MAE 5730 Final Project: Bowling Ball Dynamics

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I. The System & Analytical Model

In bowling, the optimal strategy requires the center of the ball to hit just to the side of the headpin, between the 1 and 2 or 1 and 3 pins, at a 6-degree angle. The lane is not wide enough to achieve this angle with a straight shot, so bowlers must spin the ball so that its trajectory hooks. Bowling balls and lanes are specially set up to allow bowlers to hook their shots. There are two key parameters that facilitate hooking shots. The first is the ball's inertia tensor. It makes sense to assume that bowling balls are uniformly dense spheres. However, bowling balls have custom weight blocks at their center that are either axisymmetric or asymmetric. The second is the friction between the ball and the lane, which is determined by the roughness of the surface of the bowling ball and the amount of lane oil. Lane oil varies with position in the lane, and while many different patterns are used, generally lanes have no oil on the final 1/3 and less oil near the edges. More friction between the ball and the lane causes the ball to dissipate energy faster and hook quicker, and these areas of higher friction prompt the ball to transition from sliding to hooking and then rolling. Bowling balls are under 7.26 kg (16 pounds) with a diameter of around .216 meters (8.5 inches). Bowling lanes are 1.0668 meters (42 inches) wide and 18.288 meters (60 feet) long. The sliding friction coefficient is typically 0.04 in oiled portions and 0.2 in unoiled portions of the lane. Typical initial conditions are $\|v_0\| = 8\,{}^{m}/_{s}$ and $\|\omega_0\| = 30\,{}^{rad}/_{s}$. v_0 has a relatively large x component and relatively small y component. ω_0 is oriented roughly parallel to the velocity, so that the ball appears to be spinning towards the center of the lane. Given these realistic parameters and initial conditions, the ball should start around the center of the lane, head towards the edge of the lane, and finally hook back towards the center in the final third. Initially, the ball should be slipping. After the ball hooks, it should transition to rolling without slipping.

