# APMO 1989 – Problems and Solutions

**Problem 1**
Let $x_1, x_2, \ldots, x_n$ be positive real numbers, and let

$$S = x_1 + x_2 + \cdots + x_n.$$

Prove that

$$(1+x_1)(1+x_2)\cdots(1+x_n) \le 1 + S + \frac{S^2}{2!} + \frac{S^3}{3!} + \cdots + \frac{S^n}{n!}.$$

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**Solution 1**
Let $\sigma_k$ be the *kth symmetric polynomial*, namely

$$\sigma_k = \sum_{\substack{|S|=k \\ S\subseteq\{1,2,\ldots,n\}}} \prod_{i\in S} x_i,$$

and more explicitly

$$\sigma_1 = S, \quad \sigma_2 = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n, \quad \text{and so on.}$$

Then

$$(1+x_1)(1+x_2)\cdots(1+x_n) = 1 + \sigma_1 + \sigma_2 + \cdots + \sigma_n.$$

The expansion of

$$S^k = (x_1+x_2+\cdots+x_n)^k = \underbrace{(x_1+x_2+\cdots+x_n)(x_1+x_2+\cdots+x_n)\cdots(x_1+x_2+\cdots+x_n)}_{k \text{ times}}$$

has at least $k!$ occurrences of $\prod_{i\in S} x_i$ for each subset $S$ with $k$ indices from $\{1,2,\ldots,n\}$. In fact, if $\pi$ is a permutation of $S$, we can choose each $x_{\pi(i)}$ from the $i$th factor of $(x_1+x_2+\cdots+x_n)^k$. Then each term appears at least $k!$ times, and

$$S^k \ge k!\sigma_k \iff \sigma_k \le \frac{S^k}{k!}.$$

Summing the obtained inequalities for $k = 1, 2, \ldots, n$ yields the result.

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**Solution 2**
By AM-GM,

$$(1+x_1)(1+x_2)\cdots(1+x_n) \le \left(\frac{(1+x_1)+(1+x_2)+\cdots+(1+x_n)}{n}\right)^n = \left(1+\frac{S}{n}\right)^n.$$

By the binomial theorem,

$$\left(1+\frac{S}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{S}{n}\right)^k = \sum_{k=0} \frac{1}{k!}\frac{n(n-1)\ldots(n-k+1)}{n^k}S^k \le \sum_{k=0}^{n} \frac{S^k}{k!},$$

and the result follows.

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*Comment: Maclaurin's inequality* states that

$$\frac{\sigma_1}{n} \ge \sqrt{\frac{\sigma_2}{\binom{n}{2}}} \ge \cdots \ge \sqrt[k]{\frac{\sigma_k}{\binom{n}{k}}} \ge \cdots \ge \sqrt[n]{\frac{\sigma_n}{\binom{n}{n}}}.$$

Then $\sigma_k \le \binom{n}{k}\frac{S^k}{n^k} = \frac{1}{k!}\frac{n(n-1)\ldots(n-k+1)}{n^k}S^k \le \frac{S^k}{k!}$.

**Problem 2**
Prove that the equation
$$6(6a^2 + 3b^2 + c^2) = 5n^2$$
has no solutions in integers except $a = b = c = n = 0$.

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**Solution**
We can suppose without loss of generality that $a, b, c, n \geq 0$. Let $(a, b, c, n)$ be a solution with minimum sum $a + b + c + n$. Suppose, for the sake of contradiction, that $a + b + c + n > 0$. Since 6 divides $5n^2$, $n$ is a multiple of 6. Let $n = 6n_0$. Then the equation reduces to

$$6a^2 + 3b^2 + c^2 = 30n_0^2.$$

The number $c$ is a multiple of 3, so let $c = 3c_0$. The equation now reduces to

$$2a^2 + b^2 + 3c_0^2 = 10n_0^2.$$

Now look at the equation modulo 8:

$$b^2 + 3c_0^2 \equiv 2(n_0^2 - a^2) \pmod 8.$$

Integers $b$ and $c_0$ have the same parity. Either way, since $x^2$ is congruent to 0 or 1 modulo 4, $b^2 + 3c_0^2$ is a multiple of 4, so $n_0^2 - a^2 = (n_0 - a)(n_0 + a)$ is even, and therefore also a multiple of 4, since $n_0 - a$ and $n_0 + a$ have the same parity. Hence $2(n_0^2 - a^2)$ is a multiple of 8, and

$$b^2 + 3c_0^2 \equiv 0 \pmod 8.$$

If $b$ and $c_0$ are both odd, $b^2 + 3c_0^2 \equiv 4 \pmod 8$, which is impossible. Then $b$ and $c_0$ are both even. Let $b = 2b_0$ and $c_0 = 2c_1$, and we find

$$a^2 + 2b_0^2 + 6c_1^2 = 5n_0^2.$$

Look at the last equation modulo 8:

$$a^2 + 3n_0^2 \equiv 2(c_1^2 - b_0^2) \pmod 8.$$

A similar argument shows that $a$ and $n_0$ are both even.
We have proven that $a, b, c, n$ are all even. Then, dividing the original equation by 4 we find

$$6(6(a/2)^2 + 3(b/2)^2 + (c/2)^2) = 5(n/2)^2,$$

and we find that $(a/2, b/2, c/2, n/2)$ is a new solution with smaller sum. This is a contradiction, and the only solution is $(a, b, c, n) = (0, 0, 0, 0)$.

