3. (6 points) Evaluate the improper integral

$$\int_0^4 \frac{1}{(x-3)^2}\,dx$$

or explain why it does not converge.

$$\int_0^4 \frac{1}{(x-3)^2}\,dx = \int_0^3 \frac{1}{(x-3)^2} + \int_3^4 \frac{1}{(x-3)^2}$$

$$\Rightarrow \lim_{b\to 3^-}\int_0^b \frac{1}{(x-3)^2} + \lim_{b\to 3^+}\int_b^4 \frac{1}{(x-3)^2}$$

$$\Rightarrow \lim_{b\to 3^-} \frac{-1}{(x-3)}\Big|_0^b + \lim_{b\to 3^+} \frac{-1}{x-3}\Big|_b^4$$

$$\Rightarrow \lim_{b\to 3^-} \frac{-1}{b-3} + \frac{1}{3} + \lim_{b\to 3^+} -1 + \frac{1}{b-3}$$

As $\lim_{b\to 3^-} b-3 = 0$. Then $\lim_{b\to 3^-} \frac{-1}{b-3} = \pm\infty$.

$$\boxed{\therefore \lim_{b\to 3^-}\int_0^b \frac{1}{(x-3)^2} \text{ is divergent.}}$$

As this integral is divergent, $\int_0^4 \frac{1}{(x-3)^2}$ is divergent.

* you can also show that $\lim_{b\to 3^+} -1+\frac{1}{b-3} = \pm\infty$. Thus $\lim_{b\to 3^+}\int_b^4 \frac{1}{(x-3)^2} = \pm\infty$.

2. Evaluate

$$\int \frac{\sin\theta}{\sqrt{2-\cos^2\theta}}\,d\theta$$

____________________(10)

3. Evaluate

$$\int_0^\infty \frac{e^t}{e^{2t}+3e^t+2}\,dt$$

____________________(10)

① Rewrite

$$\lim_{b\to\infty}\int_0^b \frac{e^t}{e^{2t}+3e^t+2}$$

② Solve

$u=e^t$

$du=e^t dt$

$$\lim_{b\to\infty}\int_1^{e^b}\frac{1}{u^2+3u+2} = \lim_{b\to\infty}\int_1^{e^b}\frac{1}{(u+2)(u+1)}$$

$$\frac{A}{u+2}+\frac{B}{u+1}=\frac{1}{(u+2)(u+1)} \Rightarrow A(u+1)+B(u+2)=1$$

$u=-1: B=1$

$u=-2: -A=1 \Rightarrow A=-1$

$$= \lim_{b\to\infty}\int_1^{e^b}\frac{-1}{u+2}+\int_1^{e^b}\frac{1}{u+1}$$

$$= \lim_{b\to\infty} -\ln|u+2|+\ln|u+1|\,\Big|_1^{e^b} = \lim_{b\to\infty}\ln\left|\frac{u+1}{u+2}\right|\,\Big|_1^{e^b} = \boxed{0+\ln|2/3|}$$

Converges

3. (10 points) Consider the improper integral $\int_1^{\infty} \frac{\tan^{-1}(x)}{x^2}\,dx$.

Evaluate the integral (if it converges) or explain carefully why it does not converge.

If it converges, give your answer in exact form.

① Rewrite

$$\lim_{b\to\infty} \int_1^b \frac{\tan^{-1}(x)}{x^2}\,dx$$

② Solve integral:

$$\lim_{b\to\infty} \int_1^b \frac{\tan^{-1}(x)}{x^2} \qquad u = \tan^{-1}(x) \qquad dv = x^{-2}\,dx$$

$$du = \frac{1}{1+x^2}\,dx \qquad v = -x^{-1}$$

$$= \lim_{b\to\infty} \frac{-\tan^{-1}(x)}{x}\Big|_1^b + \int_1^b \frac{1}{x(1+x^2)}$$

Partial Fractions:

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2} \Rightarrow 1 = (1+x^2)A + (Bx+C)x$$

