Statically Typed String Sanitation Inside a Python (Technical Report)

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Abstract

This report contains supporting evidence for claims put forth and explained in the paper "Statically Typed String Sanitation Inside a Python" [1], including proofs of lemmas and theorems asserted in the paper, examples, additional discussion of the paper's technical content, and errata.

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Keywords: type systems; regular languages; input sanitation; string sanitation

Contents

List of Figures

1 Terminology and Notation

Theorems and lemmas appearing in [1] are numbered correspondingly, while supporting facts appearing only in the Technical Report are lettered.

2 Regular Expressions

The syntax of regular expressions over some alphabet Σ is shown in Figure 1.

Assumption A (Regular Expression Congruences)**.** *We assume regular expressions are implicitly identified up to the following congruences:*

$$\begin{aligned}
\epsilon \cdot r &\equiv r \\
r \cdot \epsilon &\equiv r \\
(r_1 \cdot r_2) \cdot r_3 &\equiv r_1 \cdot (r_2 \cdot r_3) \\
r_1 + r_2 &\equiv r_2 + r_1 \\
(r_1 + r_2) + r_3 &\equiv r_1 + (r_2 + r_3) \\
\epsilon^* &\equiv \epsilon
\end{aligned}$$

Assumption B (Properties of Regular Languages)**.** *We assume the following properties:*

1. *If $s_1 \in \mathcal{L}\{r_1\}$ and $s_2 \in \mathcal{L}\{r_2\}$ then $s_1 s_2 \in \mathcal{L}\{r_1 \cdot r_2\}$.*
2. *For all strings s and regular expressions r, either $s \in \mathcal{L}\{r\}$ or $s \notin \mathcal{L}\{r\}$.*
3. *Regular languages are closed under reversal.*

3 λ_{RS}

The syntax of λ_{RS} is specified in Figure 2. The static semantics is specified in Figure 4.

3.1 Head and Tail Operations

The following correctness conditions must hold for any definition of $\mathsf{lhead}(r)$ and $\mathsf{ltail}(r)$.

Condition C (Correctness of Head)**.** *If $c_1 s' \in \mathcal{L}\{r\}$, then $c_1 \in \mathcal{L}\{\mathsf{lhead}(r)\}$.*

Condition D (Correctness of Tail)**.** *If $c_1 s' \in \mathcal{L}\{r\}$ then $s' \in \mathcal{L}\{\mathsf{ltail}(r)\}$.*

For example, we conjecture (but do not here prove) that the definitions below satisfy these conditions. Note that these are slightly amended relative to the published paper.

Definition 1 (Definition of $\mathsf{lhead}(r)$)**.** We first define an auxiliary relation that determines the set of characters that the head might be, tracking the remainder of any sequences that appear:

$$\begin{aligned}
\mathsf{lhead}(\epsilon, \epsilon) &= \emptyset \\
\mathsf{lhead}(\epsilon, r') &= \mathsf{lhead}(r', \epsilon) \\
\mathsf{lhead}(a, r') &= \{a\} \\
\mathsf{lhead}(r_1 \cdot r_2, r') &= \mathsf{lhead}(r_1, r_2 \cdot r') \\
\mathsf{lhead}(r_1 + r_2, r') &= \mathsf{lhead}(r_1, r') \cup \mathsf{lhead}(r_2, r') \\
\mathsf{lhead}(r^*, r') &= \mathsf{lhead}(r, \epsilon) \cup \mathsf{lhead}(r', \epsilon)
\end{aligned}$$

We define $\mathsf{lhead}(r) = a_1 + a_2 + ... + a_i$ iff $\mathsf{lhead}(r, \epsilon) = \{a_1, a_2, ..., a_i\}$.

Definition 2 (Brzozowski's Derivative)**.** The *derivative of r with respect to s* is denoted by $\delta_s(r)$ and is $\delta_s(r) = \{t | st \in \mathcal{L}\{r\}\}$.

Definition 3 (Definition of $\mathsf{ltail}(r)$)**.** If $\mathsf{lhead}(r, \epsilon) = \{a_1, a_2, ..., a_i\}$, then we define $\mathsf{ltail}(r) = \delta_{a_1}(r) + \delta_{a_2}(r) + ... + \delta_{a_i}(r)$.

3.2 Replacement

The following correctness condition must hold for any definition of $\texttt{lreplace}(r, r_1, r_2)$.

Condition E (Replacement Correctness)**.** *If $s_1 \in \mathcal{L}\{r_1\}$ and $s_2 \in \mathcal{L}\{r_2\}$ then*

$$\mathsf{replace}(r; s_1; s_2) \in \mathcal{L}\{\texttt{lreplace}(r, r_1, r_2)\}$$

We do not give a particular definition for $\texttt{lreplace}(r, r_1, r_2)$ here.

3.3 Small Step Semantics of λ_{RS}

Figure 7 specifies a small-step operational semantics for λ_{RS}.

Lemma F (Canonical Forms)**.** *If $\emptyset \vdash v : \sigma$ then:*

1. *If $\sigma = \mathsf{stringin}[r]$ then $v = \mathsf{rstr}[s]$ and $s \in \mathcal{L}\{r\}$.*
2. *If $\sigma = \sigma_1 \to \sigma_2$ then $v = \lambda x.e'$.*

Proof. By inspection of the static and dynamic semantics. □

Lemma G (Progress)**.** *If $\emptyset \vdash e : \sigma$ either $e = v$ for some v or $e \mapsto e'$ for some e'.*

Proof. The proof proceeds by rule induction on the derivation of $\emptyset \vdash e : \sigma$.

> λ **fragment**. Cases SS-T-Var, SS-T-Abs, and SS-T-App are exactly as in a proof of progress for the simply typed lambda calculus.
>
> **S-T-Stringin-I**. Suppose $\emptyset \vdash \mathsf{rstr}[s] : \mathsf{stringin}[s]$. Then $e = \mathsf{rstr}[s]$.

S-T-Concat. Suppose $\emptyset \vdash \mathsf{rconcat}(e_1; e_2) : \mathsf{stringin}[r_1 \cdot r_2]$ and $\emptyset \vdash e_1 : \mathsf{stringin}[r_1]$ and $\emptyset \vdash e_2 : \mathsf{stringin}[r_2]$. By induction, $e_1 \mapsto e_1'$ or $e_1 = v_1$ and similarly, $e_2 \mapsto e_2'$ or $e_2 = v_2$. If e_1 steps, then SS-E-Concat-Left applies and so $\mathsf{rconcat}(e_1; e_2) \mapsto \mathsf{rconcat}(e_1'; e_2)$. Similarly, if e_2 steps then e steps by SS-E-Concat-Right.

In the remaining case, $e_1 = v_1$ and $e_2 = v_2$. But then it follows by Canonical Forms that $e_1 = \mathsf{rstr}[s_1]$ and $e_2 = \mathsf{rstr}[s_2]$. Finally, by SS-E-Concat, $\mathsf{rconcat}(\mathsf{rstr}[s_1]; \mathsf{rstr}[s_2]) \mapsto \mathsf{rstr}[s_1 s_2]$.

S-T-Case. Suppose $e = \mathsf{rstrcase}(e_1; e_2; x, y.e_3)$ and $\emptyset \vdash e_1 : \mathsf{stringin}[r]$. By induction and Canonical Forms it follows that $e_1 \mapsto e_1'$ or $e_1 = \mathsf{rstr}[s]$. In the former case, e steps by S-E-Case-Left. In the latter case, note that $s = \epsilon$ or $s = at$ for some string t. If $s = \epsilon$ then e steps by S-E-Case-ϵ-Val, and if $s = at$ the e steps by S-E-Case-Concat.

S-T-Replace. Suppose $e = \mathsf{rreplace}[r](e_1; e_2)$, $\emptyset \vdash e : \mathsf{stringin}[\mathtt{lreplace}(r, r_1, r_2)]$ and:

$$\emptyset \vdash e_1 : \mathsf{stringin}[r_1] \tag{1}$$

$$\emptyset \vdash e_2 : \mathsf{stringin}[r_2] \tag{2}$$

By induction on (1), $e_1 \mapsto e_1'$ or $e_1 = v_1$ for some e_1'. If $e_1 \mapsto e_1'$ then e steps by SS-E-Replace-Left. Similarly, if e_2 steps then e steps by SS-E-Replace-Right. The only remaining case is where $e_1 = v_1$ and also $e_2 = v_2$. By Canonical Forms, $e_1 = \mathsf{rstr}[s_1]$ and $e_2 = \mathsf{rstr}[s_2]$. Therefore, $e \mapsto \mathsf{rstr}[\mathsf{replace}(r; s_1; s_2)]$ by SS-E-Replace.

