L4: Inverse Laplace Transforms & Using them to Solve ODEs

ELEC 341 | Systems and Control | Spring 2026

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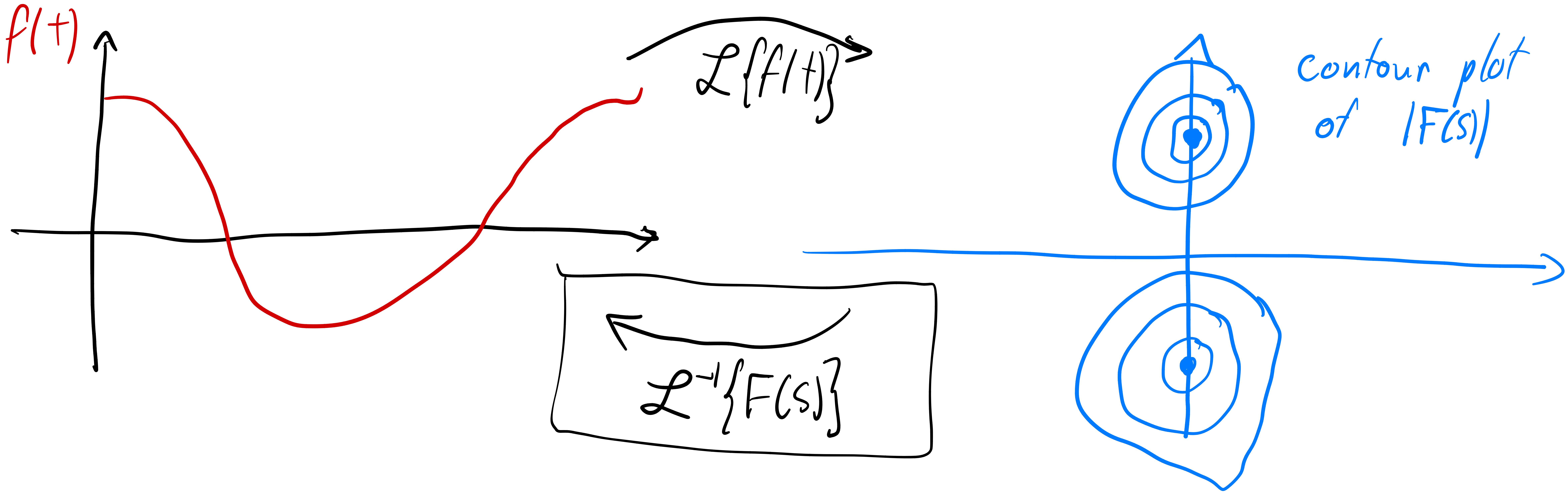
Laplace transforms

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

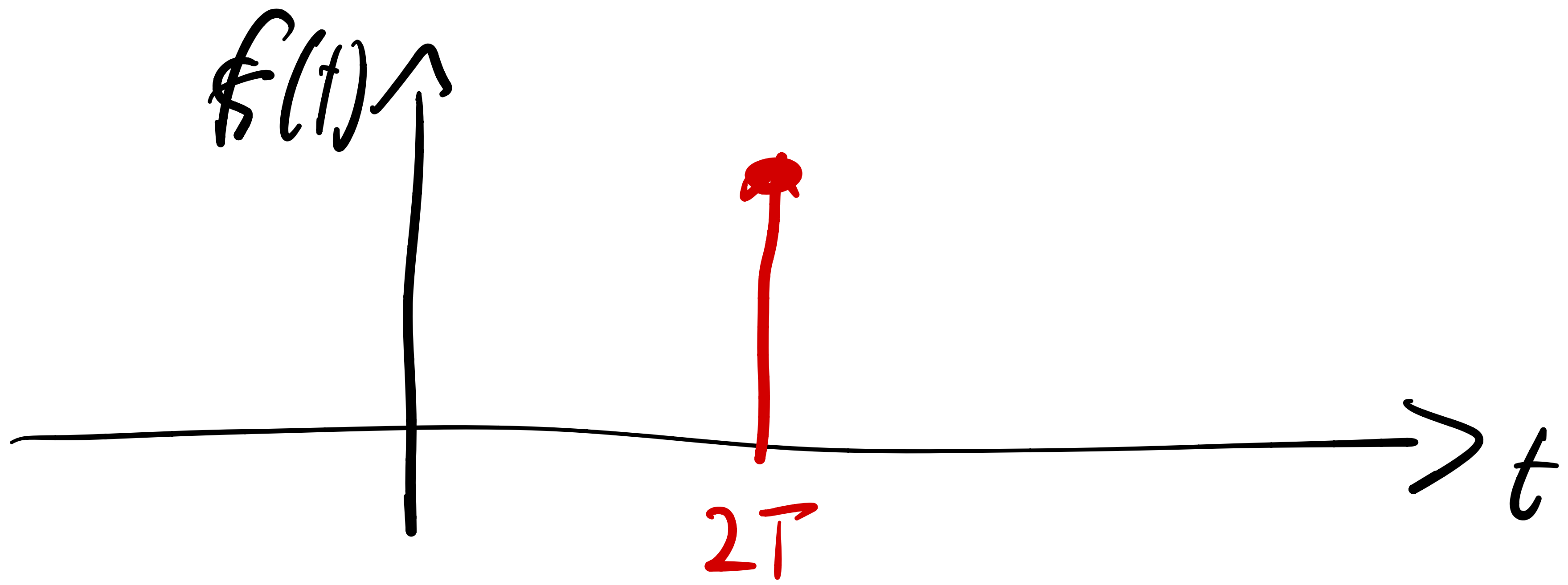


why? Easier to solve differential equations

$\mathcal{L}$ transforms diffeqs into algebraic equations.

Example

$$\mathcal{L}\{\delta(t-2T)\} = ?$$



Time shift property: $\mathcal{L}\{f(t-2T)\} = e^{-2Ts}F(s)$

$$\boxed{\therefore\ \mathcal{L}\{\delta(t-2T)\} = e^{-2Ts}}$$

Example

$$\mathcal{L}\{t\sin(2t)u(t)\} = ?$$

$$= \mathcal{L}\left\{\frac{t}{2j}\left(e^{2jt} - e^{-2jt}\right)\right\}$$

$$= \frac{1}{2j}\left[\mathcal{L}\{te^{2jt}\} - \mathcal{L}\{te^{-2jt}\}\right]$$

