1. (10 total points) Evaluate the following definite integrals.

(a) (5 points) $\displaystyle\int_0^{\pi/10} \sin^2(5x)\cos^3(5x)\,dx$

$u = \sin(5x) \quad du = 5\cos(5x)$

$$\int_0^1 \tfrac{1}{5}u^2\cos^2(5x)\,du = \int_0^1 \tfrac{1}{5}u^2(1-u^2)\,du = \int_0^1 \tfrac{1}{5}(u^2-u^4)\,du$$

$$\tfrac{1}{5}\left(\tfrac{1}{3}u^3 - \tfrac{1}{5}u^5\right)\Big|_0^1 = \tfrac{1}{5}\left(\tfrac{1}{3}-\tfrac{1}{5}\right) = \tfrac{1}{5}\left(\tfrac{2}{15}\right) = \boxed{\tfrac{2}{75}}$$

(b) (5 points) $\displaystyle\int_1^3 \frac{x^2}{x^2-2x+5}\,dx$

Partial fractions

Long division

$$\begin{array}{r|l} & 1 \\ x^2-2x+5 & x^2+0x+0 \\ & -(x^2-2x+5) \\ \hline & 2x-5 \end{array}$$

$$\int_1^3 1 + \frac{2x-5}{x^2-2x+5}\,dx = \int_1^3 1 + \frac{2x-5}{(x-1)^2+4}\,dx$$

$u = x-1 \quad du = dx$

$u+1 = x$

$$= \int_1^3 1\,dx + \int_0^2 \frac{2u}{u^2+4} - \int_0^2 \frac{3}{u^2+4}$$

$v = u^2+4$

$dv = 2u\,du$

$$= \int_1^3 1\,dx + \int_4^8 \frac{1}{v} - \int_0^2 \frac{3}{u^2+4} = x\Big|_1^3 + \ln|v|\Big|_4^8 - \frac{3}{2}\arctan\left(\frac{u}{2}\right)\Big|_0^2$$

$$= \boxed{2 + \ln(8) - \ln(4) - \frac{3\pi}{8}}$$

2. (10 total points) Evaluate the following indefinite integrals.

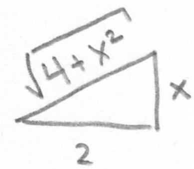
(a) (5 points) $\int \frac{dx}{x^2\sqrt{4+x^2}}$ $\quad x = 2\tan\theta \quad dx = 2\sec^2\theta\, d\theta$

$$\int \frac{\cancel{2}\sec^{\cancel{2}}\theta\, d\theta}{4\tan^2\theta\ \cancel{2\sec\theta}} = \int \frac{\sec\theta\, d\theta}{4\tan^2\theta} = \int \frac{1}{4\cos\theta\left(\frac{\sin^2\theta}{\cos^2\theta}\right)}\, d\theta$$

$$= \int \frac{\cos\theta}{4\sin^2\theta}\, d\theta \qquad u = \sin\theta \quad du = \cos\theta\, d\theta$$

$$= \int \frac{1}{4u^2}\, du = \frac{-1}{4} u^{-1} + C$$

$$= \frac{-1}{4\sin\theta} + C \qquad \frac{x}{2} = \tan\theta$$



$$= \boxed{\frac{-\sqrt{4+x^2}}{4x} + C}$$

(b) (5 points) $\int \frac{x^{1/4}+1}{x(x^{1/4}-2)^2}\, dx$ $\quad u = x^{1/4} \quad u^4 = x \quad 4u^3\, du = dx$

$$\int \frac{(u+1)4\cancel{u^3}\, du}{u^{\cancel{4}}(u-2)^2} = \int \frac{4(u+1)}{u(u-2)^2}\, du$$

$$\frac{A}{u} + \frac{B}{(u-2)} + \frac{C}{(u-2)^2} = \frac{4(u-1)}{u(u-2)^2}$$

$A(u-2)^2 + Bu(u-2) + Cu = 4(u+1)$

$u=2$: $2C = 12$ $\boxed{C=6}$

$u=0$: $4A = 4$ $\boxed{A=1}$

$u=1$: $-1 - B + 6 = 8$

$\boxed{B=-1}$

$$= \int \frac{1}{u} + \frac{-1}{u-2} + \frac{6}{(u-2)^2} \qquad v = u-2 \quad dv = du$$

$$= \ln|u| - \ln|u-2| + \int \frac{6}{v^2}\, dv$$

$$\boxed{= \ln|x^{1/4}| - \ln|x^{1/4}-2| - \frac{6}{x^{1/4}-2} + C}$$

3. (10 points) Determine whether the following improper integral converges and, if so, evaluate it. Justify your answer.