S-T-SafeCoerce. Suppose that $\emptyset \vdash \mathsf{rcoerce}[r](e_1) : \mathsf{stringin}[r]$. and $\emptyset \vdash e_1 : \mathsf{stringin}[r']$ for $\mathcal{L}\{r'\} \subseteq \mathcal{L}\{r\}$.By induction, $e_1 = v_1$ or $e_1 \mapsto e_1'$ for some e_1'. If $e_1 \mapsto e_1'$ then e steps by SS-E-SafeCoerce-Step. Otherwise, $e_1 = v$ and by Canonical Forms $e_1 = \mathsf{rstr}[s]$. In this case, $e = \mathsf{rcoerce}[r](\mathsf{rstr}[s]) \mapsto \mathsf{rstr}[s]$ by SS-E-SafeCoerce.

S-T-Check Suppose that $\emptyset \vdash \mathsf{rcheck}[r](e_0; x.e_1; e_2) : \mathsf{stringin}[r]$ and:

$$\emptyset \vdash e_0 : \mathsf{stringin}[r_0] \tag{3}$$

$$\emptyset, x : \mathsf{stringin}[r] \vdash e_1 : \sigma \tag{4}$$

$$\emptyset \vdash e_2 : \sigma \tag{5}$$

By induction, $e_0 \mapsto e_0'$ or $e_0 = v$. In the former case e steps by SS-E-Check-StepLeft. Otherwise, $e_0 = \mathsf{rstr}[s]$ by Canonical Forms. By Lemma B part 2, either $s \in \mathcal{L}\{r_0\}$ or $s \notin \mathcal{L}\{r_0\}$. In the former case e takes a step by SS-E-Check-Ok. In the latter case e takes a step by SS-E-Check-NotOk.

□

Assumption H (Substitution)**.** *If* $\Psi, x : \sigma' \vdash e : \sigma$ *and* $\Psi \vdash e' : \sigma'$*, then* $\Psi \vdash [e'/x]e : \sigma$.

Lemma I (Preservation for Small Step Semantics)**.** *If* $\emptyset \vdash e : \sigma$ *and* $e \mapsto e'$ *then* $\emptyset \vdash e' : \sigma$.

Proof. By induction on the derivation of $e \mapsto e'$ and $\emptyset \vdash e : \sigma$.

λ **fragment**. Cases SS-E-AppLeft, SS-E-AppRight, and SS-E-AppAbs are exactly as in a proof of type safety for the simply typed lambda calculus.

SS-E-Concat-Left. Suppose $e = \mathsf{rconcat}(e_1; e_2) \mapsto \mathsf{rconcat}(e_1'; e_2)$ and $e_1 \mapsto e_1'$. The only rule that applies is S-T-Concat, so $\emptyset \vdash e_1 : \mathsf{stringin}[r_1]$ and $\emptyset \vdash e_2 : \mathsf{stringin}[r_2]$. By induction, $\emptyset \vdash e_1' : \mathsf{stringin}[r_1]$. Therefore, by S-T-Concat, $\emptyset \vdash \mathsf{rconcat}(e_1'; e_2) : \mathsf{stringin}[r_1 r_2]$.

SS-E-Concat-Right. Suppose $e = \mathsf{rconcat}(e_1; e_2) \mapsto \mathsf{rconcat}(e_1; e_2')$ and $e_2 \mapsto e_2'$. The only rule that applies is S-T-Concat, so $\emptyset \vdash e_1 : \mathsf{stringin}[r_1]$ and $\emptyset \vdash e_2 : \mathsf{stringin}[r_2]$. By induction, $\emptyset \vdash e_2' : \mathsf{stringin}[r_2]$. Therefore, by S-T-Concat, $\emptyset \vdash \mathsf{rconcat}(e_1; e_2') : \mathsf{stringin}[r_1 r_2]$.

SS-E-Concat. Suppose $\mathsf{rconcat}(\mathsf{rstr}[s_1]; \mathsf{rstr}[s_2]) \mapsto \mathsf{rstr}[s_1 s_2]$. The only applicable rule is S-T-Concat, so $\emptyset \vdash \mathsf{rstr}[s_1] : \mathsf{stringin}[r_1]$ and $\emptyset \vdash \mathsf{rstr}[s_2] : \mathsf{stringin}[r_2]$ and $\emptyset \vdash \mathsf{rconcat}(\mathsf{rstr}[s_1]; \mathsf{rstr}[s_2]) : \mathsf{stringin}[r_1 \cdot r_2]$. By Canonical Forms, $s_1 \in \mathcal{L}\{r_1\}$ and $s_2 \in \mathcal{L}\{r_2\}$ from which it follows by Lemma B that $s_1 s_2 \in \mathcal{L}\{r_1 \cdot r_2\}$. Therefore, $\emptyset \vdash \mathsf{rstr}[s_1 s_2] : \mathsf{stringin}[r_1 \cdot r_2]$ by S-T-Rstr.

S-E-Case-Left. Suppose $e \mapsto \mathsf{rstrcase}(e_1'; e_2; x, y.e_3)$ and $\emptyset \vdash e : \sigma$ and $e_1 \mapsto e_1'$. The only rule that applies is S-T-Case, so:

(6) $$\emptyset \vdash e_1 : \mathsf{stringin}[r]$$

(7) $$\emptyset \vdash e_2 : \sigma$$

(8) $$\emptyset, x : \mathsf{stringin}[\mathsf{lhead}(r)], y : \mathsf{stringin}[\mathsf{ltail}(r)] \vdash e_3 : \sigma$$

By (6) and the assumption that $e_1 \mapsto e_1'$, it follows by induction that $\emptyset \vdash e_1' : \mathsf{stringin}[r]$. This fact together with (7) and (8) implies by S-T-Case that $\emptyset \vdash \mathsf{rstrcase}(e_1'; e_2; x, y.e_3) : \sigma$.

SS-E-Case-ϵ-Val. Suppose $\mathsf{rstrcase}(e_0; e_2; x, y.e_3) \mapsto e_2$. The only rule that applies is S-T-Case, so $\emptyset \vdash e_2 : \sigma$.

SS-E-Case-Concat. Suppose that $e = \mathsf{rstrcase}(\mathsf{rstr}[as]; e_2; x, y.e_3) \mapsto [\mathsf{rstr}[a], \mathsf{rstr}[s]/x, y]e_3$ and that $\emptyset \vdash e : \sigma$. The only rule that applies is S-T-Case so:

(9) $$\emptyset \vdash \mathsf{rstr}[as] : \mathsf{stringin}[r]$$

(10) $$\emptyset \vdash e_2 : \sigma$$

(11) $$\emptyset, x : \mathsf{stringin}[\mathsf{lhead}(r)], y : \mathsf{stringin}[\mathsf{ltail}(r)] \vdash e_3 : \sigma$$

We know that $as \in \mathcal{L}\{r\}$ by Canonical Forms on (9) Therefore, $a \in \mathcal{L}\{\mathsf{lhead}(r)\}$ by Condition C and $s \in \mathcal{L}\{\mathsf{ltail}(r)\}$ by Condition D.

From these facts about a and s we know by S-T-Rstr that $\emptyset \vdash \mathsf{rstr}[a] : \mathsf{stringin}[\mathsf{lhead}(r)]$ and $\emptyset \vdash \mathsf{rstr}[s] : \mathsf{stringin}[\mathsf{ltail}(r)]$. It follows by Assumption H that $\emptyset \vdash [\mathsf{rstr}[a], \mathsf{rstr}[s]/x, y]e_3 : \sigma$.

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Cases SS-E-Replace-Left, SS-E-Replace-Right, SS-E-Check-StepLeft, SS-E-SafeCoerce-Step, SS-E-Check-StepRight.

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Case SS-E-Replace.

Suppose $e = \mathsf{rreplace}[r](\mathsf{rstr}[s_1]; \mathsf{rstr}[s_2]) \mapsto \mathsf{rstr}[\mathsf{replace}(r; s_1; s_2)]$. The only applicable rule is S-T-Replace, so

$$\emptyset \vdash \mathsf{rstr}[s_1] : \mathsf{stringin}[r_1]$$
$$\emptyset \vdash \mathsf{rstr}[s_2] : \mathsf{stringin}[r_2]$$

By conanical forms, $s_1 \in \mathcal{L}\{r_1\}$ and $s_2 \in \mathcal{L}\{r_2\}$. Therefore, $\texttt{lreplace}(r, s_1, s_2) \in \mathcal{L}\{\texttt{lreplace}(r, r_1, r_2)\}$ by Theorem E. It is finally derivable by S-T-Rstr that:

$\emptyset \vdash \mathsf{rstr}[\texttt{lreplace}(r, s_1, s_2)] : \mathsf{stringin}[\texttt{lreplace}(r, r_1, r_2)]$.

Case SS-E-SafeCoerce. Suppose that $\mathsf{rcoerce}[r](\mathsf{rstr}[s_1]) \mapsto \mathsf{rstr}[s_1]$. The only applicable rule is S-T-SafeCoerce, so $\emptyset \vdash \mathsf{rcoerce}[r](s_1) : \mathsf{stringin}[r]$. By Canonical Forms, $s \in \mathcal{L}\{r\}$. Therefore, $\emptyset \vdash \mathsf{rstr}[s] : \mathsf{stringin}[r]$.

