DSADM: specification of initial conditions

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1 Introduction

The goal is specify initial conditions in such a way that the solution be nearly in the quasi-stationary regime right from the beginning of the time integration.

With any of the four (stationary) **secondary** fields $\theta = U, \rho, \nu, \sigma$, we generate the initial field by drawing a pseudo-random sample from the spatial distribution determined by its stationary spectrum b_m^θ.

With the non-stationary random field in question ξ, the approach here is to take the unperturbed (i.e. stationary) DSADM and generate the initial field by drawing a pseudo-random sample from the spatial distribution determined by its stationary spectrum b_m^ξ.

2 Fourier transform

The spatial coordinate x is measured in meters, not in radians, so that the spectral-space basis functions are $\Psi_m(x) := \mathrm{e}^{\mathrm{i}mx/R}$, R is the radius of the circle $\mathbb{S}^1(R)$, their norms are $\sqrt{2\pi R}$ and the Fourier transform pair is

$$f(x) = \sum_{m=-\infty}^{\infty} \tilde{f}_m \, \mathrm{e}^{\mathrm{i}mx/R} \tag{1}$$

$$\tilde{f}_m = \frac{(f, \Psi_m)}{\|\Psi_m\|^2} = \frac{1}{2\pi R} \int_0^{2\pi R} f(x) \, \mathrm{e}^{-\mathrm{i}mx/R} \mathrm{d}x = \frac{1}{2\pi} \int_0^{2\pi} f(\varphi R) \, \mathrm{e}^{-\mathrm{i}m\varphi} \mathrm{d}\varphi. \tag{2}$$

For the space discrete representation of the function $f(x)$ on the regular grid with $n_{grid} \equiv n$ points, the maximal wavenumber is $m_{max} = 2\pi R/h = n_{grid}/2$, where $h = 2\pi R/n_{grid}$ is the grid spacing (mesh size), so that Eq.(1) rewrites as

$$f(x) = \sum_{m=-m_{max}+1}^{m_{max}} \tilde{f}_m \, \mathrm{e}^{\mathrm{i}mx/R}. \tag{3}$$

From Eq.(2), the discrete forward FFT on the circle of radius R is the standard discrete FFT on the unit circle applied to the function defined on the unit circle: $f_R(\varphi) := f(\varphi R)$, where $0 \le \varphi \le 2\pi$. Technically, we define f on the regular grid: $\{f_j\}_{j=1}^n$ on $\mathbb{S}^1(R)$ and then simply apply the DFFT for the vector $\mathbf{f}$ with the entries f_j.

3 The general stationary model

$$\frac{\partial \xi}{\partial t} + U\frac{\partial \xi}{\partial s} + \rho\xi - \nu\frac{\partial^2 \xi}{\partial s^2} = \sigma\alpha, \tag{4}$$

4 Generation of the initial field

4.1 Theory

We expand the initial field $\xi^0(s) \equiv z(s)$ in the Fourier series

$$z(s) = \sum_{m=-n/2}^{n/2} \tilde{z}_m \, \mathrm{e}^{\mathrm{i}ms/R} \tag{5}$$

and require that the

$$\mathsf{Var}\,\tilde{z}_m = b_m, \tag{6}$$

implying that the probability distribution of the initial field equals the stationary probability distribution of ξ. The stationary spectral variances b_m are

$$b_m = \frac{a^2\sigma^2}{2} \cdot \frac{1}{\rho + \frac{\nu}{R^2}m^2} = \frac{a^2\sigma^2}{2\rho}\frac{1}{1+(\frac{Lm}{R})^2}, \tag{7}$$

where $a = 1/\sqrt{2\pi R}$.

4.2 Implementation

4.2.1 Discrete Fourier transform

First, we note that if the physical-space function $z(s)$ in Eq.(5) is defined on a regular *grid* (i.e. as the vector $\{z(s_j) \equiv z_j\}_{j=1}^n$), then we have to expand it in a bit different set of basis functions. Specifically, in the set of tentative basis functions $\Psi_m(.)$ defined as

$$\Psi_m(s_j) = \mathrm{e}^{\mathrm{i}ms_j/R} = \mathrm{e}^{\mathrm{i}m\frac{2\pi}{n}j} \tag{8}$$

with m ranging from $m = -n/2$ to $m = n/2$ does *not* constitute a basis because $\Psi_{-n/2}(.)$ and $\Psi_{+n/2}(.)$, being evaluated on the grid (with $s_j = (j-1)h$, where $h = 2\pi R/n$ and $j = 1, \dots, n$), coincide. Indeed, $\Psi_{\pm n/2}(s_j) = \mathrm{e}^{\pm\mathrm{i}\frac{n}{2}\frac{2\pi}{n}j} = \mathrm{e}^{\pm\mathrm{i}\pi j} = (-1)^j$. Therefore the set of the basis functions on a grid may contain either $\Psi_{-n/2}(.)$ or $\Psi_{+n/2}(.)$. We arbitrarily decide to retain $\Psi_{+n/2}(.)$, which implies that the gridded random field spectral expansion (the inverse discrete Fourier transform) becomes

$$z_j = \sum_{m=-n/2+1}^{n/2} \tilde{z}_m \cdot \mathrm{e}^{\mathrm{i}m\frac{2\pi}{n}j}. \tag{9}$$

Note that the number of the spectral-space coefficients $\tilde{z}_m$ is in Eq.(9) equal to n (the number of grid points).

