Review STT441

I. Basic Probability

1. Define Uncertainty

Experiment $\longrightarrow$ {outcomes} $\longrightarrow$ Event $(E \subset \Omega)$

$\uparrow$

Ω (Sample Space)

2. Define Probability $\rightarrow$ "P" $\rightarrow f$

Δ : The degree of uncertainty

(i) $0 \le P(A) \le 1$

(ii) $P(S) = 1$

(iii) $A, B,\ A \cap B = \phi \quad P(A \cap B) = 0.$

$P(A \cup B) = P(A) + P(B)$

$\boxed{\sigma\text{-algebra}}$

II. Random Variable (x, y) $\quad f(x)$

1. $X : \Omega \to \mathbb{R}$ map, function $\quad \omega \quad \Omega$

$X(\omega) = 1 \quad := X(\boxed{\cdot}) \rightarrow 1$

$\Uparrow$

$A : X \le 4$

$\omega_1 \quad \omega_2 \quad \cdots \omega_4$

$\boxed{\cdot} \downarrow 1 \quad \boxed{\cdot\cdot} \downarrow 2 \quad - \quad - \quad \boxed{\vdots\vdots} \downarrow 6$

$A :$

$$\boxed{P(X \in A) = P(X^{-1}(A)) = P(\{\underline{\omega} : X(\omega) \in A\})}$$

$\downarrow$

Codomain

$D \xrightarrow{f} R$

$f(d)$

$f : D \to R$

$f(D)$

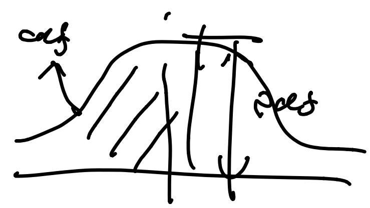
2 Properties

① Expectation: $E(X) = \begin{cases} \sum_{\Omega} x\, p(x) & \text{pmf} \\ \int_{\Omega} x f(x)\,dx & \text{pdf} \end{cases}$

② Variance:

$$E[(X-\mu)^2] = \sigma^2 = EX^2 - E^2X$$

$\mu = E(X)$



③ 2 functions: $\Rightarrow X$

a). cdf: (cumulative $\boxed{\text{distribution function}}$)

$X \Rightarrow F_X(x) = P(X \le x)$ $\quad F_X(4) = P(X \le 4)$

$X \rightarrow$ function $\quad x \in \{X\}$

b) pmf: $P(X=x) = F_X(x) - F_X(x-1)$

pdf: $f_X(x) = \frac{d}{dx} F_X(x)$

III. More about Random Variable

133 $\begin{cases} \text{Integer (上)} \\ \text{Series (下)} \end{cases}$

1. Moment generating function. (Tool)

$M_X(t) := E(e^{tx})$ $\qquad E(X^n)$: nth of X

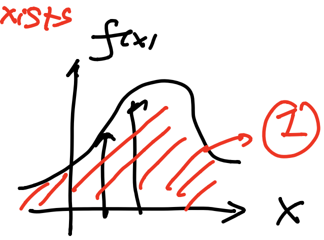
$\Rightarrow$ generate moments $\quad E(X) = M_X'(0)$

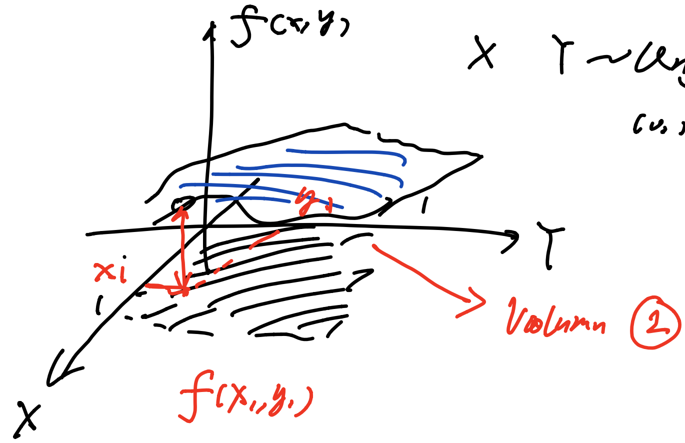
$\Rightarrow M_{X_n}(t) \rightarrow M_X(t) \Rightarrow F_{X_n}(t) = F_X(t)$ *

↳ MGF DN Always Exists

2. Joint Distribution

① X, Y.





