Revisiting the Approximate Carathéodory Problem via the Frank-Wolfe Algorithm

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Joint work with Sebastian Pokutta

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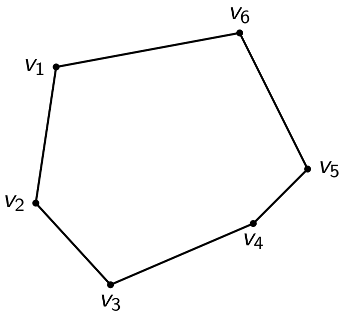

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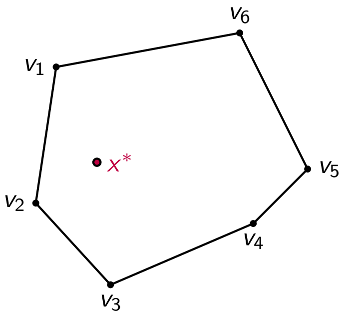

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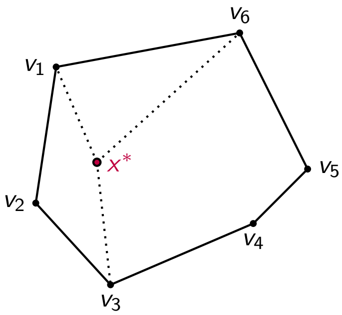

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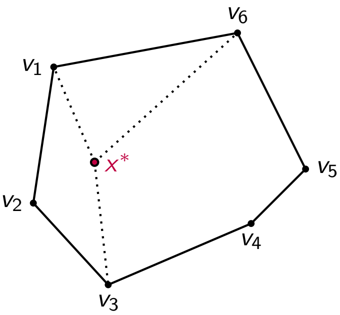

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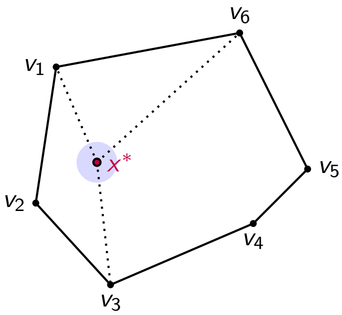

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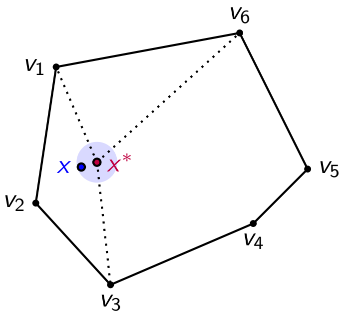

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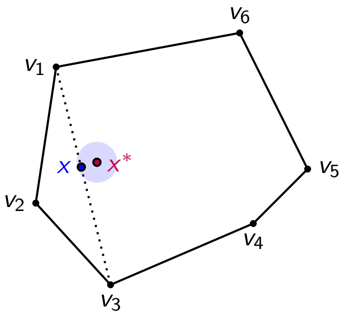

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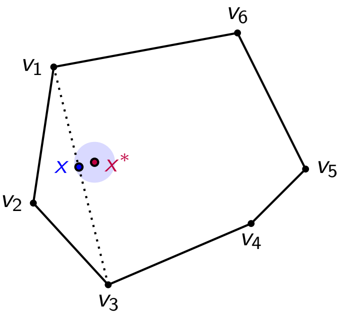
- Define the *sparsity* of x as the minimum number of vertices necessary to form x as a convex combination



The approximate Carathéodory problem

Problem

Find $x \in \mathrm{conv}(\mathcal{V})$ with high sparsity satisfying $\|x - x^*\|_p \leqslant \epsilon$.

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- Can we solve $\min\limits_{x \in \mathrm{conv}(\mathcal{V})} \|x - x^*\|_p$ by sequentially picking up vertices?

