# L3: Laplace Transforms

ELEC 341 | Systems and Control | Spring 2026

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# Last class

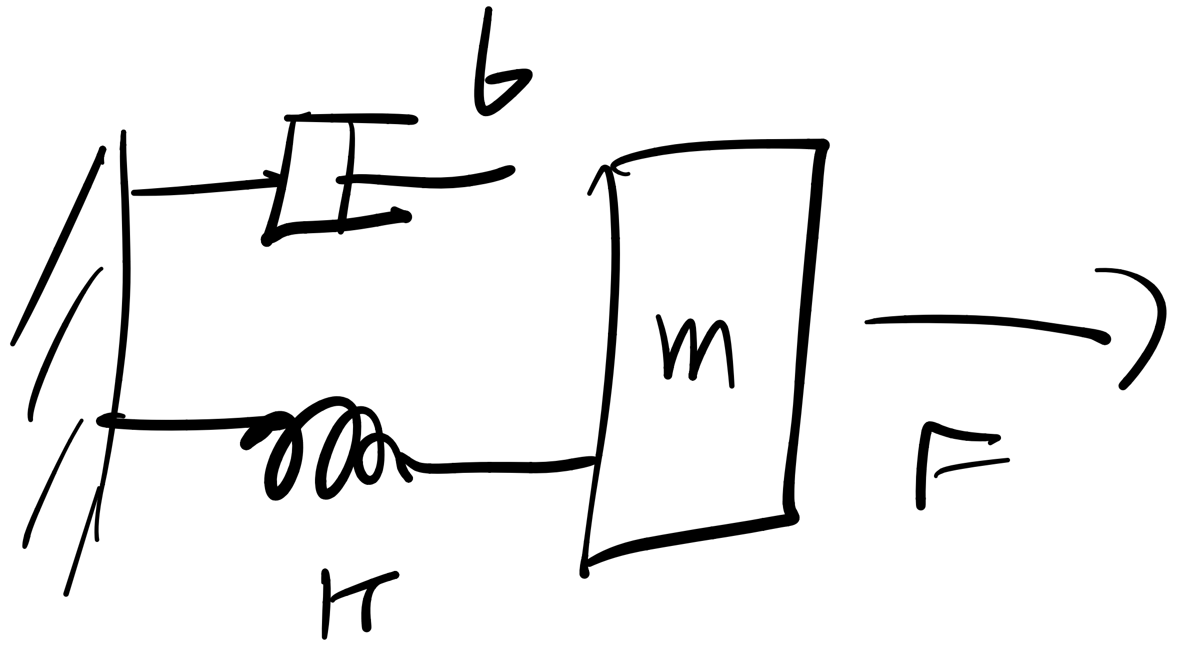
Static vs. dynamic

Deterministic vs. stochastic

Linear vs. Non-linear

SISO vs. MIMO

ODEs as models of dynamic systems




$m\ddot{x} + b\dot{x} + kx = F$

State space Models

$\dot{x} = Ax + Bu$

$y = Cx + Du$

LTI systems $\Longleftrightarrow$ 1. Linear $S(\alpha_1 u_1 + \alpha_2 u_2)$

$= \alpha_1 y_1 + \alpha_2 y_2$

2. Time invariant

$S(u(t-\tau)) = y(t-\tau) \quad \forall \tau \in \mathbb{R}$

# Question

Consider a system where the output is the input multiplied by $t$.

$$y(t) = tu(t)$$

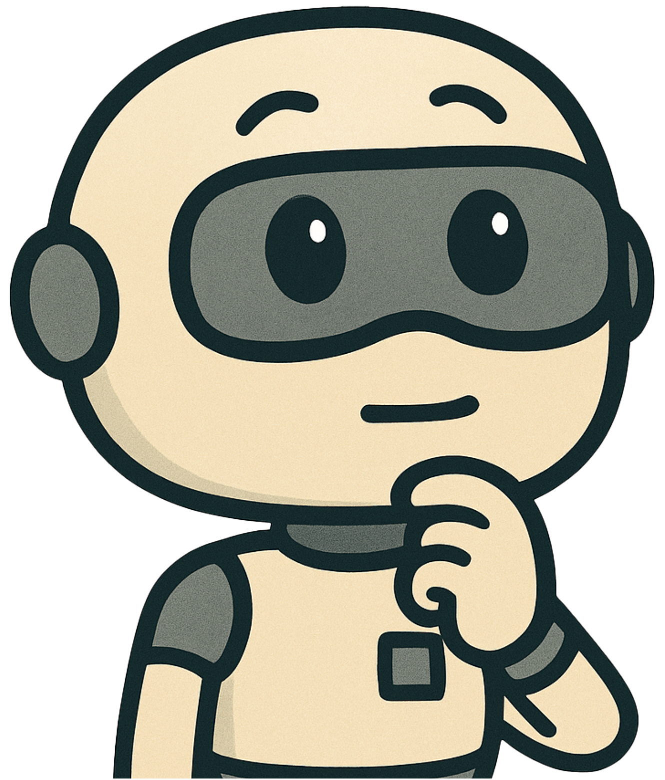
Is this system LTI?

A: Yes, because the input u(t) can be shifted.

B: Yes, because it is linear.

C: No, because the system behavior changes over time.

D: No, because it is not causal.



# The Laplace Transform

$$\boxed{F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt, \quad s \in \mathbb{C}}$$

**what?** Transforms functions of time $f(t)$, into functions of the "s-Domain" $F(s)$.

**why** Doing so transforms differential equations into algebraic equations.

↳ Algebra is easier than differential equations.

↳ Solutions in s-Domain <u>highlight</u> fundamental properties of systems.

↳ We can easily connect subsystems.

$$\xrightarrow{r} \boxed{C} \xrightarrow{u} \boxed{P} \xrightarrow{y} \qquad \longrightarrow \boxed{C * P} \longrightarrow$$

# The Laplace Transform

"Time Domain"

**Differential equation**

$$\ddot{x}(t) + 12\dot{x}(t) + 32x(t) = 32u(t)$$

$$X(s) = \mathcal{L}(x(t))$$

Laplace transform

"S-Domain"

