L5: Transfer Functions and Block Diagrams

ELEC 341 | Systems and Control | Spring 2026

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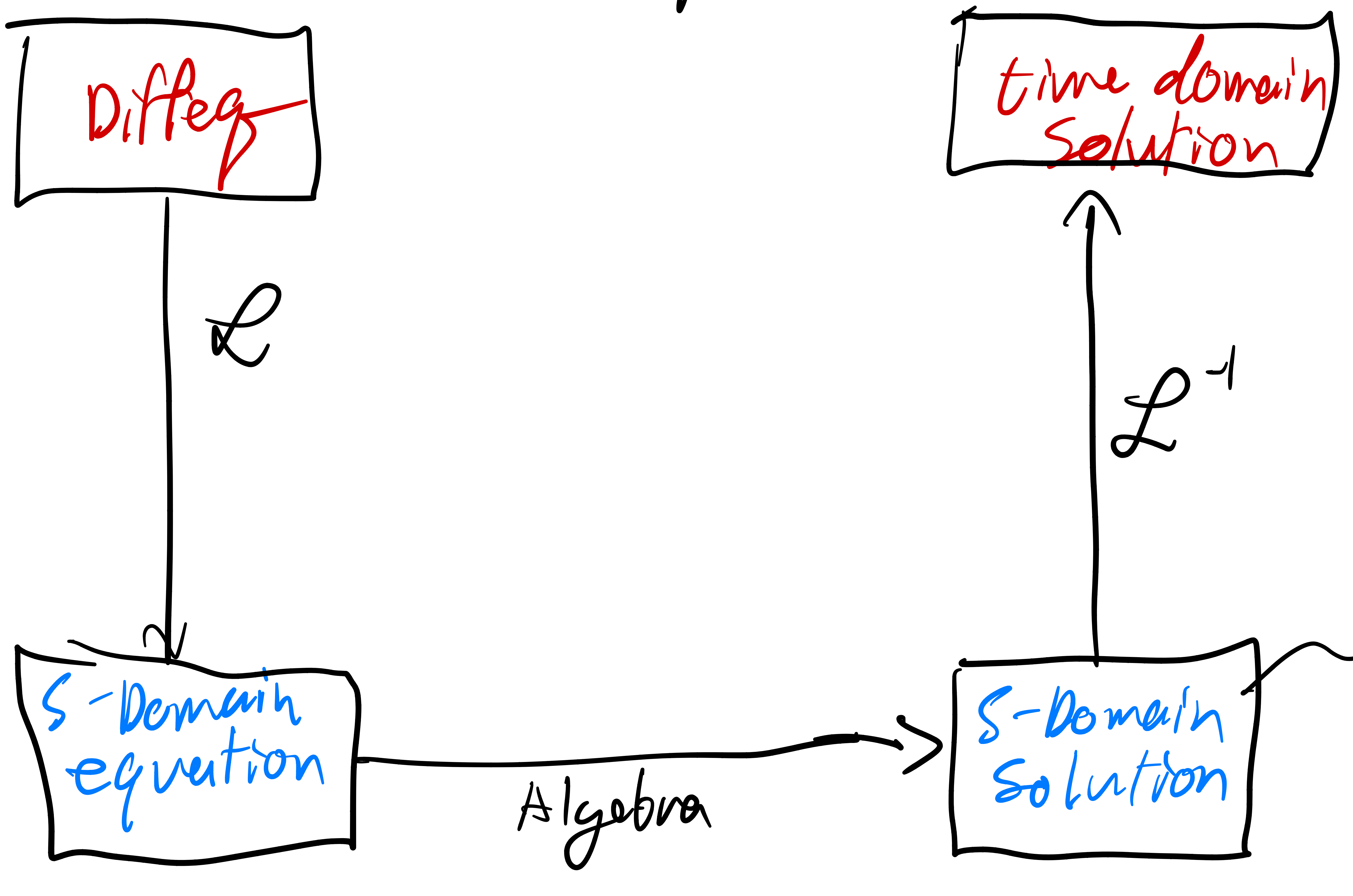
Last Class: Inverse Laplace Transforms & Solns to ODEs.



often $Y(s) = \dfrac{N(s)}{D(s)}$

Today: More analysis of S-Domain relationships between inputs and outputs.

Transfer functions: A general way of describing systems in the S-Domain.

Time Domain $\quad r(t) \longrightarrow \boxed{g} \xrightarrow{y(t)}$

$y(t) = g(r(t))$ represents solution of ODE for some input $u(t)$. Hard to write explicitly.

S-Domain $\quad R(s) \longrightarrow \boxed{G} \xrightarrow{Y(s)}$

$$Y(s) = G(s)R(s)$$

or $G(s) = \dfrac{Y(s)}{R(s)}$

The transfer function $G(s)$ is defined as

$$G(s) = \frac{Y(s)}{R(s)} \begin{matrix} \sim \text{output} \\ \sim \text{input} \end{matrix}$$

↳ Note: Transfer functions define systems in a <u>general way</u>. They are <u>not</u> dependent on the <u>specific input</u> $R(s)$.

Properties of transfer functions of LTI systems

- For LTI systems, $G(s)$ can be expressed as $\frac{N(s)}{D(s)}$ a rational function
- The roots of $N(s)$ (i.e. solutions to $N(s)=0$) are the system's zeros.
- The roots of $D(s)$ are the system's poles.

E.g. $G(s) = \frac{s+1}{s^2+\omega^2} = \frac{s+1}{(s+j\omega)(s-j\omega)}$ zero at $s=-1$, poles at $s=\pm j\omega$.

Intuition: Locations of poles and zeros in S-plane will dictate how the system responds to a particular "test input".

Examples RLC circuit

Goal: Find the transfer function $\frac{V_c(s)}{V(s)}$.

KVL $v = v_L + v_R + v_C$

$$v(t) = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau$$

Solution 1 use $i(t) = \frac{dq(t)}{dt}$ to rewrite the equation.

$$v(t) = L\ddot{q}(t) + R\dot{q}(t) + \frac{1}{C}q(t)$$ also know $q(t) = Cv_c(t)$

$$\Longrightarrow v(t) = LC\ddot{v}_c(t) + CR\dot{v}_c(t) + v_c(t)$$

taking $\mathcal{L}$ of both sides (assuming zero I.C.s)

$$V(s) = (LCs^2 + RCs + 1)V_c(s) \Longrightarrow \boxed{\frac{V_c(s)}{V(s)} = \frac{1}{s^2LC + sRC + 1}}$$

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Solution 2 Consider the current-voltage transfer functions of individual components.

Resistor: $v_R(t) = Ri(t) \Longrightarrow \frac{V_R(s)}{I(s)} = R(s)$

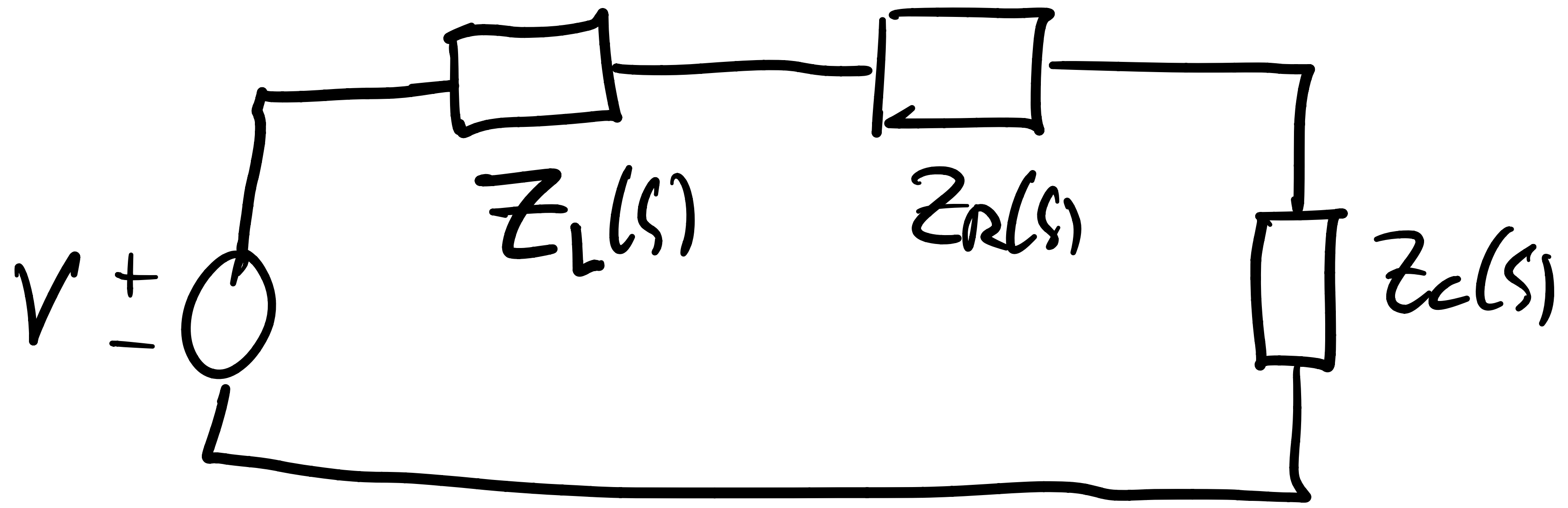
Inductor $v_L(t) = L\frac{di}{dt} \Longrightarrow V_L(s) = Ls\,I(s)$

