Question 1: (15 points)

1- Let $A \in \mathbb{R}^{m\times n}$, B$\in \mathbb{R}^{r\times s}$, $C \in \mathbb{R}^{n\times p}$, and $D \in \mathbb{R}^{s\times t}$. Show that

$$(A \otimes B)(C \otimes D) = AC \otimes BD \ \in \mathbb{R}^{mr\times pt}$$

2- Let $A \in \mathbb{R}^{m\times n}$, B $\in \mathbb{R}^{p\times q}$, $X \in \mathbb{R}^{q\times n}$, and $Y \in \mathbb{R}^{p\times m}$. Show that

$$\big(vec(Y)\big)^T (A \otimes B) vec(X) = \text{trace}(A^T Y^T B X)$$

Question 2: (30 points)

Hyperspectral images provide higher spectral resolution than typical RGB images by including per-pixel irradiance measurements in a number of narrow bands of wavelength in the visible spectrum. However, the classification of hyperspectral images is a challenging task for several reasons such as the presence of redundant features, the high dimensionality of the data, and noise. Our goal is to use Tucker decomposition to learn a low-rank structure of the data and use it to build a spectral classifier for hyperspectral images. The data set is the Indian Pines which was acquired from the AVIRIS sensor. It has 20-m spatial resolutions and 10-nm spectral resolutions covering a spectrum range of 400–2500 nm. The image scene of size 145×145 pixels was used. The data set has 200 bands after bad band removal, and 16 classes of ground objects are to be classified. "IndianPines.csv" contains the vectorized images ($X \in \mathbb{R}^{21025\times 200}$), and the last column represents the classes (Labels $\in \mathbb{R}^{21025}$).

1- Randomly split the data into 10% training and 90% test. Use the training data to build a multi-class support vector machine. Report the accuracy and confusion matrix on the test set.

2- Form a third order tensor $\mathcal{X} \in \mathbb{R}^{145\times 145\times 200}$. Calculate Tucker decomposition of $\mathcal{X}$ with rank $R = R_1 = R_2 = R_3$ varies from 5 to 140 with step 5. Reconstruct the data tensor $\big(\hat{\mathcal{X}} \in \mathbb{R}^{145\times 145\times 200}\big)$ and matricize it $\big(\hat{X} \in \mathbb{R}^{21025\times 200}\big)$ by vectorizing each frontal slice. Randomly split $\hat{X}$ into 10% training and 90% test. Use the training data to build a multi-class support vector machine. Plot rank versus accuracy. Report the confusion matrix for the optimal rank. In terms of accuracy, how much performance your classifier has gained in comparison to part 1? Why?

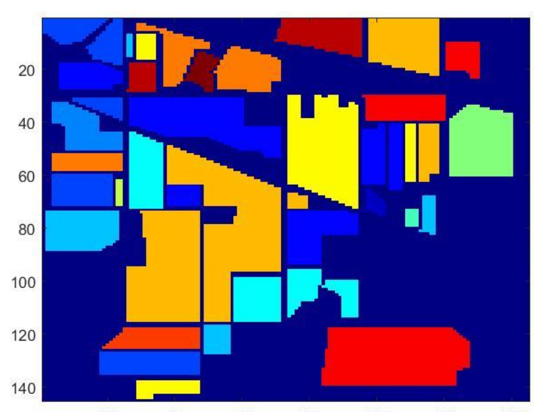

Figure 1. The groundtruth classes for the Indian Pines scene.

Table 1. Groundtruth classes for the Indian Pines scene and their respective samples number

#	**Class**	**Samples**
1	Alfalfa	46
2	Corn-notill	1428
3	Corn-mintill	830
4	Corn	237
5	Grass-pasture	483
6	Grass-trees	730
7	Grass-pasture-mowed	28
8	Hay-windrowed	478
9	Oats	20
10	Soybean-notill	972
11	Soybean-mintill	2455
12	Soybean-clean	593
13	Wheat	205
14	Woods	1265
15	Buildings-Grass-Trees-Drives	386
16	Stone-Steel-Towers	93

Question 3: (30 points)

In this question, you develop an algorithm to perform Tucker decomposition based on a numerical threshold, rather than some predefined rank. The algorithm takes the tensor $(\mathcal{X})$ and the threshold value γ as inputs and computes Tucker decomposition with the optimal rank.

$$\frac{\sum_{i=1}^{n_0} \sigma_i}{\sum_{i=1}^{n_k} \sigma_i} \geq \gamma$$

Where n_0 is the minimum number of singular values (σ_i) that allow the inequality to hold for mode-k. The factor matrix U_k is obtained from a truncated SVD of the mode-k unfolding of the tensor. Use the same threshold value γ for all modes. “CaltechFaces.zip” contains 450 images. The size of each image is 592×896 pixels. Read the images and form a third order tensor $(\mathcal{X} \in \mathbb{R}^{592\times896\times450})$. Let the threshold value γ varies from 0.1 to 0.99 with step 0.05. Plot compression versus relative error.

$$\text{Relative Error} = \frac{\|\mathcal{X} - \text{Reconstructed Tensor}\|}{\|\mathcal{X}\|}$$

Question 4: (25 points)

Amperometric sensors have been used to detect a wide range of electroactive gasses, vapors, and liquids. “Question4.zip” contains measurements of 40 silicon-based amperometric microelectrochemical gas sensor over 100 seconds for 400 trials. Read the data and form a third order tensor ($\mathcal{X} \in \mathbb{R}^{40\times100\times400}$). Perform CP decomposition and find the optimal rank using AIC method. Plot rank versus AIC and report your optimal rank (R). Plot the loading matrices (A, B, C).