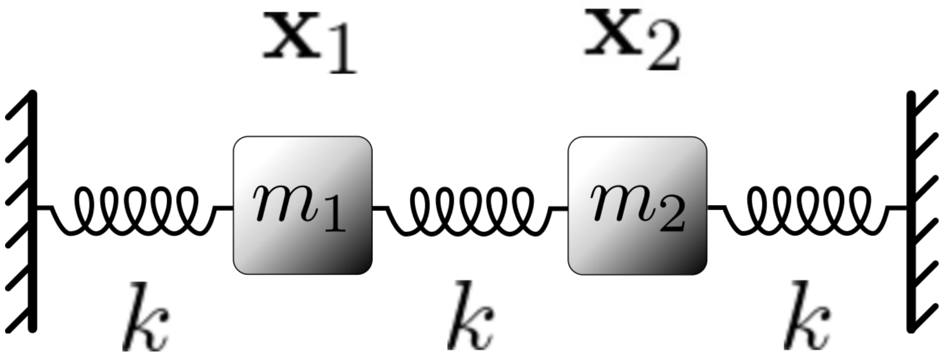
Coupled Simple Harmonics Oscillators (SHOs) & Strings

Outline

- 2 Coupled SHOs
- N Coupled SHOs
- Infinite Limit
- Strings (Continuous Limit)

2 Coupled SHOs Quick Review!



Equation of Motion

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$$

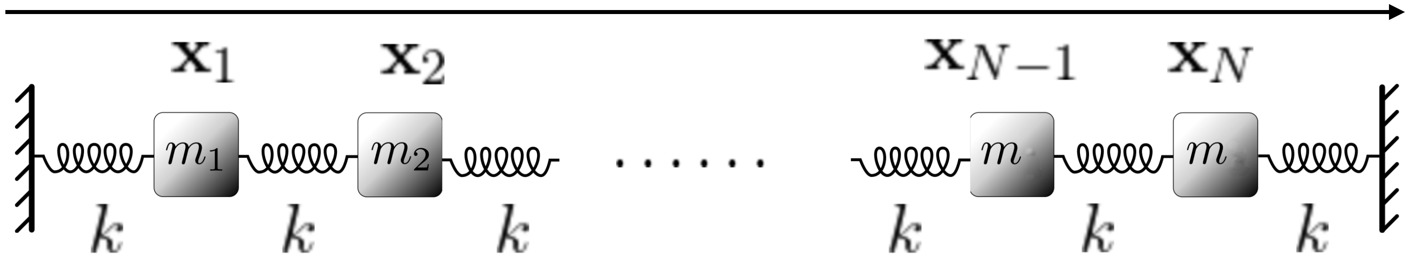
$$\ddot{\mathbf{x}} = -\mathbf{M}^{-1}\mathbf{K}\mathbf{x}$$

Position Vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Mass Matrix $\mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$

Spring Constant Matrix $\mathbf{K} = k\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

N Coupled SHOs



Position Vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{pmatrix}$

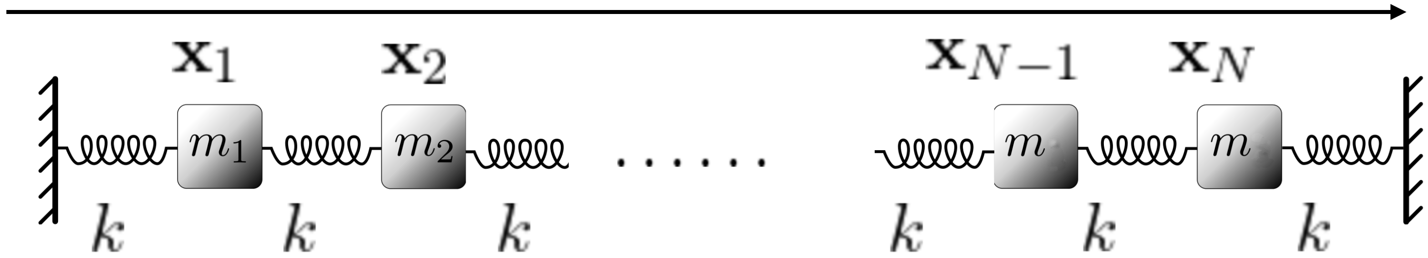
Spring Constant Matrix $\mathbf{K} = ???$