The Frank-Wolfe algorithm

Frank & Wolfe [1956], Levitin & Polyak [1966]

Algorithm Frank-Wolfe (FW)

$x_0 \in \mathcal{V}$
for $t = 0$ **to** $T - 1$ **do**
    $v_t \leftarrow \underset{v \in \mathcal{V}}{\arg\min} \langle \nabla f(x_t), v \rangle$
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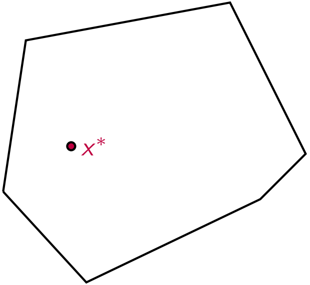

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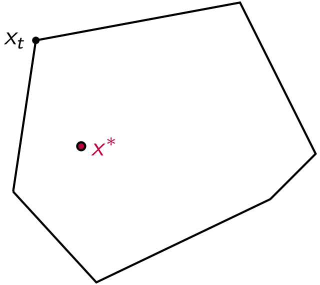

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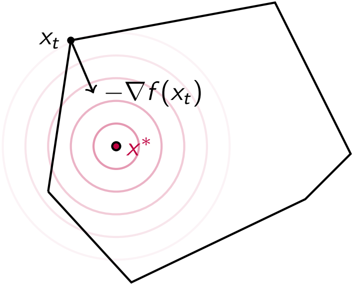

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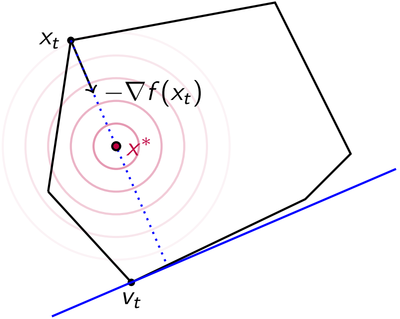

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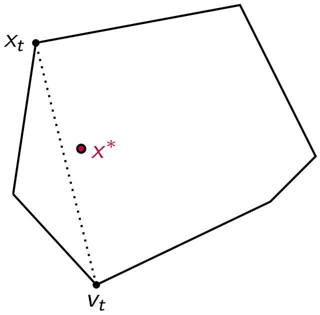

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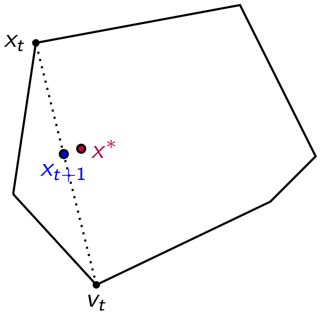

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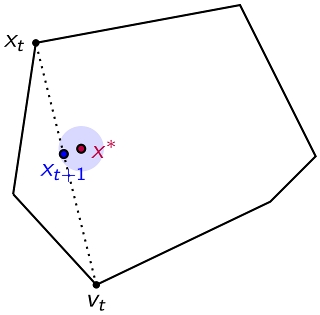

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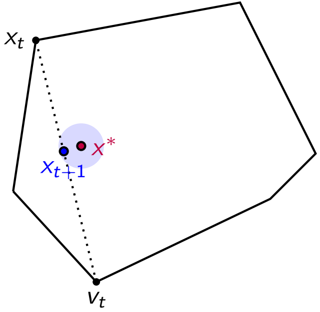

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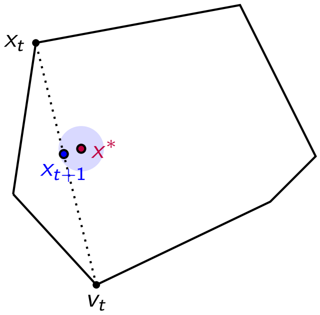

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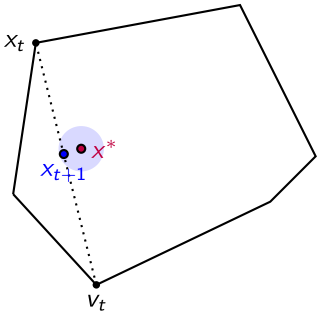

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- This is like a greedy method for the approximate Carathéodory problem!

