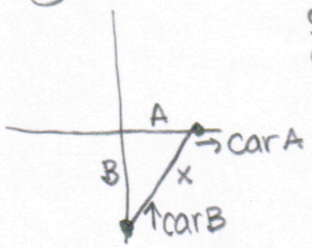
(12 points) Car A is traveling east **away** from an intersection at 20 miles per hour. Car B is traveling north **toward** the same intersection at 15 miles per hour.

(a) [10pts] At what rate is the distance between the two cars changing when car A is 0.3 miles and car B is 0.4 miles from the intersection?

①



$\frac{dA}{dt} = 20, \ \frac{dB}{dt} = -15$

Goal: Find dx/dt when $A = .3$ & $B = .4$

② Relationship:

$x^2 = A^2 + B^2$

$\Rightarrow x = \sqrt{A^2 + B^2}$

③ Derivative

$$\frac{dx}{dt} = \frac{2A\,dA/dt + 2B\,dB/dt}{2\sqrt{A^2+B^2}}$$

④ Plug in $A = .3$, $B = .4$ and $\frac{dA}{dt} = 20$, $\frac{dB}{dt} = -15$

$$\frac{dx}{dt} = \frac{2(.3)(20) + 2(.4)(-15)}{2\sqrt{.3^2 + .4^2}} = 0$$

$\boxed{\text{Hence, the rate the distance between the two cars is 0 mi/hr.}}$

(b) [2pts] Immediately afterwards, is the distance between the cars increasing or decreasing?

Use 2nd derivative Test

$$\frac{d^2x}{dt^2} = \frac{\left(2(dA/dt)^2 + 2A\frac{d^2A}{dt^2} + 2\left(\frac{dB}{dt}\right)^2 + 2B\frac{d^2B}{dt^2}\right)2\sqrt{A^2+B^2} - \left(2A\frac{dA}{dt} + 2B\frac{dB}{dt}\right)\frac{2A\frac{dA}{dt} + 2B}{2\sqrt{A^2+B^2}}}{A^2+B^2}$$

Now $\frac{d^2A}{dt^2}, \frac{d^2B}{dt^2} = 0$,

$$\frac{d^2x}{dt^2} = \frac{(20^2 + (-15)^2)\sqrt{.3^2+.4^2} - (.3(20) + .4(-15))\left(\frac{2(.3)(20) + 2(.4)(-15)}{2\sqrt{.3^2+.4^2}}\right)}{.3^2 + .4^2}$$

$\frac{d^2x}{dt^2} > 0 \Rightarrow$ concave up

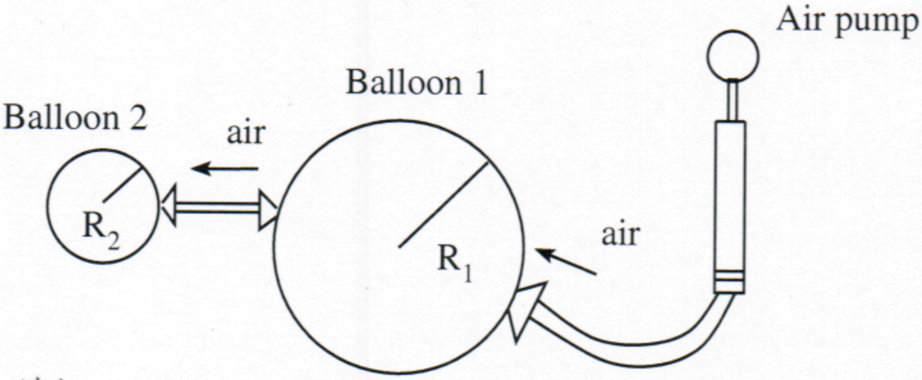
$\boxed{\text{Hence, the cars are increasing.}}$

6. (12 points) Balloon 1 is linked by a large tube to an air pump and by a smaller tube to Balloon 2 (see picture). The radius of Balloon 1 is R_1 and the radius of Balloon 2 is R_2.

