L2: Mathematical Tools for Modeling

ELEC 341 | Systems and Control | Spring 2026
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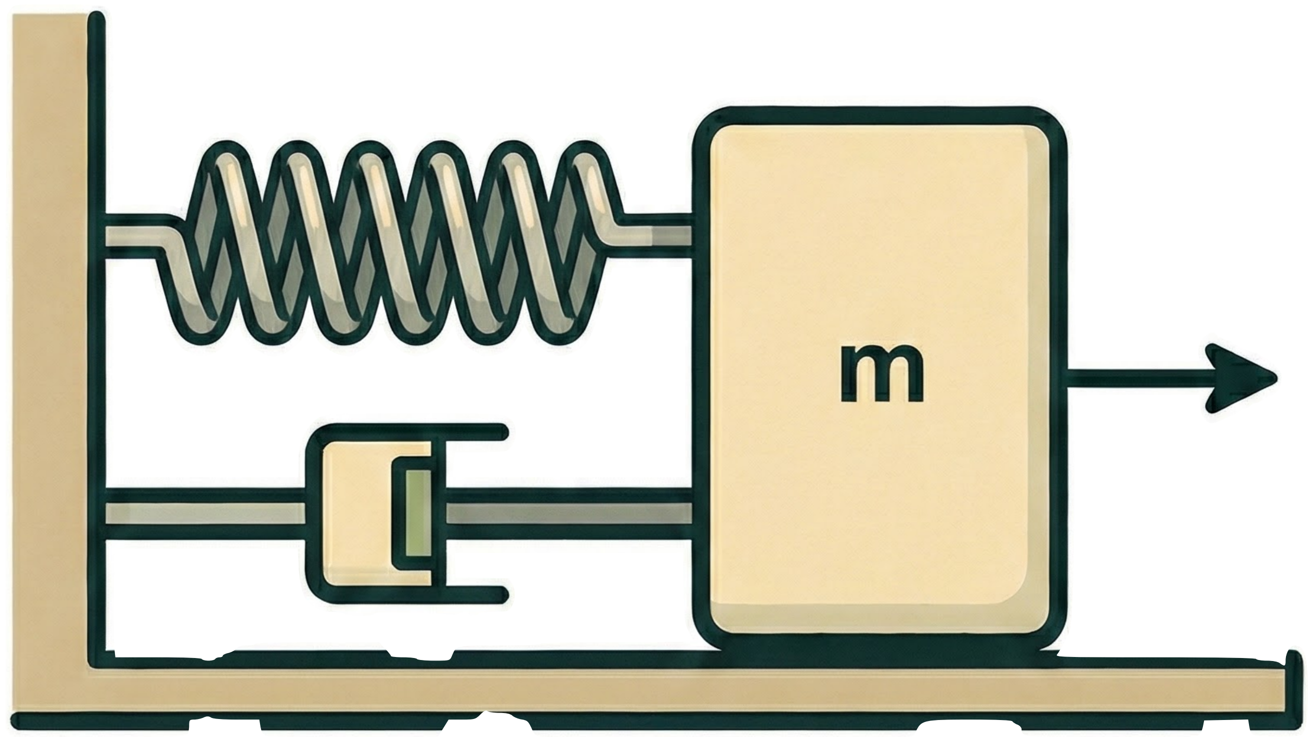
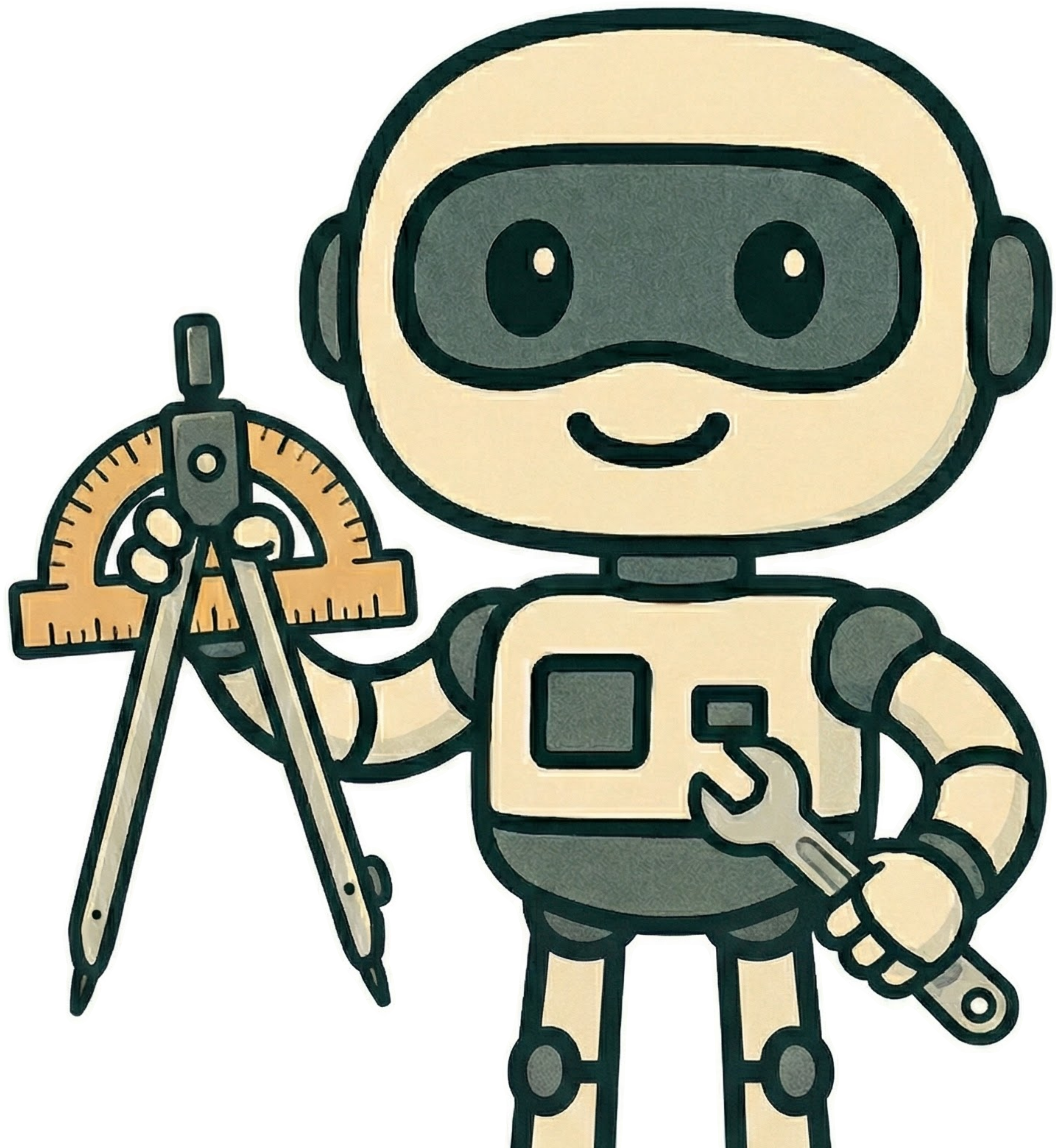