**S-Domain algebraic equation.**

$$u = \frac{1}{s}$$

$$s^2 X(s) + 12 s X(s) + 32 X(s) = 32 U(s)$$

Algebra

**S-Domain solution**

$$X(s) = \frac{1}{s} + \frac{1}{s+8} - \frac{1}{s+4}$$

Inverse Laplace

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$

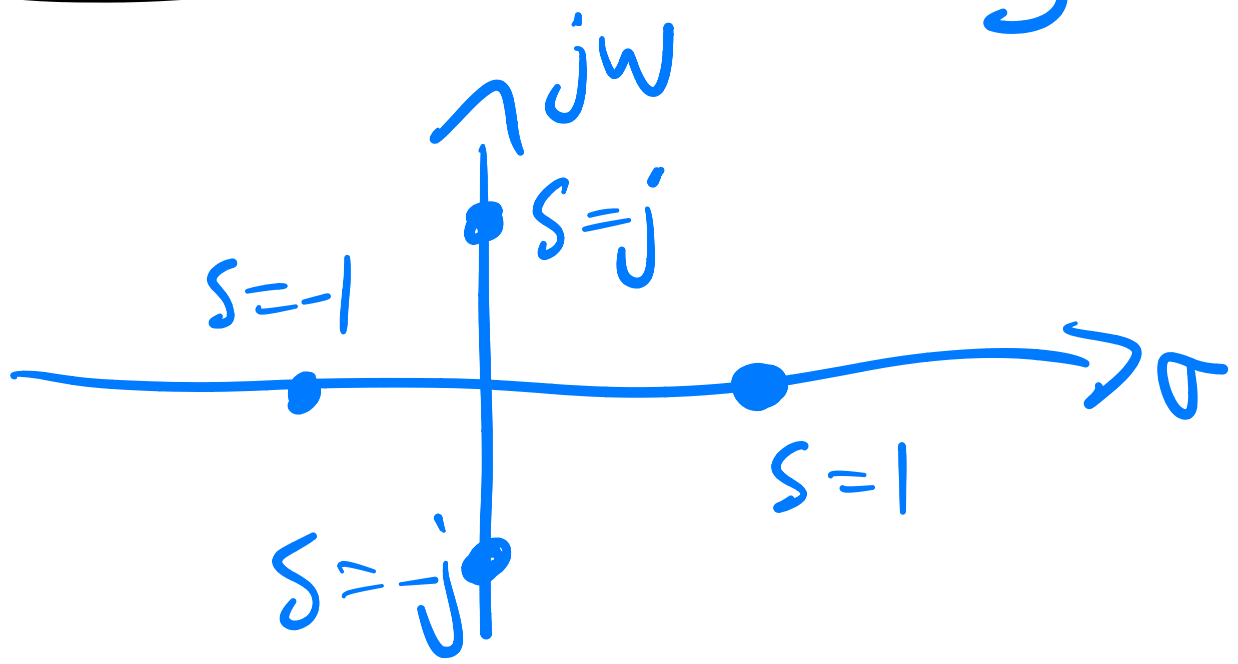
**Time Domain Solution**

$$x(t) = (1 - 2e^{-4t} + e^{-8t})\, u(t)$$

# Complex variables & Functions

**Recall:** $j$ is the "imaginary number" defined as $j^2 = -1$, $j = \sqrt{-1}$

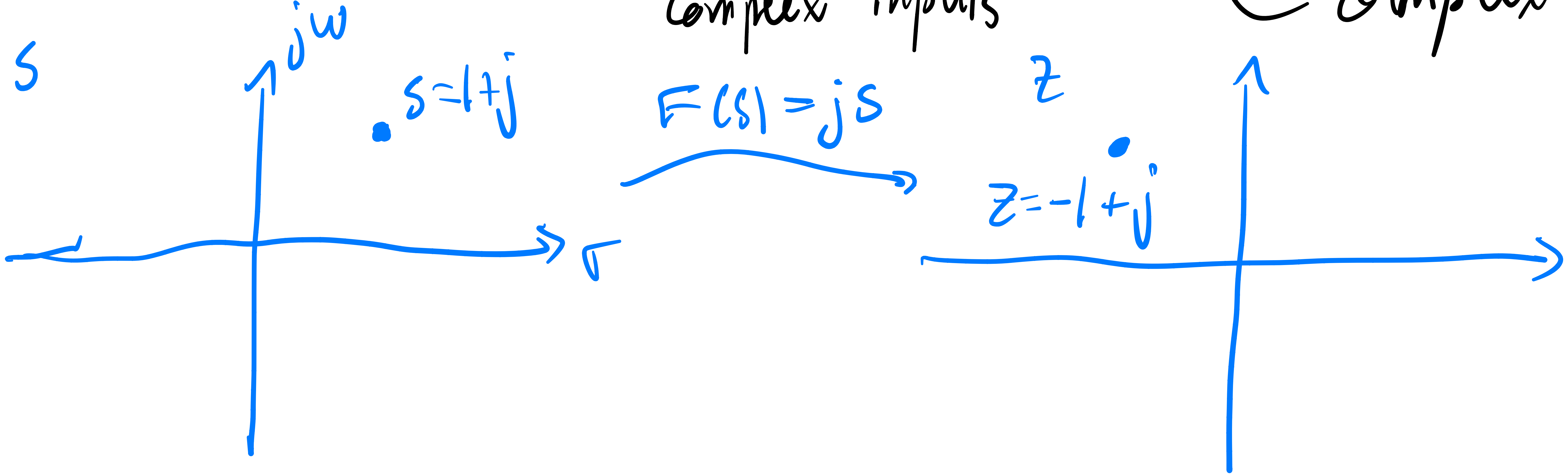
**Complex numbers** $\quad s = \sigma + j\omega, \quad \sigma, \omega \in \mathbb{R}$




A complex function $F(s)$ maps $s \in \mathbb{C}$ to $z \in \mathbb{C}$.

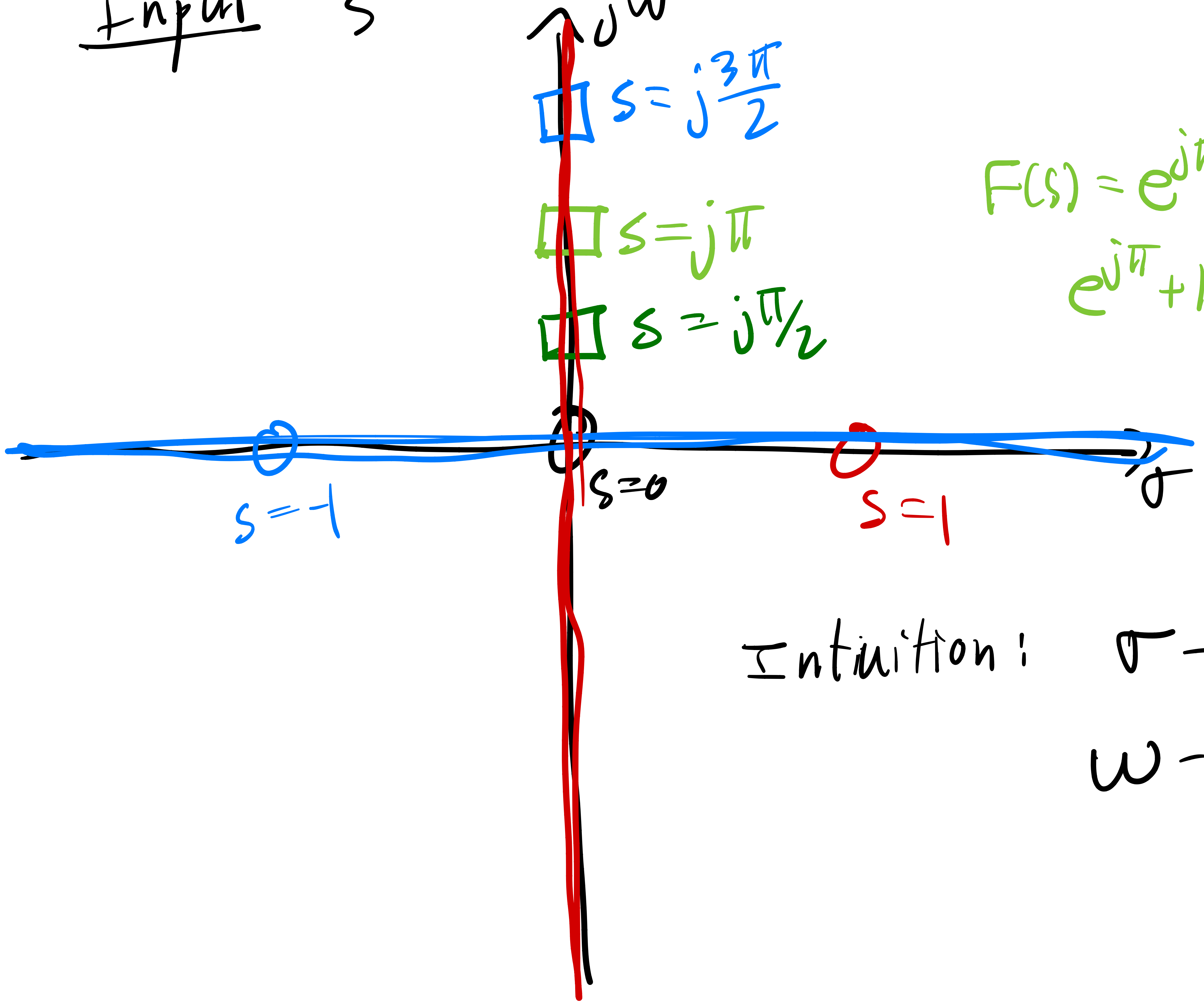
complex inputs

complex outputs.



