Statically Typed String Sanitation Inside a Python (Technical Report)

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Abstract

This report contains supporting evidence for claims put forth and explained in the paper "Statically Typed String Sanitation Inside a Python" [**?**], including proofs of lemmas and theorems asserted in the paper, examples, additional discussion of the paper's technical content, and errata.

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Keywords: type systems; regular languages; input sanitation; string sanitation

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1 Terminology and Notation

Theorems and lemmas appearing in [**?**] are numbered correspondingly, while supporting facts appearing only in the Technical Report are lettered. Throughout this technical report, we use a small step semantics corresponding to the big step semantics given in [**?**].

2 Regular Expressions

The syntax of regular expressions over some alphabet Σ is shown in Figure **??**.

Assumption A (Regular Expression Congruences)**.** *We assume regular expressions are implicitly identified up to the following congruences:*

$$
\begin{aligned}
\epsilon \cdot r &\equiv r \\
r \cdot \epsilon &\equiv r \\
(r_1 \cdot r_2) \cdot r_3 &\equiv r_1 \cdot (r_2 \cdot r_3) \\
r_1 + r_2 &\equiv r_2 + r_1 \\
(r_1 + r_2) + r_3 &\equiv r_1 + (r_2 + r_3) \\
\epsilon^* &\equiv \epsilon
\end{aligned}
$$

Assumption B (Properties of Regular Languages)**.** *We assume the following properties:*

1. *If $s_1 \in \mathcal{L}\{r_1\}$ and $s_2 \in \mathcal{L}\{r_2\}$ then $s_1 s_2 \in \mathcal{L}\{r_1 \cdot r_2\}$.*
2. *For all strings s and regular expressions r, either $s \in \mathcal{L}\{r\}$ or $s \notin \mathcal{L}\{r\}$.*
3. *Regular languages are closed under reversal.*

3 λ_{RS}

The syntax of λ_{RS} is specified in Figure **??**. The static semantics is specified in Figure **??**.

3.1 Head and Tail Operations

The following correctness conditions must hold for any definition of $\mathsf{lhead}(r)$ and $\mathsf{ltail}(r)$.

Condition C (Correctness of Head)**.** *If $c_1 s' \in \mathcal{L}\{r\}$, then $c_1 \in \mathcal{L}\{\mathsf{lhead}(r)\}$.*

Condition D (Correctness of Tail)**.** *If $c_1 s' \in \mathcal{L}\{r\}$ then $s' \in \mathcal{L}\{\mathsf{ltail}(r)\}$.*

For example, we conjecture (but do not here prove) that the definitions below satisfy these conditions. Note that these are slightly amended relative to the published paper.

Definition 1 (Definition of $\mathsf{lhead}(r)$)**.** We first define an auxiliary relation that determines the set of characters that the head might be, tracking the remainder of any sequences that appear:

$$\begin{aligned}
\mathsf{lhead}(\epsilon, \epsilon) &= \emptyset \\
\mathsf{lhead}(\epsilon, r') &= \mathsf{lhead}(r', \epsilon) \\
\mathsf{lhead}(a, r') &= \{a\} \\
\mathsf{lhead}(r_1 \cdot r_2, r') &= \mathsf{lhead}(r_1, r_2 \cdot r') \\
\mathsf{lhead}(r_1 + r_2, r') &= \mathsf{lhead}(r_1, r') \cup \mathsf{lhead}(r_2, r') \\
\mathsf{lhead}(r^*, r') &= \mathsf{lhead}(r, \epsilon) \cup \mathsf{lhead}(r', \epsilon)
\end{aligned}$$

We define $\mathsf{lhead}(r) = a_1 + a_2 + ... + a_i$ iff $\mathsf{lhead}(r, \epsilon) = \{a_1, a_2, ..., a_i\}$.

Definition 2 (Brzozowski's Derivative)**.** The *derivative of* r *with respect to* s is denoted by $\delta_s(r)$ and is $\delta_s(r) = \{t | st \in \mathcal{L}\{r\}\}$.

Definition 3 (Definition of $\mathsf{ltail}(r)$)**.** If $\mathsf{lhead}(r, \epsilon) = \{a_1, a_2, ..., a_i\}$, then we define $\mathsf{ltail}(r) = \delta_{a_1}(r) + \delta_{a_2}(r) + ... + \delta_{a_i}(r)$.

3.2 Replacement

The following correctness condition must hold for any definition of $\texttt{lreplace}(r, r_1, r_2)$.

Condition E (Replacement Correctness)**.** *If* $s_1 \in \mathcal{L}\{r_1\}$ *and* $s_2 \in \mathcal{L}\{r_2\}$ *then*

$$\mathsf{replace}(r; s_1; s_2) \in \mathcal{L}\{\texttt{lreplace}(r, r_1, r_2)\}$$

We do not give a particular definition for $\texttt{lreplace}(r, r_1, r_2)$ here.

3.3 Small Step Semantics of λ_{RS}

Figure **??** specifies a small-step operational semantics for λ_{RS}.

Lemma F (Canonical Forms)**.** *If* $\emptyset \vdash v : \sigma$ *then:*

1. *If* $\sigma = \mathsf{stringin}[r]$ *then* $v = \mathsf{rstr}[s]$ *and* $s \in \mathcal{L}\{r\}$.
2. *If* $\sigma = \sigma_1 \to \sigma_2$ *then* $v = \lambda x.e'$.

Proof. By inspection of the static and dynamic semantics. □

Lemma G (Progress)**.** *If* $\emptyset \vdash e : \sigma$ *either* $e = v$ *for some* v *or* $e \mapsto e'$ *for some* e'.

Proof. The proof proceeds by rule induction on the derivation of $\emptyset \vdash e : \sigma$.

λ **fragment**. Cases SS-T-Var, SS-T-Abs, and SS-T-App are exactly as in a proof of progress for the simply typed lambda calculus.

S-T-Stringin-I. Suppose $\emptyset \vdash \mathsf{rstr}[s] : \mathsf{stringin}[s]$. Then $e = \mathsf{rstr}[s]$.

S-T-Concat. Suppose $\emptyset \vdash \mathsf{rconcat}(e_1; e_2) : \mathsf{stringin}[r_1 \cdot r_2]$ and $\emptyset \vdash e_1 : \mathsf{stringin}[r_1]$ and $\emptyset \vdash e_2 : \mathsf{stringin}[r_2]$. By induction, $e_1 \mapsto e_1'$ or $e_1 = v_1$ and similarly, $e_2 \mapsto e_2'$ or $e_2 = v_2$. If e_1 steps, then SS-E-Concat-Left applies and so $\mathsf{rconcat}(e_1; e_2) \mapsto \mathsf{rconcat}(e_1'; e_2)$. Similarly, if e_2 steps then e steps by SS-E-Concat-Right.

In the remaining case, $e_1 = v_1$ and $e_2 = v_2$. But then it follows by Canonical Forms that $e_1 = \mathsf{rstr}[s_1]$ and $e_2 = \mathsf{rstr}[s_2]$. Finally, by SS-E-Concat, $\mathsf{rconcat}(\mathsf{rstr}[s_1]; \mathsf{rstr}[s_2]) \mapsto \mathsf{rstr}[s_1 s_2]$.

S-T-Case. Suppose $e = \mathsf{rstrcase}(e_1; e_2; x, y.e_3)$ and $\emptyset \vdash e_1 : \mathsf{stringin}[r]$. By induction and Canonical Forms it follows that $e_1 \mapsto e_1'$ or $e_1 = \mathsf{rstr}[s]$. In the former case, e steps by S-E-Case-Left. In the latter case, note that $s = \epsilon$ or $s = at$ for some string t. If $s = \epsilon$ then e steps by S-E-Case-ϵ-Val, and if $s = at$ the e steps by S-E-Case-Concat.