Case SS-E-Check-Ok. Suppose $\mathsf{rcheck}[r](\mathsf{rstr}[s]; x.e_1; e_2) \mapsto [\mathsf{rstr}[s]/x]e_1$, $s \in \mathcal{L}\{r\}$, and $\emptyset \vdash \mathsf{rcheck}[r](\mathsf{rstr}[s]; x.e_1; e_2) : \sigma$. By inversion of S-T-Check, $x : \mathsf{stringin}[r] \vdash e_1 : \sigma$. Note that $s \in \mathcal{L}\{r\}$ implies that $s : \mathsf{stringin}[r]$ by S-T-RStr. Therefore, $\emptyset \vdash [\mathsf{rstr}[s]/x]e_1 : \sigma$.

Case SS-E-Check-NotOk. Suppose $\mathsf{rcheck}[r](\mathsf{rstr}[s]; x.e_1; e_2) \mapsto e_2$, $s \notin \mathcal{L}\{r\}$, and $\emptyset \vdash \mathsf{rcheck}[r](\mathsf{rstr}[s]; x.e_1; e_2) : \sigma$. The only applicable rule is S-T-Check, so $\emptyset \vdash e_2 : \sigma$.

□

Theorem J (Type Safety for small step semantics.)**.** *If* $\emptyset \vdash e : \sigma$ *then either* e val *or* $e \mapsto^* e'$ *and* $\emptyset \vdash e' : \sigma$.

Proof. Follows directly from progress and preservation. □

3.3.1 Semantic Correspondence between Big and Small Step Semantics for λ_{RS}

Before extending the previous theorem to the big step semantics, we first establish a correspondence between the big step semantics in Figure 7 and the small step semantics in Figure 5.

Lemma K. *If* $e \Downarrow v$ *and* $e \mapsto e'$ *then* $e' \Downarrow v$.

Proof. By induction on the structure of e.

Case $e = e_1(e_2)$. The only applicable rule is S-E-App, so $e_1 \Downarrow \lambda x.e_3$ and $e_2 \Downarrow v_2$ such that $[v_2/x]e_3 \Downarrow v$.

The term $e = e_1(e_2)$ may step by three rules.

First, if e steps by L-E-AppLeft then $e_1(e_2) \mapsto e_1'(e_2)$. By induction, $e_1' \Downarrow \lambda x.e_3$. By S-E-App, $e_1'(e_2) \Downarrow \lambda x.e_3$.

Second, if e steps by L-E-AppRight then $e_1(e_2) \mapsto e_1(e_2')$. By induction, $e_2' \Downarrow v_2$. By S-E-App, $e_1(e_2') \Downarrow \lambda x.e_3$.

Third, if e steps by L-E-AppAbs then $e = (\lambda x.e_3)(v_2) \mapsto [v_2/x]e_3$. By induction, $e = (\lambda x.e')v_2 \Downarrow v$.

□

Theorem L (Semantic Correspondence for λ_{RS} (Part I))**.** *If* $e \Downarrow v$ *then* $e \mapsto^* v$.

Proof. We proceed by structural induction on e.

Case $e = \lambda x.e_1$. The only applicable rule is S-E-Abs, so $v = \lambda x.e_1$. Note that $\lambda x.e_1 \mapsto^* \lambda x.e_1$ by RT-Refl.

Case $e = e_1(e_2)$. The only applicable rule is S-E-App. By inversion:

$$\begin{aligned} e_1 &\Downarrow \lambda x.e_1' \\ e_2 &\Downarrow v_2 \\ [v_2/x]e_1' &\Downarrow v \end{aligned}$$

From which it follows by induction that:

$$\begin{aligned} e_1 &\mapsto^* \lambda x.e_1' \\ e_2 &\mapsto^* v_2 \\ [v_2/x]e_1' &\mapsto^* v \end{aligned}$$

If $e_1 = \lambda x.e_1'$ and $e_2 = v_2$ (henceforth the reflexive case) then $e \mapsto [v_2/x]e_1'$ and the conclusion follows by RT-Trans.

If $e_1 \mapsto \lambda x.e_1'$ then $e_1(e_2) \mapsto (\lambda x.e_1')(e_2) \mapsto [e_2/x]e_1'$ and the conclusion follows by two applications of RT-Trans.

If $e_1 \mapsto^k \lambda x.e_2'$ then $e_1 \mapsto e'$ so $e_1(e_2) \mapsto e'(e_2)$ and by Lemma K $e'(e_2) \mapsto^* v$. So $e_1(e_2) \mapsto e'(e_2) \mapsto^* v$; it follows by RT-Trans that $e_1(e_2) \mapsto v$.

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Note that the following rule is derivable by repeating applications of the left and right compatibility rules for application:

$$\text{L*-App}\ \frac{e_1 \mapsto^* e_1' \qquad e_2 \mapsto^* e_2'}{e_1(e_2) \mapsto^* e_1'(e_2')}$$

From these facts and L-AppAbs, we may establish that $e_1(e_2) \mapsto^* (\lambda x.e_2)(v_2) \mapsto [v_2/x]e_2$. Note that $[v_2/x]e_2 \mapsto^* v$, so by RT-Trans it follows that $e = e_1(e_2) \mapsto^* v$.

Case $e = \mathsf{rstr}[s]$. The only applicable rule is S-E-RStr, so $v = \mathsf{rstr}[s]$. By RT-Refl, $\mathsf{rstr}[s] \mapsto^* \mathsf{rstr}[s]$.

Case $e = \mathsf{rconcat}(e_1; e_2)$. The only applicable rule is S-E-Concat, so $v = \mathsf{rstr}[s_1 s_2]$. By inversion, $e_1 \Downarrow \mathsf{rstr}[s_1]$ and $e_2 \Downarrow \mathsf{rstr}[s_2]$. By induction, $e_1 \mapsto^* \mathsf{rstr}[s_1]$ and $e_2 \mapsto^* \mathsf{rstr}[s_2]$. Note that the rule following is derivable:

$$\text{SS-E-Concat-LR*}\ \frac{e_1 \mapsto^* e_1' \qquad e_2 \mapsto^* e_2'}{\mathsf{rconcat}(e_1; e_2) \mapsto^* \mathsf{rconcat}(e_1'; e_2')}$$

From these facts, it follows that $\mathsf{rconcat}(e_1; e_2) \mapsto^* \mathsf{rconcat}(\mathsf{rstr}[s_1]; \mathsf{rstr}[s_2])$. Finally, $\mathsf{rconcat}(\mathsf{rstr}[s_1]; \mathsf{rstr}[s_2]) \mapsto \mathsf{rstr}[s_1 s_2]$ by SS-E-Concat. By RT-Step, it follows that $\mathsf{rconcat}(e_1; e_2) \mapsto^* \mathsf{rstr}[s_1 s_2]$.

Case $e = \mathsf{rstrcase}(e_1; e_2; x, y.e_3)$.

There are two subcases. For the first, suppose $\mathsf{rstrcase}(e_1; e_2; x, y.e_3) \Downarrow v$ was finally derived by S-E-Case-ϵ. By inversion:

$$e_1 \Downarrow \mathsf{rstr}[\epsilon]$$
$$e_2 \Downarrow v$$

from which it follows by induction that:

$$e_1 \mapsto^* \mathsf{rstr}[\epsilon]$$
$$e_2 \mapsto^* v$$

Note that the following rule is derivable:

$$\text{SS-E-CASE-LR*}\quad \frac{e_1 \mapsto^* e_1' \qquad e_2 \mapsto^* e_2'}{\mathsf{rstrcase}(e_1; e_2; x, y.e_3) \mapsto^* \mathsf{rstrcase}(e_1'; e_2'; x, y.e_3)}$$

From these facts is follows that $e \mapsto^* \mathsf{rstrcase}(\mathsf{rstr}[\epsilon]; v; x, y.e_3)$. By S-E-Case-$\epsilon$-Val and RT-Step it follows that $e \mapsto^* v$.

Now consider the other case where $\mathsf{rstrcase}(e_1; e_2; x, y.e_3) \Downarrow v$ was finally derived by S-E-Case-Concat. By inversion, $e_1 \Downarrow \mathsf{rstr}[as]$ and $[\mathsf{rstr}[a], \mathsf{rstr}[s]/x, y]e_3 \Downarrow v$. From these facts it follows by induction that $e_1 \mapsto^* \mathsf{rstr}[as]$ and $[\mathsf{rstr}[a], \mathsf{rstr}[s]/x, y]e_3 \mapsto^* v$.

By the first of these facts, it is derivable via SS-E-Case-LR* that $e \mapsto^* \mathsf{rstrcase}(e_1'; \mathsf{rstr}[as]; x, y.e_3)$. SE-E-Case-Concat applies to this form, so by RT-Step we know $e \mapsto^* [\mathsf{rstr}[a], \mathsf{rstr}[s]/x, y]e_3$. Recall that $[\mathsf{rstr}[a], \mathsf{rstr}[s]/x, y]e_3 \mapsto^* v$, so by RT-Trans we finally derive $e \mapsto^* v$.