$x=0$: $1 = A$

$1 = A + Ax^2 + Bx^2 + Cx$

$0x^2 = x^2 + Bx^2$

$0 = 1 + B$ $\boxed{B=-1}$

$0 = Cx$ $C = 0$

$$\therefore \int_1^b \frac{1}{x(1+x^2)} = \int \frac{1}{x} + \int \frac{-x}{1+x^2} \qquad u = 1+x^2, \quad du = 2x\,dx$$

$$= \ln|x| + {}^{-1}/_{2}\ln|1+x^2|.$$

③

$$\therefore \lim_{b\to\infty} \frac{-\tan^{-1}(x)}{x}\Big|_1^b + \ln|x| - {}^{1}/_{2}\ln|1+x^2|\Big|_1^b$$

$$= 0 + \frac{\pi/4}{1} + \lim_{b\to\infty} \ln\left|\frac{x}{(1+x^2)^{1/2}}\right|\Big|_1^b$$

$$\frac{x}{\sqrt{1+x^2}}\frac{(1/x)}{(1/x)} = \frac{1}{\frac{\sqrt{1+x^2}}{x}} = \frac{1}{\frac{\sqrt{1+x^2}}{\sqrt{x^2}}} \quad \frac{1}{\sqrt{\frac{1}{x^2}+1}}$$

$$= 0 + \pi/4 + \lim_{b\to\infty} \ln\left|\frac{1}{\sqrt{1/x^2+1}}\right|\Big|_1^b = \frac{\pi}{4} + \ln|1| + \ln\left|\frac{1}{\sqrt{2}}\right|$$

$\therefore$ Converges

4. (8 points) Determine whether the following improper integral is convergent or divergent. Evaluate it if it is convergent.

$$\int_0^\infty x^3 e^{-x^2}\,dx$$

$$\int_0^\infty x^3 e^{-x^2}\,dx = \lim_{b\to\infty}\int_0^b x^3 e^{-x^2}\,dx$$

Let $u = -x^2$ $du = -2x\,dx$

$$\lim_{b\to\infty}\int_0^b x^3 e^{-x^2} = \lim_{b\to\infty}\int_0^{-b^2} \frac{-1}{2}x^2 e^{u}\,du$$

$$= \lim_{b\to\infty}\int_0^{-b^2} \frac{1}{2}u e^{u}\,du$$

$v = 1/2\,u$ $dw = e^u$

$dv = 1/2\,du$ $w = e^u$

$$= \lim_{b\to\infty} \frac{1}{2}ue^u \Big|_0^{-b^2} - \int_0^{-b^2}\frac{1}{2}e^u$$

$$= \lim_{b\to\infty} \frac{1}{2}ue^u \Big|_0^{-b^2} - \frac{1}{2}e^u\Big|_0^{-b^2}$$

$$= \lim_{b\to\infty} \frac{1}{2}(-b^2)e^{-b^2} - \frac{1}{2}e^{-b^2} + 1/2$$

$$= \lim_{b\to\infty} \frac{-1}{2}b^2e^{-b^2} - 0 + 1/2$$

$$\lim_{b\to\infty}\frac{-1}{2}\frac{b^2}{e^{b^2}} \overset{\text{L'Hop}}{=} \lim_{b\to\infty}\frac{-1}{2}\frac{2\cancel{b}}{\cancel{2b}e^{b^2}} = \lim_{b\to\infty}\frac{-1}{2e^{b^2}} = 0.$$

$\therefore \int_0^\infty x^3 e^{-x^2}\,dx$ converges.

6. (10 total points) Consider the *improper* integral

$$\int_0^1 x^k \ln x \, dx,$$

where k is a constant.

(a) (4 points) Does this improper integral converge when $k=-1$? Justify your answer.

$$\lim_{b\to 0^+} \int_b^1 \frac{\ln(x)}{x}\,dx \qquad u=\ln(x) \quad du=\frac{1}{x}dx$$

$$\lim_{b\to 0^+} \int_{\ln(b)}^0 1\,du = \lim_{b\to 0^+} u\Big|_{\ln(b)}^0 = \lim_{b\to 0^+} -\ln(b) = +\infty$$

Diverges

(b) (6 points) Determine the values of $k \neq -1$ for which the improper integral above converges. Justify your answer.