Equation of Motion

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$$

Mass Matrix $\mathbf{M} = \begin{pmatrix} m_1 & 0 & \dots & 0 & 0 \\ 0 & m_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & m_{N-1} & 0 \\ 0 & 0 & \dots & 0 & m_N \end{pmatrix}$

Spring Constant Matrix Quiz



$$\text{Spring Constant Matrix } \mathbf{K} * k^{-1} =$$

A) $\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$

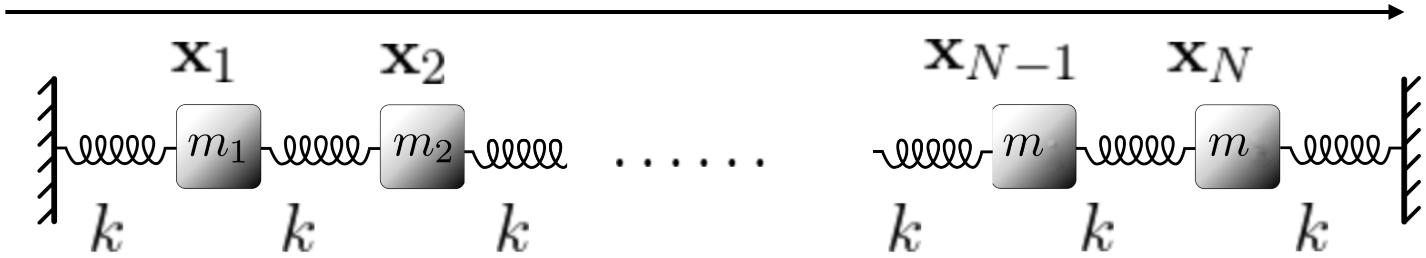
B) $\begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 2 & -1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$

C) $\begin{pmatrix} N & -(N-1) & 0 & \dots & 0 & 0 & 0 \\ -(N-1) & N & -(N-1) & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -(N-1) & N & -(N-1) \\ 0 & 0 & 0 & \dots & 0 & -(N-1) & N \end{pmatrix}$

D) $\begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$

Infinite Limit

$$\lim_{N\to\infty}$$



Position Vector $\mathbf{x} = \begin{pmatrix} \vdots \\ x_{j-1} \\ x_j \\ x_{j+1} \\ \vdots \end{pmatrix}$

Mass Matrix $\mathbf{M} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \dots & m_{j-1} & 0 & 0 & \dots \\ \dots & 0 & m_j & 0 & \dots \\ \dots & 0 & 0 & m_{j+1} & \dots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$

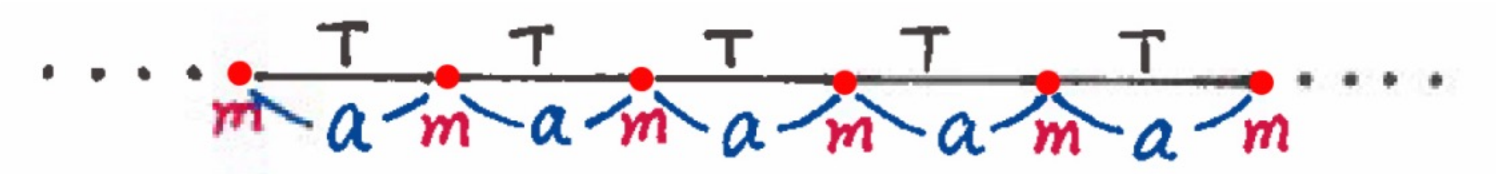
Equation of Motion

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$$

$$\ddot{\mathbf{x}} = -\mathbf{M}^{-1}\mathbf{K}\mathbf{x}$$

Spring Constant Matrix $\mathbf{K} = k \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \dots & -2 & 1 & 0 & \dots \\ \dots & 1 & -2 & 1 & \dots \\ \dots & 0 & 1 & -2 & \dots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$

