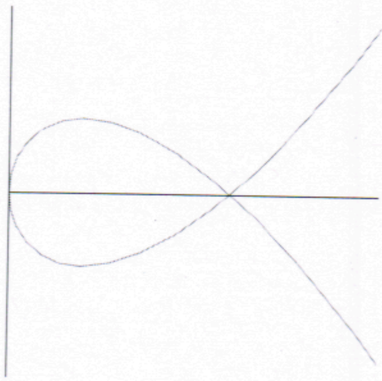
5. (12 points) The location of a particle moving in the plane at time t seconds is given by these parametric equations:

$$x = (t-2)^2 \qquad\qquad y = (t-2)^3 - 3(t-2).$$

The path is graphed below for $0 \le t \le 4$.



(a) Find all of the times when the particle crosses the x axis.

When crosses the x-axis, $y=0$

$0 = (t-2)^3 - 3(t-2)$

$0 = (t-2)\left[(t-2)^2 - 3\right]$

$0 = (t-2)(t^2 - 4t + 1)$

$t=2$ OR $\frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$

$\boxed{\text{So } t = 2 \text{ or } 2 \pm \sqrt{3}}$

(b) Find the equation of the tangent line to the path the first time the particle crosses the x axis. The first time it crosses the x-axis is $t = 2-\sqrt{3}$

This is the a) Method

① Take derivative

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3(t-2)^2 - 3}{2(t-2)}$

② Plug in point $t = 2-\sqrt{3}$

$\frac{dy}{dx} = \frac{3(2-\sqrt{3}-2)^2 - 3}{2(2-\sqrt{3}-2)} = \frac{3(\sqrt{3})^2 - 3}{2(-\sqrt{3})} = \frac{6}{-2\sqrt{3}} = \frac{-3}{\sqrt{3}}$

③ Plug into eq. of line w/ the pt $t = 2-\sqrt{3}$

$x = (2-\sqrt{3}-2)^2 = 3$

$y = (2-\sqrt{3}-2)^3 - 3(2-\sqrt{3}-2)$

$= -3\sqrt{3} + 3\sqrt{3} = 0$

$\boxed{y = \frac{-3}{\sqrt{3}}(x-3)}$

(c) Find the equation of the tangent line to the path the last time the particle crosses the x axis.

① Take derivative $dy/dx = \frac{3(t-2)^2 - 3}{2(t-2)}$

② Plug in $t = 2+\sqrt{3}$

$dy/dx = \frac{3(2+\sqrt{3}-2)^2 - 3}{2(2+\sqrt{3}-2)} = \frac{9-3}{2\sqrt{3}} = \frac{3}{\sqrt{3}}$

③ $x = (2+\sqrt{3}-2)^2 = 3$

$y = 0$

so

$\boxed{y = \frac{3}{\sqrt{3}}(x-3)}$

6. (12 points) Consider the curve in the plane defined by the equation

$$y^3 - 2y^2 - x^2 + 3xy = 0.$$

(a) How many points on the curve have x-coordinate 0? Show that $(0, 2)$ is one of them.

If $x=0$, then $y^3-2y^2=0 \Rightarrow y^2(y-2)=0$ so $y=0$ and 2

There are 2 pts w/ x-coord. 0

$(0,2)$ and $(0,0)$.

(b) Find $\frac{dy}{dx}$ in terms of x and y.

$3y^2 dy/dx - 4y\, dy/dx - 2x + 3y + 3x\, dy/dx = 0$

Put all dy/dx on one side and everything else on the other

$3y^2 dy/dx - 4y\, dy/dx + 3x\, dy/dx = 2x - 3y$

$dy/dx(3y^2 - 4y + 3x) = 2x - 3y$

$$\boxed{\frac{dy}{dx} = \frac{2x-3y}{3y^2-4y+3x}}$$

(c) Find the equation of the tangent line to the curve at $(0, 2)$.

This is the at Method

① Take derivative: (Part b)

$$dy/dx = \frac{2x-3y}{3y^2-4y+3x}$$

② Plug in $(0,2)$ into dy/dx

$$dy/dx = \frac{2(0)-3(2)}{3(2^2)-4(2)+3(0)} = \frac{-6}{4} = \frac{-3}{2}$$

③ Plug into eq. of line

$$Y = \frac{-3}{2}(x) + 2$$

8. (12 points) A particle moves through the plane along the curve C defined by the parametric equations $x(t) = 3t^2 - 4t$, $y(t) = t^2 + 4t + 4$, where $t \geq 0$. Let $P(t) = (x(t), y(t))$ be the location of the particle at time t.

(a) [6pts] Find the equation of the tangent line to the curve C at time $t = 1$.

