The rebalancing process of red-black trees

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Agenda

- Definition
- Basic assumptions
- Insert a new node and then rebalance the tree
- Remove a node and then rebalance the tree

Note

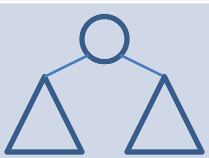
- Make sure that you have read the chapter 12 of the book 《Introduction to Algorithm》(third edition) before you continue reading this article;
- I do not talk about how to insert/remove a node into/from a binary tree, you may find details in the foregoing chapter（12.3 Insertion and deletion）;
- The purpose of the article is to give more logic to the rebalancing process to make it more comprehensible.

Definition

- Red-black tree
  1. Each node is either red or black;
  2. If a node is red, then both its children are black;
  3. Every path from a given node to any of its descendant NIL nodes goes through the same number of black nodes;
  4. The root is black;
  5. All leaves (NIL) are black.

Basic assumptions

- A binary tree is a recursive data structure, we can use a triangle to represent it and a small circle to represent a node:

Figure	Meaning
	A node in a binary tree.
	A binary tree, sub-tree, or empty tree. Note: it can be used to represent a tree or sub-tree in which there is only one node.
	A binary tree or sub-tree which has a root node and two sub-trees (a non-empty tree).

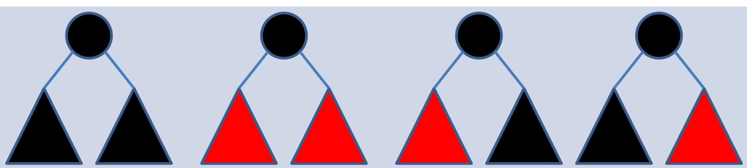
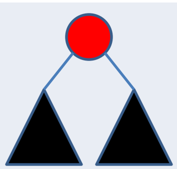
Basic assumptions (continue…)

- Each node in a Red-Black tree is either red or black, and if a node is red, then both its children are black, so:

Figure	Meaning
	A black node in a red-black tree.
	A red node in a red-black tree.
	A red-black tree or sub-tree whose root node is black, or an empty red-black tree.
	A red-black sub-tree whose root node is red.

Basic assumptions (continue…)

- At the previous abstraction level, we can say there are only two different red-black sub-trees;
- But if we unfold them a bit, we can get five different red-black sub-trees (empty red-black trees or sub-trees are excluded because if we can unfold them, they must not be empty):

Figure	Meaning
	A red-black tree or sub-tree whose root node is black.
	A red-black sub-tree whose root node is red.

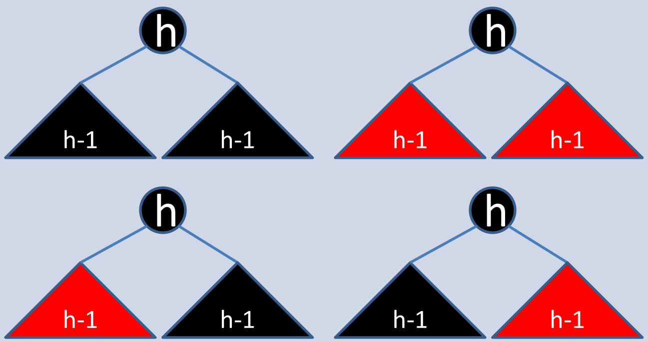
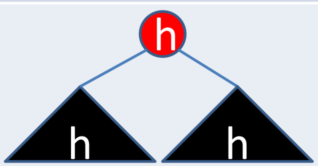
Basic assumptions (continue…)

- Every path from a given node to any of its descendant NIL nodes goes through the same number of black nodes. That is the **black depth** of a node, so:

Figure	Meaning
	A red-black tree or sub-tree whose root node is black and the black depth of the root node is h (h >= 0). When h > 0, the root node is a normal node. When h == 0, it is an empty red-black tree or sub-tree. We can say there is only a NIL node in it.
	A red-black sub-tree whose root node is red and the black depth of the root node is h (h >= 0, if h == 0, there is only a red node in it).

Basic assumptions (continue…)

- The following six figures can represent all different red-black trees, sub-trees or empty trees:

Figure	Meaning
	A red-black tree or sub-tree whose root node is black, the black depth of the root node is h (h > 0), and the black depth of its two child nodes is h-1.
	A red-black sub-tree whose root node is red, the black depth of the root node is h (h >= 0), and the black depth of its two child nodes is h too.
	An empty red-black tree.

Basic assumptions (continue…)

- If the black depth of a node X is h, we can say the black depth of the corresponding sub-tree (tree or empty tree) whose root node is the node X is h too;
- The root of a red-black is black, so:

Figure	Meaning
	The figure can be used to represent a red-black tree, sub-tree or empty tree.
	The figure can be used to represent a red-black sub-tree (or in some intermediate states, the original root node is replaced with a red node).

Basic assumptions (continue…)

- How to represent a NIL node:

Figure	Meaning
	A NIL node, whose black depth is 0. Usually they are omitted in our pictures.
	An empty red-black tree, which can be used to represent a NIL node too. Usually they are omitted in our pictures.

- A NIL node is actually a null pointer in our program.

Basic assumptions (continue…)

- Dyeing a normal black node red will decrease its black depth from h to h-1;
- Dyeing a red node black will increase it black depth from h to h+1;
- If we select a NIL node and replace it with a new red node (its black depth is 0), we can find at the very position we replace the root node 0 of the empty sub-tree with a red node 0.

Insert and then rebalance

- We always insert a red node into a red-black(or **we replace a NIL node with a red node**).
- At first, there is an empty red-black tree. It is easy to insert a red node into it:

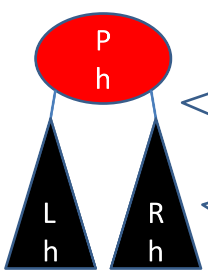
- When there is a non-empty red-black tree, at first we select an empty sub-tree and then replace the NIL node of the sub-tree with a red node. We call the root node of the very sub-tree the node **N** (now it is red but at first it was black). if N's parent is black, the process is finished; if N's parent is red, the rule 2 is broken (note: the rule 2 is only broken in the upper level sub-tree where N's parent is the root node and N is a child sub-tree. No other rules are broken).

Insert and then rebalance (continue…)

- We can say the foregoing operation is the base case of a recursive process (will give more details later);
- Generally speaking, given a sub-tree whose root node's color has been changed from black to red (**by dyeing it red or replacing it with a red node**), we call the root node of the very sub-tree the node **N.** If N's parent is black, the inserting and then rebalancing process is finished; if N's parent is red, the rule 2 is broken (note: the rule 2 is only broken in the upper level sub-tree where N's parent is the root node and N is a child sub-tree. No other rules are broken);
- The foregoing operation may generate nonconforming sub-trees, we need to rebalance them.

Insert and then rebalance (continue…)

- First we need to know how many such nonconforming sub-trees could be generated from the color change;
- Because of the precondition: the color of the root node of a sub-tree is changed from black to red and its parent is red, we only need to consider this sub-tree:

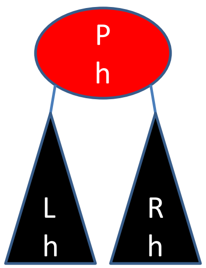


The letters P, L, R are the name of the nodes (L and R are the name of the root node of the two child sub-trees respectively). **h** is their black depth. (Note: h >= 0)

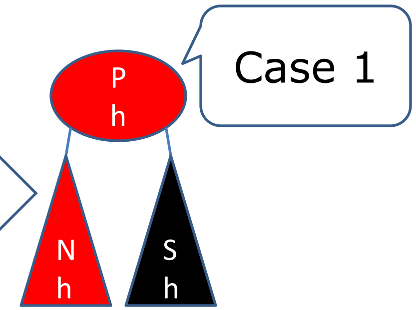
The letter L and R are the name of sub-trees too. When we say the sub-tree P, it includes the node P and the two child sub-trees L and R.

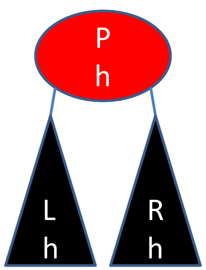
- After the color change is finished, we get two nonconforming sub-trees to rebalance:

Insert and then rebalance
(continue…)

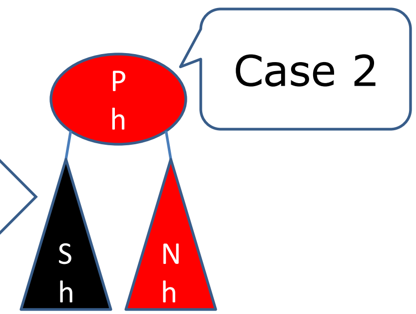


We change the color of the root node of the sub-tree L from black to red. After that, L’s root node is the node N, the node R is its sibling.





We change the color of the root node of the sub-tree R from black to red. After that, R’s root node is the node N, the node L is its sibling.

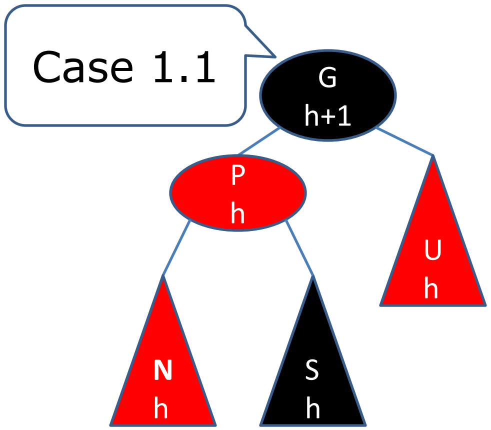


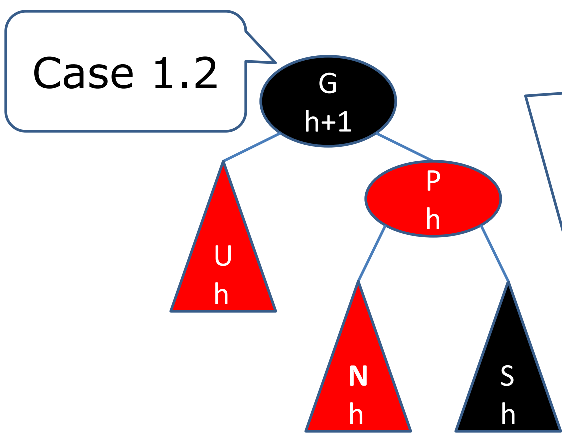
Insert and then rebalance (continue…)

- We do not know how to rebalance the following nonconforming sub-trees:

- But after we add the grandparent and the uncle of the node N into the pictures, we get eight nonconforming sub-trees.

Insert and then rebalance (continue…)

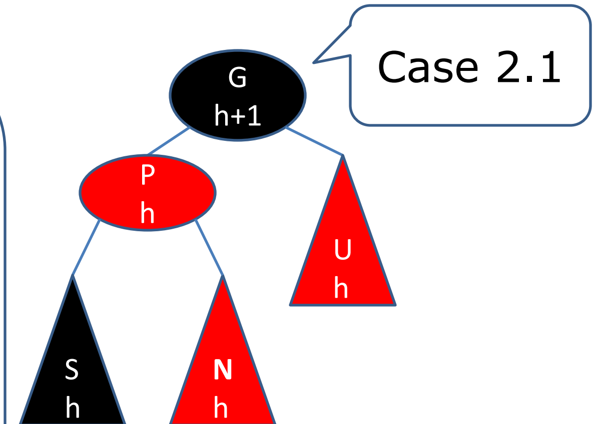


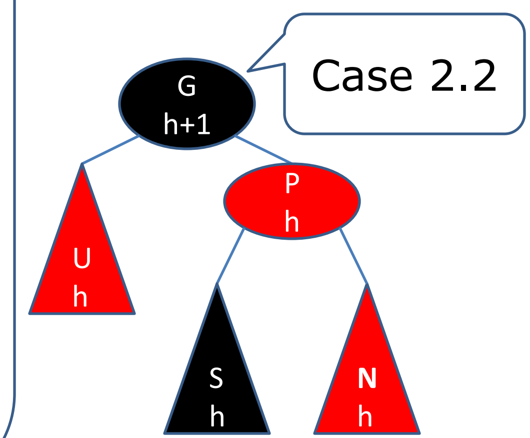


The letters G, P, U, N, S are the name of the nodes or sub-trees, the expression (h[+|-]*number*) below is their black depth.

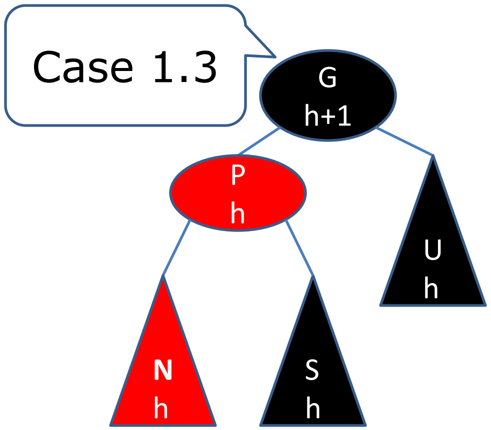
1. **G**: grandparent
2. **P**: parent
3. **U**: uncle
4. **N**: new node (whose color has been changed from black to red)
5. **S**: sibling

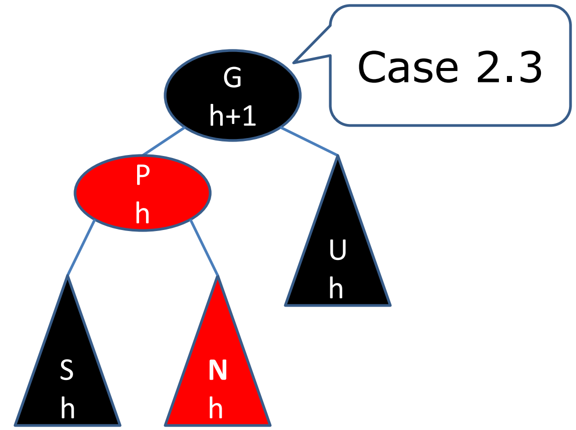
(Note: h >= 0)



