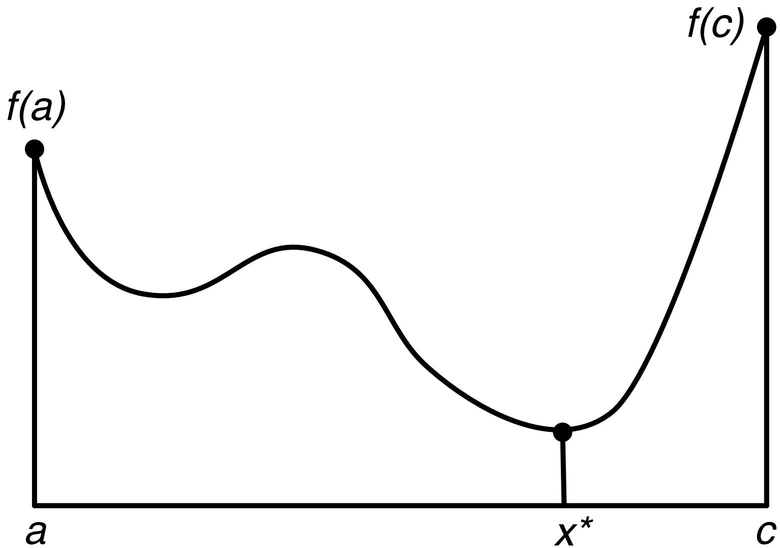
Finding the minimum of $f(x)$ via the Golden Section Search

CS 330

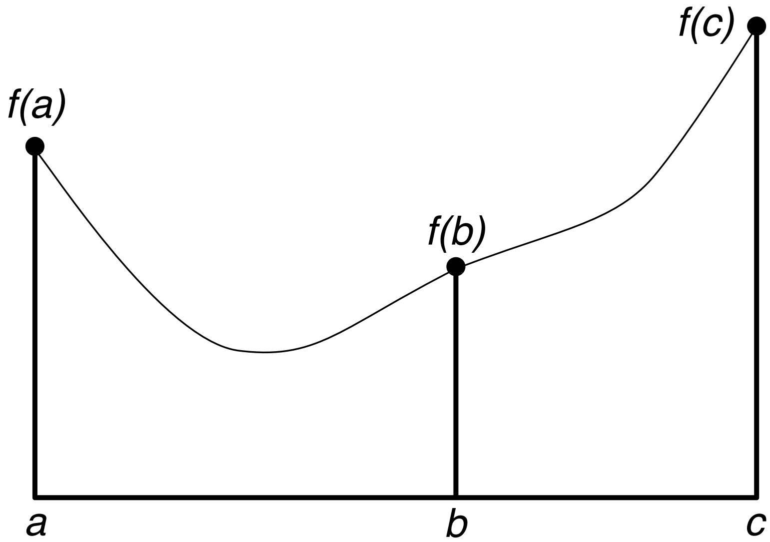
October 30, 2012

The mimimum of a one-variable function $f(x)$.



$$x^* = \underset{a \le x \le c}{\text{argmin}}\, f(x)$$

Bracketed Minimum

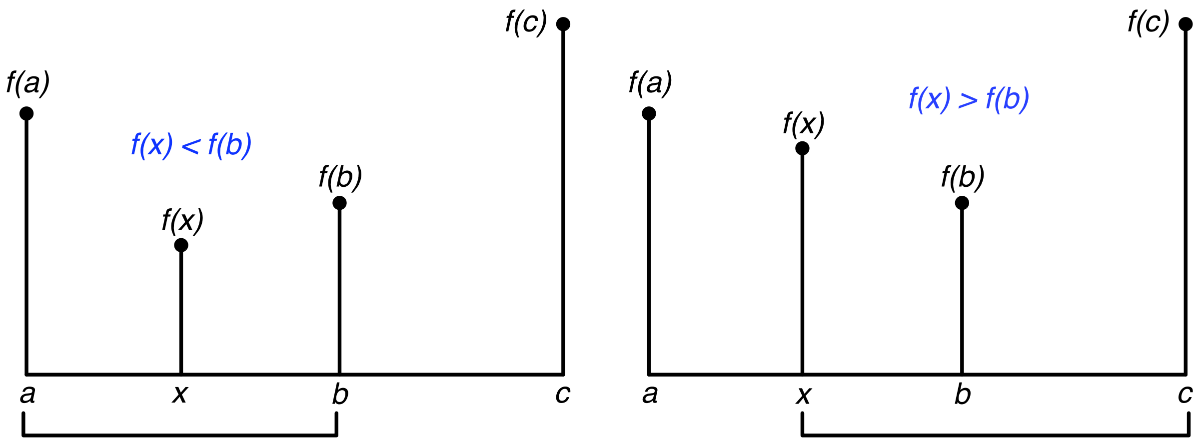


(a, b, c) *brackets* a minimum if $a < b < c$,

$f(b) < f(a)$, and $f(b) < f(c)$.

b is best approximation to minimum value.

