Your Name

Solutions

Your Signature

Student ID #

Quiz Section

Professor's Name

TA's Name

- This exam is closed book. You may use one 8.5″ × 11″ sheet of handwritten notes (both sides OK). Do not share notes. No photocopied materials are allowed.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- A scientific calculator is allowed, but graphing calculators are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- You may use any of the 20 integrals in the table on p. 484 of the text (p. 506 if you have the 5th edition of Stewart) without deriving them. **Show your work in evaluating any other integrals, even if they are on your note sheet**.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 9 pages, plus this cover sheet. Please make sure that your exam is complete.

| Question | Points | Score |
|---|---|---|
| 1 | 12 | |
| 2 | 12 | |
| 3 | 10 | |
| 4 | 12 | |
| 5 | 10 | |

| Question | Points | Score |
|---|---|---|
| 6 | 10 | |
| 7 | 12 | |
| 8 | 10 | |
| 9 | 12 | |
| Total | 100 | |

1. (12 total points) Evaluate the following indefinite integrals.

(a) (6 points) $\int 2x\ln(x+5)\,dx$

① Use integration by parts

$u=\ln(x+5) \quad dv=2x\,dx$

$du=\frac{1}{x+5}dx \quad v=x^2$

$$\int 2x\ln(x+5) = x^2\ln(x+5) - \underbrace{\int \frac{x^2}{x+5}dx}_{ⓐ}$$

② To solve ⓐ let $u=x+5$, $du=dx$

$$-\int\frac{x^2}{x+5} = -\int\frac{(u-5)^2}{u}du = -\int\frac{u^2-10u+25}{u}du$$

$$= \int -u+10-\frac{25}{u} = -\frac{1}{2}u^2+10u-25\ln|u|+C$$

$$= -\frac{1}{2}(x+5)^2+10(x+5)-25\ln|x+5|+C$$

③ Putting everything together:

$$\boxed{\int 2x\ln(x+5) = x^2\ln(x+5)-\frac{1}{2}(x+5)^2+10(x+5)-25\ln(x+5)+C}$$

you don't need |·| b/c $\ln(x+5)$ is only true for $x>-5$

(b) (6 points) $\int x^3\sqrt{9+4x^2}\,dx$

Let $u=9+4x^2 \quad du=8x\,dx$

$$\int x^3\sqrt{9+4x^2}\,dx = \int\frac{1}{8}x^2\sqrt{u}\,du = \frac{1}{8}\int\left(\frac{u-9}{4}\right)\sqrt{u}\,du$$

$$=\frac{1}{32}\int u^{3/2}-9u^{1/2}\,du$$

$$=\frac{1}{32}\left[\frac{2}{5}u^{5/2}-6u^{3/2}\right]+C$$

$$=\frac{1}{32}\left[\frac{2}{5}(9+4x^2)^{5/2}-6(9+4x^2)^{3/2}\right]+C$$

$$\boxed{=\frac{1}{80}(9+4x^2)^{5/2}-\frac{3}{16}(9+4x^2)^{3/2}+C}$$

2. (12 total points) Evaluate the following definite integrals.

(a) (6 points) $\int_1^3 \frac{1}{x^2+x^3}\,dx$

$$\int_1^3 \frac{1}{x^2+x^3}\,dx = \int_1^3 \frac{1}{x^2(1+x)}\,dx$$

Using Partial Fractions,

$$\frac{1}{x^2(1+x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1+x}$$

$$\Rightarrow 1 = Ax(1+x) + B(1+x) + Cx^2$$

Plug in $x=0$, $1=B$

$x=-1$, $1=C$

$x=1$, $1 = A(2) + 1(2) + 1$

$\Rightarrow A=-1$

$$\therefore \int_1^3 \frac{1}{x^2(1+x)} = \int_1^3 \frac{-1}{x} + \int_1^3 \frac{1}{x^2} + \int_1^3 \frac{1}{1+x}$$

$$= -\ln(x)\Big|_1^3 - \frac{1}{x}\Big|_1^3 + \ln(1+x)\Big|_1^3$$

$\therefore = -\ln(3) + 2/3 + \ln(4) - \ln($

$$\boxed{= \frac{2}{3} + \ln\left(\frac{2}{3}\right)}$$

(b) (6 points) $\int_3^{9/2} \frac{x-2}{\sqrt{6x-x^2}}\,dx$

Use Trig Substitution.