Strings



Draw The Force Diagram

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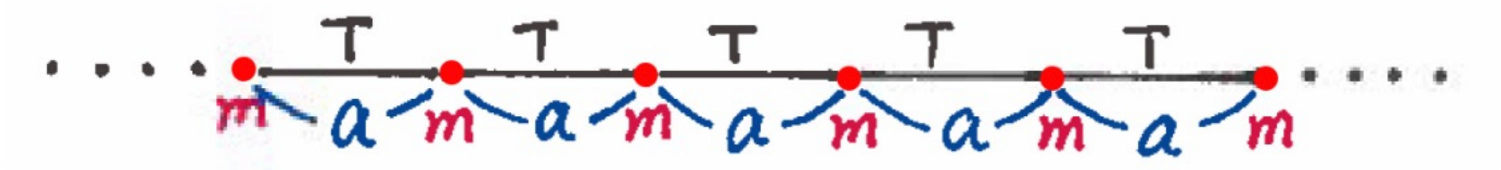
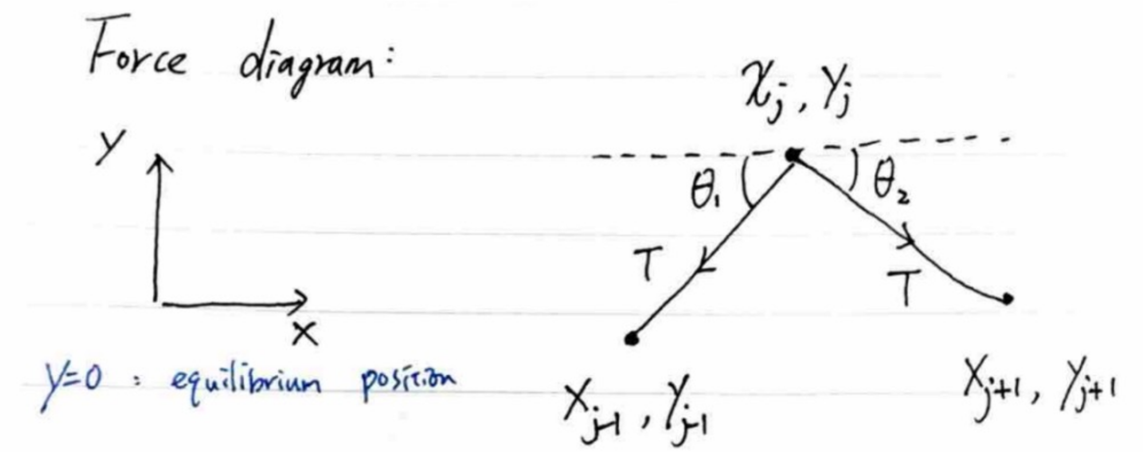
Write the Equations of Motions

Strings: Force Diagram





Equation of Motion

$$m\ddot{x}_j = -T + T = 0 \quad \text{(No motion in the horizontal direction)}$$

$$m\ddot{y}_j = -T(\sin\theta_1 + \sin\theta_2)$$

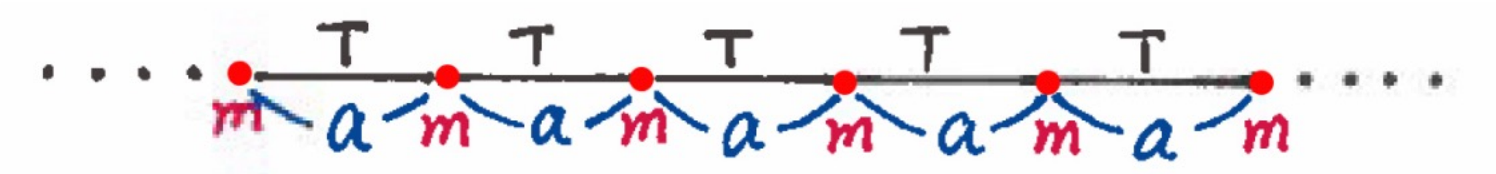
$$\approx -T\left(\frac{y_j - y_{j-1}}{a} + \frac{y_j - y_{j+1}}{a}\right)$$

