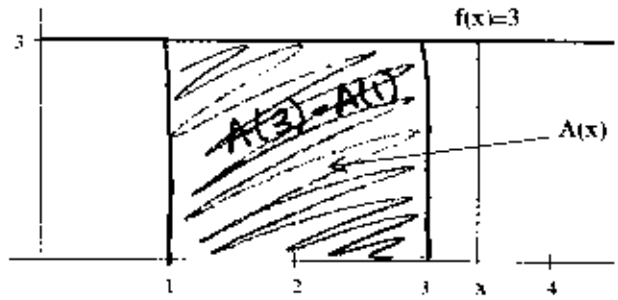
Name Solutions ______ Quiz Section ______

Introduction

Many interesting and useful functions can be defined as the area under some other function. There is a very nice relationship between the original function and the area function. We will explore that relationship in this worksheet.

Area Functions

1a Define $A(x)$ to be the **area** bounded by the x-axis and the function $f(x) = 3$ between the y-axis and the vertical line at x. (See the diagram.)



$A(1) =$ $1 \cdot 3 = 3$ $A(2) =$ $2 \cdot 3 = 6$

$A(3) =$ $3 \cdot 3 = 9$ $A(4) =$ $4 \cdot 3 = 12$

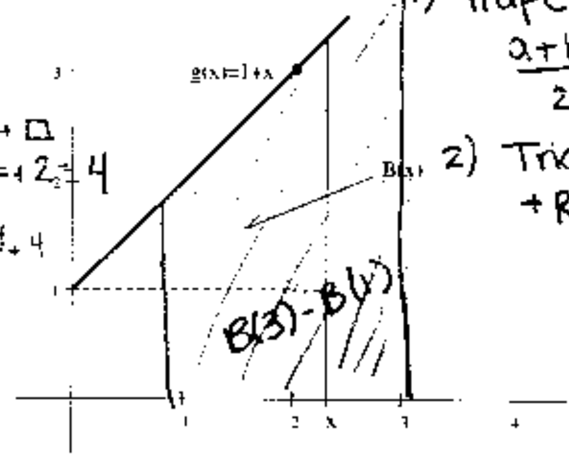
and, in general,

$A(x) =$ $3x$ (a formula)

Shade the region whose area is $A(3) - A(1)$.

Two approaches
1) Trapezoid $\frac{a+b}{2} \cdot h$
2) Triangle + Rectangle

1b Define $B(x)$ to be the **area** bounded by the x-axis and the function $g(x) = 1 + x$ between the y-axis and the vertical line at x. (See the diagram.)



Trap | Δ + □ | Trap | Δ + □

$B(1) =$ $1 \cdot \frac{1+2}{2} = \frac{3}{2}$ | $\frac{1}{2} + 1 = \frac{3}{2}$ $B(2) =$ $2 \cdot \frac{1+3}{2} = 4$ | $\frac{2 \cdot 2}{2} + 2 = 4$

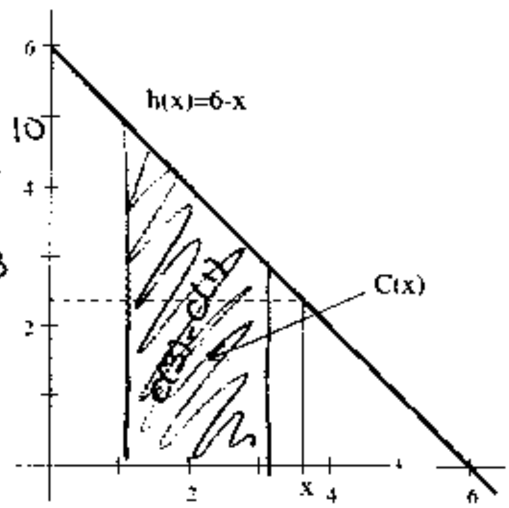
$B(3) =$ $3 \cdot \frac{1+4}{2} = \frac{15}{2} = 7.5$ | $\frac{3 \cdot 3}{2} + 3$ $B(4) =$ $4 \cdot \frac{1+5}{2} = 12$ | $\frac{4 \cdot 4}{2} + 4$

and, in general,

$B(x) =$ $\frac{x^2}{2} + x$ or $x\left(\frac{1+1+x}{2}\right)$ (a formula)
(Hint: think triangle + rectangle) or trapezoid

Shade the region whose area is $B(3) - B(1)$.

1c Define $C(x)$ to be the **area** bounded by the x-axis and the function $h(x) = 6 - x$ between the y-axis and the vertical line at x. (See the diagram.)



