Time-reversal refocusing
in homogeneous and randomly heterogeneous media

The goal is to show that time-reversal refocusing can be enhanced when the waves propagate in a randomly heterogeneous medium rather than in a homogeneous medium. We will use the paraxial wave equation rather than the full wave equation, which is simpler to simulate.

1) Paraxial approximation in a homogeneous medium.

In the paraxial approximation of the Helmholtz equation in a homogeneous medium with background velocity c_o, the time-harmonic wave field $\hat{u}$ (with frequency ω) is expressed as $\hat{u}(\boldsymbol{x}) = \phi(\boldsymbol{x})e^{ikz}$ where $k = \omega/c_o$ is the wavenumber, $\boldsymbol{x} = (x, z) \in \mathbb{R}^2$, ϕ represents the complex-valued amplitude which modulates the plane wave represented by the exponential factor. Then under a suitable assumption, ϕ approximately solves the Schrödinger equation $2ik\partial_z\phi + \partial_x^2\phi = 0$.

We consider a time-harmonic initial condition (with frequency ω) in the plane $z = 0$ whose transverse profile is a Gaussian with radius r_0:

$$\phi_0(x) = \exp\Big(-\frac{x^2}{r_0^2}\Big).$$

Solve the Schrödinger equation from the plane $z = 0$ to the plane $z = L$

$$\partial_z\phi = \frac{i}{2k}\partial_x^2\phi, \qquad \phi(z = 0, x) = \phi_0(x),$$

using the Fourier method (use `fft`):

$$\begin{array}{ccc} \phi(z=0,x) & \xrightarrow{\text{DFT}} & \hat{\phi}(z=0,\kappa) \\ {\scriptstyle \text{Schr.}}\downarrow & & \downarrow {\scriptstyle \text{Explicit ODE}} \\ \phi(z,x) & \xleftarrow{\text{IFT}} & \hat{\phi}(z,\kappa) \end{array}$$

Compute numerically the transmitted wave profile $\phi_t(x) = \phi(L, x)$ for an initial Gaussian beam with radius $r_0 = 2$, a grid of 2^{10} points over the interval $[-x_{\max}/2, x_{\max}/2]$, with $x_{\max} = 60$, $k = \omega = 1$ (we assume $c_o = 1$), and $L = 10$.

Check theoretically that the transmitted wave profile $\phi_t(x)$ is:

$$\phi_t(x) = \frac{r_0}{r_t}\exp\Big(-\frac{x^2}{r_t^2}\Big), \qquad r_t = r_0\Big(1 + 2i\frac{L}{kr_0^2}\Big)^{1/2}.$$

Compare the square modulus of the numerical transmitted wave with the theoretical profile

$$|\phi_t(x)|^2 = \frac{r_0}{R_t}\exp\Big(-\frac{2x^2}{R_t^2}\Big), \qquad R_t = r_0\Big(1 + \frac{4L^2}{k^2r_0^4}\Big)^{1/2}.$$

2) Time reversal for time-harmonic waves in a homogeneous medium.

Consider the (compactly supported) time-reversal mirror in the plane $z = L$:

$$\chi_M(x) = \Big(1 - \big(\frac{x}{2r_M}\big)^2\Big)^2 \mathbf{1}_{[-2r_M, 2r_M]}(x).$$

Carry out a time-reversal experiment: Transmit the time-reversed wave ϕ_t (i.e. its complex conjugate) from the time-reversal mirror at $z = L$ up to the plane $z = 2L$:

$$\partial_z \phi^{tr} = \frac{i}{2k}\partial_x^2 \phi^{tr}, \qquad \phi^{tr}(z = L, x) = \overline{\phi_t(x)}\chi_M(x).$$

Compute the refocused wave $\phi^{tr}(z, x)$ in the plane $z = 2L$ for different values for the radius r_M (from 2 to 20) of the mirror. Note how refocusing becomes poor when r_M becomes small.

Consider a "Gaussian time-reversal mirror" in the plane $z = L$, with a truncation function of the form

$$\chi_M(x) = \exp\Big(-\frac{x^2}{r_M^2}\Big).$$

Show analytically that the refocused wave profile $\phi_r^{tr}(x) = \phi^{tr}(z = 2L, x)$ is given by

$$\begin{aligned}
\phi_r^{tr}(x) =& \frac{1}{a_{tr}} \exp\Big(-\frac{x^2}{r_{tr}^2}\Big),\\
r_{tr}^2 =& \Big(\frac{1}{r_M^2} + \frac{1}{r_0^2 - 2i\frac{L}{k}}\Big)^{-1} + 2i\frac{L}{k}, \qquad a_{tr} = \Big(1 + \frac{4L^2}{k^2 r_0^2 r_M^2} + 2i\frac{L}{k r_M^2}\Big)^{1/2}.
\end{aligned}$$

Check also this result numerically.

3) Paraxial approximation in a random medium.

