DSADM: computation of model parameters from external parameters

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1 Specification of model parameters for the advection-diffusion-decay model with *constant* coefficients

Let the spatio-temporal random field $\eta(t,s)$ be a solution to the advection-diffusion-decay equation

$$\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial s} + \rho\eta - \nu\frac{\partial^2 \eta}{\partial s^2} = \sigma\alpha(t,s), \tag{1}$$

with the constant coefficients (model parameters) U, ρ, ν, σ. These four model parameters are computed from the *external parameters* U (advection velocity), L (spatial length scale), V_{char} (characteristic velocity), and $\mathsf{SD}(\eta)$ (the standard deviation of η in the stationary regime) or $\kappa = \exp[\mathsf{SD}(\eta)]$ as follows.

1. U is specified directly.

2. Compute the *time scale* $T = L/V_{\text{char}}$.

3. Compute ρ from the definition of the time scale

$$T = \frac{1}{\rho}\frac{\sum_m [1 + (\frac{Lm}{R})^2]^{-2}}{\sum_m [1 + (\frac{Lm}{R})^2]^{-1}} \tag{2}$$

   (where the summation is from $m = -n/2 + 1$ to $m = n/2$):

$$\rho = \frac{1}{T}\frac{\sum_m [1 + (\frac{Lm}{R})^2]^{-2}}{\sum_m [1 + (\frac{Lm}{R})^2]^{-1}} \tag{3}$$

4. Compute ν from the definition of the spatial length scale

$$L = \sqrt{\frac{\nu}{\rho}}. \tag{4}$$

   as

$$\nu = \rho L^2. \tag{5}$$

5. Finally, compute σ from the equation for the stationary field variance:

$$(\mathsf{SD}(\eta))^2 = \frac{a^2\sigma^2}{2} \sum_{m=-n/2+1}^{n/2} \frac{1}{\rho + \frac{\nu}{R^2} m^2} = \frac{a^2\sigma^2}{2\rho} \sum_{m=-n/2+1}^{n/2} [1 + (\frac{Lm}{R})^2]^{-1}. \quad (6)$$

as

$$\sigma = \frac{\mathsf{SD}(\eta)}{a} \sqrt{\frac{2\rho}{\sum_m [1 + (\frac{Lm}{R})^2]^{-1}}}, \quad (7)$$

where $\mathsf{SD}(\eta)$ is specified either directly (for θ_1^*, which corresponds to the secondary field U and for the unperturbed model for ξ, see below) or as

$$\mathsf{SD}(\eta) = \log \kappa \quad (8)$$

(for the pre-secondary fields θ_i^*, for $i = 1, 2, 3$, which correspond to the secondary fields ρ, ν, σ, see below).

Thus, we have defined the mapping from the set of external parameters $(U, L, V_{\mathrm{char}}, \mathsf{SD})$ to the set of internal (model) parameters (U, ρ, ν, σ).

2 DSADM: structure and model parameters level by level

- At the highest (third) level of the hierarchy are the four (asymptotically stationary) **pre-secondary** (generating) Gaussian fields $\theta_1^*, \theta_2^*, \theta_3^*, \theta_4^*$.

  Each θ_i^* $(i = 1, \ldots, 4)$ is a solution to the advection-diffusion-decay equation

$$\frac{\partial \theta_i^*(t,s)}{\partial t} + U_i \frac{\partial \theta_i^*(t,s)}{\partial s} + \rho_i \theta_i^*(t,s) - \nu_i \frac{\partial^2 \theta_i^*(t,s)}{\partial s^2} = \sigma_i \alpha_i(t,s), \quad (9)$$

  with the constant coefficients (model parameters) $U_i, \rho_i, \nu_i, \sigma_i$.

  Each of the four pre-secondary fields is, thus, governed by four parameters, which gives 16 model parameters in total at the third level of the hierarchy.

- At the second level of the hierarchy are four (asymptotically stationary) **secondary** (or coefficient) fields $U(t,s), \rho(t,s), \nu(t,s), \sigma(t,s)$ computed as the *pointwise transformed* pre-secondary fields:

$$U(t,s) = \bar{U} + \theta_1^*(t,s) \quad (10)$$

$$\rho(t,s) = \bar{\rho} \cdot [(1 + \varepsilon_\rho) \cdot g(\theta_2^*(t,s)) - \varepsilon_\rho] \quad (11)$$

  and

$$\nu(t,s) = \bar{\nu} \cdot [(1 + \varepsilon_\nu) \cdot g(\theta_3^*(t,s)) - \varepsilon_\nu] \quad (12)$$

$$\sigma(t,s) = \bar{\sigma} \cdot g(\theta_4^*), \quad (13)$$

  where $\bar{U}, \bar{\sigma}, \bar{\rho}, \bar{\nu}, \varepsilon_\rho, \varepsilon_\nu$ are the additional 6 model parameters.

- At the first level of the hierarchy is the random field in question $\xi(t,s)$ computed as the solution to the advection-diffusion-decay equation whose coefficients are the secondary fields $U(t,s), \rho(t,s), \nu(t,s), \sigma(t,s)$:

$$\frac{\partial \xi(t,s)}{\partial t} + U(t,s)\,\frac{\partial \xi(t,s)}{\partial s} + \rho(t,s)\,\xi(t,s) - \nu(t,s)\,\frac{\partial^2 \xi(t,s)}{\partial s^2} = \sigma(t,s)\,\alpha(t,s). \quad (14)$$

  At the first level, there are no more model parameters.

In total, there are $16+6=22$ model parameters.

3 DSADM: specification of model parameters

- The third-level model parameters are specified from the respective external parameters as described in section 1 above.

  To reduce the list of the external parameters, we postulate that all pre-secondary fields share the same advection velocity U^*, the same spatial length scale, L^*, and the same characteristic velocity $V^*_{\rm char}$. What is different for the four pre-secondary fields is their variances: $\mathsf{SD}(U)$, $\mathsf{SD}(\rho) = \log\kappa_\rho$, $\mathsf{SD}(\nu) = \log\kappa_\nu$, $\mathsf{SD}(\sigma) = \log\kappa_\sigma$.

  In total, we have 7 external parameters at the third level of the hierarchy.

- The second-level parameters $\bar U, \bar\sigma, \bar\rho, \bar\nu$, are specified by considering the "unperturbed" DSADM

$$\frac{\partial \xi}{\partial t} + \bar U \frac{\partial \xi}{\partial s} + \bar\rho\xi - \bar\nu \frac{\partial^2 \xi}{\partial s^2} = \bar\sigma\alpha \quad (15)$$

  and following the recipe described in section 1. The respective external parameters are $\bar U$ (unperturbed advection velocity), $\bar L$ (unperturbed spatial length scale), $\bar V_{\rm char}$ (unperturbed characteristic velocity, which is postulated to coincide with the characteristic velocity of the pre-secondary fields, $\bar V_{\rm char} = V^*_{\rm char}$), and $\mathsf{SD}(\xi)$. This adds 3 new external parameters to the list.

  From Eqs.(10)–(13), it follows that the secondary fields become equal to their respective unperturbed values if the variances of the respective pre-secondary fields are zero (note that $g(0)=1$), hence the term "unperturbed".

  The remaining 2 second-level model parameters $\varepsilon_\rho, \varepsilon_\nu$ are the small positive constants introduced to allow for sporadic negative values of ρ and ν, which lead to intermittent instability of the model for ξ (because negative decay coefficient or negative diffusion imply amplification of the solution over time).

  Summarizing, there are $7+3+2=12$ external parameters.