1. Ce.: $X_1, \cdots X_n \cdots$ independent $E(X_i) = \mu_i$ ($Var(X_i) = \sigma_i^2$)

show if $\frac{\sum \sigma_i^2}{n^2} \to 0$, $\boxed{\frac{\sum \mu_i}{n} \to \mu}$

then $\bar{X} \xrightarrow{p} \mu$.

Review: ① LLN: X_i $E(X_i) = \mu$ $Var(X_i) = \sigma^2$ $\bar{X_n} \xrightarrow{P} \mu$

$X = \mu$ $P(X=\mu)=1$

② 2 Inequality: $\begin{cases} ① & X \geq 0,\ P(X > a) \leq \frac{E(X)}{a} \\ ② & P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}. \end{cases}$

Even unknown distribution

Known μ, σ^2

Bound

$$P(|\bar{X}_n - \mu| > \varepsilon) \longrightarrow 0 \text{ as } n \to \infty$$

$\mu_{\bar{X}_n} = E(\bar{X}_n)$

$$P(|\bar{X}_n - E(\bar{X}_n)| > \varepsilon) \leq \frac{Var(\bar{X}_n)}{\varepsilon^2} \quad \text{as } n \to \infty$$

$$E(\bar{X}_n) = E\left(\frac{\sum X_i}{n}\right) \to \mu$$

$$Var(\bar{X}_n) = Var\left(\frac{\sum X_i}{n}\right) = \frac{1}{n^2} \sum Var(X_i) = \frac{\sum \sigma_i^2}{n^2} \to 0$$

$$P(|\bar{X}_n - E(\bar{X}_n)| > \varepsilon) \leq \frac{Var(\bar{X}_n)}{\varepsilon^2} \quad \text{as } n \to \infty$$

$$\Leftrightarrow 0 \leq P(|\bar{X}_n - \mu| > \varepsilon) \leq \frac{0}{\varepsilon^2}$$

$$\Leftrightarrow P(|\bar{X}_n - \mu| > \varepsilon) \longrightarrow 0 \text{ as } n \to \infty$$

2. (a) $\int_{-2}^{2} e^{-\frac{x^2}{2}} dx$

LLN: X_i $\quad \bar{X}_n \longrightarrow E(X)$

Let $g(X) = e^{-\frac{x^2}{2}}$ $\quad E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x) dx$

$= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} f(x) dx$

$f(x) = \frac{1}{b-a} = \frac{1}{4}$

X : Uniform$(-2, 2)$ $\rightsquigarrow$ n $\quad \frac{\sum g(X_i)}{n} \approx E(g(X))$

$Y = g(X) \to Y_i \longrightarrow E(Y) = E(g(X))$

$Y_i = g(X_i)$

$\int_{-2}^{2} e^{-\frac{x^2}{2}} \cdot \frac{1}{4} dx = \frac{\sum g(X_i)}{n}$ $\quad Y_i = g(X_i)$

$\Rightarrow \boxed{\int_{-2}^{2} e^{-\frac{x^2}{2}} dx} = 4 \cdot \frac{\sum g(X_i)}{n} = \boxed{4 \bar{Y}_n}$

(b) $Var(4\bar{Y}_n) = 16\, Var(\bar{Y}_n)$

$= 16\, Var\left(\frac{\sum Y_i}{n}\right)$

$= 16 \frac{1}{n^2} \cdot Var(\sum Y_i) = 16 \frac{1}{n^2} \cdot n\sigma^2 = 16 \cdot \frac{\sigma^2}{n}$

$= 16 \frac{\sigma^2}{n} \approx \boxed{16 \frac{S^2}{n}}$

$\sigma^2 = Var(Y_i)$ $\quad S^2_{Y_n}$

3. $X \geq 0$ $\quad$ $X_i \sim Poisson(\lambda)$ $\lambda = 1$

a). $P(X > a) \leq \frac{E(X)}{a}$ $\quad$ $E(X_i) = \lambda = 1$

$Y = \sum_{i=1}^{20} X_i$ $\quad$ $P(Y > 15) \leq \frac{E(Y)}{15} = \frac{20}{15}$

$E(Y) = E(\sum^{20} X_i) = 20$

b).