**Problem 3**

Let $A_1, A_2, A_3$ be three points in the plane, and for convenience,let $A_4 = A_1$, $A_5 = A_2$. For $n = 1, 2$, and 3, suppose that $B_n$ is the midpoint of $A_nA_{n+1}$, and suppose that $C_n$ is the midpoint of $A_nB_n$. Suppose that $A_nC_{n+1}$ and $B_nA_{n+2}$ meet at $D_n$, and that $A_nB_{n+1}$ and $C_nA_{n+2}$ meet at $E_n$. Calculate the ratio of the area of triangle $D_1D_2D_3$ to the area of triangle $E_1E_2E_3$.

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*Answer:* $\dfrac{25}{49}$.

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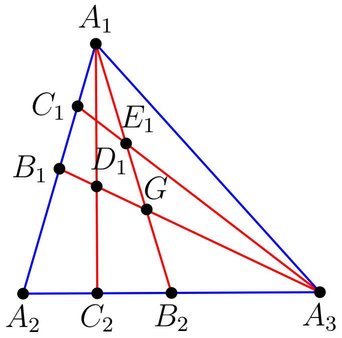
**Solution**

Let $G$ be the centroid of triangle $ABC$, and also the intersection point of $A_1B_2$, $A_2B_3$, and $A_3B_1$.




By Menelao's theorem on triangle $B_1A_2A_3$ and line $A_1D_1C_2$,

$$\frac{A_1B_1}{A_1A_2} \cdot \frac{D_1A_3}{D_1B_1} \cdot \frac{C_2A_2}{C_2A_3} = 1 \iff \frac{D_1A_3}{D_1B_1} = 2 \cdot 3 = 6 \iff \frac{D_1B_1}{A_3B_1} = \frac{1}{7}.$$

Since $A_3G = \frac{2}{3}A_3B_1$, if $A_3B_1 = 21t$ then $GA_3 = 14t$, $D_1B_1 = \frac{21t}{7} = 3t$, $A_3D_1 = 18t$, and $GD_1 = A_3D_1 - A_3G = 18t - 14t = 4t$, and

$$\frac{GD_1}{GA_3} = \frac{4}{14} = \frac{2}{7}.$$

Similar results hold for the other medians, therefore $D_1D_2D_3$ and $A_1A_2A_3$ are homothetic with center $G$ and ratio $-\frac{2}{7}$.

By Menelao's theorem on triangle $A_1A_2B_2$ and line $C_1E_1A_3$,

$$\frac{C_1A_1}{C_1A_2} \cdot \frac{E_1B_2}{E_1A_1} \cdot \frac{A_3A_2}{A_3B_2} = 1 \iff \frac{E_1B_2}{E_1A_1} = 3 \cdot \frac{1}{2} = \frac{3}{2} \iff \frac{A_1E_1}{A_1B_2} = \frac{2}{5}.$$

If $A_1B_2 = 15u$, then $A_1G = \frac{2}{3} \cdot 15u = 10u$ and $GE_1 = A_1G - A_1E_1 = 10u - \frac{2}{5} \cdot 15u = 4u$, and

$$\frac{GE_1}{GA_1} = \frac{4}{10} = \frac{2}{5}.$$

Similar results hold for the other medians, therefore $E_1E_2E_3$ and $A_1A_2A_3$ are homothetic with center $G$ and ratio $\frac{2}{5}$.

Then $D_1D_2D_3$ and $E_1E_2E_3$ are homothetic with center $G$ and ratio $-\frac{2}{7} : \frac{2}{5} = -\frac{5}{7}$, and the ratio of their area is $\left(\frac{5}{7}\right)^2 = \frac{25}{49}$.

**Problem 4**

Let $S$ be a set consisting of $m$ pairs $(a,b)$ of positive integers with the property that $1 \leq a < b \leq n$. Show that there are at least

$$4m\frac{(m-\frac{n^2}{4})}{3n}$$

triples $(a,b,c)$ such that $(a,b)$, $(a,c)$, and $(b,c)$ belong to $S$.