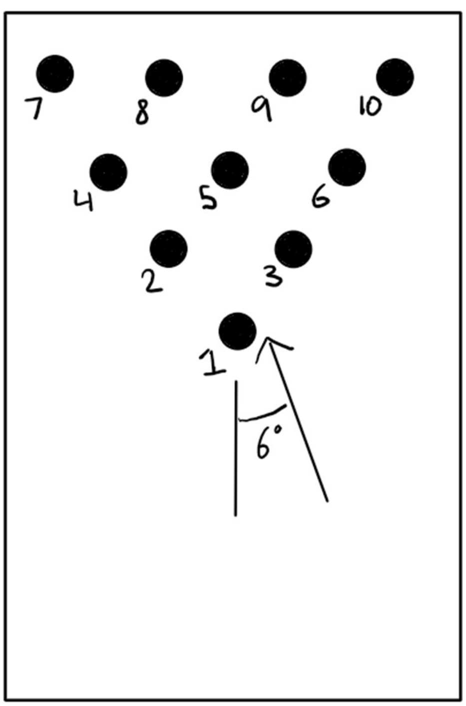

Figure 1: Sample trajectory

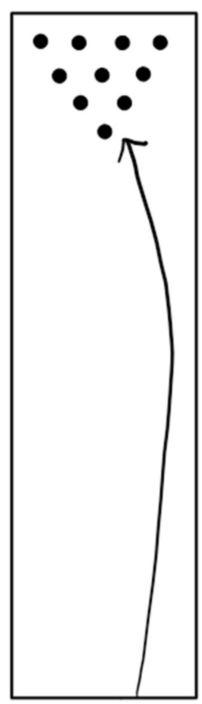

Figure 2: Sample trajectory

Modern bowling balls have centers of mass that are offset from their geometric center, which allows even more hook. For this analysis, I've assumed that the center of mass is located at the geometric center. r_C is the position of the contact point C relative to the center of mass and has magnitude R. $\vec{F}_C$ is the reaction force of the lane acting on the ball. It has a normal component in the $\vec{n}_3$ direction, F_N, and a frictional component f with components f_x and f_y in the $\vec{n}_1$ and $\vec{n}_2$ directions, respectively. f acts opposite the velocity of the contact point relative to the origin. $x, y,$ and z represent the position of the center of mass in the $\vec{n}_1, \vec{n}_2,$ and $\vec{n}_3$ directions. $\phi, \theta,$ and ψ represent a 3-1-3 Euler angle rotation from the N coordinates to the B coordinates. When slipping, the system is only constrained by F_N such that $\ddot{z} = 0$. When rolling, the system has two additional no-slip constraints: $\dot{x} + R\omega_y = 0$ and $\dot{y} - R\omega_x = 0$, where ω_x and ω_y are functions of the Euler angles.