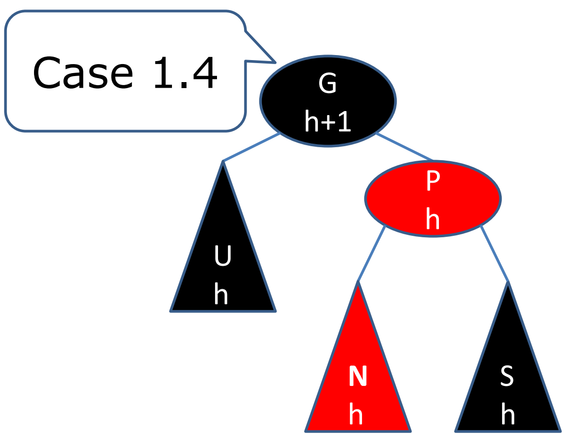


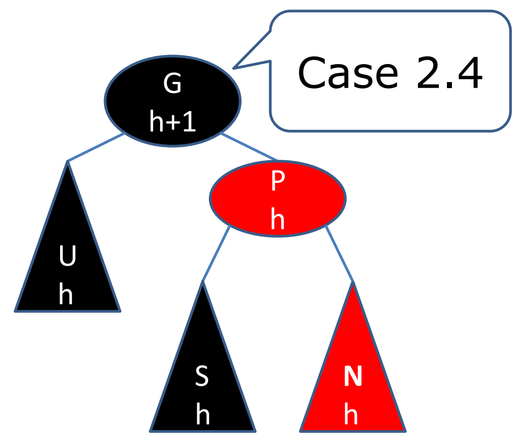
Insert and then rebalance
(continue…)





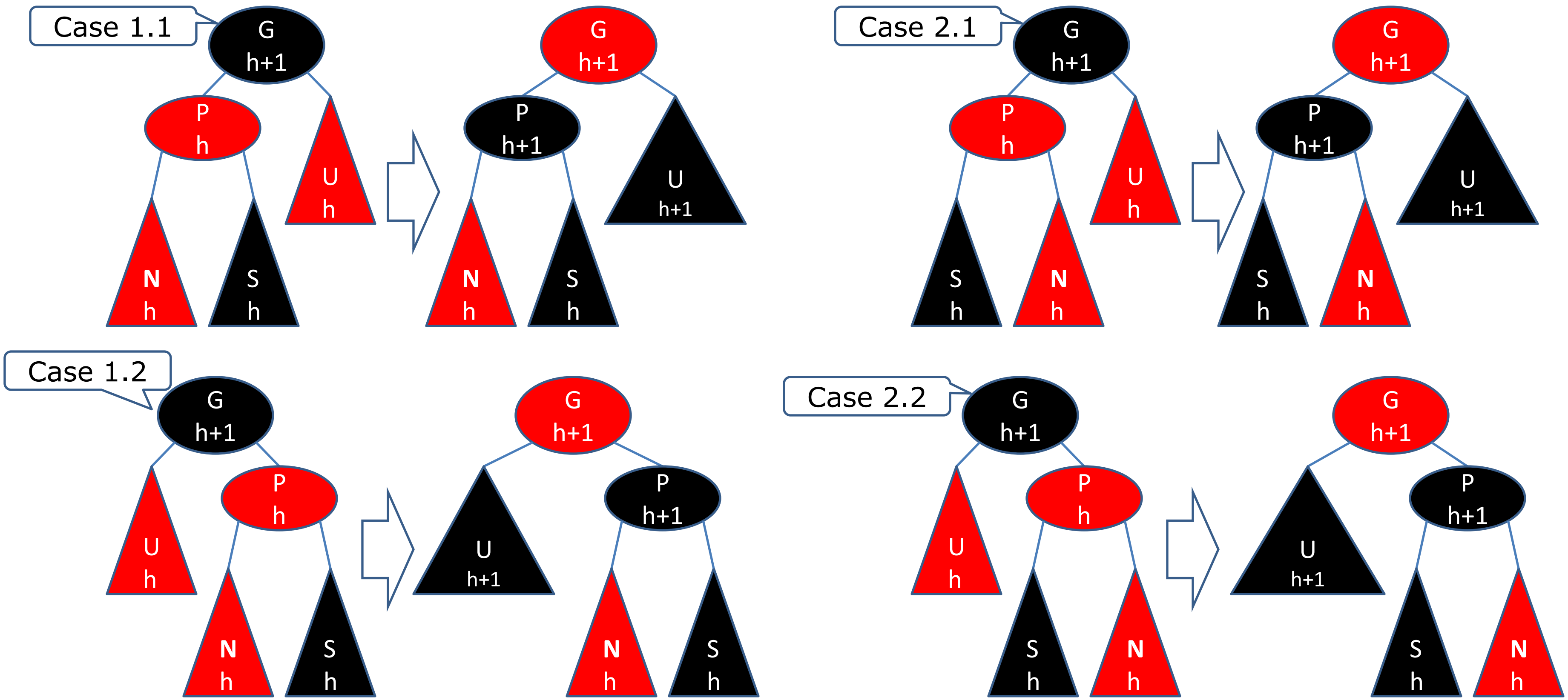
Note: h >= 0





Insert and then rebalance (continue…)

- The method to rebalance the sub-trees 1.1, 1.2, 2.1, 2.2 is as below:

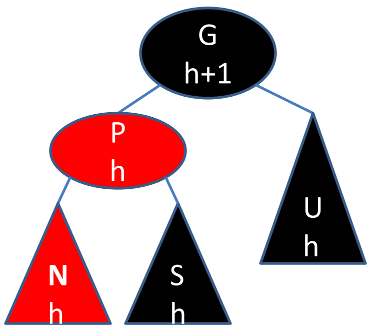


Insert and then rebalance (continue…)

- We dye the nodes P, U black and dye the node G red, all five rules are kept in the sub-trees;
- But the root node G of the sub-trees is changed from black to red, we need to check:
  - Whether the node is the root node of the whole tree, if it is, the rule 4 is broken;
  - Whether the node G's parent is red, if it is, the rule 2 is broken in the upper lever sub-tree.
- Apparently, the process is recursive.

Insert and then rebalance (continue…)

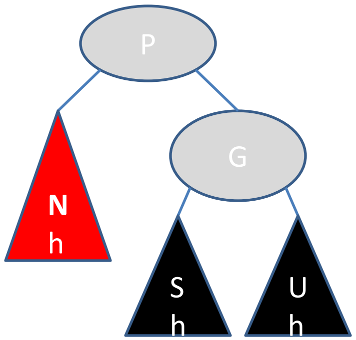
- The method to rebalance the sub-tree 1.3 is as below (note: h >= 0):



- The term "rotation" is used to describe the method to rebalance such a sub-tree, but I think we should treat it as a jigsaw puzzle: we have five pieces, how do we use them to rebuild a conforming red-black sub-tree?

Insert and then rebalance
(continue...)

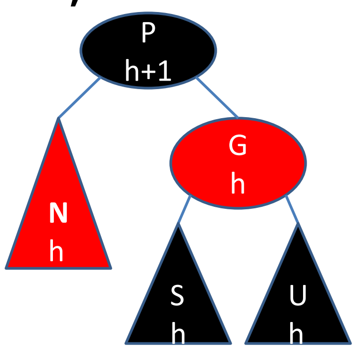
- First we can create a binary search tree like this:



- The binary search tree has the following traits:
  - all the sub-trees N, S and U are conforming red-black sub-trees (no rule is broken in them);
  - the black depth of all the sub-trees N, S and U is h;
  - until now the color of the nodes P and G is not determined.

Insert and then rebalance
(continue…)

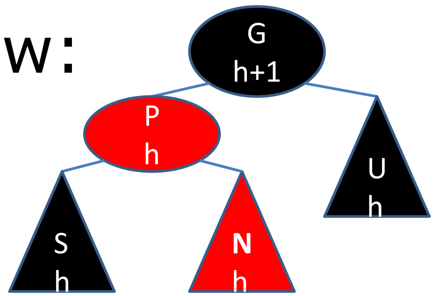
- Before we dyed the node N red, the black depth of the root node G of the sub-tree was h+1, that means that we should let the P's black depth be h+1, so we can color the nodes P and G in this way:



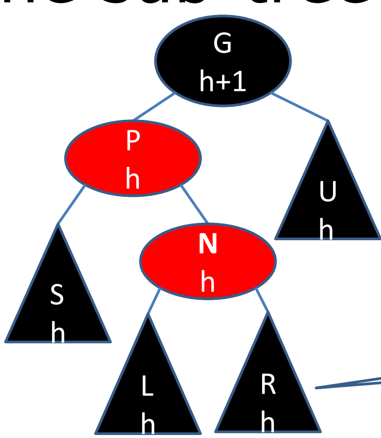
- Then we use the five pieces to rebuild a new red-black sub-tree, the black depth of the root node of the new sub-tree is still h+1, the root node is black, no rule is broken. So the rebalancing process is finished for the case.

Insert and then rebalance (continue...)

- The method to rebalance the sub-tree 2.3 is as below:



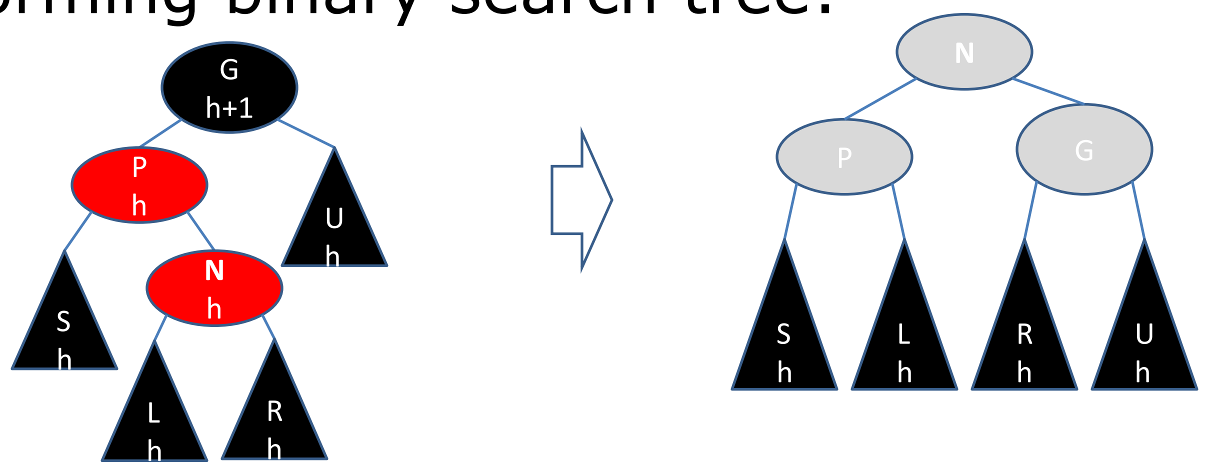
- First we unfold the node N (or the sub-tree N) a bit (we can do this even there is only a red node in the sub-tree N):



The nodes L and R are the children of the node N, and their color must be black, their black depth must be h because the node N is the root node of a conforming sub-tree. (Note: h >= 0)

Insert and then rebalance (continue…)

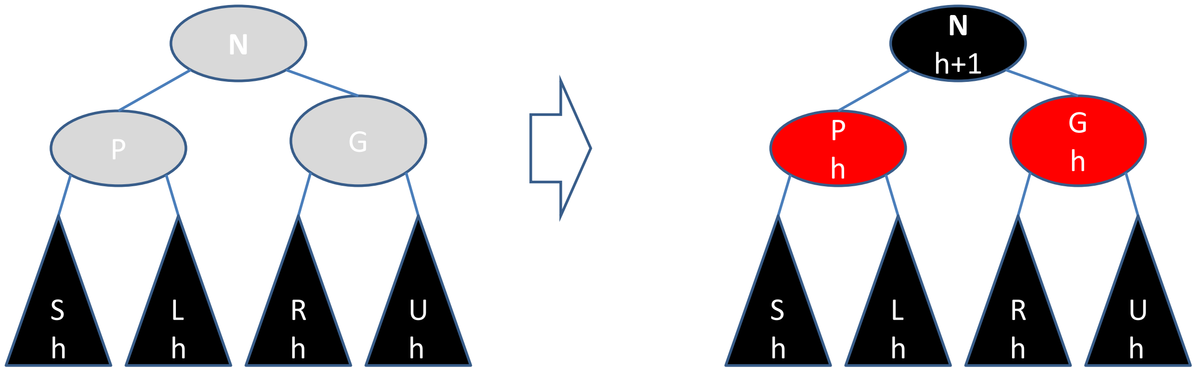
- The term "double rotation" is used to describe the method to rebalance such a sub-tree, but I think we should treat it as a jigsaw puzzle: we have seven pieces, how do we use them to rebuild a conforming red-black sub-tree?
- We can reorganize the seven pieces to get such a conforming binary search tree:



Insert and then rebalance (continue…)

- The binary search tree has the following traits:
  - all the sub-trees S, L, R and U are conforming red-black sub-trees (no rule is broken in them);
  - the black depth of all the sub-trees S, L, R and U is h;
  - until now the color of the nodes N, P and G is not determined.
- Before we dyed the node N red, the black depth of the root node G of the sub-tree was h+1, that means that we should let the N’s black depth be h+1, so we can color the nodes N, P and G in this way:

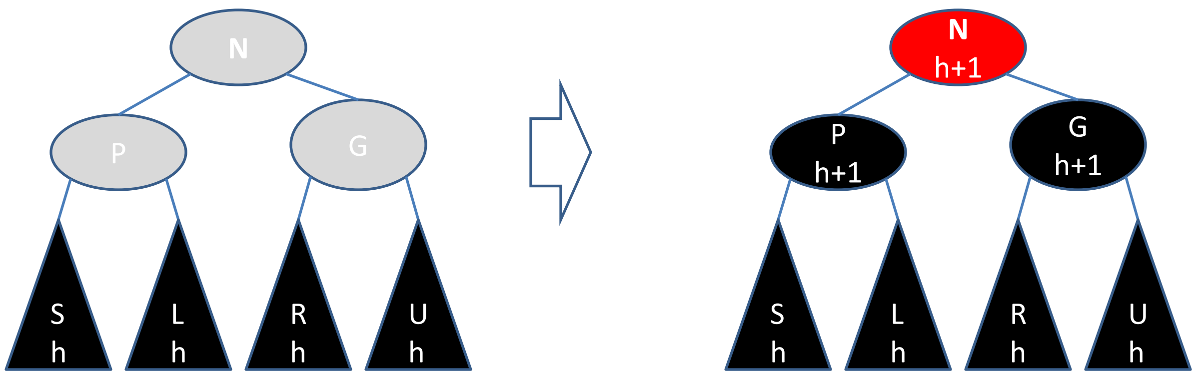
Insert and then rebalance (continue…)



- Now we use the seven pieces to rebuild a new red-black sub-tree, its root node is N with the black depth h+1;
- No rule is broken in the new red-black sub-tree, and certainly no rule is broken in any other sub-tree, it means that the rebalancing process is finished for the case.