**Example** Consider $F(s) = e^{s} = e^{\sigma + j\omega} = e^{\sigma} e^{j\omega} = e^{\sigma}\left(j\sin(\omega) + \cos(\omega)\right)$

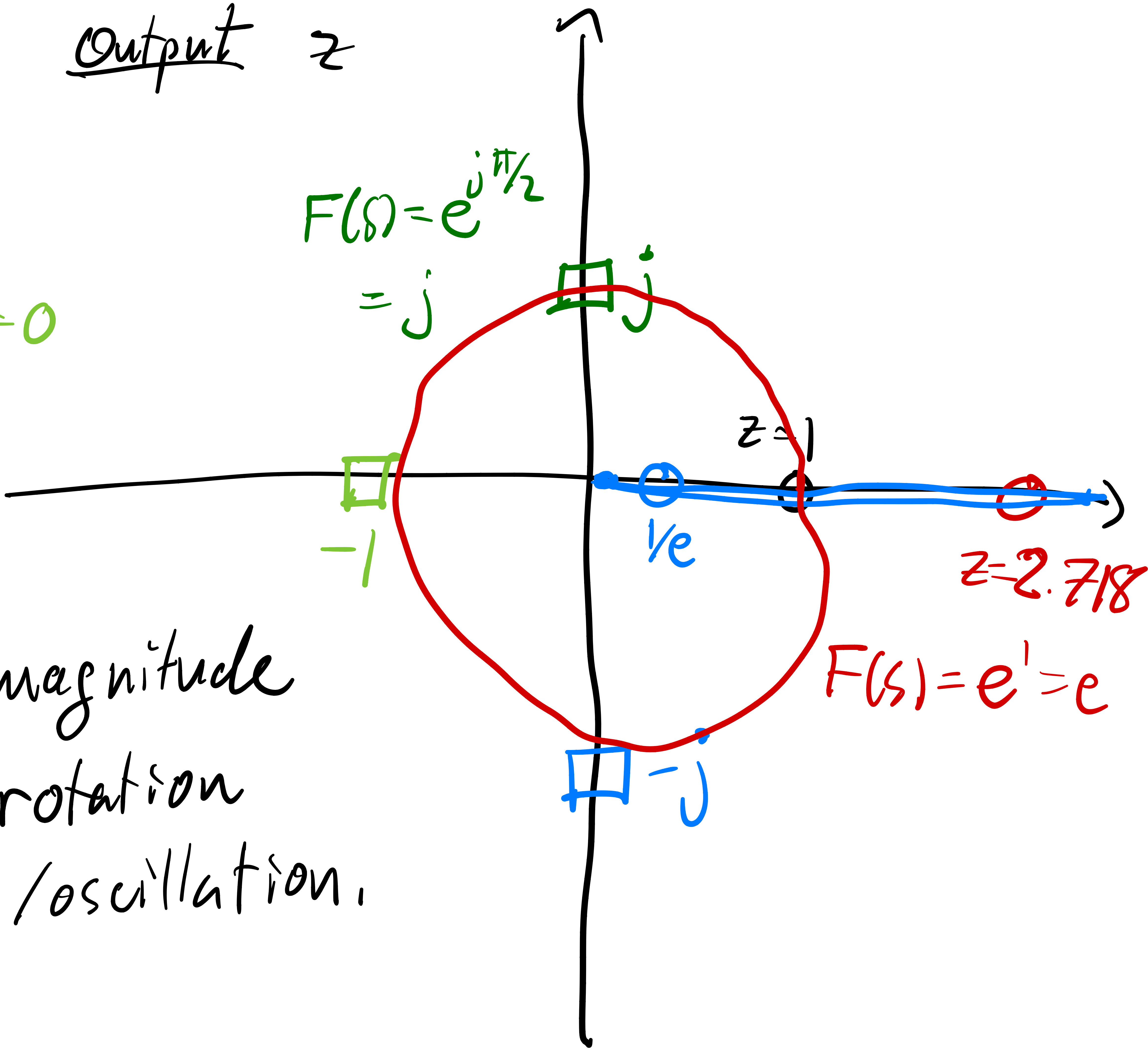
**Input** $s$




$F(s) = e^{j\pi}$

$e^{j\pi} + 1 = 0$

**Output** $z$




Intuition: $\sigma$ – magnitude

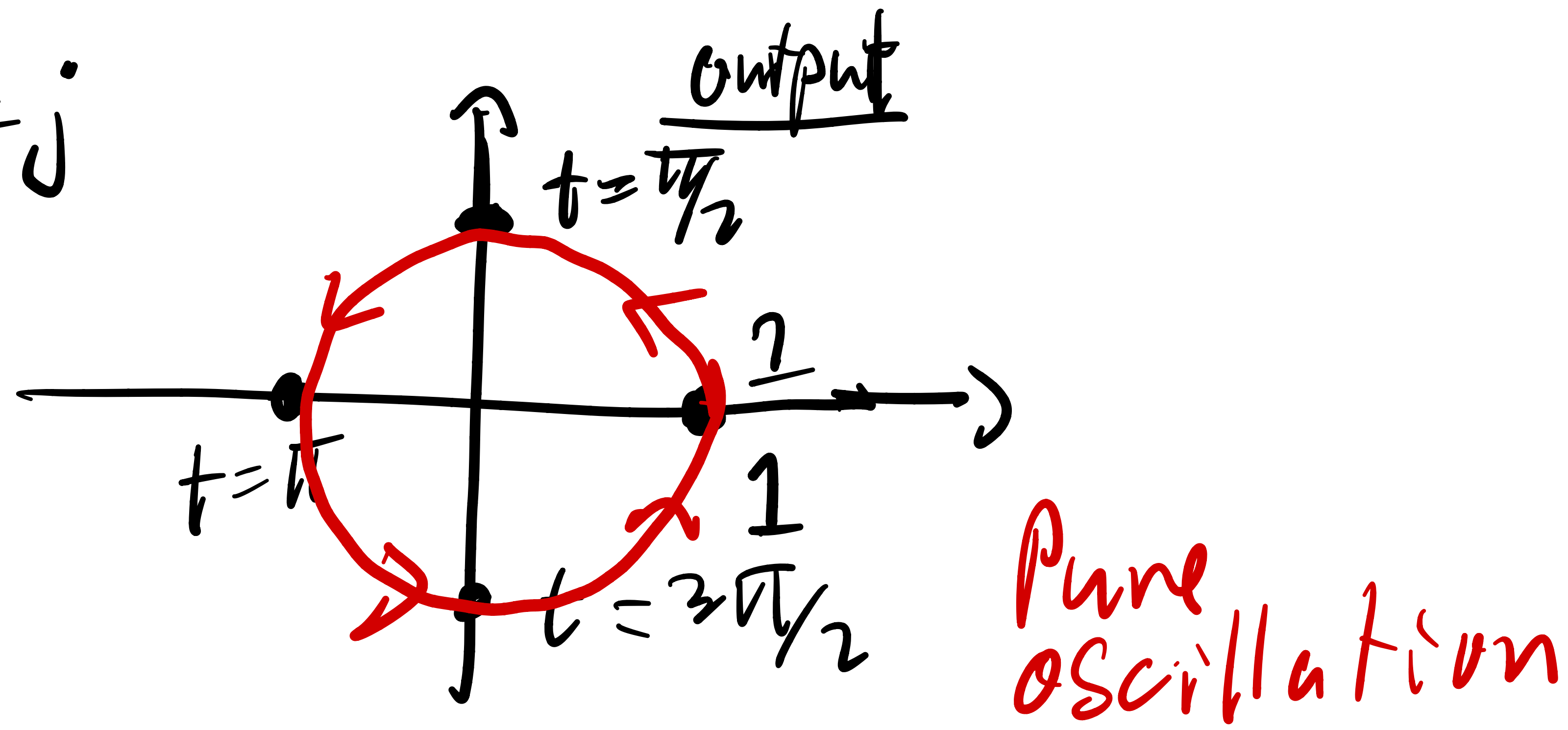
$\omega$ – rotation / oscillation.

