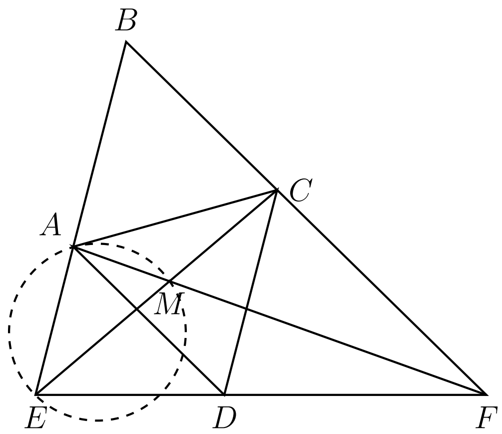
APMO 1993 – Problems and Solutions

Problem 1

Let $ABCD$ be a quadrilateral such that all sides have equal length and angle $\angle ABC$ is 60 degrees. Let ℓ be a line passing through D and not intersecting the quadrilateral (except at D). Let E and F be the points of intersection of ℓ with AB and BC respectively. Let M be the point of intersection of CE and AF.
Prove that $CA^2 = CM \times CE$.

Solution



Triangles AED and CDF are similar, because $AD \parallel CF$ and $AE \parallel CD$. Thus, since ABC and ACD are equilateral triangles,

$$\frac{AE}{CD} = \frac{AD}{CF} \iff \frac{AE}{AC} = \frac{AC}{CF}.$$

The last equality combined with

$$\angle EAC = 180^\circ - \angle BAC = 120^\circ = \angle ACF$$

shows that triangles EAC and ACF are also similar. Therefore $\angle CAM = \angle CAF = \angle AEC$, which implies that line AC is tangent to the circumcircle of AME. By the power of a point, $CA^2 = CM \cdot CE$, and we are done.

Problem 2
Find the total number of different integer values the function

$$f(x) = [x] + [2x] + \left[\frac{5x}{3}\right] + [3x] + [4x]$$

takes for real numbers x with $0 \le x \le 100$.
Note: $[t]$ is the largest integer that does not exceed t.

Answer: 734.

Solution
Note that, since $[x+n] = [x] + n$ for any integer n,

$$f(x+3) = [x+3] + [2(x+3)] + \left[\frac{5(x+3)}{3}\right] + [3(x+3)] + [4(x+3)] = f(x) + 35,$$

one only needs to investigate the interval $[0, 3)$.
The numbers in this interval at which at least one of the real numbers $x, 2x, \frac{5x}{3}, 3x, 4x$ is an integer are

- $0, 1, 2$ for x;
- $\frac{n}{2}$, $0 \le n \le 5$ for $2x$;
- $\frac{3n}{5}$, $0 \le n \le 4$ for $\frac{5x}{3}$;
- $\frac{n}{3}$, $0 \le n \le 8$ for $3x$;
- $\frac{n}{4}$, $0 \le n \le 11$ for $4x$.

Of these numbers there are

- 3 integers $(0, 1, 2)$;
- 3 irreducible fractions with 2 as denominator (the numerators are $1, 3, 5$);
- 6 irreducible fractions with 3 as denominator (the numerators are $1, 2, 4, 5, 7, 8$);
- 6 irreducible fractions with 4 as denominator (the numerators are $1, 3, 5, 7, 9, 11, 13, 15$);
- 4 irreducible fractions with 5 as denominator (the numerators are $3, 6, 9, 12$).

Therefore $f(x)$ changes values 22 per interval. Since $100 = 33 \cdot 3 + 1$, there are $33 \cdot 22$ changes in $[0, 99)$. Finally, there are 8 more changes in $[99, 100]$: 99, 100, $99\frac{1}{2}$, $99\frac{1}{3}$, $99\frac{2}{3}$, $99\frac{1}{4}$, $99\frac{3}{4}$, $99\frac{3}{5}$. The total is then $33 \cdot 22 + 8 = 734$.

Comment: A more careful inspection shows that the range of f are the numbers congruent modulo 35 to one of

$$0, 1, 2, 4, 5, 6, 7, 11, 12, 13, 14, 16, 17, 18, 19, 23, 24, 25, 26, 28, 29, 30$$

in the interval $[0, f(100)] = [0, 1166]$. Since $1166 \equiv 11 \pmod{35}$, this comprises 33 cycles plus the 8 numbers in the previous list.

Problem 3
Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \quad \text{and} \quad g(x) = c_{n+1}x^{n+1} + c_n x^n + \cdots + c_0$$

be non-zero polynomials with real coefficients such that $g(x) = (x+r)f(x)$ for some real number r. If $a = \max(|a_n|, \ldots, |a_0|)$ and $c = \max(|c_{n+1}|, \ldots, |c_0|)$, prove that $\frac{a}{c} \leq n + 1$.

