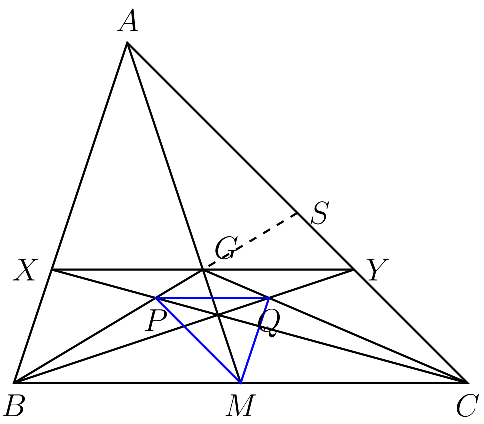
APMO 1991 – Problems and Solutions

Problem 1
Let G be the centroid of triangle ABC and M be the midpoint of BC. Let X be on AB and Y on AC such that the points X, Y, and G are collinear and XY and BC are parallel. Suppose that XC and GB intersect at Q and YB and GC intersect at P. Show that triangle MPQ is similar to triangle ABC.

Solution 1
Let S be the midpoint of AC; so BS is a median and contains the centroid G.

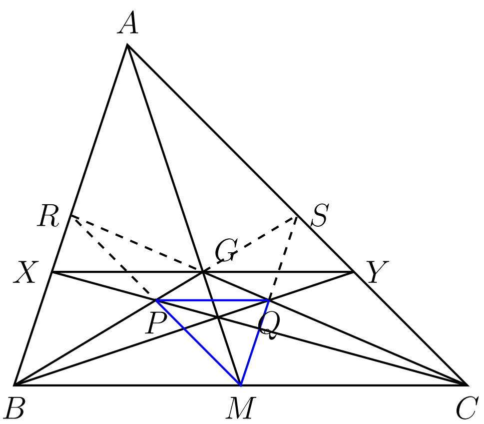


It is well known that $\frac{AG}{AM} = \frac{2}{3}$; thus the ratio of the similarity between AXY and ABC is $\frac{2}{3}$. Hence $GX = \frac{1}{2}XY = \frac{1}{3}BC$.
Now look at the similarity between triangles PBC and PGX:

$$\frac{PG}{PB} = \frac{GX}{BC} = \frac{1}{3} \implies PB = 3PG \implies PB = \frac{3}{4}BG = \frac{3}{4} \cdot \frac{2}{3}BS = \frac{1}{2}BS.$$

Finally, since $\frac{BM}{BC} = \frac{BP}{BS}$, MP is a midline in BCS. Therefore $MP = \frac{1}{2}CS = \frac{1}{4}AC$ and $MP \parallel AC$. Similarly, $MQ = \frac{1}{4}AB$ and $MQ \parallel AB$. This is sufficient to establish that MPQ and ABC are similar (with similarity ratio $\frac{1}{4}$).

Solution 2
Let R and S be the midpoints of AB and AC, respectively. Since G is the centroid, it lies in the medians BS and CR.



Due to the similarity between triangles PBC and PGX (which is true because $GX \parallel BC$), there is an inverse homothety with center P and ratio $-\frac{XG}{BC} = \frac{XY}{2BC}$ that takes B to G and C to X. This homothety takes the midpoint M of BC to the midpoint K of GX.

Now consider the homothety that takes B to X and C to G. This new homothety, with ratio $\frac{XY}{2BC}$, also takes M to K. Hence lines BX (which contains side AB), CG (which contains the median CR), and MK have a common point, which is R. Thus P lies on midline MR.
The same reasoning proves that Q lies on midline MS. Since all homothety ratios are the same, $\frac{MP}{MR} = \frac{MQ}{MS}$, which shows that MPQ is similar to MRS, which in turn is similar to ABC, and we are done.

Problem 2

Suppose there are 997 points given in a plane. If every two points are joined by a line segment with its midpoint coloured in red, show that there are at least 1991 red points in the plane. Can you find a special case with exactly 1991 red points?