$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = u(t)$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$$S(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 y_1 + \alpha_2 y_2$$



Systems theory

<u>what is a system?</u>

- A relationship between inputs u and outputs y

$$u \longrightarrow \boxed{S} \longrightarrow y \qquad y = S(u)$$

- Typically in controls, $u(t)$ and $y(t)$ are functions of time $t > 0$.

$$u : [0, \infty) \longrightarrow \mathbb{R} \quad \text{and} \quad y : [0, \infty) \longrightarrow \mathbb{R}$$

- System is a relationship between signals/functions.
  ↳ "Functional" that maps functions to functions.

- Where do we draw the box $\boxed{S}$?
  ↳ engineering judgement!

Types of systems

Generic definition $u \longrightarrow \boxed{S} \longrightarrow y$ purposefully vague!

To make things more specific, let's start classifying.

1. Static or dynamic	2. Deterministic vs. stochastic
3. Linear vs. non-linear	4. SISO vs. MIMO

Types of systems

Static vs. dynamic systems

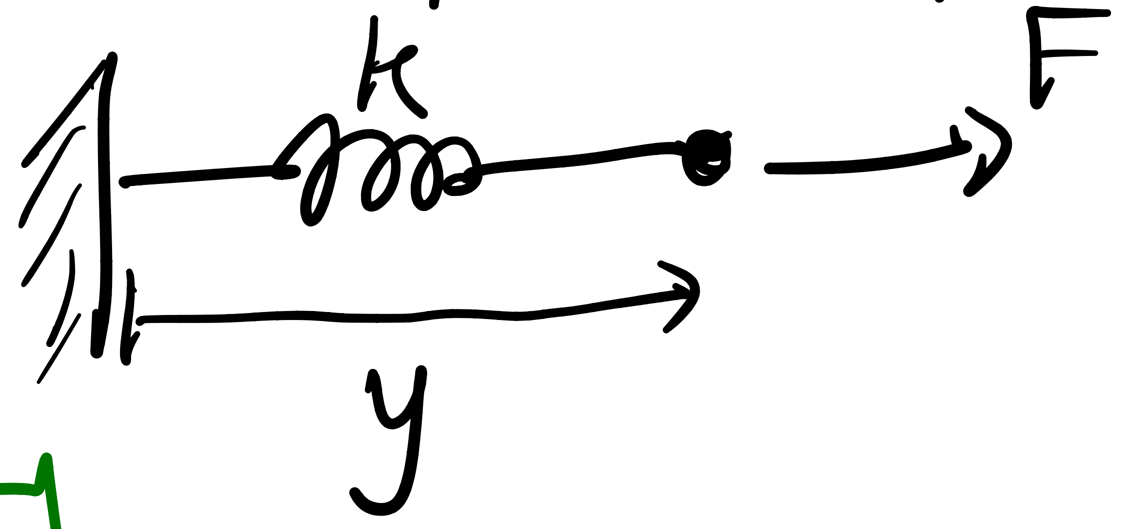
↳ Mathematically, static ⟶ algebra

dynamic ⟶ diffeqs.

Static systems are memoryless (stationary/not changing)

The output y depends only on the input u right now.

E.g.



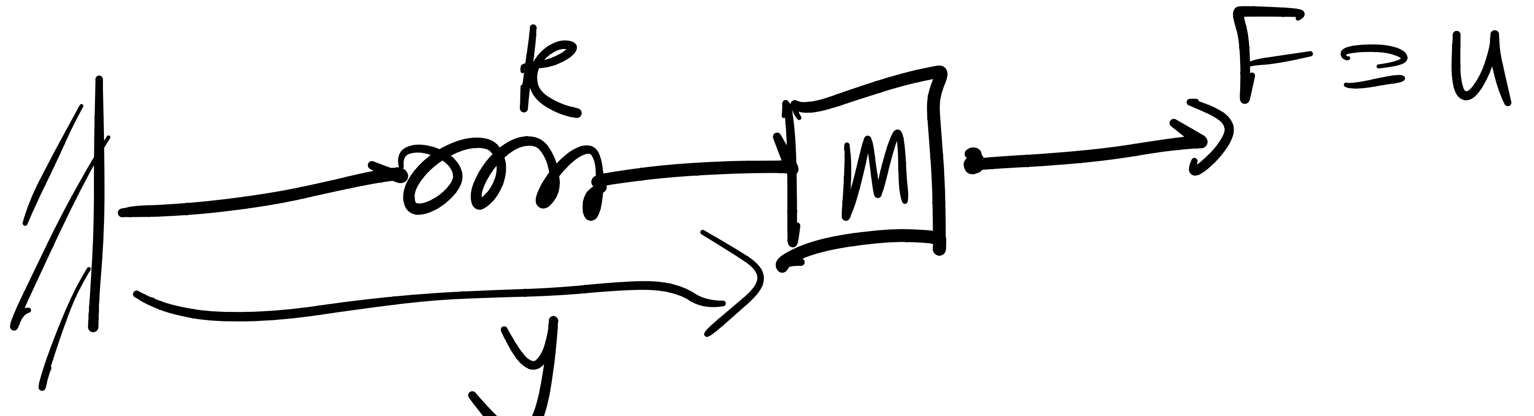
$\Rightarrow$ $u = F$

$$\boxed{y = u/k}$$

Dynamic system has memory

The output y depends on the current u, and a history of past u/y values

E.g. Add a mass $\Rightarrow$ system has inertia



$$\boxed{m\ddot{y} + ky = u}$$

Diffeq.

Types of systems

Deterministic vs. Stochastic

Is the system predictable?

$\boxed{\text{Deterministic:}}$ Repeatable. Same input $u(t)$, and same initial conditions $y(0)$, and maybe $\dot{y}(0)$, $\ddot{y}(0)$, etc.

$\Rightarrow$ we have the same output $y(t)$.

E.g. Light switch, Vending machine, most examples so far,

imagine dropping a bowling ball off a building

u = height, y = time to reach ground.

Stochastic systems Random. Same inputs $u(t) \longrightarrow$ sometimes different $y(t)$.

E.g. quantum systems, slot machine, dropping a piece of paper off building.