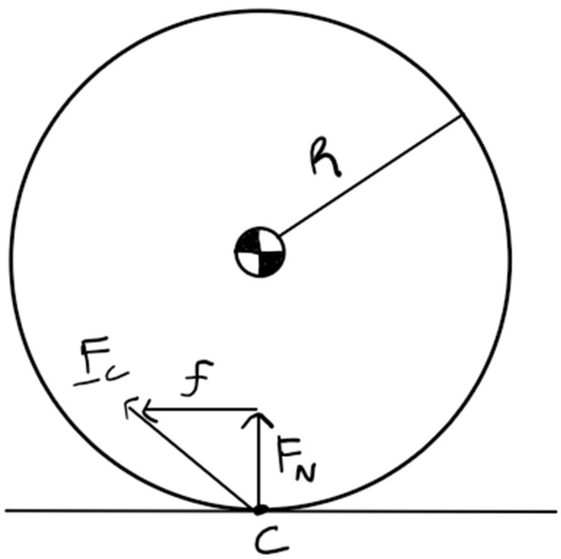

Figure 3: Free Body Diagram

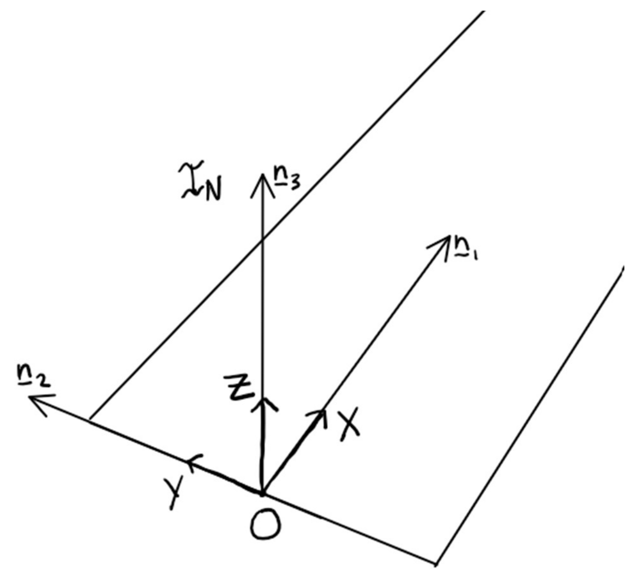

Figure 4: Translational coordinates x, y, and z

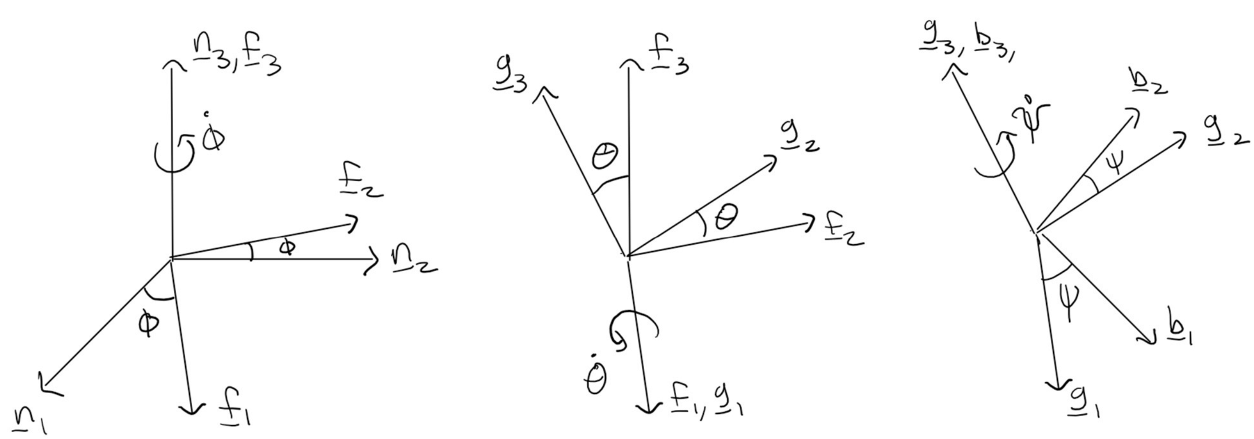

Figure 5: 3-1-3 Euler angle rotation from N to B through intermediate coordinates F and G

II. Equations of Motion

In the following analysis, g is the acceleration due to gravity, R is the radius of the bowling ball (so $\vec{r}_C = [0 \ \ 0 \ -R]^T$), μ is the sliding friction coefficient, m is the mass of the bowling ball, and I is the inertia tensor.

A. DAE Approach

The set of coordinates is $x, y, z, \phi, \theta, \psi$.

Slipping—LMB

There is one reaction, F_N. The DAE solution should be of the form $Au = B$, where A is 7x7 and B is 7x1. The system has 5 degrees of freedom. The first 3 equations come from linear momentum balance in the x, y, and z directions, done in the inertial N coordinate system.

$$\boldsymbol{F_N \mu \hat{v}_{O/C} + F_N \vec{n}_3 - mg\vec{n}_3 = m\vec{a}} \quad \mathbf{(1)}$$

$\hat{v}_{O/C}$, derived below, is a unit vector pointing in the direction of the velocity of the floor relative to the contact point C. $\vec{a}$ is acceleration of the x, y, and z coordinates.

$$\boldsymbol{\vec{a} = [\ddot{x} \;\; \ddot{y} \;\; \ddot{z}]^T}$$

$$\omega_x = \dot{\theta}\cos(\phi) + \dot{\psi}\sin(\theta)\sin(\phi)$$
$$\omega_y = \dot{\theta}\sin(\phi) + \dot{\psi}\sin(\theta)\cos(\phi)$$
$$\omega_z = \dot{\phi} + \dot{\psi}\cos(\theta)$$

$$\vec{v}_{C/G} = \vec{\omega}_N \times \vec{r}_C$$
$$\vec{v}_{G/O} = [\dot{x} \;\; \dot{y} \;\; \dot{z}]^T$$
$$\vec{v}_{C/O} = v_{C/G} + v_{G/O}$$
$$\vec{\omega}_N = [\omega_x \;\; \omega_y \;\; \omega_z]$$
$$\boldsymbol{\hat{v}_{O/C} = \frac{-\vec{v}_{C/O}}{\|\vec{v}_{C/O}\|}}$$

Slipping—AMB

The next 3 equations come from angular momentum balance about each body axis, done in the body fixed B coordinate system.

$$\boldsymbol{Q_{BN}\left(\vec{r}_C \times F_N \mu \hat{v}_{O/C}\right) = I\vec{\alpha}_B + \vec{\omega}_B \times I\vec{\omega}_B} \quad \mathbf{(2)}$$

Q_{BN} is a rotation matrix from N coordinates to B coordinates, so the left-hand side of equation 2 is the applied moment from $\vec{F}_C$ expressed in the B coordinate system. $\vec{\omega}_B$ and $\vec{\alpha}_B$ are the angular rates and angular accelerations with respect to the body coordinate system.