Insert and then rebalance (continue…)

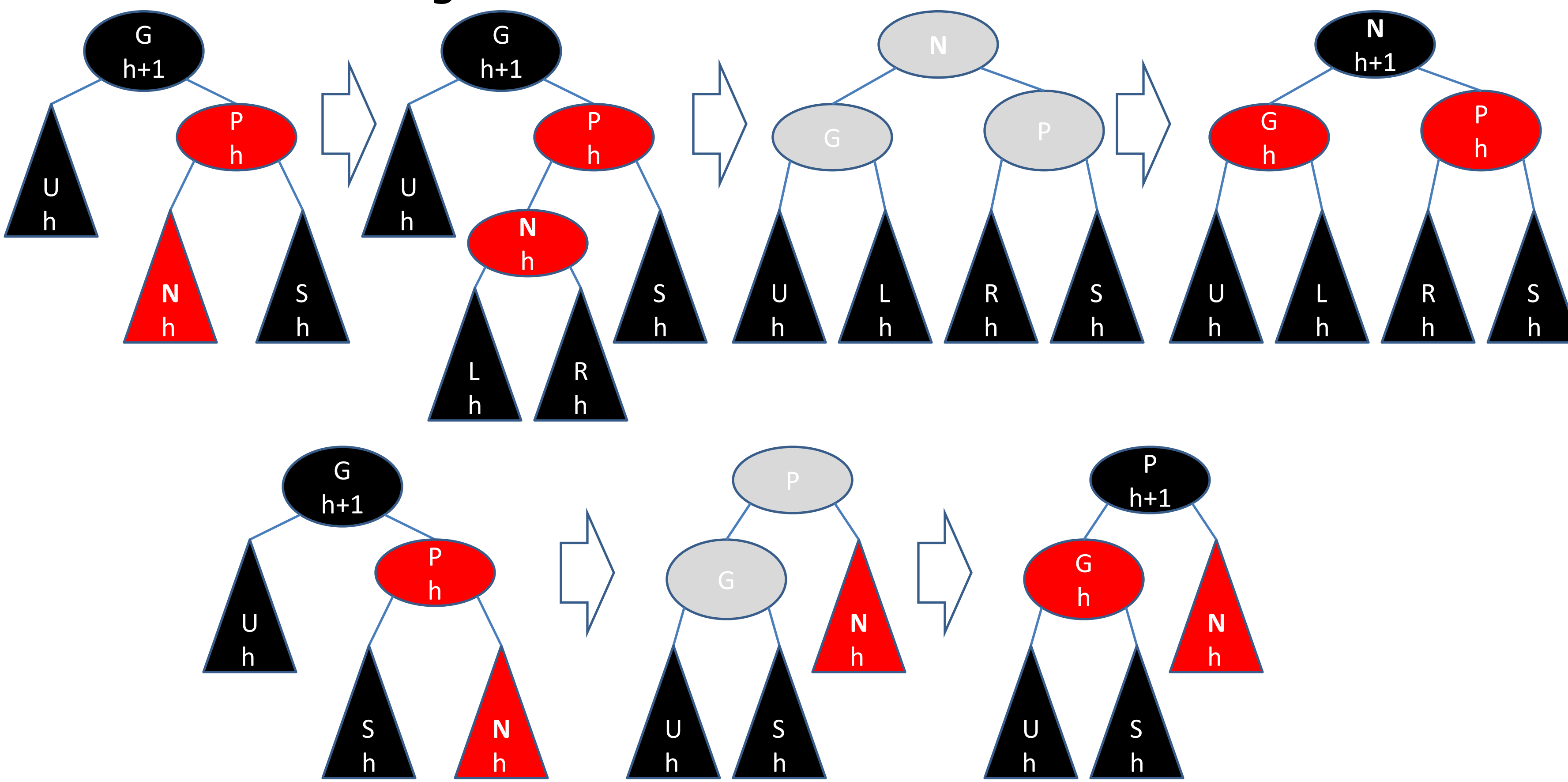
- BTW, we can color the nodes P, G and N in other way:



- The new sub-tree does not break any rule, its black depth is h+1, but its new root node N is red, the previous root node G is black. So if we select color the three nodes in this way, the rebalancing process continues;
- For performance we do not do that.

Insert and then rebalance
(continue…)

- We can use the similar method to rebalance the nonconforming sub-trees 1.4 and 2.4:



Insert and then rebalance
(continue…)

- In summary, the inserting and then rebalancing process is recursive:
  - the step 1: given a sub-tree whose root node's color has been changed from black to red, we call its root node the node **N**. Note: this sub-tree is a conforming red-black sub-tree;
  - the step 2: if the node **N** is the root node of the whole tree, we dye it black, and then the process is finished;
  - the step 3: if its parent is black, the process is finished;
  - the step 4: if its parent is red too, we get eight different nonconforming sub-trees to rebalance:
    - For the sub-trees 1.1, 1.2, 2.1, 2.2, we can dye three nodes in them a different color in order to get conforming sub-trees. But the color of the root node of the very sub-trees is changed from black to red, the root node becomes the node **N**, we return to the step 1;
    - For the sub-trees 1.3, 1.4, 2.3, 2.4, we use the foregoing method to rebalance them to get conforming sub-trees, and then the process is finished.
  - the base case is: we select an empty tree or sub-tree and replace its black root (NIL) node with a new red node.

Remove and then rebalance

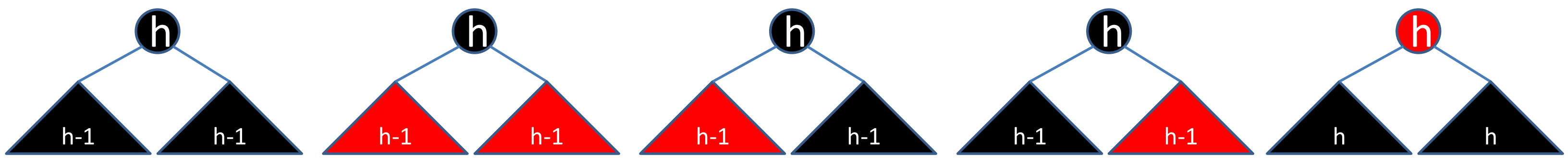
- We always remove such a node from a non-empty tree: its children are two NIL nodes;
- If the node is red, it is finished because no rule is broken;
- If the node is black and it is not the last node, the rule 3 is broken;
- If the node is black and it is the last node, we will get an empty red-black tree;
- Removing such a black node is like decreasing its black depth from h (1) to h-1 (0).

Remove and then rebalance (continue…)

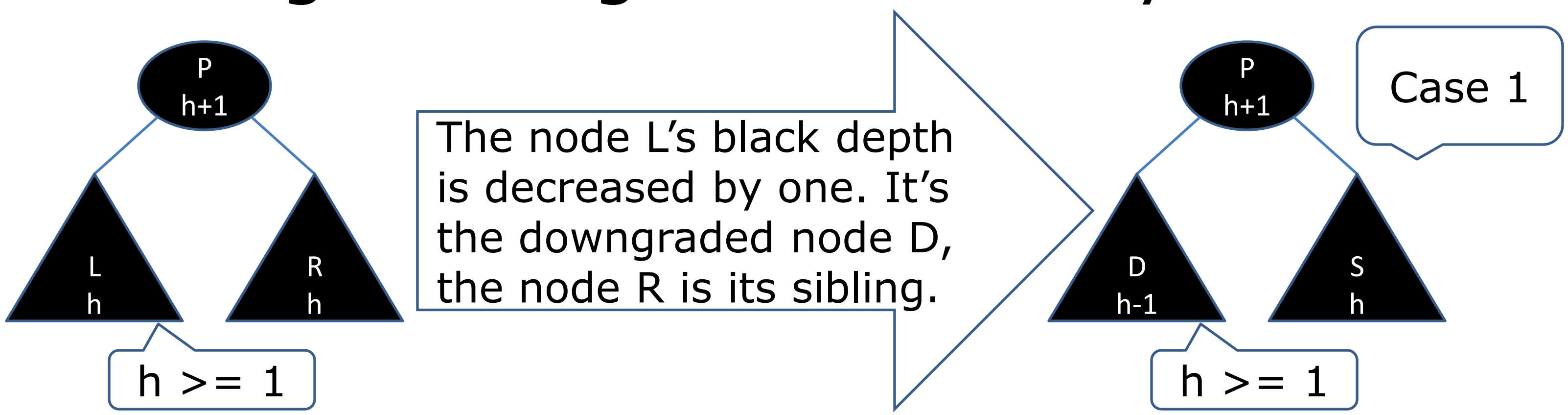
- The removing and then rebalancing process is recursive too, the base case is: a black node D with two NIL child nodes is removed, or **the node D's black depth is decreased from h** (1) **to h-1** (0), then the rule 3 is broken if it is not the last node, it causes that we need to rebalance one of many sub-trees;
- How many?

Remove and then rebalance (continue…)

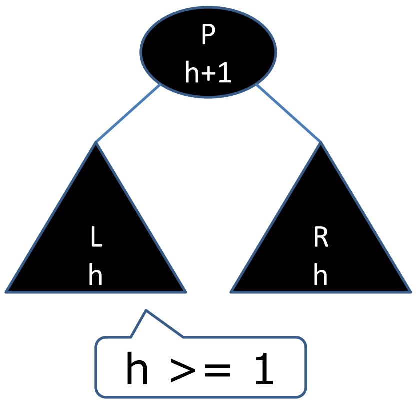
- There are only five different non-empty sub-trees:



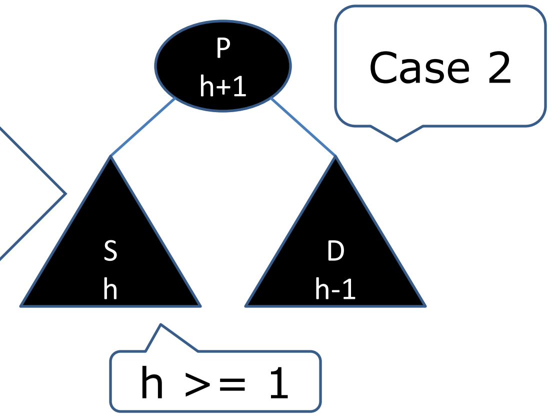
- We will go through them one bye one:

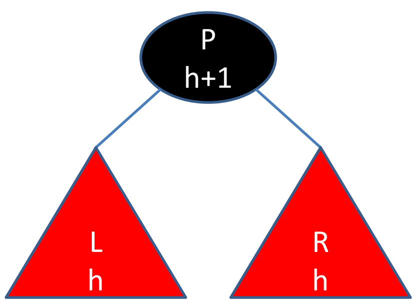


Remove and then rebalance
(continue…)



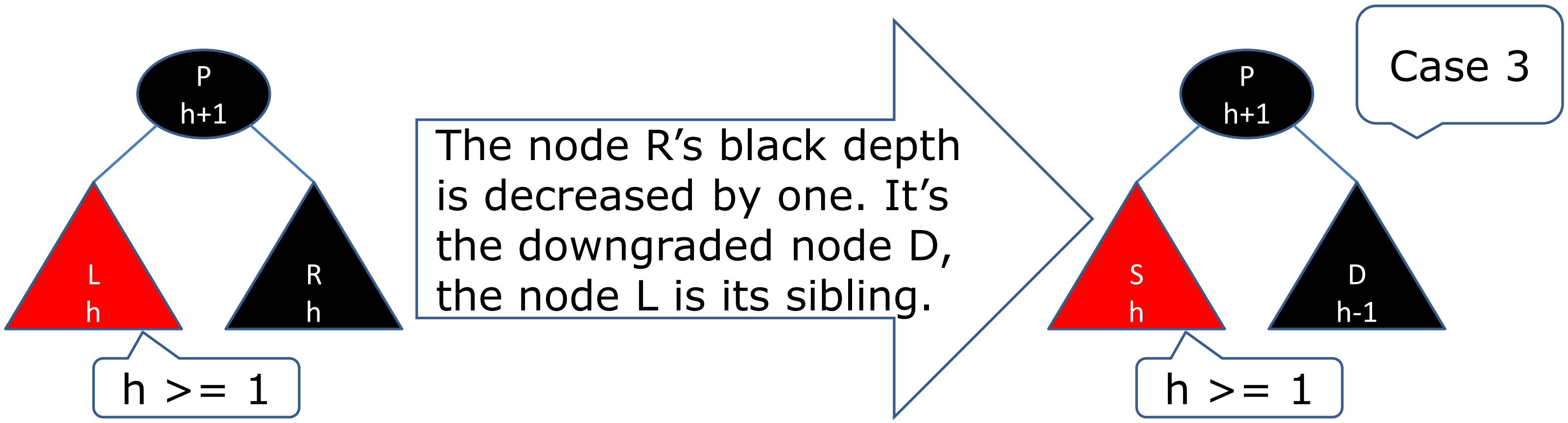
The node R's black depth is decreased by one. It's the downgraded node D, the node L is its sibling.

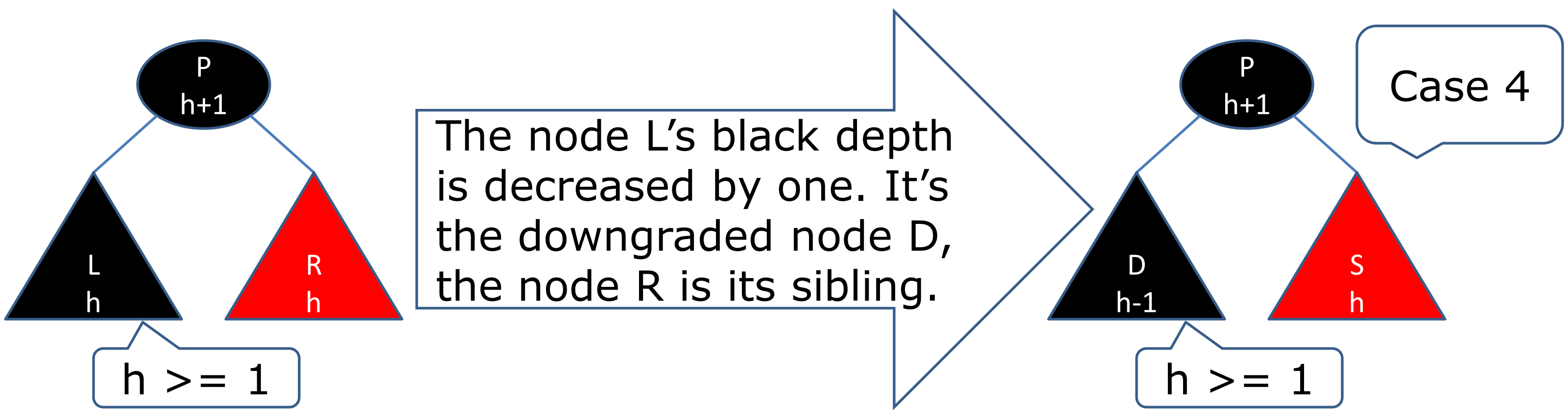




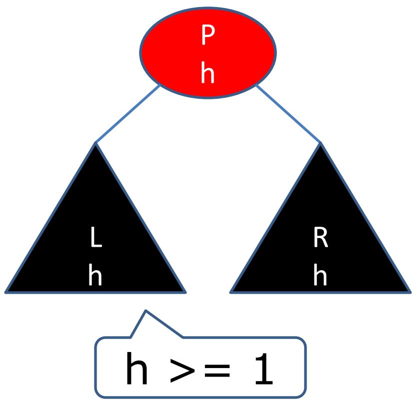
The nodes L and R are red, they cannot be the candidate of the downgraded node.

