3. (8 points) Evaluate the following integral. Simplify your answer where possible.

$$\int_1^\infty \frac{1+e^x}{e^x(1-e^x)}dx$$

① Rewrite

$$\lim_{b\to\infty}\int_1^b \frac{1+e^x}{e^x(1-e^x)}dx$$

② Solve Int.

$u=e^x \quad du=e^x dx = u\,dx$

$$\lim_{b\to\infty}\int_{e^1}^{e^b} \frac{1+u}{u^2(1-u)}$$

By Partial Fract,

$$\frac{1+u}{u^2(1-u)} = \frac{A}{u}+\frac{B}{u^2}+\frac{C}{1-u}$$

$\Rightarrow 1+u = Au(1-u)+B(1-u)+Cu^2$

$u=0: \ 1=B$

$u=1: \ 2=C$

$u=-1 \quad 0=A(-2)+2+2$

$A=2$

$$\Rightarrow \lim_{b\to\infty}\int_{e^1}^{e^b} \frac{2}{u}+\frac{1}{u^2}+\frac{2}{1-u}\,du = \lim_{b\to\infty} 2\ln|u|-\frac{1}{u}-2\ln|1-u|\Big|_{e^1}^{e^b}$$

$$= \lim_{b\to\infty} 2\ln\left|\frac{u}{1-u}\right|-\frac{1}{u}\Big|_{e^1}^{e^b} = \lim_{b\to\infty} 2\ln\left|\frac{e^b}{1-e^b}\right|-\frac{1}{e^b}-2\ln\left|\frac{e^1}{1-e}\right| + 1/e$$

③ Solve limit

$$2\ln\left|\lim_{b\to\infty}\frac{e^b}{1-e^b}\right| - 0 - 2\ln\left|\frac{e}{1-e}\right|+\frac{1}{e}$$

$$= 2\ln|-1| - 2\ln\left|\frac{e}{1-e}\right| + 1/e = \boxed{-2\ln\left|\frac{e}{1-e}\right|+\frac{1}{e}}$$

3. (8 points) Evaluate the integral

$$\int_1^\infty \frac{\sin(1/x)}{x^2}\,dx.$$

If this integral does not converge, explain why.

① Rewrite

$$\lim_{b\to\infty}\int_1^b \frac{\sin(1/x)}{x^2}\,dx$$

② Solve

$$\int_1^b \sin(1/x)/x^2\,dx \qquad u = 1/x \quad du = -1/x^2\,dx$$

$$= \int_1^{1/b} -\sin(u)\,du = \cos(u)\Big|_1^{1/b}$$

$$\Rightarrow \lim_{b\to\infty} \cos(u)\Big|_1^{1/b} = \lim_{b\to\infty} \cos(1/b) - \cos(1)$$

$$= \cos(0) - \cos(1)$$

$$\boxed{= 1 - \cos(1).}$$

3. (8 points) Evaluate the integral

$$\int_2^\infty x^5 e^{-x^3}\,dx.$$

① Rewrite

$$\lim_{b\to\infty}\int_2^b x^5 e^{-x^3}\,dx$$

② Solve Integral

$u = x^3$

$du = 3x^2 dx$

$$\lim_{b\to\infty}\int_8^{b^3} \frac{1}{3}x^3 e^{-u} = \lim_{b\to\infty}\int_8^{b^3}\frac{1}{3}ue^{-u}\,du$$

By IBP, $w = \frac{1}{3}u$ $dv = e^{-u}du$

$dw = \frac{1}{3}du$ $v = -e^{-u}$.

$$\lim_{b\to\infty} -\frac{1}{3}ue^{-u}\Big|_8^{b^3} + \frac{1}{3}\int_8^{b^3} e^{-u}\,du = \lim_{b\to\infty} -\frac{1}{3}ue^{-u}\Big|_8^{b^3} - \frac{1}{3}e^{-u}\Big|_8^{b^3}$$

$$= \lim_{b\to\infty} \frac{-1}{3}b^3e^{-b^3} + \frac{1}{3}8e^{-8} - \frac{1}{3}e^{-b^3} + \frac{1}{3}e^{-8}$$

③ Solve limit

$$\left(\lim_{b\to\infty}\frac{-1/3b^3}{e^{b^3}}\right) + \frac{8}{3}e^{-8} + \frac{1}{3}e^{-8} + 0$$

By L'Hop,

$$\left(\lim_{b\to\infty}\frac{-b^2}{3b^2e^{b^3}}\right) + 3e^{-8}$$

$$= 0 + \boxed{3e^{-8}}$$

3. (8 points) Consider the improper integral

$$\int_1^5 \frac{dx}{x^2\sqrt{25-x^2}}.$$

Evaluate this integral or explain why it does not converge.