$$\boldsymbol{Q_{BN}} = \begin{matrix} \cos(\phi)\cos(\psi) - \sin(\phi)\cos(\theta)\sin(\psi) & -\cos(\phi)\sin(\psi) - \sin(\phi)\cos(\theta)\cos(\psi) & \sin(\theta)\sin(\psi) \\ -\cos(\phi)\sin(\psi) - \sin(\phi)\cos(\theta)\cos(\psi) & -\sin(\phi)\sin(\psi) + \cos(\phi)\cos(\theta)\cos(\psi) & \sin(\theta)\cos(\psi) \\ \sin(\phi)\sin(\theta) & -\cos(\phi)\sin(\theta) & \cos(\theta) \end{matrix}$$

$$\omega_1 = \dot{\phi}\sin(\theta)\sin(\psi) + \dot{\theta}\cos(\psi)$$
$$\omega_2 = \dot{\phi}\sin(\theta)\cos(\psi) - \dot{\theta}\sin(\psi)$$
$$\omega_3 = \dot{\phi}\cos(\theta) + \dot{\psi}$$
$$\boldsymbol{\vec{\omega}_B = [\omega_1 \;\; \omega_2 \;\; \omega_3]^T}$$

$$\alpha_1 = \frac{d}{dt}\omega_1$$

$$\alpha_2 = \frac{d}{dt}\omega_2$$
$$\alpha_3 = \frac{d}{dt}\omega_3$$
$$\vec{\boldsymbol{\alpha}}_{\boldsymbol{B}} = [\boldsymbol{\alpha_1}\ \ \boldsymbol{\alpha_2}\ \ \boldsymbol{\alpha_3}]^{\boldsymbol{T}}$$

Slipping—Constraint

The seventh and final equation is the following constraint equation:

$$\ddot{\boldsymbol{z}} = \boldsymbol{0} \quad \textbf{(3)}$$

Rolling—LMB

Now, there are 3 reaction forces, F_N, f_x, and f_y, where f_x and f_y are the components of the friction force in the $\vec{n}_1$ and $\vec{n}_2$ directions, respectively. The DAE solution should be of the form $Au = B$, where A is 9x9 and B is 9x1. The system has 3 degrees of freedom. The first 3 equations again come from linear momentum balance in the x, y, and z directions, done in the inertial N coordinate system.

$$\boldsymbol{F_N}\vec{\boldsymbol{n}}_{\boldsymbol{3}} - \boldsymbol{mg}\vec{\boldsymbol{n}}_{\boldsymbol{3}} = \boldsymbol{m}\vec{\boldsymbol{a}} \quad \textbf{(4)}$$

This equation is the same as the slipping case, except $\hat{v}_{O/C} = 0$.

Rolling—AMB

The next 3 equations again come from angular momentum balance about each body axis, done in the body fixed B coordinate system.

$$\boldsymbol{Q_{BN}}\left(\vec{\boldsymbol{r}}_{\boldsymbol{C}} \times \vec{\boldsymbol{f}}\right) = \boldsymbol{I}\vec{\boldsymbol{\alpha}}_{\boldsymbol{B}} + \vec{\boldsymbol{\omega}}_{\boldsymbol{B}} \times \boldsymbol{I}\vec{\boldsymbol{\omega}}_{\boldsymbol{B}} \quad \textbf{(5)}$$

This is the same as the slipping case except the applied moment is expressed in terms of the new reaction force $\vec{f}$.

$$\vec{\boldsymbol{f}} = \left[\boldsymbol{f_x}\ \ \boldsymbol{f_y}\ \ \boldsymbol{0}\right]^{\boldsymbol{T}}$$

Rolling—Constraints

The last 3 equations are the following constraint equations, where ω_x and ω_y are functions of the coordinates ϕ, θ, and ψ, and their relationship is described above.

