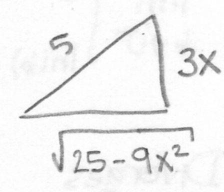
1. (10 total points) Evaluate the following indefinite integrals.

(a) (5 points) $\displaystyle\int \frac{1}{x^2\sqrt{25-9x^2}}\,dx$ $\quad x = \frac{5}{3}\sin\theta \quad dx = 5/3 \cos\theta\, d\theta$

$$\int \frac{5/3\cos\theta\, d\theta}{\frac{25}{9}\sin^2\theta\sqrt{25-25\sin^2\theta}} = \frac{1}{3}\int \frac{9}{25}\csc^2\theta\, d\theta$$

$$= 3/25\int \csc^2\theta\, d\theta = \frac{-3}{25}\cot\theta + C$$

$$\frac{3x}{5} = \sin\theta$$



$$= \frac{-3}{25}\cdot\frac{\sqrt{25-9x^2}}{3x} + C$$

$$= \boxed{\frac{-\sqrt{25-9x^2}}{25x} + C}$$

(b) (5 points) $\displaystyle\int \frac{x+1}{x^3+4x^2+5x}\,dx$

$$x(x^2+4x+5) = x^3+4x^2+5x$$

$$\frac{A}{x} + \frac{Bx+C}{x^2+4x+5} = \frac{x+1}{x^3+4x^2+5x} \qquad A(x^2+4x+5) + x(Bx+C) = x+1$$

$$x=0 \quad 5A = 1 \Rightarrow \boxed{A = 1/5}$$

$$Bx^2 + \frac{1}{5}x^2 = 0 \Rightarrow \boxed{B = -1/5}$$

$$4/5 + C = 1 \Rightarrow \boxed{C = 1/5}$$

$$\int \frac{1/5}{x} + \frac{1}{5}\int\frac{-x+1}{x^2+4x+5}\,dx \qquad (x+2)^2+1$$

$$= \int\frac{1/5}{x} + \frac{1}{5}\int\frac{1-x}{(x+2)^2+1}\,dx \qquad u = x+2 \quad du = dx \qquad u-2 = x$$

$$\frac{1}{5}\int\frac{3-u}{u^2+1}\,du \Rightarrow \frac{1}{5}\int\frac{3}{u^2+1} - \frac{1}{5}\int\frac{u}{u^2+1} \qquad v = u^2+1 \quad dv = 2u\,du$$

$$-\frac{1}{10}\int\frac{1}{v}\,dv$$

$$= \boxed{\frac{1}{5}\ln|x| + \frac{3}{5}\arctan(x+2) - 1/10\ln|(x+2)^2+1| + C}$$

2. (10 total points)

(a) (5 points) Evaluate the definite integral $\int_0^4 \frac{1}{(1+\sqrt{x})^3}\,dx$. Give your answer in exact form.

$u=\sqrt{x} \quad u^2=x \quad 2u\,du=dx$

$\int_0^2 \frac{2u}{(1+u)^3}\,du \quad v=1+u \quad dv=du \quad \int_1^3 \frac{2(v-1)}{v^3}\,dv = \int_1^3 2v^{-2}-2v^{-3}\,dv$

$= \frac{-2}{v}+\frac{1}{v^2}\Big|_1^3 = \frac{-2}{3}+\frac{1}{9}+2-1 = \boxed{4/9}$

(b) (5 points) Find the average value of the function $f(x)=\frac{\ln(2x)}{(2x)^{1/3}}$ on the interval $\left[\frac{1}{2}, 32\right]$.

Give your answer in exact form.