$$= \frac{1}{2j}\left\{\frac{1}{(s-2j)^2} - \frac{1}{(s+2j)^2}\right\}$$

$$\boxed{= \frac{4s}{(s^2+4)^2}}$$

Question

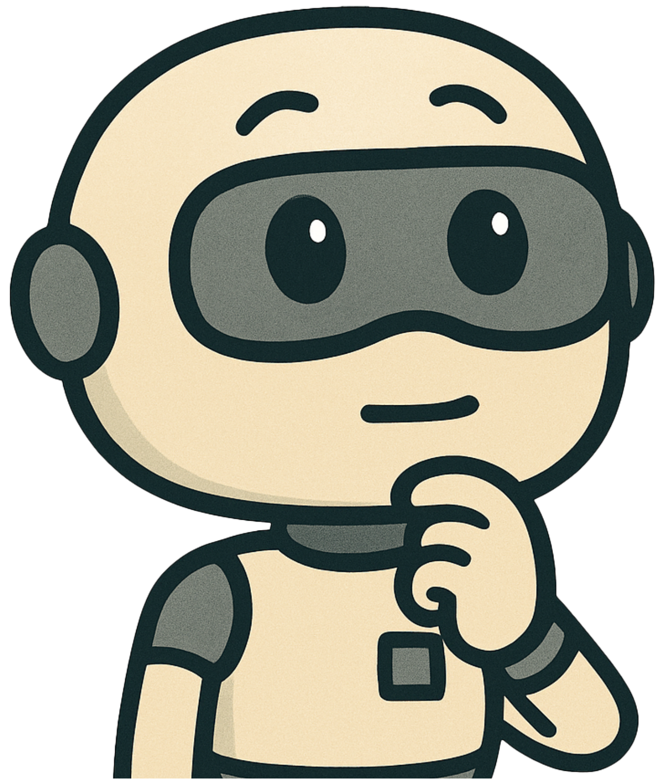
You are given the function $g(t) = u(t-3)$, which is a unit step function delayed by 3 seconds. What is its Laplace transform?

A: $\frac{1}{s+3}$

B: $\frac{e^{-3s}}{s}$

C: e^{-3s}

D: $\frac{1}{s} - 3$



Question

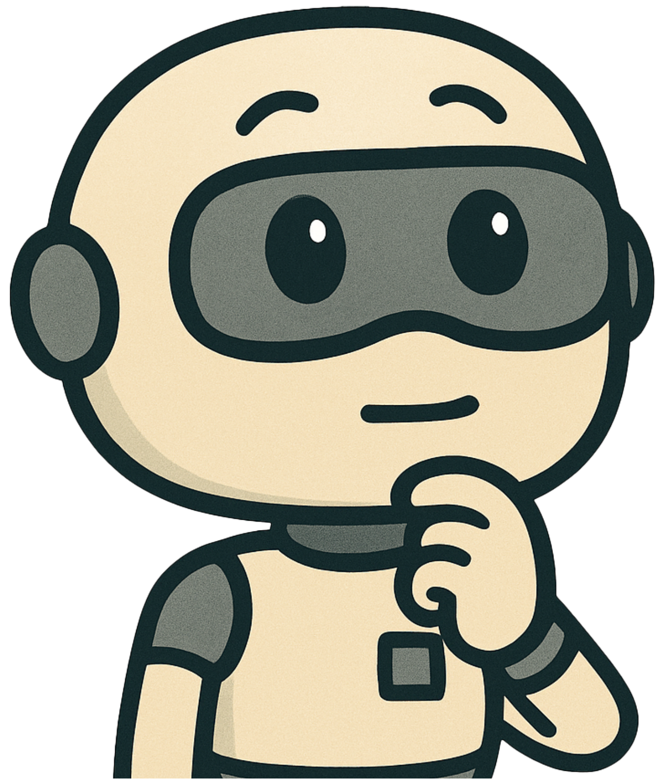
A classmate tries to find the final value of $f(t)$ for the transform $\frac{1}{s-2}$ by computing $\lim_{s \to 0} sF(s) = 0$. Is this result valid?

A: Yes, the final value is zero.

B: No, the final value theorem doesn't hold.

C: No, you must take the limit as $s \to \infty$.

D: No, the final value is $-\frac{1}{2}$.



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$\frac{1}{s-2} \Rightarrow s=2$

