

# DSADM: Stationarity of random space-time covariances

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## 1 Introduction

Here we provide a sketch of the proof of the statement that the space-time covariances  $\gamma(t, t', s, s')$  are stationary in space-time random processes.

## 2 Recursive equations for the field covariances

Recall the equations for the simultaneous time-discrete field covariances  $\mathbf{\Gamma}_k$ ,

$$\mathbf{\Gamma}_k = \mathbf{F}(\underline{\boldsymbol{\theta}}_k) \mathbf{\Gamma}_{k-1} \mathbf{F}(\underline{\boldsymbol{\theta}}_k)^\top + \mathbf{Q}(\underline{\boldsymbol{\theta}}_k), \quad (1)$$

and the lagged field covariances  $\mathbf{\Gamma}_{kj}$ ,

$$\mathbf{\Gamma}_{kj} = \mathbf{F}(\underline{\boldsymbol{\theta}}_k) \mathbf{\Gamma}_{k-1,j}. \quad (2)$$

## 3 Explicit equations for the field covariances

### 3.1 Computation

Equations (1) and (2) allow us to express all covariances  $\mathbf{\Gamma}_k \equiv \mathbf{\Gamma}_{kk}$  and  $\mathbf{\Gamma}_{kj}$  (for  $j < k$ ) from  $\mathbf{\Gamma}_0$  and  $\mathbf{Q}(\underline{\boldsymbol{\theta}}_{1:k})$  as follows.

Let's prove this by induction. This is, certainly, true for  $k = 1$ . Assume this is true for some  $k$  and prove that this holds also for  $k + 1$ . Indeed, knowing  $\mathbf{\Gamma}_k$ , we compute  $\mathbf{\Gamma}_{k+1,k+1}$  from Eq.(1). Then, for any  $j < k + 1$ , we compute  $\mathbf{\Gamma}_{k+1,j}$  using Eq.(2):

$$\mathbf{\Gamma}_{k+1,j} = \mathbf{F}(\underline{\boldsymbol{\theta}}_{k+1}) \mathbf{\Gamma}_{kj}. \quad (3)$$

This completes the proof.

### 3.2 Explicit equations

Take the simultaneous covariances  $\mathbf{\Gamma}_k$  and  $k$  large enough, apply Eq.(1) recursively, and use the fact that  $\mathbf{F}$  is a contraction (in the sense that  $\|\mathbf{F}_k\| \leq \mu < 1$ , where  $\|\cdot\|$  is the matrix norm induced by the *maximal* vector norm), so that the series

$$\mathbf{\Gamma}_k = \mathbf{Q}(\underline{\theta}_k) + \mathbf{F}(\underline{\theta}_k) \mathbf{Q}(\underline{\theta}_{k-1}) \mathbf{F}(\underline{\theta}_k)^\top + \mathbf{F}(\underline{\theta}_k) \mathbf{F}(\underline{\theta}_{k-1}) \mathbf{Q}(\underline{\theta}_{k-2}) \mathbf{F}(\underline{\theta}_{k-1})^\top \mathbf{F}(\underline{\theta}_k)^\top + \dots \quad (4)$$

is pathwise convergent.

Indeed, since  $\mathbf{Q} = \mathbf{F} \mathbf{\Sigma}^2 \mathbf{F}^\top / (\Delta s \Delta t)$ ,  $\exists C > 0$ :  $\|\mathbf{Q}\| \leq \|\mathbf{F}\|^2 \sup(\sigma^2) / (\Delta s \Delta t) < C$  pathwise. Then, with  $\|\mathbf{F}_k\| \leq \mu < 1$ , it follows that the terms of the series Eq.(4) exponentially decay pathwise. Hence the pathwise convergence of the series Eq.(4). QED.

## 4 Stationarity

Since the probability distributions of any collection of  $\underline{\theta}_{j_m}$  (for  $m = 1, \dots, M$ ) are invariant to translations over  $k$  and rotations over the spatial coordinate, Eq.(4) immediately implies that so are the probability distributions of any collection of the spatial field covariances  $\mathbf{\Gamma}_k$ . An extension to the lagged field covariances is straightforward.

This shows that  $\gamma(t, t', s, s')$  are (strongly) stationary in space-time random processes.

## 5 Conclusion

The solution to DSADM is a non-stationary in space-time random field, while its space-time covariances are stationary in space-time random processes.