$Avg_{val} = \frac{1}{32-\frac{1}{2}}\int_{1/2}^{32} \frac{\ln(2x)}{(2x)^{1/3}}\,dx \quad u=(2x)^{1/3} \quad u^3=2x \quad 3u^2du=2dx$

$= \frac{1}{63}\int_1^4 \frac{\ln(u^3)\cdot 3u^{\cancel{2}}}{\cancel{u}}\,du \quad$ IBP $\quad w=\ln(u^3) \quad dv=u\,du \quad dw=\frac{1}{u^3}\cdot 3u^2\,du \quad v=\frac{1}{2}u^2$

$= \frac{3}{63}\left[\frac{1}{2}u^2\ln(u^3)\Big|_1^4 - \frac{3}{2}\int_1^4 u\,du\right]$

$= \frac{3}{63}\left[8\ln(64) - \frac{3}{4}u^2\Big|_1^4\right] = \frac{3}{63}\left[8\ln(64) - 12 + \frac{3}{4}\right]$

$= \boxed{\frac{16}{7}\ln(2) - \frac{15}{28}}$

3. (10 points) Does the improper integral $\int_{-3}^{\infty} xe^{-x}\,dx$ converge or diverge? Justify your answer. If it converges, evaluate the integral and give your answer in exact form.

$$\lim_{b\to\infty}\int_{-3}^{b} xe^{-x}\,dx \qquad u=x \quad dv=e^{-x}dx \qquad du=dx \quad v=-e^{-x}$$

$$=\lim_{b\to\infty}\left(-xe^{-x}\Big|_{-3}^{b}+\int_{-3}^{b}e^{-x}\,dx\right)$$

$$=\lim_{b\to\infty} -be^{-b}-3e^{3}-e^{-x}\Big|_{-3}^{b} = \lim_{b\to\infty} -be^{-b}-3e^{3}-\underset{\downarrow \atop 0}{e^{-b}}+e^{3}$$

L'Hop

$$\lim_{b\to\infty}\frac{-b}{e^{b}} = \lim_{b\to\infty}\frac{-1}{e^{b}} = 0$$

$$\Rightarrow = -3e^{3}+e^{3}=\boxed{-2e^{3}}$$

Converges

4. (10 total points) Starting at time $t = 0$, a car moves along the x-axis with velocity

$$v(t) = 20t^2 - 30t \text{ miles/hr},$$

where t is time in hours. Answer the following questions. Include units in your answers.

(a) (5 points) At time $t = 3$ hours, how far is the car from its starting location?

$$\int_0^3 20t^2 - 30t \, dt = \frac{20}{3}t^3 - \frac{30}{2}t^2 \Big|_0^3$$

$$= 180 - \frac{270}{2} = 180 - 135 = \boxed{45 \text{ miles}}$$

(b) (5 points) Unfortunately, the car's gas mileage is only 6 miles per gallon. How many total gallons of gas are used from time $t = 0$ to time $t = 3$ hours?

$$\text{Total distance} = \int_0^3 |20t^2 - 30t| \, dt \qquad t(20t - 30) = 0 \quad t = 0 \quad t = 30/20$$

$$= \int_0^{3/2} -(20t^2 - 30t)\,dt + \int_{3/2}^3 20t^2 - 30t \, dt$$

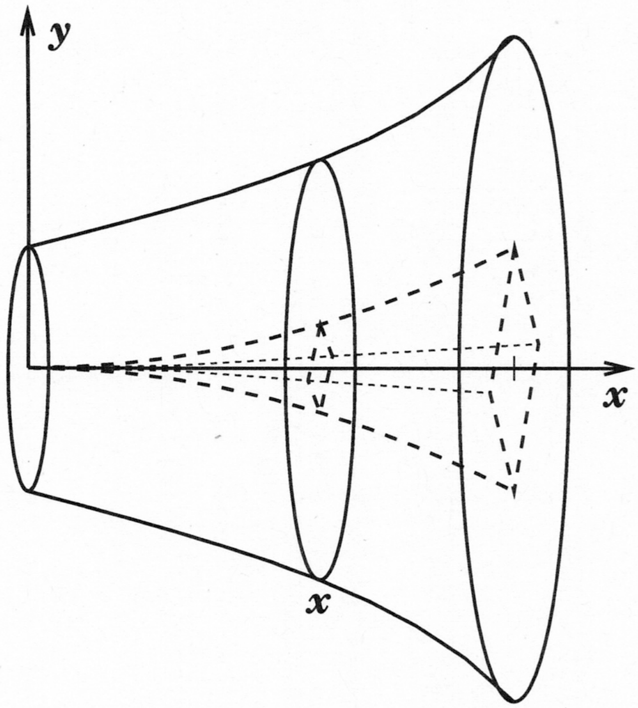
$$= \int_0^{3/2} 30t - 20t^2 + \int_{3/2}^3 20t^2 - 30t \, dt$$

$$= 15t^2 - \frac{20}{3}t^3 \Big|_0^{3/2} + \frac{20}{3}t^3 - 15t^2 \Big|_{3/2}^3$$

