Chapter 44

Satellite orbits

In this chapter we will consider the problem of building up larger programs from smaller ones. The idea is this: if you have one very big problem to solve, it is easier to break it down into smaller pieces and then put all of the pieces together rather than trying to solve everything all at once. This is a formed of **systems analysis** called **top-down development**.

We will apply this to the study of a particular application, namely, that of predicting the trajectory of a satellite in orbit around the earth. We will first study the physics, then look at the basic algorithm, breaking it down into pieces.

However, don't expect to learn any software engineering here. We are going to hack out the solution by implementing the physics as directly and quickly as we can.

Overview of the Problem

We want to be able to predict the **sub-satellite position** (or **ground track**), on the surface of the earth, of a near-earth orbiting satellite (such as the International Space Station) as a function of time. We will accomplish this goal using slightly-perturbed Keplerian motion, and by taking into account the rotation of the Earth. Here is a summary of the basic facts from physics.

- According to **Kepler's theory** (which we will later derive from Newton's theory using a point-mass approximation for the earth) the satellite's orbit is **an ellipse with one focus of the ellipse at the center of the Earth**.

- **The ellipse is constrained to lie in a plane** that is tilted at some angle with Earth's equatorial plane.

- The Earth is actually slightly blimpy, and is not a point mass. This causes a torque to be exerted on the satellite. **As a result, the orbit plane rotates about the Earth's polar axis**. This rotation causes the orbit vector to be perturbed.

- Because **the Earth rotates** under the plane of the orbit, **the ground track drifts west**. To see this, consider the plane of the orbit. A typical orbital period is about 100 minutes. Consider where the orbit plane crosses the surface of the Earth; without rotation, it will return to the same spot 100 minutes later. But because the Earth is rotating, the orbit will appear to drift westward.

So how would we put these pieces together to predict the location at some time $t = 1$, given the location at some time $t = 0$? Here is our "Basic Algorithm:"

Algorithm 44.1 Orbital algorithm, pass 1

```
input: Initial state vector (position, velocity of satellite)
for each time step Δt do
    1) Determine satellite movement in the 2D orbital plane using Kepler.
    2) Figure out how much the orbit plane has rotated.
    3) Convert the 2D position in the orbit to a 3D Earth-centered xyz.
    4) Convert the Earth-centered xyz to latitude/longitude, accounting for Earth
       rotation.
end for
return
```

Orbits are Elliptical

Elliptical orbits[1] are predicted by Newton's law of gravity. Let r_S and r_E be the position of the satellite and the position of the earth in some inertial coordinate frame, and denote their respective masses by m and M. Then Newton's law of gravity, in combination with Newton's second law of motion says that the force on the satellite by the Earth is given by

$$m\mathbf{r}_S'' = -GmM\frac{\mathbf{r}_S - \mathbf{r}_E}{|\mathbf{r}_E - \mathbf{r}_S|^3} \tag{44.1}$$

while the equal and opposite force on the Earth, by the satellite, is given by

$$M\mathbf{r}_E'' = -GmM\frac{\mathbf{r}_E - \mathbf{r}_S}{|\mathbf{r}_E - \mathbf{r}_S|^3} \tag{44.2}$$

where the derivative is taken with respect to time. Divide the first equation by m, the second by M, and subtract, to get

$$\mathbf{r}'' = -G(M+m)\frac{\mathbf{r}}{|\mathbf{r}|^3} \tag{44.3}$$

where

$$\mathbf{r} = \mathbf{r}_S - \mathbf{r}_E \tag{44.4}$$

Define the **reduced mass of the system** as

$$\mu = G(M+m) \approx GM \approx 3.986 \times 10^{14}\ \mathrm{m}^3/\mathrm{sec}^2 \tag{44.5}$$

The approximation is valid because $M \approx 5 \times 10^{24}$ kg and the heaviest satellites sent into orbit are 10,000 kg, so that $m \ll M$ is reasonable.

The **fundamental equation of motion** is then, from equations 44.3 and 44.4,

$$\mathbf{r}'' = \frac{d^2\mathbf{r}}{dt^2} = -\frac{\mu\hat{\mathbf{r}}}{r^2} \tag{44.6}$$

where $\hat{\mathbf{r}}$ is a unit vector in the same direction as $\mathbf{r}$. Taking the cross product of (44.6)

[1] Technically, orbits are conic sections, which include parabolas and hyperbolas, but we will ignore that distinction for now and only focus on near-earth circular orbits.

with $\mathbf{r}$ gives

$$\mathbf{r} \times \mathbf{r}'' = -\mathbf{r} \times \left(\frac{\mu\hat{\mathbf{r}}}{r^2}\right) = \mathbf{0} \tag{44.7}$$

because the cross product of any vector with itself is zero:

$$\mathbf{r} \times \mathbf{r} = \mathbf{0} \tag{44.8}$$

Next, we consider the following derivative, which we can calculate with the product rule:

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{r}') = \mathbf{r} \times \mathbf{r}'' + \mathbf{r}' \times \mathbf{r}' \tag{44.9}$$

The first term is zero by (44.7), and the second term is zero because it is a cross product of a vector with itself. This gives

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{r}') = \mathbf{0} \tag{44.10}$$

Define the **angular momentum density vector** as

$$\mathbf{h} = \mathbf{r} \times \mathbf{r}' \tag{44.11}$$

Thus

$$\frac{d\mathbf{h}}{dt} = \mathbf{0} \tag{44.12}$$

This gives us conservation of angular momentum.