Capacitor: $v_c(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau = V_c(s) = \frac{1}{sC}I(s)$

Define impedance as

$$Z(s) = \frac{V(s)}{I(s)}$$

Replace all components with impedance values



Now, KVL reads as

$$V(s) = (Z_L(s) + Z_R + Z_c) I(s)$$

$$\frac{V_c}{I} = \frac{1}{Cs} = Z_c$$

$$\Rightarrow V(s) = \left(Ls + R + \frac{1}{Cs}\right) I(s)$$

$$\Rightarrow \frac{I(s)}{V(s)} = \frac{1}{Ls + R + \frac{1}{Cs}}$$

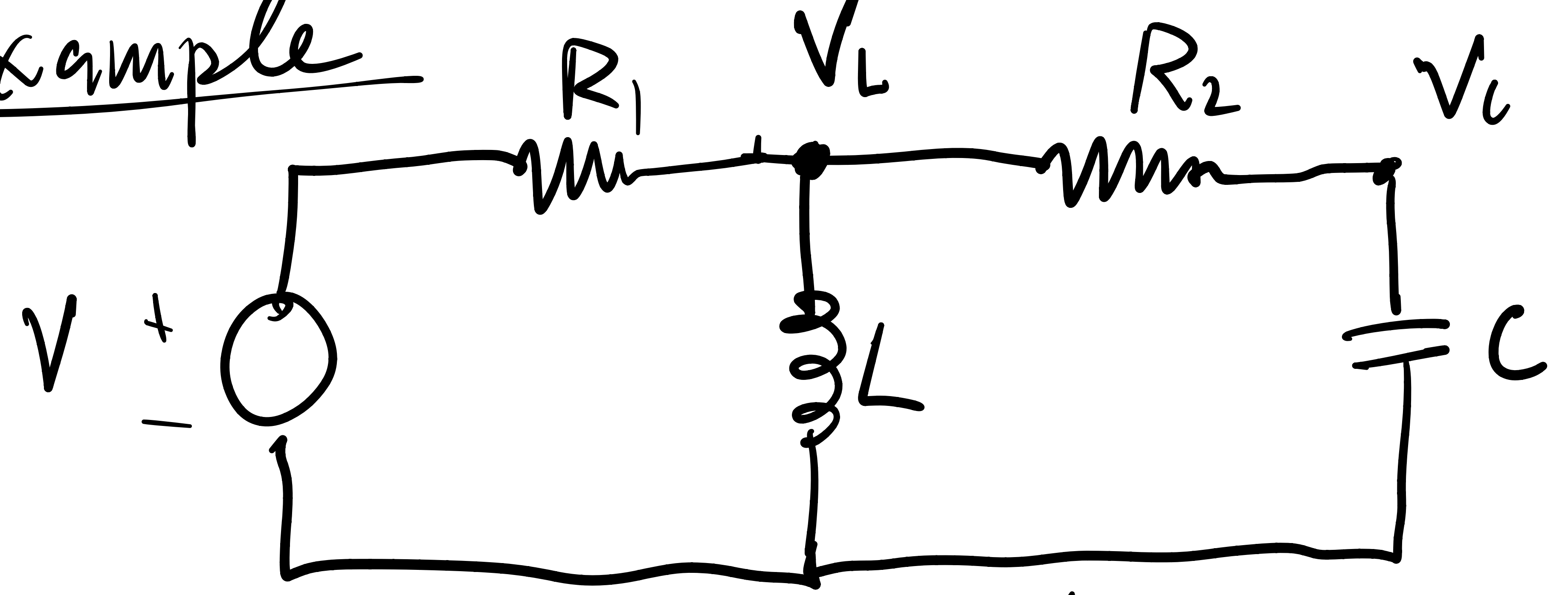
and $I(s) = V_c(s)\, Cs$

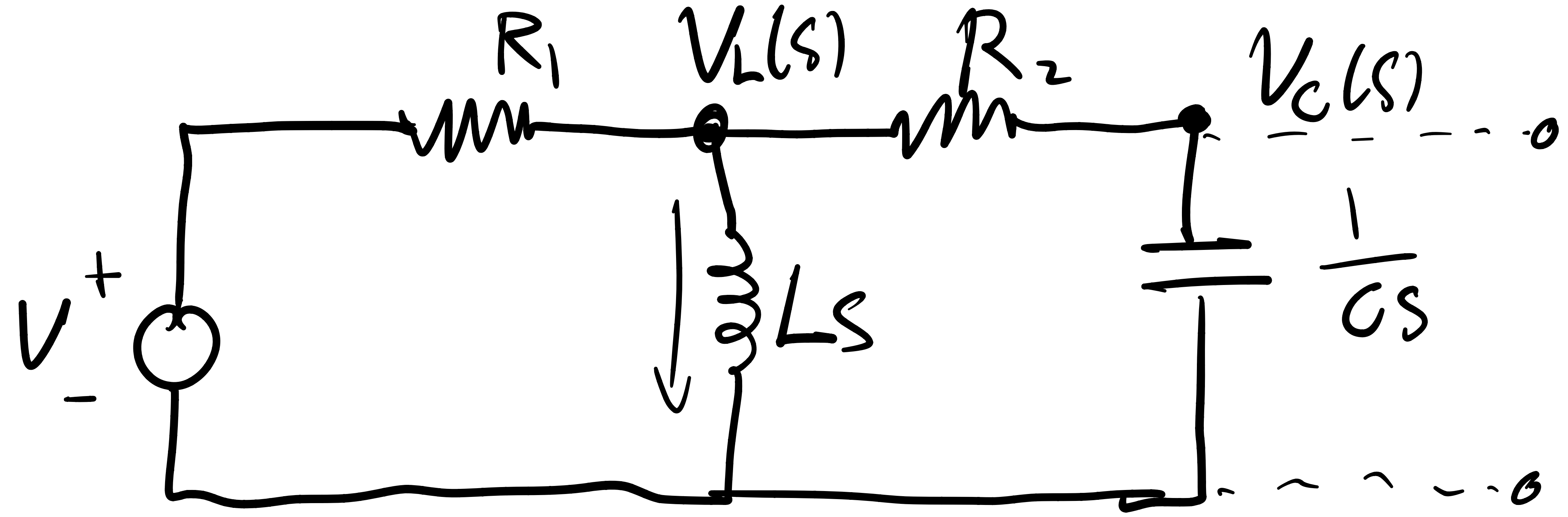
$$\Rightarrow \boxed{\frac{V_c(s)}{V(s)} = \frac{1}{LCs^2 + RCs + 1}}$$

In general,

- Replace all circuit elements with impedance values $Z(s)$
- treat them all as you would resistors.
- Combine elements in series by adding $Z_{tot} = Z_1 + Z_2 + \dots + Z_n$
- Combine parallel elements using $Z_{parallel} = \dfrac{1}{1/Z_1 + 1/Z_2 + \dots + 1/Z_n}$
- Solve for equations generated with node analysis (KCL) and mesh analysis (KVL) to find transfer function of interest.

Example





$\frac{V_c(s)}{V(s)}$?