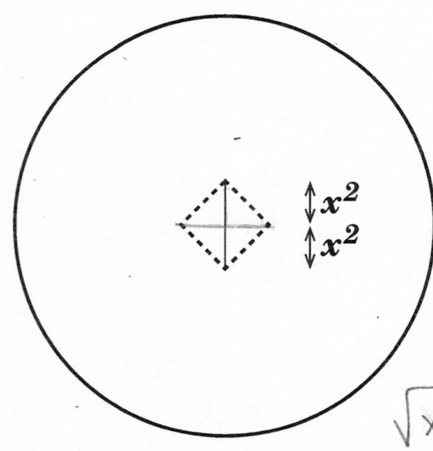
$$= \frac{45}{4} + \frac{225}{4} = \frac{135}{2}$$

$$\therefore \text{Total gallons } \frac{135/2}{6} = \boxed{45/4}$$

5. (10 points) The region R is bounded by the curve $y = 1 + \tan\left(\frac{\pi}{3}x\right)$, the lines $x = 0$ and $x = 1$, and the x-axis. A solid is obtained by rotating R around the x-axis and then carving out a hole. For $0 \le x \le 1$ the vertical cross-section of the hole at x is a square whose diagonal joins the points (x, x^2) and $(x, -x^2)$. Find the volume of the solid. Give your answer in exact form.



cross–section at x



$\sqrt{x^4 + x^4}$

$\sqrt{2x^4} = \sqrt{2}x^2$

side length.

outer area $= \pi\left(1 + \tan(\pi/3\, x)\right)^2$

Inner area $= 2(x^2)^2$

$$\int_0^1 \pi\left(1+\tan\left(\tfrac{\pi}{3}x\right)\right)^2 - 2x^4\,dx = \left(\int_0^1 \pi + 2\pi\tan\left(\tfrac{\pi}{3}x\right) + \pi\tan^2(\pi/3\,x) - 2x^4\,dx\right)$$

$$= \int_0^1 \pi + 2\pi\underbrace{\frac{\sin(\pi/3\,x)}{\cos(\pi/3\,x)}} + \pi\left(\sec^2\left(\tfrac{\pi}{3}x\right) - 1\right) - 2x^4\,dx$$

$u = \cos(\pi/3\,x)$

$du = -\pi/3 \sin(\pi/3\,x)$

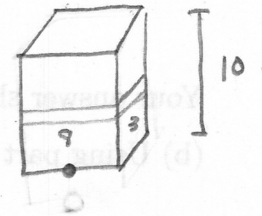
$$= \pi - 6\int_1^{1/2} \frac{1}{u} + \pi\int_0^1 \sec^2(\pi/3\,x) - 1 - \int_0^1 2x^4$$

$$= \pi - 6\ln|u|\Big|_1^{1/2} + 3\tan(\pi/3\,x) - \pi x\Big|_0^1 - \tfrac{2}{5}x^5\Big|_0^1$$

$$= \cancel{\pi} - 6\ln(1/2) + 3\sqrt{3} - \cancel{\pi} - \tfrac{2}{5} = \boxed{-6\ln(1/2) + 3\sqrt{3} - 2/5}$$

6. (10 total points) Two laborers dig a hole in the ground 10 feet deep, shoveling the dirt up to the top of the hole. The horizontal cross-section of the hole is a rectangle of length 8 feet and width 3 feet. The dirt weighs 100 pounds per cubic foot.

(a) (5 points) What is the total work (in ft-lb) done by the two laborers? (The dirt removed from the hole is cleared away by other laborers.)