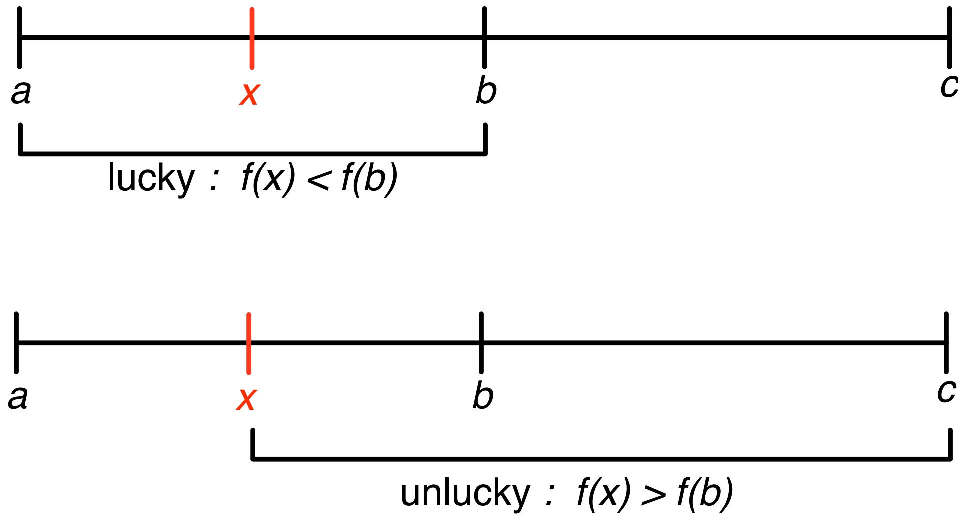
Choosing the next smaller bracket



Choose new point $x \in [a, b]$ (or $x \in [b, c]$).
If $f(x) \leq f(b)$ then new bracket is (a, x, b),
else new bracket is (x, b, c).

How to choose x?

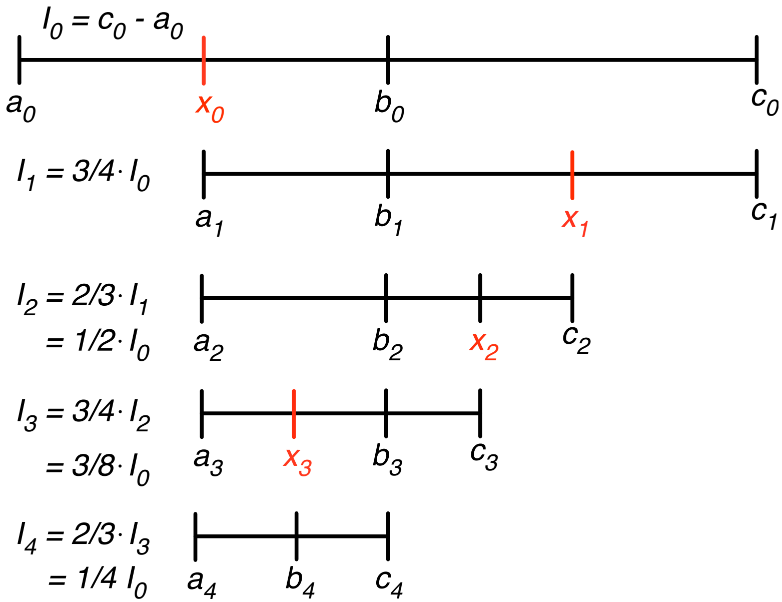
First attempt : split largest interval in half



Best case: split interval in half.

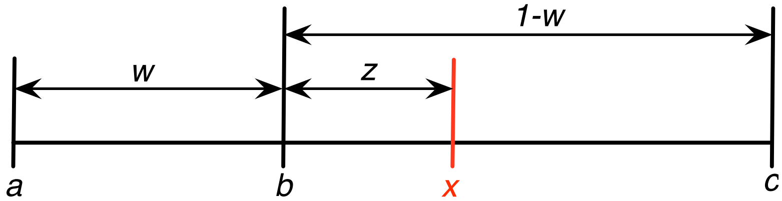
Worst case: interval reduced by 3/4.

Sequence of splitting largest interval in half assuming worst case at each step



Interval reduced by 3/4 on odd steps, and by 2/3 on even steps (halved every 2 steps).

