$\boxed{3}$ (7 points) A bag of sand originally weighs 160 lbs. It is lifted at a constant rate of 4 ft/min. The sand leaks out of the bag at a constant rate so that when it has been lifted 20 ft only half the sand is left. How much work is done lifting the bag 20 ft?

① Find mass as a function of y height

(Don't have) lbs is a force

② Find Force as a funct. of y height above ground

Need to find lbs/ft 80 since 4f/min and at y = 20ft only 80lbs.

we know $\frac{4ft}{min} \cdot t = 20 \Rightarrow t = 5\text{mins}$.

$\therefore \frac{80lbs}{5min} = \frac{16lbs}{min}$ Therefore $\frac{16lbs}{min} \cdot \frac{min}{4ft} = \frac{4lbs}{ft}$

Hence Force at height y is

$160 - \frac{4lbs}{ft} \cdot y = \text{Force}$

③ Distance is always dy

④ Bounds range of y

y = 0 to 20.

⑤ $W = \int_0^{20} 160 - 4y \, dy$

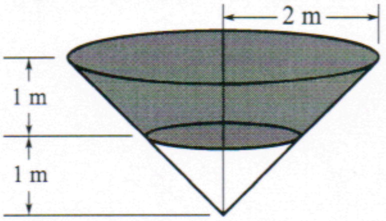
$= 160y - 2y^2 \Big|_0^{20}$

$\boxed{= 2400 \text{ ft-lbs}}$

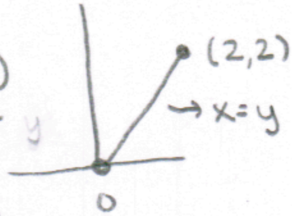
$\boxed{4}$ (7 points) Determine if the improper integral $\int_0^1 x^3 \ln(x)\, dx$ is convergent or divergent. If it is convergent, evaluate it.

5. (12 total points) You want to dig a hole in the ground in the shape of an inverted circular cone with height 2 m and base radius 2 m. The dirt in the hole has density $\rho = 1676$ kg/m^3, and the acceleration due to gravity is $g = 9.8$ m/sec^2.

(a) (6 points) Find the work required to remove the *top* 1 meter of dirt from the hole (moving it up to ground level). Give your answer in decimal form.



① Volume of slice:
(use washer method)
→ rotating $x=y$ about y-axis



W/ washers $\text{Vol} = \pi R^2\,dy$
Vol of slice $= \pi(y^2)\,dy$

② Find mass
mass = Vol · density
$= 1676\pi y^2\,dy$

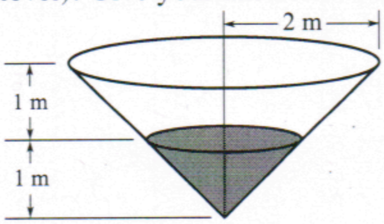
③ Find Force
Force = mass · accel.
$= 1676\pi y^2 (9.8)\,dy$

④ Find distance: At height y, how far to move slice
$2-y$

⑤ Bounds: Dirt you are moving (B/c top 1 meter)
moving $y=2$ to $y=1$

⑥ Work $= \int_1^2 9.8 \cdot 1676\pi y^2 (2-y)\,dy$
$= \int_1^2 16424.8\pi (2y^2 - y^3)\,dy$
$= 16424.8\pi \left(\frac{2}{3}y^3 - \frac{1}{4}y^4\right)\Big|_1^2$
$\boxed{= \frac{225{,}841}{15}\pi}$ 15,056.066

(b) (6 points) Find the work required to remove the *bottom* 1 meter of dirt (moving it up to ground level). Give your answer in decimal form.



① Vol. of slice (same as above)
$= \pi y^2\,dy$

② Mass
$= 1676\pi y^2\,dy$

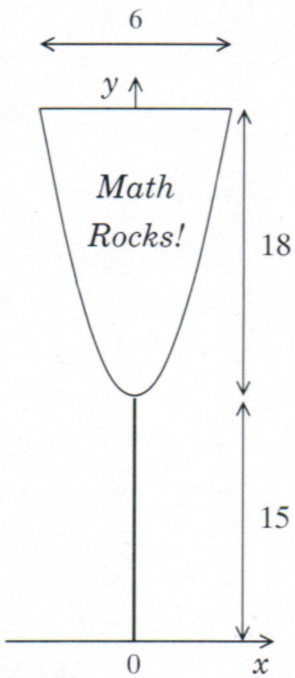
③ Force
$= 1676\pi y^2 (9.8)\,dy$

④ Distance (same as above)
$2-y$

⑤ Bounds: moving <u>bottom</u> half
so $y = 1$ to 0

⑥ Work
$\int_0^1 1676\pi (9.8) y^2 (2-y)\,dy$
$= \int_0^1 16424.8\pi (2y^2 - y^3)\,dy$
$= 16424.8\pi \left(\frac{2}{3}y^3 - \frac{1}{4}y^4\right)\Big|_0^1$
$\boxed{= \frac{20{,}531}{3}\pi}$

6. (10 points)



A flat math billboard is in the shape of (what else?) a parabola. Its top side is 6 feet wide and the billboard is 18 feet high, measured from the lowest to the highest point. It is mounted on a pole and the lowest point of the billboard is 15 feet above the ground.

Before it was mounted on the pole, the billboard was originally lying flat on the ground. The billboard weighs 3 pounds per square foot. Set up a definite integral for the work done in lifting this billboard up to where it now stands. Evaluate the integral and find the work done.

(*Hint: Slice the billboard in strips parallel to the straight edge.*)

ⓐ Find equation of parabola
vertex at $(0,15)$ and goes through $(3,33)$

$\therefore\ ax^2+15=y$

$a(3^2)+15=33 \Rightarrow a=2$

Thus $y=2x^2+15$

ⓑ Area of a slice is (always in y)

$2\sqrt{\frac{y-15}{2}}=x$ Need to multiply by 2 to get total width,

$2\sqrt{\frac{y-15}{2}}$

$\therefore$ Area of slice: $2\sqrt{\frac{y-15}{2}}\cdot dy$

ⓒ Force of a slice
Since 3lbs/ft

Force $=6\sqrt{\frac{y-15}{2}}\,dy$

ⓓ Distance (at height y how far to move slice)
y = distance

ⓔ Bounds: $y=15$ to 18

ⓕ Work $=\int_{15}^{18} 6y\sqrt{\frac{y-15}{2}}\,dy$ $\quad u=\frac{y-15}{2}\quad du=\frac{1}{2}dy$

$=\int_0^{3/2} 12(2u+15)\sqrt{u}\,du$

$=\int_0^{3/2} 12(2u^{3/2}+15u^{1/2})\,du$

$=12\left(\frac{4}{5}u^{5/2}+\frac{30}{3}u^{3/2}\right)\Big|_0^{3/2}$

$\boxed{=246.90856}$