Air is being pumped in Balloon 1 at the constant rate of $101\text{cm}^3/\text{minute}$ and air is leaking out of Balloon 1 (and into Balloon 2) at a total rate equal to π times the rate of change of R_1, in $\text{cm}^3/\text{minute}$.

At time t_0 measurements say that $R_1 = 5$ and $R_2 = 2$.

Calculate the rate of change of R_2 at that time.



Volume of Balloon 1 $= 4/3\pi R_1^3 \Rightarrow \frac{dV_1}{dt} = 4\pi (R_1)^2 \cdot dR_1/dt$

$\frac{dV_1}{dt}$ = Rate in - Rate out $\Rightarrow 4\pi(R_1)^2\frac{dR_1}{dt} = 101 - \pi\frac{dR_1}{dt}$

Volume of Ball. 2 $= 4/3\pi R_2^3 \Rightarrow 4\pi R_2^2 \cdot \frac{dR_2}{dt} = \frac{dV_2}{dt} = \pi R_1/dt$

$\Rightarrow 4R_2^2\frac{dR_2}{dt} = \frac{dR_1}{dt}$

Plugging this in, we get that

$$4\pi(R_1)^2 \cdot 4R_2^2\frac{dR_2}{dt} = 101 - \pi 4R_2^2\frac{dR_2}{dt}$$

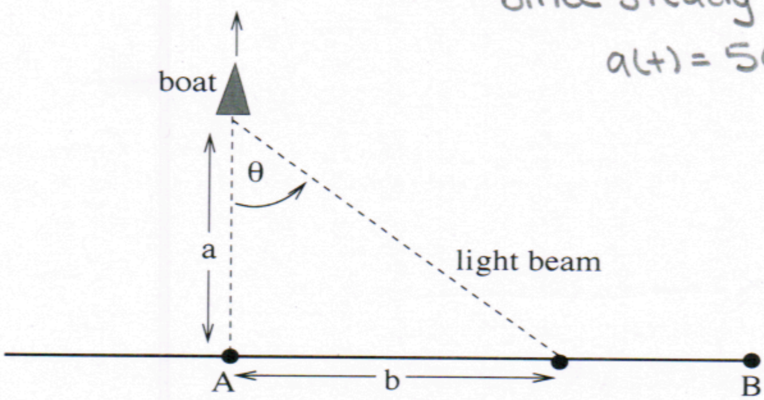
$$\Rightarrow 4\pi(5^2) \cdot 4(2^2)\frac{dR_2}{dt} = 101 - \pi 4(2^2)\frac{dR_2}{dt}$$

$$\boxed{\frac{dR_2}{dt} = \frac{101}{1616\pi}}$$

2. (12 points) At time $t = 0$ a boat starts out from point A on a straight shoreline and goes directly outward at right angles to the shore at a steady speed of 50 ft/sec. You are located at point B a distance 1500 feet down the shore from A. On the boat there is a light that produces a slowly rotating beam. Let θ denote the angle that the beam makes with the line from the boat to A, and let b denote the distance from A to the point on the shore where the beam hits. Let a denote the distance of the boat from A.

Since Steady

$a(t) = 50t \Rightarrow da/dt = 50$



1500ft

(a) Find a formula relating θ, a and b.

$\tan(\theta) = b/a$

(b) Suppose that the beam hits you at time $t = 40$ sec, and at that instant the point where it hits the shore is moving along the shore at 150 ft/sec. Find how fast the beam is rotating; make sure to include units in your final answer.

Goal: find $d\theta/dt$ when $t=40$, $b=1500$ and $db/dt = 150$

Relationship: $\tan\theta = b/a$

$$\sec^2(\theta)\, d\theta/dt = \frac{db/dt\, a(t) - da/dt\, b(t)}{a(t)^2}$$

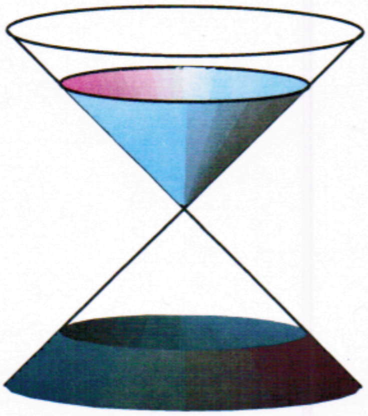
$\Rightarrow$ Note we need to find θ and $a(t)$. At $t=40$, $a(t) = 50(40) = 2000$.

