A finite Gaussian scale mixture has positive excess kurtosis

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1 Kurtosis

Let

$$\xi | I \sim N(0, \sigma_I^2)$$

and let I be the discrete random variable that takes the value of i with probability w_i.

Then

$$\mathsf{E}\,\xi = \mathsf{E}\,\mathsf{E}\,(\xi|I) = 0$$

$$\mathsf{E}\,\xi^2 = \mathsf{Var}\,\xi = \mathsf{E}\,\mathsf{E}\,(\xi^2|I) = \sum w_i \sigma_i^2.$$

$$\mathsf{E}\,\xi^3 = \mathsf{E}\,\mathsf{E}\,(\xi^3|I) = 0$$

$$\mathsf{E}\,\xi^4 = \mathsf{E}\,\mathsf{E}\,(\xi^4|I) = 3\sum w_i \sigma_i^4,$$

where 3 is the Gaussian Kurtosis.

Now, the kurtosis is

$$\mathrm{Ku} = \frac{\mathsf{E}\,\xi^4}{(\mathsf{Var}\,\xi)^2} = \frac{3\sum w_i \sigma_i^4}{(\sum w_i \sigma_i^2)^2} \tag{1}$$

Here, because $f(x) := x^2$ is convex (i.e. convex downward), then, by definition, $\forall w_i$, $i = 1, 2, \dots$ such that $\sum w_i = 1$, we have (due to the Jensen inequality):

$$(\sum w_i x_i)^2 \le \sum w_i x_i^2. \tag{2}$$

With $x_i = \sigma_i^2$, we obtain

$$(\sum w_i \sigma_i^2)^2 \leq \sum w_i \sigma_i^4. \tag{3}$$

Therefore, from Eq.(1),

$$\boxed{\mathrm{Ku} \geq 3}$$

QED.

Note that positive excess kurtosis is often associated with thick tails but this is not a necessity.