X Y ~ Unifm (0,1)

$\boxed{234}$

② cdf

$$P(X \le x, Y \le y) = F_{X,Y}(x,y)$$

③ pdf

$$f(x,y) = \frac{d}{dx}\frac{d}{dy} F_{x,y}$$

$$P(a < X < b,\ c < Y < d) = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Marginal Density:

$$P(X \le x, \underline{Y \le y}) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(x,y)\,dx\,dy$$

$$P(X \le x) = P(X \le x, -\infty < Y < \infty)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{x} f(x,y)\,dx\,dy = F_X(x)$$

$$f_X(x) = P(X = x) = \frac{\partial}{\partial x} F_X(x)$$

3. Covariance

① $\begin{cases} Cov(X,Y) = E(X-\mu_X)(Y-\mu_Y) \\ Var(X) = E(X-\mu_X)^2 = E(X-\mu_X)(X-\mu_X) = Cov(X,X) \end{cases}$

② X, Y independent $\Rightarrow cov(X,Y) = 0$

$\not\Leftarrow$

$$Var(X+Y) = \boxed{Var(X) + Var(Y) + 2Cov(X,Y)}$$

441 $\quad X, \; Y$

$$\begin{pmatrix} X\,X & Y\,Y \\ X\,Y & Y\,X \end{pmatrix}$$

$cov(Y,X) = cov(X,Y)$

Probability Concept (Zero)

Random Varible $\begin{cases} \text{Expec} \\ \text{Varian} \end{cases} \Rightarrow 133 \begin{cases} \text{Int} \\ \text{Series} \end{cases}$

Joint Dist $\begin{cases} \text{Expec} \\ \text{Covariance} \end{cases} \Rightarrow 234 \Rightarrow \text{Int}$

Functn

$\downarrow$

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Limit Theorem

I. Two Inequality $(\mu, \sigma^2) \Rightarrow \boxed{P(X \le a)}$

$E(X) = \int x f(x)\,dx$

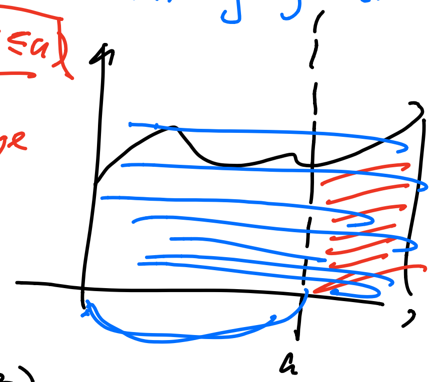
Range

1. Markov's Ineq.

If $X \ge 0$, $P(X \ge a) \le \frac{E(X)}{a}$

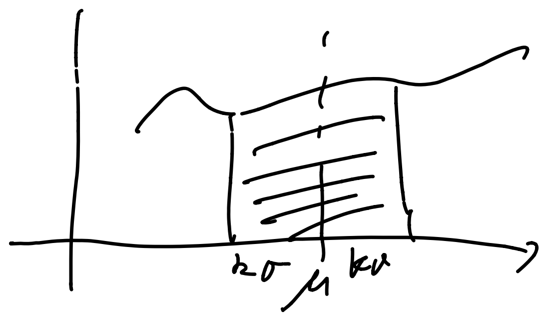
Bernolli Ber(p)

Pf.