1 Complete the square. $-(x^2-6x) = -(x-3)^2+9$

$$\therefore \int_3^{9/2} \frac{x-2}{\sqrt{9-(x-3)^2}}$$

let $x-3 = 3\sin\theta$

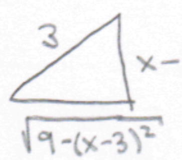
$dx = 3\cos\theta\, d\theta$

$$\Rightarrow \int \frac{(3\sin\theta+1)(3\cos\theta)}{3\cos\theta}\,d\theta = \int 3\sin\theta + 1\,d\theta$$

$$= -3\cos\theta + \theta + C$$

2 Make the Triangle

$\frac{x-3}{3} = \sin\theta = \frac{o}{h}$



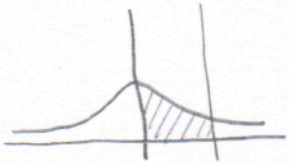

$$\therefore -3\cos\theta + \theta = -\frac{3\sqrt{9-(x-3)^2}}{3} + \sin^{-1}\left(\frac{x-3}{3}\right)\Bigg|_3^{9/2}$$

$$= -\sqrt{\frac{27}{4}} + \frac{\pi}{6} + 3 = \boxed{\frac{-3\sqrt{3}}{2} + \frac{\pi}{6} + 3}$$

3. (10 total points) Let $\mathscr{R}$ be the region bounded on top by $y = \dfrac{1}{\sqrt{1+x^2}}$, on the bottom by the $x$-axis, on the left by the $y$-axis, and on the right by the line $x = B$.

(a) (3 points) Find (in terms of $B$) the volume of the solid obtained by rotating $\mathscr{R}$ about the $x$-axis.

① Draw a Picture




② Shells or Washers?
Since $y = \frac{1}{\sqrt{1+x^2}}$ and rotating horizontally, use Washers

③ Set up Integral

$$\int_0^B \left(\frac{1}{\sqrt{1+x^2}}\right)^2 \pi\,dx = \int_0^B \frac{\pi\,dx}{1+x^2}$$

$$= \pi \tan^{-1}(x) \Big|_0^B$$

$$\boxed{= \pi \tan^{-1}(B)}$$

(b) (2 points) Does the volume in part (a) have a finite limit as $B \to \infty$? If so, what is that limit?

Yes, recall that $\lim_{B\to\infty} \tan^{-1}(B) = \pi/2$. Hence,

$$\boxed{\lim_{B\to\infty} \pi \tan^{-1}(B) = \frac{\pi^2}{2}}$$

(c) (3 points) Find (in terms of $B$) the volume of the solid obtained by rotating $\mathscr{R}$ about the $y$-axis.

② Washers or Shells?
Since $y = \frac{1}{\sqrt{1+x^2}}$ and rotating vertically, then use shells.

③ Set up integral and solve.

$$\int_0^B 2\pi \frac{x}{\sqrt{1+x^2}}\,dx \qquad u = 1+x^2, \quad du = 2x\,dx$$

$$= \int_1^{1+B^2} \frac{\pi}{\sqrt{u}}\,du = 2\pi u^{1/2} \Big|_1^{1+B^2}$$

$$\boxed{= 2\pi\sqrt{1+B^2} - 2\pi}$$

(d) (2 points) Does the volume in part (c) have a finite limit as $B \to \infty$? If so, what is that limit?

No, $\lim_{B\to\infty} \sqrt{1+B^2} = \infty$.

4. (12 total points) Xander, an indestructible robot, has jet boots that provide enough force to give him a vertical acceleration of 10.3 m/s$^2$ (if there were no gravity), and they have enough fuel to do so for 8 seconds. Suppose he uses them to lift himself straight up, starting at rest on the ground. Because the acceleration due to gravity is 9.8 m/s$^2$ in the downward direction, his net acceleration in the positive vertical direction during these 8 seconds is $10.3 - 9.8 = 0.5$ m/s$^2$.

(a) (5 points) At what height above the ground will Xander run out of fuel?