Types of systems

Linear vs. Non-linear

A system is linear iff it satisfies the principle of superposition.

$$S(u_1) = y_1, \quad S(u_2) = y_2 \implies S(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 y_1 + \alpha_2 y_2$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$.

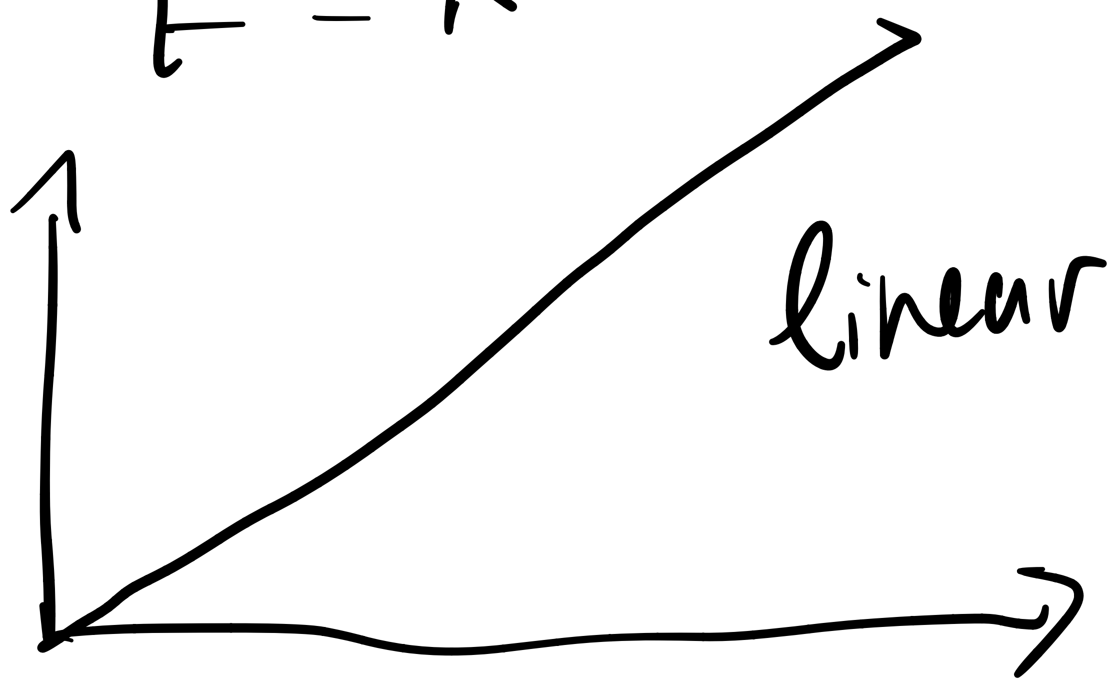
This implies two properties:

1. Additivity: Response to sum of inputs equals sum of individual responses.

2. Homogeneity: If sale input by $\alpha \Rightarrow$ scale output by α.

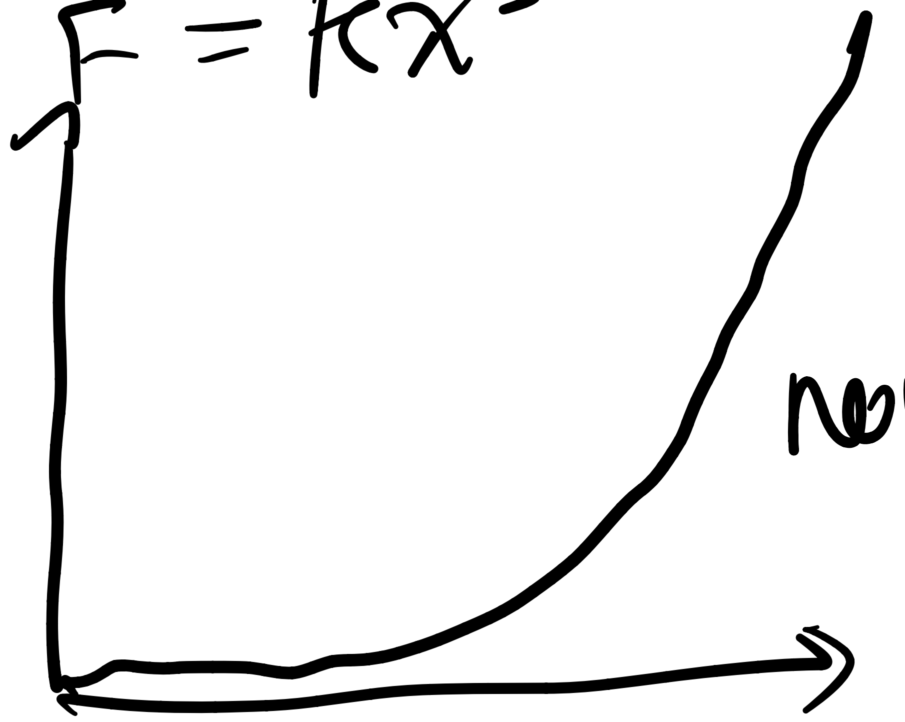
E.g.