2. Chebyshev's Inequality (Robust)

$$P(|X - \mu_X| \ge k) \le \frac{\sigma^2}{k^2} \iff P\left(\left|\frac{X-\mu}{\sigma}\right| < k\right) \ge \frac{1}{k^2}$$



II. Two Stochastic Convergence

1. Deterministic Convergence

① Convergence of Sequence (295)

$X_n : \frac{1}{1}, \frac{1}{2}, \cdots, \frac{1}{n} \to 0$

Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$, $\forall n > N$ $|x_n - 0| < \varepsilon$.

② Convergence of functions (320)

$f_1 : \Omega \to \mathbb{R}$ as $n \to \infty$, $f_n \to \infty$.

$f_2 : \Omega \to \mathbb{R}$ $|f_n(x) - f(x)| < \varepsilon$ $\boxed{\forall x \in \Omega}$

$\vdots$

$f_n : \Omega \to \mathbb{R}$

$\vdots$ ∞

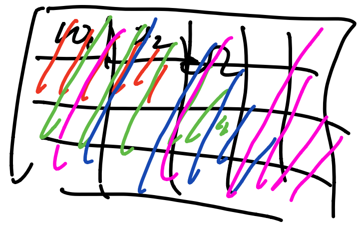
2. 2 Sto Converg. ω: <u>scenario</u>

$X: \Omega \to \mathbb{R}$

$|X_n(\omega) - \underline{X(\omega)}| < \varepsilon$, $\forall \omega \in \Omega$

(so)

Ω



① Convergence in Probability.

$A: \{\omega : X_n(\omega) \to X(\omega)\}$

as $n \to \infty$. $P(A) = 1$

$P(|X_n - X| > \varepsilon) = 0 \Longleftrightarrow P(|X_n - X| \le \varepsilon) = 1$ as $n \to \infty$

$X_n \xrightarrow{P} X$

② Convergence in Distribution (MGF)

$X_n \xrightarrow{D} X$ if $F_{X_n}(t) \to F_X(t)$ $\forall t$

$\Longleftarrow$ $M_{X_n}(t) \to M_X(t)$

a) $Bin(n, p) \longrightarrow Poisson(\lambda)$ $\lambda = np$ as $n \to \infty$.

b) $Poisson(\lambda) \longrightarrow Normal(\mu, \sigma)$ as $\lambda \to \infty$.

a): $(1 - p_n + p_n e^t)^n \longrightarrow \exp(\lambda(e^t - 1))$

$p_n = \frac{\lambda}{n}$

III. Two Limit Theorem.

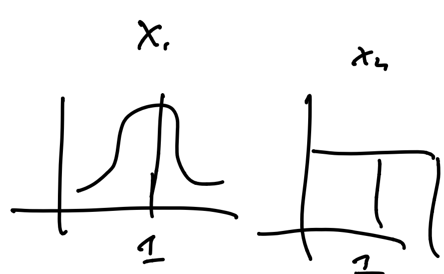
1. LLN (Convergen in Probability) $\Rightarrow$ <u>Monte Carlo Method</u>

$X_1, X_2, \dots, X_n$ independent

$E(X_i) = \mu$ $Var(X_i) = \sigma^2$

$\bar{X}_n = \frac{\sum X_i}{n}$

$\bar{X}_n \xrightarrow{P} \mu$ as $n \to \infty$



2. CLT (Convergen in Distribution)

$X_1, X_2, \cdots, X_n, \cdots \overset{iid}{\sim} X$ · $E(X_i) = \mu$ $Var(X_i) = \sigma^2$

$S_n = \sum X_i$

$P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x\right) \longrightarrow \Phi(x)$ → Normal Distn cdf

$N(\mu, \sigma^2)$

$\Updownarrow$

$P(\bar{X} \le x) \longrightarrow \Phi'(x) \sim N(\mu, \frac{\sigma^2}{n})$

$\bar{X} \xrightarrow{D} N(\mu, \frac{\sigma^2}{n})$ $\frac{\sigma^2}{n}$

$X \sim$ Unif

$\sim$ Bin

$n \to \infty$, $\frac{\sum X_i}{n} \sim N(\mu, \frac{\sigma^2}{n})$