Sum currents going into junctions:

$$(1)\quad \frac{V_L(s)-V(s)}{R_1}+\frac{V_L(s)}{Ls}+\frac{V_L(s)-V_c(s)}{R_2}=0$$

$$(2)\quad \frac{V_c(s)-V_L(s)}{R_2}+CsV_c(s)=0$$

Rearrange

$$(1)\quad V_L(s)\left(\frac{1}{R_1}+\frac{1}{Ls}+\frac{1}{R_2}\right)-\frac{V_c(s)}{R_2}=\frac{V(s)}{R_1}$$

$$(2)\quad -\frac{1}{R_2}V_L(s)+\left(Cs+1/R_2\right)V_c(s)=0$$

$$\Longrightarrow V_L(s)=(R_2Cs+1)V_c(s)$$

Plug $V_L(s)$ into equation (1) $\Rightarrow$ $\left\{(R_2Cs+1)V_c(s)\right\}\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{Ls}\right)-\frac{V_c(s)}{R_2}=\frac{V(s)}{R_2}$

$$\Longrightarrow V_c(s)\left[(R_2Cs+1)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+(R_2Cs+1)\left(\frac{1}{Ls}\right)-\frac{1}{R_2}\right]=\frac{V(s)}{R_1}$$

$$V_c(s)\left[(R_2Cs+1)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+(R_2Cs+1)\left(\frac{1}{Ls}\right)-\frac{1}{R_2}\right]=\frac{V(s)}{R_1}$$

$$V_c(s)\left[\frac{R_2Cs}{R_1}+\frac{1}{R_1}+Cs+\cancel{\frac{1}{R_2}}+\frac{R_2C}{L}+\frac{1}{Ls}-\cancel{\frac{1}{R_2}}\right]=\frac{V(s)}{R_1}$$

$$V_c(s)\left[s\left(\frac{R_2C}{R_1}+C\right)+\left(\frac{1}{R_1}+\frac{R_2C}{L}\right)+\frac{1}{Ls}\right]=\frac{V(s)}{R_1}$$

$$\frac{V_c(s)}{V(s)}=\frac{1}{R_1\left[C\left(\frac{R_1+R_2}{R_1}\right)s+\frac{L+R_2R_1C}{R_1L}+\frac{1}{Ls}\right]}$$

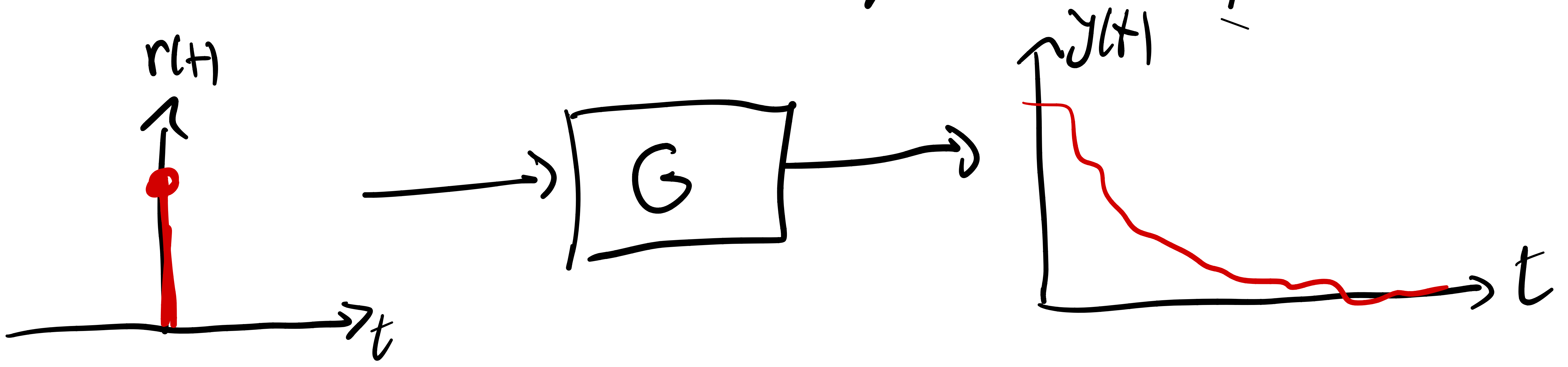
$$\boxed{\frac{V_c}{V(s)}=\frac{Ls}{LC(R_1+R_2)s^2+(L+R_2R_1C)s+R_1}}$$

What are transfer functions?

The algebraic definition $G(s) = \frac{Y(s)}{R(s)} \Longleftrightarrow Y(s) = G(s)R(s)$

↳ why does the ratio of $\frac{\text{system output}}{\text{system input}}$ characterize the system?

Another view: The transfer function is the Laplace transform of the system's impulse response.



$r(t) = \delta(t) \Rightarrow \mathcal{L}\{r(t)\} = 1$

and $Y(s) = G(s) \cdot 1$

$\Rightarrow y(t) = \mathcal{L}^{-1}\{G(s)\} = g(t)$

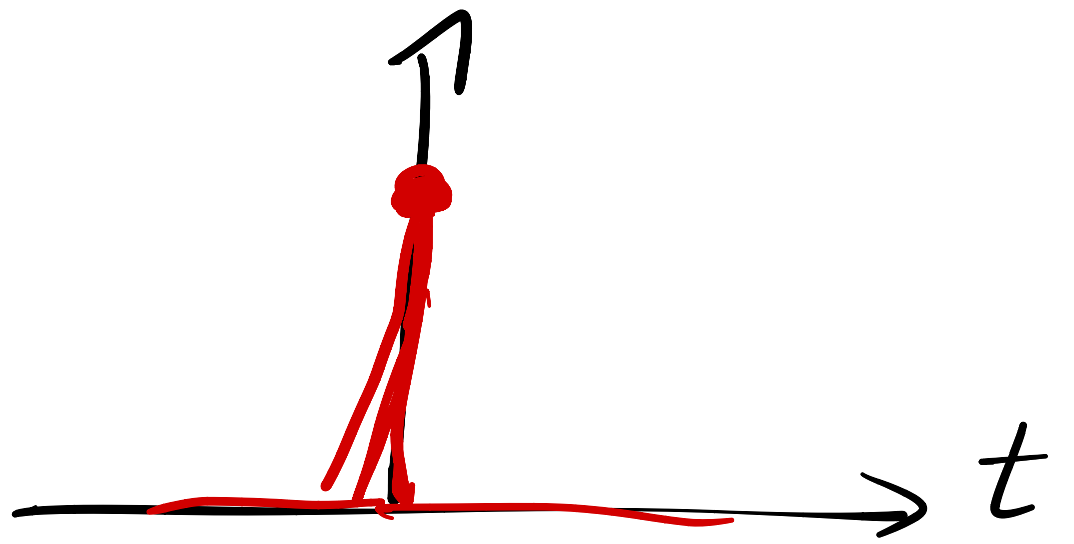
The transfer function/impulse response is like the system's "DNA". It tells how the system respons to a "kick".

Impulse Response of LTI systems

LTI properties $\Longrightarrow$ If we know how the system responds to $\delta(t)$, then we know how it responds to <u>any</u> $x(t)$.

<u>Reminder</u>

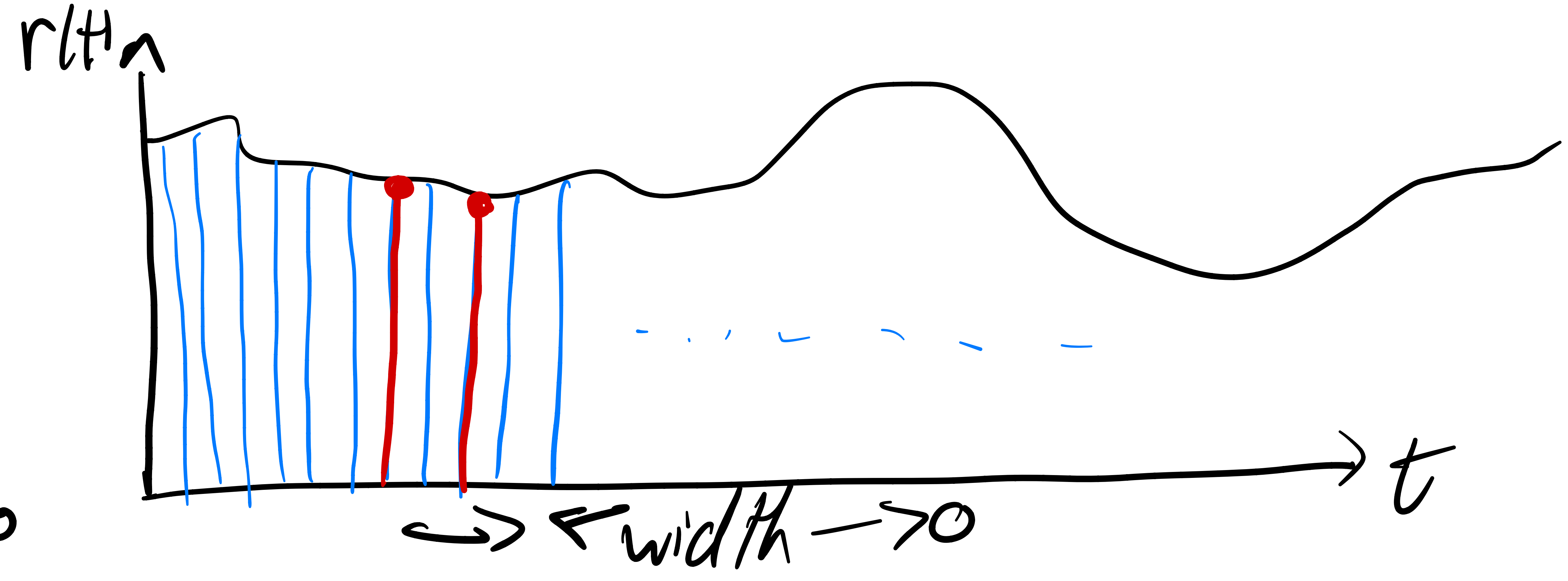
$$\delta(t) = \begin{cases} \infty & t=0 \\ 0 & t\neq 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)\,dt = \underline{1}$$



<u>Sampling property:</u> $$\int_{-\infty}^{\infty} f(t)\,\delta(t-\tau)\,dt = f(\tau)$$

<u>Define:</u> Impulse response $h(t)$ is the output of the system when the input is $\delta(t)$.