$F = kx$



linear

$F = kx^3$

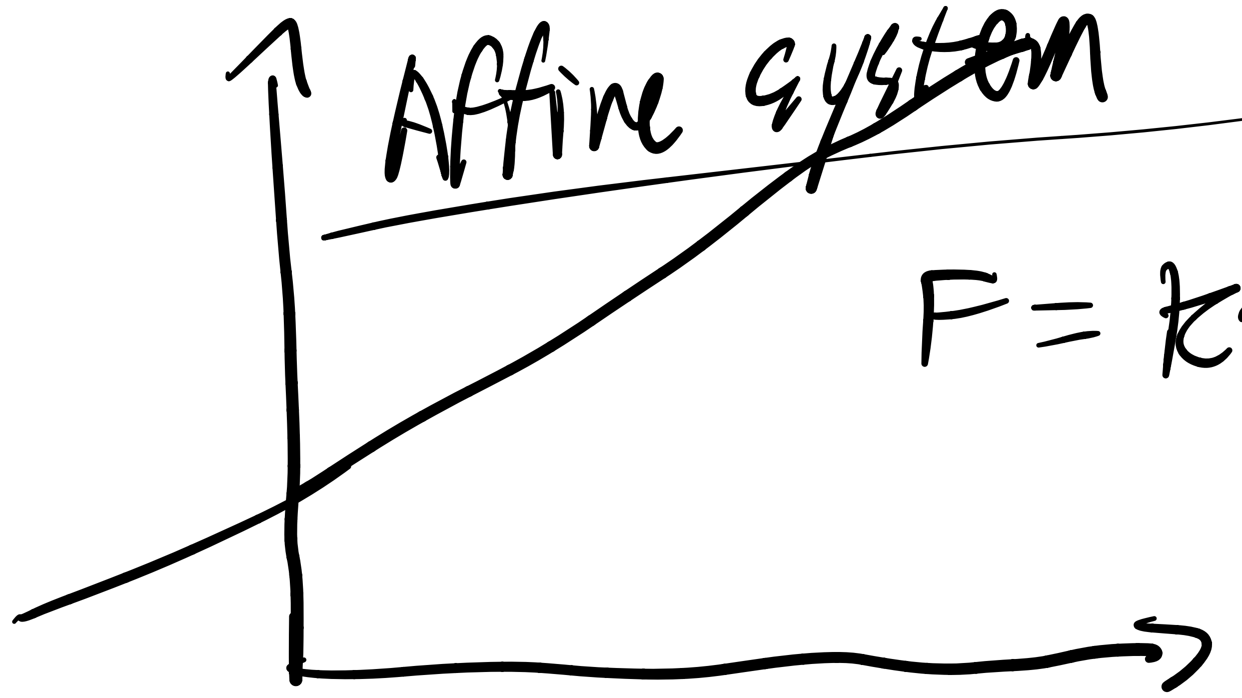


non-linear

Not-linear

Affine system

$F = kx + b$

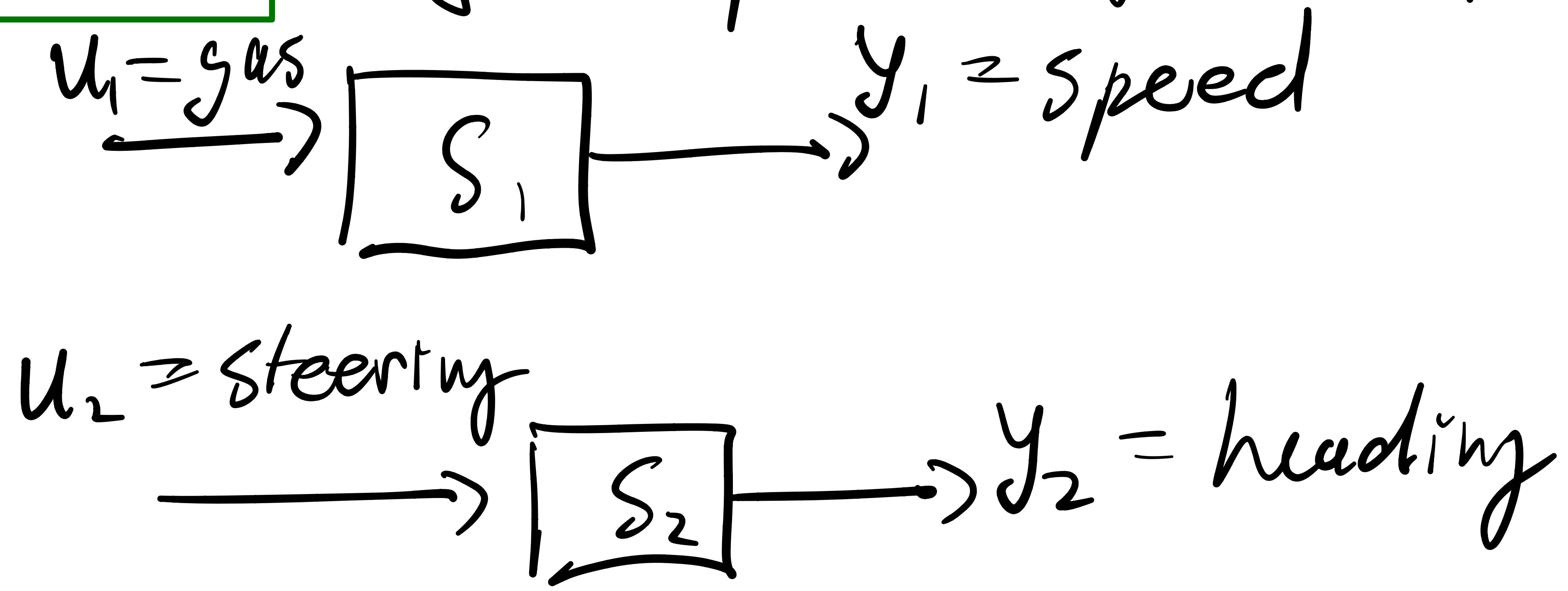


Types of systems

SISO vs. MIMO

SISO: Single-input single output

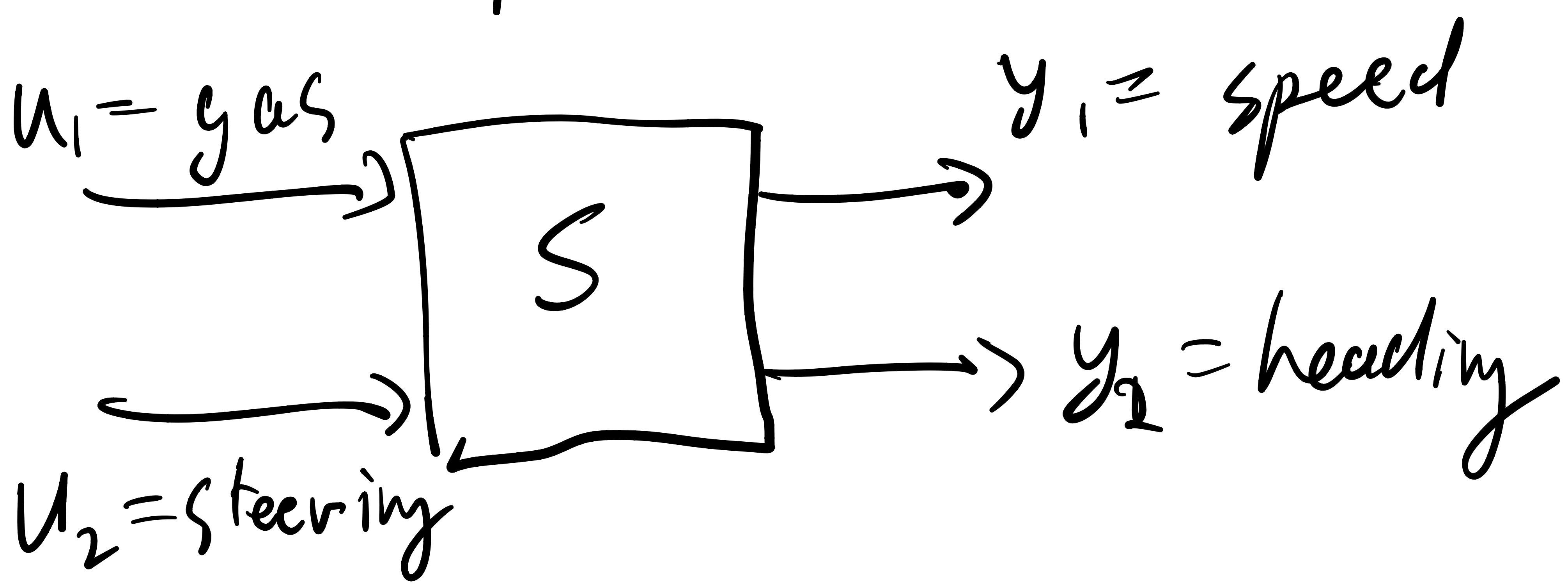
car:



Treated as two independent systems.