① Rewrite

$$\lim_{b\to 5^-} \int_1^b \frac{dx}{x^2\sqrt{25-x^2}}\, dx$$

② Solve

$x = 5\sin\theta \quad d\cancel{x} = 5\cos\theta\, d\theta$

$$\lim_{b\to 5^-} \int_{\sin^{-1}(1/5)}^{\sin^{-1}(b/5)} \frac{5\cos\theta\, d\theta}{25\sin^2\theta\sqrt{25(1-\sin^2\theta)}}$$

$$= \lim_{b\to 5^-} \int_{\sin^{-1}(1/5)}^{\sin^{-1}(b/5)} \frac{1}{25\sin^2\theta}\, d\theta$$

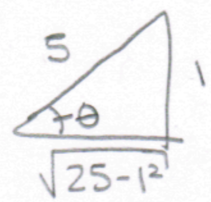
$$= \lim_{b\to 5^-} \int_{\sin^{-1}(1/5)}^{\sin^{-1}(b/5)} \frac{1}{25}\csc^2\theta\, d\theta$$

$$= \lim_{b\to 5} \frac{1}{25}\cot\theta \Big|_{\sin^{-1}(1/5)}^{\sin^{-1}(b/5)}$$

$$= \lim_{b\to 5} \frac{1}{25}\left[\cot(\sin^{-1}(b/5)) - \cot(\sin^{-1}(1/5))\right]$$

③ Solve Limit

$$= \frac{1}{25}\left[\cot(\pi/2) - \cot(\sin^{-1}(1/5))\right]$$



$$= \frac{1}{25}\left[0 - \sqrt{25-1}/1\right]$$

$$\boxed{= \frac{1}{25}\cdot\sqrt{24}}$$

3. (6 points) Consider the improper integral

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx.$$

Evaluate this integral or explain why it does not converge.

① Rewrite

$$\lim_{b\to\infty}\int_1^b \frac{\ln(x)}{x^2}\,dx$$

② Solve Int

By IBP

$u=\ln(x)$ $dv=1/x^2\,dx$

$du=\frac{1}{x}dx$ $v=\frac{-1}{x}$

$$\lim_{b\to\infty} -\frac{\ln(x)}{x}\Big|_1^b + \int_1^b \frac{1}{x^2}\,dx$$

$$\Rightarrow \lim_{b\to\infty} \frac{-\ln(x)}{x}\Big|_1^b - \frac{1}{x}\Big|_1^b$$

$$= \lim_{b\to\infty} \frac{-\ln(b)}{b} + 0 - \frac{1}{b} + 1$$

③ Solve limits

$$= \left(\lim_{b\to\infty} \frac{-\ln(b)}{b}\right) - 0 + 1$$

$$\overset{\text{L'Hop}}{=} \lim_{b\to\infty} \frac{-1/b}{1} - 0 + 1 = \boxed{1}$$

3. (8 points) Consider the improper integral

$$\int_1^\infty \frac{1}{x+x^2}\,dx.$$

Evaluate this integral or explain why it does not converge.

① Rewrite

$$\lim_{b\to\infty}\int_1^b \frac{1}{x+x^2}\,dx$$

② Solve Int.

$$\lim_{b\to\infty}\int_1^b \frac{1}{x(1+x)}\,dx$$

Partial Fractions,

$$\frac{1}{x(1+x)} = \frac{A}{x} + \frac{B}{1+x} \Rightarrow 1 = A(1+x) + Bx$$

$$x=0:\ 1=A$$

$$x=-1:\ 1=-B \Rightarrow -1=B$$

$$\therefore \lim_{b\to\infty}\int_1^b \frac{1}{x} + \frac{-1}{1+x} = \lim_{b\to\infty} \ln|x| - \ln|x+1|\Big|_1^b$$

$$= \lim_{b\to\infty} \ln\left|\frac{b}{b+1}\right| - \ln\left(\frac{1}{2}\right)$$

③ Solve limit

$$= \ln\left(\lim_{b\to\infty} \frac{b\left(\frac{1}{b}\right)}{b+1\left(\frac{1}{b}\right)}\right) - \ln\left(\frac{1}{2}\right)$$

$$= \ln\left(\lim_{b\to\infty} \frac{1}{1+1/b}\right) - \ln(1/2)$$

$$= \ln(1) - \ln(1/2)$$

$$\boxed{= -\ln(1/2)}$$