Law of Conservation of Angular Momentum

$$\mathbf{h} = \mathbf{c}(\text{constant}) \tag{44.13}$$

Take the cross product of the fundamental equation of motion (44.6) with the angular momentum vector,

$$\mathbf{r}'' \times \mathbf{h} = -\left(\frac{\mu\hat{\mathbf{r}}}{r^2}\right) \times (\mathbf{r} \times \mathbf{r}') \tag{44.14}$$

To evaluate the vector triple product $\hat{\mathbf{r}} \times (\mathbf{r} \times \mathbf{r}')$ we will use the "BAC-CAB" identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \tag{44.15}$$

hence

$$\hat{\mathbf{r}} \times (\mathbf{r} \times \mathbf{r}') = \mathbf{r}(\hat{\mathbf{r}} \cdot \mathbf{r}') - \mathbf{r}'(\hat{\mathbf{r}} \cdot \mathbf{r}) \tag{44.16}$$

Since $\mathbf{r} = r\hat{\mathbf{r}}$,

$$\hat{\mathbf{r}} \times (\mathbf{r} \times \mathbf{r}') = \frac{1}{r}(\mathbf{r}(\mathbf{r} \cdot \mathbf{r}') - r^2\mathbf{r}') \tag{44.17}$$

Hence from equation (44.14)

$$\mathbf{r}'' \times \mathbf{h} = -\left(\frac{\mu}{r^3}\right)(\mathbf{r}(\mathbf{r} \cdot \mathbf{r}') - r^2\mathbf{r}') \tag{44.18}$$

Since r is the magnitude of $\mathbf{r}$, then r' is the rate of change of $\mathbf{r}$ in a direction parallel to

$\mathbf{r}$. Hence

$$r' = \frac{dr}{dt} = \hat{\mathbf{r}} \cdot \mathbf{r}' = \frac{1}{r}\mathbf{r} \cdot \mathbf{r}' \tag{44.19}$$

Substituting this into (44.18) gives

$$\mathbf{r}'' \times \mathbf{h} = -\left(\frac{\mu}{r^3}\right)(\mathbf{r}rr' - r^2\mathbf{r}') = -\left(\frac{\mu}{r^2}\right)(\mathbf{r}r' - r\mathbf{r}') \tag{44.20}$$

But by the quotient rule

$$\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \frac{r\mathbf{r}' - \mathbf{r}r'}{r^2} \tag{44.21}$$

Hence

$$\mathbf{r}'' \times \mathbf{h} = \mu \frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) \tag{44.22}$$

We can rewrite this as

$$\mathbf{v}' \times \mathbf{h} = \mu \frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) \tag{44.23}$$

where $\mathbf{v} = \mathbf{r}'$. Reversing the order of the cross product,

$$\mathbf{h} \times \mathbf{v}' = -\mu \frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) \tag{44.24}$$

Multiply both sides of the equation by dt, and integrate

$$\int \mathbf{h} \times \mathbf{v}'\, dt = -\mu \int \frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) dt \tag{44.25}$$

Since $\mathbf{h}$ is a constant we can pull it out of the integral on the left. Further, we can write $\mathbf{v}' = d\mathbf{v}/dt$ so that

$$\mathbf{h} \times \int \frac{d\mathbf{v}}{dt}\, dt = -\mu \int \frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) dt \tag{44.26}$$

By the fundamental theorem of calculus,

$$-\mathbf{h} \times \mathbf{v} = \mu \frac{\mathbf{r}}{r} + \mathbf{C} \tag{44.27}$$

where $\mathbf{C}$ is a (vector) constant of integration. The standard notation is define an **eccentricity vector** $\mathbf{e} = \mathbf{C}/\mu$, so that

$$\mathbf{v} \times \mathbf{h} = \mu\left(\frac{\mathbf{r}}{r} + \mathbf{e}\right) \tag{44.28}$$

The reason for the name eccentricity will become apparent later. Taking the dot product of (44.28) with $\mathbf{r}$ gives

$$(\mathbf{v} \times \mathbf{h}) \cdot \mathbf{r} = \mu\left(\frac{\mathbf{r}}{r} + \mathbf{e}\right) \cdot \mathbf{r} = \mu(r + \mathbf{r} \cdot \mathbf{e}) \tag{44.29}$$

Using the vector identity $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$, and then substituting the definition of angular momentum ($\mathbf{h} = \mathbf{r} \times \mathbf{v}$, from equation (44.11)),

$$(\mathbf{v} \times \mathbf{h}) \cdot \mathbf{r} = (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{h} = \mathbf{h} \cdot \mathbf{h} = h^2 \tag{44.30}$$

where h is the constant magnitude of the angular momentum per unit mass. Substituting (44.30) into (44.29) gives us

$$h^2 = \mu(r + \mathbf{r} \cdot \mathbf{e}) \tag{44.31}$$

Define θ as the angle between $\mathbf{r}$ and the constant vector $\mathbf{e}$. Then

$$h^2 = \mu(r + re\cos\theta) = \mu r(1 + e\cos\theta) \tag{44.32}$$

Solving for r

$$r = \frac{h^2/\mu}{1 + e\cos\theta} \tag{44.33}$$

This is the equation from analytic geometry for an ellipse with **semi-parameter** $p = h^2/\mu$ and eccentricity e, in polar coordinates. The semi-parameter is more commonly written in terms of the semi-major axis and eccentricity as

$$p = a(1 - e^2) \tag{44.34}$$

Thus we get the following result.