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$$

$= \lim_{b\to\infty} \int_1^b \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$ $\quad u=\sqrt{x}$, $du = \frac{1}{2}x^{-1/2}\,dx$

$= \lim_{b\to\infty} \int_1^{\sqrt{b}} 2e^{-u}\,du$

$= \lim_{b\to\infty} -2e^{-u}\Big|_1^{\sqrt{b}} = \lim_{b\to\infty} \underbrace{-2e^{-\sqrt{b}}}_{\to 0} + 2e^{-1} = \boxed{\frac{2}{e}}$

$\left[\lim_{x\to\infty} e^{-x} = 0\right]$

4. (10 points) The position $s(t)$ of an object moving along a straight line is given by the formula

$$s(t) = \int_{t^2}^{t} e^{\sin x}\, dx.$$

Find a formula for the acceleration $a(t)$ of the object as a function of time t.

FTC

① Break up integral to get into correct form

$$s(t) = \int_{t^2}^{0} e^{\sin x}\, dx + \int_{0}^{t} e^{\sin x}\, dx$$

$$= -\int_{0}^{t^2} e^{\sin x}\, dx + \int_{0}^{t} e^{\sin x}\, dx$$

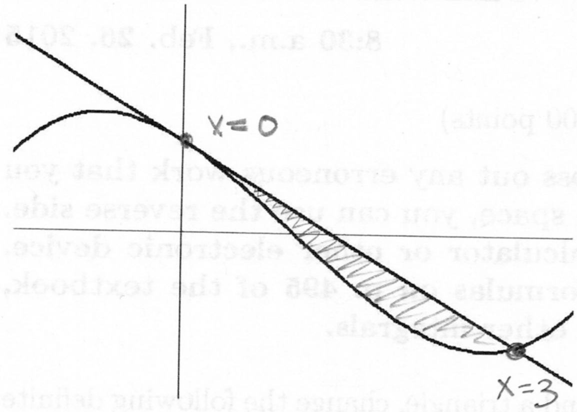
② Apply FTC

$$s'(t) = -e^{\sin(t^2)} \cdot 2t + e^{\sin(t)}$$

For acceleration,

$$\boxed{s''(t) = -e^{\sin(t^2)} \cdot [\cos(t^2) \cdot 4t^2] - 2e^{\sin(t^2)} + e^{\sin(t)} \cdot \cos(t)}$$

5. (10 points) In the graph below, the line is tangent to the curve $y = x^3 - 3x^2 - 6x + 8$ at the point on the y-axis. Find the area enclosed between the curve and the line as shown in the figure.



Find the eq. of the line

$y' = 3x^2 - 6x - 6$ $\qquad y(0) = 8$

$y'(0) = -6$

$\therefore$ equation of tangent line

$y = -6x + 8$

$-6x + 8 = x^3 - 3x^2 - 6x + 8$

$0 = x^3 - 3x^2 \Rightarrow x^2(x-3) = 0 \quad x = 0,\ x = 3$

$\Rightarrow$ Area $= \int_0^3 -6x + 8 - [x^3 - 3x^2 - 6x + 8]\,dx$

$= \int_0^3 -x^3 + 3x^2\,dx = -\frac{1}{4}x^4 + x^3 \Big|_0^3$

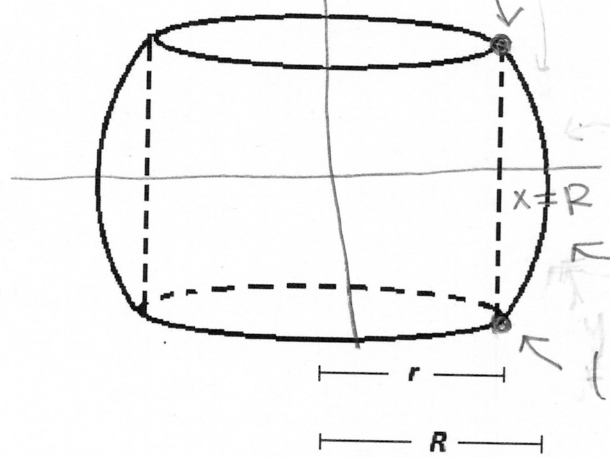
$= -\frac{81}{4} + 27 = \boxed{27/4}$

6. (10 points) A manufacturer drills a round hole of radius r through the center of a metal sphere of radius R. Find the volume of the remaining metal "bead."