$$\ddot{\boldsymbol{z}} = \boldsymbol{0} \quad \textbf{(6)}$$

$$\frac{\boldsymbol{d}}{\boldsymbol{dt}}\left(\dot{\boldsymbol{x}} + \boldsymbol{R\omega_y}\right) = \boldsymbol{0} \quad \textbf{(7)}$$

$$\frac{\boldsymbol{d}}{\boldsymbol{dt}}\left(\dot{\boldsymbol{y}} - \boldsymbol{R\omega_x}\right) = \boldsymbol{0} \quad \textbf{(8)}$$

The no-slip constraints given by equations 7 and 8 are derived by setting the velocity of the contact point with respect to the origin to 0:

$$\vec{v}_{G/O} = [\dot{x}\ \ \dot{y}\ \ \dot{z}]^T$$
$$\vec{v}_{C/G} = \vec{\omega}_N \times \vec{r}_C$$

$$\vec{v}_{C/O} = \vec{v}_{C/G} + \vec{v}_{G/O} = 0$$

$$\boldsymbol{\dot{x} + R\omega_y = 0}$$
$$\boldsymbol{\dot{y} - R\omega_x = 0}$$

B. Euler-Lagrange Approach

The generalized coordinates are $x, y, z, \phi, \theta, \psi$. For both the slipping and rolling case, the Lagrangian is composed of the same terms. There is translational and rotational kinetic energy and gravitational potential energy. The energy terms and Lagrangian are:

$$E_k = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$
$$E_P = mgz$$

$$\boldsymbol{L = E_k - E_p}$$

The Jacobian is used to evaluate $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q}$, where $J(f, v)$ is shorthand for the Jacobian matrix of function f with respect to v.

$$\boldsymbol{J(J(L,\dot{q}),q)\dot{q} + J(J(L,\dot{q}),\dot{q})\ddot{q} - J(L,q)^T} \quad \mathbf{(9)}$$

Constraints—Normal Reaction

In both sliding and rolling, there is a holonomic constraint from the normal force F_N. It is:

$$z = R$$

This forms the constraint function:

$$h(q) = z - R$$

The generalized force vector $\vec{Q}_N$ in terms of Lagrange multiplier λ_h is:

$$Q_{N,i} = \lambda_h \frac{\partial h}{\partial q_i}$$
$$\boldsymbol{\vec{Q}_N = [0 \;\; 0 \;\; \lambda_h \;\; 0 \;\; 0 \;\; 0]^T} \quad \mathbf{(10)}$$

Constraints—No Slip

In the rolling case, there are two non-holonomic no slip constraints. They are:

$$\dot{x} + R\omega_y = 0$$
$$\dot{y} - R\omega_x = 0$$

These form the constraint functions:

$$f(\dot{q}, q) = \dot{x} + R\dot{\theta}\sin(\phi) - R\dot{\psi}\sin(\theta)\cos(\phi)$$
$$g(\dot{q}, q) = \dot{y} - R\dot{\theta}\cos(\phi) - R\dot{\psi}\sin(\theta)\sin(\phi)$$

Putting them into differential form:

$$f(\dot{q},q)dt = dx + R\sin(\phi)\,d\theta - R\sin(\theta)\cos(\phi)d\psi$$
$$g(\dot{q},q)dt = dy - R\cos(\phi)\,d\theta - R\sin(\theta)\sin(\phi)d\psi$$

$$\begin{array}{cc} f_x = 1 & g_x = 0 \\ f_y = 0 & g_y = 1 \\ f_z = 0 & g_z = 0 \\ f_\phi = 0 & g_\phi = 0 \\ f_\theta = Rsin(\phi) & g_\theta = -Rcos(\phi) \\ f_\psi = -Rsin(\theta)\cos(\phi) & g_\psi = -Rsin(\theta)\sin(\phi) \end{array}$$

$$Q_i = \lambda_f f_i + \lambda_g g_i$$

$$\vec{\boldsymbol{Q}}_{\boldsymbol{NS}} = \left[\boldsymbol{Q_x}\ \ \boldsymbol{Q_y}\ \ \boldsymbol{Q_z}\ \ \boldsymbol{Q_\phi}\ \ \boldsymbol{Q_\theta}\ \ \boldsymbol{Q_\psi}\right]^{\boldsymbol{T}}$$