Orbit Position in Polar Coordinates

The distance r from the center of the earth is

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} \tag{44.35}$$

Here a is semi-major axis, e is eccentricity, θ is central angle measured from the point of closest approach, and $h^2 = \mu a(1 - e^2)$.

The Vis-Viva Equation

The potential energy for a satellite of mass m in the earth's gravity is

$$E_{potential} = -\frac{\mu m}{r} \tag{44.36}$$

where $\mu = GM$, as defined previously, and the kinetic energy is

$$E_{kinetic} = \frac{1}{2}mv^2 \tag{44.37}$$

where v is the velocity, as defined in the previous section. By the **law of energy conservation** the total energy $\mathcal{E}$ is a constant

$$\mathcal{E} = E_{kinetic} + E_{potential} = \frac{1}{2}mv^2 - \frac{\mu m}{r} \tag{44.38}$$

Let r_1, v_1 and r_2, v_2 be the position and velocity of a satellite at two different points in its orbit. Then by energy conservation,

$$\frac{1}{2}mv_1^2 - \frac{\mu m}{r_1} = \frac{1}{2}mv_2^2 - \frac{\mu m}{r_2} \tag{44.39}$$

Canceling out the common factor of m,

$$\frac{v_1^2}{2} - \frac{\mu}{r_1} = \frac{v_2^2}{2} - \frac{\mu}{r_2} \tag{44.40}$$

From equation 44.35, at $\theta = 0$, $r(0) = a(1-e)$. This distance is called **perigee**, because it is the closest point to the origin.[2] Furthermore, at perigee the velocity and the radius are perpendicular to one another, so the magnitude of the angular momentum is $h = r_{perigee}v_{perigee}$. From the equation in the discussion following (44.35)

$$h^2 = \mu a(1-e^2) = r_{perigee}^2 v_{perigee}^2 \tag{44.41}$$

Hence, since $r_{perigee} = a(1-e)$,

$$v_{perigee}^2 = \frac{\mu a(1-e^2)}{a^2(1-e)^2} = \frac{\mu}{a}\frac{1+e}{1-e} \tag{44.42}$$

If we let r_1 be any point on the orbit, and r_2 be perigee, then equation (44.40) gives us

$$\frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu}{2a}\frac{1+e}{1-e} - \frac{\mu}{a(1-e)} = \frac{\mu}{2a}\left[\frac{1+e}{1-e} - \frac{2}{1-e}\right] = -\frac{\mu}{2a} \tag{44.43}$$

Solving for v^2 gives the **Vis-Viva Equation**

Vis-Viva Equation

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \tag{44.44}$$

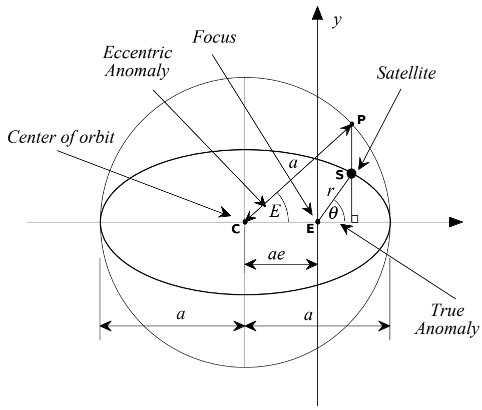
where v is the satellite velocity, r its distance from the center of the Earth, a the orbital semi-major axis, $\mu = GM$, G is Newton's universal constant of gravitation, and M is the mass of the earth.

Keplerian Orbits

We have shown that in the absence of any outside forces, when a satellite orbiting about the Earth is treated like a point mass we end up with elliptical orbits. This is the essence of Keplerian orbital dynamics. Kepler's orbital description provides a reasonable first order description of planetary motion and, in fact, only slight adjustments are necessary to get extremely accurate predictions of satellite orbits. Our description of elliptical motion in a plane is given by figure 44.1. We know from equation 44.35 that if we place the focus of the ellipse at the origin and the perigee (nearest point on the orbit to the

[2] It is only called perigee when the central body is Earth. If the central body is, eg., the sun, moon or Jupiter, we use the terms *perihelion*, *perilune*, or *perijove*. The general term is *periapsis*.

Figure 44.1: Description of an elliptical orbit. The origin is off-centered from the center of the ellipse by a distance ae along the x-axis.



focus) on the positive x axis, then

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \tag{44.45}$$

where θ, called the **true anomaly**, is the ordinary polar angular coordinate. The center of the ellipse in displaced from the origin by a distance ae to the left in this figure.

Suppose the satellite has polar coordinates (r, θ). Then in Cartesian coordinates,

$$(x, y) = (r\cos\theta,\ r\sin\theta) \tag{44.46}$$

Form the following construction, as illustrated in figure 44.1

1. Construct a circle of radius a circumscribing the ellipse.
2. Drop a perpendicular line ℓ from the satellite's location **S** to the x axis.
3. Denote by **P** the intersection of ℓ and the circle.
4. Draw a line segment from **P** to the center **C** of the circle.
5. The angle $E = \angle$**PCE** (**E** is the focus) is called the **eccentric anomaly**.