Remove and then rebalance
(continue…)

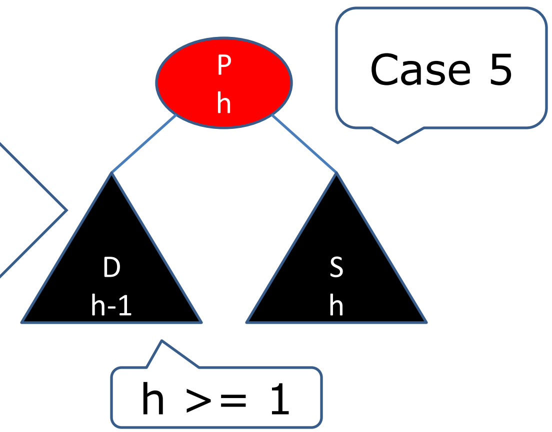


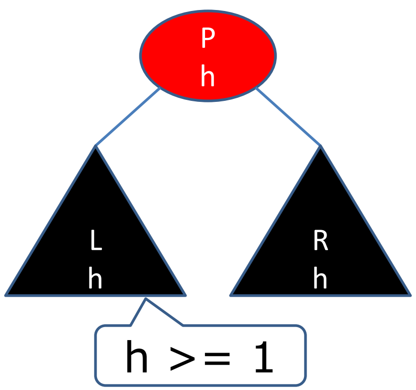


Remove and then rebalance
(continue…)

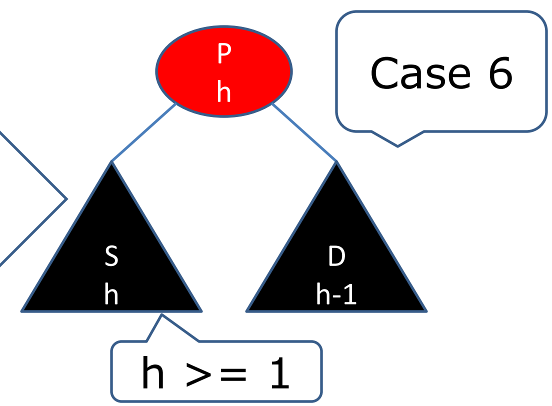


The node L's black depth is decreased by one. It's the downgraded node D, the node R is its sibling.



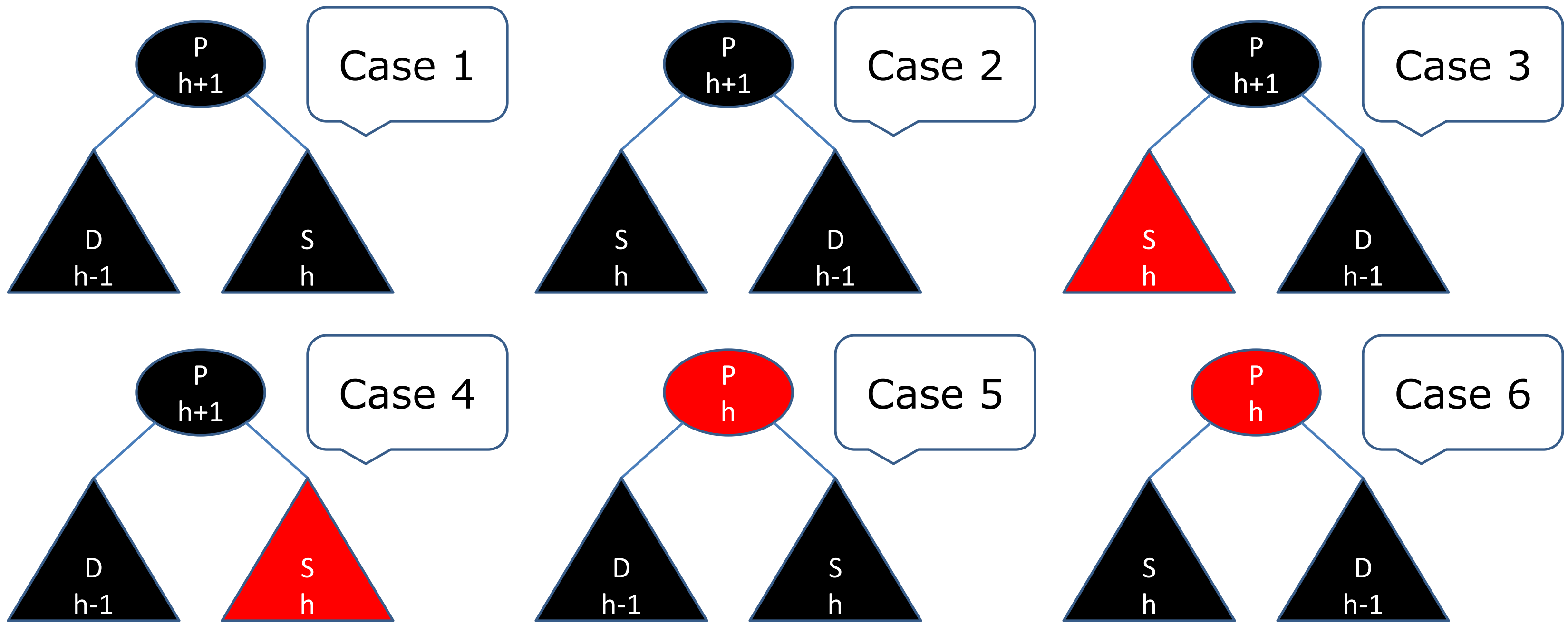


The node R's black depth is decreased by one. It's the downgraded node D, the node L is its sibling.



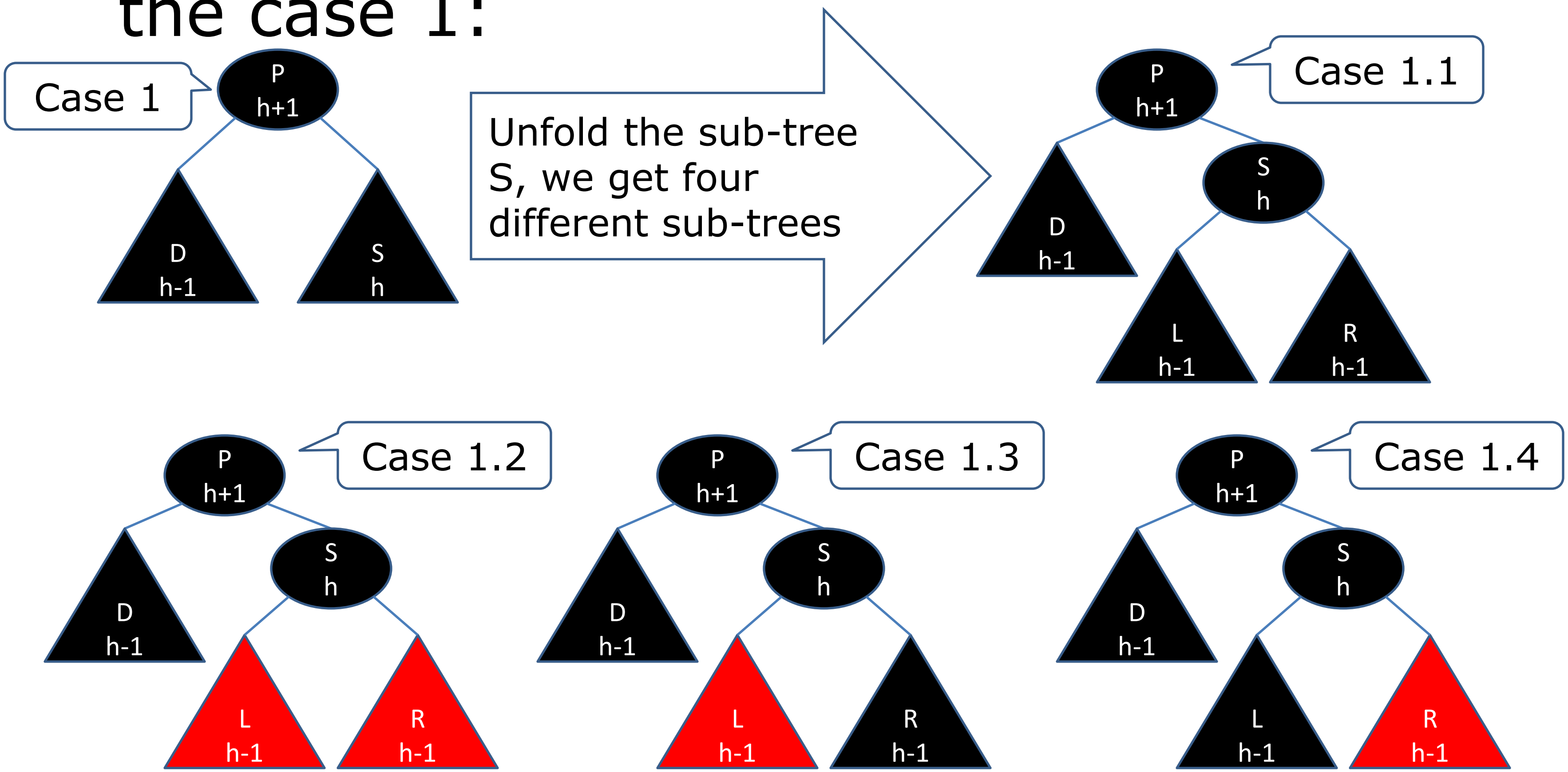
Remove and then rebalance (continue…)

- Now we get six different nonconforming red-black sub-trees to rebalance. We do not know how to do it, we need to bring more nodes into consideration.



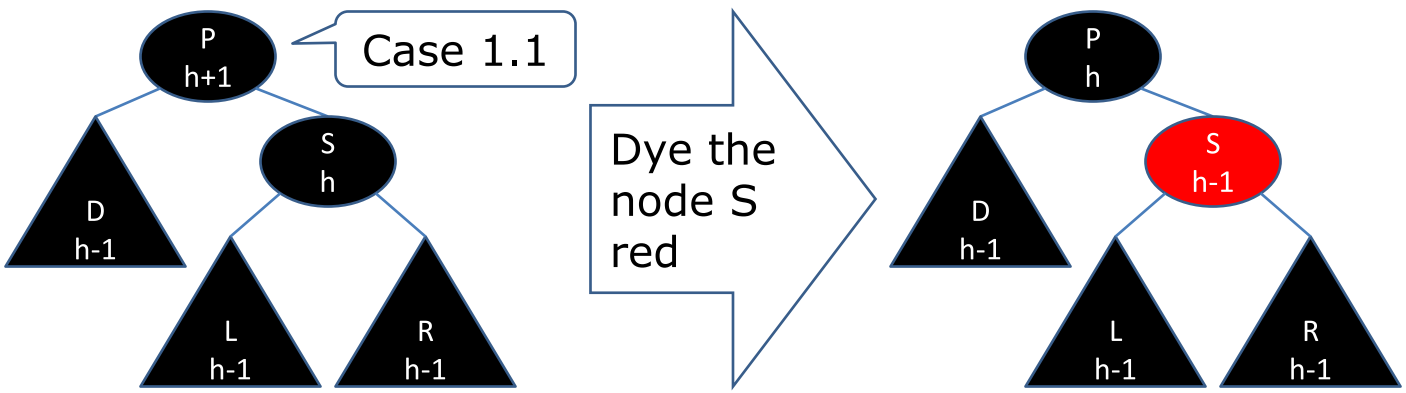
Remove and then rebalance
(continue…)

- The method to rebalance the sub-tree in the case 1:



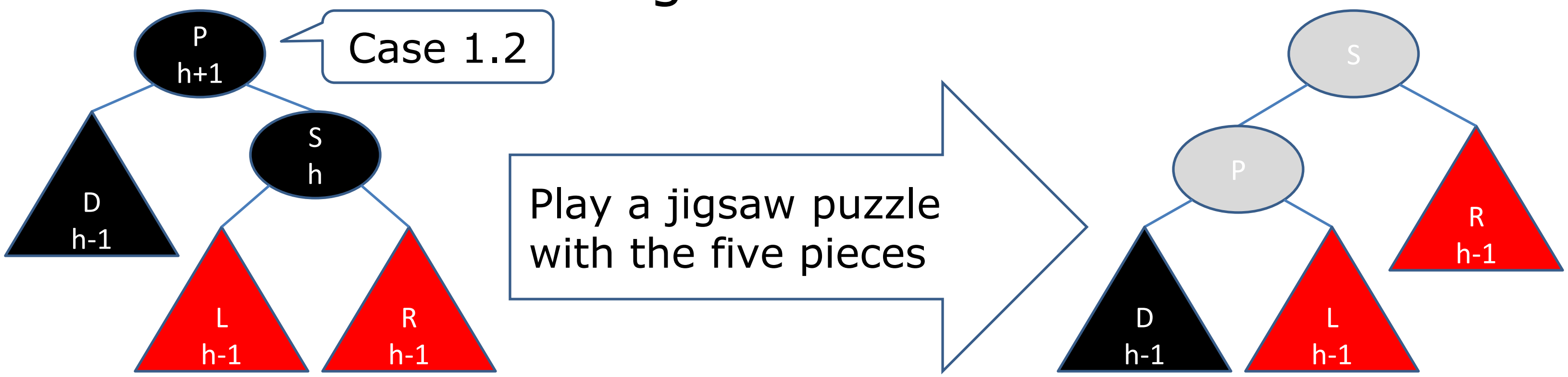
Remove and then rebalance (continue…)

- For the nonconforming red-black tree 1.1, we can dye the node S red, and then we get a conforming red-black tree with the black depth h: the black depth of the root node P is decreased by one. If P is not the root node of the whole tree, it becomes the node **D** and then the recursive process continues.