The At Method

① Find slope: $dy/dx = \dfrac{dy/dt}{dx/dt} = \dfrac{2t+4}{6t-4}$

② Plug in point into derivative

$$dy/dx = \frac{2(1)+4}{6(1)-4} = \frac{6}{2} = 3$$

③ Plug into Eq. of line

$x(1) = 3-4 = -1$ $\quad y = 1+4+4 = 9$

$$\boxed{y = 3(x+1) - 9}$$

(b) [6pts] Find the time(s) when a tangent line to the curve at $P(t)$ passes through the point $(2, 0)$.

$\boxed{\text{The time is } t = 1/4}$

① Find slope: $dy/dx = \dfrac{2t+4}{6t-4}$

② Pretend Point $(3t^2-4t,\ t^2+4t+4)$

③ Write Eq. of line

$$y = \frac{2t+4}{6t-4}\left(x - (3t^2-4t)\right) + t^2+4t+4$$

④ Plug in point given and solve for t

$$0 = \frac{2t+4}{6t-4}(2-3t^2+4t) + t^2+4t+4$$

$$0 = \frac{4t-6t^3+8t^2+8-12t^2+16t}{6t-4} + t^2+4t+4$$

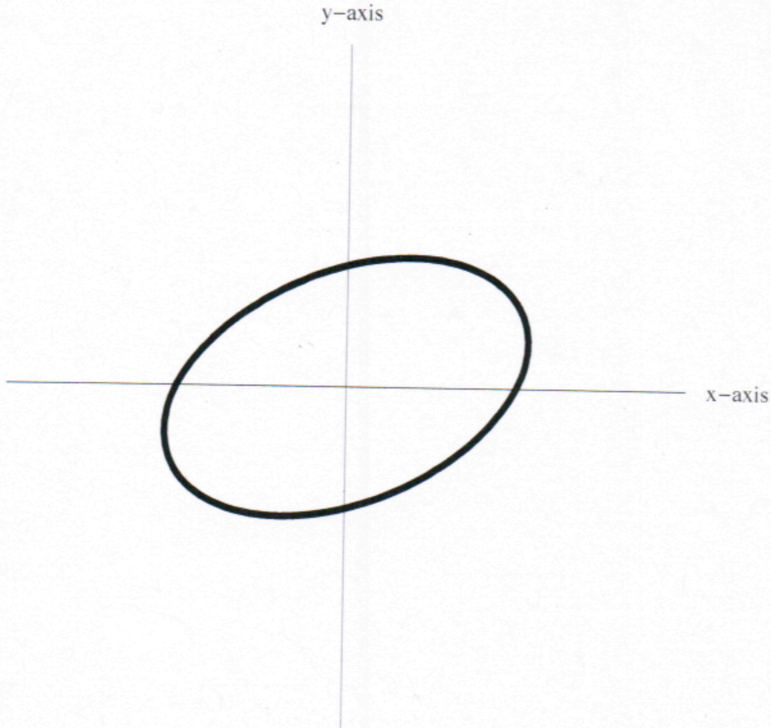
$$0 = \frac{\cancel{-6t^3}-4t^2+20t+8+\cancel{6t^3}+24t^2+24t-4t^2-16t-16}{6t-4}$$

$$0 = \frac{16t^2+28t-8}{6t-4} \Rightarrow 0 = 16t^2+28t-8$$

$$\frac{-28 \pm \sqrt{784-16(4)(-8)}}{2(16)} = \frac{-28 \pm 36}{32}$$

$t = 1/4$ or -2 $\quad$ Must be $\underline{t = 1/4}$

5. (10 points) The graph of the equation $x^2 - xy + 2y^2 = 4$ is a tilted ellipse, as pictured below.



(a) Find a formula for the implicit derivative $\frac{dy}{dx}$.

$$2x - y - x\,dy/dx + 4y\,dy/dx = 0$$
$$4y\,dy/dx - x\,dy/dx = y - 2x$$
$$\frac{dy}{dx}(4y - x) = y - 2x$$

$$\boxed{\frac{dy}{dx} = \frac{y-2x}{4y-x}}$$

(b) Find the coordinates of a point on the ellipse where the tangent line is parallel to the line with equation $y = x + 4$. (Note: there are two correct answers; either will be accepted.) Give your answer in exact form.