External Forces—Rayleigh's Dissipation

The Rayleigh's dissipation terms are the following:

$$Q_{R,i} = -\mu\dot{q}_i$$

$$\vec{\boldsymbol{Q}}_{\boldsymbol{R}} = -\boldsymbol{\mu}\left[\dot{\boldsymbol{x}}\ \ \dot{\boldsymbol{y}}\ \ \dot{\boldsymbol{z}}\ \ \dot{\boldsymbol{\phi}}\ \ \dot{\boldsymbol{\theta}}\ \ \dot{\boldsymbol{\psi}}\right]^{\boldsymbol{T}}$$

External Forces—Friction Force

The friction force should affect the linear and angular momentum of the bowling ball. I was not able to figure out the applied moment, but the generalized forces that affect linear momentum in x and y are:

$$Q_{EX,x} = mg\mu\hat{v}_{O/C}\cdot\vec{n}_1$$
$$Q_{EX,y} = mg\mu\hat{v}_{O/C}\cdot\vec{n}_2$$

$$\vec{\boldsymbol{Q}}_{\boldsymbol{EX}} = \left[\boldsymbol{Q}_{\boldsymbol{EX,x}}\ \ \boldsymbol{Q}_{\boldsymbol{EX,y}}\ \ \mathbf{0}\ \ \mathbf{0}\ \ \mathbf{0}\ \ \mathbf{0}\right]^{\boldsymbol{T}}$$

Equations of Motion

By setting equation 9 equal the relevant generalized forces for each case, the equations of motion are determined.

$$\boldsymbol{Slipping}: \boldsymbol{J}(\boldsymbol{J}(\boldsymbol{L},\dot{\boldsymbol{q}}),\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{J}(\boldsymbol{J}(\boldsymbol{L},\dot{\boldsymbol{q}}),\dot{\boldsymbol{q}})\ddot{\boldsymbol{q}} - \boldsymbol{J}(\boldsymbol{L},\boldsymbol{q})^{\boldsymbol{T}} = \vec{\boldsymbol{Q}}_{\boldsymbol{N}} + \vec{\boldsymbol{Q}}_{\boldsymbol{R}} + \vec{\boldsymbol{Q}}_{\boldsymbol{EX}}$$
$$\boldsymbol{Rolling}: \boldsymbol{J}(\boldsymbol{J}(\boldsymbol{L},\dot{\boldsymbol{q}}),\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{J}(\boldsymbol{J}(\boldsymbol{L},\dot{\boldsymbol{q}}),\dot{\boldsymbol{q}})\ddot{\boldsymbol{q}} - \boldsymbol{J}(\boldsymbol{L},\boldsymbol{q})^{\boldsymbol{T}} = \vec{\boldsymbol{Q}}_{\boldsymbol{N}} + \vec{\boldsymbol{Q}}_{\boldsymbol{NS}}$$