Trap | Δ + □ | Trap | Δ + □

$C(1) =$ $1 \cdot \frac{5+6}{2}$ | $\frac{1}{2} + 5 = \frac{11}{2}$ $C(2) =$ $2 \cdot \frac{4+6}{2}$ | $\frac{2 \cdot 2}{2} + 8 = 10$

$C(3) =$ $3 \cdot \frac{6+3}{2}$ | $\frac{3 \cdot 3}{2} + 9 = \frac{27}{2}$ $C(4) =$ $4 \cdot \frac{6+2}{2} = 16$ | $\frac{4 \cdot 4}{2} + 8$

and, in general,

$C(x) =$ $\frac{x^2}{2} + (6-x)x = 6x - \frac{x^2}{2}$ or $x\left(\frac{6+6-x}{2}\right)$ (a formula)

Shade the region whose area is $C(3) - C(1)$.

functions even though $f(x)$ is constant, $g(x)$ is increasing and $h(x)$ is decreasing. (There is a difficulty with $C(x)$ when x gets larger than 6. We'll deal with that later.)

1d Now calculate the derivatives of the area functions from problems 1, 2 and 3 above:

$A'(x) =$ 3 $\qquad B'(x) =$ 1+X $\qquad C'(x) =$ 6-X

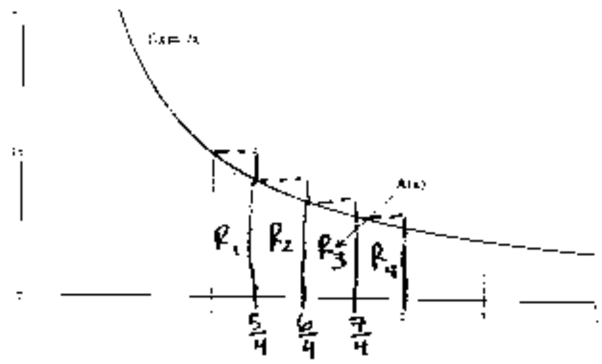
How is $A'(x)$ related to $f(x)$ in problem 1?

How is $B'(x)$ related to $g(x)$ in problem 2? all the same

How is $C'(x)$ related to $h(x)$ in problem 3?

The Natural Logarithm

2a Define $A(x)$ to be the **area** bounded by the x-axis and the function $f(x) = 1/x$ between the line $x = 1$ and the vertical line at x. (See the diagram.)



Based on your work in problem 1,

$A'(x) =$ $\frac{1}{x}$

Compute $A(1) =$ 0

Compute $A(x) =$ ln(X)

2b So the area under $f(x) = 1/x$ between $x = 1$ and $x = 2$ is equal to $\ln(2)$. Outline this area on the graph. We'll use estimates of this area to compute approximations of $\ln(2)$.

2c Slice the area up into 4 pieces by drawing 3 evenly spaced vertical lines from the x-axis up to the curve.

2d Using the left side of each slice as the height, sketch in 4 rectangles on your graph. What are the x-coordinates of the sides of the rectangles? Plug these x-coordinates into $f(x) = 1/x$ to compute the heights of the rectangles. Find the areas of the 4 rectangles and add them up. This is your first approximation of the area under the curve, and $\ln(2)$. Is it an over-estimate or an under-estimate?

Overestimate as rectangles above the curve

$$\ln(2) \approx 1 \cdot 1/4 + \frac{4}{5} \cdot \frac{1}{4} + \frac{4}{6} \cdot \frac{1}{4} + \frac{4}{7} \cdot \frac{1}{4} = \frac{319}{420} \approx 0.7595$$

area of these rectangles and add them up. This is your second approximation of the area under the curve, and ln(2). Is it an over-estimate or an under-estimate?

Right hand approx – underestimate

$$\ln(2) \approx \frac{4}{5}\cdot\frac{1}{4} + \frac{4}{6}\cdot\frac{1}{4} + \frac{4}{7}\cdot\frac{1}{4} + \frac{4}{8}\cdot\frac{1}{4}$$

$$= \frac{533}{840} \approx 0.6345$$

2f Take the average of your two estimates to get a new estimate of ln(2). How does it compare with the value given by your calculator?

$$\frac{\frac{319}{420} + \frac{533}{840}}{2} = \frac{1171}{1680} \approx 0.697$$

2g Use the midpoint of each slice to determine the height and sketch in the resulting 4 rectangles. Use them to approximate ln(2). Can you tell if you are getting an over-estimate or and under-estimate? Which of your four estimates gives you the closest answer to the value given by your calculator?

overestimate

$$\ln(2) \approx \frac{1}{4}\cdot\frac{8}{9} + \frac{1}{4}\cdot\frac{8}{11} + \frac{1}{4}\cdot\frac{8}{13} + 1/4\cdot\frac{8}{15} = \frac{4448}{6435} \approx 0.69121$$

The midpt is closest since $\ln(2) \approx 0.693147$