1. (10 total points) Evaluate the following indefinite integrals.

(a) (5 points) $\displaystyle\int \frac{x}{\sqrt{x+2}}\,dx$

$u=\sqrt{x+2}$ $\quad u^2 = x+2$

$2u\,du = dx$

$$\int \frac{(u^2-2)2\cancel{u}}{\cancel{u}}\,du = 2\left[\tfrac{1}{3}u^3 - 2u\right] + C$$

$$= \boxed{\tfrac{2}{3}\left(\sqrt{x+2}\right)^3 - 4\left(\sqrt{x+2}\right) + C}$$

(b) (5 points) $\displaystyle\int e^{2x}\sec(e^{2x})\tan^3(e^{2x})\,dx$

$u = e^{2x}$ $\quad du = 2e^{2x}dx$

$$\tfrac{1}{2}\int \sec(u)\tan^3(u)\,du = \tfrac{1}{2}\int \sec(u)\tan(u)\tan^2(u)\,du$$

$v = \sec(u)$ $\quad dv = \sec u \tan u\, du$

$$= \tfrac{1}{2}\int \tan^2(u)\,dv = \tfrac{1}{2}\int v^2 - 1\,dv = \tfrac{1}{2}\left(\tfrac{1}{3}v^3 - v\right) + C$$

$$\boxed{= \tfrac{1}{6}\left(\sec^3(e^{2x})\right) - \tfrac{1}{2}\left(\sec(e^{2x})\right) + C}$$

2. (10 total points) Evaluate the following definite integrals.

(a) (5 points) $\int_1^2 \frac{\ln x}{x^3}\,dx$ IBP $u = \ln(x)$ $dv = x^{-3}dx$

$du = \frac{1}{x}dx$ $v = \frac{-1}{2}x^{-2}$

$$= \frac{-\ln(x)}{2x^2}\Big|_1^2 + \frac{1}{2}\int_1^2 x^{-3}\,dx$$

$$= \frac{-\ln(x)}{2x^2}\Big|_1^2 - \frac{1}{4}x^{-2}\Big|_1^2 = \frac{-\ln(2)}{8} - \frac{1}{16} + \frac{1}{4}$$

$$= \boxed{\frac{-\ln(2)}{8} + \frac{3}{16}}$$

(b) (5 points) $\int_2^3 \sqrt{4x - x^2}\,dx$

Complete the square $4x - x^2 = -(x^2 - 4x) = -[(x-2)^2 - 4]$

$\int_2^3 \sqrt{4-(x-2)^2}\,dx$ $x - 2 = 2\sin\theta$ $-\pi/2 \le \theta \le \pi/2$

$dx = 2\cos\theta\,d\theta$

$$\int_0^{\pi/6} \sqrt{4 - 4\sin^2\theta}\cdot 2\cos\theta\,d\theta$$

$$= \int_0^{\pi/6} 4\cos^2\theta\,d\theta = \int_0^{\pi/6} 2(1+\cos(2\theta))\,d\theta$$

$$= 2\left[\theta + \frac{1}{2}\sin(2\theta)\right]\Big|_0^{\pi/6}$$

$$= 2\left[\frac{\pi}{6} + \frac{1}{2}\sin\left(\frac{\pi}{3}\right)\right] = \boxed{\frac{\pi}{3} + \frac{\sqrt{3}}{2}}$$

3. (10 points) Find the area under the curve

$$y = \frac{1}{\sqrt{|2x - x^2|}}$$

and above the x-axis, for $-1 \le x \le 1$. *CAUTION*: The integral you will get is an improper integral. Be sure to treat your integral as an improper integral, and justify your answer.

Find zeros of $2x - x^2 \Rightarrow 0 = x(2-x)$ $x = 0$ and $x = 2$

	$-1 < x < 0$	$0 < x < 2$
$2x - x^2$	$-$	$+$

$$\int_{-1}^{1} \frac{1}{\sqrt{|2x-x^2|}}\,dx = \int_{-1}^{0} \frac{1}{\sqrt{-(2x-x^2)}} + \int_{0}^{1} \frac{1}{\sqrt{2x-x^2}}\,dx$$

$$\Rightarrow \lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{\sqrt{x^2-2x}}\,dx + \lim_{a\to 0^+} \int_{a}^{1} \frac{1}{\sqrt{-(x^2-2x)}}\,dx$$

Complete the sq. $x^2 - 2x = (x-1)^2 - 1$

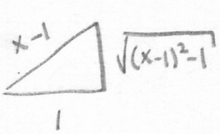
$$\Rightarrow \lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{\sqrt{(x-1)^2-1}}\,dx + \lim_{a\to 0^+} \int_{a}^{1} \frac{1}{\sqrt{1-(x-1)^2}}\,dx$$