III. Animation Still Frames

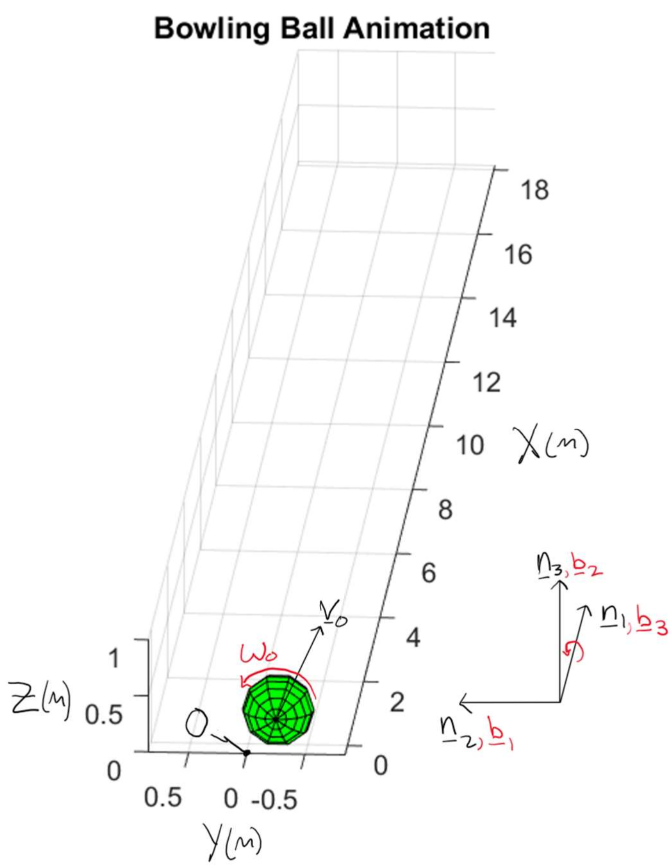

Figure 6: Animation at $t = 0$

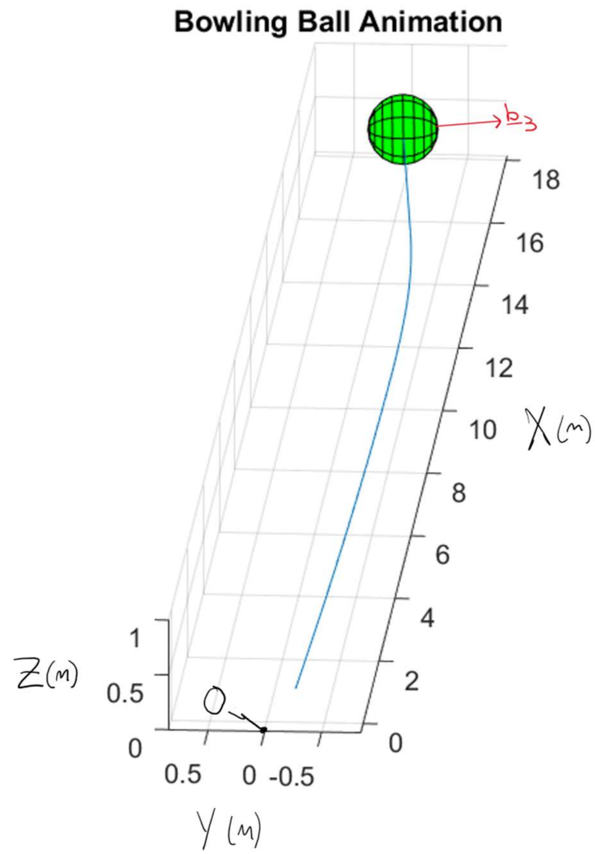

Figure 7: Animation at $t = t_f = 2.5\ s$

IV. Discussion

This simulation demonstrates the importance of friction in bowling. If the lanes are not oiled in the appropriate pattern, it's not possible achieve the optimal hooked trajectory that gives the highest chance of bowling a strike. The ball initially slides and approaches the side of the lane, gaining a better angle to attack the pins. Once it's time for the trajectory to hook, friction must increase. Once the velocity of the contact point with respect to the origin goes to 0, the ball is rolling and its final trajectory is set.