Case $e = \mathsf{rreplace}[r](e_1; e_2)$. There is only one applicable rule, so $v = \mathsf{rstr}[s]$ and by inversion it follows that:

$$e_1 \Downarrow \mathsf{rstr}[s_1]$$
$$e_2 \Downarrow \mathsf{rstr}[s_2]$$

From which it follows by induction that:

$$e_1 \mapsto^* \mathsf{rstr}[s_1]$$
$$e_2 \mapsto^* \mathsf{rstr}[s_2]$$

Furthermore, $\mathsf{replace}(r; s_1; s_2) = s$ by induction. Note that the following rule is derivable:

$$\text{SS-E-REPLACE-LR*}\quad \frac{e_1 \mapsto^* e_1' \qquad e_2 \mapsto^* e_2'}{\mathsf{rreplace}[r](e_1; e_2) \mapsto^* \mathsf{rreplace}[r](e_1'; e_2')}$$

From these facts, $\mathsf{rreplace}[r](e_1; e_2) \mapsto^* \mathsf{rreplace}[r](\mathsf{rstr}[s_1]; \mathsf{rstr}[s_2])$.

Finally, $\mathsf{rreplace}[r](\mathsf{rstr}[s_1]; \mathsf{rstr}[s_2]) \mapsto \mathsf{replace}(r; s_1; s_2)$.

From these two facts we know via RT-Step that $\mathsf{rreplace}[r](e_1; e_2) \mapsto^* \mathsf{rreplace}[r](e_1; e_2)$. Recall that $\mathsf{replace}(r; s_1; s_2) = s$, from which the conclusion follows.

Case $e = \mathsf{rcoerce}[r](e_1)$. In this case $e \Downarrow v$ is only finally derivable via S-E-SafeCoerce. Therefore, $v = \mathsf{rstr}[s]$ and by inversion $e_1 \Downarrow \mathsf{rstr}[s]$. By induction, $e_1 \mapsto^* \mathsf{rstr}[s]$.

The following rule is derivable:

$$\frac{e \mapsto^* e'}{\mathsf{rcoerce}[r](e) \mapsto^* \mathsf{rcoerce}[r](e')}\ \text{SS-E-SafeCoerce-Step}$$

Applying this rule at $e_1 \mapsto^* \mathsf{rstr}[s]$ derives $\mathsf{rcoerce}[r](e_1) \mapsto^* \mathsf{rcoerce}[r](\mathsf{rstr}[s])$. In the final step, $\mathsf{rcoerce}[r](\mathsf{rstr}[s]) \mapsto \mathsf{rstr}[s]$ by SS-E-SafeCoerce. From this fact, we may derive via RT-Trans that $e \mapsto^* \mathsf{rstr}[s]$ as required.

Case $e = \mathsf{rcheck}[r](e_1; x.e_2; e_3)$.

Note that the rule following is derivable:

$$\frac{e_1 \mapsto^* e_1' \qquad e_3 \mapsto^* e_3'}{\mathsf{rcheck}[r](e_1; x.e_2; e_3) \mapsto^* \mathsf{rcheck}[r](e_1'; x.e_2; e_3')}\ \text{SS-E-Check-Step}$$

There are two ways to finally derive $e \Downarrow v$. In both cases, $e_1 \Downarrow \mathsf{rstr}[s]$ by inversion. Therefore, in both cases, $e_1 \mapsto^* \mathsf{rstr}[s]$ by induction and so $e \mapsto^* \mathsf{rcheck}[r](\mathsf{rstr}[s]; x.e_2; e_3)$ by SS-E-Check-Step.

Suppose $e \Downarrow v$ is finally derived via SS-E-Check-Ok. By the facts mentioned above and SS-E-Check-Step, $e \mapsto^* \mathsf{rcheck}[r](\mathsf{rstr}[s]; x.e_2; e_2)$. Note that by inversion $s \in \mathcal{L}\{r\}$. Therefore, SS-E-Check-Ok applies and so by RT-Trans $e \mapsto^* [\mathsf{rstr}[s]/x]e_1$. By inversion, $[\mathsf{rstr}[s]/x]e_1 \Downarrow v$. Therefore, by induction and RT-Step $e \mapsto^* v$ as required.

Suppose that $e \Downarrow v$ is instead finally derived via SS-E-Check-NotOk. By inversion, $e_3 \Downarrow v$ and by induction $e_3 \mapsto^* v$. From these facts at SS-E-Check-Step, it is derivable that $e \mapsto^* \mathsf{rcheck}[r](\mathsf{rstr}[s]; x.e_2; v)$.

Also by inversion, $s \notin \mathcal{L}\{r\}$ and so SS-E-Check-NotOk applies. Therefore, $\mathsf{rcheck}[r](\mathsf{rstr}[s]; x.e_2; v) \mapsto v$.

The conclusion $e \mapsto^* v$ follows from these facts by RT-Step.

□

Theorem M (Semantic Correspondence for λ_{RS} (Part II))**.** *If* $\emptyset \vdash e : \sigma$, $e \mapsto^* v$ *and* v val *then* $e \Downarrow v$.

Proof. The proof proceeds by structural induction on e.

Case $e = \mathsf{concat}(e_1; e_2)$. By inversion, $\emptyset \vdash e_1 : \mathsf{stringin}[r_1]$. By Type Safety, Canonical Forms and Termination it follows that $e_1 \mapsto^* \mathsf{rstr}[s_1]$ for some s_1. By induction, $e_1 \Downarrow \mathsf{rstr}[s_1]$.

Similarly, $e_2 \mapsto^* \mathsf{rstr}[s_2]$ and $e_2 \Downarrow \mathsf{rstr}[s_2]$.

Note that $\mathsf{concat}(e_1; e_2) \mapsto^* \mathsf{concat}(\mathsf{rstr}[s_1]; \mathsf{rstr}[s_2]) \mapsto \mathsf{rstr}[s_1 s_2]$ by SS-E-Concat-LR* and S-E-Concat. Therefore, $e \mapsto^* \mathsf{rstr}[s_1 s_2]$ by RT-Step. So it suffices to show that $e \Downarrow \mathsf{rstr}[s_1 s_2]$.

Finally, $e \Downarrow \mathsf{rstr}[s_1 s_2]$ follows via S-E-Concat from the facts that $e_1 \Downarrow \mathsf{rstr}[s_1]$ and $e_2 \Downarrow \mathsf{rstr}[s_2]$. This completes the case.

Case $e = \mathsf{rreplace}[r](e_1; e_2)$. By inversion of S-T-Replace, $\emptyset \vdash e_1 : \mathsf{stringin}[r_1]$ for some r_1. It follows by Type Safety, Termination and Canonical Forms that $e_1 \mapsto^* \mathsf{rstr}[s_1]$. By induction, $e_1 \Downarrow \mathsf{rstr}[s_1]$.

Similarly, $e_2 \mapsto^* \mathsf{rstr}[s_2]$ and $e_2 \Downarrow \mathsf{rstr}[s_2]$.

Note that $e \mapsto^* \mathsf{rreplace}[r](\mathsf{rstr}[s_1]; \mathsf{rstr}[s_2]) \mapsto \mathsf{rstr}[\mathsf{replace}(r; s_1; s_2)]$ by SS-Replace-LR* and SS-E-Replace. Therefore $e \mapsto^* \mathsf{rstr}[\mathsf{replace}(r; s_1; s_2)]$ by RT-Step.

It suffices to show $e \Downarrow \mathsf{rstr}[\mathsf{replace}(r; s_1; s_2)]$, which follows by S-E-Replace from the facts that $e_1 \Downarrow \mathsf{rstr}[s_1]$ and $e_2 \Downarrow \mathsf{rstr}[s_2]$.

Case $e = \mathsf{rstrcase}(e_1; e_2; x.y.e_3)$. By inversion, $\emptyset \vdash e_1 : \mathsf{stringin}[r]$ and $e_2 : \sigma$. By Type Safety, Canonical Forms and Termination $e_1 \mapsto^* \mathsf{stringin}[s_1]$ and by induction $e_1 \Downarrow \mathsf{stringin}[s_1]$. Similarly, $e_2 \mapsto^* v_2$ and $\emptyset \vdash e_2 \Downarrow v_2$.

By SS-E-Case-LR*, $\mathsf{rstrcase}(e_1; e_2; x, y.e_3) \mapsto^* \mathsf{rstrcase}(v_1; v_2; x, y.e_3)$.

Note that either $s_1 = \epsilon$ or $s_1 = as$ because we define strings as either empty or finite sequences of characters. We proceed by cases.

If $s_1 = \epsilon$ then $\mathsf{rstrcase}(v; v_2; x, y.e_3) \mapsto v_2$ by SS-E-Case-ϵ. Therefore, by RT-Step, $e \mapsto^* v_2$. Recall $e_1 \Downarrow \mathsf{rstr}[\epsilon]$ and $e_2 \Downarrow v_2$, which is enough to establish by S-E-Case-ϵ that $e \Downarrow v_2$.