Solution
Expanding $(x+r)f(x)$, we find that $c_{n+1} = a_n$, $c_k = a_{k-1} + ra_k$ for $k = 1, 2, \ldots, n$, and $c_0 = ra_0$. Consider three cases:

- $r = 0$. Then $c_0 = 0$ and $c_k = a_{k-1}$ for $k = 1, 2, \ldots, n$, and $a = c \implies \frac{a}{c} = 1 < n + 1$.

- $|r| \geq 1$. Then

$$|a_0| = \left|\frac{c_0}{r}\right| \leq c,$$
$$|a_1| = \left|\frac{c_1 - a_0}{r}\right| \leq |c_1| + |a_0| \leq 2c,$$

  and inductively if $|a_k| \leq (k + 1)c$

$$|a_{k+1}| = \left|\frac{c_{k+1} - a_k}{r}\right| \leq |c_{k+1}| + |a_k| \leq c + (k + 1)c = (k + 2)c.$$

  Therefore, $|a_k| \leq (k + 1)c \leq (n + 1)c$ for all k, and $a \leq (n + 1)c \iff \frac{a}{c} \leq n + 1$.

- $0 < |r| < 1$. Now work *backwards*: $|a_n| = |c_{n+1}| \leq c$,

$$|a_{n-1}| = |c_n - ra_n| \leq |c_n| + |ra_n| < c + c = 2c,$$

  and inductively if $|a_{n-k}| \leq (k + 1)c$

$$|a_{n-k-1}| = |c_{n-k} - ra_{n-k}| \leq |c_{n-k}| + |ra_{n-k}| < c + (k + 1)c = (k + 2)c.$$

  Therefore, $|a_{n-k}| \leq (k + 1)c \leq (n + 1)c$ for all k, and $a \leq (n + 1)c$ again.

Problem 4
Determine all positive integers n for which the equation

$$x^n + (2+x)^n + (2-x)^n = 0$$

has an integer as a solution.

Answer: $n = 1$.

Solution
If n is even, $x^2 + (2+x)^n + (2-x)^n > 0$, so n is odd.
For $n = 1$, the equation reduces to $x + (2+x) + (2-x) = 0$, which has the unique solution $x = -4$.
For $n > 1$, notice that x is even, because x, $2-x$, and $2+x$ have all the same parity. Let $x = 2y$, so the equation reduces to

$$y^n + (1+y)^n + (1-y)^n = 0.$$

Looking at this equation modulo 2 yields that $y + (1+y) + (1-y) = y + 2$ is even, so y is even.
Using the factorization

$$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + \cdots + b^{n-1}) \quad \text{for } n \text{ odd},$$

which has a sum of n terms as the second factor, the equation is now equivalent to

$$y^n + (1+y+1-y)((1+y)^{n-1} - (1+y)^{n-2}(1-y) + \cdots + (1-y)^{n-1}) = 0,$$

or

$$y^n = -2((1+y)^{n-1} - (1+y)^{n-2}(1-y) + \cdots + (1-y)^{n-1}).$$

Each of the n terms in the second factor is odd, and n is odd, so the second factor is odd. Therefore, y^n has only one factor 2, which is a contradiction to the fact that, y being even, y^n has at least $n > 1$ factors 2. Hence there are no solutions if $n > 1$.

Problem 5
Let $P_1, P_2, \ldots, P_{1993} = P_0$ be distinct points in the xy-plane with the following properties:

(i) both coordinates of P_i are integers, for $i = 1, 2, \ldots, 1993$;

(ii) there is no point other than P_i and P_{i+1} on the line segment joining P_i with P_{i+1} whose coordinates are both integers, for $i = 0, 1, \ldots, 1992$.

Prove that for some i, $0 \leq i \leq 1992$, there exists a point Q with coordinates (q_x, q_y) on the line segment joining P_i with P_{i+1} such that both $2q_x$ and $2q_y$ are odd integers.

Solution
Call a point $(x, y) \in \mathbb{Z}^2$ *even* or *odd* according to the parity of $x + y$. Since there are an odd number of points, there are two points $P_i = (a, b)$ and $P_{i+1} = (c, d)$, $0 \leq i \leq 1992$ with the same parity. This implies that $a + b + c + d$ is even. We claim that the midpoint of P_iP_{i+1} is the desired point Q.
In fact, since $a+b+c+d = (a+c)+(b+d)$ is even, a and c have the same parity if and only if b and d also have the same parity. If both happen then the midpoint of P_iP_{i+1}, $Q = \left(\frac{a+c}{2}, \frac{b+d}{2}\right)$, has integer coordinates, which violates condition (ii). Then a and c, as well as b and d, have different parities, and $2q_x = a + c$ and $2q_y = b + d$ are both odd integers.