MIMO: Multi-input multi-output

car



One single system handles both pairs of inputs/outputs.

↳ can handle coupling between u_2 and y_1 and u_1 and y_2

Types of systems

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To make things more specific, let's start classifying.

1. Static or $\boxed{\text{dynamic}}$

2. $\boxed{\text{Deterministic}}$ vs. stochastic

3. $\boxed{\text{Linear}}$ vs. non-linear

4. $\boxed{\text{SISO}}$ vs. MIMO

Question: Is this system static or dynamic?

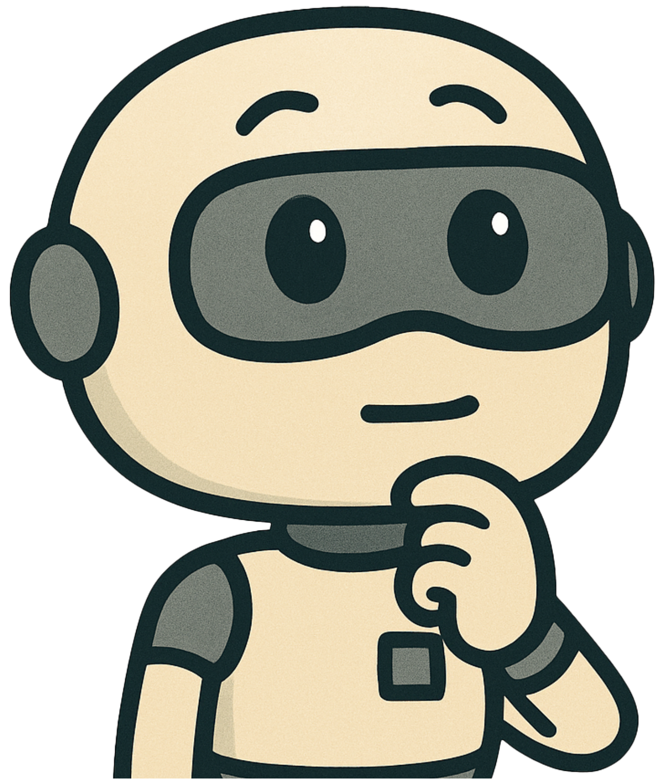
$$y(t) = 5u(t)^2 + 3u(t)$$

A: Static

B: Dynamic

C: Neither

D: It depends on the input signal



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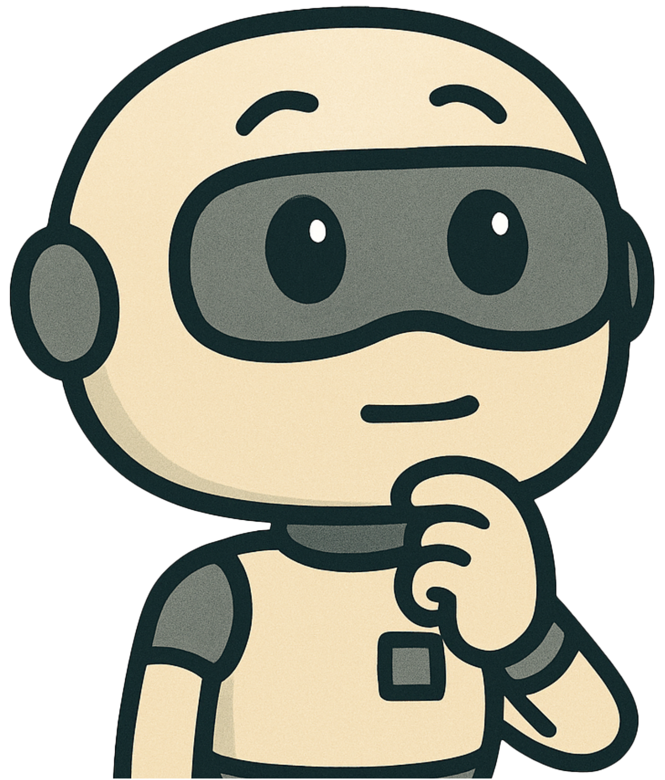


Question: Is this system Linear?

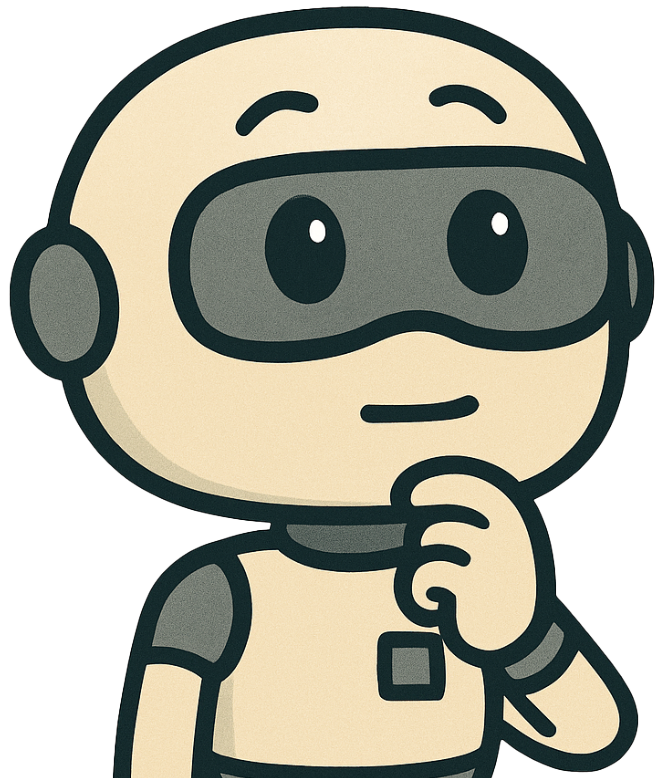
$$y(t) = 2u(t) + 5$$

A: Yes, it is the equation of a line.

B: Yes, because the slope is constant.

C: No, it fails the principle of superposition.

D: No, it is time-variant.