Consider any input $r(t)$.
Can think of this as the sum of many tiny bars.



As the width of the bars go to zero, we just have scaled, shifted versions of $\delta(t)$.

LTI system $\Rightarrow$ if we know $h(t) = G(\delta(t)) \Rightarrow \underline{G}(\underline{\delta(t-\tau) r(\tau)}) = r(\tau)\underline{h}(t-\tau)$

So, $G(r(t)) = y(t)$ is just the sum (integral) of these shifted, scaled impulse responses.

$$\boxed{y(t) = \int_{-\infty}^{\infty} r(\tau) h(t-\tau)\, d\tau = r(t) * h(t)}$$

convolution.

impulse

Laplace transform $\Rightarrow$ $\boxed{Y(s) = H(s)R(s) = G(s)R(s)}$

$$y(t) = \int_{-\infty}^{\infty} r(\tau)\, h(t-\tau)\, d\tau = r(t) * h(t)$$

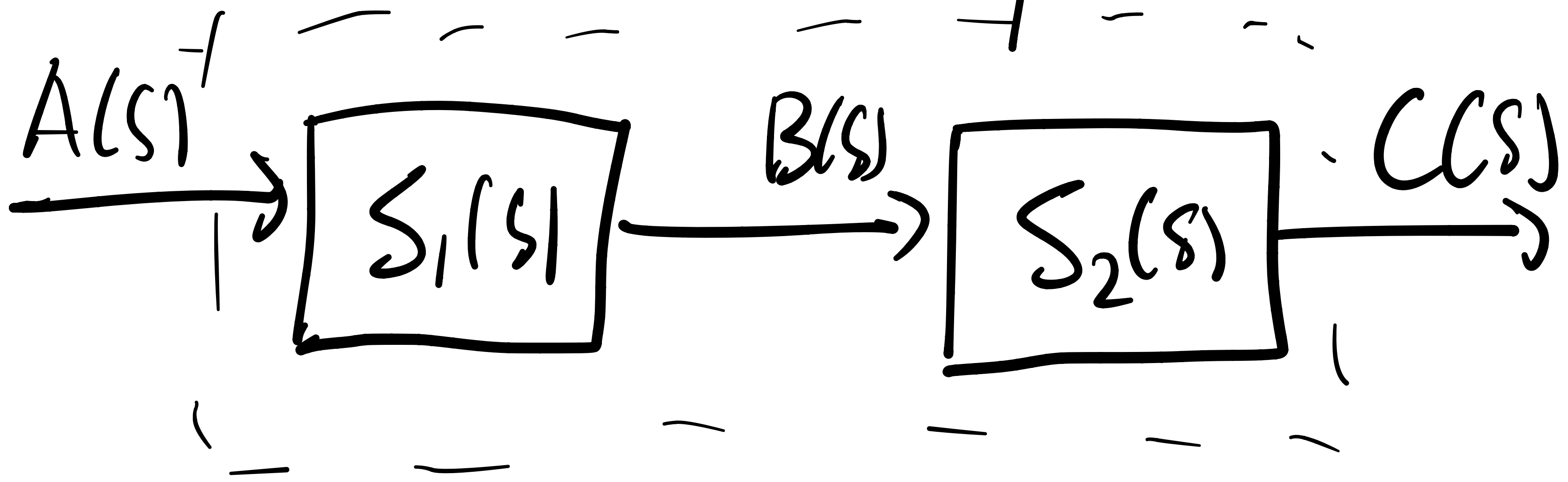
Laplace transform $\Rightarrow$ $Y(s) = H(s)\,R(s) \longrightarrow H(s) = \dfrac{Y(s)}{R(s)}$

$\Rightarrow$ impulse response $H(s)$ <u>is</u> the transfer function and multiplying <u>any</u> input $R(s)$ against $H(s)$ is equivalent to solving for $y(t)$ by taking a convolution $r(t) * h(t)$.

$\hookrightarrow$ The transfer function tells us how the system responds to a "kick" of energy.

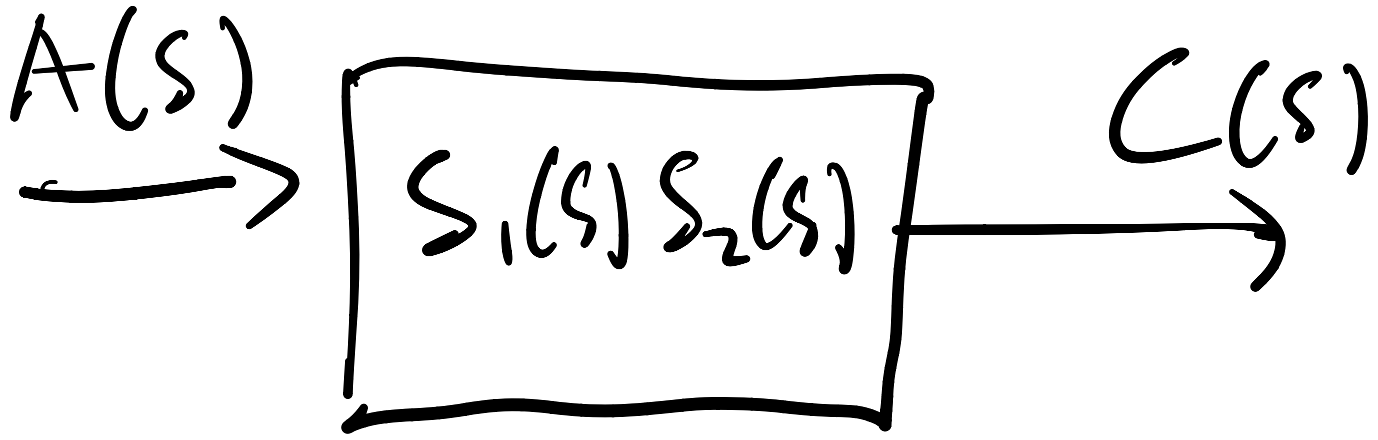
Why/How are transfer functions useful?

- They let us <u>compose</u> subsystem models, to create transfer functions for large, complex systems by combining transfer functions of smaller simpler ones.
    - ↳ Block diagrams are a visual tool for these compositional transfer function manipulations.



$$\left.\begin{array}{l} C(s) = S_2(s)\,B(s) \\ B(s) = S_1(s)\,A(s) \end{array}\right\} \Rightarrow C(s) = S_2(s)S_1(s)A(s)$$

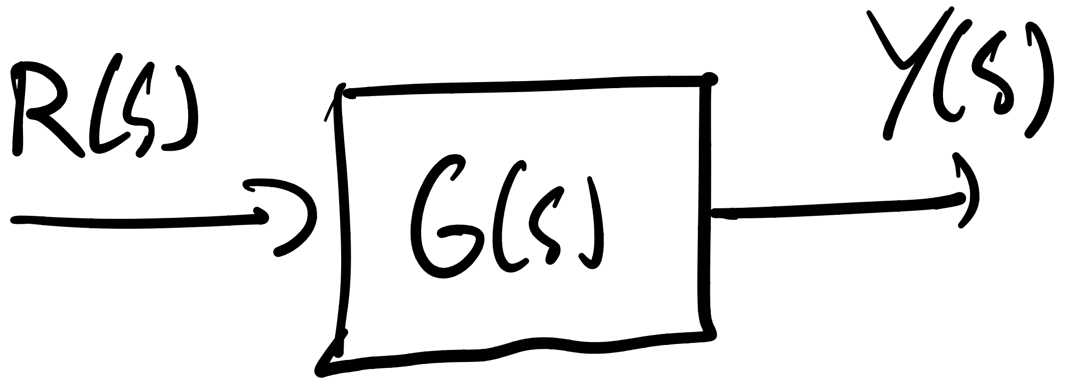
$$\Rightarrow \frac{C(s)}{A(s)} = S_2(s)S_1(s)$$