In terms of the eccentric anomaly,

$$a\cos E = ae + x \tag{44.47}$$

hence

$$x = a(\cos E - e) \tag{44.48}$$

Using (44.46) in (44.45)

$$r = \frac{a(1-e^2)}{1+ex/r} \tag{44.49}$$

Cross-multiplying and solving for r,

$$r = a(1-e^2) - ex = a(1-e^2) - ea(\cos E - e) \tag{44.50}$$
$$= a - ae^2 - ea\cos E + ae^2 = a(1 - e\cos E) \tag{44.51}$$

By the Pythagorean theorem,

$$y^2 = r^2 - x^2 = a^2(1-e\cos E)^2 - a^2(\cos E - e)^2 \tag{44.52}$$
$$= a^2(1 - 2e\cos E + e^2\cos^2 E - \cos^2 E + 2e\cos E - e^2) \tag{44.53}$$
$$= a^2(1 + e^2\cos^2 E - \cos^2 E - e^2) \tag{44.54}$$
$$= a^2(1-e^2)(1-\cos^2 E) \tag{44.55}$$

so that

$$y = a\sqrt{1-e^2}\sin E \tag{44.56}$$

We don't have to worry about getting the correct sign of y because the the sign of $\sin E$ will always give us the correct quadrant.

Differentiating (using (44.48) and (44.56))

$$\frac{dx}{dt} = -a\sin E\frac{dE}{dt} \tag{44.57}$$
$$\frac{dy}{dt} = a\sqrt{1-e^2}\cos E\frac{dE}{dt} \tag{44.58}$$

From the definition of angular momentum, $\mathbf{h} = \mathbf{r} \times \mathbf{v}$. In the coordinate frame shown, with the z axis out of the plane of the paper,

$$\mathbf{h} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & 0 \\ x' & y' & 0 \end{vmatrix} = \mathbf{k}(xy' - yx') \tag{44.59}$$

Hence

$$h = a(\cos E - e)\left(a\sqrt{1-e^2}\cos E\right)E' + (a\sqrt{1-e^2}\sin E)(a\sin E)E' \tag{44.60}$$
$$= a^2\sqrt{1-e^2}\left[(\cos E - e)\cos E + \sin^2 E\right]E' \tag{44.61}$$
$$= a^2\sqrt{1-e^2}(1 - e\cos E)E' \tag{44.62}$$

From equation 44.35, $h^2 = \mu a(1-e^2)$, hence

$$\mu a(1-e^2) = a^4(1-e^2)(1-e\cos E)^2\left(E'\right)^2 \tag{44.63}$$

After some cancellation,

$$\mu = a^3(1-e\cos E)^2\left(E'\right)^2 \tag{44.64}$$

Dividing by a^3 and taking the square root,

$$\sqrt{\frac{\mu}{a^3}} = (1 - e\cos E)E' \tag{44.65}$$

Let t_P be the time at which the satellite passes through perigee. Multiply equation (44.65) by dt and integrate from t_P:

$$\int_{t_P}^{t} \sqrt{\frac{\mu}{a^3}}\, dt = \int_{E(t_p)}^{E(t)} (1 - e\cos E) E'\, dt \tag{44.66}$$

Pulling out the constant on the left hand side and integrating it, and writing $E' dt = dE$

$$\sqrt{\frac{\mu}{a^3}}(t - t_P) = \int_{E(t_p)}^{E(t)} (1 - e\cos E)\, dE = (E - e\sin E)\Big|_0^{E(t)} = E - e\sin E \tag{44.67}$$

where the last step follows because $E(t_p) = 0$.

Mean Motion

The **mean motion** is

$$n = \sqrt{\frac{\mu}{a^3}} \tag{44.68}$$

where $\mu = GM$ and a is the semi-major axis, gives an equivalent velocity as if the satellite were moving in a circular orbit at a fixed velocity with the same period.

In terms of the mean motion, the angular position of the satellite can be found at any later time from equation (44.67) by solving **Kepler's Equation**.

Kepler's Equation

$$E - e\sin E = n(t - t_P) = M \tag{44.69}$$

Where E is the eccentric anomaly, e is the orbital eccentricity, M is mean anomaly, t_P is the time of periapsis passage, and n is the mean motion.

The number M, defined by the last equal sign of (44.69) is called the **Mean Anomaly**. The Mean Anomaly is an equivalent angle that changes linearly in time.

Mean Anomaly

The **mean anomaly** is an equivalent angle that changes linearly in time.

If we let τ be the period of the satellite, then the eccentric anomaly will be 2π. This gives

$$2\pi = n\tau = \tau\sqrt{\frac{\mu}{a^3}} \tag{44.70}$$

Squaring both sides of the equation and rearranging gives Kepler's third law.

Kepler's Third Law of Planetary Motion

$$\frac{4\pi^2}{\mu}a^3 = \tau^2 \tag{44.71}$$

This shows that Kepler's famous result, that the square of the period is proportional to the cube of the semi-major axis, follows from Newton's laws of motion.

Now we are able to produce an algorithm that predicts the position of a satellite in its orbital plane. Given the orbital elements a, e, and time of perigee passage t_p, we calculate the position of the satellite in the plane of the ellipse using algorithm 44.2.

Algorithm 44.2 Algorithm **OrbitPosition** for motion in a Keplerian orbit.

input: Orbital elements $\mathbf{v}$ that include: a (semi-major axis); e (orbital eccentricity); t_p (time of perigee passage); t (current time); ϵ (a very small number, tolerance, say 10^{-10}

$n \leftarrow \sqrt{mu/a^3}$
$M \leftarrow n(t - t_p)$
$E \leftarrow M$
while $|E - M| > \epsilon$ **do**
  $E \leftarrow M + e \sin E$ ▷ Fixed point iteration for E
end while
$x \leftarrow a(\cos E - e)$
$y \leftarrow a\sqrt{1 - e^2} \sin E$
return (x, y) as **OrbitPosition**($\mathbf{v}$, t)

We want to define a function **OrbitPosition**($\mathbf{v}$, t) that will take as its input a Keplerian input vector $\mathbf{v} = (a, e, i, \Omega, \omega, M)$ at some time $t = 0$, and determine the position and (x, y), as measured in the plane of the orbit, a time t later.

