Practice Midterm 1

Compute the following integrals

1. $\int \frac{e^x}{e^{2x}+2e^x+1}\,dx$
2. ~~$\int_{28}^{65} \sqrt{x^2-2x+1}\,dx$~~ $\int_{28}^{65} \frac{1}{\sqrt[3]{x^2-2x+1}}\,dx$
3. $\int x^7(x^4-1)^{2012}\,dx$
4. $\int xe^{x^2}\sec^2(e^{x^2})+\tan(x)\,dx$

① $\int \frac{e^x}{e^{2x}+2e^x+1}\,dx$ $\quad u=e^x$, $du/e^x=dx$ $\Rightarrow \int \frac{1}{u^2+2u+1}\,du = \int \frac{1}{(u+1)^2}\,du$

Let $v=u+1$, $dv=du$ $\quad \int \frac{1}{v^2}\,dv = \frac{-1}{v}+C = \frac{-1}{u+1}+C = \boxed{\frac{-1}{e^x+1}+C}$

② $\int_{28}^{65} \frac{1}{\sqrt[3]{x^2-2x+1}}\,dx = \int_{28}^{65} \frac{1}{(x-1)^{2/3}}\,dx$ $\quad$ Let $u=x-1$, $du=dx$, $u=65-1=64$, $u=28-1=27$ $\quad \int_{27}^{64} \frac{1}{u^{2/3}}\,du$

$= 3u^{1/3}\Big|_{27}^{64} = 3(64^{1/3})-3(27^{1/3}) = 3(4)-3(3) = \boxed{3}$

③ $\int x^7(x^4-1)^{2012}\,dx$ $\quad u=x^4-1$, $\frac{du}{dx}=4x^3$, $\frac{du}{4x^3}=dx$ $\quad \int x^7 u^{2012}\frac{du}{4x^3} = \int x^4 u^{2012}\,du$

Wait... $u=x^4-1$, $u+1=x^4$ $\quad \int (u+1)u^{2012}\,du = \int u^{2013}+u^{2012}\,du = \frac{1}{2014}u^{2014}+\frac{1}{2013}u^{2013}+C$

$\boxed{= \frac{1}{2014}(x^4-1)^{2014}+\frac{1}{2013}(x^4-1)^{2013}+C}$

④ $\int xe^{x^2}\sec^2(e^{x^2})+\tan x\,dx = \underbrace{\int xe^{x^2}\sec^2(e^{x^2})\,dx}_{ⓐ} + \underbrace{\int \tan x\,dx}_{ⓑ}$

ⓐ $= \int xe^{x^2}\sec^2(e^{x^2})\,dx$ $\quad u=e^{x^2}$, $\frac{du}{dx}=e^{x^2}\cdot 2x \Rightarrow \frac{du}{e^{x^2}2x}=dx$ $\quad \int \frac{1}{2}\sec^2 u\,du = \frac{1}{2}\tan(u)+C$

$= \frac{1}{2}\tan(e^{x^2})+C$

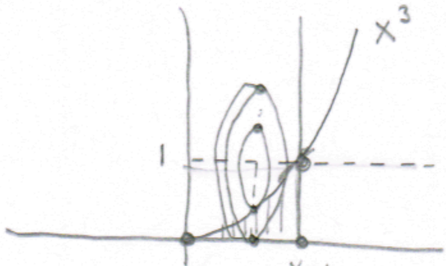
ⓑ $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$ $\quad u=\cos x$, $du/{-\sin x}=dx$ $\quad \int \frac{-1}{u}\,du = -\ln|u|+C = -\ln|\cos x|+C$

$\therefore \int xe^{x^2}\sec^2(e^{x^2})+\tan x = \boxed{\frac{1}{2}\tan(e^{x^2})-\ln|\cos x|+C}$

Let R be the region bounded by $y = x^3$, $y = 0$, $x = 1$.

1. Set up the integral to find the volume rotated about $y = 1$ using discs/washers.
2. Find the volume rotated about $y = 1$ using shells.

① ① Graph (Note that rotating horizontally and washers so need $y = \ldots$)
* Everything in x's.