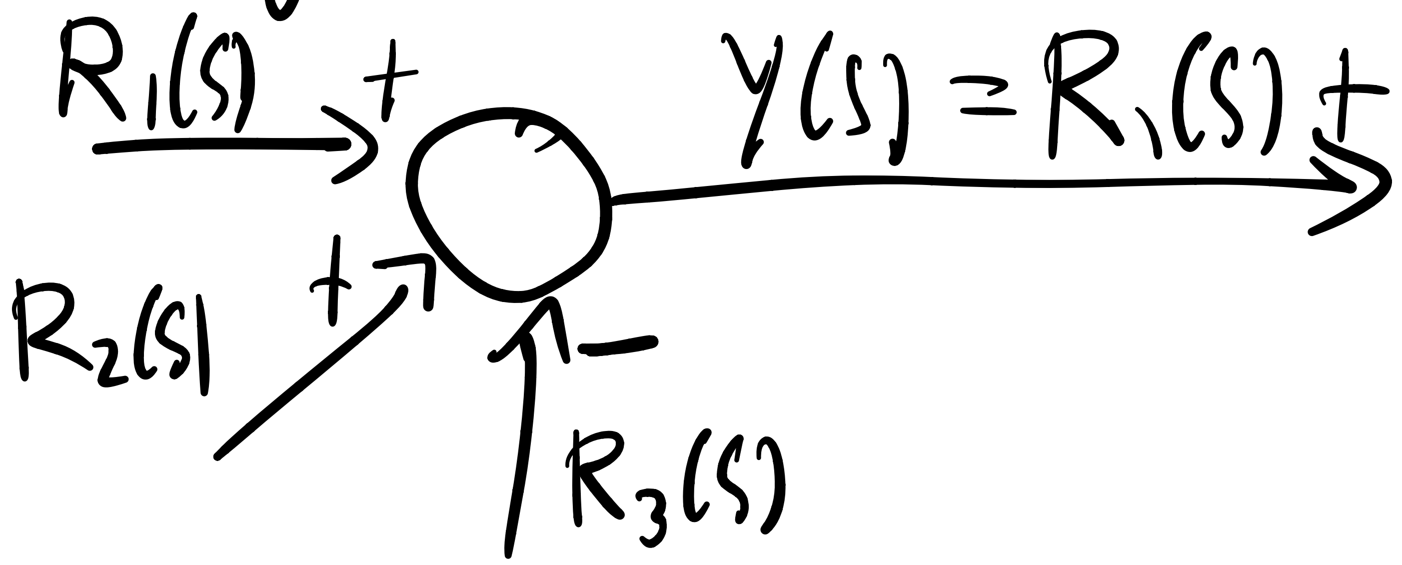
Block diagram components

signals

systems

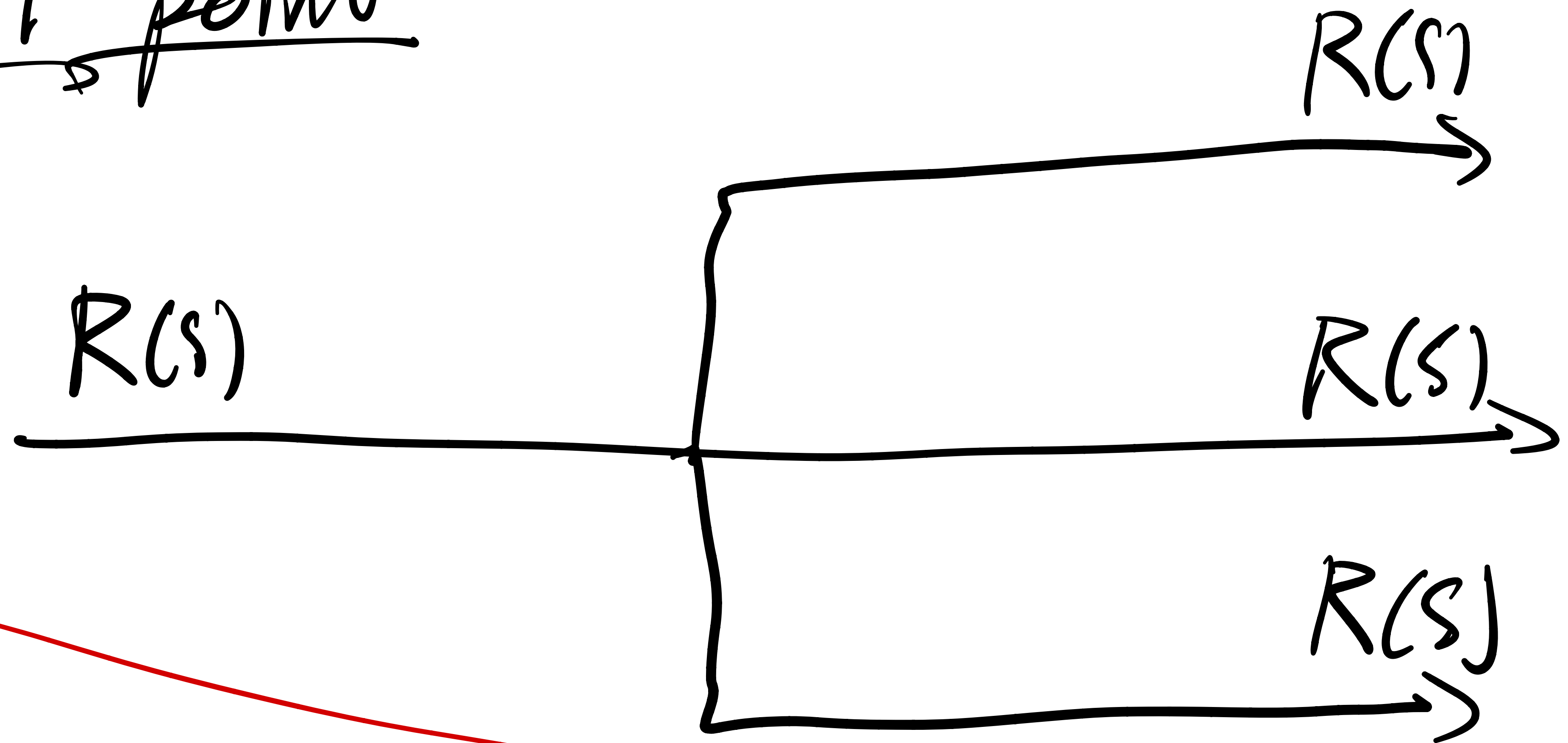


Summing junctions

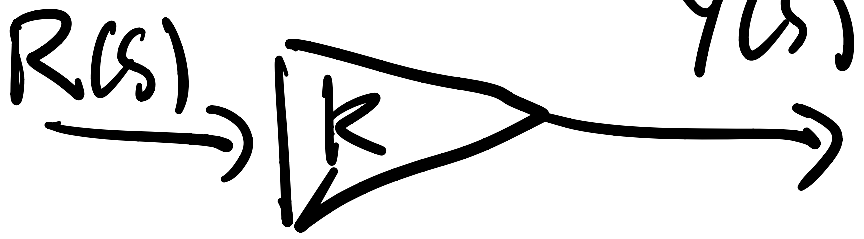


$Y(s) = R_1(s) + R_2(s) - R_3(s)$

Pickoff point

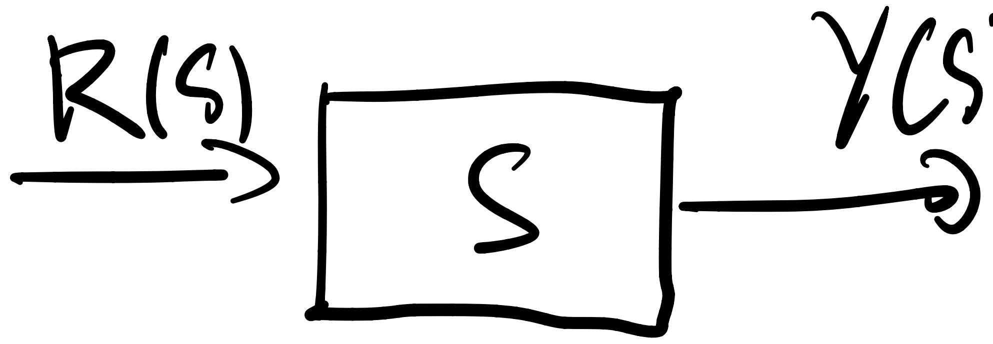


Gain



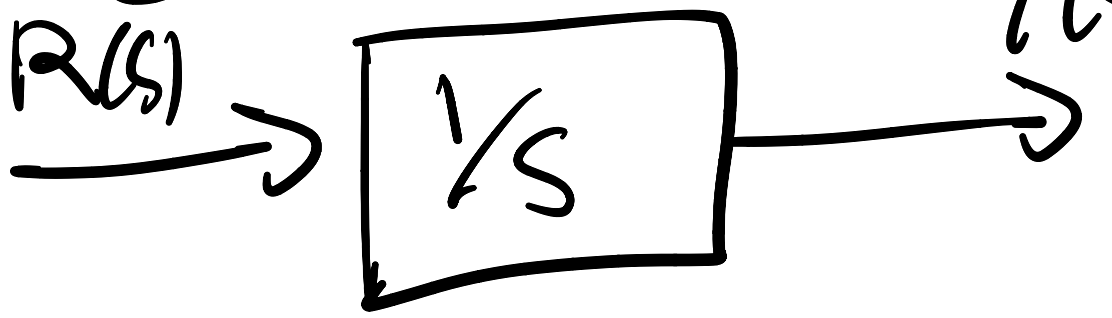