① $a(t) = .5$

$v(t) = .5t + C$

$s(t) = .25t^2 + Ct + D$

② Initial conditions:

starting from rest $\Rightarrow v(0) = 0$

starts from ground $\Rightarrow s(0) = 0$

$\therefore v(0) = 0 = .5(0) + C \Rightarrow C = 0$

$s(0) = 0 = .25(0^2) + D \Rightarrow D = 0$

$\therefore v(t) = .5t$

$s(t) = .25t^2$

The height when Xander runs out of fuel is at $t=8$ so

$\boxed{s(8) = .25(8)^2 = 16 \text{ meters}}$

(b) (2 points) What is his velocity when he runs out of fuel?

The vel. when he runs out of fuel is

$\boxed{v(8) = .5(8) = 4 \text{ m/s}}$

(c) (5 points) After he runs out of fuel, his acceleration is the acceleration due to gravity. How long after he runs out of fuel will he hit the ground? Give your answer (in seconds) in decimal form, correct to at least the second digit after the decimal point.

We restart the clock when Xander runs out of fuel. Therefore, at $t=0$, $v(0) = 4$ and $s(0) = 16$ and $a(t) = -9.8$.

① The new eq. are

$a(t) = -9.8$

$v(t) = -9.8t + C$

$v(0) = 4 = -9.8(0) + C$

$\Rightarrow C = 4$

so $v(t) = -9.8t + 4$

$s(t) = -4.9t^2 + 4t + D$

$s(0) = -4.9(0^2) + 4(0) + D = 16$

$\Rightarrow D = 16$

$s(t) = -4.9t^2 + 4t + 16$.

② When hits ground $\Rightarrow s(t) = 0$

$\therefore 0 = -4.9t^2 + 4t + 16$

$$t = \frac{-4 \pm \sqrt{16 - 4(16)(-4.9)}}{2(-4.9)}$$

$t = 2.26$ secs or $-1.44$ secs

Hence,

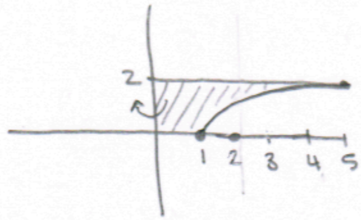
$\boxed{t = 2.26 \text{ secs}}$

5. (10 points) Let $\mathscr{R}$ be the region bounded by the $x$-axis, the $y$-axis, the line $y = 2$, and the curve $y = \sqrt{x-1}$. A container is built in the shape of the solid of revolution formed by rotating $\mathscr{R}$ about the $y$-axis, and the container is filled with cold-pressed olive oil. Find the work done in pumping all of the olive oil to the top of the container. (Units on the $x$- and $y$-axes are meters, the density of cold-pressed olive oil is 915 kg/m$^3$, and the acceleration due to gravity is 9.8 m/s$^2$.)
Give your answer (in joules = kg · m/s$^2$ · m) in decimal form, correct to the nearest joule.

① Volume of a slice

$y = \sqrt{x-1} \Rightarrow y^2+1 = x$

$\therefore$ volume of 1 slice $= \pi(y^2+1)^2 dy$




② Find Force

Mass $= 915 \cdot \pi (y^2+1)^2 dy$

Force $= (9.8)(915)\pi (y^2+1)^2 dy$

③ Distance is $2-y$.

④ Set up integral

$$\int_0^2 (2-y)(9.8)(915)\pi(y^2+1)^2 dy = 8967\pi \int_0^2 (2-y)(y^2+1)^2 dy$$

$$= 8967\pi \int_0^2 (2-y)(y^4+2y^2+1)\, dy$$

$$= 8967\pi \int_0^2 -y^5+2y^4-2y^3+4y^2-y+2$$

$$= 8967\pi \left(\frac{-1}{6}y^6 + \frac{2}{5}y^5 - \frac{1}{2}y^4 + \frac{4}{3}y^3 - \frac{1}{2}y^2 + 2y\right)\Big|_0^2$$

$$\boxed{\begin{aligned} &= 60{,}975.6\pi \\ &\approx 191{,}560.497 \end{aligned}}$$

6. (10 total points) Let $\mathscr{R}$ be the region bounded on top by the line $y = 1$, on the left by the $y$-axis, and on the right by the curve $x = g(y)$ for $0 \le y \le 1$, where $g(y)$ is a continuous function with $g(0) = 0$ and $g(y) \ge 0$ for $0 \le y \le 1$. No formula is known for $g(y)$, but we do have the following table of values:

| $y$ | 0 | 0.25 | 0.50 | 0.75 | 1.00 |
|---|---|---|---|---|---|
| $x = g(y)$ | 0 | 0.70 | 1.30 | 1.70 | 2.00 |

(a) (3 points) Set up an integral (in terms of $g(y)$) for the volume of the solid obtained by rotating the region $\mathscr{R}$ about the $x$-axis.