$x - 1 = \sec\theta$
$dx = \sec\theta\tan\theta$

$x - 1 = \sin\theta$
$dx = \cos\theta\, d\theta$

$$\hookrightarrow \int \frac{\cos\theta}{\cos\theta}\,d\theta$$
$$= \theta + C$$
$$= \arcsin(x-1) + C$$

$$\int \frac{\sec\theta \cancel{\tan\theta}\,d\theta}{\cancel{\tan\theta}} = \ln|\sec\theta + \tan\theta| + C$$



$$\frac{x-1}{1} = \frac{h}{a}$$

$$\lim_{a\to 0^+} \arcsin(x-1)\Big|_a^1$$

$$= \lim_{a\to 0^+} \underset{0}{\arcsin(0)} - \arcsin(a-1)$$

$$\lim_{b\to 0^-} \ln\left|x - 1 + \sqrt{(x-1)^2-1}\right|\Big|_{-1}^{b}$$

$$\lim_{b\to 0^-} \ln\left|b-1+\sqrt{(b-1)^2-1}\right| - \ln|-2+\sqrt{3}|$$

$$= -\ln|-2+\sqrt{3}|$$

$$= -\arcsin(-1) = -\left(\frac{-\pi}{2}\right) = \frac{\pi}{2}$$

$$\therefore \int_{-1}^{1} \frac{1}{\sqrt{|2x-x^2|}}\,dx = \boxed{\frac{\pi}{2} - \ln|-2+\sqrt{3}|}$$

4. (10 points) Let $A(t)$ denote the area under the curve $y = \sqrt{1-x^3}$ and above the x-axis, between the vertical lines $x = t$ and $x = 2t$.
Find the value of t for which $A(t)$ is a maximum on the interval $0 \le t \le 1/2$.
Justify that your answer gives the maximum.

FTC

$$A(t) = \int_t^{2t} \sqrt{1-x^3}\,dx$$

① Rewrite

$$A(t) = \int_t^0 \sqrt{1-x^3}\,dx + \int_0^{2t} \sqrt{1-x^3}\,dx$$

$$= \int_0^{2t} \sqrt{1-x^3}\,dx - \int_0^{t} \sqrt{1-x^3}\,dx$$

② Apply FTC

$$A'(t) = \sqrt{1-(2t)^3} \cdot 2 - \sqrt{1-t^3}$$

To find max. set $A'(t) = 0$

$$0 = 2\sqrt{1-8t^3} - \sqrt{1-t^3}$$

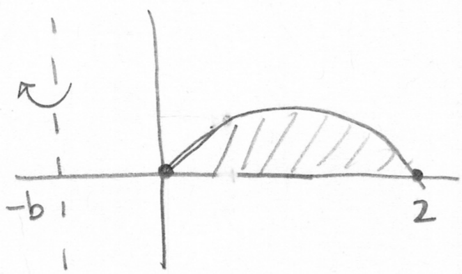
$$\sqrt{1-t^3} = 2\sqrt{1-8t^3}$$

$$1-t^3 = 4(1-8t^3)$$

$$31t^3 = 3$$

$$\boxed{t = \left(\frac{3}{31}\right)^{1/3}}$$

5. (10 points) The region under the curve $y = 2x - x^2$ and above the x-axis is rotated around the line $x = -b$, where b is a positive constant. Find the value of b for which the volume of the solid so obtained is 10π.



$0 = 2x - x^2 = x(2-x)$

Use Shells (in terms of x)

$$2\pi\int_0^2 (2x-x^2)(x+b)\,dx = 10\pi$$

$$= 2\pi\int_0^2 2x^2 - x^3 + 2bx - bx^2\,dx$$

$$= 2\pi\left[\frac{2}{3}x^3 - \frac{1}{4}x^4 + bx^2 - \frac{b}{3}x^3\right]\Big|_0^2$$

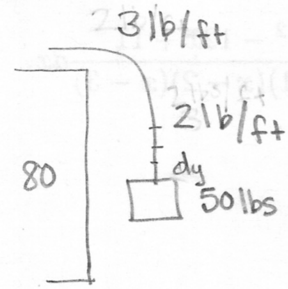
$$= 2\pi\left[\frac{16}{3} - 4 + 4b - \frac{8}{3}b\right] = 10\pi$$

$$\Rightarrow \frac{4}{3} + \frac{4}{3}b = 5$$

$$4 + 4b = 15$$

$$4b = 11 \qquad \boxed{b = \frac{11}{4}}$$

6. (10 points) An 80-ft cable is used to lift 50 pounds of coal up a mine shaft 80 ft deep. The bottom half of the cable weighs 2 pounds per foot and the top half of the cable weighs 3 pounds per foot. Find the work done in foot-pounds.



y = distance from the bottom

Work done for rope

$$\int_{40}^{80} 3(80-y)\,dy + \int_{0}^{40} 2(80-y)\,dy$$

$$= 240y - \frac{3}{2}y^2 \Big|_{40}^{80} + 160y - \frac{1}{2}y^2 \Big|_{0}^{40}$$

$$= 7{,}200$$

Work done for coal

$$\int_0^{80} 50\,dy = 50 \cdot 80 = 4000$$

$$\Rightarrow \text{Total Work} = 7200 + 4000 = \boxed{11{,}200}$$

7. (10 total points) This problem gives one way to find a rational number that approximates π.

