# DSADM: Stationarity of random space-time covariances

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## 1 Introduction

Here we provide a sketch of the proof of the statement that the space-time covariances $\gamma(t, t', s, s')$ are stationary in space-time random processes.

## 2 Recursive equations for the field covariances

Recall the equations for the simultaneous time-discrete field covariances $\mathbf{\Gamma}_k$,

$$\mathbf{\Gamma}_k = \mathbf{F}(\underline{\boldsymbol{\theta}}_k)\, \mathbf{\Gamma}_{k-1}\, \mathbf{F}(\underline{\boldsymbol{\theta}}_k)^\top + \mathbf{Q}(\underline{\boldsymbol{\theta}}_k), \tag{1}$$

and the lagged field covariances $\mathbf{\Gamma}_{kj}$,

$$\mathbf{\Gamma}_{kj} = \mathbf{F}(\underline{\boldsymbol{\theta}}_k)\, \mathbf{\Gamma}_{k-1,j}. \tag{2}$$

## 3 Explicit equations for the field covariances

### 3.1 Computation

Equations (1) and (2) allow us to express all covariances $\mathbf{\Gamma}_k \equiv \mathbf{\Gamma}_{kk}$ and $\mathbf{\Gamma}_{kj}$ (for $j < k$) from $\mathbf{\Gamma}_0$ and $\mathbf{Q}(\underline{\boldsymbol{\theta}}_{1:k})$ as follows.

Let's prove this by induction. This is, certainly, true for $k = 1$. Assume this is true for some $k$ and prove that this holds also for $k + 1$. Indeed, knowing $\mathbf{\Gamma}_k$, we compute $\mathbf{\Gamma}_{k+1,k+1}$ from Eq.(1). Then, for any $j < k + 1$, we compute $\mathbf{\Gamma}_{k+1,j}$ using Eq.(2):

$$\mathbf{\Gamma}_{k+1,j} = \mathbf{F}(\underline{\boldsymbol{\theta}}_{k+1})\, \mathbf{\Gamma}_{kj}. \tag{3}$$

This completes the proof.

### 3.2 Explicit equations

Take the simultaneous covariances $\mathbf{\Gamma}_k$ and $k$ large enough, apply Eq.(1) recursively, and use the fact that $\mathbf{F}$ is a contraction (in the sense that $\|\mathbf{F}_k\| \le \mu < 1$, where $\|.\|$ is the the matrix norm induced by the *maximal* vector norm), so that the series

$$\mathbf{\Gamma}_k = \mathbf{Q}(\underline{\boldsymbol{\theta}}_k) + \mathbf{F}(\underline{\boldsymbol{\theta}}_k)\,\mathbf{Q}(\underline{\boldsymbol{\theta}}_{k-1})\,\mathbf{F}(\underline{\boldsymbol{\theta}}_k)^\top + \mathbf{F}(\underline{\boldsymbol{\theta}}_k)\mathbf{F}(\underline{\boldsymbol{\theta}}_{k-1})\,\mathbf{Q}(\underline{\boldsymbol{\theta}}_{k-2})\,\mathbf{F}(\underline{\boldsymbol{\theta}}_{k-1})^\top\mathbf{F}(\underline{\boldsymbol{\theta}}_k)^\top + \dots \quad (4)$$

is pathwise convergent.

Indeed, since $\mathbf{Q} = \mathbf{F}\mathbf{\Sigma}^2\mathbf{F}^\top/(\Delta s\Delta t)$, $\exists C > 0$: $\|\mathbf{Q}\| \le \|\mathbf{F}\|^2 \sup(\sigma(t,s)^2)/(\Delta s\Delta t) < C$ pathwise (note that $\Sigma_{ii}(t) = \sigma(t,s_i)$ is bounded above because $\sigma = g_\sigma(\sigma^*)$ and the transformation function $g_\sigma$ is bounded). Then, with $\|\mathbf{F}_k\| \le \mu < 1$, it follows that the terms of the series Eq.(4) exponentially decay pathwise. Hence the pathwise convergence of the series Eq.(4). QED.

## 4 Stationarity

Since the probability distributions of any collection of $\underline{\boldsymbol{\theta}}_{j_m}$ (for $m = 1,\dots,M$) are invariant to translations over $k$ and rotations over the spatial coordinate, Eq.(4) immediately implies that so are the probability distributions of any collection of the spatial field covariances $\mathbf{\Gamma}_k$. An extension to the lagged field covariances is straightforward.

This shows that $\gamma(t,t',s,s')$ are (strongly) stationary in space-time random processes.

## 5 Conclusion

The solution to DSADM is a non-stationary in space-time random field, while its space-time covariances are stationary in space-time random processes.