Note that the x coordinate here is the coordinate parallel to the $\mathbf{p}$ axis (the axis through the center of the orbit to perigee), and the y coordinate is the coordinate parallel to the $\mathbf{q}$ axis (in the plane of the axis, through the focus, normal to $\mathbf{p}$), so it might be better to call these numbers (p, q) rather than (x, y).

We can now revise algorithm 44.1. For input, NASA provides orbital elements in two forms, either with the mean anomaly M or the time of periapsis passage t_p. If t_p is given, the mean anomaly can be recovered from equations 44.68 and 44.69.

Out of the Plane: Kepler's Elements

We have shown that in the absence of external perturbations the orbit is an ellipse. We also now know how to calculate the trajectory in that ellipse. The next step to to orient that ellipse in three dimensions.

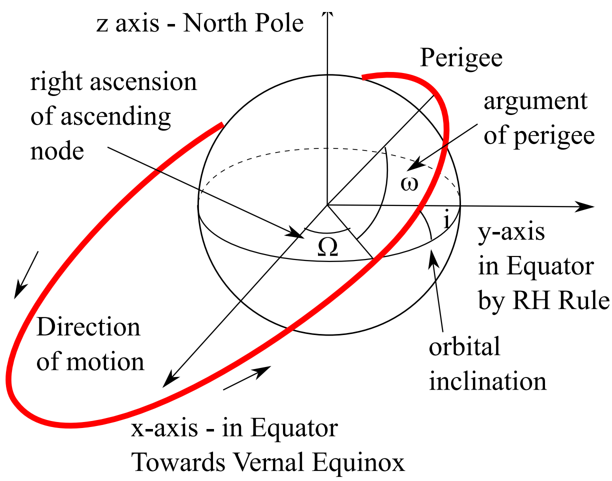
Because the problem is three dimensional, every vector has three components. Because Newton's law is a second order differential equation, there are two initial conditions because two integrations are performed. So two constants are required in each coordinate

Algorithm 44.3 Orbital algorithm, pass 2

input: Orbital element vector $\mathbf{v} = (a, e, i, \Omega, \omega, M)$

```
for Each time point t do
    1) (p, q) ← OrbitPosition(v, t)
    2) Figure out how much the orbit plane has rotated.
    3) Convert the 2D position in the orbit to a 3D Earth-centered xyz.
    4) Convert the Earth-centered xyz to latitude/longitude, accounting for Earth
       rotation.
end for
return
```

Figure 44.2: Definition of the Keplerian elements used to orient an orbital plane in space.



for a total of six constants. In Cartesian coordinates we could start with the coordinates (x, y, z) and the velocity (x', y', z') for our six numbers.

Instead of using position and velocity for his six coordinates, Kepler defined a set of six parameters based on the geometry of the problems. Three of these parameters, a, e, and t_p, describe the position of the satellite within the ellipse.

The other three parameters tell the orientation of the ellipse in space (see figure 44.2). In this coordinate system, the origin is at the center of the Earth, but:

1. The **x** axis points to the intersection of the Earth's equatorial plane and its orbital plane, called the first point in Aries or the **vernal equinox**.
2. The **z** axis points through the north pole.
3. The **y** axis is 90 degrees lies in the plane of the Earth's equator in such a way that the three axes make a right-handed coordinate system.

In this frame, we now define three additional parameters.

Table 44.1. Typical Orbital Elements for the International Space Station

Cartesian Elements (J2K)

Element	Value
x, meters	-5107606.49
y, meters	-1741563.23
z, meters	-4118273.08
x', m/sec	4677.962842
y', m/sec	4677.962842
z', m/sec	-3800.652800

Kepler Elements (M50)

Element	Value
a, meters	6780663.07
e, dimensionless	.0011495
i, degrees	51.52894
ω, degrees	38.42846
Ω, degrees	341.20455
M, degrees	191.97036

Epoch: 2015:044:12:00:00.000 UTC. Cartesian elements are in an inertial mean of year 2000 frame of reference, and the Kepler elements are the mean (averaged) elements in an inertial mean of year 1950 frame of reference.

Source: http://spaceflight.nasa.gov/realdata/elements/

1. i, the **orbital inclination**, or angle between the plane of the orbit an the earth's equitorial plane;
2. Ω, the **right ascension of ascending node**, the angle measured along the equator from the x-axis to the point where the orbital plane intersects the equatorial plane, going from south to north; and
3. ω, the **argument of perigee**, the angle measured in the plane of the orbit from the equatorial plane to the line from the center of the earth to perigee.

The vector $\mathbf{v}=(a, e, t_P, i, \Omega, \omega)$ gives us the initial conditions of the orbit. We can replace t_P by M and (mathematically) get the equivalent information from $\mathbf{v}_1 = (a, e, i, M, \Omega, \omega)$. The parameter t_p is physically measurable, while M must be calculated using Kepler's equation, so the first form is typically considered more reliable. NASA provides the elements in both forms and the conversion can be made using equations 44.68 and 44.69.

We already know how to calculate the position within the plane of the orbit. We need to know how to calculate the transformation matrix between the coordinates in the $(\mathbf{p}, \mathbf{q})$ plane and the **xyz** frame we have just defined. We will do this by first extending the $(\mathbf{p}, \mathbf{q})$ plane into a 3D coordinate frame.

