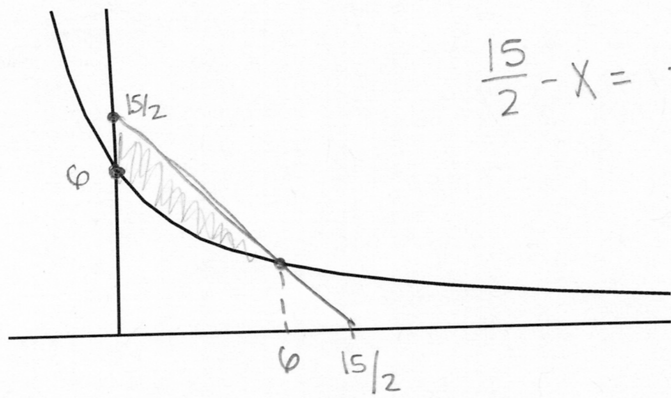
points) Let R be the region *in the first quadrant* which is *above* the hyperbola $y = \dfrac{12}{x+2}$ and *below* the line $y = \dfrac{15}{2} - x$. Sketch the region R and find its area.

The graph of the hyperbola is given below to get you started.



$$\frac{15}{2} - x = \frac{12}{x+2} \Rightarrow \left(\frac{15}{2} - x\right)(x+2) = 12$$

$$\frac{15}{2}x + 15 - x^2 - 2x = 12$$

$$-x^2 + \frac{11}{2}x + 3 = 0$$

$$\frac{-\frac{11}{2} \pm \sqrt{\left(\frac{11}{2}\right)^2 - 4(-1)(3)}}{2(-1)}$$

$$= \frac{-\frac{11}{2} \pm \frac{13}{2}}{-2} = \frac{-\frac{24}{2}}{-2} = \frac{-12}{-2} = 6$$

$$\int_0^6 \frac{15}{2} - x - \frac{12}{x+2}\,dx$$

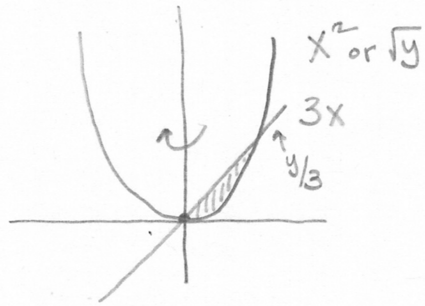
$$= \frac{15}{2}x - \frac{1}{2}x^2 - 12\ln|x+2| \Big|_0^6$$

$$= 45 - 18 - 12\ln(8) + 12(2)$$

$$= \boxed{27 - 24\ln(2)}$$

4. (12 total points) Consider the region R bounded between the parabola $y = x^2$ and the line $y = 3x$.

(a) (2 points) Sketch the region.



$x^2 - 3x = 0$

$x(x-3) = 0$

$\Rightarrow x = 0, 3$

(b) (4 points) Find the volume of the solid obtained by rotating the region R about the y-axis. (Set up a definite integral AND evaluate it.)

Shells

$$2\pi \int_0^3 (3x - x^2)x\,dx = 2\pi \int_0^3 3x^2 - x^3\,dx$$

$$= 2\pi \left[x^3 - \tfrac{1}{4}x^4\right]\Big|_0^3 = 2\pi\left[27 - \tfrac{81}{4}\right] = 2\pi(27/4)$$

$$= \boxed{\frac{27\pi}{2}}$$

(c) (6 points) Set up two separate definite integrals for the volume obtained by rotating the region R about the *horizontal* line $y = 10$. In the first integral use the method of *shells*, and in the second integral use the method of *washers*. DO NOT EVALUATE EITHER INTEGRAL.

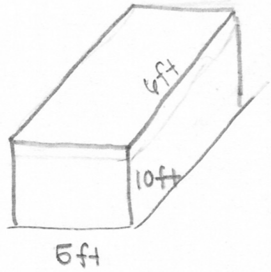
Shells: Every in terms of y

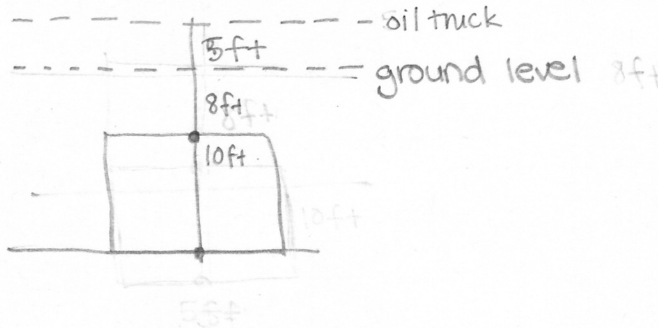
$$\boxed{2\pi \int_0^9 (\sqrt{y} - y/3)(10 - y)}$$

Washers: in terms of x

$$\boxed{\pi \int_0^3 (10 - x^2)^2 - (10 - 3x)^2\,dx}$$

5. (10 points) A rectangular oil tank has height 10 feet, length 5 feet, and width 6 feet. The top of the tank lies 8 feet below ground level. It is currently filled with oil whose weight is 70 pounds per cubic foot. Find the work required to empty the oil tank by pumping all of the oil out of the tank and into an oil truck 5 feet above the ground.





① Volume of a slice

$5 \cdot 6 \cdot dy = 30dy$

② Force of a slice

$70 \cdot 30dy = 2{,}100\,dy$

③ Distance $(23-y)$

④ Bounds 0 to 10

$$\int_0^{10} 2100(23-y)\,dy = 48{,}300y - 1050y^2 \Big|_0^{10}$$

$$= 48300(10) - 1050(100)$$

$$= \boxed{378{,}000 \text{ ft-lbs}}$$

7. (10 total points) You notice a bug flying around the room. The distance $s(t)$ (in feet) that the bug has traveled after t seconds is given by

$$s(t) = \int_0^{3t^2} \sqrt{2x^2 + \cos^2(\pi x)}\, dx.$$

Give units for your answer to both parts below.