Now, Consider $F(s) = e^{st}$ for some $t > 0$

$$F(s) = e^{(\sigma + j\omega)t} = e^{\sigma t}(\cos(\omega t) + j\sin(\omega t))$$

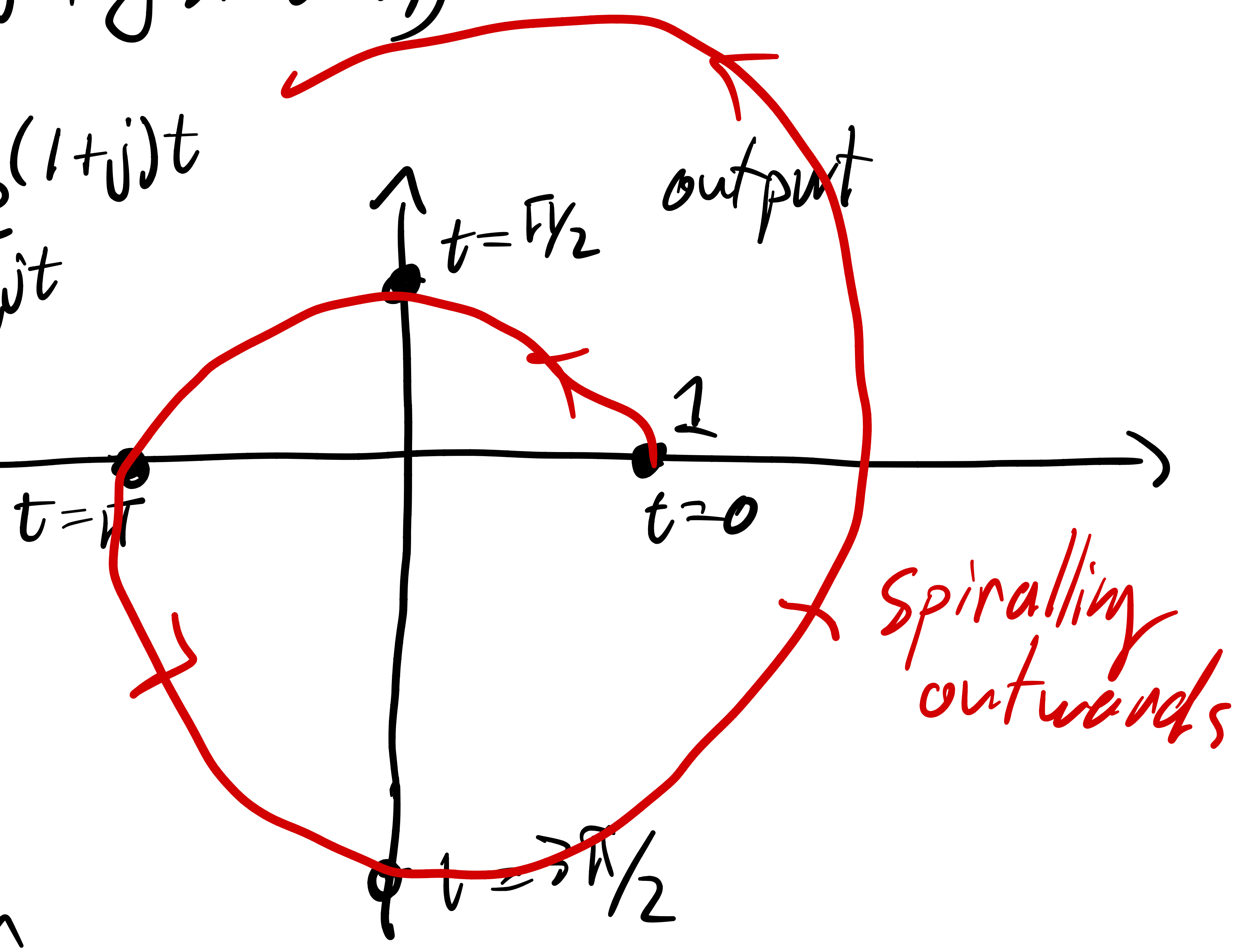
$s = j$

$e^{jt}$

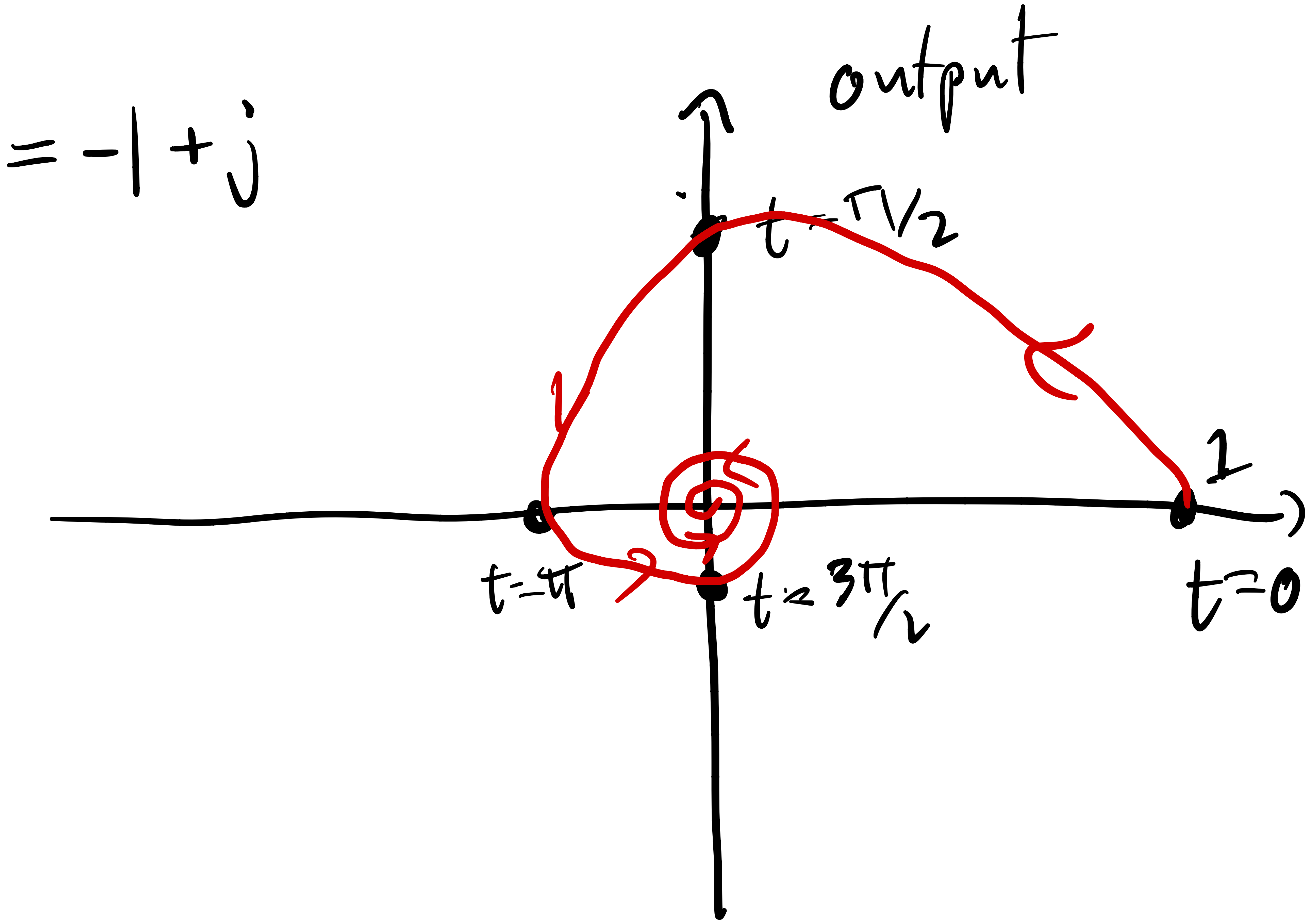



$s = 1 + j \Rightarrow e^{(1+j)t}$

$e^{t} e^{jt}$




$s = -1 + j$




<u>Physical intuition</u>

- $e^{st}$ is an oscillator in time, $t > 0$.
- $s = \sigma + j\omega$ is a <u>parameter</u> that dictates the properties of oscillation.
- $\sigma > 0 \Rightarrow$ oscillator magnitude grows
  $\sigma < 0 \Rightarrow$ decay to origin.
  $\omega \Rightarrow$ frequency + direction of oscillation

What does this look like when considering the input plane?