$\frac{S_n - n\mu}{\sigma\sqrt{n}} \sim N(0,1)$ $\quad$ $E(X_i) = \mu$ $\quad$ $\sigma^2 = Var(X_i)$

$P\left\{ \frac{S_n - n\mu}{\sigma\sqrt{n}} \leq a \right\} = \boxed{\Phi(a)}$

$X_i \overset{iid}{\sim} Poisson(1)$

$E(X_i) = \lambda = 1$

$P(S_{20} > 15) = 1 - P(S_{20} \leq 15)$ $\quad$ $Var(X_i) = \lambda = 1 = \sigma$

$= 1 - P\left(\frac{S_{20} - n\mu}{\sigma\sqrt{n}} \leq \frac{15 - n\mu}{\sigma\sqrt{n}} \right)$

$= 1 - \Phi\left(\frac{15 - n\mu}{\sigma\sqrt{n}} \right)$

μ $\quad$ $= 1 - \Phi\left(\frac{15 - 20 \cdot 1}{1 \cdot \sqrt{20}} \right)$

$= 1 - \Phi\left(\frac{-5}{\sqrt{20}} \right)$

5. $\begin{cases} Bin(n,p) \xrightarrow{n \to \infty} Poisson(\lambda) & \lambda = np. \\ Poisson \xrightarrow{\lambda \to \infty} N(\mu, \sigma^2) & \begin{cases} \mu = \lambda \\ \sigma^2 = \lambda \end{cases} \end{cases}$

$N \sim Poisson(10500) \sim N(10500, 10500)$ $\qquad \frac{X-\mu}{\sigma} \sim N(0,1)$

$P(N > 10{,}200) = 1 - P(N \leq 10{,}200)$

$= 1 - P\left(\frac{N - 10500}{\sqrt{10500}} \leq \frac{10200 - 10500}{\sqrt{10500}} \right)$

$\approx 1 - \Phi\left(\frac{10200 - 10500}{\sqrt{10500}} \right)$

4.

(ii) $X_n \xrightarrow{P} X \quad Y_n \xrightarrow{P} Y. \Rightarrow X_n + Y_n \xrightarrow{P} X+Y$

$$|(X_n+Y_n)-(X+Y)| \overset{(1)}{=} |(X_n-X)+(Y_n-Y)| \overset{(2)}{\le} |X_n-X| + |Y_n-Y|$$

$\frac{\varepsilon}{2} \quad \frac{\varepsilon}{2}$

$\boxed{P(|(X_n+Y_n)-(X+Y)|>\varepsilon)} \to 0$ as $n\to\infty$

$|(X_n+Y_n)-(X+Y)|>\varepsilon \qquad \#(A\cup B) = \#A + \#B - \#(A\cap B)$

$P(A\cup B) \le P(A)+P(B)$

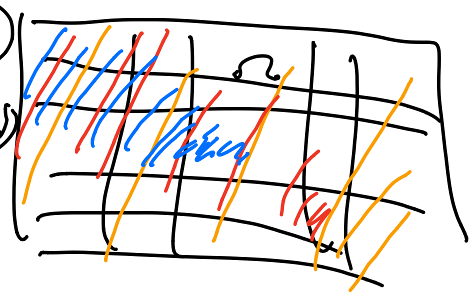
$$P(\underbrace{|(X_n+Y_n)-(X+Y)|>\varepsilon}\,)$$

$$= P(\{\omega : |(X_n(\omega)+Y_n(\omega))-(X(\omega)+Y(\omega))| > \varepsilon\})$$

$$\le P(\{|X_n-X| \ge \tfrac{\varepsilon}{2}\} \cup \{|Y_n-Y| \ge \tfrac{\varepsilon}{2}\})$$

$$\le P(\{|X_n-X| \ge \tfrac{\varepsilon}{2}\}) + P(\{|Y_n-Y| \ge \tfrac{\varepsilon}{2}\})$$

$$\to 0+0 = 0$$



(i) $X_n \xrightarrow{P} X \Rightarrow g(X_n) \xrightarrow{P} g(X) \qquad g$ is continous