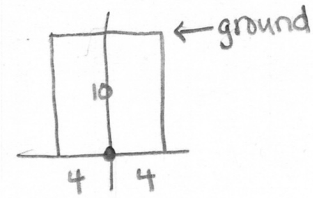


Volume of a slice

$8 \cdot 3\, dy$

Force of a slice

$100 \cdot 24 = 2400\, dy$



Distance $= 10 - y$

Bounds $= 0$ to 10

$$\int_0^{10} (10-y)(2400)\, dy = 2400\left(10y - \tfrac{1}{2}y^2\right)\Big|_0^{10}$$

$$= 2400(50) = \boxed{120{,}000 \text{ ft-lbs}}$$

(b) (5 points) Suppose the first laborer digs the hole part of the way, and then the second laborer finishes digging the hole. How deep (in ft) should the first laborer dig in order to do half of the total work? Give your answer in decimal form with at least three digits after the decimal point.

$\frac{1}{2}$ Work $= 60{,}000$

$$60000 = \int_0^{m} (10-y)(2400)\, dy = 2400\left(10y - \tfrac{1}{2}y^2\right)\Big|_0^{m}$$

$$= 2400\left(10m - \tfrac{1}{2}m^2\right)$$

$25 = 10m - \frac{1}{2}m^2 \Rightarrow 0 = -\frac{1}{2}m^2 + 10m - 25$

$$m = \frac{-10 \pm \sqrt{100 - 4(-\frac{1}{2})(-25)}}{2(-\frac{1}{2})} = \frac{-10 \pm \sqrt{50}}{-1} = \begin{matrix} 10 + \sqrt{50} \approx 17.071 \\ 10 - \sqrt{50} \approx 2.92 \end{matrix}$$

$\therefore$ First labor should dig $10 - (10 - \sqrt{50}) = \sqrt{50} = \boxed{5\sqrt{2} \approx 7.071 \text{ ft}}$

7. (10 total points) Consider the Lissajous curve given by the parametric equations

$$\begin{cases} x = \cos(5t), \\ y = \sin(2t). \end{cases}$$

(a) (5 points) Set up a definite integral for the arc length of the part of the curve for $0.10 \le t \le 0.25$. DO NOT EVALUATE THE INTEGRAL.

$x'(t) = -5\sin(5t) \quad y' = 2\cos(2t)$

$$\text{Arclength} = \int_{.1}^{.25} \sqrt{25\sin^2(5t) + 4\cos^2(2t)}\, dt$$

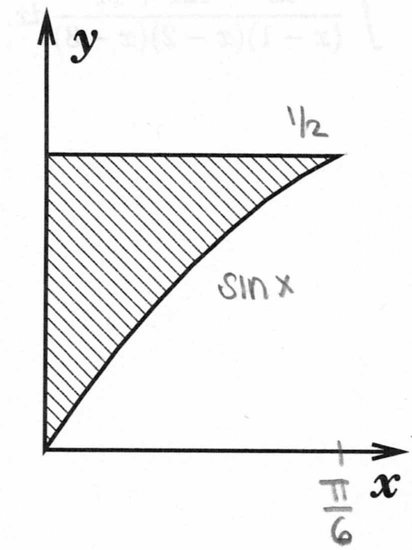
(b) (5 points) Use the Trapezoid Rule with $n = 3$ subintervals to approximate the definite integral in part (a). Give your answer in decimal form with at least four digits after the decimal point.

$$\Delta x = \frac{.25 - .1}{3} = \frac{.15}{3} = .05$$

$$\frac{.05}{2}\Big[\sqrt{25\sin^2(0.5) + 4\cos^2(.2)} + 2\sqrt{25\sin^2(.75) + 4\cos^2(.3)} + 2\sqrt{25\sin^2(1) + 4\cos^2(.4)} + \sqrt{25\sin^2(1.25) + 4\cos^2(.5)}\Big]$$

$$\approx \boxed{0.6289}$$

8. (10 points) Consider a uniform flat plate in the first quadrant that is bounded by the y-axis, the line $y = 1/2$, and the curve $y = \sin x$. Find the y-coordinate $\bar{y}$ of the center of mass of this plate. Give your answer in exact form.