S-T-Replace. Suppose $e = \mathsf{rreplace}[r](e_1; e_2)$, $\emptyset \vdash e : \mathsf{stringin}[\mathtt{lreplace}(r, r_1, r_2)]$ and:

$$\emptyset \vdash e_1 : \mathsf{stringin}[r_1] \tag{1}$$

$$\emptyset \vdash e_2 : \mathsf{stringin}[r_2] \tag{2}$$

By induction on (1), $e_1 \mapsto e_1'$ or $e_1 = v_1$ for some e_1'. If $e_1 \mapsto e_1'$ then e steps by SS-E-Replace-Left. Similarly, if e_2 steps then e steps by SS-E-Replace-Right. The only remaining case is where $e_1 = v_1$ and also $e_2 = v_2$. By Canonical Forms, $e_1 = \mathsf{rstr}[s_1]$ and $e_2 = \mathsf{rstr}[s_2]$. Therefore, $e \mapsto \mathsf{rstr}[\mathsf{replace}(r; s_1; s_2)]$ by SS-E-Replace.

S-T-SafeCoerce. Suppose that $\emptyset \vdash \mathsf{rcoerce}[r](e_1) : \mathsf{stringin}[r]$. and $\emptyset \vdash e_1 : \mathsf{stringin}[r']$ for $\mathcal{L}\{r'\} \subseteq \mathcal{L}\{r\}$.By induction, $e_1 = v_1$ or $e_1 \mapsto e_1'$ for some e_1'. If $e_1 \mapsto e_1'$ then e steps by SS-E-SafeCoerce-Step. Otherwise, $e_1 = v$ and by Canonical Forms $e_1 = \mathsf{rstr}[s]$. In this case, $e = \mathsf{rcoerce}[r](\mathsf{rstr}[s]) \mapsto \mathsf{rstr}[s]$ by SS-E-SafeCoerce.

S-T-Check Suppose that $\emptyset \vdash \mathsf{rcheck}[r](e_0; x.e_1; e_2) : \mathsf{stringin}[r]$ and:

$$\emptyset \vdash e_0 : \mathsf{stringin}[r_0] \tag{3}$$

$$\emptyset, x : \mathsf{stringin}[r] \vdash e_1 : \sigma \tag{4}$$

$$\emptyset \vdash e_2 : \sigma \tag{5}$$

By induction, $e_0 \mapsto e_0'$ or $e_0 = v$. In the former case e steps by SS-E-Check-StepLeft. Otherwise, $e_0 = \mathsf{rstr}[s]$ by Canonical Forms. By Lemma **??** part 2, either $s \in \mathcal{L}\{r_0\}$ or $s \notin \mathcal{L}\{r_0\}$. In the former case e takes a step by SS-E-Check-Ok. In the latter case e takes a step by SS-E-Check-NotOk.

□

Assumption H (Substitution)**.** *If* $\Psi, x : \sigma' \vdash e : \sigma$ *and* $\Psi \vdash e' : \sigma'$*, then* $\Psi \vdash [e'/x]e : \sigma$.

Lemma I (Preservation for Small Step Semantics)**.** *If* $\emptyset \vdash e : \sigma$ *and* $e \mapsto e'$ *then* $\emptyset \vdash e' : \sigma$.

Proof. By induction on the derivation of $e \mapsto e'$ and $\emptyset \vdash e : \sigma$.

λ **fragment**. Cases SS-E-AppLeft, SS-E-AppRight, and SS-E-AppAbs are exactly as in a proof of type safety for the simply typed lambda calculus.

S-E-Concat-Left. Suppose $e = \mathsf{rconcat}(e_1; e_2) \mapsto \mathsf{rconcat}(e_1'; e_2)$ and $e_1 \mapsto e_1'$. The only rule that applies is S-T-Concat, so $\emptyset \vdash e_1 : \mathsf{stringin}[r_1]$ and $\emptyset \vdash e_2 : \mathsf{stringin}[r_2]$. By induction, $\emptyset \vdash e_1' : \mathsf{stringin}[r_1]$. Therefore, by S-T-Concat, $\emptyset \vdash \mathsf{rconcat}(e_1'; e_2) : \mathsf{stringin}[r_1 r_2]$.

S-E-Concat-Right. Suppose $e = \mathsf{rconcat}(e_1; e_2) \mapsto \mathsf{rconcat}(e_1; e_2')$ and $e_2 \mapsto e_2'$. The only rule that applies is S-T-Concat, so $\emptyset \vdash e_1 : \mathsf{stringin}[r_1]$ and $\emptyset \vdash e_2 : \mathsf{stringin}[r_2]$. By induction, $\emptyset \vdash e_2' : \mathsf{stringin}[r_2]$. Therefore, by S-T-Concat, $\emptyset \vdash \mathsf{rconcat}(e_1; e_2') : \mathsf{stringin}[r_1 r_2]$.

S-E-Concat. Suppose $\mathsf{rconcat}(\mathsf{rstr}[s_1]; \mathsf{rstr}[s_2]) \mapsto \mathsf{rstr}[s_1 s_2]$. The only applicable rule is S-T-Concat, so $\emptyset \vdash \mathsf{rstr}[s_1] : \mathsf{stringin}[r_1]$ and $\emptyset \vdash \mathsf{rstr}[s_2] : \mathsf{stringin}[r_2]$ and $\emptyset \vdash \mathsf{rconcat}(\mathsf{rstr}[s_1]; \mathsf{rstr}[s_2]) : \mathsf{stringin}[r_1 \cdot r_2]$. By Canonical Forms, $s_1 \in \mathcal{L}\{r_1\}$ and $s_2 \in \mathcal{L}\{r_2\}$ from which it follows by Lemma **??** that $s_1 s_2 \in \mathcal{L}\{r_1 \cdot r_2\}$. Therefore, $\emptyset \vdash \mathsf{rstr}[s_1 s_2] : \mathsf{stringin}[r_1 \cdot r_2]$ by S-T-Rstr.

S-E-Case-Left. Suppose $e \mapsto \mathsf{rstrcase}(e_1'; e_2; x, y.e_3)$ and $\emptyset \vdash e : \sigma$ and $e_1 \mapsto e_1'$. The only rule that applies is S-T-Case, so:

$$\emptyset \vdash e_1 : \mathsf{stringin}[r] \tag{6}$$

$$\emptyset \vdash e_2 : \sigma \tag{7}$$

$$\emptyset, x : \mathsf{stringin}[\mathsf{lhead}(r)], y : \mathsf{stringin}[\mathsf{ltail}(r)] \vdash e_3 : \sigma \tag{8}$$

By (**??**) and the assumption that $e_1 \mapsto e_1'$, it follows by induction that $\emptyset \vdash e_1' : \mathsf{stringin}[r]$. This fact together with (**??**) and (**??**) implies by S-T-Case that $\emptyset \vdash \mathsf{rstrcase}(e_1'; e_2; x, y.e_3) : \sigma$.

S-E-Case-ϵ-Val. Suppose $\mathsf{rstrcase}(e_0; e_2; x, y.e_3) \mapsto e_2$. The only rule that applies is S-T-Case, so $\emptyset \vdash e_2 : \sigma$.

S-E-Case-Concat. Suppose that $e = \mathsf{rstrcase}(\mathsf{rstr}[as]; e_2; x, y.e_3) \mapsto [\mathsf{rstr}[a], \mathsf{rstr}[s]/x, y]e_3$ and that $\emptyset \vdash e : \sigma$. The only rule that applies is S-T-Case so:

$$\emptyset \vdash \mathsf{rstr}[as] : \mathsf{stringin}[r] \tag{9}$$

$$\emptyset \vdash e_2 : \sigma \tag{10}$$

$$\emptyset, x : \mathsf{stringin}[\mathsf{lhead}(r)], y : \mathsf{stringin}[\mathsf{ltail}(r)] \vdash e_3 : \sigma \tag{11}$$

We know that $as \in \mathcal{L}\{r\}$ by Canonical Forms on (**??**) Therefore, $a \in \mathcal{L}\{\mathsf{lhead}(r)\}$ by Condition **??** and $s \in \mathcal{L}\{\mathsf{ltail}(r)\}$ by Condition **??**.