If $s_1 = as$ instead, then $\mathsf{rstrcase}(\mathsf{rstr}[s_1]; v_2; x, y.e_3) \mapsto [\mathsf{rstr}[a], \mathsf{rstr}[s]/x, y]e_3$ by SS-E-Case-Concat. Inversion of the typing relation satisfies the assumptions necessary to appeal to termination. Therefore,

$$[\mathsf{rstr}[a], \mathsf{rstr}[s]/x, y]e_3 \mapsto^* v \text{ for } v\ \mathsf{val}.$$

It follows by RT-Step that $e \mapsto^* v$.

Note that the substitution does not change the structure of e_3. So by induction, $[\mathsf{rstr}[a], \mathsf{rstr}[s]/x, y]e_3 \Downarrow v$. Recall that $e_1 \Downarrow \mathsf{rstr}[s_1]$ and so by S-E-Case it follows that $e \Downarrow [a, s/x, y]e_3 \Downarrow v$.

□

The cases for coercion and checking are straightforward.

3.4 Extension of Safety for Small Step Semantics

Theorem 4 (Type Safety)**.** *If* $\emptyset \vdash e : \sigma$ *and* $e \Downarrow e'$ *then* $\emptyset \vdash e' : \sigma$.

Proof. If $\emptyset \vdash e : \sigma$ then $e \mapsto^* e'$. Therefore, $e \Downarrow e'$ by part 2 of the semantic correspondence theorem. Since $\emptyset \vdash e : \sigma$ and $e \mapsto^* e'$, it follows that $\emptyset \vdash e' : \sigma$ by type safety for the small step semantics. □

3.4.1 The Security Theorem

Theorem 5 (Correctness of Input Sanitation for λ_{RS})**.** *If* $\emptyset \vdash e : \mathsf{stringin}[r]$ *and* $e \Downarrow \mathsf{rstr}[s]$ *then* $s \in \mathcal{L}\{r\}$.

Proof. If $\emptyset \vdash e : \mathsf{stringin}[r]$ and $e \Downarrow \mathsf{rstr}[s]$ then $\emptyset \vdash \mathsf{rstr}[s] : \mathsf{stringin}[r]$ by Type Safety. By Caonical Forms, $s \in \mathcal{L}\{r\}$. □

4 Proofs of Lemmas and Theorems About λ_P

This section follows the same structure as the safety proof for λ_{RS} – we prove type safety for a small-step semantics, prove a semantic correspondence, and then transfer the safety result to the big-step semantics in the paper.

Our proofs here are more sketchy than in the λ_{RS} section both because the proof method is already demonstrated, and becaues λ_P is a fairly straightforward extension of λ for which these results are already estbalished.

Lemma 6 (Canonical Forms for Target Language)**.**

- *If* $\emptyset \vdash \iota : \mathsf{regex}$ *then* $\iota \Downarrow \mathsf{rx}[r]$ *such that* r *is a well-formed regular expression.*
- *If* $\emptyset \vdash \iota : \mathsf{string}$ *then* $\iota \Downarrow \mathsf{str}[s]$.

Theorem 7 (Progress)**.** *If* $\emptyset \vdash \iota : \tau$ *either* $\iota = \dot{v}$ *or* $\iota \mapsto \iota'$ *for some* ι'.

Proof. The proof proceeds by induction on the typing assumption. We elide contexts where context has obviously not changed.

Consider only the string and regex (non-λ) fragments of λ_P.

Regular expressions are always values.

For strings, the concatenation cases are trivial.

P-T-Case. Suppose $\emptyset \vdash \mathsf{strcase}(\iota_1; \iota_2; x, y.\iota_3)$. By inversion, $\iota_1 : \mathsf{string}$ and so either $\iota_1 \mapsto \iota_1'$ or by canonical forms, $\iota_1 = \mathsf{str}[s_1]$. Similarly, $\iota_2 \mapsto \iota_2'$ or else $\iota_2 = \mathsf{str}[s_2]$. In the former cases, progress occurs via the compatibility rules. in the case where both are string values, progress occurs via the case concatenation rule.

P-T-Replace. Suppose $\emptyset \vdash \mathsf{replace}(\iota_1; \iota_2; \iota_3)$. By inversion, $\iota_1 : \mathsf{regex}$ and so by canonical forms $\iota_1 = \mathsf{rx}[r]$. By inversion, $\iota_2 : \mathsf{string}$ and so by induction either $\iota_2 \mapsto \iota_2'$ or else $\iota_2 = \mathsf{str}[s_2]$ for some string s_2. Similarly, either ι_3 steps or else $\iota_3 = \mathsf{str}[s_3]$. In case any steps occur, progress occurs. In the remaining case, PP-E-Replace applies and so progress occurs.

P-T-Check. Finally, suppose $\emptyset \vdash \check{\iota_x}\iota_1\iota_2\iota_3$. In case any of these step, then progress occurs. In the remaining cases, applications of inversion and canonical forms for each ι_x and ι_1 implies that the term at hand equals $\mathsf{rx}\check{[}r\check{]}\mathsf{str}[s]\iota_2\iota_3$, which evaluates to either ι_2 or ι_3.

□

Theorem 8 (Preservation)**.** *If* $\emptyset \vdash \iota : \tau$ *and* $\iota \mapsto \iota'$ *then* $\emptyset \vdash \iota' : \tau$.

Proof. □

Fill in if there's time. Otherwise there's nothing interesting going on.

Theorem 9 (Safety for Small-Step Semantics)**.** *If* $\iota \Downarrow \dot{v}$ *and* $\iota \mapsto \iota'$ *then* $\iota' \Downarrow \dot{v}$.

Proof. A direct result from progress and preservation. □

We only prove semantic correspondence in one direction; again, whereas λ_{RS} proofs were detailed, here we provide a less verbosy proof.

Theorem 10 (Semantic Correspondence)**.** *If* $\iota \mapsto^* \iota'$ *then* $\iota \Downarrow \iota'$.

Proof. By induction on the structure of ι.

□

Theorem 11 (Safety for λ_P)**.** *If $\emptyset \vdash \iota : \tau$ then $\iota \Downarrow \dot{v}$ and $\emptyset \vdash \dot{v} : \tau$.*

Proof. If $\emptyset \vdash \iota : \tau$ then $\iota \mapsto^* \iota'$. Therefore, $\iota \Downarrow \iota'$ by part 2 of the semantic correspondence theorem.
Since $\emptyset \vdash \iota : \tau$ and $\iota \mapsto^* \iota'$, it follows that $\emptyset \vdash \iota' : \tau$ by type safety for the small step semantics.

□

We can also define canonical forms for regular expressions and strings in the usual way:

5 Proofs and Lemmas and Theorems About Translation

Theorem 12 (Translation Correctness)**.** *If $\Theta \vdash e : \sigma$ then there exists an ι such that $[\![e]\!] = \iota$ and $[\![\Theta]\!] \vdash \iota : [\![\sigma]\!]$. Furthermore, $e \Downarrow v$ and $\iota \Downarrow \dot{v}$ such that $[\![v]\!] = \dot{v}$.*

Proof. We present a proof by induction on the derivation that $\Theta \vdash e : \sigma$. we write $e \rightsquigarrow \iota$ as shorthand for the final property.

Case $e = \mathsf{rstr}[s]$. Suppose $\Theta \vdash \mathsf{rstr}[s] : \sigma$.

By examination the syntactic structure of conclusions in the relation S-T, we know this is true just in case $\sigma = \mathsf{stringin}[r]$ for some r such that $s \in \mathcal{L}\{r\}$; and of course, there is always such an r.

There are no free variables in $\mathsf{rstr}[s]$, so we might as well proceed from the fact that $\emptyset \vdash \mathsf{rstr}[s] : \mathsf{stringin}[r]$.

By definition of the translation ($[\![\cdot]\!]$) the following statements hold:

$$[\![\mathsf{rstr}[s]]\!] = \mathsf{string}s \tag{12}$$

$$[\![\mathsf{stringin}[r]]\!] = \mathsf{string} \tag{13}$$

$$[\![\emptyset]\!] = \emptyset \tag{14}$$

Note that $\emptyset \vdash \mathsf{string}s : \mathsf{string}$ by P-T-Str. Recall that contexts are standard and, in particular, can be weakened. So since $[\![\Theta]\!]$ is either a weakening of $\emptyset$ or $\emptyset$ itself, $[\![\Theta]\!] \vdash \mathsf{str}[s] : \mathsf{string}$ by weakening.

Summarily, $\mathsf{string}s$ is a term of λ_P such that $[\![\Theta]\!] \vdash \mathsf{string}s : [\![\sigma]\!]$

It remains to be shown that there exist $v, \dot{v}$ such that $\mathsf{rstr}[s] \Downarrow v$, $\mathsf{string}s \Downarrow \dot{v}$, and $[\![v]\!] = \dot{v}$. But this is immediate because each term evaluates to itself and we have already established the equality.