Solution

Embed the points in the cartesian plane such that no two points have the same y-coordinate. Let $P_1, P_2, \ldots, P_{997}$ be the points and $y_1 < y_2 < \ldots < y_{997}$ be their respective y-coordinates. Then the y-coordinate of the midpoint of P_iP_{i+1}, $i = 1, 2, \ldots, 996$ is $\frac{y_i+y_{i+1}}{2}$ and the y-coordinate of the midpoint of P_iP_{i+2}, $i = 1, 2, \ldots, 995$ is $\frac{y_i+y_{i+2}}{2}$. Since

$$\frac{y_1 + y_2}{2} < \frac{y_1 + y_3}{2} < \frac{y_2 + y_3}{2} < \frac{y_2 + y_4}{2} < \cdots < \frac{y_{995} + y_{997}}{2} < \frac{y_{996} + y_{997}}{2},$$

there are at least $996 + 995 = 1991$ distinct midpoints, and therefore at least 1991 red points. The equality case happens if we take $P_i = (0, 2i)$, $i = 1, 2, \ldots, 997$. The midpoints are $(0, i+j)$, $1 \le i < j \le 997$, which are the points $(0, k)$ with $1 + 2 = 3 \le k \le 996 + 997 = 1993$, a total of $1993 - 3 + 1 = 1991$ red points.

Problem 3
Let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$. Show that

$$\frac{a_1^2}{a_1 + b_1} + \frac{a_2^2}{a_2 + b_2} + \cdots + \frac{a_n^2}{a_n + b_n} \geq \frac{a_1 + a_2 + \cdots + a_n}{2}.$$

Solution
By the Cauchy-Schwartz inequality,

$$\left(\frac{a_1^2}{a_1 + b_1} + \frac{a_2^2}{a_2 + b_2} + \cdots + \frac{a_n^2}{a_n + b_n}\right)((a_1+b_1)+(a_2+b_2)+\cdots+(a_n+b_n)) \geq (a_1+a_2+\cdots+a_n)^2.$$

Since $((a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n)) = 2(a_1 + a_2 + \cdots + a_n)$,

$$\frac{a_1^2}{a_1 + b_1} + \frac{a_2^2}{a_2 + b_2} + \cdots + \frac{a_n^2}{a_n + b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{2(a_1 + a_2 + \cdots + a_n)} = \frac{a_1 + a_2 + \cdots + a_n}{2}.$$

Problem 4
During a break, n children at school sit in a circle around their teacher to play a game. The teacher walks clockwise close to the children and hands out candies to some of them according to the following rule. He selects one child and gives him a candy, then he skips the next child and gives a candy to the next one, then he skips 2 and gives a candy to the next one, then he skips 3, and soon. Determine the values of n for which eventually, perhaps after many rounds, all children will have at least one candy each.

Answer: All powers of 2.

Solution 1
Number the children from 0 to $n-1$. Then the teacher hands candy to children in positions $f(x) = 1 + 2 + \cdots + x \bmod n = \frac{x(x+1)}{2} \bmod n$. Our task is to find the range of $f\colon \mathbb{Z}_n \to \mathbb{Z}_n$, and to verify whether the range is $\mathbb{Z}_n$, that is, whether f is a bijection.
If $n = 2^a m$, $m > 1$ odd, look at $f(x)$ modulo m. Since m is odd, $m \mid f(x) \iff m \mid x(x+1)$. Then, for instance, $f(x) \equiv 0 \pmod m$ for $x = 0$ and $x = m-1$. This means that $f(x)$ is not a bijection modulo m, and there exists t such that $f(x) \not\equiv t \pmod m$ for all x. By the Chinese Remainder Theorem,

$$f(x) \equiv t \pmod n \iff \begin{cases} f(x) \equiv t \pmod{2^a} \\ f(x) \equiv t \pmod m \end{cases}$$

Therefore, f is not a bijection modulo n.
If $n = 2^a$, then

$$f(x) - f(y) = \frac{1}{2}(x(x+1) - y(y+1)) = \frac{1}{2}(x^2 - y^2 + x - y) = \frac{(x-y)(x+y+1)}{2}.$$

and

$$f(x) \equiv f(y) \pmod{2^a} \iff (x-y)(x+y+1) \equiv 0 \pmod{2^{a+1}}. \qquad (*)$$

If x and y have the same parity, $x+y+1$ is odd and $(*)$ is equivalent to $x \equiv y \pmod{2^{a+1}}$. If x and y have different parity,

$$(*) \iff x + y + 1 \equiv 0 \pmod{2^{a+1}}.$$

However, $1 \le x + y + 1 \le 2(2^a - 1) + 1 = 2^{a+1} - 1$, so $x + y + 1$ is not a multiple of 2^{a+1}. Therefore f is a bijection if n is a power of 2.