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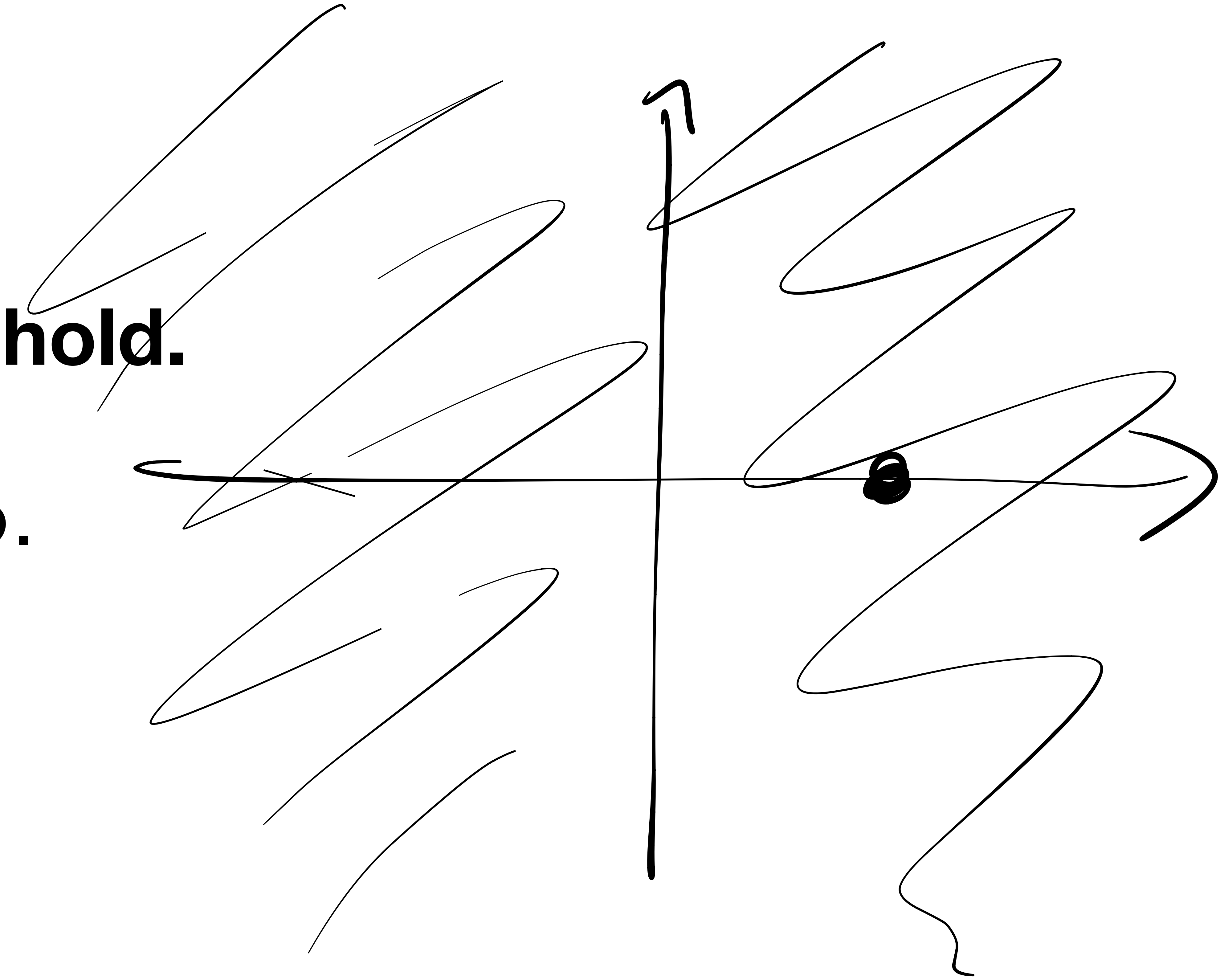
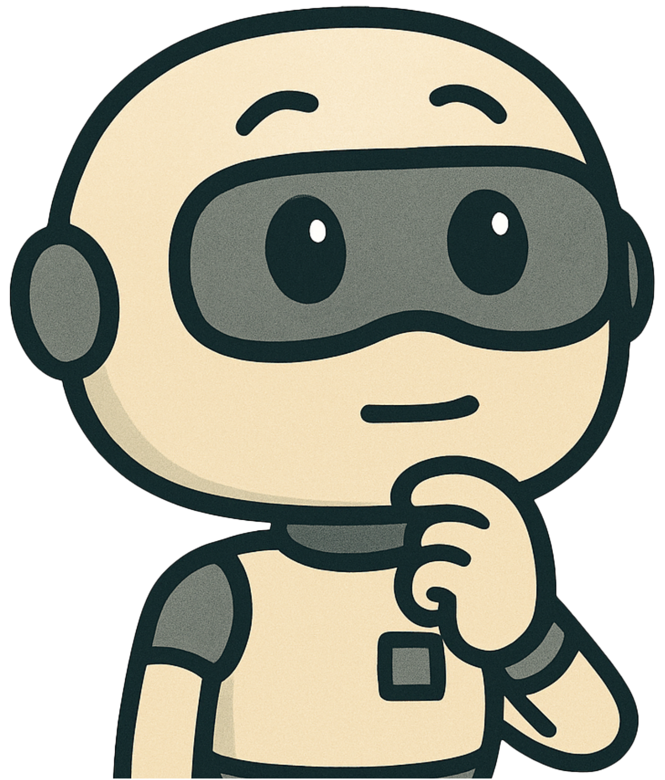
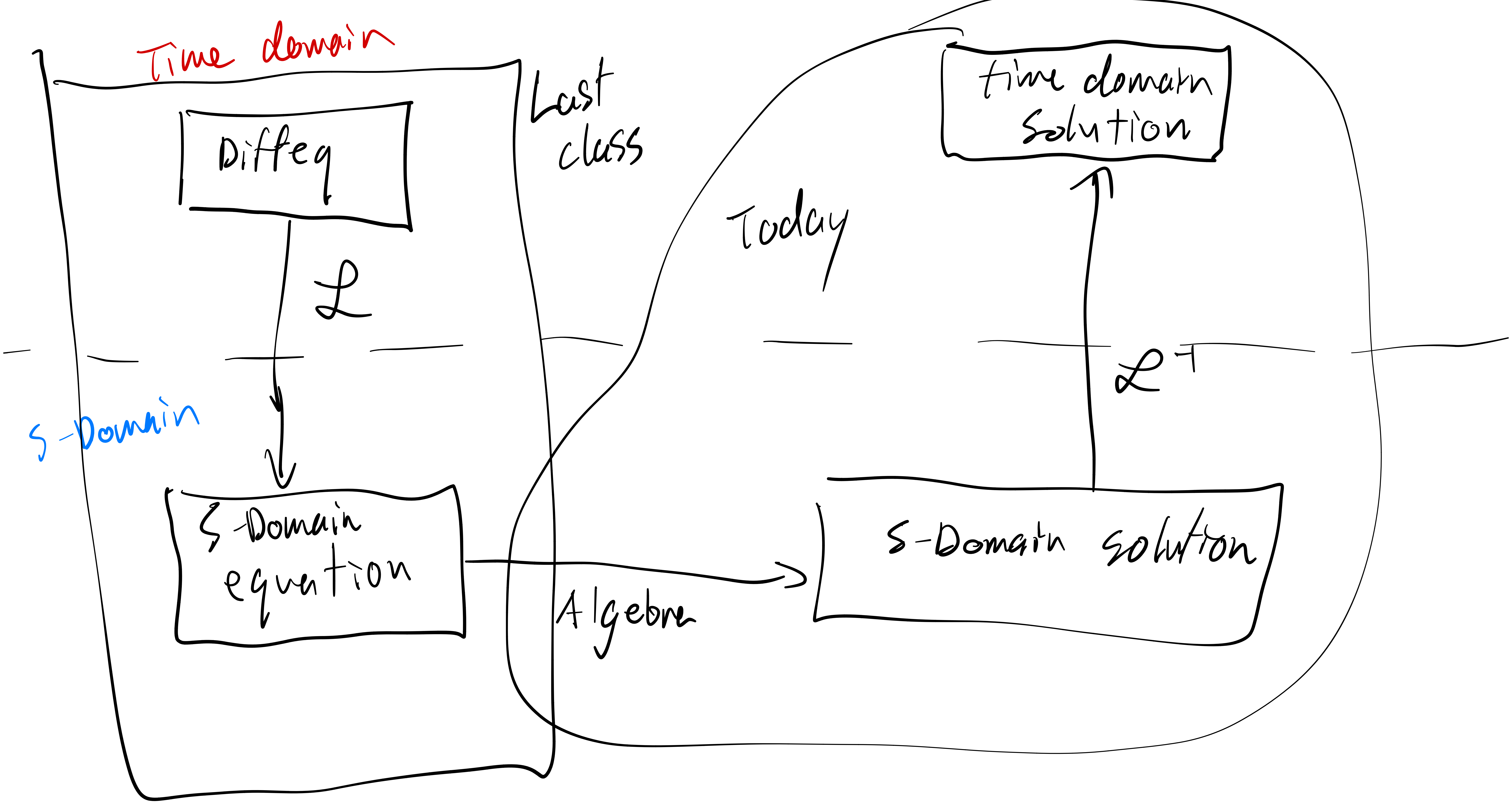






Time domain
Last class
Diffeq
$\mathcal{L}$
S-Domain
S-Domain equation
Algebra
Today
time domain solution
$\mathcal{L}^{-1}$
S-Domain solution

Using the Laplace transform to solve ODEs

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 5u(t)$$

Step input

ICs $y(0) = -1$, $\dot{y}(0) = 2$

$$\mathcal{L}\{ \quad '' \quad \} = \mathcal{L}\{ '' \}$$

$$\mathcal{L}\{\ddot{y}\} + 3\mathcal{L}\{\dot{y}(t)\} + 2\mathcal{L}\{y(t)\} = 5\mathcal{L}\{u(t)\}$$

Note: $Y(s) = \mathcal{L}(y(t))$

$$(s^2Y(s) - sy(0) - \dot{y}(0)) + 3(sY(s) - y(0)) + 2Y(s) = \frac{5}{s}$$

Plug in ICs.