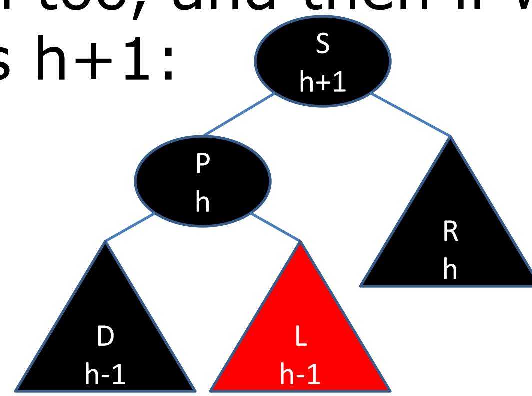


Remove and then rebalance (continue…)

- For the nonconforming red-black tree 1.2:

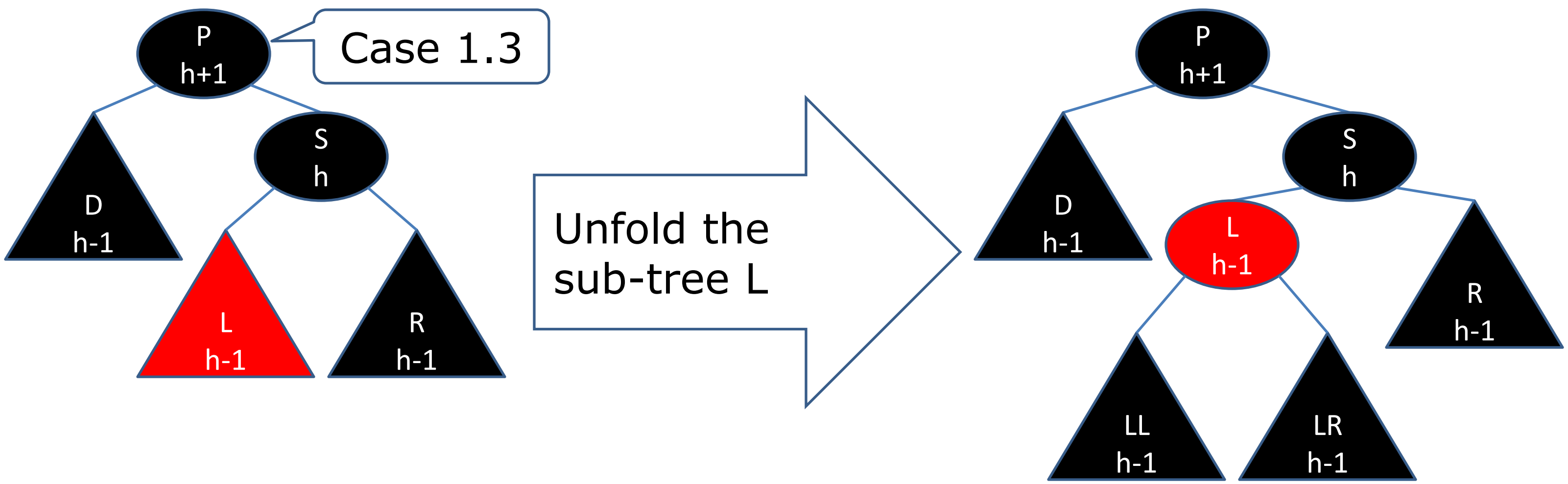


- Because L is red, so P must be black, then P’s black depth will be h, it causes that we must dye R black, so R’s black depth will be h too, and then if we dye S black, S’s black depth is h+1:

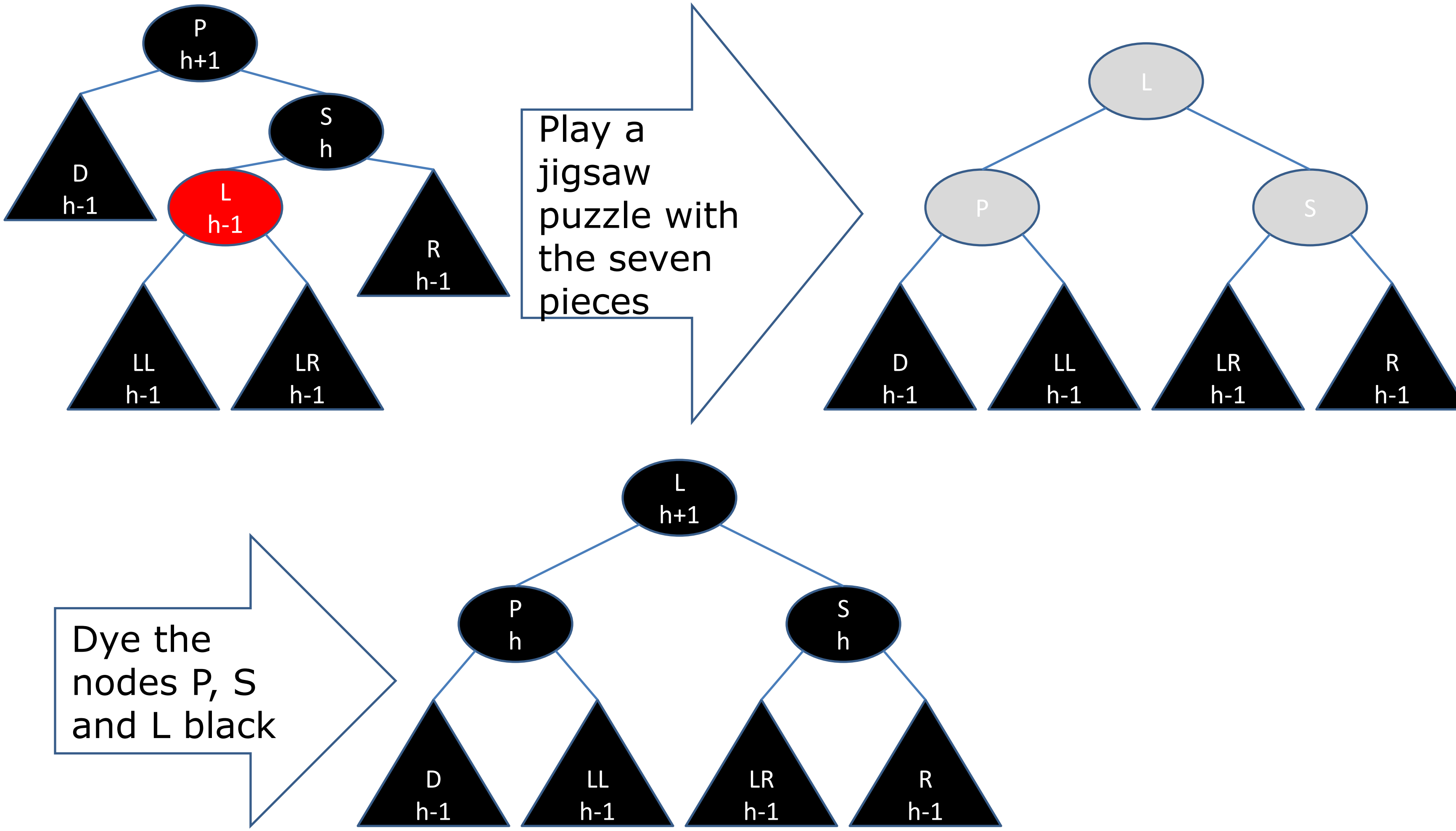


Remove and then rebalance (continue…)

- Our method resolve the case 1.2, no more action is required;
- For the nonconforming red-black tree 1.3:

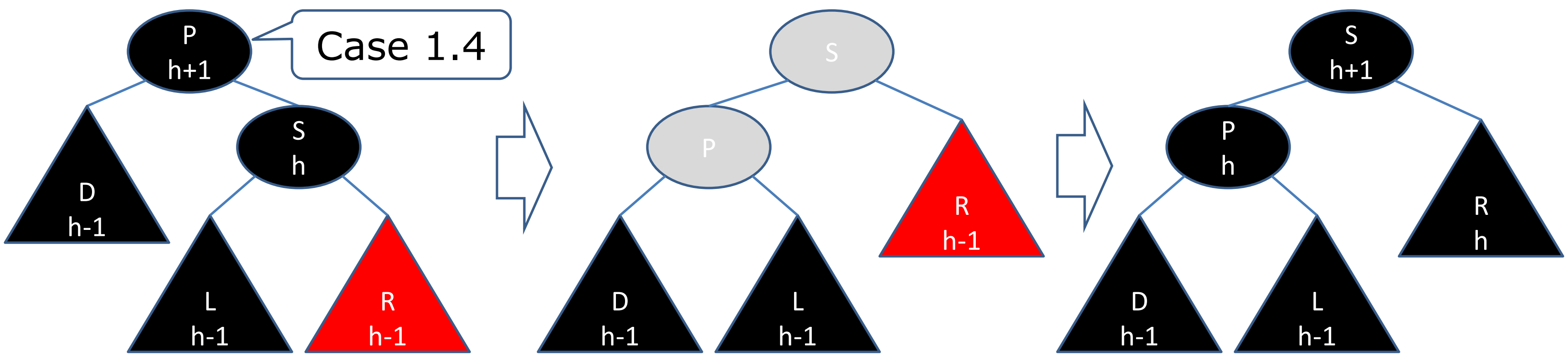


Remove and then rebalance (continue…)



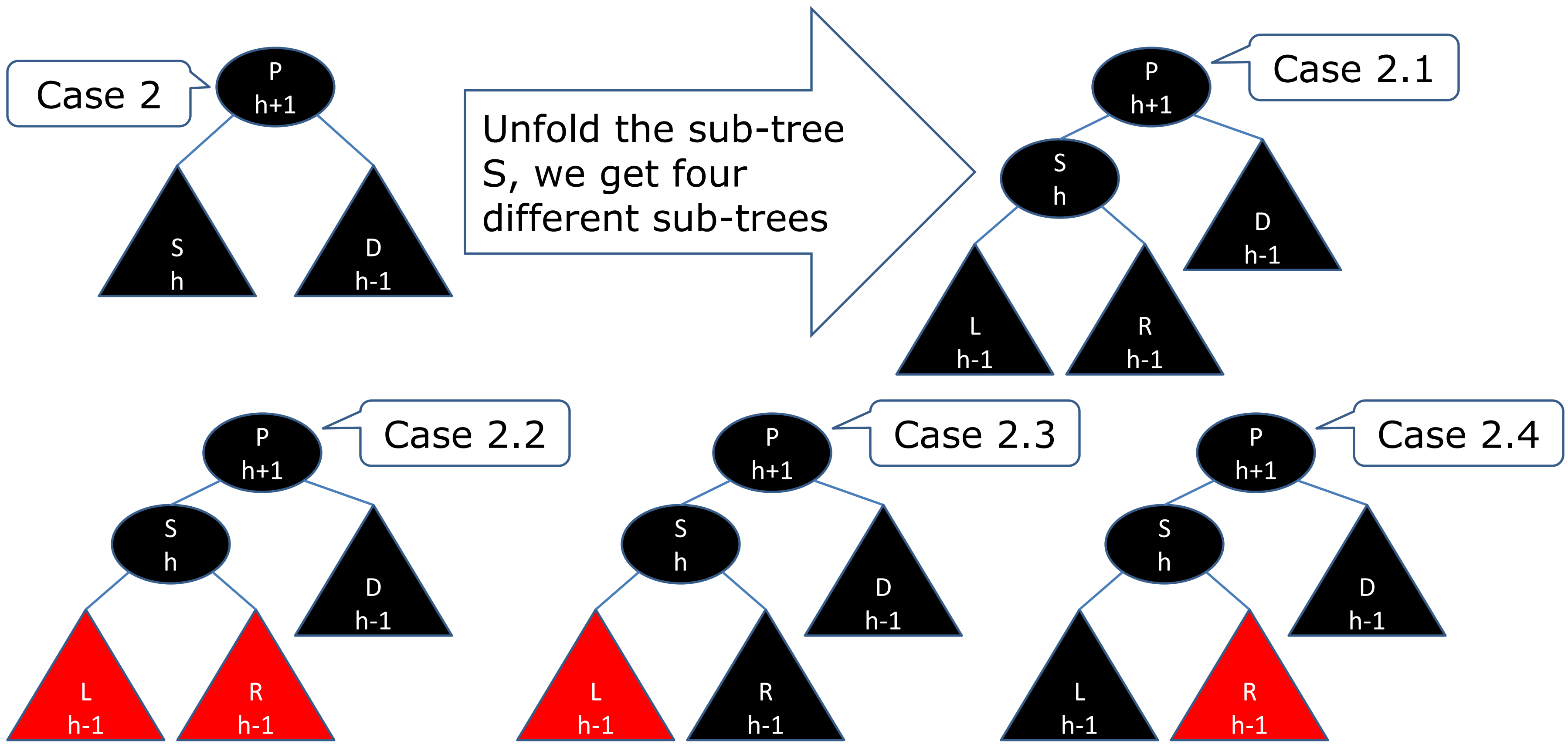
Remove and then rebalance (continue…)

- Our method resolve the case 1.3, no more action is required;
- For the nonconforming red-black tree 1.4, we can use the method which is similar to the method 1.2 to resolve it:

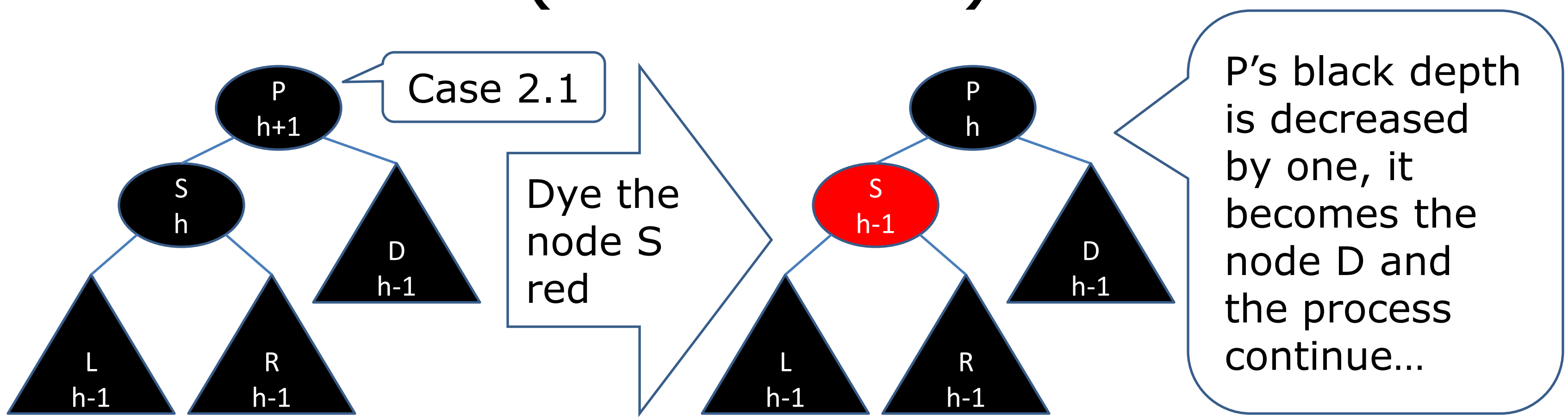


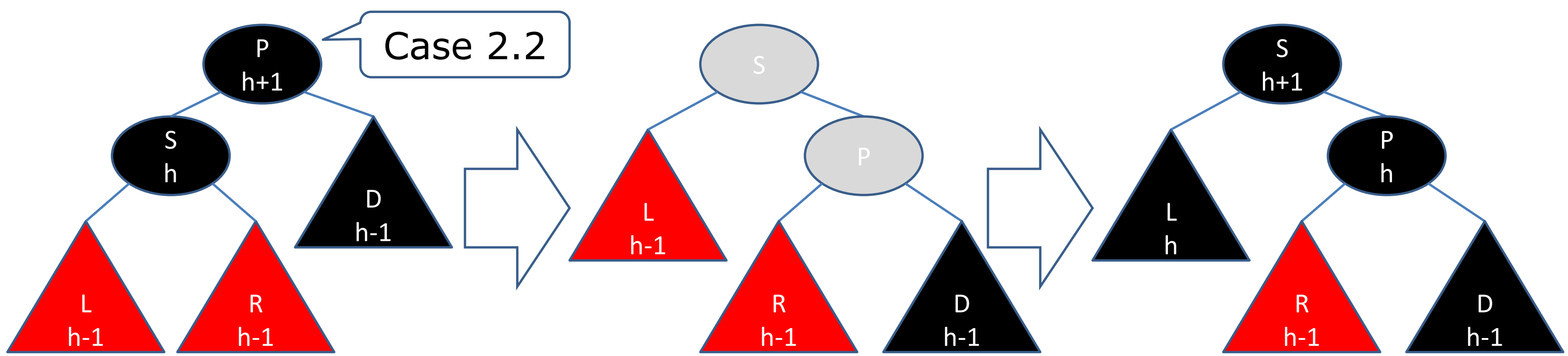
Remove and then rebalance (continue…)

- We can use the similar method to rebalance the sub-tree in the case 2:

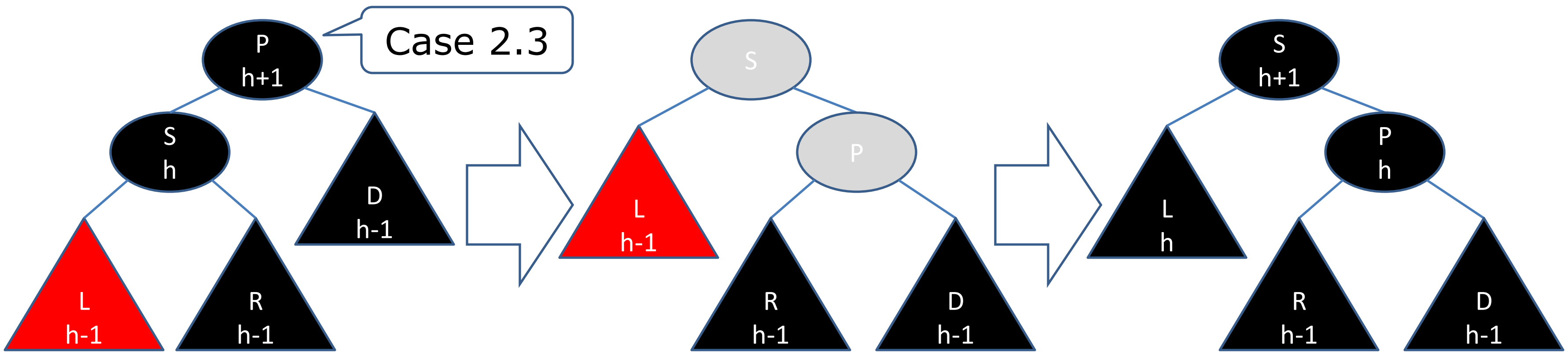


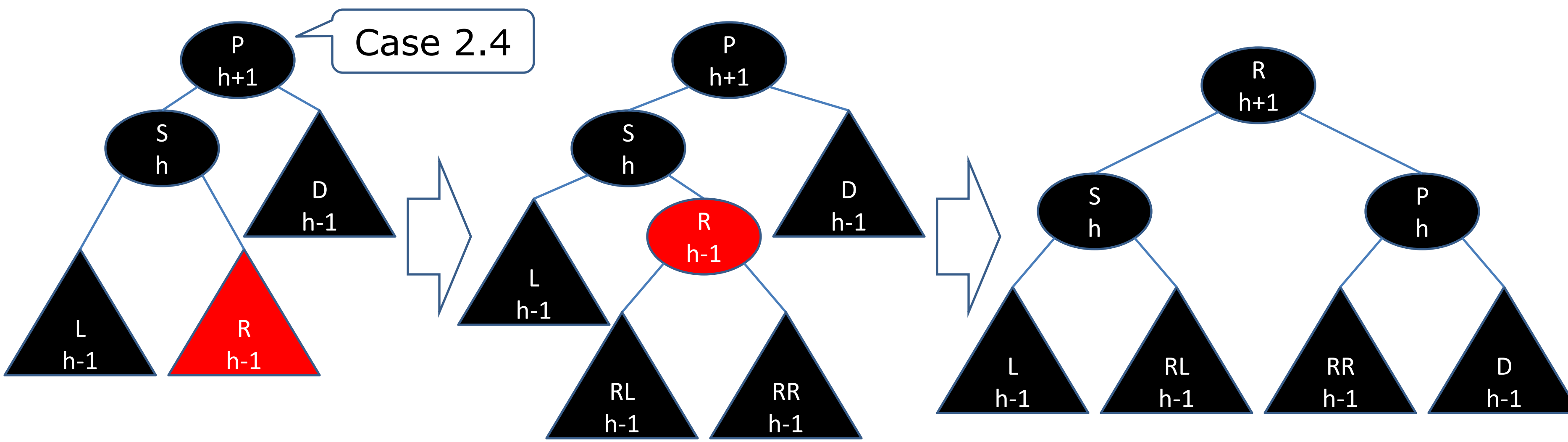
Remove and then rebalance (continue…)





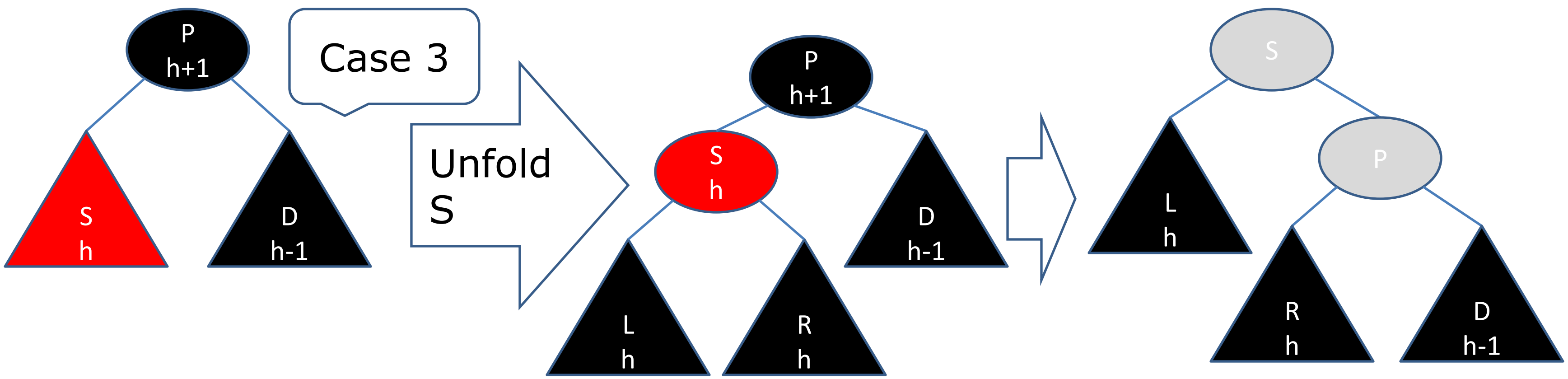
Remove and then rebalance (continue…)





Remove and then rebalance (continue…)

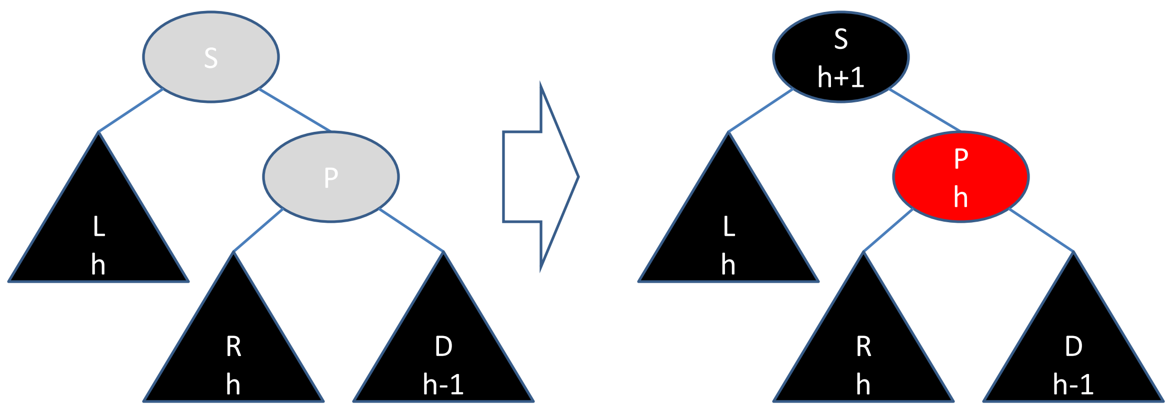
- The method to rebalance the sub-tree in the case 3:



- We unfold S and rotate the sub-tree, we still get a nonconforming red-black tree;
- But maybe we can transform it into other case.

Remove and then rebalance (continue…)

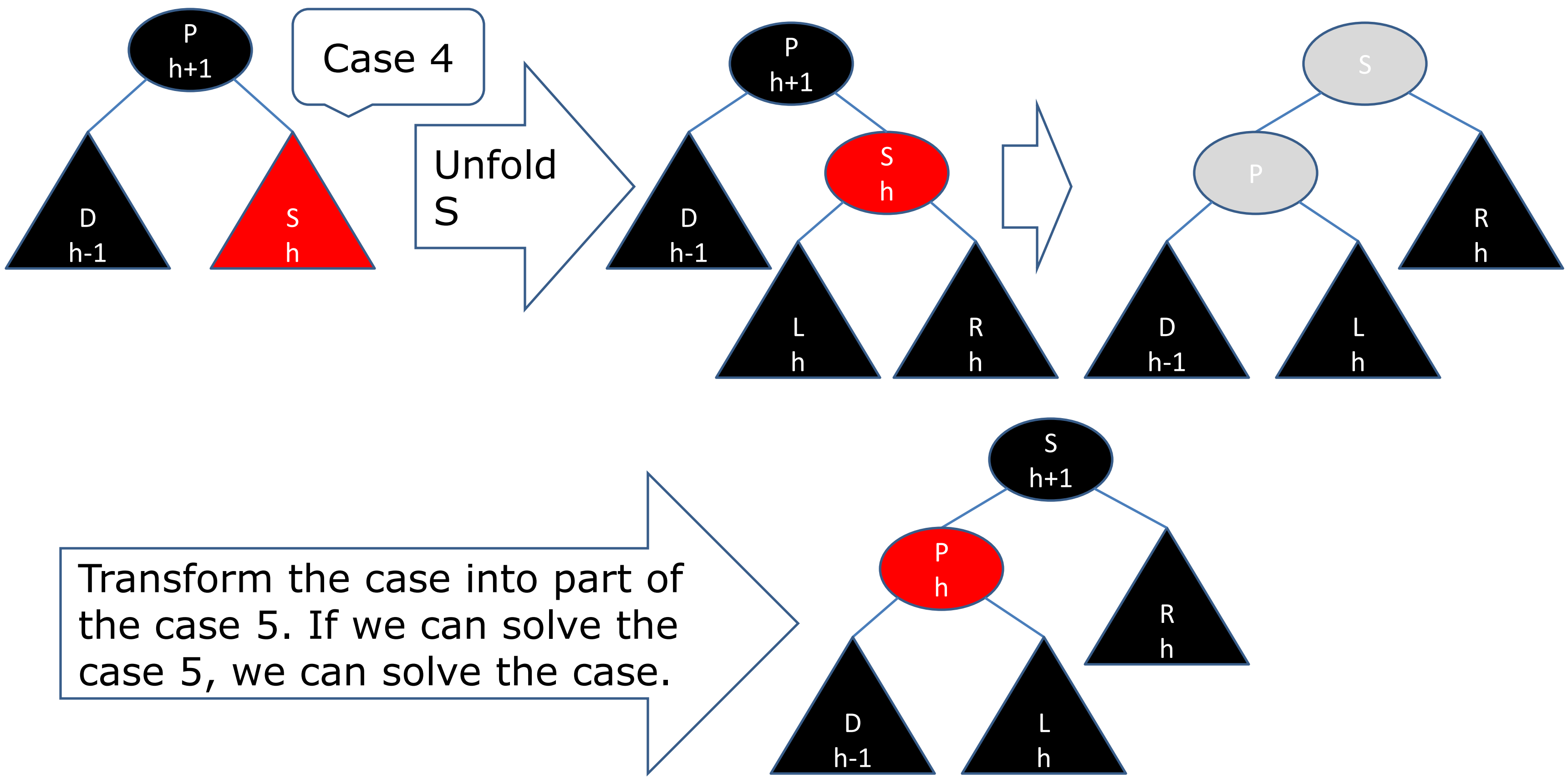
- If we dye P red, suppose that P's black depth is h; dye S black, S's black depth is h+1:



- And then it becomes part of the case 6. If we can resolve the case 6, we can resolve this case.