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**Solution**

Call a triple $(a,b,c)$ *good* if and only if $(a,b)$, $(a,c)$, and $(b,c)$ all belong to $S$. For $i$ in $\{1,2,\ldots,n\}$, let $d_i$ be the number of pairs in $S$ that contain $i$, and let $D_i$ be the set of numbers paired with $i$ in $S$ (so $|D_i| = d_i$). Consider a pair $(i,j) \in S$. Our goal is to estimate the number of integers $k$ such that any permutation of $\{i,j,k\}$ is good, that is, $|D_i \cap D_j|$. Note that $i \notin D_i$ and $j \notin D_j$, so $i,j \notin D_i \cap D_j$; thus any $k \in D_i \cap D_j$ is different from both $i$ and $j$, and $\{i,j,k\}$ has three elements as required. Now, since $D_i \cup D_j \in \{1,2,\ldots,n\}$,

$$D_i \cap D_j = |D_i| + |D_j| - |D_i \cup D_j| \leq d_i + d_j - n.$$

Summing all the results, and having in mind that each good triple is counted three times (one for each two of the three numbers), the number of good triples $T$ is at least

$$T \geq \frac{1}{3}\sum_{(i,j)\in S}(d_i + d_j - n).$$

Each term $d_i$ appears each time $i$ is in a pair from $S$, that is, $d_i$ times; there are $m$ pairs in $S$, so $n$ is subtracted $m$ times. By the Cauchy-Schwartz inequality

$$T \geq \frac{1}{3}\left(\sum_{i=1}^{n} d_i^2 - mn\right) \geq \frac{1}{3}\left(\frac{\left(\sum_{i=1}^{n} d_i\right)^2}{n} - mn\right).$$

Finally, the sum $\sum_{i=1}^{n} d_i$ is $2m$, since $d_i$ counts the number of pairs containing $i$, and each pair $(i,j)$ is counted twice: once in $d_i$ and once in $d_j$. Therefore

$$T \geq \frac{1}{3}\left(\frac{(2m)^2}{n} - mn\right) = 4m\frac{(m-\frac{n^2}{4})}{3n}.$$

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*Comment:* This is a celebrated graph theory fact named *Goodman's bound*, after A. M. Goodman's method published in 1959. The generalized version of the problem is still studied until today.

**Problem 5**
Determine all functions $f$ from the reals to the reals for which

(1) $f(x)$ is strictly increasing,

(2) $f(x)+g(x)=2x$ for all real $x$, where $g(x)$ is the composition inverse function to $f(x)$.

(*Note*: $f$ and $g$ are said to be composition inverses if $f(g(x))=x$ and $g(f(x))=x$ for all real $x$.)

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*Answer:* $f(x)=x+c$, $c\in\mathbb{R}$ constant.

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**Solution**
Denote by $f_n$ the $n$th iterate of $f$, that is, $f_n(x)=\underbrace{f(f(\ldots f}_{n\text{ times}}(x)))$.
Plug $x\to f_{n+1}(x)$ in (2): since $g(f_{n+1}(x))=g(f(f_n(x)))=f_n(x)$,

$$f_{n+2}(x)+f_n(x)=2f_{n+1}(x),$$

that is,

$$f_{n+2}(x)-f_{n+1}(x)=f_{n+1}(x)-f_n(x).$$

Therefore $f_n(x)-f_{n-1}(x)$ does not depend on $n$, and is equal to $f(x)-x$. Summing the corresponding results for smaller values of $n$ we find

$$f_n(x)-x=n(f(x)-x).$$

Since $g$ has the same properties as $f$,

$$g_n(x)-x=n(g(x)-x)=-n(f(x)-x).$$

Finally, $g$ is also increasing, because since $f$ is increasing $g(x)>g(y) \implies f(g(x))>f(g(y)) \implies x>y$. An induction proves that $f_n$ and $g_n$ are also increasing functions.
Let $x>y$ be real numbers. Since $f_n$ and $g_n$ are increasing,

$$x+n(f(x)-x)>y+n(f(y)-y) \iff n[(f(x)-x)-(f(y)-y)]>y-x$$

and

$$x-n(f(x)-x)>y-n(f(y)-y) \iff n[(f(x)-x)-(f(y)-y)]<x-y.$$

Summing it up,

$$|n[(f(x)-x)-(f(y)-y)]|<x-y \quad \text{for all } n\in\mathbb{Z}_{>0}.$$

Suppose that $a=f(x)-x$ and $b=f(y)-y$ are distinct. Then, for all positive integers $n$,

$$|n(a-b)|<x-y,$$

which is false for a sufficiently large $n$. Hence $a=b$, and $f(x)-x$ is a constant $c$ for all $x\in\mathbb{R}$, that is, $f(x)=x+c$.
It is immediate that $f(x)=x+c$ satisfies the problem, as $g(x)=x-c$.