Case $e = \mathsf{rconcat}(e_1; e_2)$. This case is an obvious appeal to induction.

Case $e = \mathsf{rstrcase}(e_1; e_2; x, y.e_3)$. This case relies on our definition of context translation.

Suppose $\Psi \vdash \mathsf{rstrcase}(e_1; e_2; x, y.e_3) : \sigma$. By inversion of the typing relation it follows that $\Psi \vdash e_1 : \mathsf{stringin}[r]$, $\Psi \vdash e_2 : \sigma$ and $\Psi, x : \mathsf{stringin}[\mathsf{lhead}(r)], y : \mathsf{stringin}[\mathsf{ltail}(r)] \vdash e_3 : \sigma$.

By induction, there exists an ι_1 such that $[\![e_1]\!] = \iota_1$, $[\![\Psi]\!] \vdash \iota_1 : [\![\sigma]\!]$, and $e_1 \rightsquigarrow \iota_1$. Similarly for e_2 and some ι_2.

By canonical forms, $e_1 \Downarrow \mathsf{rstr}[s]$ and so $\iota_1 \Downarrow \mathsf{str}[s]$ by $\rightsquigarrow$.

Choose $\iota = \mathsf{concat}(\iota_1; \iota_2)x, y.\iota_3$ and note that by the properties established via induction, $[\![e]\!] = \iota$ and $[\![\Psi]\!] \vdash \iota : [\![\sigma]\!]$.

Suppose $s = \epsilon$. Then $e \Downarrow v$ where $e_2 \Downarrow v$ and $\iota \Downarrow \dot{v}$ where $\iota_2 \Downarrow \dot{v}$. But recall that $e_2 \rightsquigarrow v_2$ and so $[\![v]\!] = \dot{v}$.

Suppose otherwise that $s = at$ for some character a and string t. Then $e \Downarrow v$ where $[a, t/x, y]e_3 \Downarrow v$. Similarly, $\iota \Downarrow \dot{v}$ where $[a, t/x, y]\iota_3 \Downarrow \dot{v}$

□

Theorem 13 (Correctness of Input Sanitation for Translated Terms)**.** *If* $[\![e]\!] = \iota$ *and* $\emptyset \vdash e : \mathsf{stringin}[r]$ *then* $\iota \Downarrow \mathsf{str}[s]$ *for* $s \in \mathcal{L}\{r\}$.

Proof. By Theorem 12 and the rules given, $\iota \Downarrow \mathsf{str}[s]$ implies that $e \Downarrow \mathsf{rstr}[s]$. Theorem 5 together with the assumption that e is well-typed implies that $s \in \mathcal{L}\{r\}$. □

References

[1] N. Fulton, C. Omar, and J. Aldrich. Statically typed string sanitation inside a python. SPLASH '14. ACM, 2014.

$$r \quad ::= \quad \epsilon \mid . \mid a \mid r \cdot r \mid r + r \mid r* \qquad\qquad a \in \Sigma$$

Figure 1: Regular expressions over the alphabet Σ.

$$\begin{array}{lll}
\sigma & ::= & \sigma \to \sigma \mid \mathsf{stringin}[r] & \text{source types} \\
e & ::= & x \mid v & \text{source terms} \\
 & \mid & \mathsf{rconcat}(e; e) \mid \mathsf{rstrcase}(e; e; x, y.e) & s \in \Sigma^* \\
 & \mid & \mathsf{rreplace}[r](e; e) \mid \mathsf{rcoerce}[r](e) \mid \mathsf{rcheck}[r](e; x.e; e) & \\
v & ::= & \lambda x.e \mid \mathsf{rstr}[s] & \text{source values}
\end{array}$$

Figure 2: Syntax of λ_{RS}.

$$\begin{array}{lll}
\tau & ::= & \tau \to \tau \mid \mathsf{string} \mid \mathsf{regex} & \text{target types} \\
\iota & ::= & x \mid \dot{v} & \text{target terms} \\
 & \mid & \mathsf{concat}(\iota; \iota) \mid \mathsf{strcase}(\iota; \iota; x, y.\iota) & \\
 & \mid & \mathsf{rx}[r] \mid \mathsf{replace}(\iota; \iota; \iota) \mid \mathsf{check}(\iota; \iota; \iota; \iota) & \\
\dot{v} & ::= & \lambda x.\iota \mid \mathsf{str}[s] \mid \mathsf{rx}[r] & \text{target values}
\end{array}$$

Figure 3: Syntax for the target language, λ_P, containing strings and statically constructed regular expressions.

$\boxed{\Psi \vdash e : \sigma}$ $\quad \Psi ::= \emptyset \mid \Psi, x : \sigma$

$$\text{S-T-VAR}\ \frac{x : \sigma \in \Psi}{\Psi \vdash x : \sigma} \qquad \text{S-T-ABS}\ \frac{\Psi, x : \sigma_1 \vdash e : \sigma_2}{\Psi \vdash \lambda x.e : \sigma_1 \to \sigma_2} \qquad \text{S-T-APP}\ \frac{\Psi \vdash e_1 : \sigma_2 \to \sigma \qquad \Psi \vdash e_2 : \sigma_2}{\Psi \vdash e_1(e_2) : \sigma}$$

$$\text{S-T-STRINGIN-I}\ \frac{s \in \mathcal{L}\{r\}}{\Psi \vdash \mathsf{rstr}[s] : \mathsf{stringin}[r]}$$

$$\text{S-T-CONCAT}\ \frac{\Psi \vdash e_1 : \mathsf{stringin}[r_1] \qquad \Psi \vdash e_2 : \mathsf{stringin}[r_2]}{\Psi \vdash \mathsf{rconcat}(e_1; e_2) : \mathsf{stringin}[r_1 \cdot r_2]}$$

$$\text{S-T-CASE}\ \frac{\Psi \vdash e_1 : \mathsf{stringin}[r] \qquad \Psi \vdash e_2 : \sigma \qquad \Psi, x : \mathsf{stringin}[\mathsf{lhead}(r)], y : \mathsf{stringin}[\mathsf{ltail}(r)] \vdash e_3 : \sigma}{\Psi \vdash \mathsf{rstrcase}(e_1; e_2; x, y.e_3) : \sigma}$$

$$\text{S-T-REPLACE}\ \frac{\Psi \vdash e_1 : \mathsf{stringin}[r_1] \qquad \Psi \vdash e_2 : \mathsf{stringin}[r_2]}{\Psi \vdash \mathsf{rreplace}[r](e_1; e_2) : \mathsf{stringin}[\mathtt{lreplace}(r, r_1, r_2)]} \qquad \text{S-T-SAFECOERCE}\ \frac{\Psi \vdash e : \mathsf{stringin}[r'] \qquad \mathcal{L}\{r'\} \subseteq \mathcal{L}\{r\}}{\Psi \vdash \mathsf{rcoerce}[r](e) : \mathsf{stringin}[r]}$$

$$\text{S-T-CHECK}\ \frac{\Psi \vdash e_0 : \mathsf{stringin}[r_0] \qquad \Psi, x : \mathsf{stringin}[r] \vdash e_1 : \sigma \qquad \Psi \vdash e_2 : \sigma}{\Psi \vdash \mathsf{rcheck}[r](e_0; x.e_1; e_2) : \sigma}$$

Figure 4: Typing rules for λ_{RS}. The typing context Ψ is standard.

$\boxed{e \Downarrow v}$

$$\text{S-E-Abs}\ \frac{}{\lambda x.e \Downarrow \lambda x.e} \qquad \text{S-E-App}\ \frac{e_1 \Downarrow \lambda x.e_3 \qquad e_2 \Downarrow v_2 \qquad [v_2/x]e_3 \Downarrow v}{e_1(e_2) \Downarrow v} \qquad \text{S-E-RStr}\ \frac{}{\mathsf{rstr}[s] \Downarrow \mathsf{rstr}[s]} \qquad \text{S-E-Concat}\ \frac{e_1 \Downarrow \mathsf{rstr}[s_1] \qquad e_2 \Downarrow \mathsf{rstr}[s_2]}{\mathsf{rconcat}(e_1; e_2) \Downarrow \mathsf{rstr}[s_1 s_2]}$$

$$\text{S-E-Case-}\epsilon\ \frac{e_1 \Downarrow \mathsf{rstr}[\epsilon] \qquad e_2 \Downarrow v_2}{\mathsf{rstrcase}(e_1; e_2; x, y.e_3) \Downarrow v_2} \qquad \text{S-E-Case-Concat}\ \frac{e_1 \Downarrow \mathsf{rstr}[as] \qquad [\mathsf{rstr}[a], \mathsf{rstr}[s]/x, y]e_3 \Downarrow v_3}{\mathsf{rstrcase}(e_1; e_2; x, y.e_3) \Downarrow v_3}$$

$$\text{S-E-Replace}\ \frac{e_1 \Downarrow \mathsf{rstr}[s_1] \qquad e_2 \Downarrow \mathsf{rstr}[s_2] \qquad \mathsf{replace}(r; s_1; s_2) = s}{\mathsf{rreplace}[r](e_1; e_2) \Downarrow \mathsf{rstr}[s]} \qquad \text{S-E-SafeCoerce}\ \frac{e \Downarrow \mathsf{rstr}[s]}{\mathsf{rcoerce}[r](e) \Downarrow \mathsf{rstr}[s]}$$

$$\text{S-E-Check-Ok}\ \frac{e \Downarrow \mathsf{rstr}[s] \qquad s \in \mathcal{L}\{r\} \qquad [\mathsf{rstr}[s]/x]e_1 \Downarrow v}{\mathsf{rcheck}[r](e; x.e_1; e_2) \Downarrow v} \qquad \text{S-E-Check-NotOk}\ \frac{e \Downarrow \mathsf{rstr}[s] \qquad s \notin \mathcal{L}\{r\} \qquad e_2 \Downarrow v}{\mathsf{rcheck}[r](e; x.e_1; e_2) \Downarrow v}$$

Figure 5: Big step semantics for λ_{RS}.