$$s^2Y(s) + s - 2 + 3sY(s) + 3 + 2Y(s) = 5/s$$

$$(s^2 + 3s + 2)Y(s) + s + 1 = 5/s$$

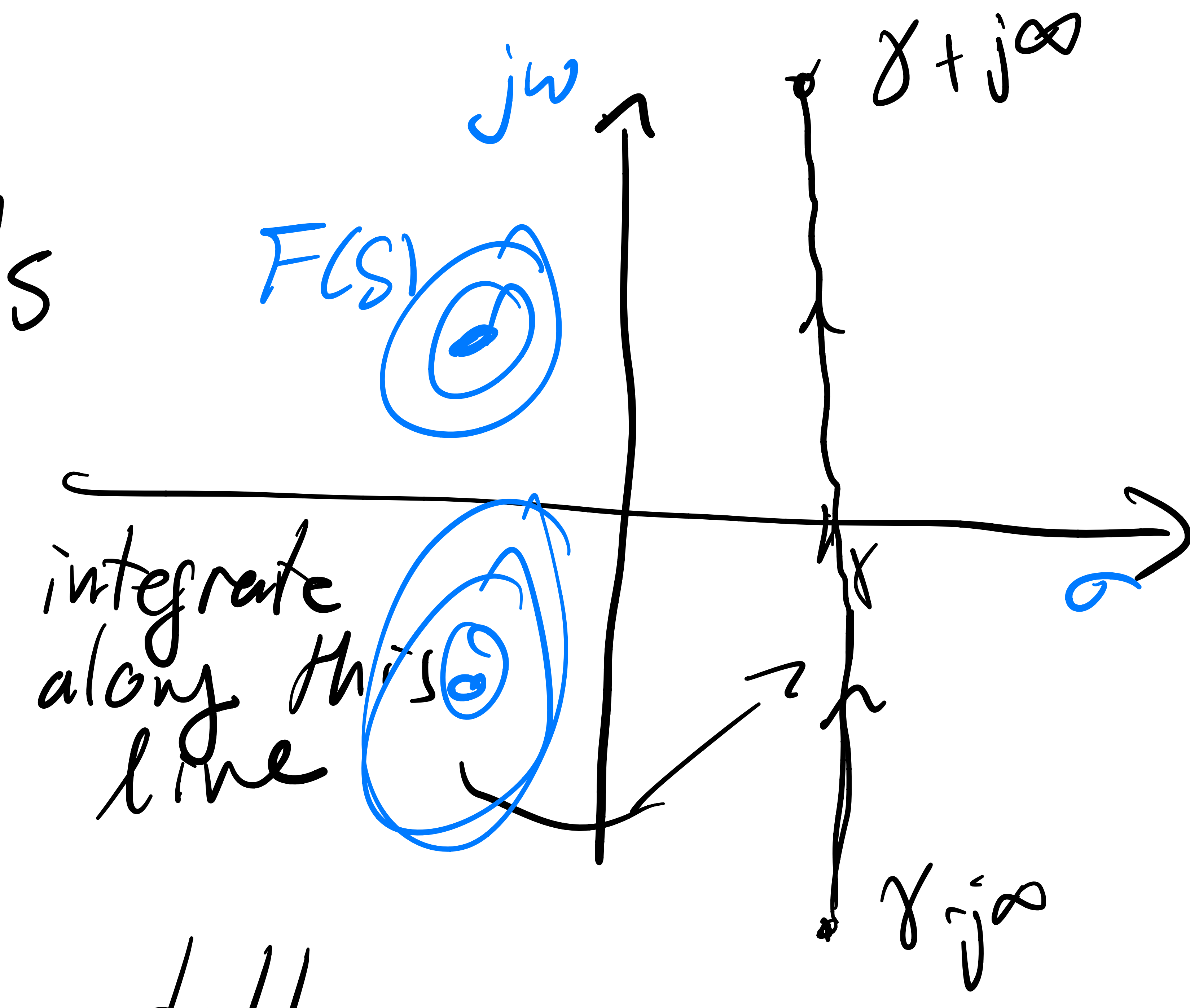
$$\Rightarrow \boxed{Y(s) = \frac{5 - s^2 - s}{s(s^2+3s+2)}}$$

Solution $y(t)$ to ODE

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

Inverse Laplace Transform

$$f(t)u(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{\gamma - j\infty}^{\gamma + j\infty} F(s)\, e^{st}\, ds$$



- To evaluate, look at Laplace transform table
  ↳ combine key properties.

Inverse LT properties

1. Linearity $\quad \mathcal{L}^{-1}\{\alpha_1 F_1(s) + \alpha_2 F_2(s)\} = \alpha_1 \mathcal{L}^{-1}\{F_1(s)\} + \alpha_2 \mathcal{L}^{-1}\{F_2(s)\}$

2. Frequency shift $\quad \mathcal{L}^{-1}\{F(s+a)\} = e^{-at} f(t), \quad a \in \mathbb{C}$

3. Time Delay $\quad \mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t-a)\, u(t-a)$

Example 1 Find the ILT of $F(s) = \frac{1}{s+3}$

From our table, we know $\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$

So, $\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}u(t)$

Example 2 Find the ILT of $F(s) = \frac{1}{(s+3)^2}$

- We know, that $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = tu(t)$

Frequency shift property $\mathcal{L}^{-1}\{F(s+a)\} = e^{-at}f(t)$

So, $\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\} = e^{-3t}tu(t)$

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 5u(t), \qquad \text{ICs: } y(0) = -1,\ \dot{y}(0) = 2$$

$$\Big\downarrow \mathcal{L}$$

$$Y(s) = \frac{5 - s^2 - s}{s(s^2+3s+2)}$$

to solve for $y(t)$, we need $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

How do we take $\mathcal{L}^{-1}\left\{\dfrac{5-s^2-s}{s(s^2+3s+2)}\right\}$?

Inverse Laplace Transform via Partial fraction expansion

- Note! Control systems (LTI systems) often result in "Transfer functions" (output equations) that are ratios of polynomials.

$$Y(s) = \frac{N(s)}{D(s)}$$ where $N(s)$ and $D(s)$ are polynomials in s.

- "Partial fraction expansion" Let's express this ratio of polynomials as:

$$Y(s) = \frac{N(s)}{D(s)} = \frac{R_1}{s+p_1} + \frac{R_2}{s+p_2} + \dots + \frac{R_n}{s+p_n},$$

$n = \#$ of roots counted with multiplicity of $D(s)$.

where $p_i \in \mathbb{R}$ are defined by the roots of polynomial $D(s)$. (The poles).

and $R_i \in \mathbb{R}$, are unknown coefficients called the "Residues".