Assume $k \neq -1$

$$\int_0^1 x^k \ln(x)\,dx = \lim_{a\to 0^+} \int_a^1 x^k \ln(x)\,dx \qquad u=\ln(x) \quad dv = x^k dx \quad du=\frac{1}{x}dx \quad v = x^{k+1}/_{k+1}$$

$$= \lim_{a\to 0^+} \frac{x^{k+1}\ln(x)}{k+1}\Big|_a^1 - \int_a^1 \frac{1}{k+1} x^k\,dx$$

$$= \lim_{a\to 0^+} -\frac{a^{k+1}\ln(a)}{k+1} - \frac{1}{(k+1)^2} x^{k+1}\Big|_a^1 = \lim_{a\to 0^+} \frac{-1}{(k+1)^2} + \frac{a^{k+1}}{(k+1)^2} - \frac{a^{k+1}\ln(a)}{k+1}$$

If $k+1<0$ [i.e. $k<-1$], then $\lim_{a\to 0^+} \frac{a^{k+1}}{(k+1)^2} = \infty$ and

$$\lim_{a\to 0^+} \frac{-a^{k+1}\ln(a)}{k+1} = +\infty. \Rightarrow \lim_{a\to 0^+} \frac{-1}{(k+1)^2} + \frac{a^{k+1}}{(k+1)^2} - \frac{a^{k+1}\ln(a)}{k+1} = \infty$$

diverges

If $k+1>0$ [i.e. $k>-1$], then $\lim_{a\to 0^+} \frac{a^{k+1}}{(k+1)^2} = 0$ and

$$\lim_{a\to 0^+} \frac{-a^{k+1}\ln(a)}{k+1} = \lim_{a\to 0^+} \frac{-\ln(a)}{(k+1)a^{-(k+1)}} \underset{\text{L'Hopital}}{=} \lim_{a\to 0^+} \frac{-1/a}{-(k+1)^2 a^{-k-2}} = \lim_{a\to 0^+} \frac{a^{k+1}}{(k+1)^2} = 0$$

$$\therefore \lim_{a\to 0^+} \frac{-1}{(k+1)^2} + \frac{a^{k+1}}{(k+1)^2} - \frac{a^{k+1}\ln(a)}{k+1} = \frac{-1}{(k+1)^2}$$ so it converges

3. (8 points) Evaluate the improper integral

$$\int_0^\infty \frac{1}{\sqrt{x}(1+x)}\,dx.$$

Be sure to indicate the limit(s) you are taking to evaluate the integral because it is an improper integral.

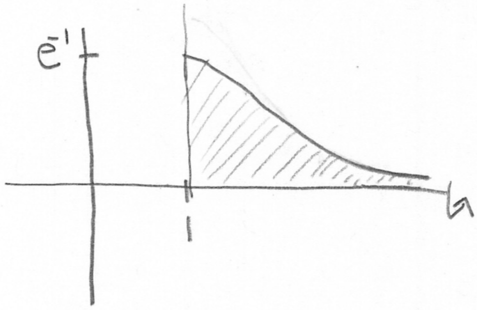
$$\int_0^\infty \frac{1}{\sqrt{x}(1+x)}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{\sqrt{x}(1+x)}\,dx \qquad u=\sqrt{x}, \quad u^2 = x \Rightarrow 2u\,du = dx$$

$$= \lim_{b\to\infty}\int_0^{\sqrt{b}} \frac{2u}{u(1+u^2)}\,du = \lim_{b\to\infty}\int_0^{\sqrt{b}} \frac{2}{(1+u^2)}\,du$$

$$= \lim_{b\to\infty} 2\arctan(u)\Big|_0^{\sqrt{b}} = \lim_{b\to\infty} 2\arctan(\sqrt{b}) - 2\arctan(0)$$

$$= \lim_{b\to\infty} 2(\pi/2) = \pi$$

3. (10 points) Let R be the region that is between the curve $y = \sqrt{x}e^{-x^2}$ and the x-axis, is bounded on the left by the line $x = 1$, and extends infinitely far out to the right. Let S be the solid obtained by rotating R around the x-axis.
Does S have finite volume? If so, find it, and give your answer in exact form.



use Washers

$$\pi \int_1^\infty (\sqrt{x}e^{-x^2})^2 dx = \pi\left(\int_1^\infty xe^{-2x^2} dx\right)$$

$$= \lim_{b\to\infty} \pi \int_1^b xe^{-2x^2} dx \qquad u = 2x^2, \quad du = 4x\,dx$$

$$= \lim_{b\to\infty} \frac{\pi}{4} \int_2^{2b^2} e^{-u} du$$

$$= \lim_{b\to\infty} -\frac{\pi}{4} e^{-u} \Big|_2^{2b^2}$$

$$= \lim_{b\to\infty} \underbrace{-\frac{\pi}{4} e^{-2b^2}}_{\to 0} + \frac{\pi}{4} e^{-2}$$

$$\boxed{\frac{\pi}{4} e^{-2}}$$