Name________________________________

Math 125 **First Midterm** **8:30 Jan. 30, 2014**

(6 problems, 80 minutes, 100 points)

1. (30 points) Evaluate the following definite and indefinite integrals. Please show your work clearly, and simplify when possible.

(a) $\displaystyle\int_{-1}^{2} \frac{x^5\,dx}{\sqrt{x^3+17}}$; (b) $\displaystyle\int \frac{e^{\sin(\pi x)}\cos(\pi x)dx}{1+e^{2\sin(\pi x)}}$.

(a) Let $u = x^3+17$ $du = 3x^2 dx$

Change bounds $u = (-1)^3+17 = 16$

$u = (2^3)+17 = 25$

$x^3 = u-17$

$\int_{16}^{25} \frac{1}{3}\frac{x^3}{\sqrt{u}}du = \int_{16}^{25}\frac{1}{3}\frac{u-17}{\sqrt{u}}du$

$= \int_{16}^{25}\frac{1}{3}(u^{1/2}-17u^{-1/2})du$

$= \frac{2}{9}u^{3/2} - \frac{34}{3}u^{1/2}\Big|_{16}^{25}$

$= \frac{2}{9}5^3 - \frac{34}{3}(5) - \left(\frac{2}{9}4^3 - \frac{34}{3}(4)\right)$

$= \boxed{\frac{20}{9}}$

(b) $u = \sin(\pi x)$

$du = \cos(\pi x)\cdot\pi\, dx$

$= \int \frac{1}{\pi}\frac{e^u}{1+e^{2u}}du$

$v = e^u$

$dv = e^u du$

$= \int \frac{1}{\pi}\frac{1}{1+v^2}dv$

$= \frac{1}{\pi}\arctan(v) + C$

$\boxed{= \frac{1}{\pi}\arctan\left(e^{\sin(\pi x)}\right) + C}$

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2. (14 points) In this problem use the trapezoid rule (which is the same as the average of the left and right Riemann sums). Suppose you have a table of velocities in m/sec of a fast bicycle A and a slower one B. At time $t = 0$, A passes B. The table has the velocities $v_0^A, \ldots, v_4^A$ of A at 15-sec intervals for a minute, and similarly for B. Using the trapezoid rule, write an expression in terms of the v-values for the distance in meters that A is ahead of B after the minute is over.

t	$v_A(t)$	$v_B(t)$
0	v_0^A	v_0^B
15	v_1^A	v_1^B
30	v_2^A	v_2^B
45	v_3^A	v_3^B
60	v_4^A	v_4^B

The distance A over B $= $ initial distance bicycle A is ahead of bicycle B $+ \int_0^{60} V_A(t) - V_B(t)\,dt$

(This is 0 in the problem)

Right Riemann Sum

$$\text{Trapezoid} = \frac{1}{2}\Big[15\left[(V_1^A - V_1^B) + (V_2^A - V_2^B) + (V_3^A - V_B^B) + (V_4^A - V_4^B)\right]$$
$$+ 15\left[(V_0^A - V_0^B) + (V_1^A \cdot V_1^B) + (V_2^A - V_2^B) + (V_3^A - V_3^B)\right]\Big]$$

left Riemann Sum

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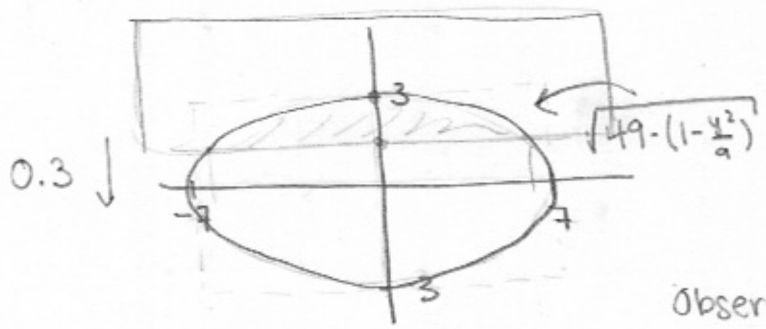
3. (10 points) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$, has major axis $2a$ ("longest diameter"), has minor axis $2b$ ("shortest diameter"), and has area πab. Suppose that the ellipse

$$\frac{x^2}{49} + \frac{y^2}{9} = 1$$

is being covered by a big rectangle whose lower horizontal side is being pulled downward at 0.3 units/sec. At the instant when the lower side of the rectangle crosses the x-axis, what is the rate at which the area of the ellipse is being covered?

$$\frac{x^2}{49} = 1 - y^2/9$$



$$x = \pm\sqrt{49\,(1 - y^2/9)}$$

Observe that

$3 - 0.3t$ is the lower limit

$$\int_{3-0.3t}^{3} 2\sqrt{49(1-y^2/9)}\,dy$$

$$\Rightarrow -\int_{3}^{3-0.3t} 2\sqrt{49(1-y^2/9)}\,dy$$

By the FTC,

$$\frac{d}{dt}\int_{3}^{3-0.3t} 2\sqrt{49(1-y^2/9)}\,dy = -2\sqrt{49\left(1-\frac{(3-0.3t)^2}{9}\right)} \cdot (-0.3)$$

The instant when the rectangle hits $y=0$ is precisely at

$3 - 0.3t = 0 \Rightarrow 3 = \frac{3}{10}t \Rightarrow t = 10$

$\therefore$ The rate at which the area of the ellipse is being covered is

$$-2\sqrt{49\left(1-\frac{(3-0.3(10))^2}{9}\right)}\,(-0.3) = 0.6\sqrt{49} = \boxed{4.2\ \frac{m^2}{sec}}$$

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4. (24 points) In the xy-plane draw the circle $x^2 + y^2 = 100$ and the line segment joining the points $(0, 10)$ and $(6, 8)$. Let R be the region in the circle that's above the line segment. Write each of the following in terms of definite integrals. Do **NOT** evaluate the integrals.