Solution 2
We give a full description of a_n, the size of the range of f.
Since congruences modulo n are defined, via Chinese Remainder Theorem, by congruences modulo p^α for all prime divisors p of n and α being the number of factors p in the factorization of n, $a_n = \prod_{p^\alpha \| n} a_{p^\alpha}$.
Refer to the first solution to check the case $p = 2$: $a_{2^\alpha} = 2^\alpha$.
For an odd prime p,

$$f(x) = \frac{x(x+1)}{2} = \frac{(2x+1)^2 - 1}{8},$$

and since p is odd, there is a bijection between the range of f and the quadratic residues modulo p^α, namely $t \mapsto 8t + 1$. So a_{p^α} is the number of quadratic residues modulo p^α.
Let g be a primitive root of p^α. Then there are $\frac{1}{2}\phi(p^\alpha) = \frac{p-1}{2} \cdot p^{\alpha-1}$ quadratic residues that are coprime with p: $1, g^2, g^4, \ldots, g^{\phi(p^n)-2}$. If p divides a quadratic residue kp, that is, $x^2 \equiv kp \pmod{p^\alpha}$, $\alpha \ge 2$, then p divides x and, therefore, also k. Hence p^2 divides this quadratic residue, and these quadratic residues are p^2 times each quadratic residue of $p^{\alpha-2}$. Thus

$$a_{p^\alpha} = \frac{p-1}{2} \cdot p^{\alpha-1} + a_{p^{\alpha-2}}.$$

Since $a_p = \frac{p-1}{2} + 1$ and $a_{p^2} = \frac{p-1}{2} \cdot p + 1$, telescoping yields

$$a_{p^{2t}} = \frac{p-1}{2}(p^{2t-1} + p^{2t-3} + \cdots + p) + 1 = \frac{p(p^{2t}-1)}{2(p+1)} + 1$$

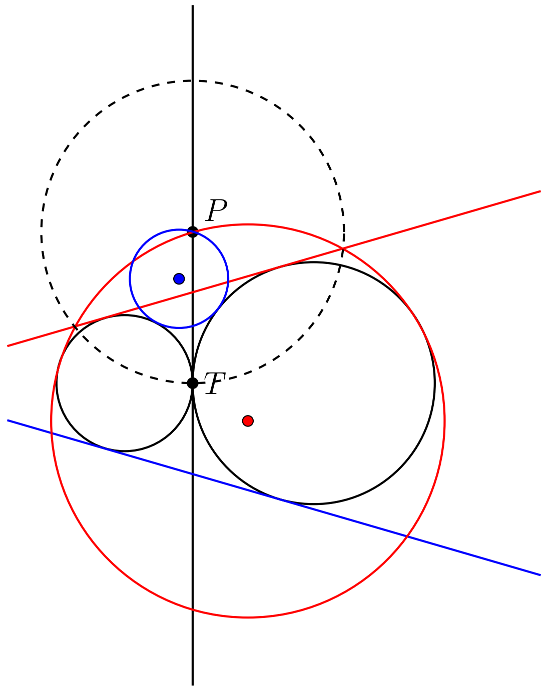
and

$$a_{p^{2t-1}} = \frac{p-1}{2}(p^{2t-2} + p^{2t-4} + \cdots + 1) + 1 = \frac{p^{2t}-1}{2(p+1)} + 1$$

Now the problem is immediate: if n is divisible by an odd prime p, $a_{p^\alpha} < p^\alpha$ for all α, and since $a_t \le t$ for all t, $a_n < n$.

Problem 5
Given are two tangent circles and a point P on their common tangent perpendicular to the lines joining their centres. Construct with ruler and compass all the circles that are tangent to these two circles and pass through the point P.

Solution
Throughout this problem, we will assume that the given circles are *externally* tangent, since the problem does not have a solution otherwise.
Let Γ_1 and Γ_2 be the given circles and T be their tangency point. Suppose ω is a circle that is tangent to Γ_1 and Γ_2 and passes through P.
Now invert about point P, with radius PT. Let any line through P that cuts Γ_1 do so at points X and Y. The power of P with respect to Γ_1 is $PT^2 = PX \cdot PY$, so X and Y are swapped by this inversion. Therefore Γ_1 is mapped to itself in this inversion. The same applies to Γ_2. Since circle ω passes through P, it is mapped to a line tangent to the images of Γ_1 (itself) and Γ_2 (also itself), that is, a common tangent line. This common tangent cannot be PT, as PT is also mapped to itself. Since Γ_1 and Γ_2 have exactly other two common tangent lines, there are two solutions: the inverses of the tangent lines.



