09/09/20 JL Seminar

Contents

1 Introduction

1.1. Our goal is to understand irred. adm. repn. of $\mathrm{GL}_n(F)$ for $n = 2$, F local narc.

1.1 Writing conventions

1.2. I will be using many shorthands, generally following a "syllabic abbreviation", i.e.

- ext. : extension. With first three letters for the type of extensions.
  - alg./sep. : algebraic/separable
- cplt./cpt./td.: complete/compact/totally disconnected.
- wrt./narc. : with respect to/ non-archimedean.

In general, the context (ctx) should make it clear what I'm talking about.

1.2 Notation

1.3. On matrices. We follow [JL70] with minor modification. Let $G_F := \mathrm{GL}_2(F)$ we describe several *sbgps*

- $K_F := \mathrm{GL}_2(\mathcal{O}_F)$, is also a[1] max. cpt. open sbgrp.
- Z_F is center of G_F consisting of scalar matrices, hence iso. to $F^\times$.
- D_F be sbgrp. of matrices of the from $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$

[1]is this *the*?

- B_F is sbgrp. of matrices of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, also known as *Borel subgrp.*
- N_F is sbgrp. of matrices of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. We thus have an identification

$$F \xrightarrow{\simeq} N_F, x \mapsto n_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

- A_F is subgrp. of diagonal matrices.
- C_F is subgrp. of matrices of the form $\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$

1.4. Mapping spaces. Let X be a space, $V \in \mathrm{Vect}_{\mathbb{C}}$.

- $\mathrm{Map}(X, V)$ is the *set* of V-valued fncs.
- $\mathrm{Map}^{\infty}(X, V)$ " loc. const. V-valued fncs.
- $\mathrm{Map}_c^{\infty}(X, V)$ " loc. const. cptly supported V-value fncs.

Remark 1.5. When $V = \mathbb{C}$, we often omit the V. The second and third type are also called *smooth* and *schwartz* functions respectively, denoted as $C^{\infty}(X, V)$ and $\mathcal{S}(X, V)$ in [JL70].

2 Overview

2.1. [PS83, 13] The method of constructing repns consists of three stages.

1. Use general methods to construct representations of D_F.
2. Then we "jump" to B_F an induce characters from B_F to G.
3. The last is to explore those repns that do not appear. (hardest).

2.2. Whittaker models come about at step 1. These correspond to induced representations from N_F.

2.1 Structure on subgroups

2.3. Structure of B_F.

- B_F is a solvable grp [2], whose normal abelian gp is U_F
- N_F and D_F and normal subgroup of B_F.
- We have the followin two decompositions for B_F

$$B_F = D_F \rtimes Z_F = N_F \rtimes A_F$$

2.4. Structure of D_F.

- $D_F = N_F \rtimes C_F$.
- The action of C_F on N_F is by conjugation of $F^\times$ on F^+, i.e.

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha\beta \\ 0 & 1 \end{pmatrix}$$

2.2 Kirillov model

2.5. Kirillov representation of D_F. It $V \subset \mathrm{Map}(F^\times, \mathbb{C})$, *complex valued functions*, on which D_F operates by

$$\pi \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \xi(x) = \psi_F(bx)\xi(ax)$$

then π is a *Kirillov representation*. This also restircts to an action on $\mathrm{Map}^\infty(F^\times, \mathbb{C}), \mathrm{Map}_c^\infty(F^\times, \mathbb{C})$. We denote this repn. as

$$(\psi_F, \mathrm{Map}(F^\times))$$

Definition 2.6. A *Kirillov model* of (π, V), is an equiv. repn. of G_F on a subspace of $V' \subset \mathrm{Map}(F^\times)$ such that the canonical inclusion $D_F \hookrightarrow G_F$ identifies $\mathrm{Res}_{D_F}^{G_F} V'$ as a submodule of $(\psi_F, \mathrm{Map}(F^\times))$. Here

$$\mathrm{Res}_{D_F}^{G_F} : \mathrm{Rep}(G_F) \to \mathrm{Rep}(D_F)$$

is the restriction functor (left adjoint to induction).

Theorem 2.7. Let (π, V) be an admissible infinite dimensional representation of G_F. Then π has a unique Kirillov model.

[2] i.e. there is a subnormal series whose factors are abelian.

Proof. Step 0. (π, V) *is a Pre-Kirillov model: we can identify* V *as a subspace of* $\mathrm{Map}^\infty(F^\times, J_\psi V)$.

Step 1. Understanding this space.

Step 2. Understanding the action of G_F. ☐

2.8. A key input in *Step 2* is understanding the structure theory of G_F, it can be decomposed to three types of matrices.

