Math 124 Review Problems

Sections 3.1-3.4, Differentiation Rules
Find the derivatives of the following:

1. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$
2. $y = \pi^x + \arctan\left(\frac{\pi}{x^2}\right)$
3. $f(x) = \ln(x\sec(x) + \sqrt{1+x^2})$
4. $y = \cos\sqrt{\sin(\tan(3x))}$
5. $f(t) = \cos^2(e^{\cos^2(t)})$

1). $y' = \frac{1}{2}\left(x + \sqrt{x+\sqrt{x}}\right)^{-1/2} \cdot \left(1 + \frac{1}{2}(x+\sqrt{x})^{-1/2} \cdot \left(1 + \frac{1}{2}(x)^{-1/2}\right)\right)$

2). $y' = \pi^x \ln(\pi) + \frac{1}{1+\left(\frac{\pi}{x^2}\right)^2} \cdot (-2\pi x^{-3})$

3). $f'(x) = \frac{1}{x\sec(x)} \cdot \left[\left(1 \cdot \sec x + x\sec x \tan x\right) + \frac{1}{2}(1+x^2)^{-1/2} \cdot 2x\right]$

4). $-\sin\left(\sqrt{\sin(\tan(3x))}\right) \cdot \frac{1}{2}(\sin(\tan(3x)))^{-1/2} \cdot \cos(\tan(3x)) \cdot \sec^2(3x) \cdot 3$

5). $f'(t) = 2\cos(e^{\cos^2 t}) \cdot -\sin(e^{\cos^2 t}) \cdot e^{\cos^2 t} \cdot 2\cos(t)(-\sin(t))$

Implicit Differentiation
Implicitely Differentiate the following:

1. $\sqrt{xy} = 1 + x^2y$

2. $1 + x = \sin(xy^2)$

3. $e^y \cos(x) = 1 + \sin(xy)$

4. $2x^3 + x^2y - xy^3 = 2$

5. $\sqrt{x+y} = 1 + x^2y^2$.

① $\frac{1}{2}(xy)^{-1/2} \cdot [y + x\, dy/dx] = 2xy + x^2 dy/dx$

$\frac{1}{2}(xy)^{-1/2} \cdot y + \frac{1}{2}x(xy)^{-1/2} dy/dx = 2xy + x^2 dy/dx$

$$\boxed{\frac{\frac{1}{2}(xy)^{-1/2}y - 2xy}{\left(x^2 - \frac{1}{2}x(xy)^{-1/2}\right)} = dy/dx}$$

② $1 = \cos(xy^2) \cdot [y^2 + x2y\, dy/dx] \Rightarrow 1 = y^2\cos(xy^2) + x2y\cos(xy^2)\, dy/dx$

$$\boxed{\Rightarrow \frac{1 - y^2\cos(xy^2)}{x2y\cos(xy^2)} = dy/dx}$$

③ $e^y dy/dx \cos(x) - e^y \sin(x) = \cos(xy)\,[y + x\, dy/dx]$

$e^y dy/dx \cos(x) - e^y \sin(x) = y\cos(xy) + x\cos(xy)\, dy/dx$

$$\boxed{dy/dx = \frac{y\cos(xy) + e^y\sin(x)}{e^y\cos(x) - x\cos(xy)}}$$

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④ $6x^2 + 2xy + x^2 dy/dx - y^3 - 3y^2x\, dy/dx = 0$

$$\boxed{\frac{6x^2 + 2xy - y^3}{3y^2x - x^2} = dy/dx}$$

⑤ $\frac{1}{2}(x+y)^{-1/2} \cdot [1 + dy/dx] = 2xy^2 + x^2 2y \frac{dy}{dx}$

$\frac{1}{2}(x+y)^{-1/2} + \frac{1}{2}(x+y)^{-1/2} dy/dx = 2xy^2 + x^2 2y \frac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = \frac{2xy^2 - \frac{1}{2}(x+y)^{-1/2}}{\frac{1}{2}(x+y)^{-1/2} - x^2 2y}}$$

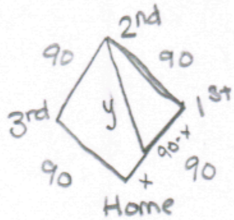
Problem 18, Section 3.9, Related Rates
A baseball diamond is a square with sides 90 feet. A batter hits the ball and runs toward first base with a speed of 24 ft/sec.

a. At what rate is his distance from 2nd base decrease when he is halfway to first base?

b. At what rate is his distance from third base increasing at the same moment

Part a)