From these facts about a and s we know by S-T-Rstr that $\emptyset \vdash \mathsf{rstr}[a] : \mathsf{stringin}[\mathsf{lhead}(r)]$ and $\emptyset \vdash \mathsf{rstr}[s] : \mathsf{stringin}[\mathsf{ltail}(r)]$. It follows by Assumption **??** that $\emptyset \vdash [\mathsf{rstr}[a], \mathsf{rstr}[s]/x, y]e_3 : \sigma$.

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Case S-E-Replace-Left. Suppose that $e = \mathsf{rreplace}[r](e_1; e_2) \mapsto \mathsf{rreplace}[r](e_1'; e_2)$ when $e_1 \mapsto e_1'$. The only rule that applies is S-T-Replace, so $\emptyset \vdash e : \mathsf{stringin}[\mathtt{lreplace}(r, r_1, r_2)]$ where:

$$\emptyset \vdash e_1 : \mathsf{stringin}[r_1]$$
$$\emptyset \vdash e_2 : \mathsf{stringin}[r_2]$$

By induction, $\emptyset \vdash e_1' : \mathsf{stringin}[r_1]$. Therefore, $\emptyset \vdash \mathsf{rreplace}[r](e_1'; e_2) : \mathsf{stringin}[\mathtt{lreplace}(r, r_1, r_2)]$ by S-T-Replace.

Case S-E-Replace-Right. Suppose that $e = \mathsf{rreplace}[r](e_1; e_2) \mapsto \mathsf{rreplace}[r](e_1'; e_2)$ when $e_1 \mapsto e_1'$. The only rule that applies is S-T-Replace, so $\emptyset \vdash e : \mathsf{stringin}[\mathtt{lreplace}(r, r_1, r_2)]$ where:

$$\emptyset \vdash e_1 : \mathsf{stringin}[r_1]$$
$$\emptyset \vdash e_2 : \mathsf{stringin}[r_2]$$

By induction, $\emptyset \vdash e_1' : \mathsf{stringin}[r_1]$. Therefore, $\emptyset \vdash \mathsf{rreplace}[r](r_1'; r_2) : \mathsf{stringin}[\mathtt{lreplace}(r, r_1, r_2)]$ by S-T-Replace.

Case S-E-Replace.

Suppose $e = \mathsf{rreplace}[r](\mathsf{rstr}[s_1]; \mathsf{rstr}[s_2]) \mapsto \mathsf{rstr}[\mathsf{replace}(r; s_1; s_2)]$. The only applicable rule is S-T-Replace, so

$$\emptyset \vdash \mathsf{rstr}[s_1] : \mathsf{stringin}[r_1]$$
$$\emptyset \vdash \mathsf{rstr}[s_2] : \mathsf{stringin}[r_2]$$

By conanical forms, $s_1 \in \mathcal{L}\{r_1\}$ and $s_2 \in \mathcal{L}\{r_2\}$. Therefore, $\mathtt{lreplace}(r, s_1, s_2) \in \mathcal{L}\{\mathtt{lreplace}(r, r_1, r_2)\}$ by Theorem **??**. It is finally derivable by S-T-Rstr that:

$\emptyset \vdash \mathsf{rstr}[\mathtt{lreplace}(r, s_1, s_2)] : \mathsf{stringin}[\mathtt{lreplace}(r, r_1, r_2)]$.

Case S-E-SafeCoerce. Suppose that $\mathsf{rcoerce}[r](\mathsf{rstr}[s_1]) \mapsto \mathsf{rstr}[s_1]$. The only applicable rule is S-T-SafeCoerce, so $\emptyset \vdash \mathsf{rcoerce}[r](s_1) : \mathsf{stringin}[r]$. By Canonical Forms, $s \in \mathcal{L}\{r\}$. Therefore, $\emptyset \vdash \mathsf{rstr}[s] : \mathsf{stringin}[r]$.

Case S-E-Check-Ok. Suppose $\mathsf{rcheck}[r](\mathsf{rstr}[s]; x.e_1; e_2) \mapsto [\mathsf{rstr}[s]/x]e_1$, $s \in \mathcal{L}\{r\}$, and $\emptyset \vdash \mathsf{rcheck}[r](\mathsf{rstr}[s]; x.e_1; e_2) : \sigma$. By inversion of S-T-Check, $x : \mathsf{stringin}[r] \vdash e_1 : \sigma$. Note that $s \in \mathcal{L}\{r\}$ implies that $s : \mathsf{stringin}[r]$ by S-T-RStr. Therefore, $\emptyset \vdash [\mathsf{rstr}[s]/x]e_1 : \sigma$.

Case S-E-Check-NotOk. Suppose $\mathsf{rcheck}[r](\mathsf{rstr}[s]; x.e_1; e_2) \mapsto e_2$, $s \notin \mathcal{L}\{r\}$, and $\emptyset \vdash \mathsf{rcheck}[r](\mathsf{rstr}[s]; x.e_1; e_2) : \sigma$. The only applicable rule is S-T-Check, so $\emptyset \vdash e_2 : \sigma$.

□

Theorem J (Type Safety for small step semantics.)**.** *If* $\emptyset \vdash e : \sigma$ *then either* e val *or* $e \mapsto^* e'$ *and* $\emptyset \vdash e' : \sigma$.

Proof. Follows directly from progress and preservation. □

3.3.1 The Security Theorem

Theorem 4 (Correctness of Input Sanitation for λ_{RS})**.** *If* $\emptyset \vdash e : \mathsf{stringin}[r]$ *and* $e \mapsto^* \mathsf{rstr}[s]$ *then* $s \in \mathcal{L}\{r\}$.

Proof. By type sfaetly, $\emptyset \vdash \mathsf{rstr}[s] : \mathsf{stringin}[r]$. Therefore, $s \in \mathcal{L}\{r\}$. □

4 Proofs of Lemmas and Theorems About λ_P

This section follows the same structure as the safety proof for λ_{RS} – we prove type safety for a small-step semantics, prove a semantic correspondence, and then transfer the safety result to the big-step semantics in the paper.

Lemma 5 (Canonical Forms for Target Language)**.**

- *If* $\emptyset \vdash \iota : \mathsf{regex}$ *then* $\iota \mapsto^* \mathsf{rx}[r]$ *such that* r *is a well-formed regular expression.*
- *If* $\emptyset \vdash \iota : \mathsf{string}$ *then* $\iota \mapsto^* \mathsf{str}[s]$.

Theorem 6 (Progress)**.** *If* $\emptyset \vdash \iota : \tau$ *either* $\iota = \dot{v}$ *or* $\iota \mapsto \iota'$ *for some* ι'.

Proof. The proof proceeds by induction on the typing assumption. Consider only the string and regex (non-λ) fragments of λ_P.

P-T-Case. Suppose $\emptyset \vdash \mathsf{strcase}(\iota_1; \iota_2; x, y.\iota_3)$. By inversion, $\iota_1 : \mathsf{string}$ and so either $\iota_1 \mapsto \iota_1'$ or by canonical forms, $\iota_1 = \mathsf{str}[s_1]$. Similarly, $\iota_2 \mapsto \iota_2'$ or else $\iota_2 = \mathsf{str}[s_2]$. In the former cases, progress occurs via the compatibility rules. in the case where both are string values, progress occurs via the case concatenation rule.

