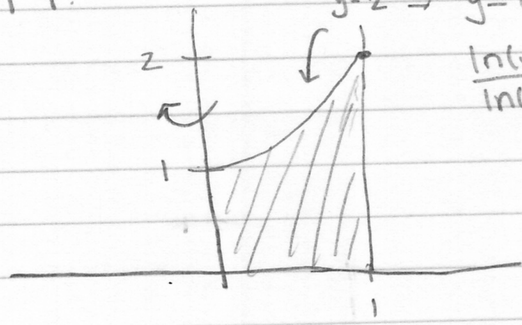
Worksheet 5

Problem 1:



$y=2^x \Rightarrow y=e^{\ln(2)x}$

$\frac{\ln(y)}{\ln(2)} = x$

a) Shell method (in terms of x)

Height $= 2^x$ radius $= x$

$$\text{Volume} = 2\pi \int_0^1 2^x \cdot x\,dx = 2\pi \int_0^1 e^{x\ln(2)} x\,dx$$

$u = x \quad dv = e^{\ln(2)x}dx$

$du = dx \quad v = \frac{1}{\ln(2)} e^{\ln(2)x}$

$$= 2\pi \left(\frac{x}{\ln(2)} e^{\ln(2)x}\right)\Big|_0^1 - \frac{2\pi}{\ln(2)} \int_0^1 e^{x\ln(2)}dx$$

$$= 2\pi\left(\frac{2}{\ln(2)}\right) - \frac{2\pi}{[\ln(2)]^2} e^{x\ln(2)}\Big|_0^1$$

$$= 2\pi\left[\frac{2}{\ln(2)} - \frac{2}{(\ln(2))^2} + \frac{1}{(\ln(2))^2}\right]$$

$$\boxed{= \frac{2\pi}{(\ln(2))^2}[2\ln(2) - 1]}$$

b) Washer Method (in terms of y)

	if $0 \le y \le 1$	if $1 \le y \le 2$
Outer radius	1	1
Inner radius	0	$\ln(y)/\ln(2)$

$$\text{Volume} = \pi \int_0^1 1^2\,dy + \pi \int_1^2 1^2 - \left(\frac{\ln(y)}{\ln(2)}\right)^2 dy$$

FOR ①,

$$\pi \int_0^1 1^2 \, dy = \pi y \Big|_0^1 = \pi$$

For ②,

$$\pi \int_1^2 1 - \frac{1}{\ln(2)^2} \ln(y)^2 \, dy = \pi y \Big|_1^2 - \pi \int_1^2 \frac{\ln(y)^2}{\ln(2)^2} \, dy$$

$$= \pi - \frac{\pi}{(\ln(2))^2} \int_1^2 \ln(y)^2 \, dy$$

$u = \ln(y)^2 \quad dv = dy$

$du = \frac{2\ln(y)}{y} \quad v = y$

$$= \pi - \frac{\pi}{(\ln(2))^2} \left[y\ln(y)^2 \Big|_1^2 - \int_1^2 2\ln(y) \, dy \right]$$

$$= -\pi + \frac{2\pi}{(\ln(2))^2} \int_1^2 \ln(y) \, dy$$

$u = \ln(y) \quad dv = dy$

$du = \frac{1}{y} dy \quad v = y$

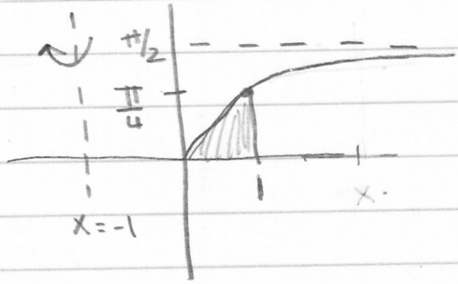
$$= -\pi + \frac{2\pi}{(\ln(2))^2} \left[y\ln(y) \Big|_1^2 - \int_1^2 dy \right]$$

$$= -\pi + \frac{2\pi}{(\ln(2))^2} \left[2\ln(2) - 1 \right]$$

$$\boxed{\text{Volume} = ① + ② = \cancel{\pi - \pi} + \frac{2\pi}{(\ln(2))^2} \left[2\ln(2) - 1 \right]}$$

Note that the shell and washer method agree.

Problem 2



Washer

$$\pi \int_0^{\pi/4} 2^2 - (\tan y + 1)^2 \, dy = \pi \int_0^{\pi/4} 4 - (\tan^2 y + 2\tan y + 1) \, dy$$

$$= \pi \int_0^{\pi/4} 3 - \tan^2 y - \frac{2\sin y}{\cos y} \, dy$$

$$= \pi \left(\underbrace{\int_0^{\pi/4} 3 \, dy}_{1} - \underbrace{\int_0^{\pi/4} \tan^2 y \, dy}_{2} - 2 \underbrace{\int_0^{\pi/4} \frac{\sin y}{\cos y} \, dy}_{3} \right)$$

$$1 = 3y \Big|_0^{\pi/4} = 3\pi/4$$

$$2 = \int_0^{\pi/4} \tan^2 y \, dy = \int_0^{\pi/4} \sec^2 y - 1 \, dy = \tan y - y \Big|_0^{\pi/4}$$

$$= 1 - \frac{\pi}{4}$$

$$3 = \int_0^{\pi/4} \frac{\sin y}{\cos y} \, dy \quad \begin{array}{l} u = \cos y \\ du = -\sin y \, dy \end{array} = \int_1^{\sqrt{2}/2} \frac{-1}{u} \, du$$

$$= -\ln|u| \Big|_1^{\sqrt{2}/2} = -\ln|\sqrt{2}/2|$$

$$\therefore \text{Volume} = \pi \left[\frac{3\pi}{4} - \left(1 - \frac{\pi}{4}\right) + 2\ln\left(\frac{\sqrt{2}}{2}\right) \right] = \pi \left[\pi - 1 + 2\ln\left(\frac{\sqrt{2}}{2}\right) \right]$$

$$\boxed{= \pi \left[\pi - 1 - \ln(2) \right]}$$

Problem 3

$\int e^{-kt}\sin(wt)\,dt$ $\quad u=\sin(wt) \quad dv=e^{-kt}\,dt$

$\quad du=w\cos(wt) \quad v=\frac{-1}{k}e^{-kt}$

$\int e^{-kt}\sin(wt) = \frac{-1}{k}\sin(wt)e^{-kt} + \frac{w}{k}\int \cos(wt)e^{-kt}\,dt$

$\quad u=\cos(wt) \quad dv=e^{-kt}\,dt$

$\quad du=-w\sin(wt) \quad v=\frac{-1}{k}e^{-kt}$

$$\int e^{-kt}\sin(wt)\,dt = \frac{-1}{k}\sin(wt)e^{-kt} - \frac{w}{k^2}\cos(wt)e^{-kt} - \frac{w^2}{k^2}\int \sin(wt)e^{-kt}\,dt$$

Bring to the other side

$$\left(1+\frac{w^2}{k^2}\right)\int e^{-kt}\sin(wt)\,dt = \frac{-1}{k}\sin(wt)e^{-kt} - \frac{w}{k^2}\cos(wt)e^{-kt}$$

$$\boxed{\int e^{-kt}\sin(wt)\,dt = \frac{1}{\left(1+\frac{w^2}{k^2}\right)}\left[\frac{-1}{k}\sin(wt)e^{-kt} - \frac{w}{k^2}\cos(wt)\,e^{-kt}\right]}$$