$$m\ddot{y}_j = \frac{T}{a}(y_{j-1} - 2y_j + y_{j+1})$$

Solution

$$y_j = \mathrm{Re}[A_j e^{i(\omega t + \phi)}]$$

Strings

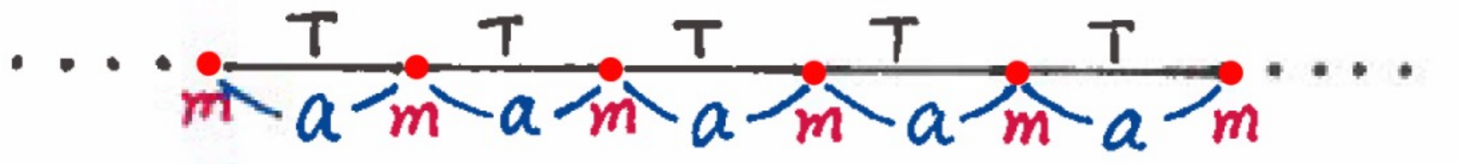


Write down the Eigenvalue Equation for ω.

Argue why $A_{j\pm1} = A_j e^{\pm ika}$

Hint: Translational Symmetry

Strings: Normal Frequencies



Equation of Motion

$$m\ddot{x}_j = -T + T = 0 \quad \text{(No motion in the horizontal direction)}$$

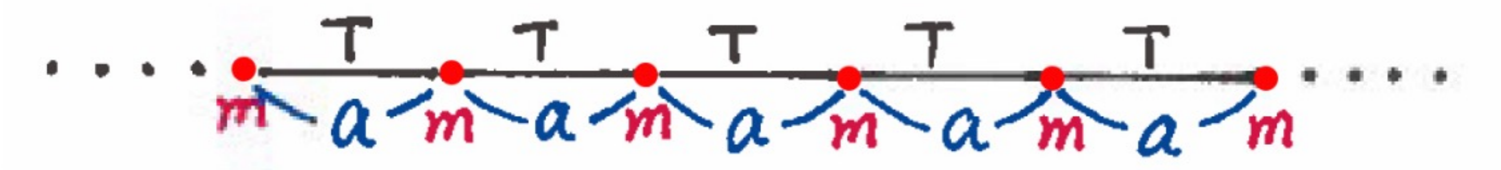
$$\begin{aligned} m\ddot{y}_j &= -T(\sin\theta_1 + \sin\theta_2) \\ &\approx -T\left(\frac{y_j - y_{j-1}}{a} + \frac{y_j - y_{j+1}}{a}\right) \\ m\ddot{y}_j &= \frac{T}{a}(y_{j-1} - 2y_j + y_{j+1}) \end{aligned}$$

Solution

$$y_j = \mathrm{Re}[A_j e^{i(\omega t + \phi)}]$$

$$\begin{aligned} \omega^2 A_j &= \frac{T}{ma}(-A_{j-1} + 2A_j - A_{j+1}) \\ \omega^2 A_j &= \frac{T}{ma}A_j(-e^{-ika} + 2 + e^{ika} \\ \omega^2 &= \frac{T}{ma}(2 - 2\cos ka) \\ &= 2\omega_0^2(1 - \cos ka) \\ \omega^2 &= 4\omega_0^2 \sin^2\left(\frac{ka}{2}\right) \end{aligned}$$