P-T-Replace. Suppose $\emptyset \vdash \mathsf{replace}(\iota_1; \iota_2; \iota_3)$. By inversion, $\iota_1 : \mathsf{regex}$ and so by canonical forms $\iota_1 = \mathsf{rx}[r]$. By inversion, $\iota_2 : \mathsf{string}$ and so by induction either $\iota_2 \mapsto \iota_2'$ or else $\iota_2 = \mathsf{str}[s_2]$ for some string s_2. Similarly, either ι_3 steps or else $\iota_3 = \mathsf{str}[s_3]$. In case any steps occur, progress occurs. In the remaining case, PP-E-Replace applies and so progress occurs.

P-T-Check. Finally, suppose $\emptyset \vdash \check{\iota_x}\iota_1\iota_2\iota_3$. In case any of these step, then progress occurs. In the remaining cases, applications of inversion and canonical forms for each ι_x and ι_1 implies that the term at hand equals $\check{\mathsf{rx}[r]}\mathsf{str}[s]\iota_2\iota_3$, which evaluates to either ι_2 or ι_3.

□

Lemma K (Substitution Lemma)**.** *If* $\theta, x : \tau \vdash \iota : \tau'$ *and* $\theta \vdash \iota' : \tau$ *then* $\theta \vdash [\iota'/x]\iota : \tau'$.

Theorem 7 (Preservation)**.** *If* $\emptyset \vdash \iota : \tau$ *and* $\iota \mapsto \iota'$ *then* $\emptyset \vdash \iota' : \tau$.

Proof. The proof proceeds by induction of the derivations of $\emptyset \vdash \iota : \tau$ and $\iota \mapsto \iota'$.
We treat only the non-lambda fragment.

Case PS-E-ConcatLeft. Suppose:

$$\begin{aligned}
&\iota = \mathsf{rconcat}(\iota_1; \iota_2) \mapsto \mathsf{rconcat}(\iota_1'; \iota_2) \\
&\emptyset \vdash \iota : \mathsf{string} \\
&\iota \mapsto \iota'
\end{aligned}$$

The only applicable typing rule is P-T-Concat, so $\emptyset \vdash \iota_1 : \mathsf{string}$ and $\emptyset \vdash \iota_2 : \mathsf{string}$. By induction, $\emptyset \vdash \iota_1' : \mathsf{string}$, so $\emptyset \vdash \mathsf{rconcat}(\iota_1'; \iota_2) : \mathsf{string}$.

Case PS-E-ConcatRight

$$\begin{aligned}
&e = \mathsf{rconcat}(e_1; e_2) \mapsto \mathsf{rconcat}(e_1; e_2') \\
&\emptyset \vdash e : \mathsf{string} \\
&\iota \mapsto \iota'
\end{aligned}$$

The only applicable typing rule is P-T-Concat, so $\emptyset \vdash \iota_1 : \mathsf{string}$ and $\emptyset \vdash \iota_2 : \mathsf{string}$. By induction, $\emptyset \vdash \iota_1' : \mathsf{string}$, so $\emptyset \vdash \mathsf{rconcat}(\iota_1; \iota_2') : \mathsf{string}$.

Case PS-E-Concat Let $e = \mathsf{rconcat}(\mathsf{rstr}[s_1]; \mathsf{rstr}[s_2]) \mapsto \mathsf{rstr}[s_1 s_2]$. The only rule that applies is P-T-Concat, so $\emptyset \vdash e : \mathsf{string}$. By canonical forms, $\emptyset \mathsf{rstr}[s_1 s_2] : \mathsf{string}$.

Case PS-E-CaseLeft Let $\iota = \mathsf{rstrcase}(\iota_1; \iota_2; x, y.\iota_3) \mapsto \mathsf{rstrcase}(\iota_1'; \iota_2; x, y.\iota_3)$ when $\iota_1 \mapsto \iota_1'$. The only rule that applies is P-T-Case, so $\emptyset \vdash \iota : \tau$ where:

$$\begin{aligned} \emptyset &\vdash \iota_1 : \mathsf{string} \\ \emptyset &\vdash \iota_2 : \tau \\ \emptyset, x : \mathsf{string}, y : \mathsf{string} &\vdash \iota_3 : \tau \end{aligned}$$

By induction, $\emptyset \vdash \iota_1' : \mathsf{string}$. By P-T-Case, $\emptyset \vdash \mathsf{rstrcase}(\iota_1; \iota_2; x, y.\iota_3) : \tau$.

Case PS-E-CaseEpsilon Let $\iota = \mathsf{rstrcase}(\mathsf{rstr}[\epsilon]; \iota_2; x, y.\iota_3) \mapsto \iota_2$. The only rule that applies is P-T-Case, so $\emptyset \vdash \iota : \tau$ where $\iota_2 : \tau$.

Case PS-E-Case Let $\iota = \mathsf{rstrcase}(\mathsf{rstr}[as]; \iota_2; x, y.\iota_3) \mapsto [\mathsf{rstr}[a], \mathsf{rstr}[s]/x, y]\iota_3$ The only rule that applies is P-T-Case, so $\emptyset \vdash \iota : \tau$ where:

$$\begin{aligned} \emptyset &\vdash \iota_1 : \mathsf{string} \\ \emptyset &\vdash \iota_2 : \tau \\ \emptyset, x : \mathsf{string}, y : \mathsf{string} &\vdash \iota_3 : \tau \end{aligned}$$

The result follows by the substitution lemma.

Case PS-E-ReplaceLeft Let $\iota = \mathsf{rreplace}[\iota_1](\iota_2; \iota_3) \mapsto \mathsf{rreplace}[\iota_1'](\iota_2; \iota_3)$ where $\iota_1 \mapsto \iota_1'$. The applicable typing rule is P-T-Replace, so $\emptyset \vdash \iota : \mathsf{string}$ where:

$$\begin{aligned} \emptyset &\vdash \iota_1 : \mathsf{regex} \\ \emptyset &\vdash \iota_2 : \mathsf{string} \\ \emptyset &\vdash \iota_3 : \mathsf{string} \end{aligned}$$

By induction, $\emptyset \vdash \iota_1' : \mathsf{regex}$. Therefore, $\emptyset \vdash \mathsf{rreplace}[\iota_1'](\iota_2; \iota_3)$.

Case PS-E-ReplaceMid Let $\iota = \mathsf{rreplace}[\iota_1](\iota_2; \iota_3) \mapsto \mathsf{rreplace}[\iota_1](\iota_2'; \iota_3)$ where $\iota_2 \mapsto \iota_2'$. The applicable typing rule is P-T-Replace, so $\emptyset \vdash \iota : \mathsf{string}$ where:

$$\begin{aligned} \emptyset &\vdash \iota_1 : \mathsf{regex} \\ \emptyset &\vdash \iota_2 : \mathsf{string} \\ \emptyset &\vdash \iota_3 : \mathsf{string} \end{aligned}$$

By induction, $\emptyset \vdash \iota_2' : \mathsf{string}$. Therefore, $\emptyset \vdash \mathsf{rreplace}[\iota_1](\iota_2'; \iota_3)$.

Case PS-E-ReplaceRight Let $\iota = \mathsf{rreplace}[\iota_1](\iota_2; \iota_3) \mapsto \mathsf{rreplace}[\iota_1](\iota_2; \iota_3')$ where $\iota_3 \mapsto \iota_3'$. The applicable typing rule is P-T-Replace, so $\emptyset \vdash \iota : \mathsf{string}$ where:

$$\begin{aligned} \emptyset &\vdash \iota_1 : \mathsf{regex} \\ \emptyset &\vdash \iota_2 : \mathsf{string} \\ \emptyset &\vdash \iota_3 : \mathsf{string} \end{aligned}$$

By induction, $\emptyset \vdash \iota_3' : \mathsf{string}$. Therefore, $\emptyset \vdash \mathsf{rreplace}[\iota_1](\iota_2; \iota_3')$.

