DSADM: specification of $\varepsilon_\rho, \varepsilon_\nu$ from π_ρ, π_ν

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The transformation of the Gaussian pre-secondary fields θ_2^* and θ_3^* (denoted here generically by ψ^*) to the secondary fields ρ, ν (denoted generically by ψ), respectively, is performed pointwise as follows:

$$\psi(t,s) = \bar\psi \cdot \left[(1+\varepsilon_\psi)\cdot g(\psi^*(t,s)) - \varepsilon_\psi\right], \tag{1}$$

where we require that $\bar\psi > 0$ and

$$g(z) := (1+\mathrm{e}^b)\frac{\mathrm{e}^{z-b}}{1+\mathrm{e}^{z-b}} \equiv \frac{1+\mathrm{e}^b}{1+\mathrm{e}^{b-z}} \tag{2}$$

is the scaled and shifted logistic function and b is the constant assumed to be always equal to 1.

The probability of occurrence of *negative* field values $\psi(t,s) < 0$ is quantified by the external parameter π_ψ:

$$P(\psi(t,s) < 0) = \pi_\psi. \tag{3}$$

Substituting ψ from Eq.(1) to Eq.(3) and utilizing the monotonicity of the transformation function g (see Eq.(2)) and the Gaussianity of the field ψ^*, we easily come up with a relation between π_ψ and ε_ψ. Specifically,

$$\begin{aligned} P(\psi(t,s) < 0) = P\left((1+\varepsilon_\psi)\cdot g(\psi^*(t,s)) - \varepsilon_\psi < 0\right) = \\ P\left(g(\psi^*(t,s)) < \frac{\varepsilon_\psi}{1+\varepsilon_\psi}\right) = P\left(\psi^*(t,s) < g^{-1}(\frac{\varepsilon_\psi}{1+\varepsilon_\psi})\right) \end{aligned} \tag{4}$$

Here $g^{-1}(w)$ is the inverse g function, which is easily derived by solving Eq.(2) w.r.t. z:

$$g^{-1}(w) = b - \log\left(\frac{1+\mathrm{e}^b}{w} - 1\right). \tag{5}$$

As a result, since $\psi^*(t,s) \sim \mathcal{N}(\text{mean} = 0, \text{SD} = \log\kappa_\psi) \equiv \log\kappa_\psi \cdot \mathcal{N}(0,1)$, we finally obtain

$$\pi_\psi = P\left(\mathcal{N}(0,1) < g^{-1}(\frac{\varepsilon_\psi}{1+\varepsilon_\psi})/\log\kappa_\psi\right) \equiv \Phi\left(g^{-1}(\frac{\varepsilon_\psi}{1+\varepsilon_\psi})/\log\kappa_\psi\right), \tag{6}$$

where Φ is the cumulative distribution function of the standard Gaussian random variable.

To retrieve ε_ψ from this equation, apply the standard normal *quantile function* (inverse cumulative distribution function) $Q(x)$ to both sides of Eq.(6), getting

$$Q(\pi_\psi) = g^{-1}(\frac{\varepsilon_\psi}{1+\varepsilon_\psi})/\log \kappa_\psi \tag{7}$$

$$Q(\pi_\psi)\ \log \kappa_\psi = g^{-1}(\frac{\varepsilon_\psi}{1+\varepsilon_\psi}) \tag{8}$$

$$G \equiv g[Q(\pi_\psi)\ \log \kappa_\psi] = \frac{\varepsilon_\psi}{1+\varepsilon_\psi} \tag{9}$$

$$\boxed{\varepsilon_\psi = \frac{G}{1-G} \equiv \frac{g[Q(\pi_\psi)\ \log \kappa_\psi]}{1-g[Q(\pi_\psi)\ \log \kappa_\psi]}}. \tag{10}$$