Principle Series Representation Part II.

We have constructed:

$$\begin{tikzcd} B_{\mu_1,\mu_2} \arrow[r] \arrow[dr, "\cong"'] & K_{\mu_1,\mu_2} \\ & \mathcal{F}_{\mu_1\mu_2^{-1}} \arrow[u, "\Phi_\varphi"'] \end{tikzcd}$$

a space of functions on F

$$\varphi \longmapsto x \mapsto \mu_2(x)|x|^{\frac{1}{2}}\hat{\Phi}_\varphi(x)$$

- $K_{\mu_1,\mu_2} = \left\{ \xi \in C^\infty(F^\times) \;\middle|\; \begin{array}{l} x \text{ large}, \ \xi \text{ vanishes} \\ x \text{ near } 0, \ \xi(x) = \begin{cases} |x|^{\frac{1}{2}} a\mu_1(x) + b\mu_2(x), & \mu_1\mu_2^{-1} \neq \mathbb{1}, |\,|^{-1} \\ |x|^{\frac{1}{2}}(a\mu_2(x)v(x) + b\mu_2(x)), & \mu_1\mu_2^{-1} = \mathbb{1} \\ b|x|^{\frac{1}{2}}\mu_2(x), & \mu_1\mu_2^{-1} = |\,|^{-1}. \end{cases} \end{array} \right\}$

- Map is isom if $\mu_1\mu_2^{-1} \neq |\,|^{-1}$.
  Kernel is span of $\mu_1 \circ \det |\det|^{\frac{1}{2}}$, if $\mu_1\mu_2^{-1} = |\,|^{-1}$. (Kernel of $\mathcal{F}_{\mu_1\mu_2^{-1}} \to K_{\mu_1,\mu_2}$ is const funs)

- If we transport the repn. of G on $\begin{pmatrix} P_{\mu_1,\mu_2}, \\ B_{\mu_1,\mu_2} \end{pmatrix}$ to K_{μ_1,μ_2} and call it κ_{μ_1,μ_2}, then $\kappa_{\mu_1,\mu_2}\begin{pmatrix} a & b \\ & 1 \end{pmatrix}\xi(x) = \tau_F(bx)\,\xi(ax)$.

Thm 6

① If $\mu_1\mu_2^{-1} \neq |\ |, |\ |^{-1}$, ρ_{μ_1,μ_2} irred.

② If $\mu_1\mu_2^{-1} = |\ |^{-1}$, B_{μ_1,μ_2} contains a 1-dim subrepn. generated by $\mu_1 \circ \det |\det|^{\frac{1}{2}}$. The quotient space is irred.

③ If $\mu_1\mu_2^{-1} = |\ |$, B_{μ_1,μ_2} contains a irred. subrepn. of codim 1, consisting of functions φ s.t.

$$\int_{B\backslash G} \varphi(g)\, \mu_1^{-1}(\det g)\, |\det g|^{\frac{1}{2}}\, d\bar{g} = 0$$

$$= \int_F \varphi\left(w^{-1}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) dx$$

using additive notation for characters

($B_{\mu_1,\mu_2} \times B_{-\mu_1,-\mu_2} \longrightarrow \mathbb{C}$ non-deg pairing.)

$\varphi \in$ orthog to $\longrightarrow \mu_1^{-1} \circ \det |\det|^{\frac{1}{2}}$

Pf: <u>Step 1</u>

$$\text{inv. subsp} \subset K_{\mu_1,\mu_2}$$
$$\xi \quad \circlearrowleft$$
$$\xi - \kappa_{\mu_1,\mu_2}\begin{pmatrix}1 & b\\ & 1\end{pmatrix}\xi$$

$$x \mapsto \xi(x) - \tau_F(bx)\xi(x) \in \mathcal{S}(F^\times)$$

Any non-zero inv. subsp. of K_{μ_1,μ_2} contains a non-zero fun in $\mathcal{S}(F^\times)$.

$\Rightarrow$ — — — — — — — — all of $\mathcal{S}(F^\times)$

b/c $\mathcal{S}(F^\times) \circlearrowleft \left\{\begin{pmatrix}a & b\\ & 1\end{pmatrix}\right\}$ mirabolic is irred.

Step 2

$$\begin{array}{ccccc} & \mathcal{B}_{\mu_1,\mu_2} & \longrightarrow & \mathcal{K}_{\mu_1,\mu_2} & \\ & \cup & & \cup & \\ \text{subrepn.} & V & \longrightarrow & \mathcal{K}_V & \supset \mathcal{S}(F^\times) \\ & \rotatebox{90}{$\in$} & & & \rotatebox{90}{$\in$} \\ & \varphi - \rho_{\mu_1,\mu_2}\begin{pmatrix}1 & b\\ & 1\end{pmatrix}\varphi & \xleftarrow{\text{implies}} & \xi - \kappa_{\mu_1,\mu_2}\begin{pmatrix}1 & b\\ & 1\end{pmatrix}\xi & \forall \xi \in \mathcal{K}_{\mu_1,\mu_2}, \\ & \forall \varphi \in \mathcal{B}_{\mu_1,\mu_2} & & & \end{array}$$

assumed to be nonzero.
in other words, $V \neq \mathbb{C}\mu_1 \circ \det |\det|^{\frac{1}{2}}$ if $\mu_1\mu_2^{-1} = |\,|^{-1}$.
and $V \neq 0$.