Case PS-E-Replace Let $\iota = \mathsf{rreplace}[\mathsf{rx}[r]](\mathsf{rstr}[s_2]; \mathsf{rstr}[s_3]) \mapsto \mathsf{rstr}[\mathtt{lreplace}(r, s_2, s_3)]$. The applicable typing rule is P-T-Replace, so $\emptyset \vdash \iota : \mathsf{string}$. The result follows by canonical forms.

Case PS-E-CheckLeft Let $\iota = \mathsf{rcheck}[\iota_x](\iota_1; \iota_2; \iota_3) \mapsto \mathsf{rcheck}[\iota_x'](\iota_1; \iota_2; \iota_3)$ where $\iota_x \mapsto \iota_x'$. The applicable typing rule is P-T-Check, so $\emptyset \vdash \iota : \tau$ where:

$$\emptyset \vdash \iota_x : \mathsf{regex}$$
$$\emptyset \vdash \iota_1 : \mathsf{string}$$
$$\emptyset \vdash \iota_2 : \tau$$
$$\emptyset \vdash \iota_3 : \tau$$

By induction, $\iota_x : \mathsf{regex}$. Therefore, $\emptyset \mathsf{rcheck}[\iota_x'](\iota_1; \iota_2; \iota_3) : \tau$.

Case PS-E-CheckRight Let $\iota = \mathsf{rcheck}[\iota_x](\iota_1; \iota_2; \iota_3) \mapsto \mathsf{rcheck}[\iota_x](\iota_1'; \iota_2; \iota_3)$ where $\iota_1 \mapsto \iota_1'$. The applicable typing rule is P-T-Check, so $\emptyset \vdash \iota : \tau$ where:

$$\emptyset \vdash \iota_x : \mathsf{regex}$$
$$\emptyset \vdash \iota_1 : \mathsf{string}$$
$$\emptyset \vdash \iota_2 : \tau$$
$$\emptyset \vdash \iota_3 : \tau$$

By induction, $\iota_1' : \mathsf{string}$. Therefore, $\emptyset \mathsf{rcheck}[\iota_x](\iota_1'; \iota_2; \iota_3) : \tau$.

Case PS-E-Check-Ok Let $\iota = \mathsf{rcheck}[\mathsf{rx}[r]](\mathsf{rstr}[s]; \iota_2; \iota_3) \mapsto \iota_2$ and $s \in \mathcal{L}\{r\}$. The applicable typing rule is P-T-Check, so $\emptyset \vdash \iota : \tau$ where $\emptyset \vdash \iota_2 : \tau$.

Case PS-E-Check-NotOk Let $\iota = \mathsf{rcheck}[\mathsf{rx}[r]](\mathsf{rstr}[s]; \iota_2; \iota_3) \mapsto \iota_3$ where $s \notin \mathcal{L}\{r\}$. The applicable typing rule is P-T-Check, so $\emptyset \vdash \iota : \tau$ where $\emptyset \vdash \iota_3 : \tau$.

□

5 Proofs and Lemmas and Theorems About Translation

Theorem 8 (Translation Correctness). *If $\Psi \vdash e : \sigma$ then there exists an ι such that $[\![e]\!] = \iota$ and $[\![\Psi]\!] \vdash \iota : [\![\sigma]\!]$. Furthermore, if $e \mapsto^* v$ then $\iota \mapsto^* \dot{v}$ such that $[\![v]\!] = \dot{v}$.*

Proof. We present a proof by induction on the structure of e. We write $e \rightsquigarrow \iota$ as shorthand for the final property.

This proof needs to be changed to use only the small-step semantics.

Case $e = \mathsf{rstr}[s]$. Suppose $\Theta \vdash \mathsf{rstr}[s] : \sigma$.

By examination the syntactic structure of conclusions in the relation S-T, we know this is true just in case $\sigma = \mathsf{stringin}[r]$ for some r such that $s \in \mathcal{L}\{r\}$; and of course, there is always such an r.

There are no free variables in $\mathsf{rstr}[s]$, so we might as well proceed from the fact that $\emptyset \vdash \mathsf{rstr}[s] : \mathsf{stringin}[r]$.

By definition of the translation ($[\![\cdot]\!]$) the following statements hold:

$$[\![\mathsf{rstr}[s]]\!] = \mathsf{str}[s] \tag{12}$$

$$[\![\mathsf{stringin}[r]]\!] = \mathsf{string} \tag{13}$$

$$[\![\emptyset]\!] = \emptyset \tag{14}$$

Note that $\emptyset \vdash \mathsf{str}[s] : \mathsf{string}$ by P-T-Str. Recall that contexts are standard and, in particular, can be weakened. So since $[\![\Theta]\!]$ is either a weakening of $\emptyset$ or $\emptyset$ itself, $[\![\Theta]\!] \vdash \mathsf{str}[s] : \mathsf{string}$ by weakening.

Summarily, $\mathsf{str}[s]$ is a term of λ_P such that $[\![\Theta]\!] \vdash \mathsf{str}[s] : [\![\sigma]\!]$

It remains to be shown that there exist $v, \dot{v}$ such that $\mathsf{rstr}[s] \mapsto^* v$, $\mathsf{str}[s] \mapsto^* \dot{v}$, and $[\![v]\!] = \dot{v}$. But this is immediate because each term is already a value and $s = s$.

Case $e = \mathsf{rconcat}(e_1; e_2)$. The applicable typing rule is S-T-Concat, so $\Psi \vdash \mathsf{rconcat}(e_1; e_2) : \mathsf{stringin}[r_1 \cdot r_2]$ where $\Psi \vdash e_1 : \mathsf{stringin}[r_1]$ and $\Psi \vdash e_2 : \mathsf{stringin}[r_2]$.

By induction, $e_1 \rightsquigarrow \iota_1$ and $e_2 \rightsquigarrow \iota_2$. Therefore, $[\![\Psi]\!] \vdash \mathsf{concat}(\iota_1; \iota_2)$ by P-T-Concat.

By canonical forms, $e_1 \mapsto^* \mathsf{rstr}[s_1]$ where by induction $\iota_1 \mapsto^* \mathsf{str}[s_1]$. Similarly, $e_2 \mapsto^* \mathsf{rstr}[s_2]$ and $\iota_2 \mapsto^* \mathsf{str}[s_2]$. Therefore, $e \mapsto^* \mathsf{rstr}[s_1 s_2]$ by S-E-Concat at last, and $\mathsf{concat}(\iota_1; \iota_2) \mapsto^* \mathsf{str}[s_1 s_2]$ by P-E-Concat at last. Note that $[\![\mathsf{rstr}[s_1 s_2]]\!] = \mathsf{str}[s_1 s_2]$.

Case $e = \mathsf{rstrcase}(e_1; e_2; x, y.e_3)$. This case relies on our definition of context translation.

Suppose $\Psi \vdash \mathsf{rstrcase}(e_1; e_2; x, y.e_3) : \sigma$. By inversion of the typing relation it follows that $\Psi \vdash e_1 : \mathsf{stringin}[r]$, $\Psi \vdash e_2 : \sigma$ and $\Psi, x : \mathsf{stringin}[\mathsf{lhead}(r)], y : \mathsf{stringin}[\mathsf{ltail}(r)] \vdash e_3 : \sigma$.

By induction, there exists an ι_1 such that $e_1 \mapsto \iota_1$.

By canonical forms, $e_1 \mapsto^* \mathsf{rstr}[s]$. Therefore, $\iota_1 \mapsto *\mathsf{str}[s]$ because $e_1 \rightsquigarrow \iota_1$.

Choose $\iota = \mathsf{strcase}(\iota_1; \iota_2; x, y.\iota_3)$ and note that by the properties established via induction, $[\![e]\!] = \iota$ and $[\![\Psi]\!] \vdash \iota : [\![\sigma]\!]$.

To prove the evaluation correspondence, we consider two cases for the value of s.