$\boxed{e \mapsto e}$

$$\text{SS-E-AppLeft}\ \frac{e_1 \mapsto e_1'}{e_1(e_2) \mapsto e_1'(e_2)} \qquad \text{SS-E-AppRight}\ \frac{e_2 \mapsto e_2'}{v_1 \mapsto v_1} \qquad \text{SS-E-AppAbs}\ \frac{}{(\lambda x : \tau_{11}.t_{12})v_2 \mapsto [v_2/x]t_{12}}$$

$\boxed{e \mapsto^* e}$

$$\text{RT-Refl}\ \frac{}{e \mapsto^* e} \qquad \text{RT-Trans}\ \frac{e \mapsto^* e' \qquad e' \mapsto e''}{e \mapsto^* e''}$$

Figure 6: Call-by-name small step Semantics for λ and its reflexive, transitive closure.

$\boxed{e \mapsto e}$ (Contiues figure 6)

$$\text{SS-E-CONCAT-LEFT}\ \frac{e_1 \mapsto e_1'}{\mathsf{rconcat}(e_1; e_2) \mapsto \mathsf{rconcat}(e_1'; e_2)} \qquad \text{SS-E-CONCAT-RIGHT}\ \frac{e_2 \mapsto e_2'}{\mathsf{rconcat}(v_1; e_2) \mapsto \mathsf{rconcat}(v_1; e_2')}$$

$$\text{SS-E-CONCAT}\ \frac{}{\mathsf{rconcat}(\mathsf{rstr}[s_1]; \mathsf{rstr}[s_2]) \mapsto \mathsf{rstr}[s_1 s_2]} \qquad \text{SS-E-CASE-LEFT}\ \frac{e_1 \mapsto e_1'}{\mathsf{rstrcase}(e_1; e_2; x, y.e_3) \mapsto \mathsf{rstrcase}(e_1'; e_2; x, y.e_3)}$$

$$\text{SS-E-CASE-}\epsilon\text{-VAL}\ \frac{}{\mathsf{rstrcase}(\mathsf{rstr}[\epsilon]; e_2; x.y.e_3) \mapsto e_2} \qquad \text{SS-E-CASE-CONCAT}\ \frac{}{\mathsf{rstrcase}(\mathsf{rstr}[as]; e_2; x, y.e_3) \mapsto [\mathsf{rstr}[a], \mathsf{rstr}[s]/x, y]e_3}$$

$$\text{SS-E-REPLACE-LEFT}\ \frac{e_1 \mapsto e_1'}{\mathsf{rreplace}[r](v_1; e_2) \mapsto \mathsf{rreplace}[r](v_1'; e_2)} \qquad \text{SS-E-REPLACE-RIGHT}\ \frac{e_2 \mapsto e_2'}{\mathsf{rreplace}[r](e_1; e_2) \mapsto \mathsf{rreplace}[r](e_1; e_2')}$$

$$\text{SS-E-REPLACE}\ \frac{}{\mathsf{rreplace}[r](\mathsf{rstr}[s_1]; \mathsf{rstr}[s_2]) \mapsto \mathsf{rstr}[\mathsf{replace}(r; s_1; s_2)]} \qquad \text{SS-E-SAFECOERCE-STEP}\ \frac{e \mapsto e'}{\mathsf{rcoerce}[r](e) \mapsto \mathsf{rcoerce}[r](e')}$$

$$\text{SS-E-SAFECOERCE}\ \frac{}{\mathsf{rcoerce}[r](\mathsf{rstr}[s]) \mapsto \mathsf{rstr}[s]} \qquad \text{SS-E-CHECK-STEPLEFT}\ \frac{e \mapsto e'}{\mathsf{rcheck}[r](e; x.e_1; e_2) \mapsto \mathsf{rcheck}[r](e'; x.e_1; e_2)}$$

$$\text{SS-E-CHECK-OK}\ \frac{s \in \mathcal{L}\{r\}}{\mathsf{rcheck}[r](\mathsf{rstr}[s]; x.e_1; e_2) \mapsto [\mathsf{rstr}[s]/x]e_1} \qquad \text{SS-E-CHECK-NOTOK}\ \frac{s \notin \mathcal{L}\{r\}}{\mathsf{rcheck}[r](\mathsf{rstr}[s]; x.e_1; e_2) \mapsto e_2}$$

Figure 7: Small step semantics for λ_{RS}. Extends 6.

$\boxed{\Theta \vdash \iota : \tau}$ $\quad \Theta ::= \emptyset \mid \Theta, x : \tau$

$$\textsc{P-T-Var}\ \frac{x : \tau \in \Theta}{\Theta \vdash x : \tau} \qquad \textsc{P-T-Abs}\ \frac{\Theta, x : \tau_1 \vdash \iota_2 : \tau_2}{\Theta \vdash \lambda x.\iota_2 : \tau_1 \to \tau_2} \qquad \textsc{P-T-App}\ \frac{\Theta \vdash \iota_1 : \tau_2 \to \tau \qquad \Theta \vdash \iota_2 : \tau_2}{\Theta \vdash \iota_1(\iota_2) : \iota} \qquad \textsc{P-T-String}\ \frac{}{\Theta \vdash \mathsf{str}[s] : \mathsf{string}}$$

$$\textsc{P-T-Regex}\ \frac{}{\Theta \vdash \mathsf{rx}[r] : \mathsf{regex}} \qquad \textsc{P-T-Concat}\ \frac{\Theta \vdash \iota_1 : \mathsf{string} \qquad \Theta \vdash \iota_2 : \mathsf{string}}{\Theta \vdash \mathsf{concat}(\iota_1; \iota_2) : \mathsf{string}}$$

$$\textsc{P-T-Case}\ \frac{\Theta \vdash \iota_1 : \mathsf{string} \qquad \Theta \vdash \iota_2 : \tau \qquad \Theta, x : \mathsf{string}, y : \mathsf{string} \vdash \iota_3 : \tau}{\Theta \vdash \mathsf{strcase}(\iota_1; \iota_2; x, y.\iota_3) : \tau}$$

$$\textsc{P-T-Replace}\ \frac{\Theta \vdash \iota_1 : \mathsf{regex} \qquad \Theta \vdash \iota_2 : \mathsf{string} \qquad \Theta \vdash \iota_3 : \mathsf{string}}{\Theta \vdash \mathsf{replace}(\iota_1; \iota_2; \iota_3) : \mathsf{string}}$$

$$\textsc{P-T-Check}\ \frac{\Theta \vdash \iota_r : \mathsf{regex} \qquad \Theta \vdash \iota_1 : \mathsf{string} \qquad \Theta \vdash \iota_2 : \sigma \qquad \Theta \vdash \iota_3 : \sigma}{\Theta \vdash \mathsf{check}(\iota_r; \iota_1; \iota_2; \iota_3) : \sigma}$$

Figure 8: Typing rules for λ_P. The typing context Θ is standard.