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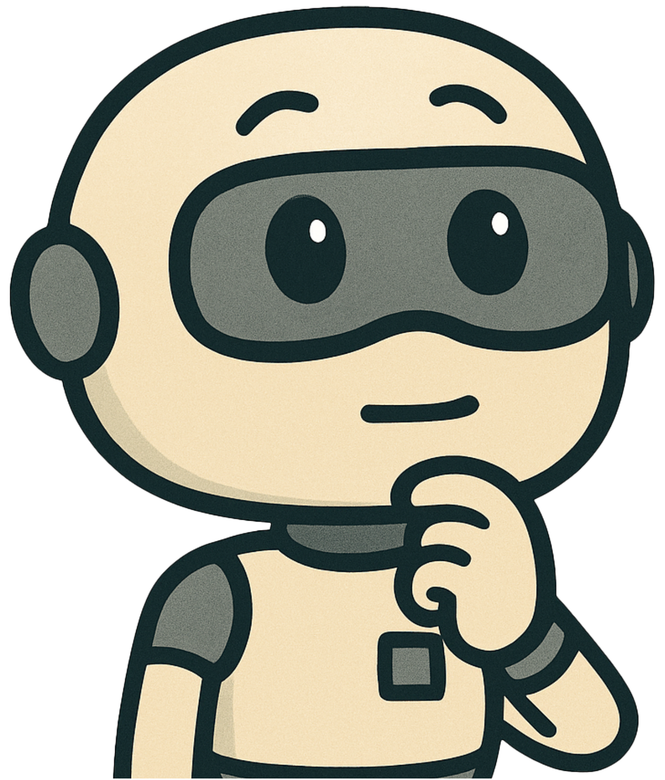


Modeling systems with ODEs

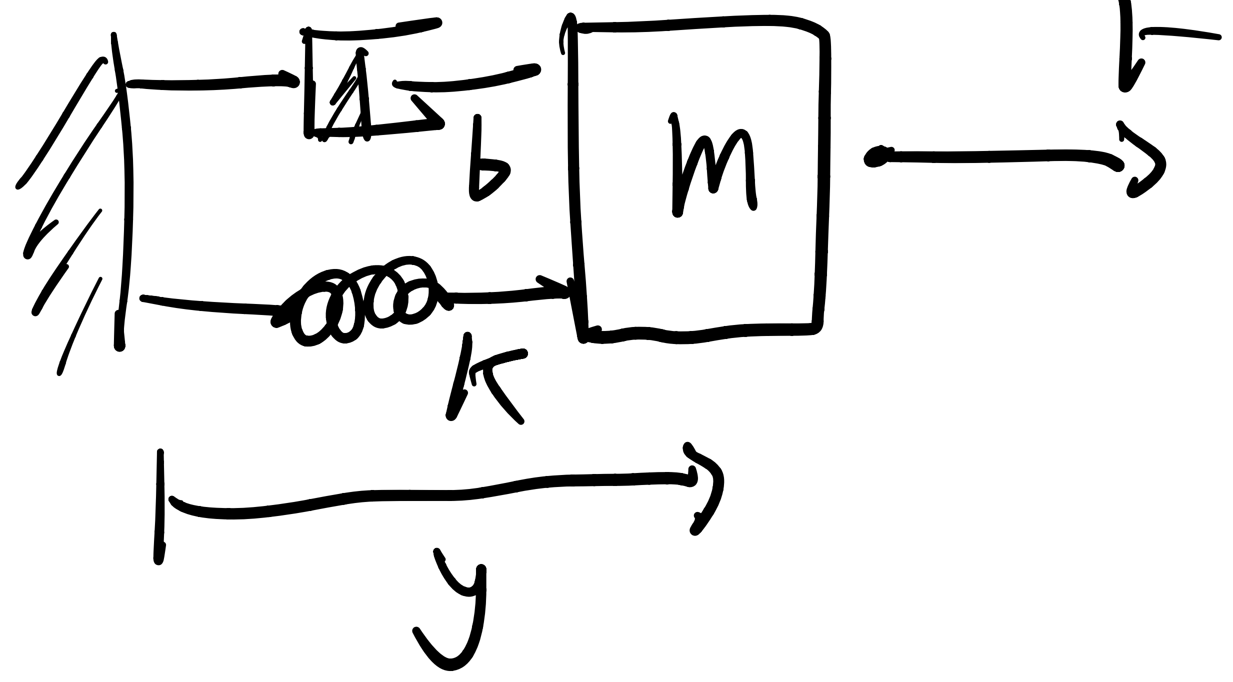
Our notion of systems is still abstract!

How do we model a system?

For physics/engineering, we typically deal with Differential Equations.

Modeling systems with ODEs

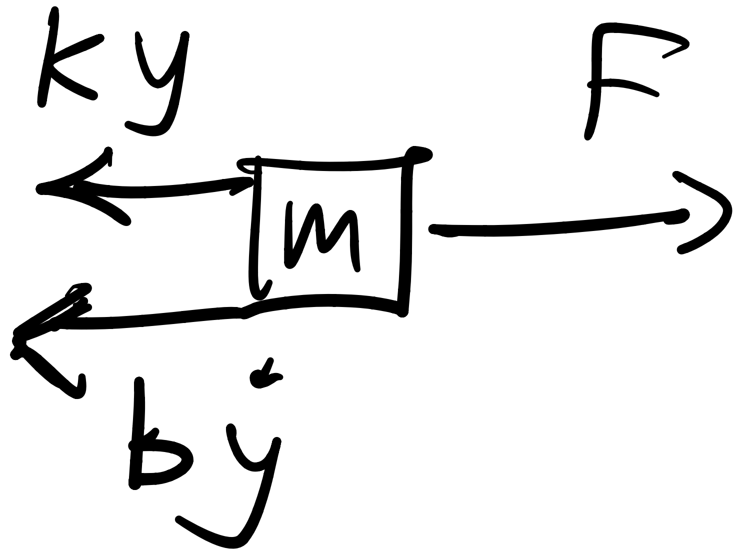
Example: Mass-spring damper



physical laws: Newton's laws $\sum F_{ext} = ma$

$\Downarrow$

Free body diagram



$$F - ky - b\dot{y} = ma$$

$$a = \ddot{y}$$

Rearranging $m\ddot{y} + b\dot{y} + ky = F$ Linear, second-order ODE.