- Diagonal.
- D_F.
- $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

We will need a generalized version of Mellin transform.

2.9. We will end with showing the equivalence of three statements for an irred. adm. inf. dim. rep. (π, V):

1. $J_\psi V$ is one dimensional.
2. π admits a unique Kirillov model.
3. π admits a unique Whittaker model.

2.3 Representations and functionals on Schwartz Space

[JL70, 2]

Definition 2.10. We define a representation (ξ_ψ, D_F) on the spaces $\mathrm{Map}(F, X)$ and $\mathrm{Map}(F^\times, X)$ by [3]

$$\left(\xi_\psi \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \phi\right)(y) = \psi(yx)\phi(ya) \tag{1}$$

This also induces action on $\mathrm{Map}^\infty(F, X), \mathrm{Map}^\infty_c(F, X)$ etc.

Lemma 2.11. [JL70, 2.13.3] Let ϕ be an element of $\mathcal{S}(F^\times)$. Then there exists

- A finite subset S of $F^\times$
- Complex numbers $\lambda_y \in S$ where
$$\sum \lambda_y = 0, \quad \sum \lambda_y \psi(y) = \phi(1)$$
- an element $\phi_0 \in \mathrm{Map}^\infty_c(F^\times)$.

such that

$$\phi = \sum_{y \in S} \lambda_y \phi_\psi(n_y)\phi_0$$

[3]Note that the action of a is on the right.

Proof. Step 1. Fourier transformation Extend ϕ to a function on F - this is still an element on $\mathcal{S}(F)$. Let ϕ' denote the Fourier transform of ϕ. [4]

Step 2. Discreteness Then the function

$$F \times F \to \mathbb{C}, \quad (y,x) \mapsto \phi'(-y)\psi(xy)$$

is loc. const. and cptly. sup.
Step 2. Evaluation □

Corollary 2.12. [JL70, 2.13.1] Let L be a linear functional on Schwartz space $\mathcal{S}(F^\times)$ satisfying

$$L(\xi_\psi(n_x)\phi) = \psi(x)L(\phi)$$

for all ϕ in $\mathcal{S}(F^\times)$ and all $x \in F$. Then there is a scalar λ such that

$$L(\phi) = \lambda\phi(1)$$

Proof. Step 0. A linear reduction. As open subgrps of top. groups are also closed, 3. of 3.3, char. fncs. 1_U, where U is an open sbgrp, lies in $\mathrm{Map}^\infty(F^\times)$ and in $\mathrm{Map}_c^\infty(F^\times) = \mathcal{S}(F^\times)$ if U is cpt.

Hence, given $\phi \in \mathcal{S}(F^\times)$, replacing subtracting by $\phi(1)1_U$, we have

$$L(\phi - \phi(1)1_U) = L(\phi) - \phi(1)L(1_U)$$

If we can prove *Step 1.* below, we have obtained the desired form with $\lambda L(1_U)$.

Step 1. Use the representation in 2.11 □

2.4 Uniqueness of Whittaker functional

2.5 Uniqueness of Kirillov model

2.13. (π, V) is as ctx. With its Kirillov model.

Proposition 2.14. Kirillov model of (π, V) is unique. [?God70, 5].

Proof. Step 0. Set up. Let (π', V') be a represemtation equivalent to (π, V), where $V' \subset \mathrm{Map}(F^\times, \mathbb{C})$ whose restriction to D_F is ψ_F. Let $A : V' \to V$ denote the iso of G_F-repn.

Step 1. Inducing new Whittaker functional.

Step 1a. Define $L\phi := (A\phi)(1)$ for $\phi \in V$. If we show that L is Whittaker functional then $A\phi = \lambda\phi$, for some $\lambda \in \mathbb{C}$. Thus $V = V'$ with $\pi(g) = \pi'(g)$ (using the fact that ϕ is also an iso.)

Step 1b. Checking that L as defined is indeed a Whittaker functional. This is a simple computational check and N_F linearity.

$$L\left(\pi\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) = \left(\pi'\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}(A\phi)\right)(1) = \psi(x)L(\phi(1))$$

□

[4] Why do we pass to $\mathcal{S}(F)$?

2.6 The Whittaker Model

Definition 2.15. Let $\mathcal{W}(\psi)$ be subspace of $\mathrm{Map}(G_F, \mathbb{C})$ st.

$$W\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) = \psi(x)W(g)$$

This is a G_F-repn via right regular action, denoted $(\rho, \mathcal{W}(\psi))$, i.e.

$$(\rho(h)W)(g) = W(gh)$$

Theorem 2.16. [JL70, 2.14] Let (π, V) be as in ctx. Then π has a unique Whittaker model.