Non-deg. pairing: Exclude the $V = 1$-dim case.

$$\begin{array}{ccccc} \mathcal{B}_{\mu_1,\mu_2} & \times & \mathcal{B}_{-\mu_1,-\mu_2} & \longrightarrow & \mathbb{C} \\ \cup & & \cup & & \\ V & & V^\perp & & \end{array}$$

Every $\psi \in V$ is fixed by $\begin{pmatrix}1 & b\\ & 1\end{pmatrix}$ $\forall b$.

$\psi\Big(\underbrace{\begin{pmatrix}* & *\\ & *\end{pmatrix} w^{-1} \begin{pmatrix}1 & *\\ & 1\end{pmatrix}}_{\text{open cell}}\Big)$ is determined by value $\psi(w^{-1})$.

$\Rightarrow$ ψ ———————— .

$\Rightarrow$ $\dim V^\perp \leq 1$. codim $V \leq 1$. (some reasoning on next page)

Such a ψ exists only when $\mu_1\mu_2^{-1} = |\ |$. In this case $\psi = \mu_1^{-1} \circ \det |\det|^{\frac{1}{2}}$ up to const.
$(\neq 0)$

Pf: $\psi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \psi\left(\begin{pmatrix} c^{-1}\det g & * \\ & c \end{pmatrix} w^{-1} \begin{pmatrix} 1 & c^{-1}d \\ & 1 \end{pmatrix}\right)$

$c \neq 0.$ $\quad = \mu_1^{-1}(c^{-1}\det g)\,\mu_2^{-1}(c)\,|c^{-2}\det g|^{\frac{1}{2}}\,\psi(w^{-1}).$

c near 0 $\quad = \mu_1(c)\,\mu_2^{-1}(c)\,\mu_1^{-1}(\det g)\,|c|^{-1}\,|\det g|^{\frac{1}{2}}\,\psi(w^{-1})$

$\psi\begin{pmatrix} a & b \\ & d \end{pmatrix} = \psi\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ //

c small

$\|$

$\mu_1^{-1}(a)\,\mu_2^{-1}(d)\,|ad^{-1}|^{\frac{1}{2}}\,\psi(1). = \mu_1(d)\,\mu_2^{-1}(d)\,|d|^{-1}\,\mu_1^{-1}(\det g)\,|\det g|^{\frac{1}{2}}\,\psi(1)$

This is possible only when $\mu_1\mu_2^{-1}|\ |^{-1} = \mathbb{1}.$

i.e. $\mu_1\mu_2^{-1} = |\ |.$

Get Jordan-Hölder series:

B_{μ_1,μ_2} irred if $\mu_1\mu_2^{-1} \neq |\ |, |\ |^{-1}$.

$B_{\mu_1,\mu_2} \supset$ 1-dim $\mu_1 \circ \det |\det|^{\frac{1}{2}}$ if $\mu_1\mu_2^{-1} = |\ |^{-1}$.

the quotient

$B_{\mu_1,\mu_2} \supset$ codim 1 irred. if $\mu_1\mu_2^{-1} = |\ |$

dual to $\mu_1^{-1} \circ \det |\det|^{\frac{1}{2}} \subset B_{-\mu_1,-\mu_2}$.

called special repn. σ_{μ_1,μ_2} $\sigma_{\mu_1,\mu_1|\ |}$ $\sigma_{\mu_1,\mu_1|\ |^{-1}}$. (in Jacquet-Langlands notation) $\square$

principal series repn

Remarks on Kirillov model.

$f, f_1, f_2 \in \mathcal{S}(F)$.

$\mu_1\mu_2^{-1} \neq |\ |, |\ |^{-1}$ $\quad \pi = \rho_{\mu_1,\mu_2}$ $\quad K(\pi) = \mathcal{K}_{\mu_1,\mu_2} \overset{\text{codim } 2}{\supset} \mathcal{S}(F^\times)$ $\quad \begin{cases} |x|^{\frac{1}{2}}\mu_1(x) f_1(x) + |x|^{\frac{1}{2}}\mu_2(x) f_2(x), & \mu_1\mu_2^{-1} \neq 1 \\ |x|^{\frac{1}{2}}\mu_2(x) f_1(x) + |x|^{\frac{1}{2}}\mu_2(x)\nu(x) f_2(x), & \mu_1\mu_2^{-1} = 1 \end{cases}$

$\mu_1\mu_2^{-1} = |\ |^{-1}$ $\quad \pi =$ special repn. $\quad K(\pi) = \mathcal{K}_{\mu_1,\mu_2} \overset{\text{codim } 1}{\supset} \mathcal{S}(F^\times)$ $\quad |x|^{\frac{1}{2}}\mu_2(x) f(x)$

$\mu_1\mu_2^{-1} = |\ |$ $\quad \pi =$ special repn. $\quad K(\pi) \overset{\text{codim } 1}{\subset} \mathcal{K}_{\mu_1,\mu_2}$ $\quad |x|^{\frac{1}{2}}\mu_1(x) f(x)$

$\cup$ codim 1

$\mathcal{S}(F^\times)$

<u>Thm 7</u>

$$\rho_{\mu_1,\mu_2} \simeq \rho_{\mu_2,\mu_1} \qquad \mu_1\mu_2^{-1} \neq |\ |, |\ |^{-1}.$$

$$\sigma_{\mu_1,\mu_2} \simeq \sigma_{\mu_2,\mu_1} \qquad \mu_1\mu_2^{-1} = |\ |, |\ |^{-1}.$$

No other equivalence.