The approximate Carathéodory problem via FW

- Apply FW to $f(x) = \|x - x^*\|_p^2$ and count the number of iterations T to achieve ϵ^2-convergence: $\|x_T - x^*\|_p^2 \leqslant \epsilon^2$

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- Replace strong convexity with a weaker condition: the PL inequality w.r.t. $\|\cdot\|$ [Polyak, 1963, Łojasiewicz, 1963]

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- For $p \geqslant 2$, $f(x) = \|x - x^*\|_p^2$ is $2(p-1)$-smooth and 2-PL w.r.t. $\|\cdot\|_p$

Sparsity bounds via FW convergence rates

Levitin & Polyak [1966], Guélat & Marcotte [1986], Jaggi [2013], Garber & Hazan [2015]

- $p \geqslant 2$
- $\mathcal{C} \subset \mathbb{R}^n$ be a compact convex set
- $\mathcal{V} \subseteq \partial\mathcal{C}$ be the compact set of interest (e.g., $\mathcal{C} = \mathrm{conv}(\mathcal{V})$)
- We want a sparse approximate convex decomposition of $x^* \in \mathrm{conv}(\mathcal{V})$

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Assumptions	**FW rate**	**Sparsity bound**
–	$\dfrac{4(p-1)D_p^2}{t+2}$	$\dfrac{4(p-1)D_p^2}{\epsilon^2} = \mathcal{O}\left(\dfrac{pD_p^2}{\epsilon^2}\right)$
$\mathcal{C}$ is S_p-strongly convex	$\dfrac{\max\{9(p-1)D_p^2, 1152(p-1)^2/S_p^2\}}{(t+2)^2}$	$\mathcal{O}\left(\dfrac{\sqrt{p}D_p + p/S_p}{\epsilon}\right)$
$x^* \in \mathrm{relint}_p(\mathcal{C})$ with radius r_p	$\left(1 - \dfrac{1}{p-1}\dfrac{r_p^2}{D_p^2}\right)^t \epsilon_0$	$\mathcal{O}\left(\dfrac{pD_p^2}{r_p^2}\ln\left(\dfrac{1}{\epsilon}\right)\right)$

Sparsity bounds via FW convergence rates

Levitin & Polyak [1966], Guélat & Marcotte [1986], Jaggi [2013], Garber & Hazan [2015]

- $p \geqslant 2$
- $\mathcal{C} \subset \mathbb{R}^n$ be a compact convex set
- $\mathcal{V} \subseteq \partial\mathcal{C}$ be the compact set of interest (e.g., $\mathcal{C} = \mathrm{conv}(\mathcal{V})$)
- We want a sparse approximate convex decomposition of $x^* \in \mathrm{conv}(\mathcal{V})$

Assumptions	**FW rate**	**Sparsity bound**
–	$\dfrac{4(p-1)D_p^2}{t+2}$	$\dfrac{4(p-1)D_p^2}{\epsilon^2} = \mathcal{O}\left(\dfrac{pD_p^2}{\epsilon^2}\right)$
$\mathcal{C}$ is S_p-strongly convex	$\dfrac{\max\{9(p-1)D_p^2, 1152(p-1)^2/S_p^2\}}{(t+2)^2}$	$\mathcal{O}\left(\dfrac{\sqrt{p}D_p + p/S_p}{\epsilon}\right)$
$x^* \in \mathrm{relint}_p(\mathcal{C})$ with radius r_p	$\left(1 - \dfrac{1}{p-1}\dfrac{r_p^2}{D_p^2}\right)^t \epsilon_0$	$\mathcal{O}\left(\dfrac{pD_p^2}{r_p^2}\ln\left(\dfrac{1}{\epsilon}\right)\right)$

- FW adapts to the geometry of the problem to yield higher sparsity

Fully-Corrective Frank-Wolfe algorithm

Algorithm Frank-Wolfe (FW)

$x_0 \in \mathcal{V}$

for $t = 0$ **to** $T - 1$ **do**

  $v_t \leftarrow \arg\min_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$

  $x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t)$

end for

Fully-Corrective Frank-Wolfe algorithm

Algorithm Frank-Wolfe (FW)

1: $x_0 \in \mathcal{V}$
2: **for** $t = 0$ **to** $T - 1$ **do**
3: $\quad v_t \leftarrow \arg\min_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$
4: $\quad x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t)$
5: **end for**

- Selected vertices in FW may be redundant, can we fix this?