Proof. Step 0. Existence. We define an injectiono of G_F-modules,

$$V \hookrightarrow \mathrm{Map}(G_F, \mathbb{C}), \quad \phi \mapsto W_\phi$$

$$W_\phi(g) := (\pi(g)\phi)(1) \tag{2}$$

whose image is in $\mathcal{W}(\psi)$. There are a few things to be checked.

1. Well defined, i.e. the image indeed lies in $\mathcal{W}(\psi)$. Now

$$W_\phi(n_x g) = (\pi n_x \pi(g)\phi)(1)) = \psi(x)(\pi(g)\phi)(1)$$

2. The maps is clearly $\mathbb{C}$ -linear. It is G_F-equivariant too:

$$W_{\pi(h)\phi}(g) = (\pi(g)\pi(h)\phi)(1) = W_\phi(gh) = (\rho(h)W_\phi)(g)$$

3. Injectivity. Note

$$W_\phi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right) = \phi(a)$$

so ϕ is zero iff W_ϕ is.

Step 1. Uniqueness. This proof imitates that of 2.14 □

3 Appendix

3.1 Topological Groups

3.1. We recall some topological notions. We let $X \in \text{Top}$, G a topological group.

Basics.

- A space X is *hom.* if given any two points $x, y \in X$, exists $f : X \to X$ such that $fx = y$.

Local compactness and connectedness.

- X is *locally cpt.* if for all $x \in X$.
- G is a *loc. cpt. grp* if it is Hausdorf and loc. cpt space.

Example 3.2. Let R be a top. ring. $\mathrm{GL}_n(R)$, $M_n(R)$ are both, top. ring, given the subspace topology in R^{n^2}.

3.3. Now let us list a whole host of properties for a topological group. G

1. If $U \subset G$, then U is open iff tU is open iff Ut is open iff U^{-1} for all $t \in G$.
2. Every nhoodo U of 1 contains an open symmetric nhood V of 1 such that $VV \subset U$.
3. Every open subgroup is also closed.
4.

Proof. 3. Let $H \subset G$ be open subgroup. G can be written as the union of cosets of H. We have the relation

$$Y = \bigcup_{x \in G \setminus H} xH$$

$$H = G \setminus Y$$

□

Proposition 3.4. [Vin08, a.4.1] Let G be a Hausdorff top. grp. Any subgroup of G which is loc. cpt. is closed.

Corollary 3.5. [Vin08, e.4.2] A Hausdorff top. grp. G is loc. cpt. and t.d. iff every nhood of 1 contains a compact open subgroup.

Remark 3.6. Importantly, for those reading the text [BZ76], these are the *l-groups*.

3.2 Smooth and admissible representations

Definition 3.7. Let G be tdlc, (π, V) a representation. V admits *no* topology.

- π is *smooth* if for any $v \in V$, stabilizer [5]

  $$\mathrm{Stab}(v) := \{g \in G : gv = v\}$$

  is an open subgrp of G. This is nonempty as e lies in the grp.

[5] This is a rather abuse of notation, but the context should make it clear.

- If π is smooth, and if for any open subgroup $U \subset G$

$$V^U = \{v \in V \,:\, gv = v \text{ for all } g \in U\} \tag{3}$$

  is fin. dim, then π is *admissible*.

3.8. Continuity. I find it more natural to interpret smooth representations as *continuous* representations. By definition, if V is given the discrete topology, then (π, V) is smooth iff it is continuous.

Proposition 3.9. Finite dimensionality. Let (π, V) be a fd. rep. of a tdlc group G Then the following are equivalent.

1. π is admissible.
2. π is smooth.
3. Kernel of π is an open subgroup.
4. π, as a map $G \to \mathrm{GL}(V)$ is continuous.

Proof. $1 \Leftrightarrow 2$ is clear from defn. $2 \Leftrightarrow 3$. Suppose $\ker \pi$ is open. Then for any $g \in \mathrm{Stab}(v)$, $g \ker \pi \subset \mathrm{Stab}(v)$ is an open hood of g. So $\mathrm{Stab}(v)$ is open. Suppose $\mathrm{Stab}(v)$ is open. Let $\{v_i\}$ be a $\mathbb{C}$-basisc of V, so

$$\ker \pi = \bigcap_1^n \mathrm{Stab}(v_i)$$

is open.

$3 \Leftrightarrow 4$.

□

3.10. Irreducible rep'ns.

- If (π, V) is a smooth or admissible rep'n, then every G-invariant subspace of V is also smooth or admissible rep'n respectively.
- A smooth representation (π, V) of G is *irreducible* if V contains no nontrivial G-invariant subspaces.

References

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