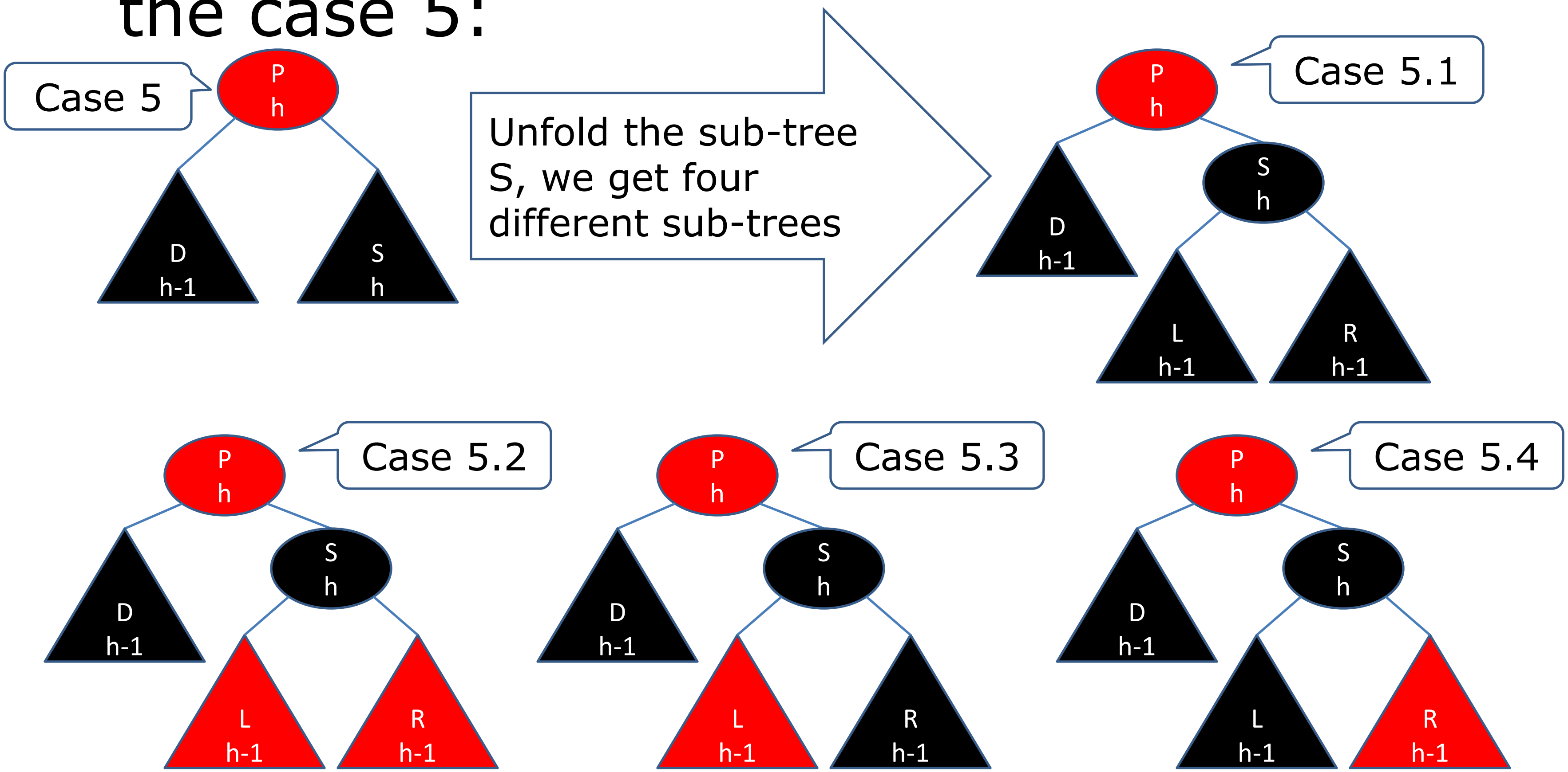
Remove and then rebalance (continue…)

- The method to rebalance the sub-tree in the case 4:



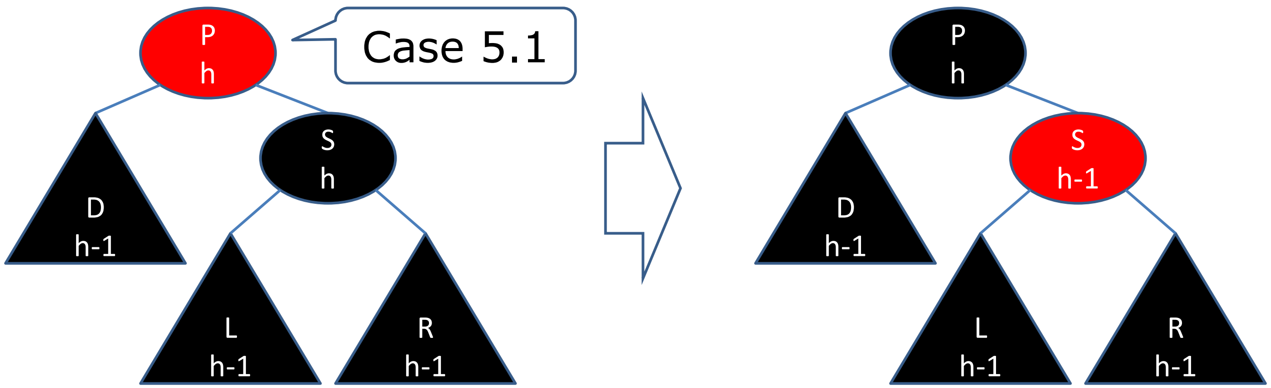
Remove and then rebalance (continue…)

- The method to rebalance the sub-tree in the case 5:



Remove and then rebalance (continue…)

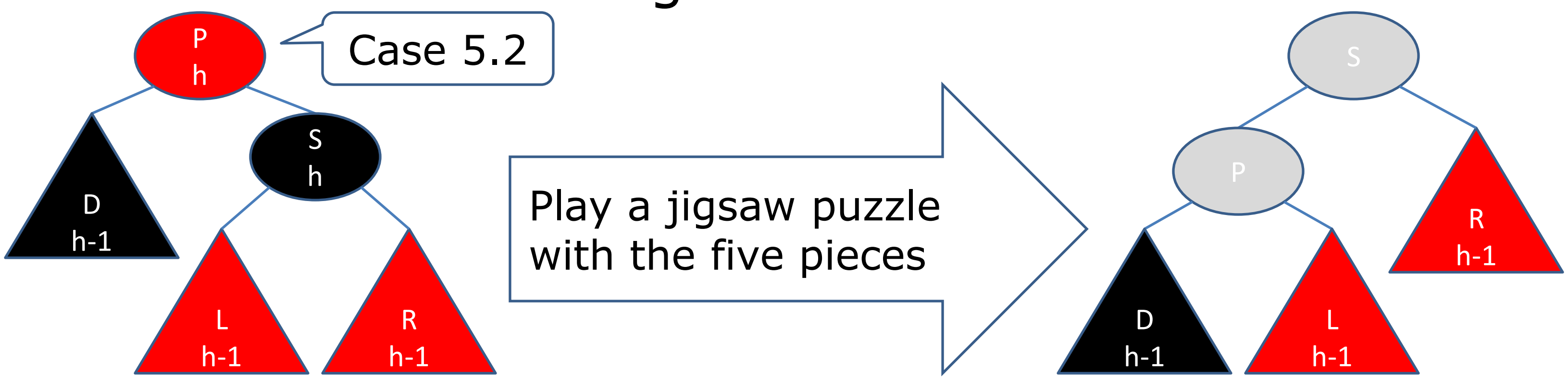
- For the nonconforming red-black tree 5.1, we can dye S red, dye P black:



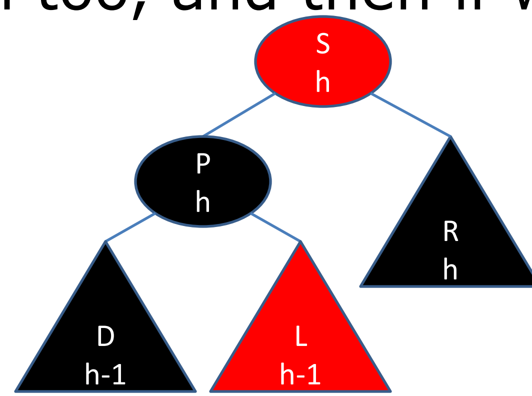
- And then we get a conforming sub-tree where no rule is broken, the black depth of the sub-tree is still h, and the color change of the root node of the sub-tree will not cause that some rules may be broken in any upper level sub-trees, so the process is finished.

Remove and then rebalance (continue…)

- For the nonconforming red-black tree 1.2:

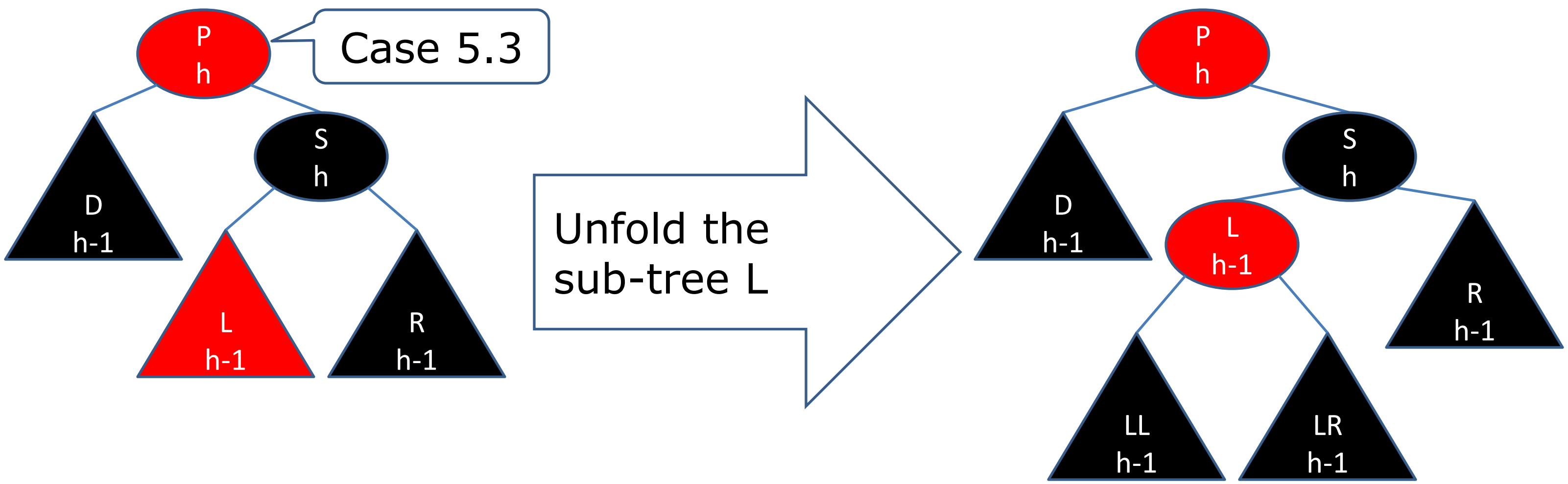


- Because L is red, so P must be black, then P’s black depth will be h, it causes that we must dye R black, so R’s black depth will be h too, and then if we dye S red, S’s black depth is h:

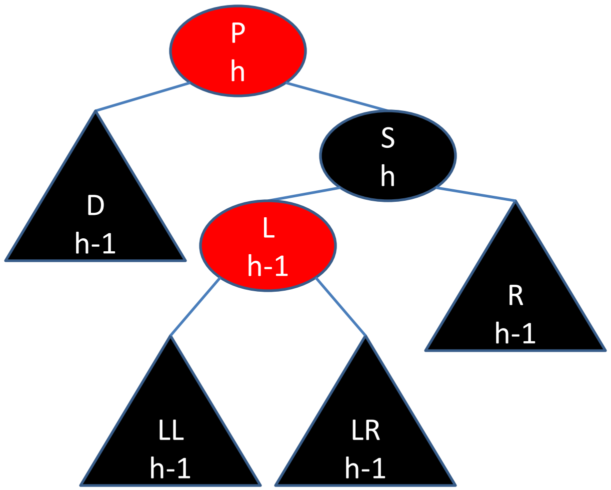


Remove and then rebalance (continue…)

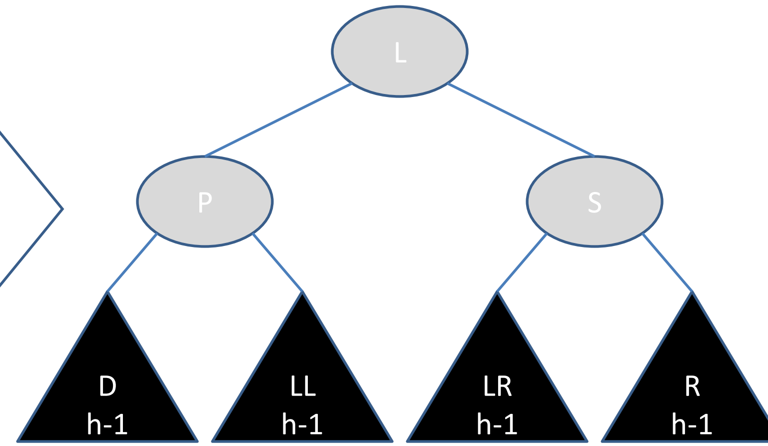
- Our method resolve the case 5.2, no more action is required;
- For the nonconforming red-black tree 5.3:



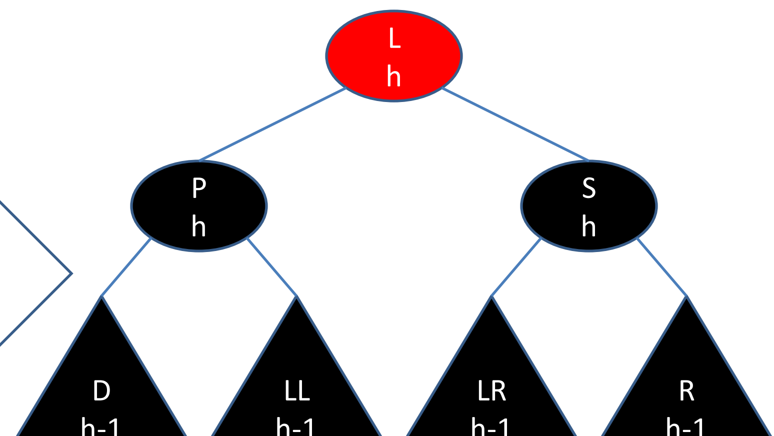
Remove and then rebalance (continue…)