If parallel, then same slope, so

$$1 = \frac{y-2x}{4y-x} \Rightarrow 4y - x = y - 2x \Rightarrow 3y = -x \Rightarrow y = \frac{-x}{3}$$

Plugging into original, we get

$$x^2 - x\left(\frac{-x}{3}\right) + 2\left(\frac{-x}{3}\right)^2 = 4$$
$$x^2 + x^2/3 + 2/9\,x^2 = 4$$
$$14/9\,x^2 = 4$$
$$x^2 = 36/14 \Rightarrow x = \pm 6/\sqrt{14}$$

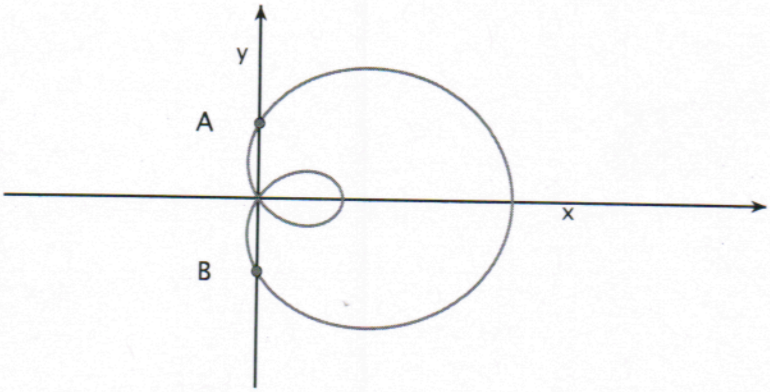
$$\boxed{\begin{array}{l}\therefore \text{The 2 points are} \\ \left(6/\sqrt{14}, \frac{-6}{3\sqrt{14}}\right) \text{ and} \\ \left(\frac{-6}{\sqrt{14}}, \frac{6}{3\sqrt{14}}\right)\end{array}}$$

5. (12 points)

The curve defined implicitly by

$$(x^2 - 2x + y^2)^2 = x^2 + y^2$$

is called a *limaçon trisectrix.* This curve is pictured below, along with the y-intercepts, labeled A and B.



Find the coordinates of the point where the tangent lines at A and B intersect.

This is the at Method

① Find slope

$$2(x^2-2x+y^2)\cdot[2x-2+2y\,dy/dx] = 2x+2y\,dy/dx$$

② Plug in given pt into slope

Points A and B are where $x=0$

$$(y^2)^2 = y^2 \Rightarrow y^4-y^2=0 \Rightarrow y^2(y^2-1)=0$$

$$\Rightarrow y=0 \text{ or } y=\pm 1$$

Thus, $A=(0,1)$ and $B=(0,-1)$

For A: $2(1^2)\cdot[-2+2\,dy/dx] = 2\,dy/dx$

$$-4+4\,dy/dx = 2\,dy/dx$$

$$2\,dy/dx = 4 \Rightarrow dy/dx = 2$$

For B: $2(1^2)[-2-2\,dy/dx] = -2\,dy/dx$

$$-4-4\,dy/dx = -2\,dy/dx \Rightarrow dy/dx = -2$$

③ Eq. of Tang. line

$$y=2(x)+1 \qquad y=-2(x)-1$$

Now find where equal

$$2x+1=-2x-1$$

$$4x=-2 \quad \boxed{x=-1/2}$$

$\boxed{\text{If } x=-1/2, \text{ then } y=2(-1/2)+1=0 \text{ so at the pt } (-1/2,0)}$

4. (10 points) Consider the curve defined by the parametric equations

$$x = \frac{1}{3}t^3 - \ln t, \quad y = \frac{81}{2}t^2 + \frac{8}{t^2} + 3,$$

where $t > 0$.

(a) Find all the horizontal tangent lines to the curve.

Horizontal Tangent line where $dy/dx = \frac{dy/dt}{dx/dt} = 0$

$\Rightarrow dy/dt = 0$ and $dx/dt \neq 0$.

$dy/dt = 81t - 16t^{-3} = 0$

$\frac{81t^4 - 16}{t^3} = 0 \Rightarrow 81t^4 - 16 = 0$

$t^4 = 16/81 \Rightarrow \boxed{t = \pm 2/3}$

At $t = 2/3$, $x = 1/3(\frac{2}{3})^3 - \ln(2/3)$ $y = 81/2(2/3)^2 + 8/(2/3)^2 + 3 = 39$

Thus, the tangent line is $\boxed{y = 39}$

(b) Find all the vertical tangent lines to the curve.

Now you want to find where vertical $\Rightarrow dy/dx =$ undefined

Since $dy/dx = \frac{dy/dt}{dx/dt}$, then vertical $\Rightarrow dx/dt = 0$.

Hence,

$dx/dt = t^2 - 1/t = 0$

$\Rightarrow \frac{t^3 - 1}{t} = 0 \Rightarrow t^3 - 1 = 0 \Rightarrow t = 1$

$x(1) = 1/3 - 0$, $y(1) = 81/2 + 8 + 3$

$\boxed{\text{Thus, the vertical tangent line is: } x = 1/3}$