$\boxed{\iota \Downarrow \dot{v}}$

$$\textsc{P-E-Abs}\ \frac{}{\lambda x.e \Downarrow \lambda x.e} \qquad \textsc{P-E-App}\ \frac{\iota_1 \Downarrow \lambda x.\iota_3 \qquad \iota_2 \Downarrow \dot{v}_2 \qquad [\dot{v}_2/x]\iota_3 \Downarrow \dot{v}_3}{\iota_1(\iota_2) \Downarrow \dot{v}_3} \qquad \textsc{P-E-Str}\ \frac{}{\mathsf{str}[s] \Downarrow \mathsf{str}[s]} \qquad \textsc{P-E-Rx}\ \frac{}{\mathsf{rx}[r] \Downarrow \mathsf{rx}[r]}$$

$$\textsc{P-E-Concat}\ \frac{\iota_1 \Downarrow \mathsf{str}[s_1] \qquad \iota_2 \Downarrow \mathsf{str}[s_2]}{\mathsf{concat}(\iota_1; \iota_2) \Downarrow \mathsf{str}[s_1 s_2]} \qquad \textsc{P-E-Case-}\epsilon\ \frac{\iota_1 \Downarrow \mathsf{str}[\epsilon] \qquad \iota_2 \Downarrow \dot{v}_2}{\mathsf{strcase}(\iota_1; \iota_2; x, y.\iota_3) \Downarrow \dot{v}_2} \qquad \textsc{P-E-Case-Concat}\ \frac{\iota_1 \Downarrow \mathsf{str}[as] \qquad [\mathsf{str}[a], \mathsf{str}[s]/x, y]\iota_3 \Downarrow \dot{v}}{\mathsf{strcase}(\iota_1; \iota_2; x, y.\iota_3) \Downarrow \dot{v}}$$

$$\textsc{P-E-Replace}\ \frac{\iota_1 \Downarrow \mathsf{rx}[r] \qquad \iota_2 \Downarrow \mathsf{str}[s_2] \qquad \iota_3 \Downarrow \mathsf{str}[s_3]}{\mathsf{replace}(\iota_1; \iota_2; \iota_3) \Downarrow \mathsf{str}[\mathsf{replace}(r; s_2; s_3)]} \qquad \textsc{P-E-Check-OK}\ \frac{\iota_x \Downarrow \mathsf{rx}[r] \qquad \iota \Downarrow \mathsf{str}[s] \qquad s \in \mathcal{L}\{r\} \qquad \iota_1 \Downarrow \dot{v}_1}{\mathsf{check}(\iota_x; \iota; \iota_1; \iota_2) \Downarrow \dot{v}_1}$$

$$\textsc{P-E-Check-NotOK}\ \frac{\iota_x \Downarrow \mathsf{rx}[r] \qquad \iota \Downarrow \mathsf{str}[s] \qquad s \notin \mathcal{L}\{r\} \qquad \iota_2 \Downarrow \dot{v}_2}{\mathsf{check}(\iota_x; \iota; \iota_1; \iota_2) \Downarrow \dot{v}_2}$$

Figure 9: Big step semantics for λ_P

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$\boxed{\iota \Downarrow \dot{v}}$

PP-E-CONCATLEFT
$$\frac{\iota_1 \mapsto \iota_1'}{\mathsf{concat}(\iota_1; \iota_2) \Downarrow \mathsf{concat}(\iota_1'; \iota_2)}$$

PP-E-CONCATRIGHT
$$\frac{\iota_2 \mapsto \iota_2'}{\mathsf{concat}(\iota_1; \iota_2) \Downarrow \mathsf{concat}(\iota_1; \iota_2')}$$

PP-E-CONCAT
$$\frac{}{\mathsf{concat}(\mathsf{str}[s_1]; \mathsf{str}[s_2]) \Downarrow \mathsf{str}[s_1 s_2]}$$

PP-E-CASELEFT
$$\frac{\iota_1 \mapsto \iota_1'}{\mathsf{strcase}(\iota_1; \iota_2; x, y.\iota_3) \mapsto \mathsf{strcase}(\iota_1'; \iota_2; x, y.\iota_3)}$$

PP-E-CASERIGHT
$$\frac{\iota_2 \mapsto \iota_2'}{\mathsf{strcase}(\dot{v}; \iota_2; x, y.\iota_3) \mapsto \mathsf{strcase}(\dot{v}; \iota_2'; x, y.\iota_3)}$$

PP-E-CASE
$$\frac{}{\mathsf{strcase}(\mathsf{str}[s_1]; \mathsf{str}[s_2]; x, y.\iota_3) \mapsto \mathsf{str}[s_1 s_2]}$$

PP-E-REPLACELEFT
$$\frac{\iota_1 \mapsto \iota_1'}{\mathsf{replace}(\iota_1; \iota_2; \iota_3) \mapsto \mathsf{replace}(\iota_1'; \iota_2; \iota_3)}$$

PP-E-REPLACEMID
$$\frac{\iota_2 \mapsto \iota_2'}{\mathsf{replace}(\mathsf{rx}[r]; \iota_2; \iota_3) \mapsto \mathsf{replace}(\mathsf{rx}[r]; \iota_2'; \iota_3)}$$

PP-E-REPLACERIGHT
$$\frac{\iota_3 \mapsto \iota_3'}{\mathsf{replace}(\mathsf{rx}[r]; \mathsf{str}[s_2]; \iota_3) \mapsto \mathsf{replace}(\mathsf{rx}[r]; \mathsf{str}[s_2]; \iota_3')}$$

PP-E-REPLACE
$$\frac{}{\mathsf{replace}(\mathsf{rx}[r]; \mathsf{str}[s_2]; \mathsf{str}[s_3]) \mapsto \mathsf{str}[\mathsf{replace}(r; s_2; s_3)]}$$

PP-E-CHECK-LEFT
$$\frac{\iota_x \mapsto \iota_x'}{\check{\iota_x} \iota \iota_1 \iota_2 \mapsto \check{\iota_x} \iota \iota_1 \iota_2}$$

PP-E-CHECK-RIGHT
$$\frac{\iota \mapsto \iota'}{\check{\mathsf{rx}}[r] \iota \iota_1 \iota_2 \mapsto \check{\mathsf{rx}}[r] \iota' \iota_1 \iota_2}$$

PP-E-CHECK-OK
$$\frac{s \in \mathcal{L}\{r\}}{\check{\mathsf{rx}}[r] \mathsf{str}[s] \iota_1 \iota_2 \mapsto \iota_1}$$

PP-E-CHECK-OK
$$\frac{s \notin \mathcal{L}\{r\}}{\check{\mathsf{rx}}[r] \mathsf{str}[s] \iota_1 \iota_2 \mapsto \iota_2}$$

Figure 10: Small step semantics for λ_P (extends L-E rules)

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$\boxed{[\![\sigma]\!] = \tau}$

$$\textsc{Tr-T-String}\ \frac{}{[\![\mathsf{stringin}[r]]\!] = \mathsf{string}} \qquad \textsc{Tr-T-Arrow}\ \frac{[\![\sigma_1]\!] = \tau_1 \qquad [\![\sigma_2]\!] = \tau_2}{[\![\sigma_1 \to \sigma_2]\!] = \tau_1 \to \tau_2}$$

$\boxed{[\![\Psi]\!] = \Theta}$

$$\textsc{Tr-T-Context-Emp}\ \frac{}{[\![\emptyset]\!] = \emptyset} \qquad \textsc{Tr-T-Context-Ext}\ \frac{[\![\Psi]\!] = \Theta \qquad [\![\sigma]\!] = \tau}{[\![\Psi, x : \sigma]\!] = \Theta, x : \tau}$$

$\boxed{[\![e]\!] = \iota}$

$$\textsc{Tr-Var}\ \frac{}{[\![x]\!] = x} \qquad \textsc{Tr-Abs}\ \frac{[\![e]\!] = \iota}{[\![\lambda x.e]\!] = \lambda x.\iota} \qquad \textsc{Tr-App}\ \frac{[\![e_1]\!] = \iota_1 \qquad [\![e_2]\!] = \iota_2}{[\![e_1(e_2)]\!] = \iota_1(\iota_2)}$$

$$\textsc{Tr-Case}\ \frac{[\![e_1]\!] = \iota_1 \qquad [\![e_2]\!] = \iota_2 \qquad [\![e_3]\!] = \iota_3}{[\![\mathsf{rstrcase}(e_1; e_2; x, y.e_3)]\!] = \mathsf{strcase}(\iota_1; \iota_2; x, y.\iota_3)}$$

$$\textsc{Tr-string}\ \frac{}{[\![\mathsf{rstr}[s]]\!] = \mathsf{str}[s]} \qquad \textsc{Tr-Concat}\ \frac{[\![e_1]\!] = \iota_1 \qquad [\![e_2]\!] = \iota_2}{[\![\mathsf{rconcat}(e_1; e_2)]\!] = \mathsf{concat}(\iota_1; \iota_2)} \qquad \textsc{Tr-Subst}\ \frac{[\![e_1]\!] = \iota_1 \qquad [\![e_2]\!] = \iota_2}{[\![\mathsf{rreplace}[r](e_1; e_2)]\!] = \mathsf{replace}(\mathsf{rx}[r]; \iota_1; \iota_2)}$$

$$\textsc{Tr-SafeCoerce}\ \frac{[\![e]\!] = \iota}{[\![\mathsf{rcoerce}[r'](e)]\!] = \iota} \qquad \textsc{Tr-Check}\ \frac{[\![e]\!] = \iota \qquad [\![e_1]\!] = \iota_1 \qquad [\![e_2]\!] = \iota_2}{[\![\mathsf{rcheck}[r](e; x.e_1; e_2)]\!] = \mathsf{check}(\mathsf{rx}[r]; \iota; (\lambda x.\iota_1)(\iota); \iota_2)}$$

Figure 11: Translation from source terms (e) to target terms (ι).