text{Rate out})\,t$$
$$= 20 + (4-2)\,t$$
$$\boxed{= 20 + 2t}$$

   (b) (3 points) Write a differential equation for the amount $y(t)$ (in lbs) of salt in the tank at time t.

$$\frac{dy}{dt} = \text{Rate in} - \text{Rate out} = 0 - \frac{y(t)}{20+2t}\cdot 2$$

pure water

$$\boxed{\frac{dy}{dt} = \frac{-2y}{20+2t}}$$

   (c) (3 points) Solve this differential equation and use the initial amount of salt in the tank to find a formula for $y(t)$.

$$\frac{dy}{-2y} = \frac{dt}{20+2t} \Rightarrow \int \frac{dy}{-2y} = \int \frac{dt}{20+2t}$$
$$\Rightarrow \frac{-1}{2}\ln(y) = \frac{1}{2}\ln(20+2t) + C$$
$$\Rightarrow \ln(y) = -\ln(20+2t) + C$$
$$y = C\,\frac{1}{20+2t}$$

as $t=0$, $y=10$

$$10 = \frac{C}{20} \Rightarrow C = 200$$

$$\boxed{\therefore\ y = \frac{200}{20+2t}}$$

   (d) (3 points) What is the amount of salt in the tank at the moment that the tank becomes full?

When tank is full $V(t) = 50 \Rightarrow 50 = 20 + 2t$ $\quad 30/2 = t$

Hence,

$$y = \frac{200}{20 + (\frac{30}{2})(2)} = \frac{200}{50} = \boxed{4\text{ lbs}}$$

9. (10 points) Find the solution of the differential equation that satisfies the given initial condition.

$$\frac{dy}{dx} = \frac{x\sin(x^2)}{y}, \qquad y(0) = -2.$$

1

$y\,dy = x\sin(x^2)\,dx \Rightarrow \int y\,dy = \int x\sin(x^2)\,dx$

$\downarrow \quad u = x^2$

$du = 2x\,dx$

$\frac{1}{2}y^2 = \frac{1}{2}\int \sin(u)\,du$

$= -\frac{1}{2}\cos(u) + C$

$= -\frac{1}{2}\cos(x^2) + C$

$y^2 = -\cos(x^2) + C$

$y = -\sqrt{C - \cos(x^2)}$

↑ * Note - sign as $y(0) = \underline{\underline{-2}}$

2 Plug in $y(0) = -2$

$-2 = -\sqrt{C - \cos(0)}$

$-2 = -\sqrt{C-1}$

$4 = C - 1$

$C = 5$

$$\boxed{\therefore\ y = -\sqrt{5 - \cos(x^2)}}$$

11. (8 points) Find the function $y(x)$ which satisfies $\frac{dy}{dx} = \frac{x(y^2+1)}{\sqrt{x^2-1}}$ such that $y = 1$ when $x = \sqrt{2}$.

① $\frac{dy}{y^2+1} = \frac{x\,dx}{\sqrt{x^2-1}} \Rightarrow \int \frac{dy}{y^2+1} = \int \frac{x\,dx}{\sqrt{x^2-1}}$

$\hookrightarrow u = x^2-1$

$du = 2x\,dx$

$\tan^{-1}(y) = \frac{1}{2}\int \frac{du}{\sqrt{u}}$

$\tan^{-1}(y) = \sqrt{u} + C$

$\tan^{-1}(y) = \sqrt{x^2-1} + C$

$y = \tan\left(\sqrt{x^2-1} + C\right)$

② $y=1,\ x=\sqrt{2}$

$1 = \tan\left(\sqrt{2-1} + C\right)$

$1 = \tan(1+C)$

$\tan^{-1}(1) = 1+C$

$\pi/4 = 1+C$

$\pi/4 - 1 = C$

$$\boxed{\therefore\ y = \tan\left(\sqrt{x^2-1} + \pi/4 - 1\right)}$$

10. (10 points) At time $t = 0$, a tank contains 100 gallons of pure gasoline. A mixture whose volume is 30% ethanol and 70% gasoline is pumped into the tank at a rate of 2 gallons per minute. The solution is kept thoroughly mixed and drains from the tank at the same rate. Find a formula for the number of gallons of ethanol in the tank after t minutes.

e = gallons of ethanol.

① $\frac{de}{dt}$ = Rate in - Rate out

$= .3(2) - \frac{e}{100} 2$

② Solve

$$\frac{de}{.6 - \frac{e}{50}} = dt \Rightarrow \int \frac{de}{.6 - e/50} = \int dt$$

$$-50 \ln|.6 - e/50| = t + C$$

$$\ln|.6 - e/50| = \frac{-t}{50} + C$$

$$.6 - e/50 = Ce^{-t/50}$$

$$.6 - Ce^{-t/50} = e/50$$

$$30 - Ce^{-t/50} = e(t)$$

③ $e(0) = 0$

↑ b/c pure gasoline, no ethanol.

$$30 - Ce^0 = 0$$

$$C = 30$$

$$\boxed{\therefore\ 30 - 30e^{-t/50} = e(t)}$$

8. (10 points) Find the solution of the differential equation that satisfies the given initial condition.

$$\frac{dP}{dt} = \sqrt{Pt}, \qquad P(4) = 1.$$

① $\frac{dP}{dt} = \sqrt{P} \cdot \sqrt{t} = \frac{dP}{\sqrt{P}} = \sqrt{t}\,dt$

$\Rightarrow \int P^{-1/2}\,dP = \int t^{1/2}\,dt$

$2\sqrt{P} = 2/3\, t^{3/2} + C$

$\sqrt{P} = 1/3\, t^{3/2} + C$

$P = (1/3\, t^{3/2} + C)^2$

② $P(4) = 1$

$1 = (1/3(4^{3/2}) + C)^2$

$1 = (8/3 + C)^2$

$1 = 8/3 + C \qquad C = -5/3$

$\boxed{\therefore\ P = (1/3\, t^{3/2} - 5/3)^2}$

9. (12 total points) In 2009 (which we take to be $t = 0$, with t in years) there are 5000 wolves in a big forest area. In the absence of hunting, the wolf population would increase at the rate of 1% per year. However, hunters are killing wolves at the steady rate of 100 wolves per year.

(a) (4 points) Write a differential equation for $W(t)$.

$$\frac{dW}{dt} = \text{Rate in} - \text{Rate out}$$
$$= .01W - 100$$

$$\boxed{\frac{dW}{dt} = .01W - 100}$$

(b) (4 points) Solve this differential equation and use the initial number of wolves to find a formula for $W(t)$.

① $\frac{dW}{.01W-100} = dt \Rightarrow \int \frac{dW}{.01W-100} = \int dt \Rightarrow 100\ln|.01W-100| = t + C$

$$\ln|.01W-100| = \frac{t}{100} + C$$
$$.01W - 100 = Ce^{t/100}$$
$$.01W = Ce^{t/100} + 100$$
$$W = Ce^{t/100} + 10000$$

② $W(0) = 5000$

$5000 = Ce^0 + 10000 \quad C = -5000$

$$\Rightarrow \boxed{W = 10{,}000 - 5000e^{t/100}}$$

(c) (4 points) In what year will the wolves in this forest area die out? Your answer should be some year in this century.

Wolves will die out when $W = 0$

$$0 = 10{,}000 - 5000e^{t/100}$$
$$\frac{10{,}000}{5000} = e^{t/100}$$
$$\ln(2) = t/100$$
$$100\ln(2) = t$$
$$t = 69.3147 \text{ years}$$

As it is 2009,

$2009 + 69.3147$

$\boxed{= 2078}$