Suppose $s = \epsilon$. Then $e \mapsto^* v$ where $e_2 \mapsto^* v$, from which it follows that $\iota \mapsto^* \dot{v}$ where $\iota_2 \mapsto^* \dot{v}$. But recall that $e_2 \rightsquigarrow v_2$ and so $[\![v]\!] = \dot{v}$.

Suppose otherwise that $s = at$ for some character a and string t. Then $e \mapsto^* v$ where $[a, t/x, y]e_3 \mapsto^* v$. Similarly, $\iota \mapsto^* \dot{v}$ where $[a, t/x, y]\iota_3 \mapsto^* \dot{v}$

Case $e = \mathsf{rreplace}[r](e_1; e_2)$. There is only one applicable typing rule, so suppose $\Psi \vdash \mathsf{rreplace}[r](e_1; e_2) : \mathsf{stringin}[\mathtt{lreplace}(r, e_1, e_2)]$. Let $\Theta = [\![\Psi]\!]$. Note that $[\![\mathsf{rreplace}[r](e_1; e_2)]\!] = \mathsf{replace}(\mathsf{rx}[r]; \iota_1; \iota_2)$ where by induction $[\![e_1]\!] = \iota_1$ and $[\![e_2]\!] = \iota_2$ such that $\Theta \vdash \iota_1$ and $\Theta \vdash \iota_2$. It follows by P-T-Replace that $\Theta \vdash \mathsf{replace}(\mathsf{rx}[r]; \iota_1; \iota_2) : \mathsf{string}$. Finally, note that $[\![\mathsf{stringin}[\mathtt{lreplace}(r, e_1, e_2)]]\!] = \mathsf{string}$.

For evaluation correspondence, note that $[\![\mathsf{rstr}[\mathtt{lreplace}(r, s_1, s_2)]]\!] = \mathsf{rstr}[\mathtt{lreplace}(r, s_1, s_2)]$ and so it suffices to show that $\mathsf{replace}(\mathsf{rx}[r]; \iota_1; \iota_2) \mapsto^* \mathsf{rstr}[r]s_1 s_2$. Note that $\mathtt{lreplace}(r, e_1, e_2) \mapsto^* \mathsf{rstr}[\mathtt{lreplace}(r, s_1, s_2)]$ where $e_1 \mapsto^* \mathsf{rstr}[s_1]$, $e_2 \mapsto^* \mathsf{rstr}[s_2]$, $r \mapsto^* r$. By induction, $\iota_1 \mapsto^* \mathsf{rstr}[s_1], \iota_2 \mapsto^* \mathsf{rstr}[s_2]$, and $\mathsf{rx}[r] \mapsto^* \mathsf{rx}[r]$. So by S-E-Replace, the sufficient condition holds.

Case $e = \mathsf{rcoerce}[r](e')$. The only applicable tpying rule is S-T-SafeCoerce, so suppose $\Psi \vdash \mathsf{rcoerce}[r](e') : \mathsf{stringin}[r]$ where $\Psi \vdash e' : \mathsf{stringin}[r']$ and $\mathcal{L}\{r'\} \subseteq \mathcal{L}\{r\}$. By induction, $e' \rightsquigarrow \iota$ for some ι. Therefore, $[\![\mathsf{rcoerce}[r](e')]\!] = \iota$ by Tr-SafeCoerce.

For evaluation correspondence, note that $e \mapsto^* v$ where $e' \mapsto^* v$. The result follows by induction because $e' \rightsquigarrow \iota$.

Case $e = \mathsf{rcheck}[r](e_1; x.e_2; e_3)$. The applicable typing rule is S-T-Check, so $\Psi \vdash e : \sigma$ where $\Psi \vdash e_1 : \mathsf{stringin}[r]$, $\Psi, x : \mathsf{stringin}[r] \vdash e_2 : \sigma$, and $\Psi \vdash e_3 : \sigma$. By induction and a corresponding substitution principle there exists $\iota_1, \iota_2, \iota_3$ such that $e_1 \rightsquigarrow \iota_1$, $e_2 \rightsquigarrow \iota_2$ in context $\Psi, s : \mathsf{stringin}[r]$, and $e_3 \rightsquigarrow \iota_3$. Choose $\iota = \mathsf{check}(\mathsf{rx}[r]; \iota_1; \lambda x.\iota_2; \iota_3)$. The result follows by induction.

□

Theorem 9 (Correctness of Input Sanitation for Translated Terms)**.** *If* $[\![e]\!] = \iota$ *and* $\emptyset \vdash e : \mathsf{stringin}[r]$ *then* $\iota \mapsto^* \mathsf{str}[s]$ *for* $s \in \mathcal{L}\{r\}$.

Proof. By **??**, $e \mapsto^* \mathsf{rstr}[s]$ where $\emptyset \vdash \mathsf{rstr}[s] : \mathsf{stringin}[r]$. Therefore, $s \in \mathcal{L}\{r\}$. Note that $[\![\cdot]\!]$ is a function and $[\![\mathsf{rstr}[s]]\!] = \mathsf{str}[s]$; therefore, by theorem **??**, $\iota \mapsto^* \mathsf{str}[s]$. □

$$r \quad ::= \quad \epsilon \mid . \mid a \mid r \cdot r \mid r + r \mid r* \qquad\qquad a \in \Sigma$$

Figure 1: Regular expressions over the alphabet Σ.

$$\begin{array}{llll}
\sigma & ::= & \sigma \to \sigma \mid \mathsf{stringin}[r] & \text{source types} \\
e & ::= & x \mid v & \text{source terms} \\
 & \mid & \mathsf{rconcat}(e; e) \mid \mathsf{rstrcase}(e; e; x, y.e) & s \in \Sigma^* \\
 & \mid & \mathsf{rreplace}[r](e; e) \mid \mathsf{rcoerce}[r](e) \mid \mathsf{rcheck}[r](e; x.e; e) & \\
v & ::= & \lambda x.e \mid \mathsf{rstr}[s] & \text{source values}
\end{array}$$

Figure 2: Syntax of λ_{RS}.

$$\begin{array}{llll}
\tau & ::= & \tau \to \tau \mid \mathsf{string} \mid \mathsf{regex} & \text{target types} \\
\iota & ::= & x \mid \dot{v} & \text{target terms} \\
 & \mid & \mathsf{concat}(\iota; \iota) \mid \mathsf{strcase}(\iota; \iota; x, y.\iota) & \\
 & \mid & \mathsf{rx}[r] \mid \mathsf{replace}(\iota; \iota; \iota) \mid \mathsf{check}(\iota; \iota; \iota; \iota) & \\
\dot{v} & ::= & \lambda x.\iota \mid \mathsf{str}[s] \mid \mathsf{rx}[r] & \text{target values}
\end{array}$$

Figure 3: Syntax for the target language, λ_P, containing strings and statically constructed regular expressions.