Fully-Corrective Frank-Wolfe algorithm

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end for

- Selected vertices in FW may be redundant, can we fix this?

Algorithm Fully-Corrective Frank-Wolfe (FCFW)

$x_0 \in \mathcal{V}$
$\mathcal{S}_0 \leftarrow \{x_0\}$
for $t = 0$ **to** $T - 1$ **do**
  $v_t \leftarrow \arg\min_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$
  $\mathcal{S}_{t+1} \leftarrow \mathcal{S}_t \cup \{v_t\}$
  $x_{t+1} \leftarrow \arg\min_{\mathrm{conv}(\mathcal{S}_{t+1})} f$
end for

Fully-Corrective Frank-Wolfe algorithm

Algorithm Frank-Wolfe (FW)

1: $x_0 \in \mathcal{V}$
2: **for** $t = 0$ **to** $T - 1$ **do**
3: $\quad v_t \leftarrow \arg\min_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$
4: $\quad x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t)$
5: **end for**

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Algorithm Fully-Corrective Frank-Wolfe (FCFW)

1: $x_0 \in \mathcal{V}$
2: $\mathcal{S}_0 \leftarrow \{x_0\}$
3: **for** $t = 0$ **to** $T - 1$ **do**
4: $\quad v_t \leftarrow \arg\min_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$
5: $\quad \mathcal{S}_{t+1} \leftarrow \mathcal{S}_t \cup \{v_t\}$
6: $\quad x_{t+1} \leftarrow \arg\min_{\mathrm{conv}(\mathcal{S}_{t+1})} f$
7: **end for**

Fully-Corrective Frank-Wolfe algorithm

Algorithm Frank-Wolfe (FW)

$x_0 \in \mathcal{V}$
for $t = 0$ **to** $T - 1$ **do**
  $v_t \leftarrow \arg\min_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$
  $x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t)$
end for

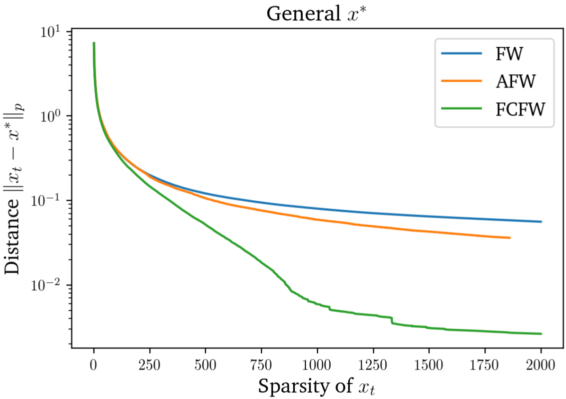
- Selected vertices in FW may be redundant, can we fix this?

Algorithm Fully-Corrective Frank-Wolfe (FCFW)

$x_0 \in \mathcal{V}$
$\mathcal{S}_0 \leftarrow \{x_0\}$
for $t = 0$ **to** $T - 1$ **do**
  $v_t \leftarrow \arg\min_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$
  $\mathcal{S}_{t+1} \leftarrow \mathcal{S}_t \cup \{v_t\}$
  $x_{t+1} \leftarrow \arg\min_{\mathrm{conv}(\mathcal{S}_{t+1})} f$
end for