1. **p** axis points from the center of the Earth towards perigee.
2. **q** axis points from the center of the Earth, in the plane of orbit, to a point on the path of the orbit that is 90 degrees advanced from the satellite's motion.
3. **w** axis points from the center of the Earth and is orthogonal to the orbit plane so as to make **pqw** a right-handed coordinate frame.

We will designate unit vectors along each of the axes in the **pqw** frame as **p**, **q**, and **w**, and unit vectors in the **xyz** frame by **i**, **j**, **k**. The following sequence of rotations will transform the **xyz** frame to the **pqw** frame:

1. Rotate by Ω (right ascension of ascending node) about the z axis. Call the new x axis x'.

2. Rotate by i (inclination) about the x'-axis. Call the new z axis z''.
3. Rotate by ω (argument of perigee) about the z'' axis.

This will transform the axes as follows:

$$\mathbf{i} \to \mathbf{p}, \quad \mathbf{j} \to \mathbf{q}, \quad \mathbf{k} \to \mathbf{w} \tag{44.72}$$

Let $\mathbf{R_v}(\theta)$ be the standard rotation matrix about the axis $\mathbf{v}$. These can be found in any standard textbook on linear algebra. Then

$$\begin{bmatrix} \mathbf{p} \\ \mathbf{q} \\ \mathbf{w} \end{bmatrix} = \mathbf{R_z}(\omega)\mathbf{R_x}(i)\mathbf{R_z}(\Omega) \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} \tag{44.73}$$

$$= \begin{bmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix} \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} \tag{44.74}$$

$$= \begin{bmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\cos i\sin\Omega & \cos i\cos\Omega & \sin i \\ \sin i\sin\Omega & -\sin i\cos\Omega & \cos i \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} \tag{44.75}$$

$$= \begin{bmatrix} \cos\omega\cos\Omega - \sin\omega\cos i\sin\Omega & \cos\omega\sin\Omega + \sin\omega\cos i\cos\Omega & \sin\omega\sin i \\ -\sin\omega\cos\Omega - \cos\omega\cos i\sin\Omega & -\sin\omega\sin\Omega + \cos\omega\cos i\cos\Omega & \sin i\cos\omega \\ \sin i\sin\Omega & -\sin i\cos\Omega & \cos i \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} \tag{44.76}$$

Once we have the satellite's position (p, q) from **OrbitPosition**($\mathbf{v}$,t), then the orbital position in Earth centered coordinates is

$$\begin{aligned} \mathbf{r} &= p\mathbf{p} + q\mathbf{q} && (44.77) \\ &= p(p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k}) + q(q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}) && (44.78) \\ &= (pp_x + qq_x)\mathbf{i} + (pp_y + qq_y)\mathbf{j} + (pp_z + qq_z)\mathbf{k} && (44.79) \end{aligned}$$

This is summarized in algorithm 44.4

Algorithm 44.4 Algorithm **PQ2XYZ** to convert from orbital position to earth centered coordinates.

input: (p, q), coordinates in (p, q) (from **OrbitPosition**)

$p_x \leftarrow \cos\omega\cos\Omega - \sin\omega\cos i\sin\Omega$
$p_y \leftarrow \cos\omega\sin\Omega + \sin\omega\cos i\cos\Omega$
$p_z \leftarrow \sin\omega\sin i$
$q_x \leftarrow -\sin\omega\cos\Omega - \cos\omega\cos i\sin\Omega$
$q_y \leftarrow -\sin\omega\sin\Omega + \cos\omega\cos i\cos\Omega$
$q_z \leftarrow \sin i\cos\omega$
$w_x \leftarrow \sin i\sin\Omega$
$w_y \leftarrow -\sin i\cos\Omega$
$w_z \leftarrow \cos i$
$\mathbf{r} \leftarrow (pp_x + qq_x)\mathbf{i} + (pp_y + qq_y)\mathbf{j} + (pp_z + qq_z)\mathbf{k}$
return $\mathbf{r}$

We are ready for a third pass at the orbit algorithm now.

Algorithm 44.5 Orbital algorithm, pass 3

input: Orbital element vector $\mathbf{v} = (a, e, i, \Omega, \omega, M)$

for At each time t **do**
  1) $(p, q) \leftarrow$ **OrbitPosition**$(\mathbf{v}, t)$
  2) Figure out how much the orbit plane has rotated.
  3) $(x, y, z) \leftarrow$ **PQ2XYZ**(p, q)
  4) Convert the Earth-centered xyz to latitude/longitude, accounting for Earth rotation.
end for
return

Perturbed Keplerian Orbits

In low earth orbits the main perturbing force that causes deviations from Keplerian motion is due to the Earth's oblateness. To be really accurate – and for some purposes this is required – there are hundreds of additional terms that need to be added to Newton's law of gravity. [3] It may also be necessary to include lunar and solar gravitational effects. The general idea is to expand the gravity field as a sum of spherical harmonics:

$$V = \frac{\mu}{r}\left[1 + \sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n} P_{nm}(\sin\phi)(c_{nm}\cos m\lambda + s_{nm}\sin m\lambda)\right] \tag{44.80}$$