$\boxed{\Psi \vdash e : \sigma}$ $\quad \Psi ::= \emptyset \mid \Psi, x : \sigma$

S-T-VAR
$$\frac{x : \sigma \in \Psi}{\Psi \vdash x : \sigma}$$

S-T-ABS
$$\frac{\Psi, x : \sigma_1 \vdash e : \sigma_2}{\Psi \vdash \lambda x.e : \sigma_1 \to \sigma_2}$$

S-T-APP
$$\frac{\Psi \vdash e_1 : \sigma_2 \to \sigma \qquad \Psi \vdash e_2 : \sigma_2}{\Psi \vdash e_1(e_2) : \sigma}$$

S-T-STRINGIN-I
$$\frac{s \in \mathcal{L}\{r\}}{\Psi \vdash \mathsf{rstr}[s] : \mathsf{stringin}[r]}$$

S-T-CONCAT
$$\frac{\Psi \vdash e_1 : \mathsf{stringin}[r_1] \qquad \Psi \vdash e_2 : \mathsf{stringin}[r_2]}{\Psi \vdash \mathsf{rconcat}(e_1; e_2) : \mathsf{stringin}[r_1 \cdot r_2]}$$

S-T-CASE
$$\frac{\Psi \vdash e_1 : \mathsf{stringin}[r] \qquad \Psi \vdash e_2 : \sigma \qquad \Psi, x : \mathsf{stringin}[\mathsf{lhead}(r)], y : \mathsf{stringin}[\mathsf{ltail}(r)] \vdash e_3 : \sigma}{\Psi \vdash \mathsf{rstrcase}(e_1; e_2; x, y.e_3) : \sigma}$$

S-T-REPLACE
$$\frac{\Psi \vdash e_1 : \mathsf{stringin}[r_1] \qquad \Psi \vdash e_2 : \mathsf{stringin}[r_2]}{\Psi \vdash \mathsf{rreplace}[r](e_1; e_2) : \mathsf{stringin}[\mathtt{lreplace}(r, r_1, r_2)]}$$

S-T-SAFECOERCE
$$\frac{\Psi \vdash e : \mathsf{stringin}[r'] \qquad \mathcal{L}\{r'\} \subseteq \mathcal{L}\{r\}}{\Psi \vdash \mathsf{rcoerce}[r](e) : \mathsf{stringin}[r]}$$

S-T-CHECK
$$\frac{\Psi \vdash e_0 : \mathsf{stringin}[r] \qquad \Psi, x : \mathsf{stringin}[r] \vdash e_1 : \sigma \qquad \Psi \vdash e_2 : \sigma}{\Psi \vdash \mathsf{rcheck}[r](e_0; x.e_1; e_2) : \sigma}$$

Figure 4: Typing rules for λ_{RS}. The typing context Ψ is standard.

$\boxed{e \mapsto e}$

$$\frac{e_1 \mapsto e_1'}{e_1(e_2) \mapsto e_1'(e_2)}\ \text{SS-E-AppLeft} \qquad \frac{e_2 \mapsto e_2'}{v_1 \mapsto v_1}\ \text{SS-E-AppRight} \qquad \frac{}{(\lambda x : \tau_{11}.t_{12})v_2 \mapsto [v_2/x]t_{12}}\ \text{SS-E-AppAbs}$$

$\boxed{e \mapsto^* e}$

$$\frac{}{e \mapsto^* e}\ \text{RT-Refl} \qquad \frac{e \mapsto^* e' \quad e' \mapsto e''}{e \mapsto^* e''}\ \text{RT-Trans}$$

Figure 5: Call-by-name small step Semantics for λ and its reflexive, transitive closure.

$\boxed{e \mapsto e}$ (Contiues figure 6)

$$\frac{e_1 \mapsto e_1'}{\mathsf{rconcat}(e_1; e_2) \mapsto \mathsf{rconcat}(e_1'; e_2)}\ \text{SS-E-Concat-Left} \qquad \frac{e_2 \mapsto e_2'}{\mathsf{rconcat}(v_1; e_2) \mapsto \mathsf{rconcat}(v_1; e_2')}\ \text{SS-E-Concat-Right}$$

$$\frac{}{\mathsf{rconcat}(\mathsf{rstr}[s_1]; \mathsf{rstr}[s_2]) \mapsto \mathsf{rstr}[s_1 s_2]}\ \text{SS-E-Concat} \qquad \frac{e_1 \mapsto e_1'}{\mathsf{rstrcase}(e_1; e_2; x, y.e_3) \mapsto \mathsf{rstrcase}(e_1'; e_2; x, y.e_3)}\ \text{SS-E-Case-Left}$$

$$\frac{}{\mathsf{rstrcase}(\mathsf{rstr}[\epsilon]; e_2; x.y.e_3) \mapsto e_2}\ \text{SS-E-Case-}\epsilon\text{-val} \qquad \frac{}{\mathsf{rstrcase}(\mathsf{rstr}[as]; e_2; x, y.e_3) \mapsto [\mathsf{rstr}[a], \mathsf{rstr}[s]/x, y]e_3}\ \text{SS-E-Case-Concat}$$

$$\frac{e_1 \mapsto e_1'}{\mathsf{rreplace}[r](v_1; e_2) \mapsto \mathsf{rreplace}[r](v_1'; e_2)}\ \text{SS-E-Replace-Left} \qquad \frac{e_2 \mapsto e_2'}{\mathsf{rreplace}[r](e_1; e_2) \mapsto \mathsf{rreplace}[r](e_1; e_2')}\ \text{SS-E-Replace-Right}$$

$$\frac{}{\mathsf{rreplace}[r](\mathsf{rstr}[s_1]; \mathsf{rstr}[s_2]) \mapsto \mathsf{rstr}[\mathsf{replace}(r; s_1; s_2)]}\ \text{SS-E-Replace} \qquad \frac{e \mapsto e'}{\mathsf{rcoerce}[r](e) \mapsto \mathsf{rcoerce}[r](e')}\ \text{SS-E-SafeCoerce-Step}$$

$$\frac{}{\mathsf{rcoerce}[r](\mathsf{rstr}[s]) \mapsto \mathsf{rstr}[s]}\ \text{SS-E-SafeCoerce} \qquad \frac{e \mapsto e'}{\mathsf{rcheck}[r](e; x.e_1; e_2) \mapsto \mathsf{rcheck}[r](e'; x.e_1; e_2)}\ \text{SS-E-Check-StepLeft}$$

$$\frac{s \in \mathcal{L}\{r\}}{\mathsf{rcheck}[r](\mathsf{rstr}[s]; x.e_1; e_2) \mapsto [\mathsf{rstr}[s]/x]e_1}\ \text{SS-E-Check-Ok} \qquad \frac{s \notin \mathcal{L}\{r\}}{\mathsf{rcheck}[r](\mathsf{rstr}[s]; x.e_1; e_2) \mapsto e_2}\ \text{SS-E-Check-NotOk}$$

Figure 6: Small step semantics for λ_{RS}. Extends **??**.

$$\boxed{\Theta \vdash \iota : \tau} \qquad \Theta ::= \emptyset \mid \Theta, x : \tau$$

$$\text{P-T-VAR}\ \frac{x : \tau \in \Theta}{\Theta \vdash x : \tau} \qquad \text{P-T-ABS}\ \frac{\Theta, x : \tau_1 \vdash \iota_2 : \tau_2}{\Theta \vdash \lambda x.\iota_2 : \tau_1 \to \tau_2} \qquad \text{P-T-APP}\ \frac{\Theta \vdash \iota_1 : \tau_2 \to \tau \qquad \Theta \vdash \iota_2 : \tau_2}{\Theta \vdash \iota_1(\iota_2) : \iota} \qquad \text{P-T-STRING}\ \frac{}{\Theta \vdash \mathsf{str}[s] : \mathsf{string}}$$

$$\text{P-T-REGEX}\ \frac{}{\Theta \vdash \mathsf{rx}[r] : \mathsf{regex}} \qquad \text{P-T-CONCAT}\ \frac{\Theta \vdash \iota_1 : \mathsf{string} \qquad \Theta \vdash \iota_2 : \mathsf{string}}{\Theta \vdash \mathsf{concat}(\iota_1; \iota_2) : \mathsf{string}}$$

$$\text{P-T-CASE}\ \frac{\Theta \vdash \iota_1 : \mathsf{string} \qquad \Theta \vdash \iota_2 : \tau \qquad \Theta, x : \mathsf{string}, y : \mathsf{string} \vdash \iota_3 : \tau}{\Theta \vdash \mathsf{strcase}(\iota_1; \iota_2; x, y.\iota_3) : \tau}$$

$$\text{P-T-REPLACE}\ \frac{\Theta \vdash \iota_1 : \mathsf{regex} \qquad \Theta \vdash \iota_2 : \mathsf{string} \qquad \Theta \vdash \iota_3 : \mathsf{string}}{\Theta \vdash \mathsf{replace}(\iota_1; \iota_2; \iota_3) : \mathsf{string}}$$

$$\text{P-T-CHECK}\ \frac{\Theta \vdash \iota_x : \mathsf{regex} \qquad \Theta \vdash \iota_1 : \mathsf{string} \qquad \Theta \vdash \iota_2 : \tau \qquad \Theta \vdash \iota_3 : \tau}{\Theta \vdash \mathsf{check}(\iota_x; \iota_1; \iota_2; \iota_3) : \tau}$$

Figure 7: Typing rules for λ_P. The typing context Θ is standard.