The respective forward discrete Fourier transform is

$$\tilde{z}_m = \frac{1}{n}\sum_{j=1}^{n} z_j \cdot \mathrm{e}^{-\mathrm{i}m\frac{2\pi}{n}j}. \tag{10}$$

4.2.2 Spectral-space simulation

Note that $\tilde{z}_m = v_m + \mathrm{i}w_m$ are *complex* numbers except z_0 and $z_{n/2}$ (which are real: $z_0 = v_0$ and $z_{n/2} = v_{n/2}$).

We simulate (note that all standard Gaussian random variables $\mathcal{N}(0,1)$ below are mutually independent)

1. $z_0 \sim \mathcal{N}(0,1) \cdot \sqrt{b_0}$
2. $z_{n/2} \sim \mathcal{N}(0,1) \cdot \sqrt{b_{n/2}}$
3. For $m = 1, \dots, \frac{n}{2} - 1$:
   $v_m \sim \mathcal{N}(0,1) \cdot \sqrt{b_m/2}$
   $w_m \sim \mathcal{N}(0,1) \cdot \sqrt{b_m/2}$
   $\tilde{z}_m = v_m + \mathrm{i}w_m$
   $z_{-m} = \overline{\tilde{z}_m}.$

And then apply the backward DFFT (following Eq.(A12) to the vector $\mathbf{z}$ arranged as in Eq.(A13).

Appendices

A DFFT in R

`fft` is in the package `stats`.

A.1 DFFT: general

Perform the forward DFFT (discrete FFT): given the "physical-space" vector $\mathbf{f}$ of length n, its **forward** DFFT is computed as follows:

$$\boxed{\tilde{f} < -\mathtt{fft}(f, inverse = FALSE)/n} \tag{A11}$$

The **backward** DFFT is

$$\boxed{f \leftarrow \mathtt{fft}(\tilde{f}, inverse = TRUE)} \tag{A12}$$

(NB: no division by n).

Note that the forward DFFT is defined with $\mathrm{e}^{-\mathrm{i}kx}$ and the division by 2π, whereas the backward DFFT is defined with $\mathrm{e}^{+\mathrm{i}kx}$.

A.2 Ordering of the physical-space and spectral-space coefficients

The complex spectral coefficients produced by `fft` are ordered as follows:

$$\boxed{\tilde{\mathbf{f}} := \tilde{f}_0,\ \tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \dots, \tilde{f}_{n/2-1},\ \tilde{f}_{n/2},\ \tilde{f}_{-n/2+1}, \dots, \tilde{f}_{-3}, \tilde{f}_{-2}, \tilde{f}_{-1}}, \tag{A13}$$

that is, from wvn=0 go to the right up to $n_{max} := n_{grid}/2 \equiv n/2$, then jump to the very left to (-nmax+1) (but not to -nmax!) and then go up till n=-1. Thus, all wavenumbers are counted only once.

Note that this order is exactly the whole circle of wavenumbers:

$$m = 0, 1, 2, \ldots, \frac{n}{2}, \frac{n}{2}+1, \frac{n}{2}+2, \ldots, n-1$$

because

$$\tilde{f}_m = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{e}^{-\mathrm{i}mx}\mathrm{d}x \leftarrow \frac{1}{2\pi}\frac{2\pi}{n}\sum_{j=0}^{n-1} f_j \mathrm{e}^{-\mathrm{i}\frac{2\pi}{n}mj} = \frac{1}{n}\sum_{j=0}^{n-1} f_j \mathrm{e}^{-\mathrm{i}\frac{2\pi}{n}mj}$$

so that, say, for m=n/2+1,

$$\tilde{f}_{n/2+1} = \frac{1}{n}\sum_{j=0}^{n-1} f_j \mathrm{e}^{-\mathrm{i}\frac{2\pi}{n}(\frac{n}{2}+1)j} = \frac{1}{n}\sum_{j=0}^{n-1} f_j \mathrm{e}^{-\mathrm{i}\frac{2\pi}{n}(-\frac{n}{2}+1)j}$$

because the arguments in the two latter e differ by $2\pi j$ (which yields no difference in the value of the complex exponent).

So, in the DFFT, both f_j and $\tilde{f}_k$ can be assumed to be running from 0 to $n-1$. Equivalently, the wavenumber can be assumed to run from 0 to $n/2$ and then from $-n/2+1$ back to 1.

A.3 Ordering if $f(x)$ is real

If in physical space, if $\mathbf{f}$ is REAL, then its DFFT is

$$\boxed{\tilde{\mathbf{f}} := \tilde{f}_0,\ \tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \ldots, \tilde{f}_{n/2-1},\ \tilde{f}_{n/2},\ \tilde{f}^*_{n/2-1}, \ldots, \tilde{f}^*_3, \tilde{f}^*_2, \tilde{f}^*_1}, \tag{A14}$$

where * denotes complex conjugation. So, except for $\tilde{f}_0$ and $\tilde{f}_{n/2}$, all other $\tilde{f}_m$ are repeated (up to conjugation) twice and the length of the (complex) data vector is equal to n.

Thus, if f is real, then $\tilde{f}$ is *symmetric* (i.e. complex-conjugate-even).

Similarly, if $\tilde{f}$ is real, then f is *symmetric*, and vice versa.

Note also that if $f(x)$ is real, then both $\tilde{f}_0$ and $\tilde{f}_{n/2}$ are real.