Question 1

1) f is convex.

Let $u = \frac{t}{x}$ $\quad 0 \leq t \leq x \quad \therefore 0 \leq u \leq 1$

$$F(x) = \frac{1}{x}\int_0^x f(ux)\,dt$$

$$= \frac{1}{x}\int_0^1 f(ux)\,x\,du$$

$$= \int_0^1 f(ux)\,du$$

For all $a, b \in \operatorname{dom} F$, $0 \leq \theta \leq 1$

$$\theta F(a) + (1-\theta)F(b)$$

$$= \int_0^1 \theta f(au)\,du + \int_0^1 (1-\theta) f(bu)\,du$$

$$= \int_0^1 \big(\underline{\theta f(au) + (1-\theta) f(bu)} \big)\,du$$

$$F(\theta a + (1-\theta) b)$$

$$= \int_0^1 \underline{f(\theta u a + (1-\theta) b u)}\,du$$

$\because f$ is convex

$\therefore \theta f(au) + (1-\theta) f(bu) \geq f(\theta a u + (1-\theta) b u)$

$\therefore \int_0^1 (\theta f(au) + (1-\theta) f(bu))\,du$

$$\geq \int_0^1 f(\theta u a + (1-\theta) b u)\,du$$

$$\therefore \theta F(a) + (1-\theta) F(b) \geq F(\theta a + (1-\theta) b)$$

$\therefore F(x)$ is convex.

2) When theta = 1, f is linear and can be either convex or concave; when 0< theta <1, f is non-convex.

$$f_\theta'(x) = \theta^{-1}\theta x^{\theta-1} = x^{\theta-1}$$

$$f_\theta''(x) = (\theta-1)x^{\theta-2}$$

$\because 0<\theta \leq 1$, $\mathrm{dom}\, f_\theta = \mathbb{R}_+$ $\therefore x^{\theta-2}$

$$f_\theta''(x) \leq 0$$

- when $\theta = 1$, $f_\theta(x)$ is a linear function, can be either convex or concave.
- when $0<\theta<1$, $f_\theta''(x) < 0$, $f_\theta(x)$ is non-convex

3) f is convex.

$$\nabla f(x) = \begin{bmatrix} \frac{2x_1}{x_2} \\ \frac{-x_1^2}{x_2^2} \end{bmatrix}$$

find quadratic form of a matrix.

$$\nabla^2 f(x) = \begin{bmatrix} \frac{2}{x_2} & \frac{-2x_1}{x_2^2} \end{bmatrix}$$

$$\begin{bmatrix} \cdots & \cdots \\ \frac{-2x_1}{x_2^2} & \frac{+2x_1^2}{x_2^3} \end{bmatrix}$$

define $v = \begin{pmatrix} a \\ b \end{pmatrix}$,

$$\nabla^2 f(x)\, v = \begin{pmatrix} \frac{2a}{x_2} - \frac{2x_1 b}{x_2^2} \\ \frac{-2ax_1}{x_2^2} + \frac{2bx_1^2}{x_2^3} \end{pmatrix}$$

$$\langle \nabla^2 f(x) v, v \rangle = \frac{2a^2}{x_2} - \frac{2abx_1}{x_2^2} + \frac{2b^2x_1^2}{x_2^3} - \frac{2abx_1}{x_2^2}$$

ex.

$$= \frac{2a^2}{x_2} + \frac{2b^2x_1^2}{x_2^3} - \frac{4abx_1}{x_2^2}$$

$\because x_2 > 0$

$$= \sqrt{\frac{2a^2}{x_2}}^{\,2} - 2\sqrt{\frac{2a^2}{x_2}}\sqrt{\frac{2b^2x_1^2}{x_2^2 \cdot x_2}} + \sqrt{\frac{2b^2x_1^2}{x_2^3}}^{\,2}$$

$$= \left(\sqrt{\frac{2a^2}{x_2}} - \sqrt{\frac{2b^2x_1^2}{x_2^3}}\right)^2 \geq 0$$

$\therefore f(x_1, x_2)$ is convex

4) f is non-convex.

The Gradient Vector and Hessian Matrix are as follows. If Hessian is positive semi-definite, its every diagonal entry has to be non-negative. However, in this case we have -e^(-x1) <0 and hence we can conclude that the Hessian is not positive semi-definite. Hence, f(x1,x2,x3) is not convex.

4. $$\nabla f(x) = \begin{pmatrix} +e^{-x_1} \\ x_3^2 \\ 2x_2x_3 \end{pmatrix}$$

symmet

$$\nabla^2 f(x) = \begin{pmatrix} -e^{-x_1} & 0 & 0 \\ 0 & 0 & 2x_3 \\ 0 & 2x_3 & 2x_2 \end{pmatrix}$$

Quadratic form of $\nabla^2 f(x)$ is:

$-x_1$