# Question

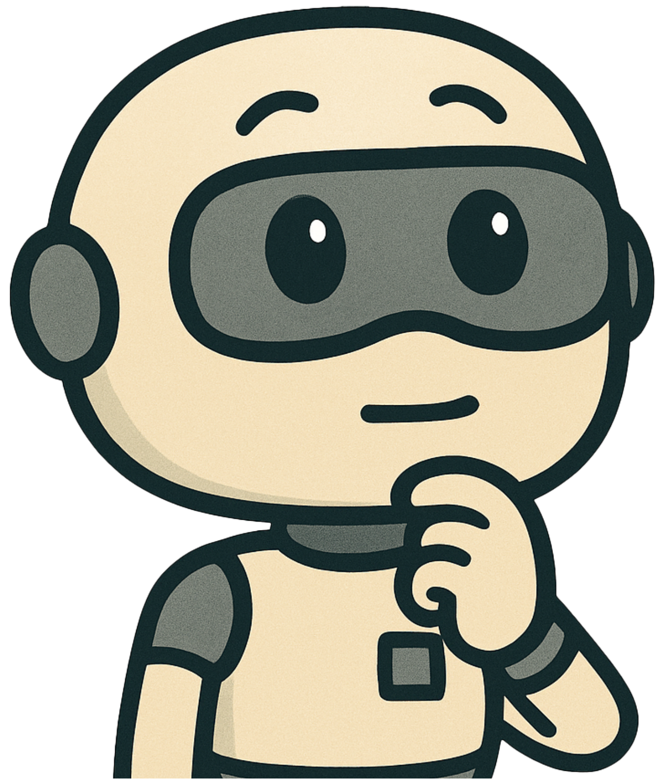
If a signal component corresponds to a pole at $s = -3 + 2j$, what does the real part $(-3)$ tell us about the signal's behavior in time?

A: It oscillates at a frequency of 3 rads/s.

B: It decays exponentially over time.

C: It grows exponentially over time.

D: It is a pure phase shift.



# Question

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A: ~~It oscillates at a frequency of 3 rads/s.~~

**B: It decays exponentially over time.**

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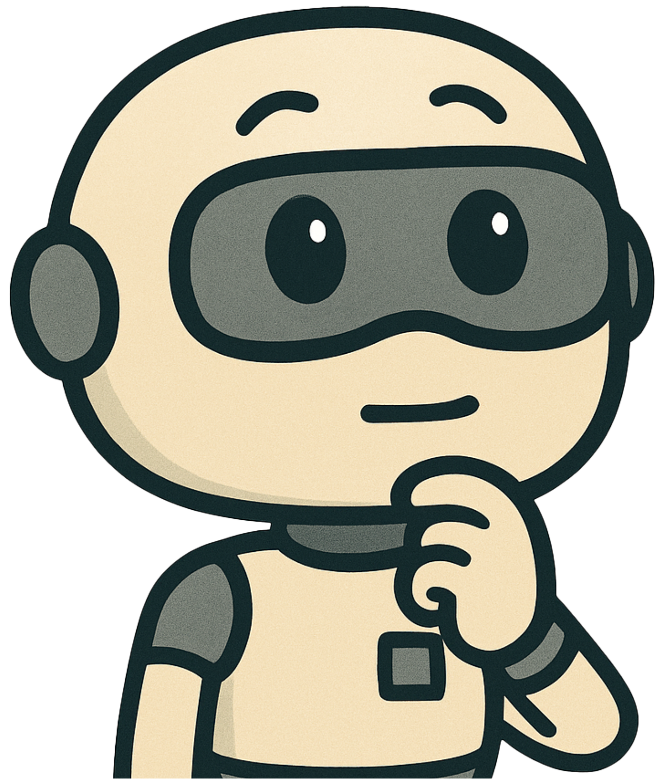


## Who cares?

Many functions we care about in engineering can be written as

$$f(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + \dots + c_n e^{s_n t}$$

Eg. $\cos(wt) = \frac{1}{2}\left(e^{jwt} + e^{-jwt}\right)$, $\sin(wt) = \frac{1}{2j}\left(e^{jwt} - e^{-jwt}\right)$

What is a Laplace transform?

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

It is a probe that scans $f(t)$ to identify its components "Modes" $c_i e^{s_i t}$.

$f(t) = \cos(t)$

$$\Rightarrow \frac{1}{2}\int_0^\infty e^{-st}\left(e^{jt} + e^{-jt}\right) dt$$

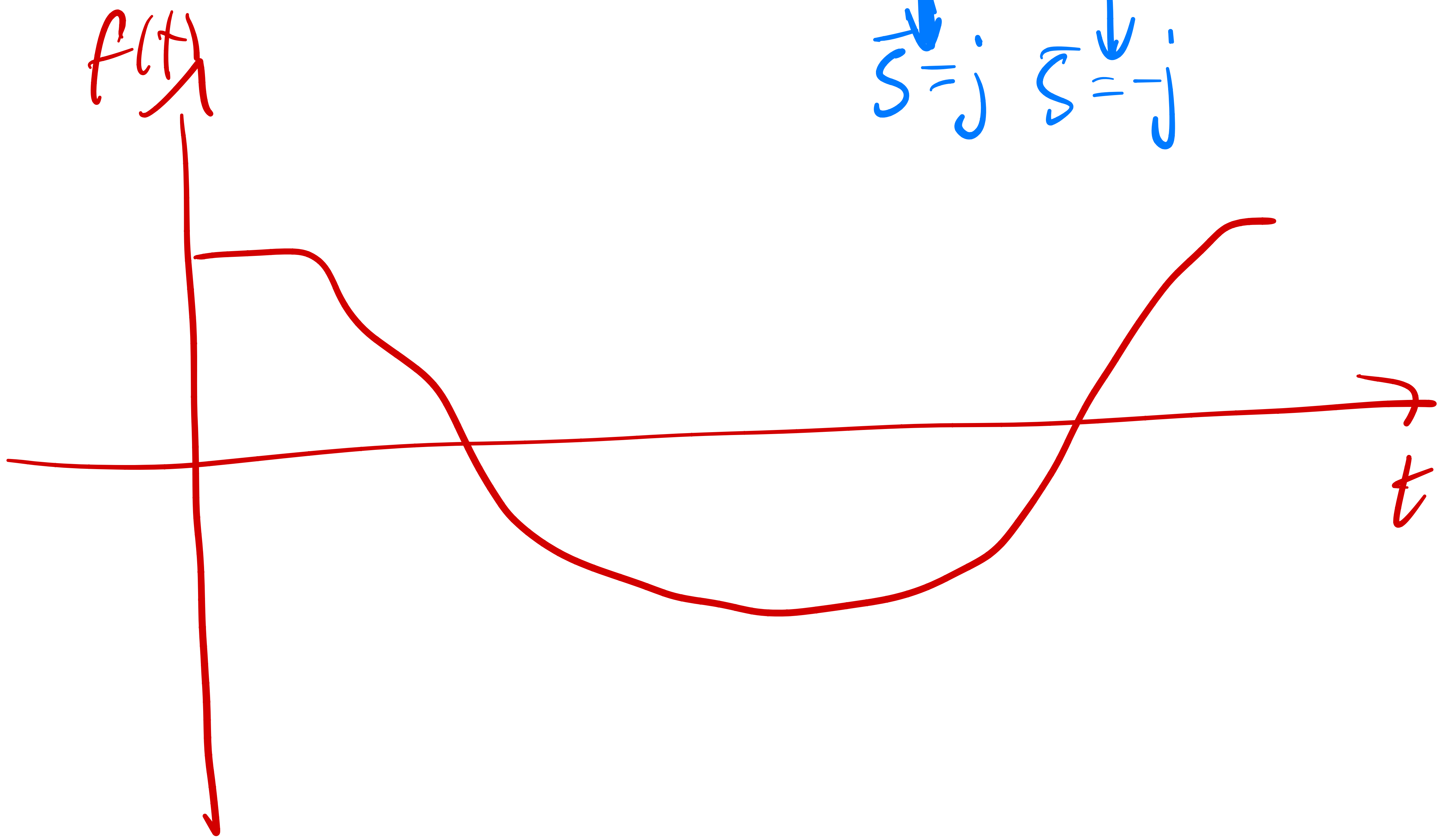
Suppose $F(s)$ is Laplace transform of $f(t)$