Strings: Continuous Limit



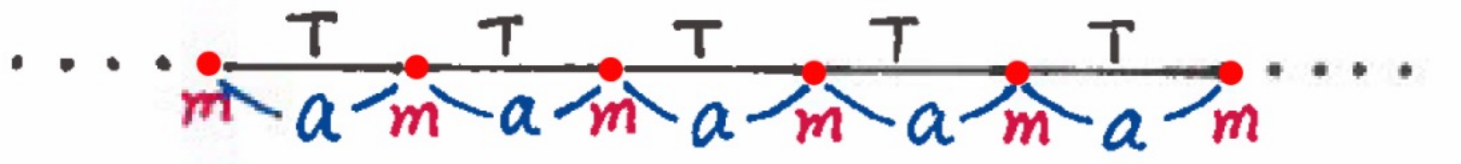
Equation of Motion

$$M^{-1}kA \;\Rightarrow\; \omega^2 A_j = \frac{T}{ma}(-A_{j-1} + 2A_j - A_{j+1})$$

In the continuous limit this equation transforms into:

$$M^{-1}kA \;\Rightarrow\; \omega^2 A(x) = \frac{T}{ma}(-A(x-a) + 2A(x) - A(x+a))$$

Strings: Continuous Limit



If we Taylor Expand:

$$A(x-a) = A(x) - aA'(x) + \frac{1}{2}a^2 A''(x) + \cdots$$
$$A(x+a) = A(x) + aA'(x) + \frac{1}{2}a^2 A''(x) + \cdots$$

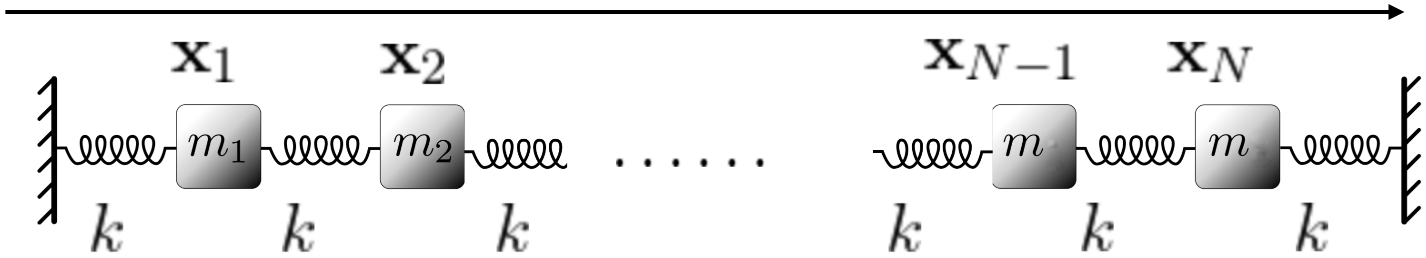
$$\Rightarrow \quad -A(x-a) + 2A(x) - A(x+a) = -\frac{\partial^2 A(x)}{\partial x^2} a^2 + \cdots$$
$$M^{-1}kA(x) = -\frac{T}{ma}\frac{\partial^2 A(x)}{\partial x^2} a^2 + \cdots$$

In the $a << $ wavelength we can ignore the a^3 and higher order terms. We define $\rho_L \equiv \frac{m}{a}$ and $M^{-1}k$ becomes an "operator":

$$M^{-1}k \to -\frac{T}{\rho_L}\frac{\partial^2}{\partial x^2}$$
$$\frac{\partial\psi(x,t)}{\partial t^2} = \frac{T}{\rho_L}\frac{\partial\psi(x,t)}{\partial x^2}$$

Symmetries

$$\lim_{N\to\infty}$$



Right-Shift Operator

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & 1 \end{pmatrix}$$

Symmetry Invariance:

$$\tilde{x}(t) = Sx(t)$$

$$\ddot{\tilde{x}}(t) = -M^{-1}k\tilde{x}(t)$$

$$\Rightarrow \quad S\ddot{x}(t) = -M^{-1}kSx(t)$$

From Equation of Motion:

$$S\ddot{x}(t) = -SM^{-1}kx(t)$$