$(r, \sqrt{R^2-r^2})$

Equation of a Sphere

$x^2+y^2=R^2$



Washers (in terms of y)

$x=\sqrt{R^2-y^2}$

$(r, -\sqrt{R^2-r^2})$

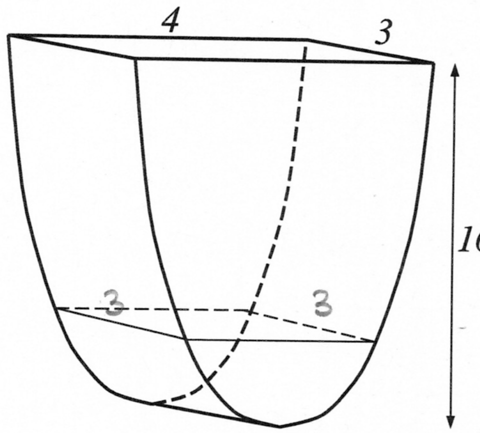
$$\text{Volume} = \pi \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} R^2-y^2-r^2 \, dy$$

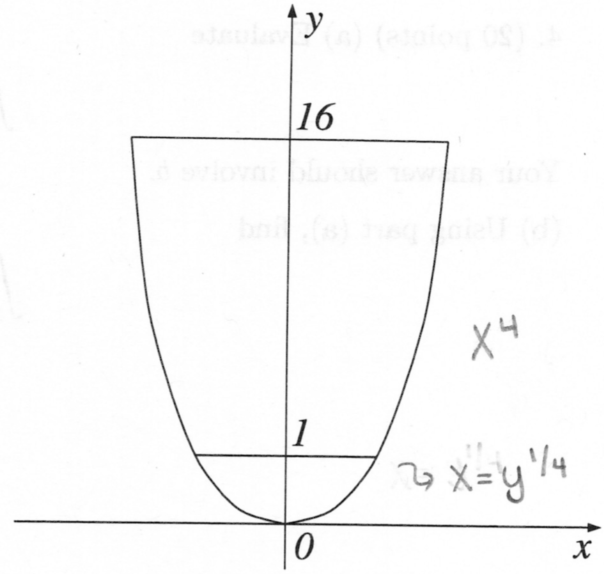
$$= \pi \left[(R^2-r^2)y - \frac{1}{3}y^3 \right] \Big|_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}}$$

$$= \pi \left[(R^2-r^2)^{3/2} - \frac{1}{3}(R^2-r^2)^{3/2} + (R^2-r^2)^{3/2} - \frac{1}{3}(R^2-r^2)^{3/2} \right]$$

$$= \boxed{\frac{4\pi}{3}(R^2-r^2)^{3/2}}$$

7. (10 points)





A tank (shown in the figure on the left) is 16 feet high, with an open rectangular top of width 3 feet and length 4 feet. Each horizontal cross-section of the tank is a rectangle of fixed width 3 feet and length that changes with height. The figure on the right shows the front face of the tank, which has the shape of the function $y = x^4$ for $-2 \le x \le 2$.

Initially, there is fluid in the tank up to a height of 1 foot. The fluid weighs 15 lb/ft^3. How much work does is take to empty the tank by pumping all of the fluid to the top of the tank?