In the paraxial approximation of the Helmholtz equation in a heterogeneous medium with velocity $c_o/n(\boldsymbol{x})$, the time-harmonic wave field $\hat{u}$ (with frequency ω) is expressed as $\hat{u}(\boldsymbol{x}) = \phi(\boldsymbol{x})e^{ikz}$ where $k = \omega/c_o$ is the homogeneous wavenumber and ϕ represents the complex-valued amplitude which modulates the plane wave represented by the exponential factor. Then under a suitable assumption, ϕ approximately solves the Schrödinger equation $2ik\partial_z\phi + \partial_x^2\phi + k^2(n^2-1)\phi = 0$.

Implement a split-step Fourier method with longitudinal step h to solve the Schrödinger equation

$$\partial_z \phi = \frac{i}{2k}\partial_x^2\phi + \frac{ik}{2}\mu(z, x)\phi, \qquad \phi(z = 0, x) = \phi_0(x) = \exp\Big(-\frac{x^2}{r_0^2}\Big).$$

Consider a random potential μ of the form:

$$\mu(z, x) = \mu_n(x), \qquad \text{if } z \in [nz_c, (n+1)z_c),$$

where $\mu_0(x)$, $\mu_1(x)$, …, $\mu_{[L/z_c]}(x)$ are independent realizations of a Gaussian process with mean zero and covariance function $\mathbb{E}[\mu_n(x)\mu_n(x')] = \sigma^2 \exp(-(x - x')^2/x_c^2)$.

Take $h = 1$, $z_c = 1$, $x_c = 4$, $\sigma = 1$.

Check numerically that the transmitted wave profile $\phi_t(x) = \phi(L, x)$ has mean:

$$\mathbb{E}[\phi_t(x)] = \frac{r_0}{r_t} \exp\Big(-\frac{x^2}{r_t^2}\Big) \exp\Big(-\frac{\gamma_0 \omega^2 L}{8}\Big),$$

with $\gamma_0 = \sigma^2 z_c$. For the estimation of the mean take the average over 100 runs with 100 independent realizations of the random medium.

4) Time reversal for time-harmonic waves in a random medium.

Consider a Gaussian time-reversal mirror in the plane $z = L$: $\chi_M(x) = \exp(-x^2/r_M^2)$.

Perform a time-reversal experiment: Transmit the time-reversed wave ϕ_t in the same random medium (i.e., the wave travels through the same medium as in 3)):

$$\partial_z \phi^{tr} = \frac{i}{2k} \partial_x^2 \phi^{tr} + \frac{ik}{2} \mu(2L - z, x) \phi^{tr}, \qquad \phi^{tr}(z = L, x) = \overline{\phi_t(x)} \chi_M(x),$$

and compute the refocused wave $\phi_r^{tr}(x) = \phi^{tr}(z = 2L, x)$ in the plane $z = 2L$. Check numerically that the mean refocused wave profile is given by

$$\mathbb{E}[\phi_r^{tr}(x)] = \frac{1}{a_{tr}} \exp\Big(-\frac{x^2}{r_{tr}^2}\Big) \exp\Big(-\frac{x^2}{r_a^2}\Big),$$

with $r_a^{-2} = \gamma_2 \omega^2 L/48$ and $\gamma_2 = 2\sigma^2 z_c / x_c^2$.

Try different values for the radius r_M of the mirror. Note that refocusing becomes significantly better in the random medium case than in the homogeneous medium case when the mirror is relatively small (say $r_M = 2$).

Perform another time-reversal experiment: Transmit the time-reversed wave ϕ_t in a homogeneous medium:

$$\partial_z \phi^{tr} = \frac{i}{2k} \partial_x^2 \phi^{tr}, \qquad \phi^{tr}(z = L, x) = \overline{\phi_t(x)} \chi_M(x),$$

and compute the refocused wave $\phi_r^{tr}(x) = \phi^{tr}(z = 2L, x)$ in the plane $z = 2L$. Check numerically that the mean refocused wave profile is given by

$$\mathbb{E}[\phi_r^{tr}(x)] = \frac{1}{a_{tr}} \exp\Big(-\frac{x^2}{r_{tr}^2}\Big) \exp\Big(-\frac{\gamma_0 \omega^2 L}{8}\Big).$$

5) Time reversal for time-dependent waves in a random medium.

Here we consider a time-dependent initial condition, whose spectrum is flat over $[\omega_0 - B, \omega_0 + B]$, with $\omega_0 = 1$ and $B = 0.75$, and whose transverse profile is a Gaussian with radius r_0.

Perform a time-reversal experiment for this wave: simply sum the frequency components computed in the previous section for a set of regularly sampled frequencies (say, 20 frequencies over $[\omega_0 - B, \omega_0 + B]$).

Observe the refocused wave profile, compare with

$$\phi_{r,theo}^{tr}(x) = \frac{1}{a_{tr}} \exp\Big(-\frac{x^2}{r_{tr}^2}\Big) \exp\Big(-\frac{x^2}{r_a^2}\Big),$$

and observe the statistical stability of the refocused wave (i.e. repeat the experiment with different realizations, and show that the refocused profile is almost independent of the realization).

References:

[1] P. Blomgren, G. Papanicolaou, and H. Zhao, Super-Resolution in Time-Reversal Acoustics, Journal of the Acoustical Society of America, Vol 111, (2002), pp. 230-248.