(a) (5 points) Find the speed $\frac{ds}{dt}$ of the bug at time $t = \frac{1}{2}$ seconds. Don't forget units.

Use FTC

$$\frac{ds}{dt} = \sqrt{2(3t^2)^2 + \cos^2(\pi \cdot 3t^2)} \cdot 6t$$

$$s'(\tfrac{1}{2}) = \sqrt{2(3/4)^2 + \cos^2(\tfrac{3\pi}{4})} \cdot 3 = \boxed{3\sqrt{13/8} \text{ ft/sec}}$$

(b) (5 points) After $t = 1$ second, the bug has traveled a distance

$$s(1) = \int_0^{3} \sqrt{2x^2 + \cos^2(\pi x)}\, dx.$$

Use Simpson's rule with $n = 4$ subintervals to approximate the value of this integral. You can either give your answer in exact form, or in decimal form with at least three digits after the decimal point. Don't forget units.

$$\Delta x = \frac{3-0}{4} = \frac{3}{4}$$

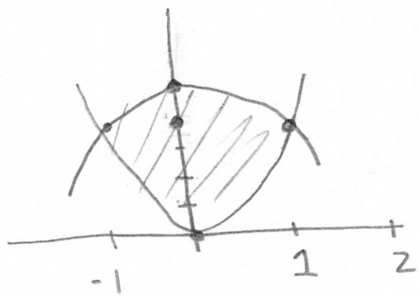
$$\frac{3}{12}\Big[\sqrt{\cos^2(0)} + 4\sqrt{2(\tfrac{3}{4})^2 + \cos^2(\tfrac{3\pi}{4})} + 2\sqrt{2(\tfrac{3}{2})^2 + \cos^2(\tfrac{3\pi}{2})}$$
$$+ 4\sqrt{2(\tfrac{9}{4})^2 + \cos^2(\tfrac{9\pi}{4})} + \sqrt{2(3)^2 + \cos^2(3\pi)}\Big]$$

$$= \frac{1}{4}\Big[1 + 4\sqrt{13/8} + 2\sqrt{9/2} + 4\sqrt{\tfrac{85}{8}} + \sqrt{19}\Big]$$

$$\boxed{\approx 6.935 \text{ ft}}$$

8. (10 total points) Consider the region $\mathscr{R}$ bounded between the curves $y = 5 - x^2$ and $y = 4x^2$.

(a) (3 points) Find the area of $\mathscr{R}$.



$$\int_{-1}^{1} 5-x^2-4x^2\,dx = \int_{-1}^{1} 5-5x^2\,dx$$

$$= 5x - \tfrac{5}{3}x^3 \Big|_{-1}^{1}$$

$$= 5 - \tfrac{5}{3} + 5 - \tfrac{5}{3}$$

$$= \boxed{20/3}$$

(b) (3 points) Find the x-coordinate $\bar{x}$ of the centroid (center of mass) of $\mathscr{R}$.

$M_y = 0$ by symmetry

$$\frac{M_y}{\text{Area}} = \boxed{\bar{x} = 0}$$

(c) (4 points) Find the y-coordinate $\bar{y}$ of the centroid (center of mass) of $\mathscr{R}$.

$$M_x = \tfrac{1}{2}\int_{-1}^{1} (5-x^2)^2 - (4x^2)^2\,dx$$

$$(5-x^2)(5-x^2) = 25 - 10x^2 + x^4$$

$$= \tfrac{1}{2}\int_{-1}^{1} 25 - 10x^2 - 15x^4\,dx$$

$$= \tfrac{1}{2}\left(25x - \tfrac{10}{3}x^3 - 3x^5\right)\Big|_{-1}^{1}$$

$$= \tfrac{1}{2}\left(25 - \tfrac{10}{3} - 3 + 25 - \tfrac{10}{3} - 3\right) = 56/3$$

$$\bar{y} = \frac{56/3}{20/3} = \frac{56}{20} = \frac{28}{10} = \boxed{14/5}$$

9. (10 points) Find the solution of the differential equation

$$\frac{dy}{dx} = \frac{x\sqrt{4-y^2}}{\sqrt{x^2-1}}$$

that satisfies $y(\sqrt{2}) = 1$. Give your answer in the form $y = f(x)$.

Separate the variables $\dfrac{dy}{\sqrt{4-y^2}} = \dfrac{x}{\sqrt{x^2-1}}\,dx$

$$\int \frac{dy}{\sqrt{4-y^2}} = \int \frac{x\,dx}{\sqrt{x^2-1}} \qquad u = x^2-1, \quad du = 2x\,dx$$

$y = 2\sin\theta$
$dy = 2\cos\theta\,d\theta$

$$\int \frac{2\cos\theta}{\sqrt{4-4\cos\theta}}\,d\theta = \int \frac{1}{2}u^{-1/2}\,du$$

$$\theta = u^{1/2} + C \Rightarrow \sin^{-1}(y/2) = \sqrt{x^2-1} + C$$

$$\sin^{-1}\left(\tfrac{1}{2}\right) = \sqrt{2-1} + C = 1 + C$$

$$\pi/6 = 1 + C \qquad C = \pi/6 - 1$$

$$\therefore\ y/2 = \sin\left(\sqrt{x^2-1} + \pi/6 - 1\right)$$

$$\boxed{y = 2\sin\left(\sqrt{x^2-1} + \pi/6 - 1\right)}$$