Minimizing the worst case



Fractional interval sizes:

$$w = \frac{b-a}{c-a}, \quad 1-w = \frac{c-b}{c-a}, \quad z = \frac{x-b}{c-a}$$

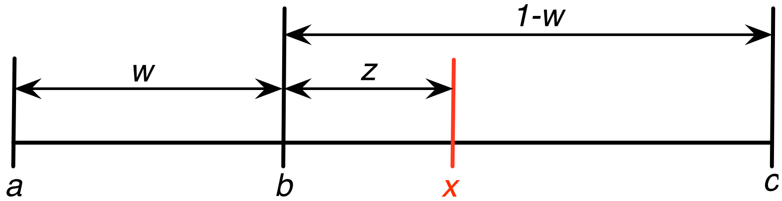
Rel. size s of next bracket:

$$s = \begin{cases} w + z = \frac{x-a}{c-a} & \text{if } (a,b,x) \text{ next bracket} \\ 1-w & \text{if } (b,x,c) \text{ next bracket} \end{cases}$$

Minimize the <u>worst case</u> by making both equal:

$$w + z \; = \; 1 - w$$

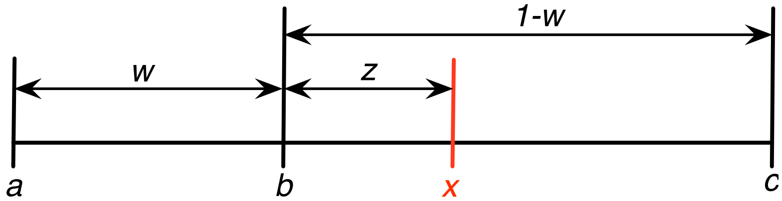
Minimizing the worst case, cont. . .



We choose z optimally and assume w was chosen optimally in the previous iteration.
z is same fraction of $[b, c]$ as w is of $[a, c]$:

$$\begin{aligned}
\frac{z}{1-w} &= w \\
w + z &= 1 - w \quad \text{(prev. slide)} \\
\frac{1-2w}{1-w} &= w \\
w^2 - 3w + 1 &= 0
\end{aligned}$$

Optimizing the worst case, cont...



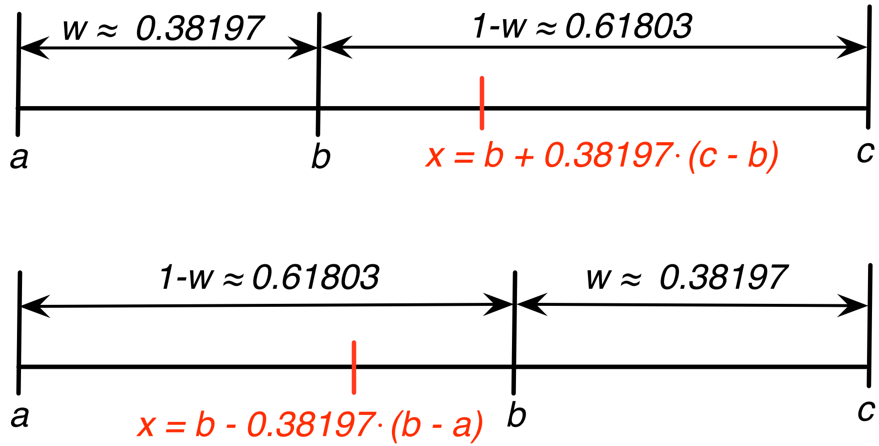
$$w^2 - 3w + 1 = 0$$

$$w = \frac{3-\sqrt{5}}{2} \approx 0.38197$$

$$1 - w = \frac{-1+\sqrt{5}}{2} = 1/\rho \approx 0.61803$$

where $\rho = (1+\sqrt{5})/2$ is the *golden ratio*.

Golden Section Search



We pick x to be 0.38197 into the larger of the two intervals. If initial intervals not "golden," later intervals eventually will be.

Termination Tolerance

Taylor series for $f(x)$ at minimal value x^* :

$$f(x) = f(x^*) + f'(x^*) \cdot (x - x^*) + \frac{f''(\xi)}{2} \cdot (x - x^*)^2$$

Since $f'(x^*) = 0$ at the minimum we have

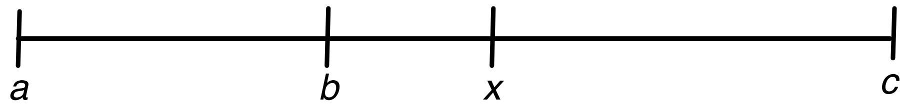
$$f(x) \approx C_1 + C_2 \cdot (x - x^*)^2.$$

If $|x - x^*| \leq \epsilon$, then

$$f(x) \approx C_1 + C_2 \cdot \epsilon^2,$$

but a change in $f \leq \epsilon^2$ is not detectable. Thus, to save us from *useless bisections*, we use $|x - x^*| \leq \sqrt{\epsilon}$ as a termination condition.

Terminating based on relative error



- Termination condition based on *relative error* below $\sqrt{\epsilon}$:

$$|c - a| \leq \sqrt{\epsilon} \cdot (|b| + |x|)$$

- single precision: $\sqrt{\epsilon} \approx 10^{-4}$
- double precision: $\sqrt{\epsilon} \approx 10^{-8}$

Compare with bisection method for finding roots

- Need initial bracket (may be hard to find).
- No derivative information necessary
  - Use faster methods in this case.
  - (Find roots of $f'(x) = 0$ if f' known).
- Robust "divide and conquer" strategy.
  - Only requires function to be defined everywhere on the interval.
  - What if function not continuous? not bounded?