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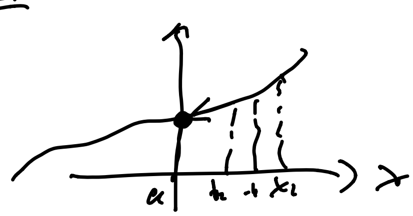
$P(|g(X_n)-g(X)|>\varepsilon) \to 0$



$\omega \in \Omega_X$

$\{\omega : |g(X_n(\omega))-g(X(\omega))| > \varepsilon\}$

$\lim (g(x_n)) \to g(a)$

$\boxed{\varepsilon}, \delta \qquad (a-\delta, a+\delta)$

$\forall x \in B_\delta(a)$

$|g(x)-g(a)| < \varepsilon.$

$A = B_\delta = \{x \in \Omega : \exists y \in \Omega,\ |x-y| < \delta,\ |g(x)-g(y)| > \varepsilon\}$.

$B \qquad A \cap B$

$\{X \notin B_\delta\} \cap \{|g(x)-g(y)| > \varepsilon\} \subset \{|X_n - X| > \delta\}$

$A^c \cap B \subset C \Rightarrow B \subset A \cup C$

$P(A \cup C)$

$x_n \xrightarrow{P} x$

$P(\{|g(x)-g(y)| > \varepsilon\})$

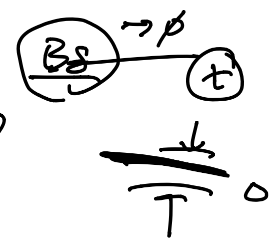
$\le \underbrace{P(B_\delta)}_{\downarrow 0} + \underbrace{P(|X_n - X| > \delta)}_{0}$

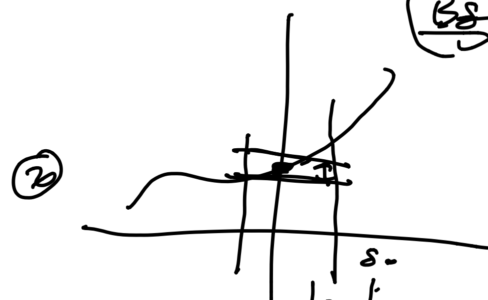
$\delta \to 0$

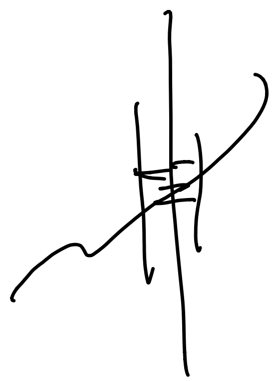
$n \to \infty$

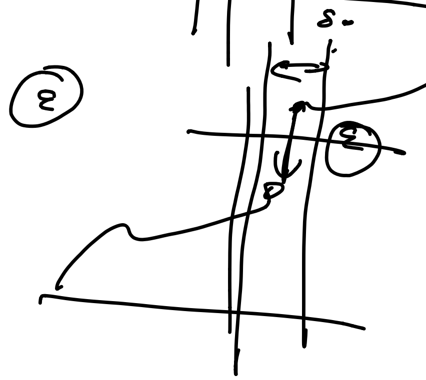
$\to 0$

$B_\delta \to \emptyset$









Given $\varepsilon > 0$, we can find a δ, s.t

$$B_\delta = \{ x \in \Omega : \exists y \in \Omega,\ |x-y| < \delta,\ |f(x) - f(y)| > \varepsilon \}.$$

$$B_\delta^c \cap \{ x : |g(X_n) - g(X)| > \varepsilon \} \subset \{ x : |X_n - X| \geq \delta \}.$$

$$\{ x : |g(X_n) - g(X)| > \varepsilon \} \subset \left(B_\delta \cup \{ x : |X_n - X| \geq \delta \} \right)$$

$$P(|g(X_n) - g(X)| > \varepsilon) \leq P(B_\delta) + P(|X_n - X| \geq \delta)$$

$$n \to \infty \qquad \to 0 + 0 = 0$$

$$\Rightarrow P(|g(X_n) - g(X)| > \varepsilon) \to 0$$

$$\Rightarrow g(X_n) \xrightarrow{P} g(X)$$