① Volume of a slice

$2y^{1/4} \cdot 3\, dy$

② Force of a slice

$15 \cdot 2y^{1/4} \cdot 3\, dy$

$\Rightarrow 90y^{1/4}\, dy$

③ Distance $(16-y)$

④ Bounds – where there is water

0 to 1

$\text{Work} = \int_0^1 90(16-y)y^{1/4}\, dy$

$= \int_0^1 90(16y^{1/4} - y^{5/4})\, dy$

$= 90\left[\frac{64}{5}y^{5/4} - \frac{4}{9}y^{9/4}\right]\Big|_0^1$

$= 90\left[64/5 - 4/9\right]$

$= 64(18) - 40$

$\boxed{= 1{,}112 \text{ ft-lbs}}$

8. (10 total points)

(a) (4 points) Write a definite integral for the arclength L of the graph of $y=x^2$ from $x=1$ to $x=3$. DO NOT EVALUATE THE INTEGRAL.

$$\text{arc length} = \int_1^3 \sqrt{1+(2x)^2}\,dx$$

$$= \int_1^3 \sqrt{1+4x^2}\,dx$$

(b) (6 points) Use Simpson's Rule with $n=4$ subintervals to approximate the definite integral in part (a). Give your answer in exact form.

$$\Delta x = \frac{3-1}{4} = \frac{1}{2}$$

$$\frac{1}{6}\left[f(1)+4f(\tfrac{3}{2})+2f(2)+4f(\tfrac{5}{2})+f(3)\right]$$

$$\boxed{=\frac{1}{6}\left[\sqrt{5}+4(\sqrt{10})+2\sqrt{17}+4\sqrt{26}+\sqrt{37}\right]}$$

9. (10 points) Find the solution of the initial value problem

$$\frac{dy}{dt} = t\sin(t)\cos^2(y), \quad y(0) = \frac{\pi}{4}.$$

Give your answer in the form $y = f(t)$.

$$\int \frac{dy}{\cos^2(y)} = \int t\sin(t)\,dt$$

IBP $u = t$ $dv = \sin(t)\,dt$
$du = dt$ $v = -\cos(t)$

$$\int \sec^2(y)\,dy = -t\cos(t) + \int \cos(t)\,dt$$

$$= -t\cos(t) + \sin(t) + C$$

$$\tan(y) = -t\cos(t) + \sin(t) + C$$

$$1 = 0 + 0 + C$$

$$\Rightarrow \boxed{y = \arctan(-t\cos(t) + \sin(t) + 1)}$$

10. (10 total points) An advertising company introduces a new product to Seattle. Let $P = P(t)$ be the number of people *in thousands* who are aware of this new product at time t in days. Seattle has a total population of 700 thousand. Initially, no one has heard of the product.

(a) (3 points) Set up a differential equation for P if it increases at a rate proportional to the number of people in Seattle still unaware of the product. Also state the initial condition for P.

$$\frac{dP}{dt} = k\underbrace{(700-P)}_{\text{\# of people who have not heard}} \qquad P(0)=0$$

(b) (5 points) After 20 days, 140 thousand people have heard of the product. Solve the differential equation for $P(t)$, and also determine the exact value of any constants in your solution.

$$\int \frac{dP}{k(700-P)} = \int dt$$

$$-\frac{1}{k}\ln|700-P| = t + C$$

$$\ln|700-P| = -kt + C$$

$$700 - P = Ce^{-kt}$$

$$P(t) = 700 - Ce^{-kt}$$

$$P(0) = 0 = 700 - Ce^{0}$$

$$C = 700$$

$$P(t) = 700 - 700e^{-kt}$$

$$P(20) = 140$$

$$140 = 700(1-e^{-k(20)})$$

$$\frac{1}{5} = 1 - e^{-20k}$$

$$\frac{4}{5} = e^{-20k}$$

$$\ln(4/5) = -20k$$

$$-\frac{1}{20}\ln(4/5) = k$$

$$\therefore$$

$$\boxed{P(t) = 700 - 700e^{\frac{1}{20}\ln(4/5)t}}$$

(c) (2 points) How long does it take before half the population has heard of the product? Give your answer in exact form.

1/2 population is 350

$$350 = 700\left(1 - e^{\frac{1}{20}\ln(4/5)t}\right)$$

$$\frac{1}{2} = 1 - e^{\frac{1}{20}\ln(4/5)t}$$

$$\ln\left(\tfrac{1}{2}\right) = \frac{1}{20}\ln(4/5)t$$

$$\boxed{t = \frac{20\ln(\frac{1}{2})}{\ln(4/5)}}$$