So what? Why write $Y(s)$ this way?

$$Y(s) = \frac{N(s)}{D(s)} = \frac{R_1}{s+p_1} + \cdots + \frac{R_n}{s+p_n}$$

- Linearity of $\mathcal{L}^{-1}$ and the fact that $\mathcal{L}^{-1}\left\{\frac{1}{s+p_i}\right\} = e^{-p_i t}$
  ↳ make it very easy to find $\mathcal{L}^{-1}\{Y(s)\}$.

$$\mathcal{L}^{-1}\{Y(s)\} = R_1 \mathcal{L}^{-1}\left\{\frac{1}{s+p_1}\right\} + \cdots + R_n \mathcal{L}^{-1}\left\{\frac{1}{s+p_n}\right\}$$

$$\boxed{\mathcal{L}^{-1}\{Y(s)\} = R_1 e^{-p_1 t} + R_2 e^{-p_2 t} + \cdots R_n e^{-p_n t}}$$

Intuition: The time response $y(t)$ of an LTI system with a rational transfer function $Y(s) = \frac{N(s)}{D(s)}$, is a sum of "modes" each defined by the location of a root $p_i \in \mathbb{C}$ of $D(s)$ in the complex plane.

- $Re(p_i) = \sigma$ corresponds to growth or decay depending on $\sigma > 0$, $\sigma < 0$
- $Im(p_i) = j\omega$ correspons to oscillations.

Computing the partial fraction expansion

How to find R_i values?

Case 1: Distinct roots (cover-up method)

$$Y(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = \frac{R_1}{s+p_1} + \frac{R_2}{s+p_2} + \cdots + \frac{R_n}{s+p_n}$$

Step 1 Write equation in terms of unknown ~~residues~~ residues, as above.

Step 2 To evaluate each R_m, multiply both sides of equation by $(s+p_m)$. Then take $s \longrightarrow -p_m$.

↳ This will isolate R_m in the equation.

Step 3 Repeat the process seperately to solve for all residues R_i.

Example Consider $\frac{N(s)}{D(s)} = \frac{2}{(s+1)(s+2)}$

Step 1 $\frac{2}{(s+1)(s+2)} = \frac{R_1}{s+1} + \frac{R_2}{s+2}$

Step 2 $\frac{2\cancel{(s+1)}}{\cancel{(s+1)}(s+2)} = \frac{R_1\cancel{(s+1)}}{\cancel{s+1}} + \frac{R_2(s+1)}{s+2} \Rightarrow R_1 = \frac{2}{s+2} - \frac{R_2(s+1)}{s+2}$

↳ This equation holds for all s.

↳ take $s \to -1$ to eliminate R_2 term.

$$R_1 = \frac{2}{(-1)+2} - \cancel{\frac{R_2(-1+1)}{-1+2}} = 2$$

Step 3 Repeat process for R_2

$$\frac{2}{s+1} = \frac{R_1(s+2)}{s+1} + R_2 \xrightarrow{\text{take } s \to -2} R_2 = -2$$

So, $\frac{2}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{2}{s+2} \Rightarrow \boxed{\mathcal{L}^{-1}\left\{\frac{2}{(s+1)(s+2)}\right\} = 2\left(e^{-t} - e^{-2t}\right)u(t)}$

Note: Cover-up Method only works for distinct roots.

↳ It won't work for roots with higher multiplicity, e.g. $D(s) = s^2$

Double root at $s=0$.

In this case, use the method of undetermined coefficients

↳ slower, but it works in all cases

Step 1 Same as before

Imagine p_1 is root of multiplicity r.

$$\frac{N(s)}{D(s)} = \frac{R_1}{s+p_1} + \frac{R_2}{(s+p_1)^2} + \dots + \frac{R_r}{(s+p_1)^r} + \frac{R_{r+1}}{(s+p_2)} + \dots + \frac{R_n}{s+p_n}$$

Step 2 Multiply both sides of eqn. by $(s+p_1)(s+p_2)\cdots(s+p_n)$

Step 3 Expand both sides, and group by like powers of s.

Step 4 Form and solve equations for R_i, by equating like powers of s.

Example $\quad \dfrac{N(s)}{D(s)} = \dfrac{2}{(s+1)(s+2)^2}$ — root of multiplicity 2

Poles at $s=-1$, $s=-2$, $s=-2$

Step 1 $\quad \dfrac{2}{(s+1)(s+2)^2} = \dfrac{R_1}{s+1} + \dfrac{R_2}{s+2} + \dfrac{R_3}{(s+2)^2}$

Step 2 $\quad \dfrac{2(s+1)(s+2)^2}{(s+1)(s+2)^2} = R_1(s+2)^2 + R_2(s+1)(s+2) + R_3(s+1)$

$$2 =$$

Step 3 $\quad 2 = R_1(s^2+4s+4) + R_2(s^2+3s+2) + R_3(s+1)$

$$\Rightarrow 2 = s^2(R_1+R_2) + s(4R_1+3R_2+R_3) + (4R_1+2R_2+R_3)$$

Step 4

$$\left.\begin{aligned} &\underline{s^2 \text{ term}} \quad 0 = R_1+R_2 \\ &\underline{s^1 \text{ term}} \quad 0 = 4R_1+3R_2+R_3 \\ &\underline{s^0 \text{ term}} \quad 2 = 4R_1+2R_2+R_3 \end{aligned}\right\} \Rightarrow R_1 = 2,\ R_2 = -2,\ R_3 = -2$$

$$\mathcal{L}^{-1}\left\{\frac{2}{(s+1)(s+2)^2}\right\} = 2\left(e^{-t} - e^{-2t} - te^{-2t}\right)u(t)$$

Question

The Laplace transform of $f(t) = tu(t)$ is $F(s) = \frac{1}{s^2}$. Based on the Frequency Shift Property, what is the inverse transform of $G(s) = \frac{1}{(s+2)^2}$?

A: $t^2 e^{-2t} u(t)$

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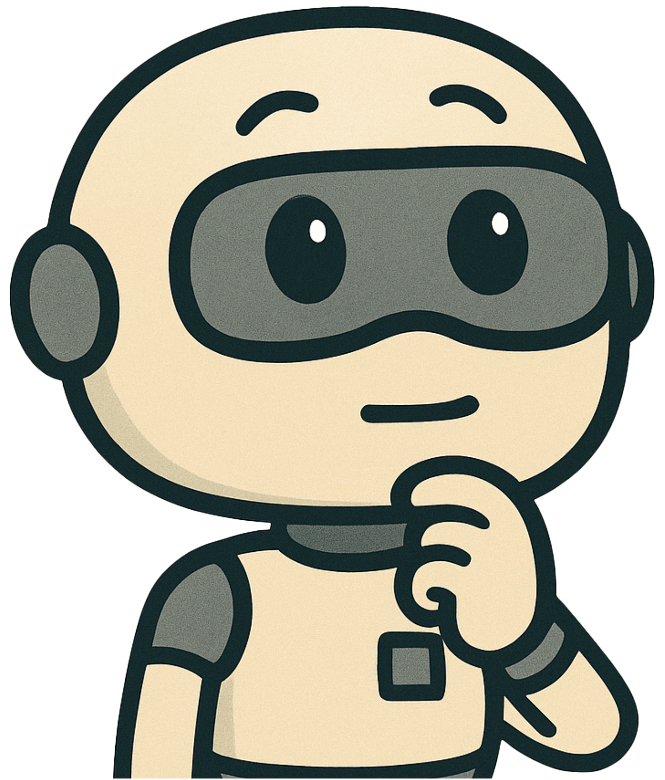