The gravitational force then becomes

$$\mathbf{F} = -\nabla V = -\mathbf{r}\frac{\partial V}{\partial r} + \frac{\boldsymbol{\phi}}{r}\frac{\partial V}{\partial \phi} + \frac{\boldsymbol{\theta}}{r\sin\phi}\frac{\partial V}{\partial \theta} \tag{44.81}$$

where P_{nm} are the associated Legendre Polynomials, and ϕ and λ are the geocentric latitude and longitude. In practice, of course, the series cannot be taken to infinity as not all terms have been measured. In the EGM2008 Gravity model the coefficients are known to spherical harmonic degree 2159.[4] In the Newtonian approximation only the first terms remains in the gravitational field,

$$V = \frac{\mu}{r} \text{ and } \mathbf{F} = -\mathbf{r}\frac{\partial V}{\partial r} = -\frac{\mu\mathbf{r}}{r^3} \tag{44.82}$$

A simpler expansion that assumes the potential is an oblate spheroid and ignores other variations is given by[5]

$$V = \frac{\mu}{r}\left[1 - \sum_{n=2}^{\infty} J_n P_n(\sin\phi)\right] \tag{44.83}$$

[3] If you are interested I've written a paper with one way of describing these deviations. See Shapiro and Bhat, GTARG, the TOPEX/Poseidon Ground Track Maintenance Maneuver Targeting Program, AIAA Aerospace Design Conference, Irvine, Feb 16-19, 1993, AIAA Paper 93-1129.

[4] see http://earth-info.nga.mil/GandG/wgs84/gravitymod/egm2008/index.html

[5] Y Kozai, Second Order Solution of Artificial Satellite Theory Without Air Drag, Astronomical Journal, 67:446 (1962).

The negative sign in the sum is a convention, and like the earlier approximation, the sum is rarely taken to very high degree. The principal perturbation is due to the J_2 effect, and we will ignore all higher order perturbations, as they are much smaller. The effect on the orbital elements is a constant rate of change, called a **secular** perturbation. The result (which we will not derive here) is

$$\frac{da}{dt} = \frac{de}{dt} = \frac{di}{dt} = 0 \tag{44.84}$$

$$\frac{d\Omega}{dt} = -\left(\frac{r_e}{a}\right)^2 \frac{3J_2 n \cos i}{2(1-e^2)^2} \tag{44.85}$$

$$\frac{d\omega}{dt} = -\left(\frac{r_e}{a}\right)^2 \frac{3J_2 n(5\cos^2 i - 1)}{4(1-e^2)^2} \tag{44.86}$$

$$\frac{dM}{dt} = n\left(1 - \left(\frac{r_e}{a}\right)^2 \frac{3J_2(1 - 3\sin^2 i \sin^2 \omega)}{2(1-e)^3}\right) \tag{44.87}$$

A good value for $J_2 \approx 0.000108263$. Here $r_e \approx 6378.140$ km is the approximate equatorial radius of the earth.

The next most important perturbation on low earth orbits is drag, which mainly affects the semi-major axis according to

$$\frac{da}{dt} = -\frac{\rho A C_D \mu a}{m}\left[1 - \frac{\omega_E}{n}\cos i\right]^2 \tag{44.88}$$

and has very little affect on the other elements. Here ρ is the atmospheric density at the altitude of the satellite; $C_D \approx 2.2$ the drag coefficient; A the satellite cross-sectional area normal to its direction of motion; and $\omega_E \approx 2\pi/86400$ radians/second the earth rotation rate.

Recalling Euler's method for solving an initial value problem, the numerical solution of

$$\frac{dy}{dt} = f(t, y) \quad y(t_i) = y_i \tag{44.89}$$

is given approximately by

$$y(t_{i+1}) = y(t_i) + \left.\frac{dy}{dt}\right|_{t=t_i} (t_{i+1} - t_i) = y(t_i) + \left.\frac{dy}{dt}\right|_{t=t_i} \Delta t \tag{44.90}$$

Based on this, a single-step in Euler's method for propagating the Kepler elements is given by **PropStep**($\mathbf{v}$, Δt) in algorithm 44.6.

This gives us the next iteration of our basic algorithm (pass 4).

Satellite Ground Track

To convert a satellite position to local sub-satellite ground track in latitude and longitude you must take into account the Earth's rotation. That is because the x axis, in which

Algorithm 44.6 Euler's method algorithm for **PropStep**.

input: Orbital element vector $\mathbf{v} = (a, e, i, \Omega, \omega, M)$; time step Δt.

1: $n \leftarrow \sqrt{\mu/a^3}$

2: $\Delta a \leftarrow -\dfrac{\rho A C_D \mu a}{m}\left[1 - \dfrac{\omega_E}{n}\cos i\right]^2 \Delta t$

3: $\Delta\Omega \leftarrow -\left(\dfrac{r_e}{a}\right)^2 \dfrac{3J_2 n \cos i}{2(1-e^2)^2}\Delta t$

4: $\Delta\omega \leftarrow -\left(\dfrac{r_e}{a}\right)^2 \dfrac{3J_2 n(5\cos^2 i - 1)}{4(1-e^2)^2}\Delta t$

5: $\Delta M \leftarrow n\left(1 - \left(\dfrac{r_e}{a}\right)^2 \dfrac{3\mu J_2(1 - 3\sin^2 i \sin^2\omega)}{2(1-e)^3}\right)\Delta t$

6: $\Delta\mathbf{v} \leftarrow (\Delta a, 0, 0, \Delta\Omega, \Delta\omega, \Delta M)$

7: $\mathbf{v} \leftarrow \mathbf{v} + \Delta\mathbf{v}$

8: **return** $\mathbf{v}$ as **PropStep**($\mathbf{v}$, Δt)

Algorithm 44.7 Orbital algorithm, pass 4

input: Orbital element vector $\mathbf{v} = (a, e, i, \Omega, \omega, M)$

1: **for** At each time t **do**

2: 1) $(p, q) \leftarrow$**OrbitPosition**($\mathbf{v}, t$)

3: 2) $\mathbf{v} \leftarrow$ **PropStep**($\mathbf{v}, \Delta t$)

4: 3) $(x, y, z) \leftarrow$ **PQ2XYZ**(p, q)

5: 4) Convert the Earth-centered xyz to latitude/longitude, accounting for Earth rotation.