$\frac{1}{2} = \sin x$

$x = \pi/6$

$$\text{Area} = \int_0^{\pi/6} \frac{1}{2} - \sin(x)\,dx = \frac{1}{2}x + \cos(x)\Big|_0^{\pi/6} = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

$$M_x = \frac{1}{2}\int_0^{\pi/6} \left[\frac{1}{4} - \sin^2(x)\right]dx = \frac{1}{2}\int_0^{\pi/6} \left[\frac{1}{4} - \frac{1}{2}(1 - \cos(2x))\right]dx$$

$$= \frac{-1}{8}x + \frac{1}{8}\sin(2x)\Big|_0^{\pi/6} = \frac{-\pi}{48} + \frac{\sqrt{3}}{16}$$

$$\bar{y} = \frac{3\sqrt{3} - \pi}{48\left[\frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1\right]} = \bar{y} = \boxed{\frac{3\sqrt{3} - \pi}{4\pi + 24\sqrt{3} - 48}}$$

9. (10 points) Find the solution of the initial value problem

$$y\,y' = e^{2x}(1+y^2), \qquad y(0) = -1.$$

Give your answer in the form $y = f(x)$.

$$\frac{y\,dy}{1+y^2} = e^{2x}\,dx \Rightarrow \int \frac{y}{1+y^2}\,dy = \int e^{2x}\,dx$$

$v = 1+y^2$

$dv = 2y\,dy$

$$= \frac{1}{2}\int \frac{1}{v}\,dv = \frac{1}{2}e^{2x} + C$$

$$\Rightarrow \frac{1}{2}\ln|1+y^2| = \frac{1}{2}e^{2x} + C$$

$$\Rightarrow \ln|1+y^2| = e^{2x} + C$$

$$\Rightarrow 1+y^2 = Ce^{e^{2x}}$$

$$2 = Ce^{e^0} = Ce^1 \Rightarrow C = \frac{2}{e}$$

$$\Rightarrow 1+y^2 = \frac{2}{e}e^{(e^{2x})}$$

$$\Rightarrow y = \pm\sqrt{\frac{2}{e}e^{(e^{2x})} - 1} \Rightarrow \boxed{y = -\sqrt{\frac{2}{e}e^{(e^{2x})} - 1}}$$

10. (10 total points) A 167°F cup of coffee is brought into a 59°F room. After two minutes, the temperature of the coffee is 149°F. Assume the temperature of the room is constant, and that the rate of change of the temperature of the coffee is proportional to the difference between the temperature of the coffee and the temperature of the room.

(a) (3 points) Set up a differential equation for the temperature $T(t)$ of the coffee (in °F) as a function of time t (in minutes) after it has been brought into the room.

$$\frac{dT}{dt} = k(T-59)$$

(b) (5 points) Find $T(t)$ by solving the differential equation in part (a). Use separation of variables and solve for any unknown constant(s).
SHOW ALL OF YOUR STEPS. DO NOT QUOTE A FORMULA.

$$\frac{dT}{k(T-59)} = dt$$

$$\frac{1}{k}\ln|T-59| = t + C$$

$$\ln|T-59| = kt + C$$

$$T - 59 = Ce^{kt}$$

$T(0) = 167 \Rightarrow 167 - 59 = Ce^0$

$108 = C$

$T(2) = 149$

$\hookrightarrow 149 - 59 = 108e^{k(2)}$

$\ln\left(\frac{5}{6}\right) = k2 \Rightarrow k = \frac{\ln(5/6)}{2} \approx -0.0911$

$$\therefore \boxed{T = 108e^{\ln(5/6)/2 \cdot t} + 59}$$

(c) (2 points) Find $T(4)$. Give your answer in exact form and simplify your answer.

$$T(4) = 108e^{\ln(5/6)\cdot 2} + 59 = 108\left(\frac{5}{6}\right)^2 + 59 = \boxed{134°}$$