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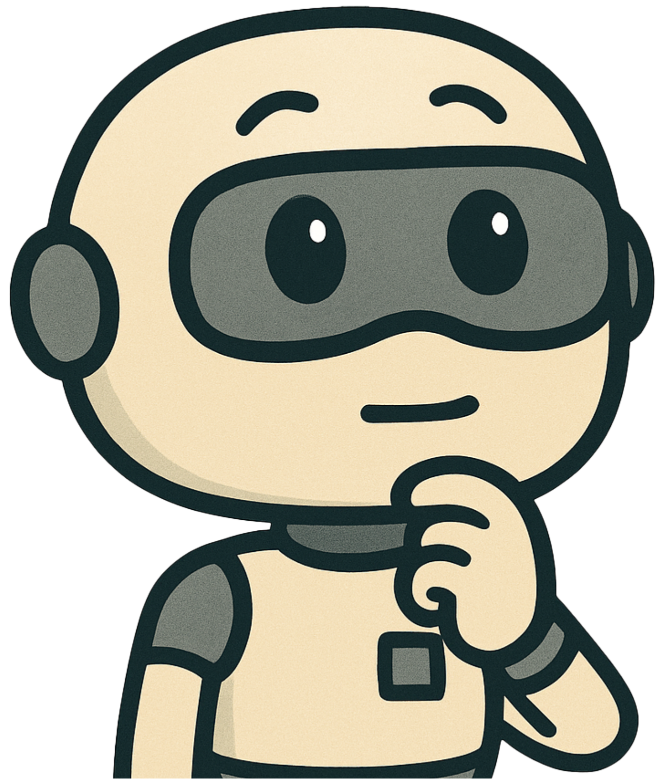


Question

Without performing the full inverse transform, look at the poles of the following system:

$$Y(s) = \frac{1}{(s+2)(s-3)}$$

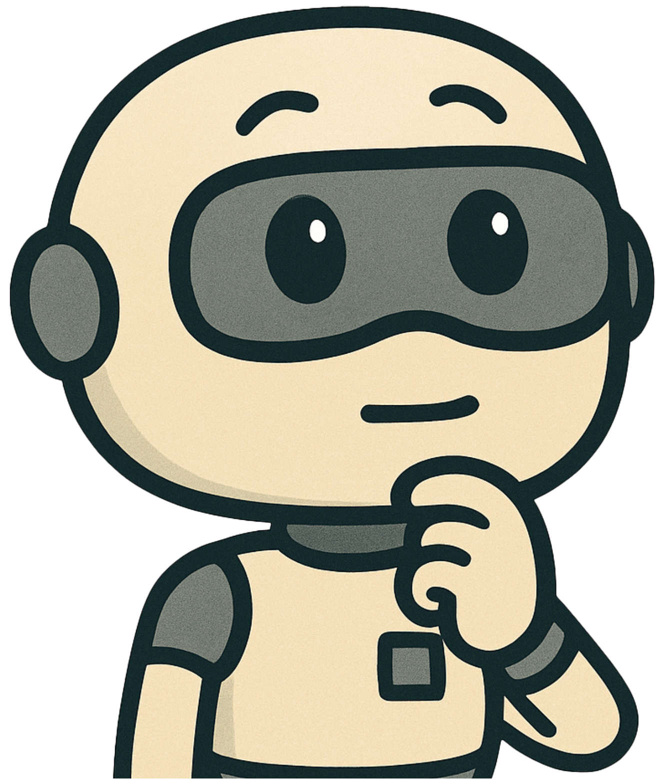
What will happen to the time-domain response $y(t)$ as $t \to \infty$?

A: It will decay to zero (stable).

B: It will oscillate forever (marginally stable).

C: It will grow to infinity (unstable).

D: It will settle to a constant non-zero value.



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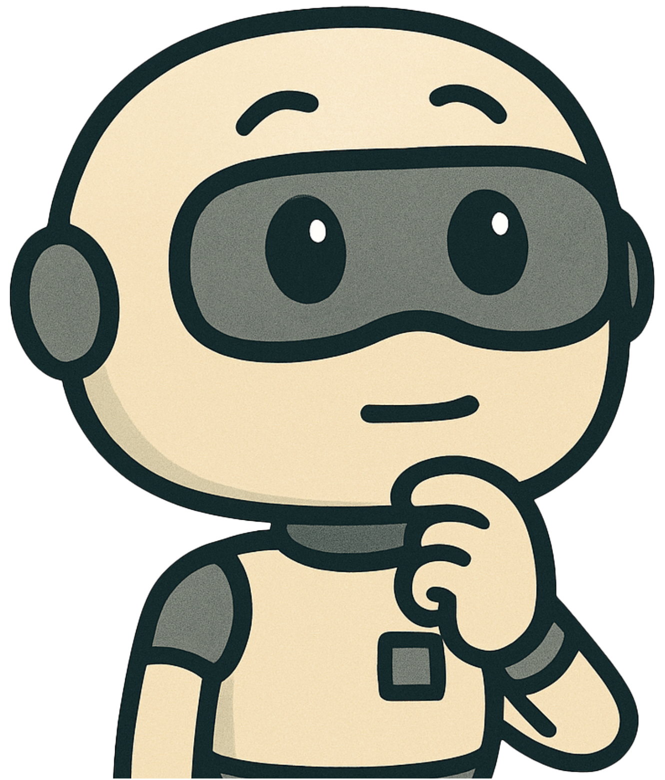
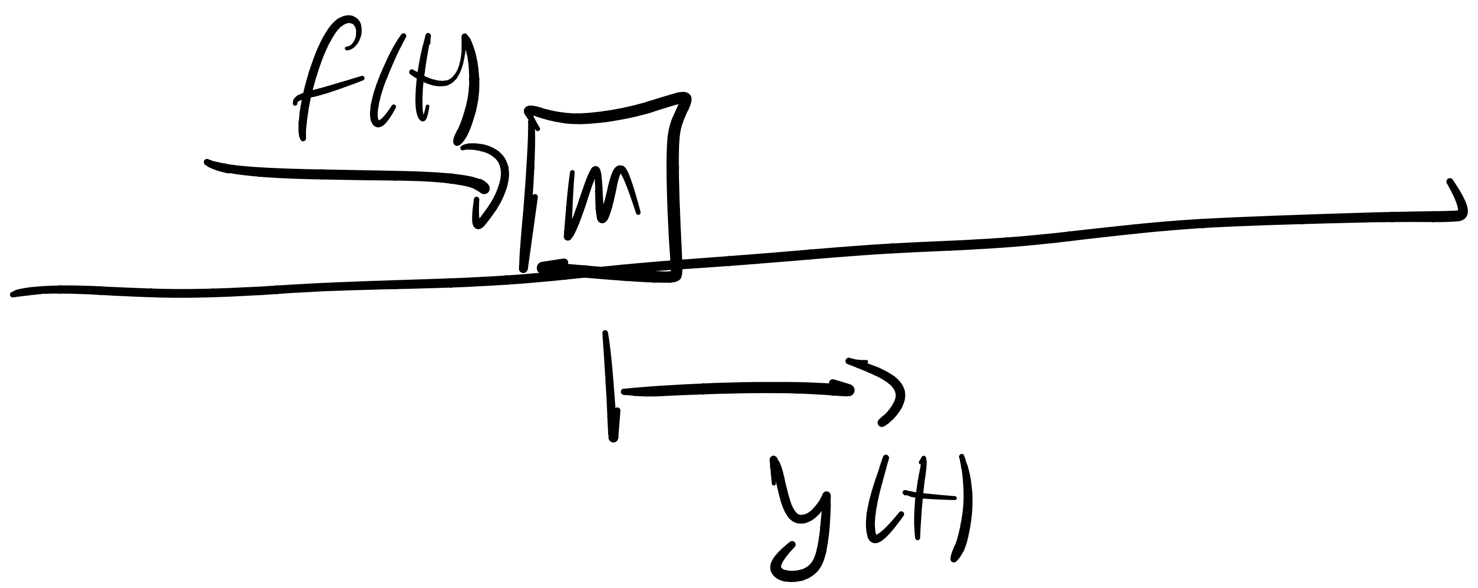