6: **end for**

7: **return**

the satellite coordinates are calculated, is fixed,[6] whereas the x axis for the longitude has a 24-hour period.

This relationship is given approximately by the formula[7]

$$\begin{aligned}\theta = {} & 100.460618375 + 36000.770053608336t + 0.0003879333t^2 \\ & + 15h + \frac{m}{4} + \frac{s}{240} \ \mathbf{mod}\ 360.0\end{aligned} \tag{44.91}$$

where t is the time since Jan. 1, 2000 in centuries, and h:m:s is the current time in hours, minutes, and seconds. The result of this formula is an angle in degrees. Then the longitude is

$$\text{longitude} = \tan^{-1}\frac{y}{x} - \theta \tag{44.92}$$

in degrees, where x and y are the satellite coordinates.

Now we have a bit of a complication, because we have to include an actual time, rather than just a relative time from the start of the calculation.

Let us create (exercise 44.9) a function **xyz2latlog** which takes as input a vector (x, y, z)

[6]Well, not really, it has a 26,000 year period, but is essentially fixed over the duration of your calculation.
[7]See P. K. Seidelmann, Explanatory Supplement to the Astronomical Almanac, US Naval Observatory, 1961.

at some absolute time (use whatever coordinates you want, such as Nov 17, 2010 at 9:43 AM), and converts it to the correct latitude and longitude. Times are typically much easier to work with if we convert them into a continuous real valued number of days from some arbitrary origin, such as Jan 1. 2000 at 12:00 AM. One way to do this is with the Python `DateTime` package.

```
>>> import datetime
>>> t=datetime(2015,1,7,10,07,14)
>>> tz=datetime.datetime(2000,1,1,0,0,0)
>>> t-tz
datetime.timedelta(5485, 36434)
>>> (t-tz).days
5485
>>> (t-tz).seconds
36434
```

Including **xyz2latlong**, our fifth iteration is given by algorithm 44.8.

Putting it all together, we can now write an Euler's method algorithm (algorithm 44.9) to predict the satellite orbit and ground track with the function **prop**.

Algorithm 44.8 Orbital algorithm, pass 5

input: Orbital element vector $\mathbf{v} = (a, e, i, \Omega, \omega, M)$
for At each time t **do**
  1) $(p, q) \leftarrow$ **OrbitPosition**$(\mathbf{v}, t)$
  2) $\mathbf{v} \leftarrow$ **PropStep**$(\mathbf{v}, \Delta t)$
  3) $(x, y, z) \leftarrow$ **PQ2XYZ**(p, q)
  4) $(\lambda, \theta) \leftarrow$ **xyz2latlong**$(x, y, z; t)$
end for
return

Algorithm 44.9 Orbit Propagation

input: Orbital element vector $\mathbf{v} = (a, e, i, \Omega, \omega, M)$; t_{start}, t_{end}, Δt.
$t \leftarrow t_{\text{start}}$
while $t < t\_\text{end}$ **do**
  $(p, q) \leftarrow$ **OrbitPosition**$(\mathbf{v}, \Delta t)$
  $\mathbf{v} \leftarrow$ **PropStep**$(\mathbf{v}, \Delta t)$
  $(x, y, z) \leftarrow$ **PQ2XYZ**$((p, q), \mathbf{v})$
  $(\lambda, \theta) \leftarrow$ **xyz2latlong**$(x, y, z; t)$
  $t \leftarrow t + \Delta t$
  Print coordinates or plot on a map.
end while

After you produce a list of latitude and longitude coordinates describing the orbit, you can plot them on a map of the world. You just have to choose a map projection and add the list of map coordinates to the map as a line plot. You can do this with the Basemap package as described in chapter 45.

Exercises

1. The mean anomaly M and eccentric anomaly E are two different angles used to measure the position of an object within the plane of its orbit. They are related by Kepler's equation $M = E - e \sin E$, where e is the orbital eccentricity. Suppose that you are given the value of e, where $0 \leq e < 1$, and that you know the value of M. Write a program to solve for E using fixed point iteration.
2. Write a program that takes as input a satellite's Keplerian elements at a given time and converts them to Earth centered elements (xyz) at the same time.
3. Write a program that takes as input a satellite's Keplerian elements at a given time, and calculates the sub-satellite latitude and longitude at the same time.
4. Write a program that takes as input a set of Keplerian orbit elements at some time t, and predicts the orbital elements at some later time t', assuming that there are no perturbations on the orbit.
5. Modify the program in the previous exercise to also calculate the latitude and longitude at a fixed interval between t and t' and print out a table of values.
6. Modify the program in exercise 4 to include J_2 perturbations.
7. Modify the program in exercise 5 to include J_2 perturbations.
8. Look up the orbital elements of the international space station. Write a computer program to figure out when it will pass nearly overhead at night during the next 3 weeks. Check your predictions online at http://spotthestation.nasa.gov/.
9. Write the algorithm for **xyz2latlong**. Hint: see the discussion starting around eq. 44.91.