$Y(s) = kR(s)$

Differentiation



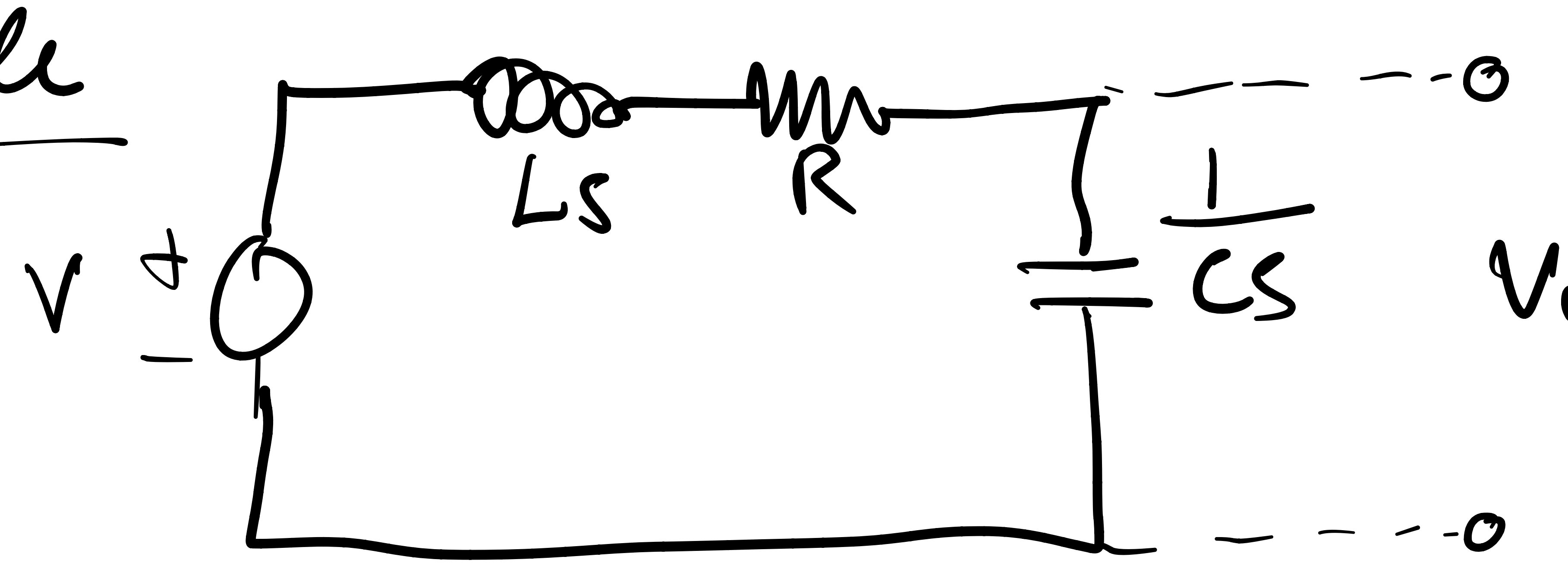
$Y(s) = sR(s)$

Integration



$Y(s) = \frac{R(s)}{s}$

Example



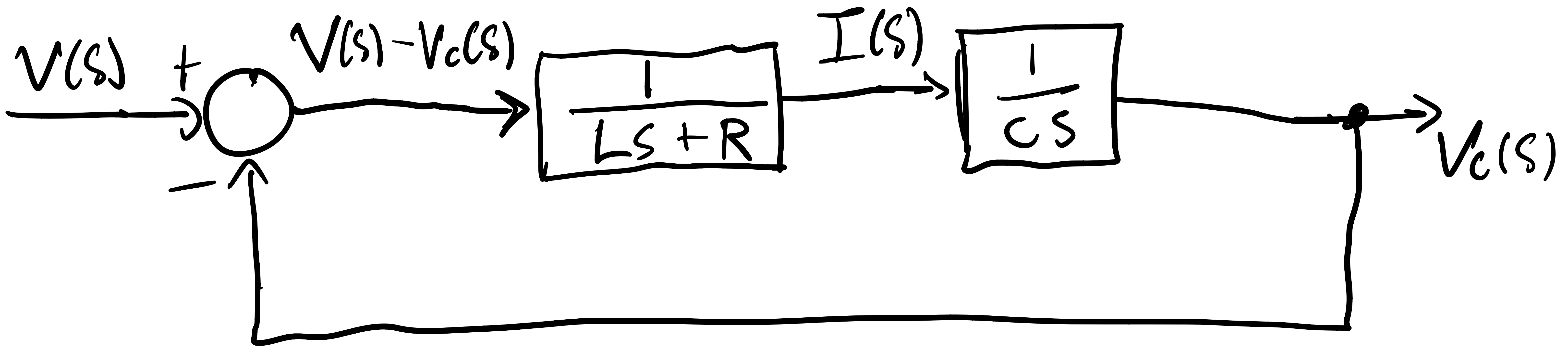
Input: $V(s)$

Output: $V_c(s)$

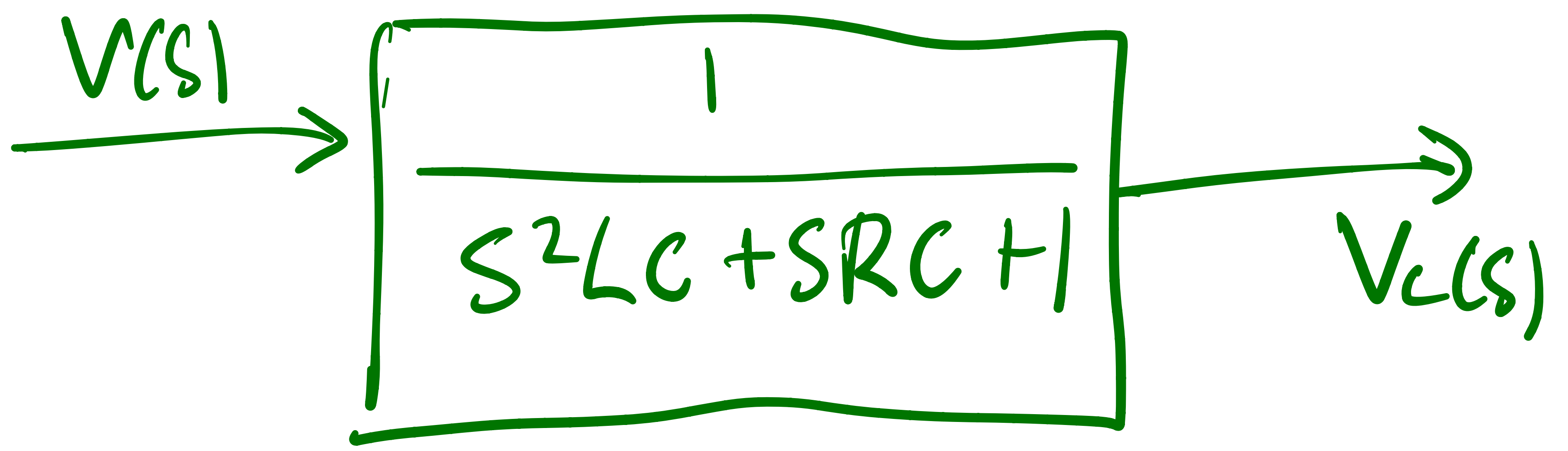
$$V(s) = Ls\,I(s) + R\,I(s) + V_c(s)$$

Define intermediate variable:

$$I(s) = \frac{V(s) - V_c(s)}{Ls + R}, \quad V_c(s) = \frac{1}{Cs} I(s)$$

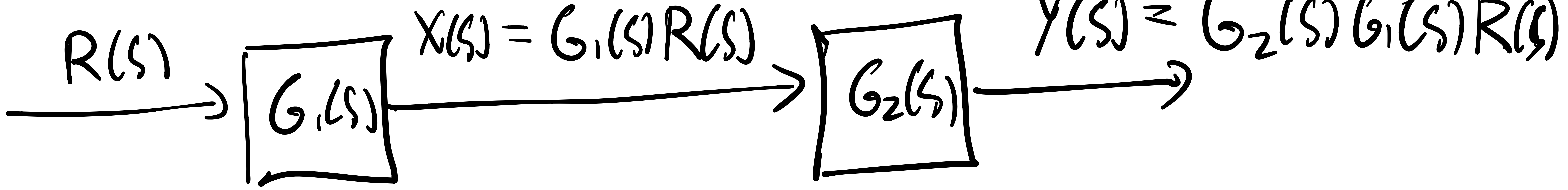


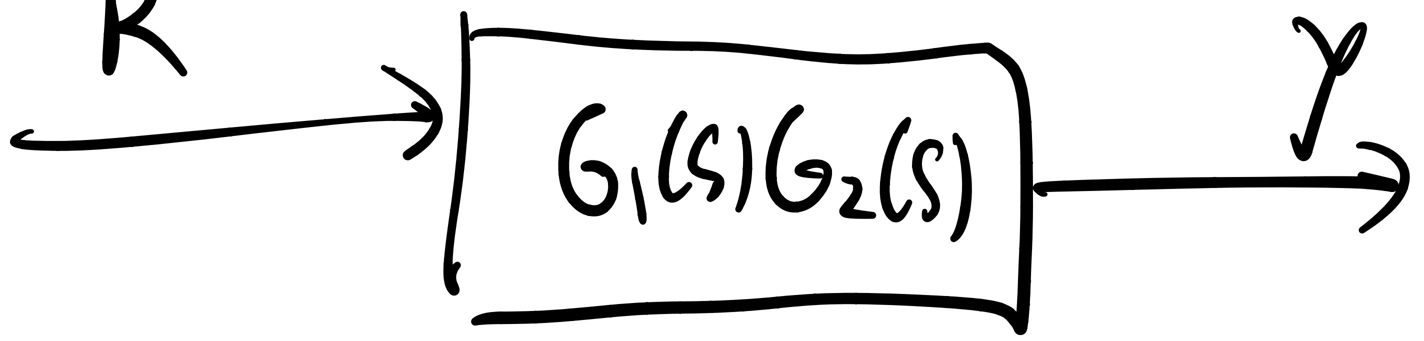
We already had solved for $\frac{V_c}{V} = \frac{1}{s^2LC + sRC + 1}$