Example Newton's 2^{nd} law



$$M\ddot{y}(t) = f(t)$$

we want to know output position $y(t)$ for a force $f(t)$.

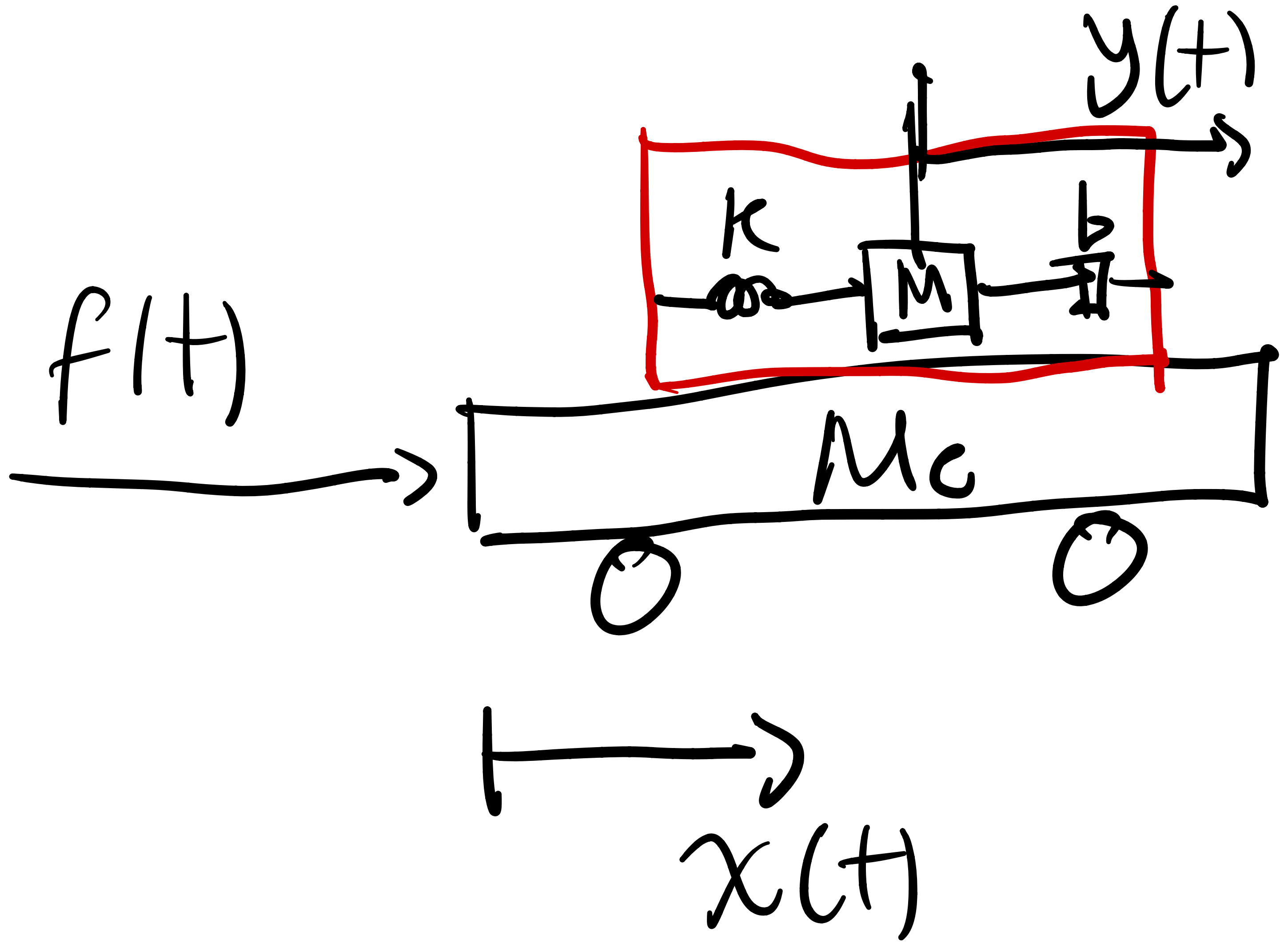
Laplace transform $\Rightarrow \mathcal{L}\{M\ddot{y}(t)\} = \mathcal{L}\{f(t)\}$

$$\Rightarrow M\left(s^2 Y(s) - sy(0) - \dot{y}(0)\right) = F(s)$$

$$\Rightarrow Y(s) = \frac{1}{Ms^2}F(s) + \frac{y(0)}{s} + \frac{\dot{y}(0)}{s^2}$$

$$y(t) = \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{Ms^2}F(s)\right\}}_{\text{"forced response"}} + \underbrace{y(0)u(t) + \dot{y}(0)\,t\,u(t)}_{\text{"IC response"}}$$

Example Car accelerometer



We want to know, is how y(t) moves when a unit step is applied to the car. Assume zero I.C.'s.