① Shells or Washers? Since $x=g(y)$ and rotating horizontally, use SHELLS.

② $\boxed{\int_0^1 y g(y) \cdot 2\pi \, dy}$

(b) (7 points) Use the trapezoid rule with $n = 4$ subintervals to estimate the integral in part (a). Give your answer in decimal form, correct to at least the second digit after the decimal point.

Trapezoid Rule

$$\frac{b-a}{2N}\left(f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)\right)$$

$$x_i = \frac{1-0}{4} \cdot i = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$$

$$\therefore \frac{1}{8}\left(f(0) + 2f(\tfrac{1}{4}) + 2f(\tfrac{1}{2}) + 2f(\tfrac{3}{4}) + f(1)\right)$$

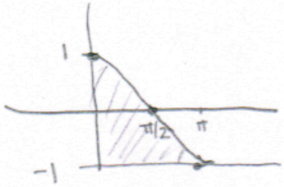
$$= \frac{1}{8}\left(0 + 2(.7)(.25)(2\pi) + 2(1.3)(.5)2\pi + 2(1.70)(.75)2\pi + 2(2\pi)\right)$$

$$= \frac{\pi}{4} \cdot \frac{31}{5}$$

$$\boxed{= \frac{31\pi}{20} \approx 4.87}$$

7. (12 points) Find the $x$- and $y$-coordinates of the center of mass of a uniform plate bounded on top by $y = \cos x$ for $0 \le x \le \pi$, on the bottom by the line $y = -1$, and on the left by the $y$-axis.
Give your answers in exact form.

① Draw a Picture




② Find Area

$$\text{Area} = \int_0^{\pi} \cos x + 1\, dx$$
$$= \sin x + x \Big|_0^{\pi}$$
$$= \pi$$

③ Find x-coordinate

$$\bar{x} = \frac{1}{A} \int x f(x)$$
$$= \frac{1}{A} \int_0^{\pi} x(\cos x + 1)\, dx$$
$$= \underbrace{\frac{1}{\pi} \int_0^{\pi} x \cos x}_{a} + \underbrace{\frac{1}{\pi} \int_0^{\pi} x}_{b}$$

To solve $(b) = \frac{1}{2\pi} x^2 \Big|_0^{\pi} = \frac{\pi}{2}$

To solve (a) Use integration by parts,

Let $u = x$ $dv = \cos x$

$du = 1dx$ $v = \sin x$

$$\therefore \tfrac{1}{\pi} \int_0^{\pi} x \cos x = \frac{1}{\pi} \left( x \sin x \Big|_0^{\pi} \right) - \frac{1}{\pi} \int_0^{\pi} \sin x\, dx$$
$$= \frac{1}{\pi} \left( 0 + \cos x \Big|_0^{\pi} \right) = \frac{1}{\pi}(-1-1) = \frac{-2}{\pi}$$

$$\boxed{\therefore \bar{x} = \pi/2 - \frac{2}{\pi}}$$

④ Find $\bar{y}$

$$\bar{y} = \frac{1}{A} \int_0^{\pi} \frac{1}{2} f(x)^2$$
$$= \frac{1}{\pi} \int_0^{\pi} \frac{1}{2} \cos^2 x - \frac{1}{2}\, dx$$
$$= \frac{1}{\pi} \int_0^{\pi} \frac{1}{4} (1 - \cos(2x)) - \frac{1}{2}\, dx$$
$$= \frac{1}{\pi} \left( \frac{1}{4} x - \frac{1}{8} \sin(2x) - \frac{1}{2} x \right) \Big|_0^{\pi}$$
$$= \frac{1}{\pi} \left( \frac{\pi}{4} - \frac{\pi}{2} \right)$$
$$= \frac{1}{\pi} \left( \frac{-\pi}{4} \right)$$
$$\boxed{= -1/4 = \bar{y}}$$

The center of mass is $\left( \frac{\pi}{2} - \frac{2}{\pi}, \frac{-1}{4} \right)$.