We proceed with the construction with the aid of some macro constructions that will be detailed later.

Step 1. Draw the common tangents to Γ_1 and Γ_2.

Step 2. For each common tangent t, draw the projection P_t of P onto t.

Step 3. Find the inverse P_1 of P_t with respect to the circle with center P and radius PT.

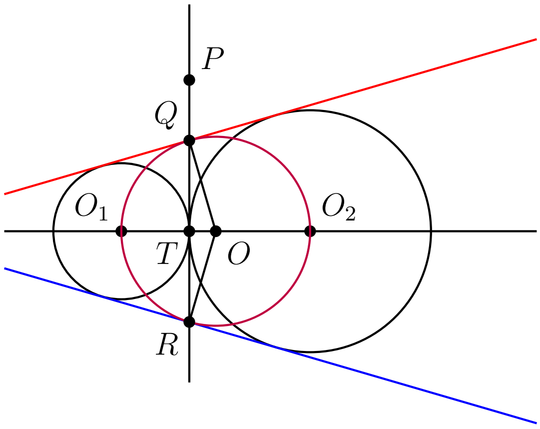
Step 4. ω_t is the circle with diameter PP_1.

Let's work out the details for steps 1 and 3. Steps 2 and 4 are immediate.

Step 1. In this particular case in which Γ_1 and Γ_2 are externally tangent, there is a small shortcut:

- Draw the circle with diameter on the two centers O_1 of Γ_1 and O_2 of Γ_2, and find its center O.

- Let this circle meet common tangent line OP at points Q, R. The required lines are the perpendicular to OQ at Q and the perpendicular to OR at R.



Let's show why this construction works. Let R_i be the radius of circle Γ_i. Note that $OQ = \frac{1}{2}O_1O_2 = \frac{R_1+R_2}{2}$, $OT = OO_1 - R_1 = \frac{|R_1-R_2|}{2}$, so

$$\sin \angle TQO = \frac{OT}{OQ} = \frac{|R_1 - R_2|}{R_1 + R_2},$$

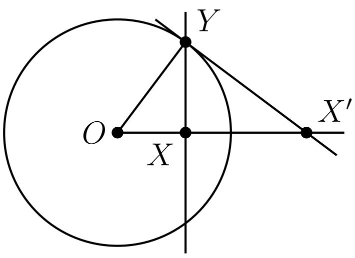
which is also the sine of the angle between O_1O_2 and the common tangent lines.
Let t be the perpendicular to OQ through Q. Then $\angle(t, O_1O_2) = \angle(OQ, QT) = \angle TQO$, and t is parallel to a common tangent line. Since

$$d(O,t) = OQ = \frac{R_1 + R_2}{2} = \frac{d(O_1,t) + d(O_2,t)}{2},$$

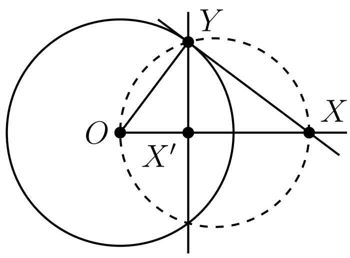
and O is the midpoint of O_1O_2, O is also at the same distance from t and the common tangent line, so these two lines coincide.

Step 3. Finding the inverse of a point X given the inversion circle Ω with center O is a well known procedure, but we describe it here for the sake of completeness.

- If X lies in Ω, then its inverse is X.
- If X lies in the interior of Ω, draw ray OX, then the perpendicular line ℓ to OX at X. Let ℓ meet Ω at a point Y. The inverse of X is the intersection X' of OX and the line perpendicular to OY at Y. This is because OYX' is a right triangle with altitude YX, and therefore $OX \cdot OX' = OY^2$.
- If X is in the exterior of Ω, draw ray OX and one of the tangent lines ℓ from X to Ω (just connect X to one of the intersections of Ω and the circle with diameter OX). Let ℓ touch Ω at a point Y. The inverse of X is the projection X' of Y onto OX. This is because OYX' is a right triangle with altitude YX', and therefore $OX \cdot OX' = OY^2$.



X is inside Ω



X is outside Ω