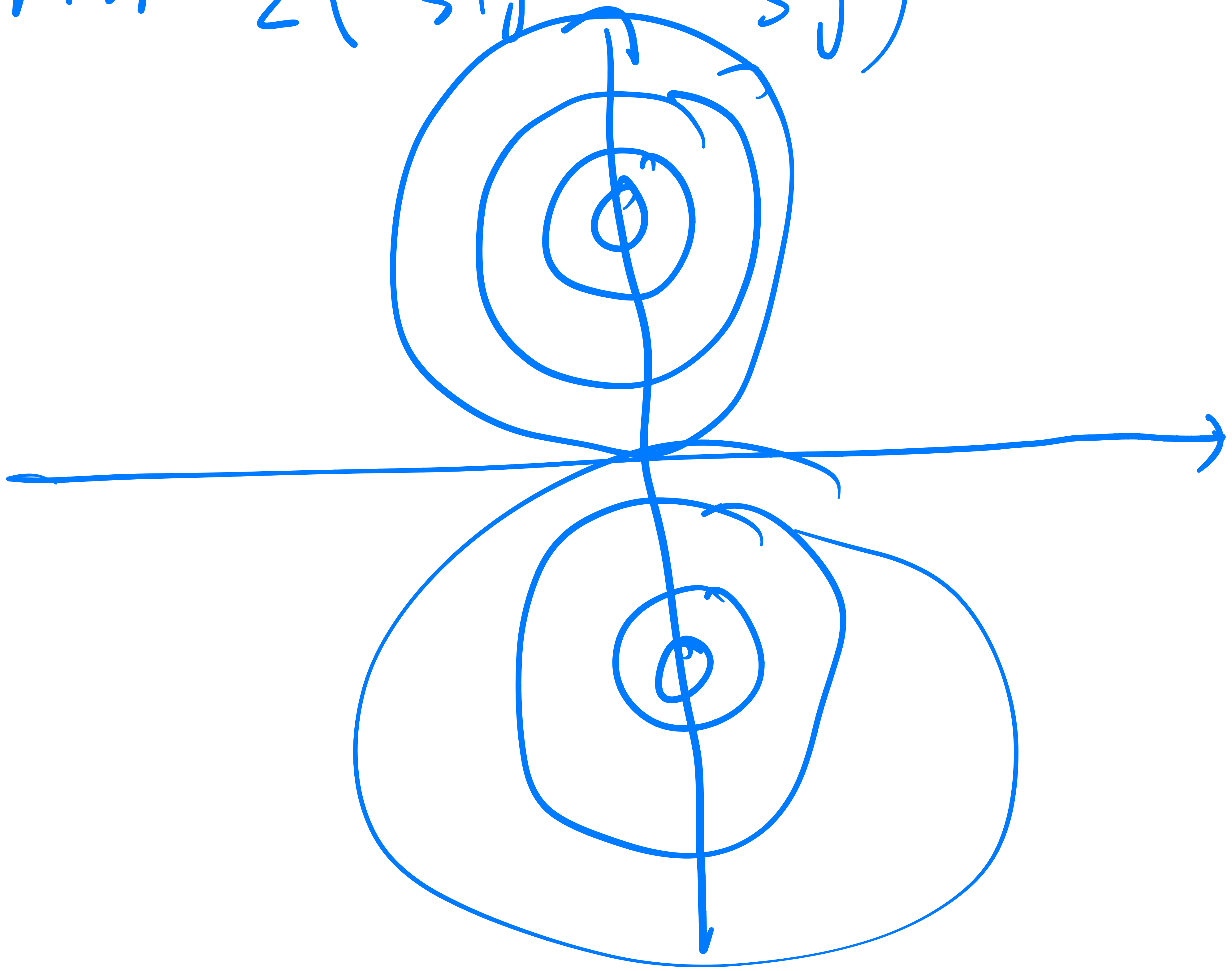
Choose a test point $\bar{s}$ in s-Domain

$|F(\bar{s})|$ will be very large if $e^{\bar{s}t}$ contributes strongly to $f(t)$.

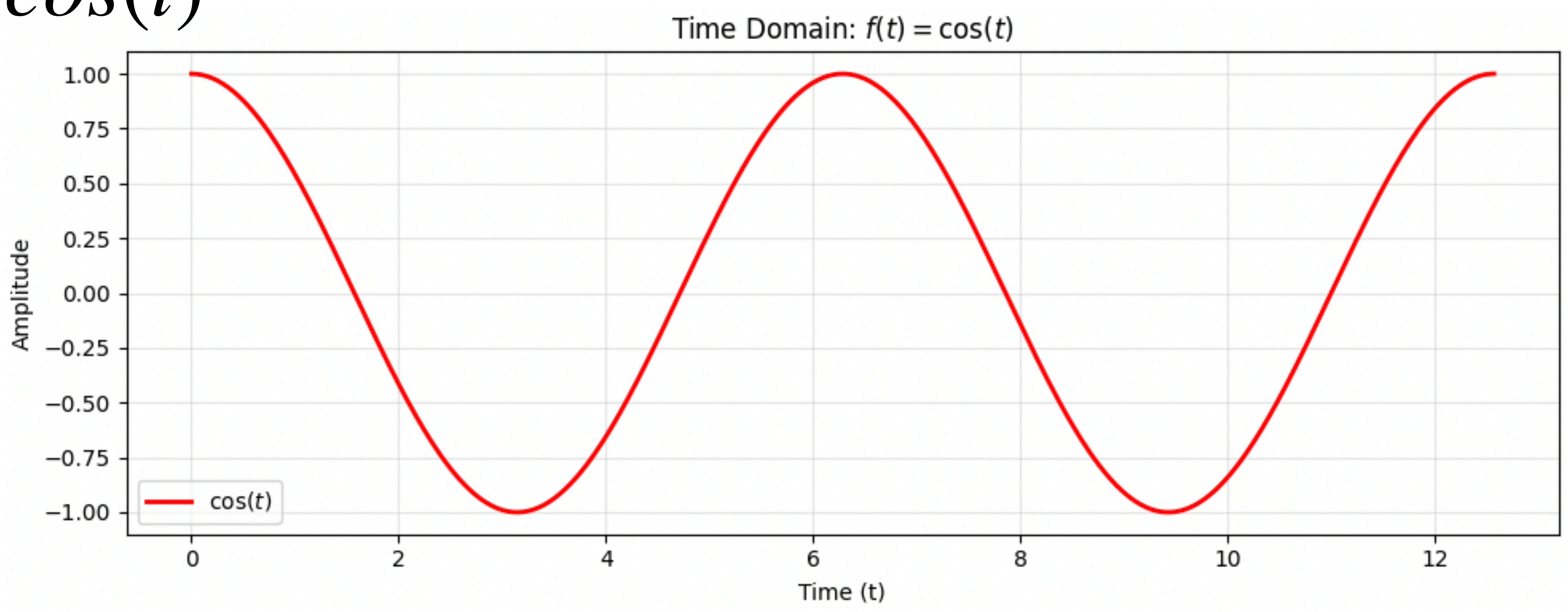
Example $f(t) = \cos(t) = \frac{1}{2}(e^{jt} + e^{-jt})$ $\quad \bar{s}=j \quad \bar{s}=-j$