(a) The volume using the washer method when R is revolved around the x-axis.

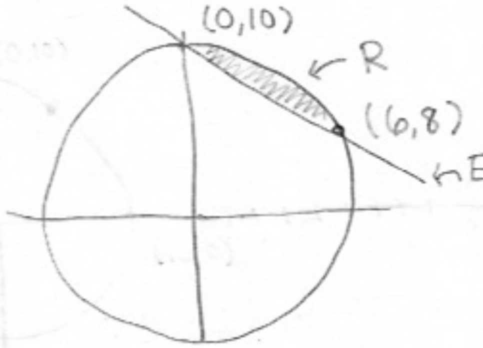
(b) The volume using the washer method when R is revolved around the y-axis.

(c) The volume using the shell method when R is revolved around the line $x = -2$.

Note

$6^2 + 8^2 = 100$

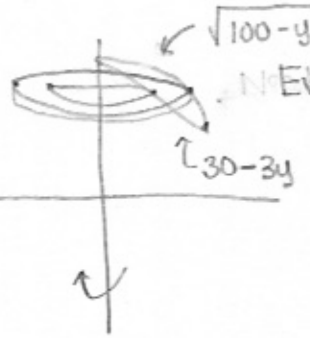
so the pt $(6,8)$ lies on the circle



Equation of line

$y = \frac{-1}{3}(x-0) + 10$

(b)



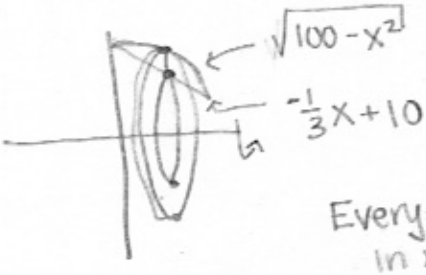
Everything in y's

Outer radius $= \sqrt{100-y^2}$

Inner radius $= 30-3y$

$$\boxed{\text{Volume} = \pi \int_8^{10} (\sqrt{100-y^2})^2 - (30-3y)^2 \, dy}$$

(a)



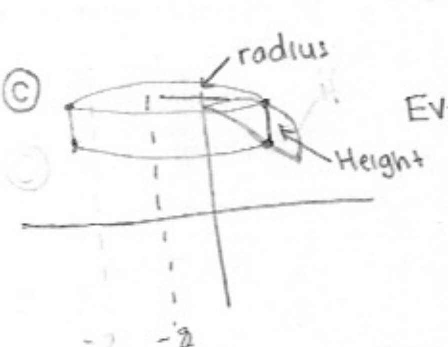
Outer radius $= \sqrt{100-x^2}$

Inner radius $= -\frac{1}{3}x + 10$

Everything in x's

$$\boxed{\text{Volume} = \pi \int_0^{6} (\sqrt{100-x^2})^2 - (-\tfrac{1}{3}x+10)^2 \, d}$$

(c)



Everything in x's

Height $= \sqrt{100-x^2} - (-\frac{1}{3}x + 10)$

Radius $= x + 2$

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$$\boxed{\text{Volume} = 2\pi \int_0^{6} (x+2)\left[\sqrt{100-x^2} - (-\tfrac{1}{3}x+10)\right] dx}$$

5. (12 points) A solid is formed as follows. Its base is the region in the xy-plane bounded by the curve $y = x^{-3/2}$, the x-axis, the line $x = 1$, and the line $x = 3$. Its cross-section by a plane perpendicular to the x-axis at x is a square resting on the xy-plane whose side is the line joining $(x, 0)$ to $(x, x^{-3/2})$. Find the volume of the solid. Please show your work clearly.

$$\text{Volume} = \int_1^3 \text{Area of square}\, dx = \int_1^3 (x^{-3/2})^2\, dx$$

$$= \int_1^3 x^{-3}\, dx$$

$$= \frac{-1}{2} x^{-2} \Big|_1^3 = \frac{-1}{2}\left[\frac{1}{9} - 1\right]$$

$$= \frac{8}{18} = \boxed{\frac{4}{9}}$$

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6. (10 points) During a certain journey of 200 miles the fuel efficiency of your car at a given point depended on several factors — road conditions, weather, traffic, etc. Let $f(x)$ denote the fuel efficiency, measured in miles per gallon, at a distance x from the start of the journey. Let

$$I = \int_0^{200} \frac{dx}{f(x)}.$$

What is the practical meaning of I, and what are the units of I?

$\frac{1}{f(x)}$ has units $\frac{1}{\frac{\text{miles}}{\text{gal}}} = \frac{\text{gal}}{\text{miles}}$

Since dx = miles, then

$$\boxed{I = \frac{\text{gal.}}{\text{miles}} \cdot \text{miles} = \text{gallons}}$$

The practical meaning of I is how much fuel it took you to drive 200 miles.