② Find Pts of Intersection

$x = 0$ and $x = 1$ from graph

$x^3 = 1 \Rightarrow x = 1$

③ Make Chart

	$0 \le x \le 1$
Outer	1
Inner	$1 - x^3$

④ Write integral

$\pi \int R^2 - r^2 = \boxed{\pi \int_0^1 1^2 - (1-x^3)^2 \, dx}$

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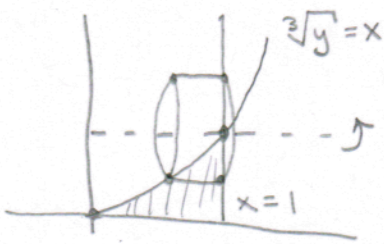
② ① graph (Note that rotating horizontally and shells need $x = \ldots$)
* Everything in y.



② Find pts of intersection

$y = 0$ and $y = 1$ from graph

③ Make Chart

	$0 \le y \le 1$
Height	$1 - \sqrt[3]{y}$
Radius	$1 - y$

④ Make integrals and solve

$2\pi \int_0^1 (1-y)(1-\sqrt[3]{y}) \, dy$

$= 2\pi \int_0^1 1 - y - y^{1/3} + y^{4/3} \, dy$

$= 2\pi \left(y - \frac{1}{2}y^2 - \frac{3}{4}y^{4/3} + \frac{3}{7}y^{7/3} \right) \Big|_0^1$

$= 2\pi \left(\frac{5}{28} \right) = \boxed{\frac{5\pi}{14}}$

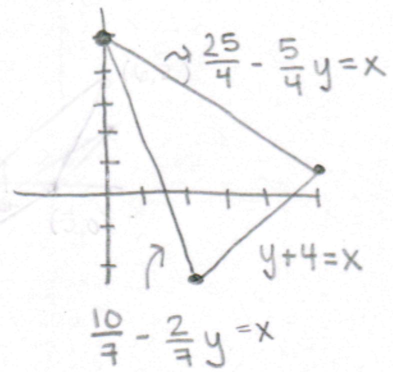
Find the area of the triangle with the given vertices

$$(0,5), \quad (2,-2) \quad \cancel{(5,1)} \quad (6,1)$$

1. With respect to y
2. With respect to x

W/ respect to y

① Graph:



Calculate eq. of lines:

$\frac{1-5}{5-0} = \frac{-4}{5} \quad y = \frac{-4}{5}x + 5$

$\frac{1+2}{5-2} = 1 \quad y = (x-5)+1$

$y = x - 4$

$\frac{5+2}{0-2} = \frac{-7}{2} \quad y = \frac{-7}{2}x + 5$

② Find pts of intersection

$y=5$, $y=-2$, and $y=1$.

③ Make Chart

	$-2 \le y \le 1$	$1 \le y \le 5$
Right	$y+4$	$25/4 - \frac{5}{4}y$
Left	$\frac{10}{7} - \frac{2}{7}y$	$\frac{10}{7} - 2/7y$

④ Set up Integrals

$\int_{-2}^{1} y+4-\frac{10}{7}+\frac{2}{7}y + \int_{1}^{5} \frac{25}{4} - \frac{5}{4}y - \frac{10}{7} + \frac{2}{7}y$

$= \int_{-2}^{1} \frac{9}{7}y + 18/7 + \int_{1}^{5} \frac{-27}{28}y + \frac{135}{28}$

$= \frac{9}{14}y^2 + \frac{18}{7}y \Big|_{-2}^{1} + \frac{-27}{56}y^2 + \frac{135}{28}y \Big|_{1}^{5}$

$= \frac{45}{14} + \frac{18}{7} + \frac{675}{56} - \frac{243}{56}$

$\boxed{= 27/2}$

W/ respect to x

① Graph: same as above

② Pts of intersection

$x=0$, $x=2$, $x=5$

③ Make chart

	$0 \le x \le 2$	$2 \le x \le 5$
Top	$-4/5x+5$	$-4/5x+5$
Bott	$-7/2x+5$	$x-4$

④ Make Integrals

$\int_{0}^{2} -4/5x + 5 + 7/2x - 5 + \int_{2}^{5} -4/5x + 5 - x + 4$

$= \int_{0}^{2} 27/10x + \int_{2}^{5} -9/5x + 9\,dx$

$= \frac{27}{20}x^{\frac{3}{2}} \Big|_{0}^{2} + \frac{-9}{10}x^2 + 9x \Big|_{2}^{5}$

$= \frac{27}{5} + \frac{45}{2} - \frac{72}{5} = \boxed{\frac{27}{2}}$

Find

$$\lim_{x \to 0} \frac{\int_{x^2}^{x^3} \cos(t^2)\, dt}{x}.$$

Recall the definition of a derivative, $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = f'(a)$.