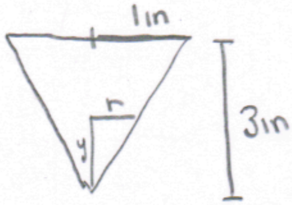
$\Rightarrow \tan(\theta) = 1500/2000$. Recall that $\tan^2\theta + 1 = \sec^2\theta \Rightarrow \left(\frac{3}{4}\right)^2 + 1 = \sec^2\theta$

Thus, $$\left(\left(\frac{3}{4}\right)^2 + 1\right)\frac{d\theta}{dt} = \frac{150(2000) - 50(1500)}{(2000)^2}$$

$$\boxed{\frac{d\theta}{dt} = .036 \text{ rad/sec}}$$

5. (12 points) An hour glass is made up of two glass cones connected at their tips. Both cones have radius 1 inch and height 3 inches. When the hourglass is flipped over sand starts falling to the lower cone.



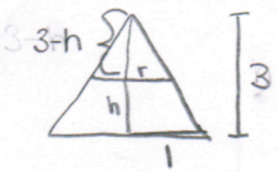


(a) When the sand remaining in the **upper cone** has height y inches, what is its volume V_1 in terms of y? Recall that the volume of a cone with radius R and height H is $V = \frac{1}{3}\pi R^2 H$.

The $V_1 = \frac{1}{3}\pi r^2 y$. By similar Triangles, $\frac{r}{1} = \frac{y}{3}$. Thus

$$V_1 = 1/3\pi (y/3)^2 y = \frac{\pi}{27} y^3.$$

(b) When the sand in the **lower cone** has reached a height of h inches, what is its volume V_2 in terms of its height h? Hint: The volume of sand is the volume of the cone minus the volume of empty space above it.



$V_2 =$ Full cone - Empty cone

$= \frac{1}{3}\pi 3 - \frac{1}{3}\pi r^2 (3-h)$

$\Rightarrow \pi - \frac{1}{3}\pi \left(\frac{3-h}{3}\right)^2 (3-h)$

$\Rightarrow \boxed{\pi - \frac{\pi}{27}(3-h)^3}$

By similar triangles, $\frac{r}{1} = \frac{3-h}{3}$

5. continued.

(c) The total volume of the sand in the hourglass is $\frac{3\pi}{4}$ cubic inches. If the height of the sand in the upper cone is decreasing at a rate of 0.1 inches per second, how fast is the height of the sand in the lower cone increasing when the sand in the lower cone is 1 inch high?

Total Volume = $V_1 + V_2$

$$\frac{3\pi}{4} = \frac{\pi}{27}y^3 + \pi - \frac{\pi}{27}(3-h)^3$$

Taking derivatives, we get that

$$0 = \frac{\pi}{9}y^2\, dy/dt + \frac{\pi}{9}(3-h)^2\frac{dh}{dt}$$

We need y so,

$$\frac{3\pi}{4} = \frac{\pi}{27}y^3 + \pi - \frac{\pi}{27}(3-1)^3$$

$$-\frac{\pi}{4} = \frac{\pi}{27}(y^3-8) \Rightarrow -\frac{27}{4} = y^3 - 8 \Rightarrow \frac{5}{4} = y^3 \quad \boxed{y = \sqrt[3]{\frac{5}{4}}}$$

Plugging $dy/dt = .1$, $h = 1$, $y = \sqrt[3]{\frac{5}{4}}$

$$0 = \pi/9\left(\sqrt[3]{\tfrac{5}{4}}\right)^2 .1 + \pi/9\,(2^2)\frac{dh}{dt}$$

$$\frac{-\pi}{90}\left(\sqrt[3]{\frac{5}{4}}\right)^2 = \frac{4\pi}{9}\frac{dh}{dt}$$

$$\boxed{\frac{dh}{dt} = \frac{1}{40}\left(\sqrt[3]{\frac{5}{4}}\right)^2}$$