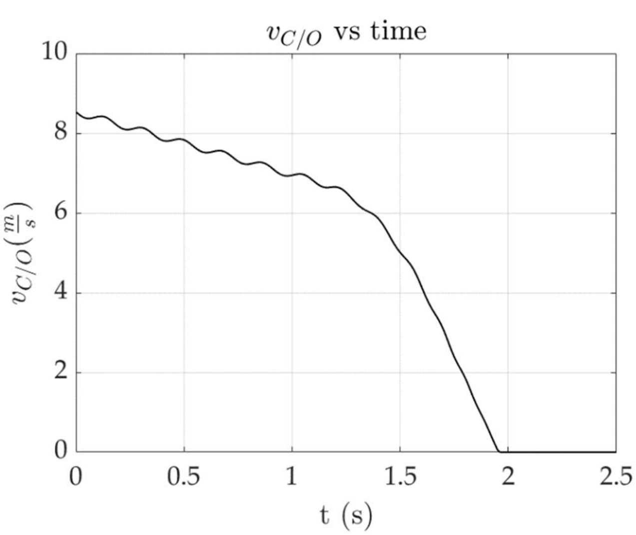

Figure 8: Velocity of the contact point w.r.t. O

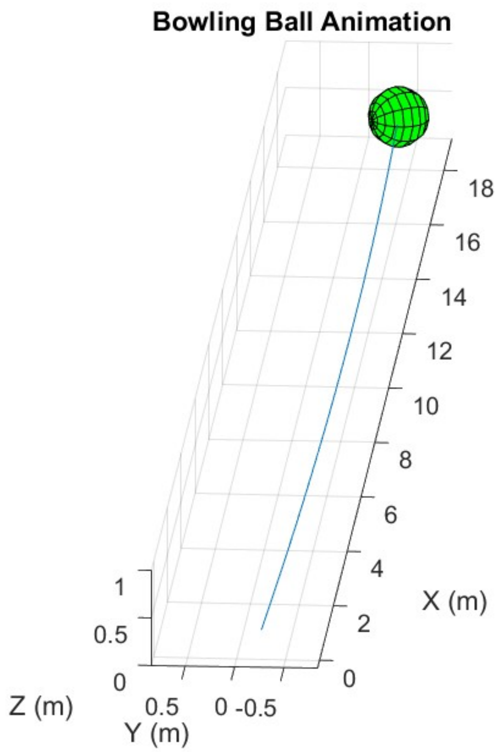

Figure 9: $\mu = .04$

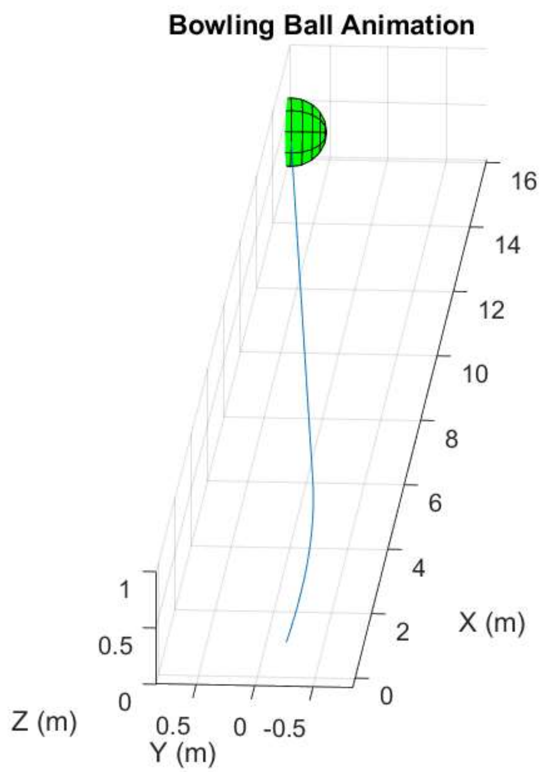

Figure 10: $\mu = .3$

As seen in the figures above, when the lane is not oiled optimally, the results vary widely. In figure 9, the coefficient of friction is $\mu = 0.04$, which is a typical value for the most oiled portion of the lane. However, if the friction never increases, the ball won't hook. On the opposite end of the spectrum, figure 10 shows what happens when the whole lane has high friction: the ball hooks too soon and misses the pins. Figure 7 demonstrates something close to the optimal oil pattern, where μ is initially around 0.04 and rises to around 0.2 or 0.3 by the final third of the lane. The two sections of different μ are also visible in figure 8.