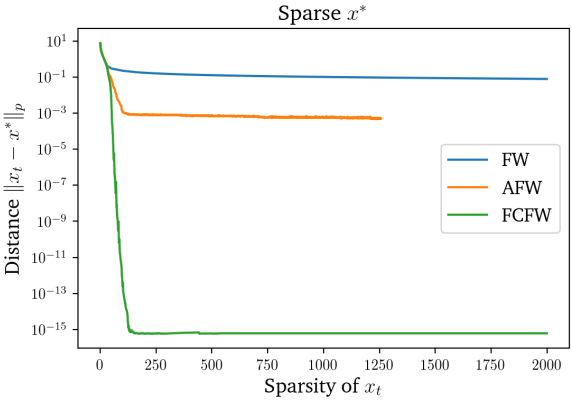
Fully-Corrective Frank-Wolfe algorithm

- We randomly generated 1000 vertices and $x^* \in \mathrm{conv}(\mathcal{V})$
- Here arbitrarily chose $p = 4$



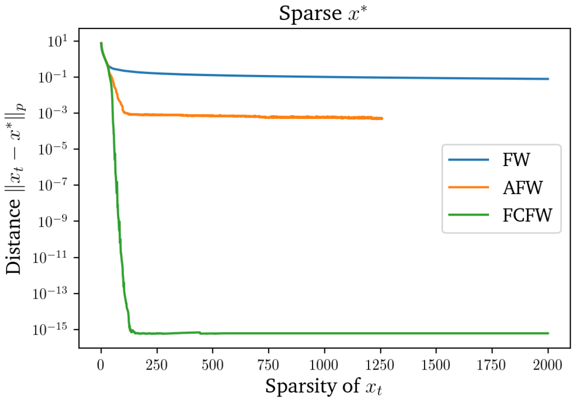
Fully-Corrective Frank-Wolfe algorithm

- Here x^* is generated by a convex combination of only 50 vertices



Fully-Corrective Frank-Wolfe algorithm

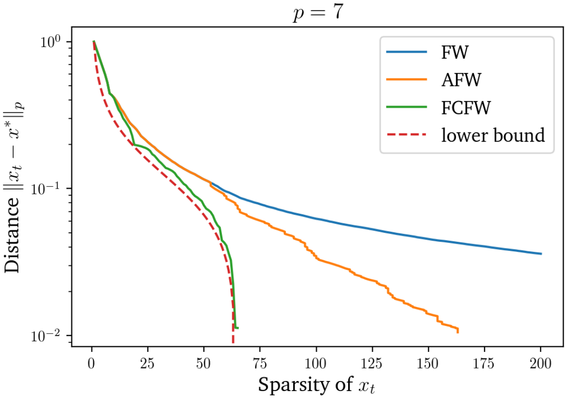
- Here x^* is generated by a convex combination of only 50 vertices



- FCFW obtains an *exact* convex decomposition of x^* once it picks up all its vertices

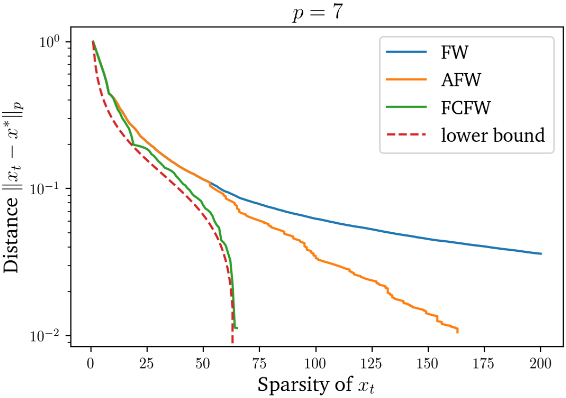
Fully-Corrective Frank-Wolfe algorithm

- FCFW matches the theoretical lower bound!



Fully-Corrective Frank-Wolfe algorithm

- FCFW matches the theoretical lower bound!



- Can we derive a precise convergence rate for FCFW?

Thank you!

https://arxiv.org/pdf/1911.04415.pdf

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