8. (10 points) Find the solution of the differential equation

$$\frac{dy}{dx} = \frac{y^2 - 2y}{x}$$

that satisfies the condition $y(1) = 1$. Solve for $y$, giving your answer in the form $y = f(x)$.

① Separate and solve

$$\frac{dy}{y^2-2y} = \frac{dx}{x} \Rightarrow \int \frac{dy}{y^2-2y} = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{dy}{y(y-2)} = \ln|x| + C \qquad (a)$$

Using Partial Fractions,

$$\frac{1}{y(y-2)} = \frac{A}{y} + \frac{B}{y-2} \Rightarrow 1 = A(y-2) + By$$

Plugging in $y=0$: $1 = A(-2) \Rightarrow A = -1/2$

$y=2$: $1 = 2B \Rightarrow B = 1/2$

$$\therefore \int \frac{dy}{y(y-2)} = \int \frac{-1/2}{y} + \int \frac{1/2}{y-2} = -\frac{1}{2}\ln|y| + 1/2 \ln|y-2|$$

$$= \frac{1}{2}\ln\left|\frac{y-2}{y}\right|$$

Hence, by (a),

$$\frac{1}{2}\ln\left|\frac{y-2}{y}\right| = \ln|x| + C$$

$$\ln\left|\frac{y-2}{y}\right| = 2\ln|x| + C$$

$$\frac{y-2}{y} = Cx^2 \Rightarrow y - 2 = Cx^2 y$$

$$y - Cx^2 y = 2$$

$$y(1 - Cx^2) = 2 \Rightarrow y = \frac{2}{1-Cx^2}$$

② Plug in initial value, $y(1) = 1$

$$1 = \frac{2}{1-C} \Rightarrow 1 - C = 2 \Rightarrow C = -1. \qquad \boxed{\therefore\ y = \frac{2}{1+x^2}}$$

9. (12 total points) Your friend wins the lottery, and gives you $P_0$ dollars to help you pay your college expenses. The money is invested in a savings account that earns 10% annual interest, compounded continuously, and you withdraw the money continuously (a pretty good approximation to reality if you make regular frequent withdrawals) at a rate of \$3600 per year.

(a) (3 points) Set up a differential equation for the amount of money $P(t)$ (in dollars) left in the account at time $t$ (in years).

Recall, Rate in - Rate out

$$\boxed{\frac{dP}{dt} = .1P(t) - 3{,}600}$$

(b) (3 points) Solve the differential equation to obtain a formula for $P(t)$. Your formula should involve $P_0$.

① Separate and solve

$$\frac{dP}{\frac{P}{10} - 3{,}600} = dt$$

$$\Rightarrow \int \frac{dP}{P/10 - 3600} = \int dt$$

$$\Rightarrow 10\ln|P/10 - 3600| = t + C$$

$$\Rightarrow \ln|P/10 - 3600| = \frac{t}{10} + C$$

$$P/10 - 3600 = Ce^{t/10}$$

$$\Rightarrow P(t) = Ce^{t/10} + 36{,}000$$

② Initial conditions: $P(0) = P_0$

$$\therefore P_0 = Ce^0 + 36000$$

$$\Rightarrow C = P_0 - 36000$$

$$\therefore \boxed{P(t) = (P_0 - 36{,}000)e^{t/10} + 36{,}000}$$

(c) (3 points) If $P_0 = \$20{,}000$, then how much money is left in the account 4 years later? Give your answer (in dollars) in decimal form, correct to the nearest cent (i.e., nearest \$0.01).

$$P(t) = -16{,}000e^{t/10} + 36{,}000$$

$$P(4) = -16{,}000e^{4/10} + 36{,}000 \boxed{= \$12{,}130.80}$$

(d) (3 points) What should $P_0$ be so that after 4 years your account has exactly \$0 left over? Give your answer (in dollars) in decimal form, correct to the nearest cent (i.e., nearest \$0.01).

Now

$$0 = (P_0 - 36{,}000)e^{4/10} + 36{,}000$$

$$\Rightarrow \frac{-36{,}000}{e^{4/10}} = P_0 - 36000$$

$$\Rightarrow \frac{-36{,}000}{e^{4/10}} + 36{,}000 = P_0 = \boxed{\$11{,}868.48}$$