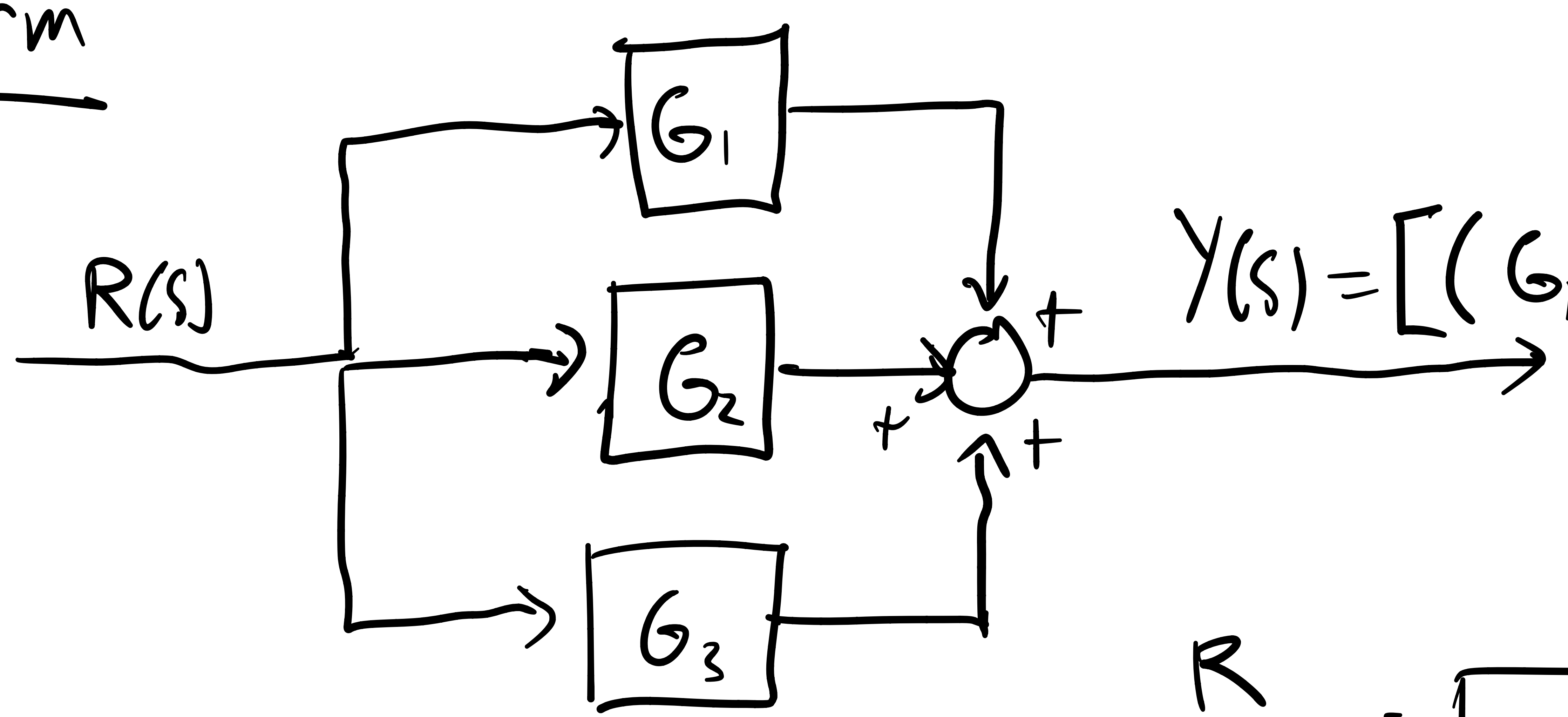
Block diagram reduction

cascade connections

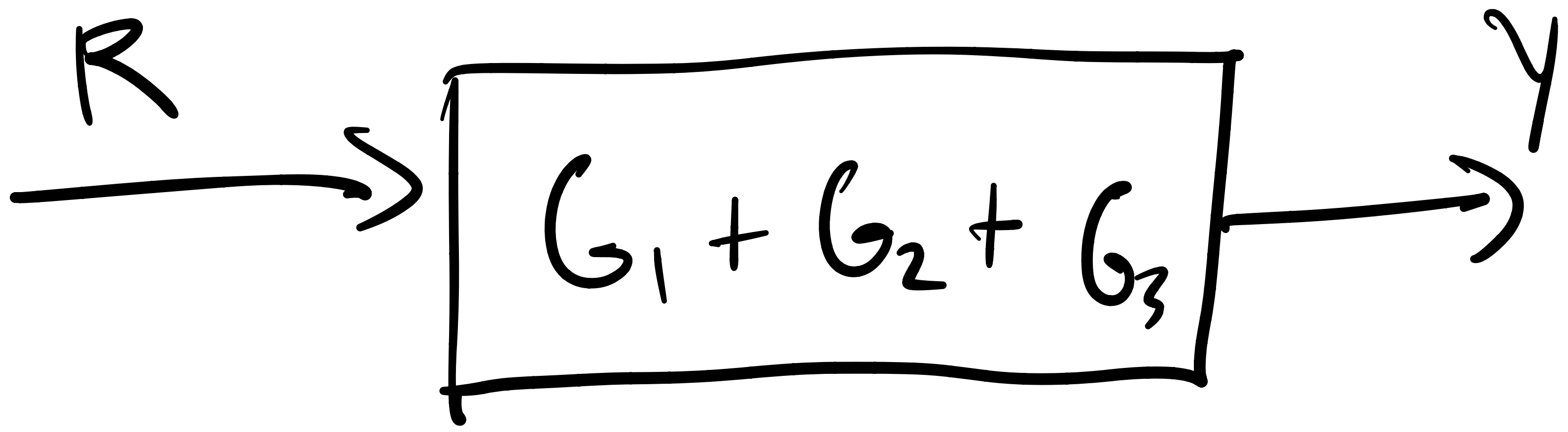




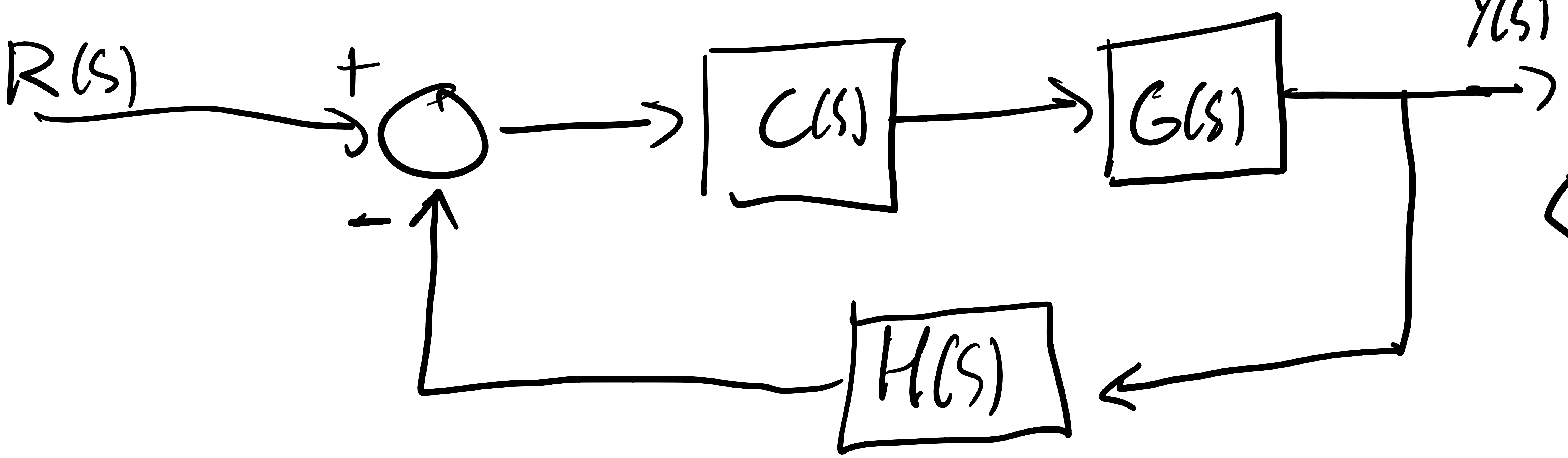
Parallel form



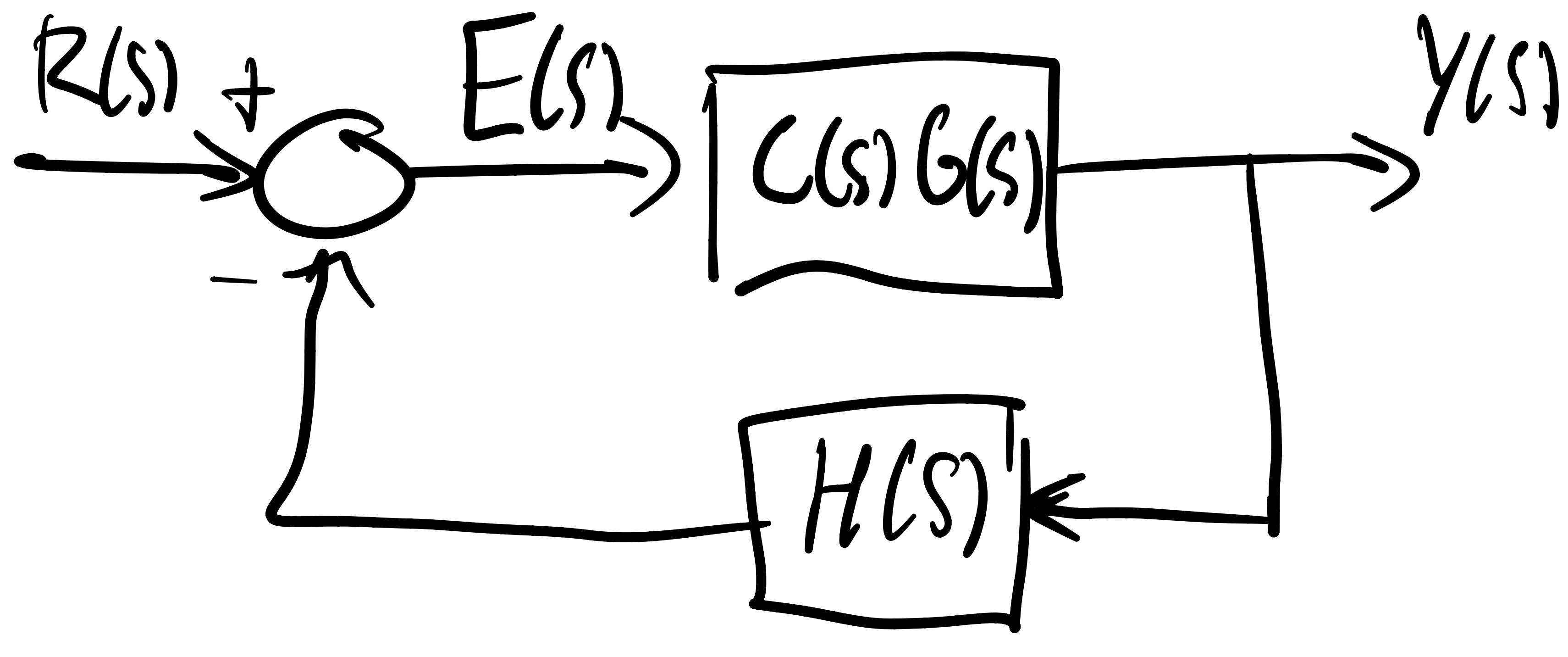
$Y(s) = [(G_1(s) + G_2(s) + G_3(s))R(s)]$



Feed back form



$\Longleftrightarrow$



$\dfrac{Y(s)}{R(s)} = ?$ Let's work this out algebraically.

$$Y(s) = C(s)G(s)E(s) = C(s)G(s)\left[R(s) - H(s)Y(s)\right]$$

$$\Rightarrow Y(s)\left(1 + C(s)G(s)H(s)\right) = C(s)G(s)R(s)$$

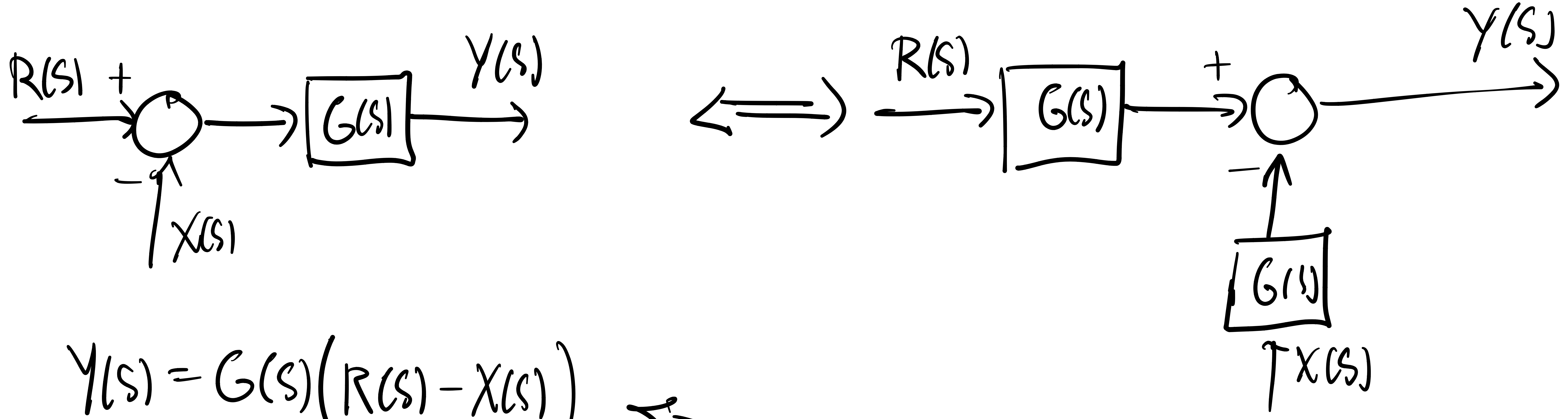
$$\Longrightarrow \boxed{\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)H(s)}}$$

Moving Blocks to create familiar forms

To simplify block diagrams: look for
- cascade connections
- Parallel forms
- feedback loops

to simplify.

However, these forms are not readily apparent.



$$Y(s) = G(s)\big(R(s) - X(s)\big)$$

$$Y(s) = R(s)G(s) - X(s)G(s) = G(s)\big(R(s) - X(s)\big)$$