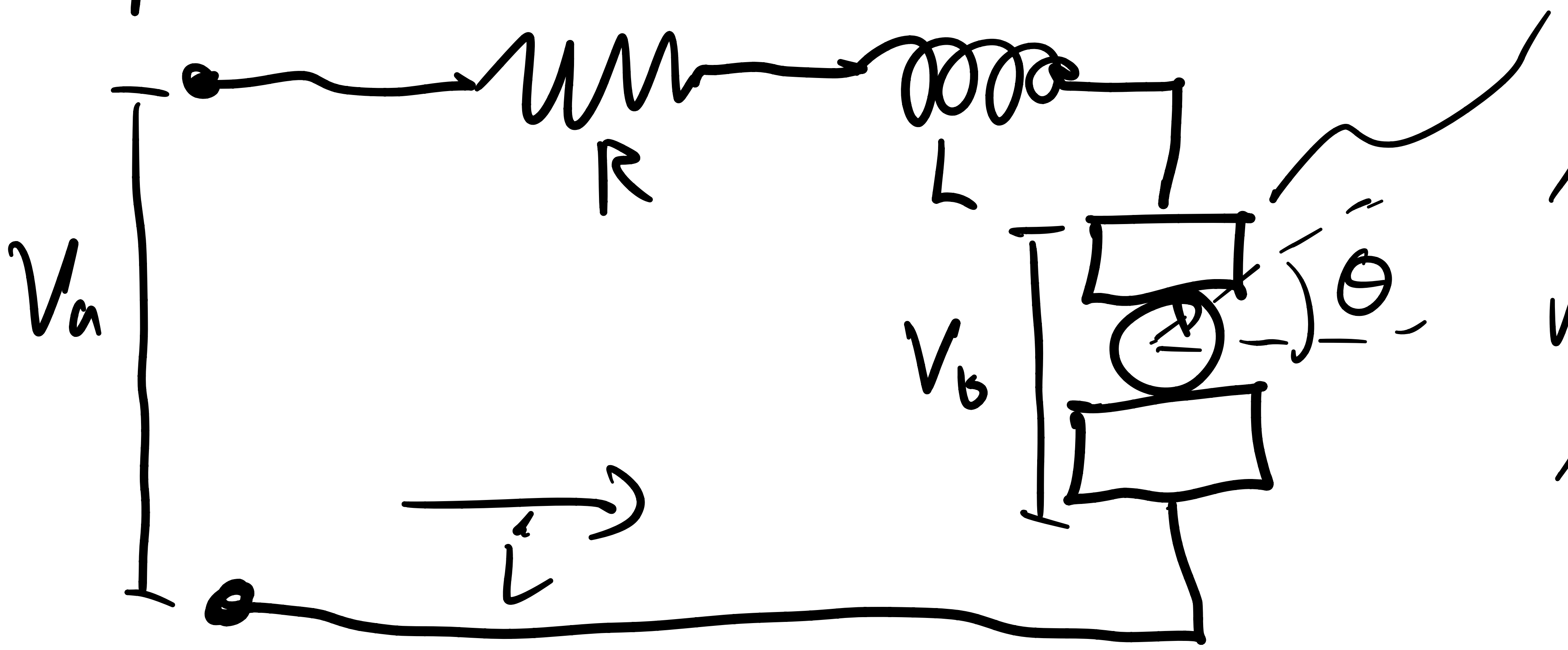
2nd order equation: Highest-order derivative is 2.

Linear: $\ddot{y}$ and $\dot{y}$ and y all appear linearly in the equation

$\hookrightarrow$ no nonlinear functions of these variables.

Modeling systems with ODEs

Example ~~Servo~~ DC motor



Motor angle θ
speed $\dot{\theta}$
load torque τ_L
motor torque τ_m
load MOI J_L
motor MOI J_m

Newton's laws: $J\ddot{\theta} = \sum \tau$

$$\sum \tau = \tau_m - \tau_L$$

Motor torque proportional to current

$$\tau_m = K_\tau i$$

Back EMF: $V_b = K_e \dot{\theta}$

Kirchoff's laws:

$$iR + L\dot{i} + V_b = V_a$$

$$\Rightarrow iR + L\dot{i} + K_e\dot{\theta} = V_a$$

State variables: $i, \theta, \dot{\theta}$

Inputs: V_a, τ_L

outputs: θ

$$\dot{\omega} = \frac{K_t}{J} i - \frac{1}{J}\tau_L$$

$$\dot{i} = -\frac{K_e}{L} w - \frac{R}{L} i + \frac{1}{L} V_a$$

State space models of systems

Real systems often result in messy, coupled, high-order ODEs.

State-Space Models are a common mathematical framework to represent linear systems.

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}}$$

$x \in \mathbb{R}^n$ Vector of state (memory) variables

$u \in \mathbb{R}^m$ Vector of inputs.

$y \in \mathbb{R}^d$ Vector of outputs.

Advantages: Transform high-order differential equations into systems (collections) of first-order differential equations.

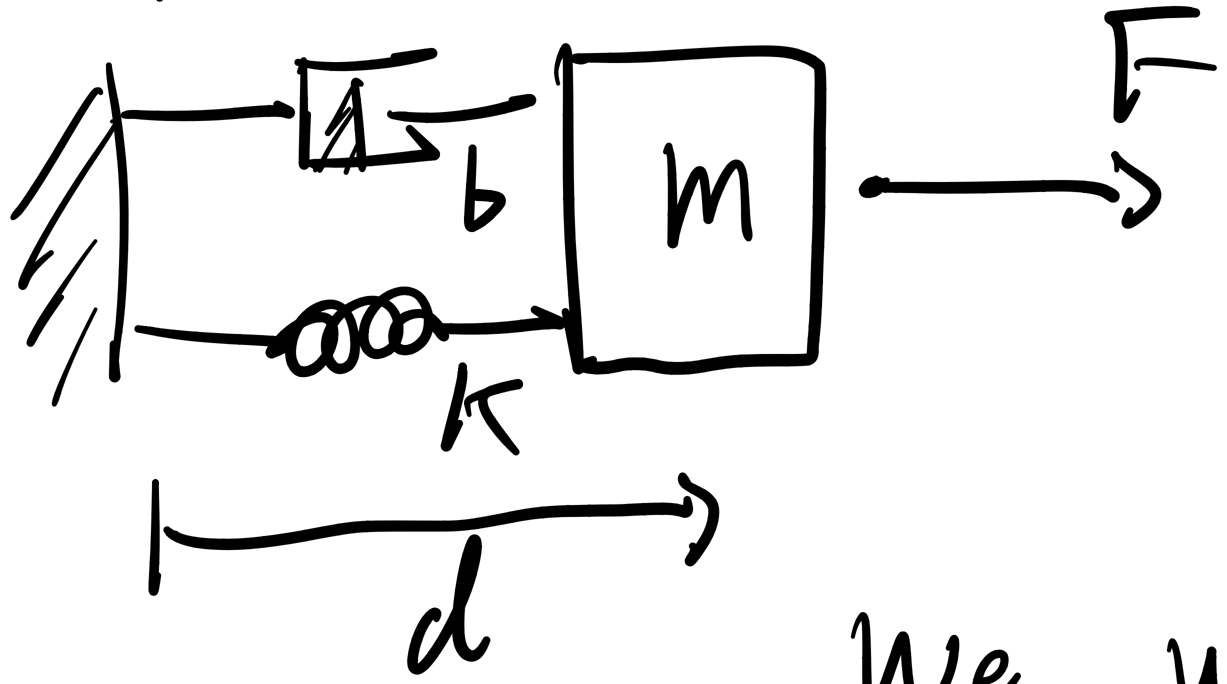
Note: Concept of state x is important.