$\boxed{\iota \mapsto \iota}$

$$\textsc{PS-E-ConcatLeft}\quad \frac{\iota_1 \mapsto \iota_1'}{\mathsf{concat}(\iota_1;\iota_2) \mapsto \mathsf{concat}(\iota_1';\iota_2)}$$

$$\textsc{PS-E-ConcatRight}\quad \frac{\iota_2 \mapsto \iota_2'}{\mathsf{concat}(\iota_1;\iota_2) \mapsto \mathsf{concat}(\iota_1;\iota_2')}$$

$$\textsc{PS-E-Concat}\quad \frac{}{\mathsf{concat}(\mathsf{str}[s_1];\mathsf{str}[s_2]) \mapsto \mathsf{str}[s_1 s_2]}$$

$$\textsc{PS-E-CaseLeft}\quad \frac{\iota_1 \mapsto \iota_1'}{\mathsf{strcase}(\iota_1;\iota_2;x,y.\iota_3) \mapsto \mathsf{strcase}(\iota_1';\iota_2;x,y.\iota_3)}$$

$$\textsc{PS-E-Case-Epsilon}\quad \frac{}{\mathsf{strcase}(\epsilon;\iota_2;x,y.\iota_3) \mapsto \iota_2}$$

$$\textsc{PS-E-Case}\quad \frac{}{\mathsf{strcase}(\mathsf{str}[as];\iota_2;x,y.\iota_3) \mapsto \mathsf{str}[as]}$$

$$\textsc{PS-E-ReplaceLeft}\quad \frac{\iota_1 \mapsto \iota_1'}{\mathsf{replace}(\iota_1;\iota_2;\iota_3) \mapsto \mathsf{replace}(\iota_1';\iota_2;\iota_3)}$$

$$\textsc{PS-E-ReplaceMid}\quad \frac{\iota_2 \mapsto \iota_2'}{\mathsf{replace}(\mathsf{rx}[r];\iota_2;\iota_3) \mapsto \mathsf{replace}(\mathsf{rx}[r];\iota_2';\iota_3)}$$

$$\textsc{PS-E-ReplaceRight}\quad \frac{\iota_3 \mapsto \iota_3'}{\mathsf{replace}(\mathsf{rx}[r];\mathsf{str}[s_2];\iota_3) \mapsto \mathsf{replace}(\mathsf{rx}[r];\mathsf{str}[s_2];\iota_3')}$$

$$\textsc{PS-E-Replace}\quad \frac{}{\mathsf{replace}(\mathsf{rx}[r];\mathsf{str}[s_2];\mathsf{str}[s_3]) \mapsto \mathsf{str}[\mathsf{replace}(r;s_2;s_3)]}$$

$$\textsc{PS-E-CheckLeft}\quad \frac{\iota_x \mapsto \iota_x'}{\mathsf{rcheck}[\iota_x](\iota;\iota_1;\iota_2) \mapsto \mathsf{rcheck}[\iota_x](\iota;\iota_1;\iota_2)}$$

$$\textsc{PS-E-CheckRight}\quad \frac{\iota \mapsto \iota'}{\mathsf{rcheck}[\mathsf{rx}[r]](\iota;\iota_1;\iota_2) \mapsto \mathsf{rcheck}[\mathsf{rx}[r]](\iota';\iota_1;\iota_2)}$$

$$\textsc{PS-E-Check-OK}\quad \frac{s \in \mathcal{L}\{r\}}{\mathsf{rcheck}[\mathsf{rx}[r]](\mathsf{str}[s];\iota_1;\iota_2) \mapsto \iota_1}$$

$$\textsc{PS-E-Check-NotOK}\quad \frac{s \notin \mathcal{L}\{r\}}{\mathsf{rcheck}[\mathsf{rx}[r]](\mathsf{str}[s];\iota_1;\iota_2) \mapsto \iota_2}$$

Figure 8: Small step semantics for λ_P (extends L-E rules)

.

$\boxed{[\![\sigma]\!] = \tau}$

$$\frac{\textsc{Tr-T-String}}{[\![\mathsf{stringin}[r]]\!] = \mathsf{string}}$$

$$\textsc{Tr-T-Arrow}\quad \frac{[\![\sigma_1]\!] = \tau_1 \qquad [\![\sigma_2]\!] = \tau_2}{[\![\sigma_1 \to \sigma_2]\!] = \tau_1 \to \tau_2}$$

$\boxed{[\![\Psi]\!] = \Theta}$

$$\frac{\textsc{Tr-T-Context-Emp}}{[\![\emptyset]\!] = \emptyset}$$

$$\textsc{Tr-T-Context-Ext}\quad \frac{[\![\Psi]\!] = \Theta \qquad [\![\sigma]\!] = \tau}{[\![\Psi, x : \sigma]\!] = \Theta, x : \tau}$$

$\boxed{[\![e]\!] = \iota}$

$$\frac{\textsc{Tr-Var}}{[\![x]\!] = x}$$

$$\textsc{Tr-Abs}\quad \frac{[\![e]\!] = \iota}{[\![\lambda x.e]\!] = \lambda x.\iota}$$

$$\textsc{Tr-App}\quad \frac{[\![e_1]\!] = \iota_1 \qquad [\![e_2]\!] = \iota_2}{[\![e_1(e_2)]\!] = \iota_1(\iota_2)}$$

$$\textsc{Tr-Case}\quad \frac{[\![e_1]\!] = \iota_1 \qquad [\![e_2]\!] = \iota_2 \qquad [\![e_3]\!] = \iota_3}{[\![\mathsf{rstrcase}(e_1; e_2; x, y.e_3)]\!] = \mathsf{strcase}(\iota_1; \iota_2; x, y.\iota_3)}$$

$$\frac{\textsc{Tr-string}}{[\![\mathsf{rstr}[s]]\!] = \mathsf{str}[s]}$$

$$\textsc{Tr-Concat}\quad \frac{[\![e_1]\!] = \iota_1 \qquad [\![e_2]\!] = \iota_2}{[\![\mathsf{rconcat}(e_1; e_2)]\!] = \mathsf{concat}(\iota_1; \iota_2)}$$

$$\textsc{Tr-Subst}\quad \frac{[\![e_1]\!] = \iota_1 \qquad [\![e_2]\!] = \iota_2}{[\![\mathsf{rreplace}[r](e_1; e_2)]\!] = \mathsf{replace}(\mathsf{rx}[r]; \iota_1; \iota_2)}$$

$$\textsc{Tr-SafeCoerce}\quad \frac{[\![e]\!] = \iota}{[\![\mathsf{rcoerce}[r'](e)]\!] = \iota}$$

$$\textsc{Tr-Check}\quad \frac{[\![e]\!] = \iota \qquad [\![e_1]\!] = \iota_1 \qquad [\![e_2]\!] = \iota_2}{[\![\mathsf{rcheck}[r](e; x.e_1; e_2)]\!] = \mathsf{check}(\mathsf{rx}[r]; \iota; (\lambda x.\iota_1)(\iota); \iota_2)}$$

Figure 9: Translation from source terms (e) to target terms (ι).