$F(s) = \frac{1}{2}\left(\frac{1}{s+j} + \frac{1}{s-j}\right)$



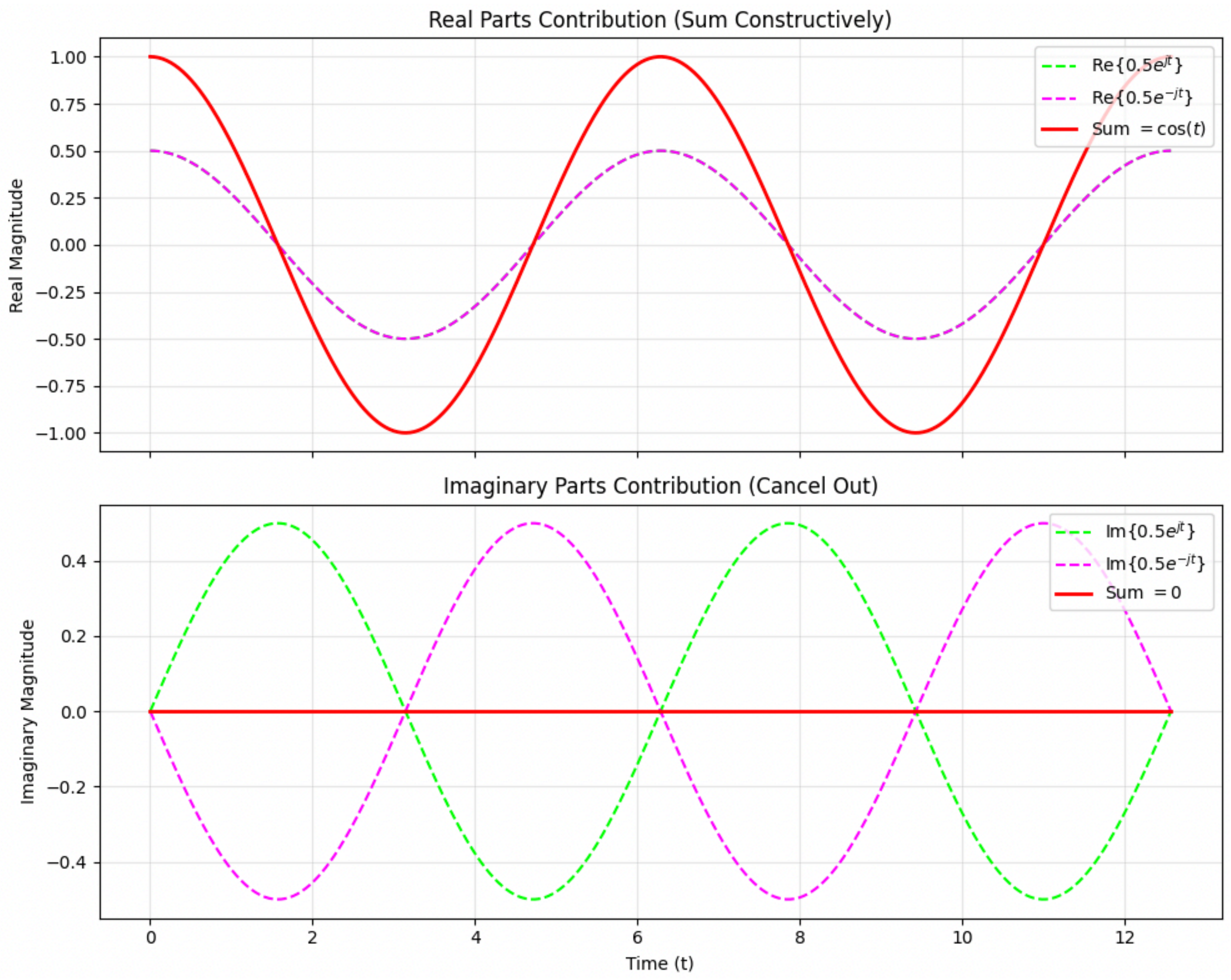
$$f(t) = cos(t)$$



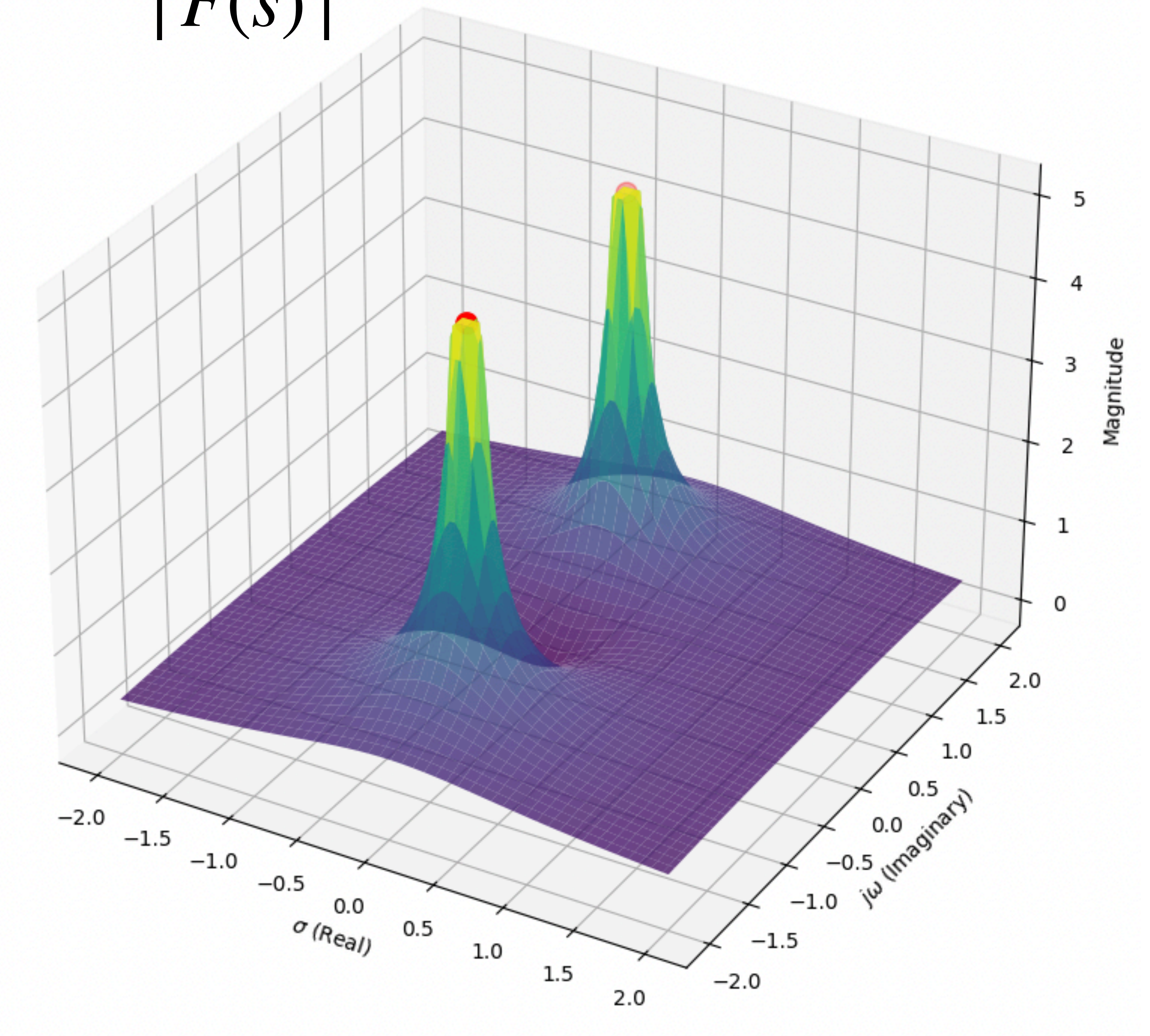

# Laplace Transform Visualization Example

$$\text{Re}\left(\frac{e^{jt} + e^{-jt}}{2}\right)$$

$$\text{Im}\left(\frac{e^{jt} + e^{-jt}}{2}\right)$$




$$|F(s)|$$



$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{st} f(t)\,dt$$

## Common Laplace transforms

| | $f(t)$ | $F(s)$ |
|---|---|---|
| dirac delta | $\delta(t)$ | $1$ |
| step input | $u(t)$ | $1/s$ |
| ramp | $t\,u(t)$ | $1/s^2$ |
| | $t^n u(t)$ | $n!/s^{n+1}$ |
| | $e^{-\alpha t} u(t)$ | $\frac{1}{s+\alpha}$ |
| | $\sin(\omega t) u(t)$ | $\omega/(s^2+\omega^2)$ |
| | $\cos(\omega t) u(t)$ | $s/(s^2+\omega^2)$ |
| | $t e^{-\alpha t} u(t)$ | $1/(s+\alpha)^2$ |

E.g. Unit impulse $\delta(t)$

$$\delta(t) = \begin{cases} \infty, & t=0 \\ 0, & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