Note: This is an aside.

State space models of systems

Example: Mass-spring damper

$$m\ddot{d} + b\dot{d} + kd = F \quad (*)$$



Rewrite $x_1 = d$ $\quad x = \begin{bmatrix} d \\ \dot{d} \end{bmatrix}$

$x_2 = \dot{d}$

We want to write $(*)$ as $\dot{x} = Ax + Bu$ $\quad A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$

We know that $\dot{x}_1 = x_2$.

rewrite $(*)$ to solve for $x_2 = \dot{d} \Rightarrow \dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}F$

$\dot{x}_2 = \ddot{d}$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}F \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} F$$

$A = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$. Suppose $y = d = x_1$

$$\Rightarrow y = \overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}^{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \end{bmatrix}}^{D} F$$

Question

We are modeling the mass-spring damper system in state space.

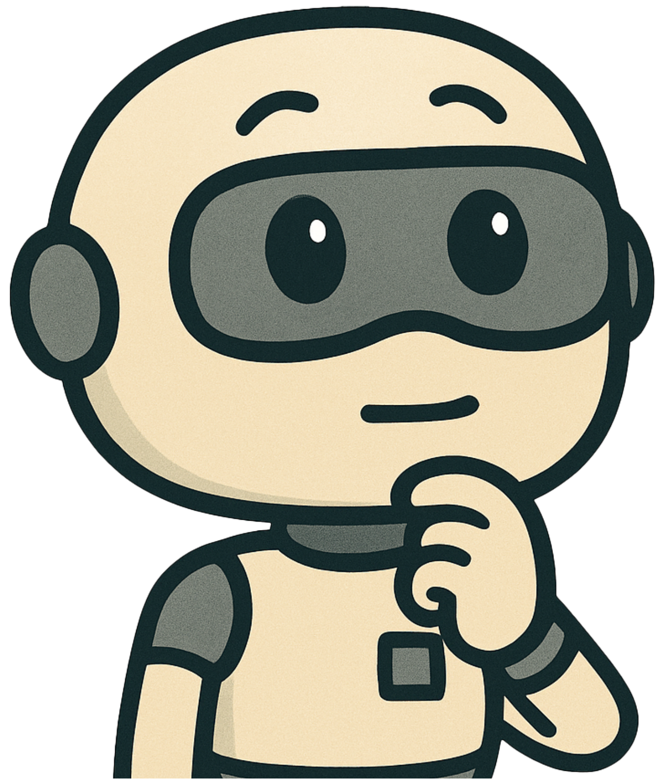
We define x as $x = \begin{bmatrix} d \\ \dot{d} \end{bmatrix} \in \mathbb{R}^2$

If our system equation is $\dot{x} = Ax + Bu$, what are the dimensions of A?

A: 1×1 B: 2×1 C: 1×2. **D: 2×2.**

$$\underbrace{\dot{x}}_{\mathbb{R}^2} = A\underbrace{x}_{\mathbb{R}^2} + \cancel{Bu}$$

$A \in \mathbb{R}^{2 \times 2}$



LTI Systems

Linear — Time invariance

In this course, restrict ourselves entirely to LTI systems.

Definition: An LTI system must have the following properties:

1. **Linearity:** S is linear iff. it has properties additivity & homogeneity.

   Additivity: $S(u_1) = y_1$, $S(u_2) = y_2 \Longrightarrow S(u_1 + u_2) = y_1 + y_2$

   Homogeneity: $S(u) = y \Longrightarrow S(\alpha u) = \alpha y$

   Can jointly define the properties as $S(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 y_1 + \alpha_2 y_2$

2. **Time invariance** Behavior of S doesn't change over time.

$$S(u(t)) = y(t) \Longrightarrow \text{for any } \tau, \quad S(u(t-\tau)) = y(t-\tau)$$

LTI Systems

Example ODE $\boxed{\dot{y}(t) + 2y(t) = u(t)}$ $S(u(t)) = y(t)$

Verify LTI by checking properties.

Linearity: Let $y_1(t) = S(u_1(t))$ and $y_2(t) = S(u_2(t))$

Consider the input $u(t) = \alpha_1 u_1(t) + \alpha_2 u_2(t)$,

Hypothesize, $y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$

Plug hypothesis into ODE. LHS.

$$\frac{d}{dt}\left(\alpha_1 y_1(t) + \alpha_2 y_2(t)\right) + 2\left(\alpha_1 y_1(t) + \alpha_2 y_2(t)\right) = \alpha_1 \dot{y}_1(t) + \alpha_2 \dot{y}_2(t) + 2\alpha_1 y_1(t) + 2\alpha_2 y_2(t)$$

$$= \alpha_1\left(\dot{y}_1(t) + 2y_1(t)\right) + \alpha_2\left(\dot{y}_2(t) + 2y_2(t)\right)$$

$$= \alpha_1 u_1(t) + \alpha_2 u_2(t)$$

$\therefore$ System is Linear

LTI Systems

$\dot{y}(t) + 2y(t) = u(t)$

Time invariance: Let $y(t)$ be the output for $u(t)$.

Consider $u'(t) = u(t-\tau)$. we want to check if $y'(t) = y(t-\tau)$ is a valid output for input $u'(t)$.

This holds by form of ODE.

for ODEs. if t only appears in y or u and doesn't appear as an independent variable in equation, then the system is time invariant.

$\therefore$ this ODE is time invariant.

$\longrightarrow$ the whole system is LTI.

example

$\dot{y}(t) + 2ty(t) = u(t)$

is time varying.

LTI Systems

Example 2. ODE $\dot{y}(t) + 2y(t)^2 = u(t)$

Let $y(t) = S(u(t))$. Consider $u'(t) = \alpha u(t)$

If the system were linear, we expect $y'(t) = \alpha y(t)$,

Let's plug $y'(t)$ into the LHS of the ODE to check.

$$\frac{d}{dt}(\alpha y(t)) + 2(\alpha y(t))^2 = \alpha \dot{y}(t) + 2\alpha^2 y(t)^2$$

LHS includes α^2

$$\neq \alpha u(t) = \alpha(\dot{y}(t) + 2y(t)^2)$$

RHS includes α

So, the system is not linear and

$\therefore$ not LTI.