Play a jigsaw puzzle with the seven pieces

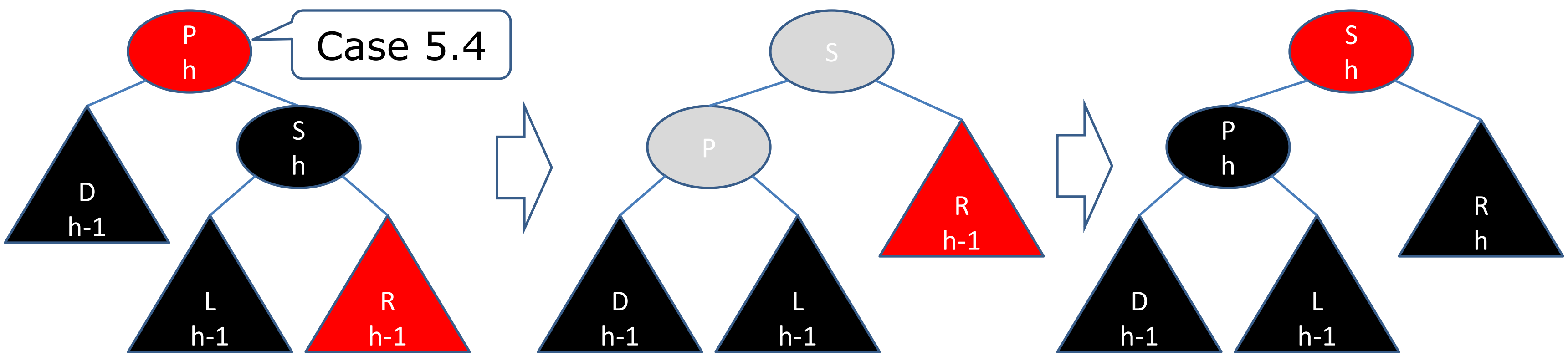


Dye the nodes P, S black, dye L red



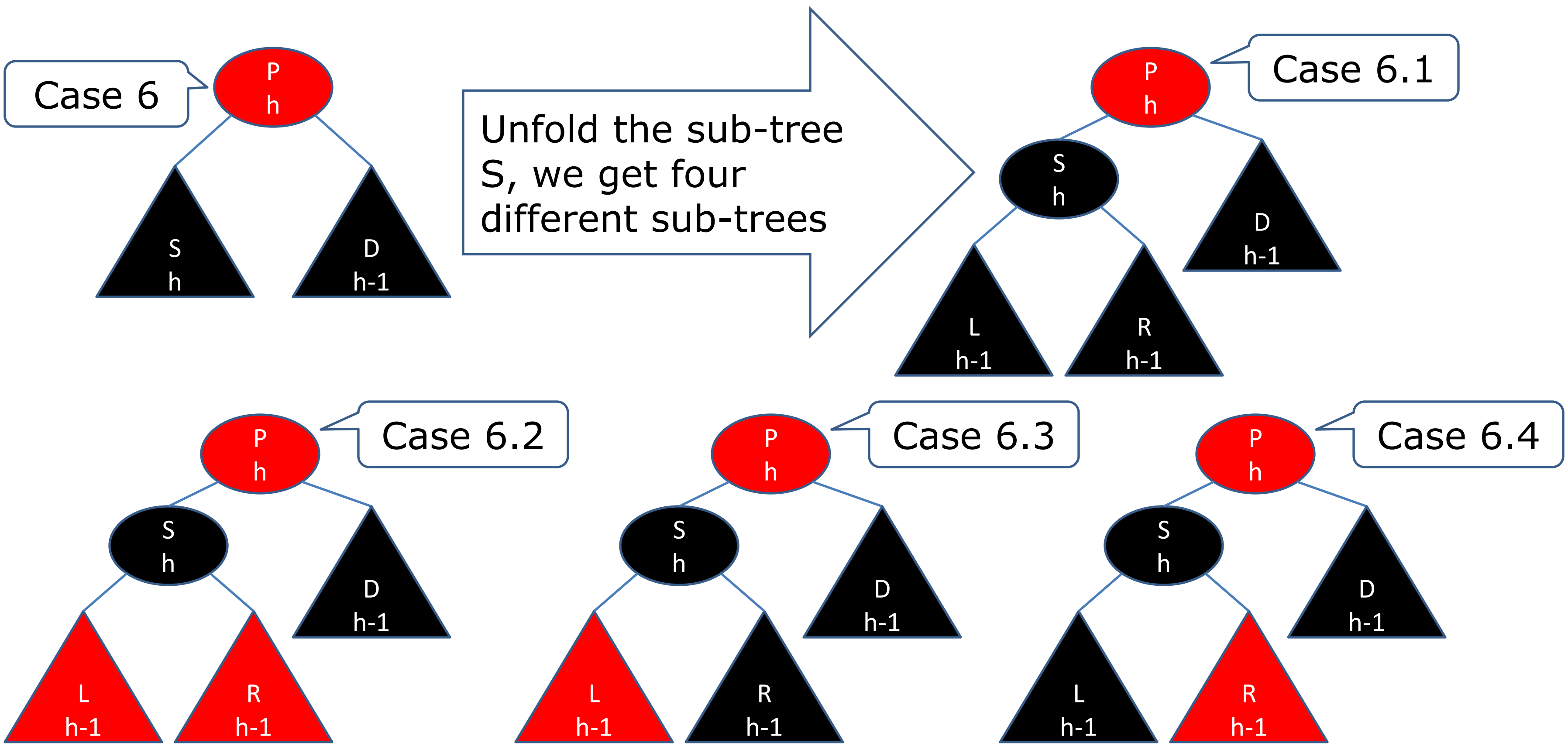
Remove and then rebalance (continue…)

- Our method resolve the case 5.3, no more action is required;
- For the nonconforming red-black tree 5.4, we can use the method which is similar to the method 5.2 to resolve it:

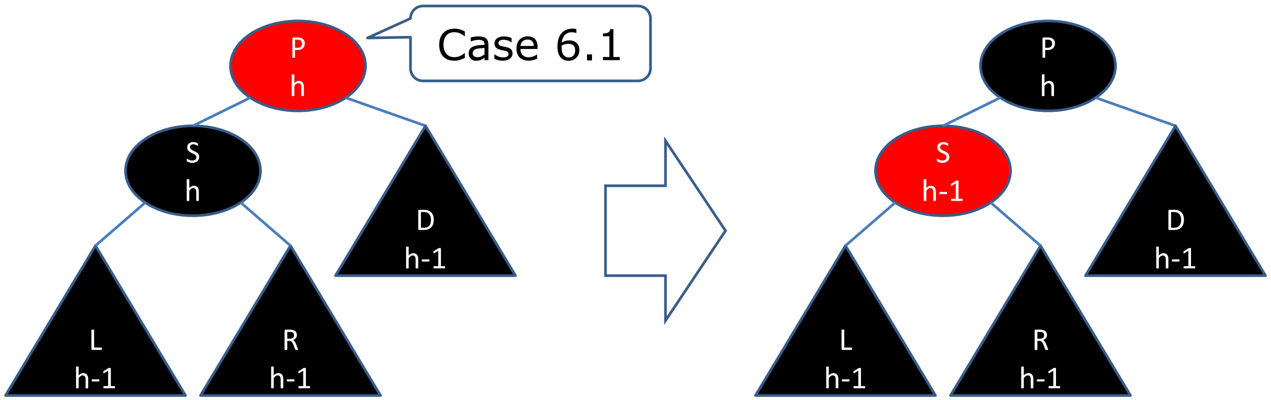


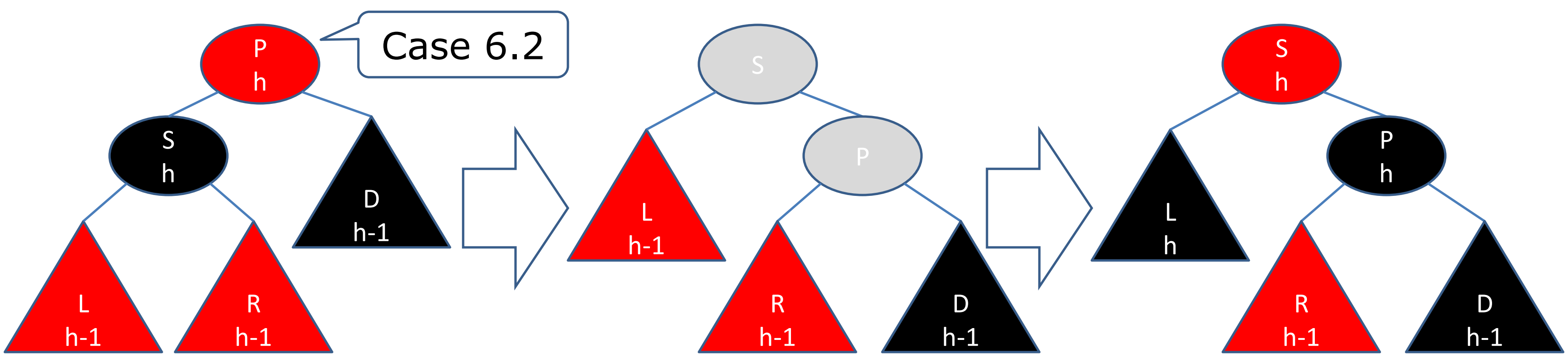
Remove and then rebalance
(continue…)

- We can use the similar method to rebalance the sub-tree in the case 6:

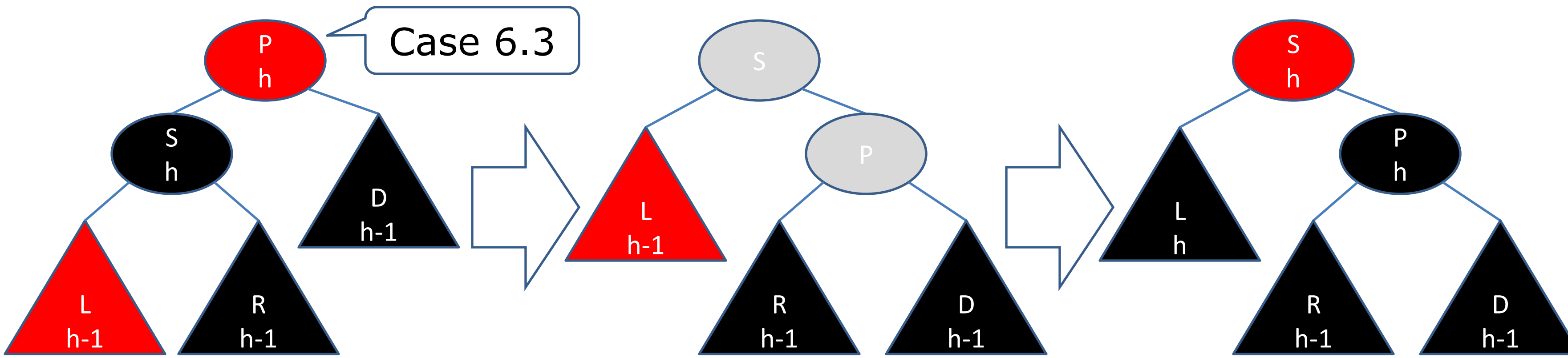


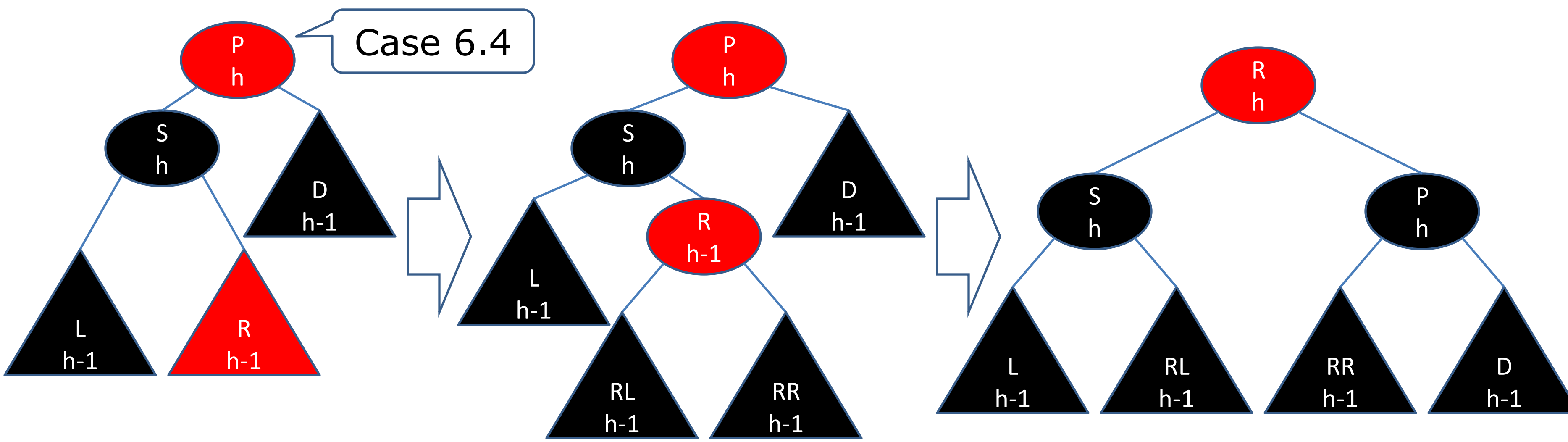
Remove and then rebalance
(continue…)





Remove and then rebalance
(continue…)





Remove and then rebalance
(continue…)

- In summary, the removing and then rebalancing process is recursive:
  - the step 1: given a black node **D** whose black depth has been decreased by one;
  - the step 2: if the node D is the root node of the whole tree, the process is finished;
  - the step 3: if D has parent, we get eighteen different nonconforming sub-trees to rebalance:
    - for the sub-trees 1.1, 2.1, we dye D’s sibling red and then the black depth of the parent of the node D is decreased by one. The parent of the node D becomes the black node D, we return to the step 1;

Remove and then rebalance (continue…)

- for the sub-trees 1.2, 1.3, 1.4, 2.2, 2.3, 2.4, we use the foregoing method to rebalance them to get conforming sub-trees, and then the process is finished;
- the sub-tree 3 can be transformed into the sub-tree 6;
- the sub-tree 4 can be transformed into the sub-tree 5;
- for the sub-trees 5.1, 6.1, we exchange the color of D's sibling and D's parent to finish the recursive process;
- for the sub-trees 5.2, 5.3, 5.4, 6.2, 6.3, 6.4, we use the foregoing method to rebalance them to get conforming sub-trees, and then the process is finished.

Remove and then rebalance (continue…)

- The base case is: we select a black node whose children are two NIL nodes and replace it with a NIL node. Then the NIL node is the black node **D**.

Code

- In this website https://github.com/cyril-gao/wheel/tree/master/Algorithms/BST, you can find C++ code and Python code which implement the algorithm.