Now in our case $a=0$, $\therefore$

$$\lim_{x \to 0} \frac{\int_{x^2}^{x^3} \cos(t^2)\,dt}{x} = f'(0)$$

① Rewrite integral: $\int_{x^2}^{x^3} \cos(t^2)\,dt = \int_0^{x^3} \cos(t^2)\,dt - \int_0^{x^2} \cos(t^2)\,dt$

② Use $f'(x) = h(g(x)) \cdot g'(x)$

$f'(x) = 3x^2\cos(x^6) - 2x\cos(x^4)$

$\boxed{\text{Need } f'(0) = 3(0^2)\cos(0) - 2(0)\cos(0) = 0}$

Find an expression for the area under the graph of $f(x)$ as a limit, where $f(x) = x\cos(x)$ and $0 \le x \le \pi/2$

Use $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(a + i\Delta x)$

① Δx

$\frac{\frac{\pi}{2} - 0}{n} = \frac{\pi}{2n}$

② $a + i\Delta x$

$0 + i\pi/2n$

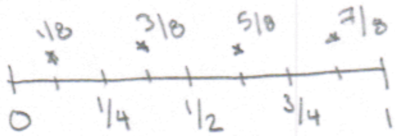
③ Write sum

$\lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(a+i\Delta x) = \boxed{\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{2n}\left(\frac{i\pi}{2n}\right)\cos\left(\frac{i\pi}{2n}\right)}$

Use Midpoint Rule with $n = 4$ to approximate the area of the region bounded above by $y = \sqrt{x^2+1}$ and bounded below by $y = 1 - x^2$ for $0 \le x \le 1$.

① Find midpts

$\Delta x = \frac{1-0}{4} = \frac{1}{4}$



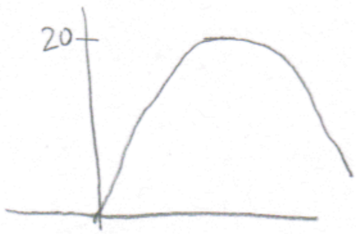
② $\Delta x\,(f(x_1) + f(x_2) + \dots)$

$\text{Area} = \frac{1}{4}\left[\sqrt{\frac{1}{8}^2+1} - \left(1-\left(\frac{1}{8}\right)^2\right) + \sqrt{\left(\frac{3}{8}\right)^2+1} - \left(1-\left(\frac{3}{8}\right)^2\right) + \sqrt{\left(\frac{5}{8}\right)^2+1} - \left(1-\left(\frac{5}{8}\right)^2\right) + \sqrt{\left(\frac{7}{8}\right)^2+1} - \left(1-\left(\frac{7}{8}\right)^2\right)\right]$

$\boxed{\approx .4740746}$

A tomato is thrown vertically upward from ground level toward the ceiling of a tall barn. The ceiling height is 20 meters. With what velocity must the tomato be thrown so that it *just reaches* the ceiling? *Assume acceleration due to gravity is* $10m/s^2$.

① Draw Picture



② Write $a(t) = -10$

$v(t) = -10t + C$

$s(t) = -5t^2 + Ct + D$

③ Initial conditions

$s(0) = 0 \Rightarrow D = 0$

$s(t) = -5t^2 + Ct$

④ when barely hits ceiling $v(t_0) = 0$

and $s(t_0) = 20$

$0 = -10t_0 + C$ $\qquad$ $20 = -5t_0^2 + Ct_0$

$\Rightarrow t_0 = C/10$ $\qquad$ $20 = -5\left(\frac{C}{10}\right)^2 + C(C/10)$

$20 = \frac{-1}{20}C^2 + C^2/10$

$20 = 1/20\, C^2$

$20 = C$

Thus Thrown upward at $\boxed{20m/s}$