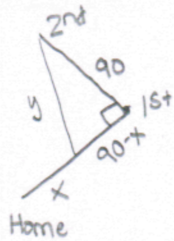
① Draw a Picture
label knowns, give variables names, and write goal



Find dy/dt when $x = 45$

Know $dx/dt = 24$

② Find Relationship (Always Look for Right Triangles)



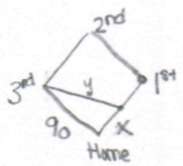
$\Rightarrow$ Pyth. Thm $y^2 = (90-x)^2 + 90^2$

③ Differentiate and plug in knowns

$$2y\,dy/dt = 2(90-x)(-dx/dt)$$
$$y\,dy/dt = 2(90-x)(-dx/dt)$$
$$\Rightarrow y\,dy/dt = 2(90-45)(-24)$$
$$\Rightarrow dy/dt = \frac{2(45)(-24)}{100.623} = \approx \boxed{-21.4662 \text{ ft/s}}$$

Note to find y
$y^2 = (90-45)^2 + 90^2$
$y = 100.623$

Part b) Repeat Part a)
① Draw Pict.



Goal Find dy/dx when $x=45$
Given $dx/dt = 24$

② Relationship

$$y^2 = x^2 + 90^2$$

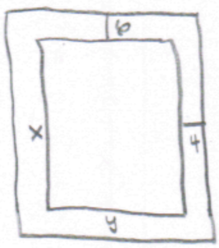
③ Differentiate

$$2y\,dy/dt = 2x\,dx/dt \Rightarrow dy/dt = \frac{x}{y}\,dx/dt$$
$$\Rightarrow dy/dt = \frac{45}{100.623} \cdot 24$$
$$= \boxed{10.733126}$$

$y^2 = 45^2 + 90^2$
$y = 100.623$

Problem 31, Section 4.7, Optimization Problem
The top and bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the area of printed material on the poster is fixed at 384 cm^2, find the dimensions of the poster with the smallest area.

① Draw Picture



$$(y+8)(x+12) = A$$

$$yx = 384$$

$$\Rightarrow x = \frac{384}{y}$$

so $$(y+8)\left(\frac{384}{y}+12\right) = A$$

$$384 + 12y + \frac{3072}{y} + 96 = A$$

② Take derivative:

$$dA/dt = 12 - \frac{3072}{y^2}$$

③ Set Equal to 0 to max/min

$$0 = 12 - \frac{3072}{y^2} = \frac{12y^2 - 3072}{y^2}$$

$$\Rightarrow 0 = 12y^2 - 3072$$

$$\Rightarrow y^2 = \frac{3072}{12}$$

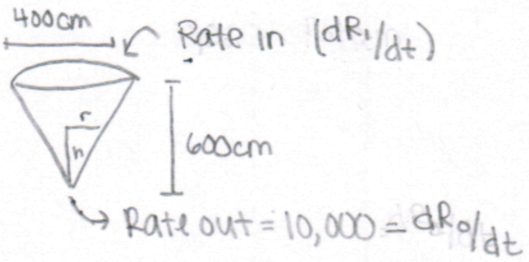
$$y = \sqrt{256} = 16$$

∴ The smallest Area is a poster w/ dimensions

$$(16+8) \text{ by } \left(\frac{384}{16}+12\right)$$

Problem 23, Section 3.9, Related Rates
Water is leaking out of an inverted conical tank at a rate of 10,000 cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.

1). Draw a Picture



Goal Find dR_I/dt when $dh/dt = 20$ and $h = 200$cm

2). Relationships: Let V = volume of water then $V = \frac{1}{3}rh$.
Also Rate in - Rate out = Rate inside is changing. Hence $dV/dt = \frac{dR_I}{dt} - \frac{dR_o}{dt}$.
Since there are 2 variables r and h, let's reduce to 1.
By similar triangles, $\frac{r}{h} = \frac{200}{600} = \frac{1}{3} \Rightarrow r = \frac{1}{3}h$.

Hence, $V = \frac{1}{3}\left(\frac{1}{3}h\right)h = \frac{1}{9}h^2$

3). Take derivative and plug in

$$\frac{dV}{dt} = \frac{1}{9}2h\,dh/dt \Rightarrow \frac{dR_I}{dt} - \frac{dR_o}{dt} = \frac{1}{9}2h\,dh/dt$$

$$\frac{dR_I}{dt} - 10{,}000 = \frac{1}{9}2(200)(20)$$

$$\boxed{\frac{dR_I}{dt} = 10{,}888.889\ cm^3/min}$$