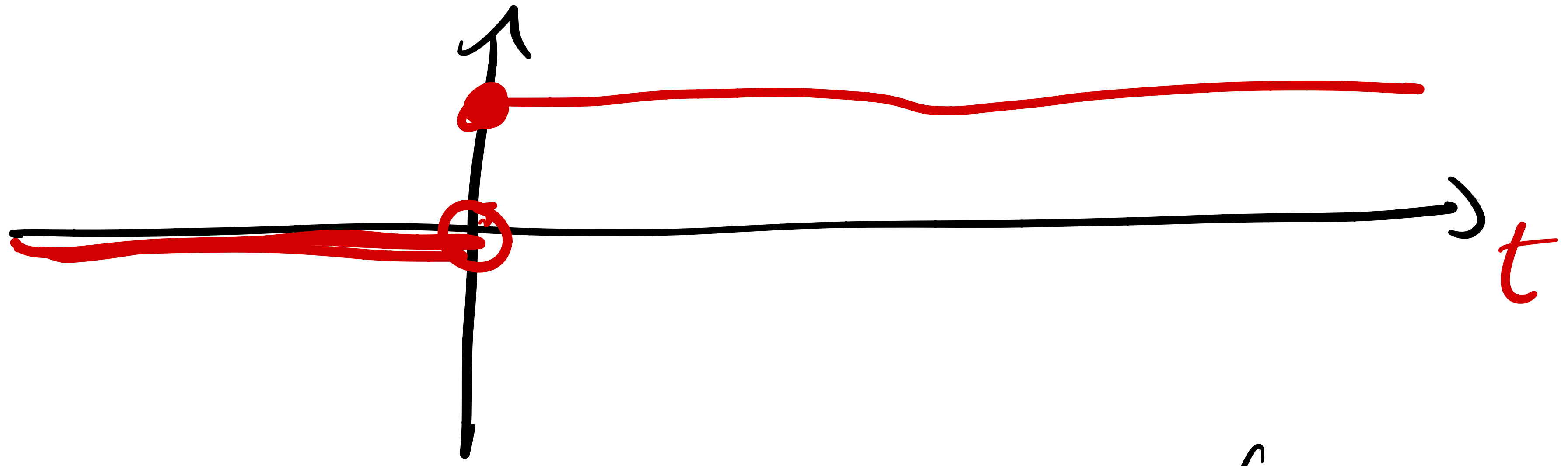
Sampling property $\int_{-\infty}^{\infty} \delta(t) f(t)\,dt = f(0)$

$F(s) = ?$

$$\boxed{F(s) = \int_0^{\infty} \delta(t) e^{-st}\,dt = e^{-s(0)} = 1}$$

E.g. Unit step

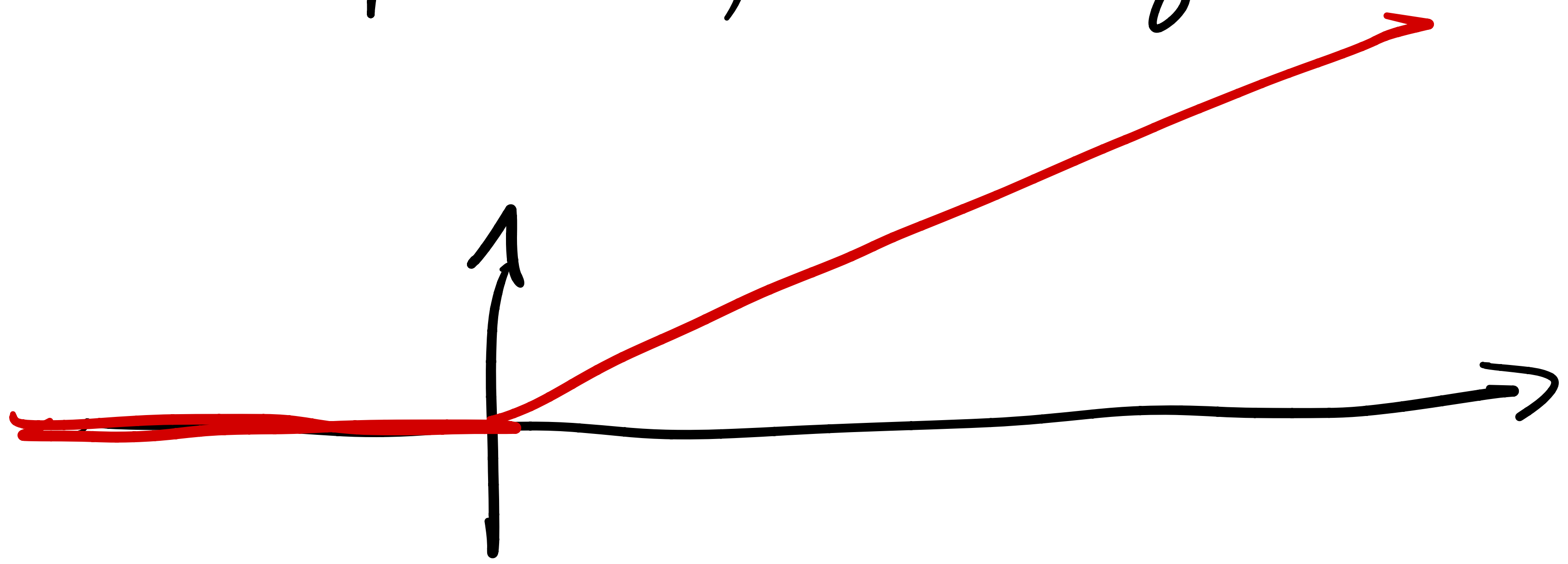
$$u(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$$




$$F(s) = \int_0^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big]_0^\infty = \frac{1}{s}$$ (Assuming real part of $s$ is positive so integral converges)

E.g. Unit ramp

$$tu(t) = \begin{cases} 0, & t<0 \\ t, & t>0 \end{cases}$$



$$F(s) = \int_0^\infty te^{-st}\,dt$$ integrate by parts

Let $u=t \Rightarrow du=dt$

$dv=e^{-st}dt \Rightarrow v=-\frac{1}{s}e^{-st}$

$$= \int_0^\infty u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$= -\frac{t}{s}e^{-st}\Big|_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}\,dt = -\frac{1}{s^2}e^{-st}\Big|_0^\infty = \boxed{\frac{1}{s^2}}$$

# Laplace Transform Properties

1. **Linearity** $\mathcal{L}\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\} = \alpha_1 F_1(s) + \alpha_2 F_2(s)$

Useful for decomposing complex transforms into simpler ones.

E.g. $\mathcal{L}\{5u(t) + 3e^{-2t}\} = 5\mathcal{L}\{u(t)\} + 3\mathcal{L}\{e^{-2t}\} = \frac{5}{s} + \frac{3}{s+2}$

2. **Time delay/phase shift** $\mathcal{L}\{f(t-\tau)\} = e^{-\tau s}F(s)$

E.g. $\mathcal{L}\{\sin(\omega(t-\tau))\,u(t-\tau)\} = e^{-\tau s}\frac{\omega}{s^2+\omega^2}$

3. <u>Differentiation</u> $\boxed{\mathcal{L}\{\dot{f}\} = sF(s) - f(0)}$ Multiplication by s!

This is how we turn ODEs into algebra!!

<u>E.g.</u> $f(t) = \dot{x}(t) \implies F(s) = \boxed{sX(s) - x(0)}$ where $X(s) = \mathcal{L}\{x(t)\}$

$f(t) = \ddot{x}(t) \implies F(s) = s\mathcal{L}\{\dot{x}(t)\} - \dot{x}(0)$

$= \boxed{s^2 X(s) - s x(0) - \dot{x}(0)}$

4. <u>Integration</u> $\boxed{\mathcal{L}\{\int_0^t f(\tau)\,d\tau\} = \frac{F(s)}{s}}$ (Division by s)

E.g. $\mathcal{L}\{t\,u(t)\} = \mathcal{L}\left\{\int_0^t u(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{u\} = \frac{1}{s^2}$

## 5. Final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$ If the poles of $sF(s)$ are in the open LHP.

not including $j\omega$ axis.

E.g. $\lim_{t\to\infty} te^{-at} = \lim_{s\to 0} \frac{s}{(s+a)^2} = 0$ for $a>0$

$$\frac{0}{(0+a)^2} \qquad \frac{1}{\frac{s^2+2sa+a^2}{s}} \qquad \frac{1}{1/s + 2a + \frac{a^2}{s}}$$

non-example $\lim_{t\to\infty} \sin(t) = \lim_{s\to 0} \frac{s}{s^2+1} = 0.$ <— clearly not!

↳ doesn't mean anything, because poles are at $\pm j\omega$ ⟹ theorem doesn't hold.

## 6. Initial value theorem

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$$

## 7. Convolutions

$$\boxed{\mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)\,d\tau\right\} = F(s)G(s)}$$

Important: • Convolutions define compositions of systems.

↳ These become multiplications in s-Domain

$$\to \boxed{C} \to \boxed{P} \to \quad \Longleftrightarrow \quad \to \boxed{CP} \to$$

## 8. Frequency shift

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$$

E.g. $\mathcal{L}\{te^{-2t}\} = \dfrac{1}{(s+2)^2}$

$\mathcal{L}\{t\} = \dfrac{1}{s^2}$