Newton's Laws

- $M \frac{d^2}{dt^2}\left(x(t) + y(t)\right) = -b\dot{y}(t) - ky(t)$
- $M_c \ddot{x}(t) = f(t) \Rightarrow \ddot{x}(t) = \frac{f(t)}{M_c}$

$$\Rightarrow M\left(\frac{1}{M_c}f(t) + \ddot{y}(t)\right) + b\dot{y}(t) + ky(t) = 0 \Rightarrow M\ddot{y}(t) + b\dot{y}(t) + ky(t) = -\frac{M}{M_c}f(t)$$

$$\mathcal{L} \Rightarrow Ms^2Y(s) + bsY(s) + kY(s) = -\frac{M}{M_c}F(s) \Rightarrow Y(s) = -\frac{M}{M_c}\left(\frac{1}{Ms^2+bs+k}\right)F(s)$$

$f(t) = \text{unit step} \Rightarrow F(s) = \frac{1}{s}$ $\therefore \; Y(s) = -\frac{M}{M_c}\left(\frac{1}{Ms^2+bs+k}\right)\cdot\frac{1}{s}$

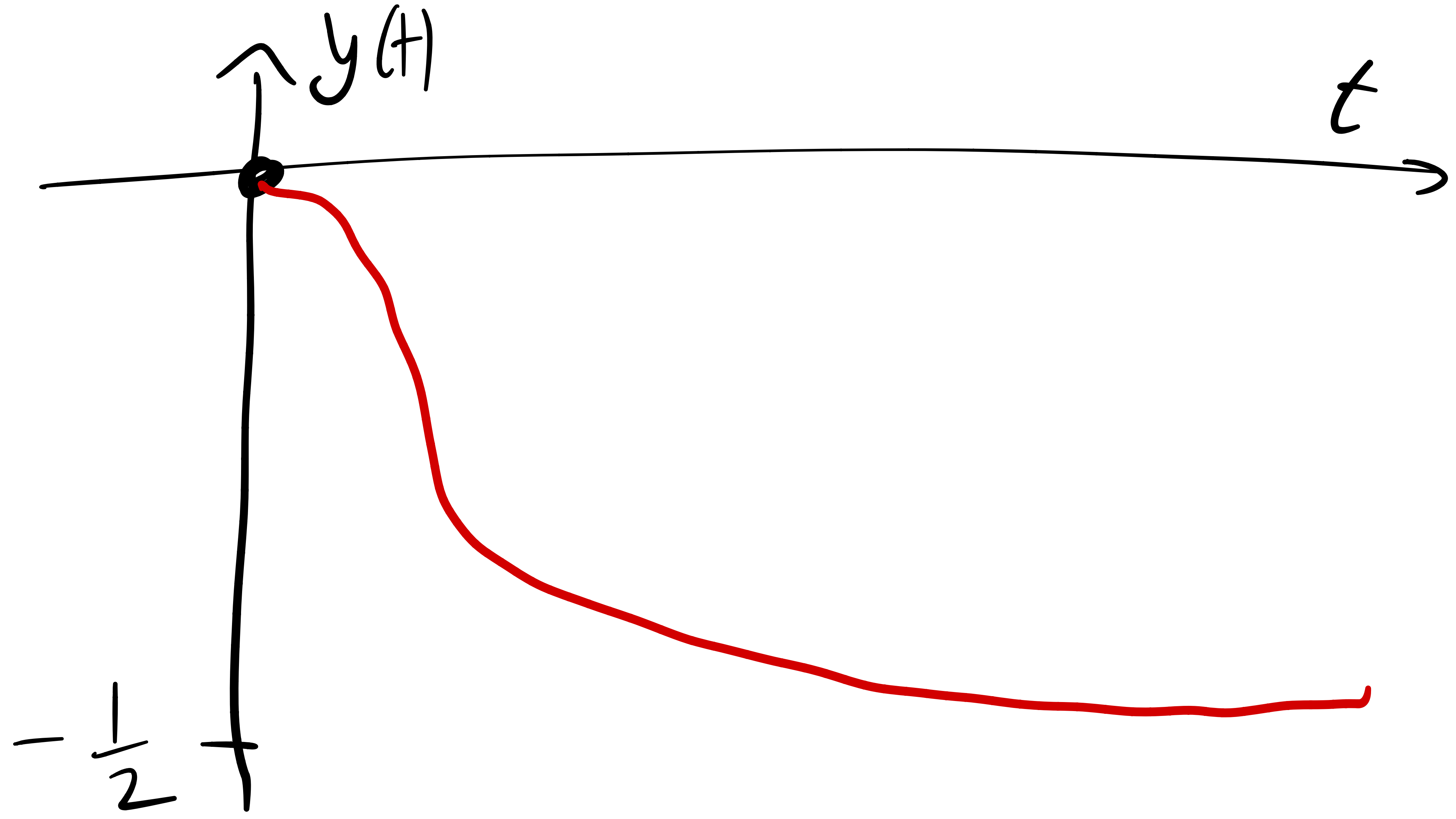
$$Y(s) = \frac{1}{M_c}\left(\frac{1}{s^2 + \left(\frac{b}{M}\right)s + \frac{k}{M}}\right) \cdot \frac{1}{s}$$ suppose $M_c = 1$, $b/M = 3$, $k/M = 2$

$$\Longrightarrow Y(s) = -\frac{1}{\underline{s^2+3s+2}} \cdot \frac{1}{s}$$ poles at $s = 0, -1, -2$

<u>PFE</u> $$-\frac{1}{s(s+1)(s+2)} = -\left(\frac{R_1}{s} + \frac{R_2}{(s+1)} + \frac{R_3}{s+2}\right)$$

$$\Longrightarrow R_1 = 1/2,\ R_2 = -1,\ R_3 = 1/2$$

$$\Longrightarrow y(t) = \left(-\frac{1}{2} + e^{-t} - \frac{1}{2}e^{-2t}\right)u(t)$$



Question

We need to find the residue R_1 for the term associated with the pole at $s = -1$:

$$Y(s) = \frac{10}{(s+1)(s+6)} = \frac{R_1}{s+1} + \frac{R_2}{s+6}$$

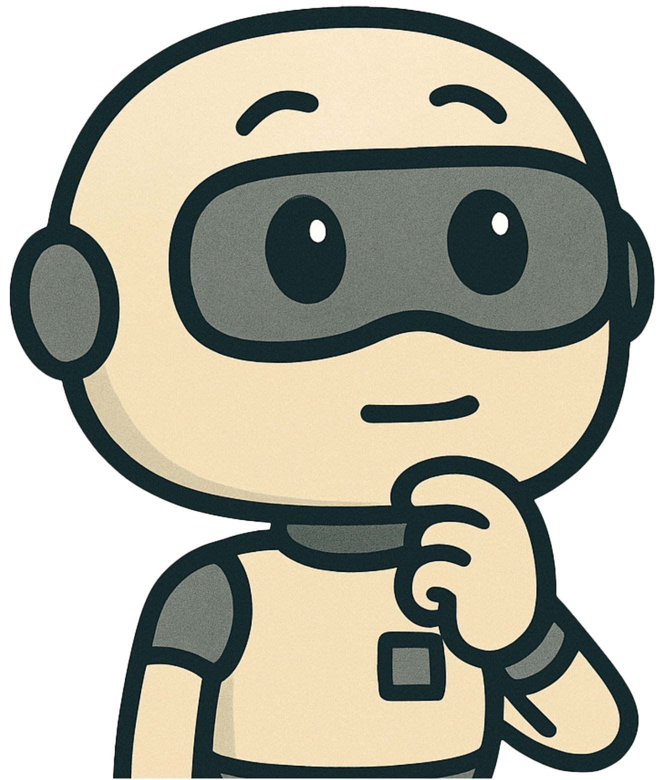
Using the cover-up method, what is the value of R_1?

A: 10/7

B: 10/5 = 2.

C: 10/1 = 10.

D: 10 / (-5) = -2.



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$$\frac{10\cancel{(s+1)}}{s+6} = R_1$$

$$\frac{10}{5} = 2$$