(a) (4 points) Show that $$\int_0^4 \frac{dy}{1+(y^2/16)} = \pi.$$

$y = 4\tan\theta$

$$\int_0^4 \frac{16\,dy}{16+y^2} = \frac{16}{4}\arctan\left(\frac{y}{4}\right)\Big|_0^4 = 4\left(\frac{\pi}{4}\right) = \pi$$

(b) (6 points) Subdivide $[0,4]$ into 4 equal subintervals and use Simpson's rule to approximate the integral in part (a). You do *NOT* have to simplify any expressions involving fractions. (For example, if you have terms that look something like $\frac{2}{1+\frac{81}{16}}$, just leave them in that form.)

$$\Delta x = \frac{4-0}{4} = 1$$

$$\frac{1}{3}\left[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)\right]$$

$$= \frac{1}{3}\left[1 + \frac{4}{1+1/16} + \frac{2}{1+1/4} + \frac{4}{1+9/16} + \frac{1}{2}\right]$$

$$= \frac{8011}{2550}$$

8. (10 points) Let $\mathscr{R}$ be the region above the x-axis, below the graph of $y = \dfrac{1}{(x+1)(x+2)}$, between $x = 1$ and $x = 2$. Find the x-coordinate of the centroid (center of mass) of the region $\mathscr{R}$.

$$\text{Area} = \int_1^2 \frac{1}{(x+1)(x+2)}\,dx \qquad \frac{A}{x+1} + \frac{B}{x+2} = \frac{1}{(x+1)(x+2)}$$

$$A(x+2) + B(x+1) = 1$$

$$x=-2: \ -B = 1 \quad \boxed{B=-1}$$

$$x = -1: \quad \boxed{A = 1}$$

$$A = \int_1^2 \frac{1}{x+1} + \frac{-1}{x+2} = \ln|x+1| - \ln|x+2| \Big|_1^2 = \ln(3) - \ln(4) - \ln(2) + \ln(3)$$

$$= 2\ln(3) - 3\ln(2)$$

$$M_y = \int_1^2 \frac{x}{(x+1)(x+2)}\,dx \qquad \frac{A}{x+1} + \frac{B}{x+2} = \frac{x}{(x+1)(x+2)}$$

$$A(x+2) + B(x+1) = x$$

$$-2: \quad -B = -2 \quad \boxed{B=2}$$

$$-1: \quad \boxed{A = -1}$$

$$= \int_1^2 \frac{-1}{x+1} + \frac{2}{x+2}\,dx = -\ln|x+1| + 2\ln|x+2| \Big|_1^2$$

$$= -\ln(3) + 2\ln(4) + \ln(2) - 2\ln(3)$$

$$= 5\ln(2) - 3\ln(3)$$

$$\therefore \boxed{\bar{x} = \frac{M_y}{\text{Area}} = \frac{2\ln(3) - 3\ln(2)}{5\ln(2) - 3\ln(3)}}$$

9. (10 points) Find the solution of the initial value problem

$$\frac{dy}{dt} = \frac{1}{ye^{t/2}}, \quad y(0) = -1.$$

Give your answer in the form $y = f(t)$.

1) Separate the variables

$$y\,dy = e^{-t/2}\,dt$$

2) Integrate both sides

$$\int y\,dy = \int e^{-t/2}\,dt$$

$$\tfrac{1}{2}y^2 = -2e^{-t/2} + C$$

3). Solve for y

$$y^2 = -4e^{-t/2} + C$$

$$y = \pm\sqrt{-4e^{-t/2} + C}$$

4). Solve for C

$$-1 = \pm\sqrt{-4 + C} \Rightarrow 1 = -4 + C \Rightarrow C = 5$$

$$\therefore \boxed{y = -\sqrt{-4e^{-t/2} + 5}}$$

10. (10 points) A tank initially contains 15 liters of pure water. Sea water with a salt concentration of 35 grams per liter is added at a rate of 2 liters per minute. In addition, pure water is added at a rate of 1 liter per minute. The solution is kept thoroughly mixed and is drained from the tank at a rate of 3 liters per minute. How much salt is in the tank after t minutes?

$Q(t)$ = amt of salt in the tank

$\frac{dQ}{dt}$ = Rate in − Rate out

Rate in: Concentration = 35g/lit Rate in of liquid = 2 lit/min

Rate in = 35g/lit * 2 lit/min = 70 g/min

Rate out: Concentration = $\frac{Q(t)}{15}$ Rate out of liquid = 3 lit/min

Rate out = $\frac{Q}{15} \cdot 3 = \frac{Q}{5}$

$$\frac{dQ}{dt} = 70 - Q/5 \Rightarrow \int \frac{dQ}{70 - Q/5} = \int dt \Rightarrow -5\ln|70 - Q/5| = t + C$$

$$70 - Q/5 = Ce^{-t/5}$$

$$350 - Ce^{-t/5} = Q(t) \qquad Q(0) = 0$$

$$C = 350$$

$$\Rightarrow \boxed{Q(t) = 350 - 350e^{-t/5}}$$