F-virtual Abelian varieties of GL_2-type

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Let F be a number field. This chapter studies F-virtual Abelian varieties of GL_2-type. These Abelian varieties themselves are not defined over F but their isogeny classes are defined over F. They are generalization of Abelian varieties of GL_2-type defined over F which are in turn generalization of elliptic curves. The Galois representations of $\mathrm{Gal}(\overline{F}/F)$ associated to F-virtual Abelian varieties of GL_2-type are projective representations of dimension 2. Thus it is expected many techniques for GL_2-type can also be applied, such as modularity results and Gross-Zagier formula. Furthermore the study of virtual Abelian varieties of GL_2-type can possibly furnish evidence for the BSD conjecture.

The simplest case of F-virtual Abelian varieties of GL_2-type consists of $\mathbb{Q}$-elliptic curves. They were first studied by Gross[3] in the CM case and by Ribet[7][8] in the non-CM case. Also Ribet generalized the notion of $\mathbb{Q}$-elliptic curves to $\mathbb{Q}$-virtual Abelian varieties of GL_2-type. Elkies studied the quotients of modular curves $X^*(N)$ that parametrize $\mathbb{Q}$-elliptic curves and computed some explicit equations of these quotients[1]. Then González and Lario[2] described those $X^*(N)$ with genus zero or one. Based on the parametrization, Quer[6] computed explicit equations of some $\mathbb{Q}$-curves.

In this chapter first we study the ℓ-adic representations associated to F-virtual Abelian varieties of GL_2-type. Then we determine the quotients of Shimura varieties that parametrize F-virtual Abelian varieties and classify them birationally in the case where the quotients are surfaces. It turns out that almost all of them are of general type. We also give examples of surfaces that are rational.

1 Abelian Varieties of GL_2-type

We start with the definition and some properties of the main object of this chapter.

Definition 1.1. Let A be an Abelian variety over some number field F and E a number field. Let $\theta : E \hookrightarrow \mathrm{End}^0 A = \mathrm{End}\, A \otimes_{\mathbb{Z}} \mathbb{Q}$ be an algebra embedding. Then the pair (A, θ) is said to be of $\mathrm{GL}_2(E)$-type if $[E : \mathbb{Q}] = \dim A$.

Remark 1.2. We will drop θ if there is no confusion. It is well-known that the Tate module $V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is a free $E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-module of rank 2. The action of $\mathrm{Gal}(\overline{F}/F)$ on $V_\ell(A)$ defines a representation with values in $\mathrm{GL}_2(E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)$ and thus the nomenclature GL_2-type.

We can also define $\mathrm{GL}_n(E)$-type if we have $\theta : E \hookrightarrow \mathrm{End}^0(A)$ with $[E : \mathbb{Q}] = 2\dim A/n$. Of course, we require $n|2\dim A$.

In the following when we say a field acting on an Abelian variety we mean the action up to isogeny.

Definition 1.3. An Abelian variety A over some number field F of dimension g is said to have sufficiently many complex multiplication (CM) if $\mathrm{End}^0(A_{\overline{F}})$, the endomorphism algebra of $A_{\overline{F}}$ contains a commutative $\mathbb{Q}$-algebra of degree $2g$. Also A over F of dimension g is said to have sufficiently many complex multiplication (CM) over F if $\mathrm{End}^0(A)$ contains a commutative $\mathbb{Q}$-algebra of degree $2g$.

Proposition 1.4. *Let A be an Abelian variety defined over a totally real field F. Then A does not have sufficiently many complex multiplication (smCM) over F.*

Proof. Suppose the contrary. Fix an embedding of F into $\overline{\mathbb{Q}}$. Suppose $E \hookrightarrow \mathrm{End}^0 A$ with E a CM-algebra and $[E : \mathbb{Q}] = 2\dim A$. Consider the CM-type coming from the action of E on $\mathrm{Lie}\, A_{\overline{\mathbb{Q}}}$. Then the reflex field E' of E is $\mathbb{Q}(\mathrm{tr}\,\Phi)$ and is a CM-field. Since E actually acts on $\mathrm{Lie}\, A/F$, we find $\mathrm{tr}\,\Phi \subset F$. Thus $E' = \mathbb{Q}(\mathrm{tr}\,\Phi)$ is contained in the totally real field F, so we get a contradiction. □

Remark 1.5. The proof, in particular, shows that if a field is embedded in $\mathrm{End}^0(A)$ for an Abelian variety A over a totally real field F, then it is of at most degree $\dim A$.

1.1 Decompostion over $\overline{F}$

Suppose that A is an Abelian variety of dimension g over some number field F of $\mathrm{GL}_2(E)$-type. Fix an embedding $F \hookrightarrow \overline{\mathbb{Q}}$. We consider the decomposition of $A_{\overline{\mathbb{Q}}}$. Since the embedding θ of a number field E into endomorphism algebra is given as part of the data, when we consider isogenies between Abelian varieties of GL_2-type we require compatibility with the given embeddings. More precisely we define:

Definition 1.6. Let θ_i be an embedding of a $\mathbb{Q}$-algebra D into the endomorphism algebra of an Abelian variety A_i/F, for $i = 1$ or 2. Then an isogeny μ between A_1 and A_2 is said to be D-equivariant or D-linear if $\mu \circ \theta_2(a) = \theta_1(a) \circ \mu$, for all $a \in D$.

To describe the factors of $A_{\overline{\mathbb{Q}}}$ we define F-virtuality:

Definition 1.7. Let A be an Abelian variety defined over $\overline{\mathbb{Q}}$ and suppose $\theta : D \xrightarrow{\sim} \mathrm{End}^0(A)$. Let F be a number field that embeds into $\overline{\mathbb{Q}}$. Then A is said to be F-virtual if $({}^{\sigma}A, {}^{\sigma}\theta)$ is D-equivariantly isogenous to (A, θ) for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$.

Remark 1.8. Note that here we assumed that θ is an isomorphism.

This definition makes it precise what it means for an Abelian variety to have isogenous class defined over F.

If $\dim A = 1$ and $F = \mathbb{Q}$, A is what is known as a $\mathbb{Q}$-curve, c.f. [3] and [8].

We analyze the endomorphism algebra of $A_{\overline{\mathbb{Q}}}$. Notice the number field E also embeds into $\mathrm{End}^0(A_{\overline{\mathbb{Q}}})$ via $\mathrm{End}^0 A \hookrightarrow \mathrm{End}^0(A_{\overline{\mathbb{Q}}})$. Let C be the commutant of E in $\mathrm{End}^0(A_{\overline{\mathbb{Q}}})$. There are two possibilities: either $E = C$ so that we are in the non-CM case or $E \subsetneq C$ so that we are in the CM case.

1.1.1 Case CM

We have $E \subsetneq C$, so $[C:\mathbb{Q}] = 2\dim A = 2g$. Hence A is CM. A priori, $A_{\overline{\mathbb{Q}}} \sim \prod B_i^{n_i}$ with B_i's pairwise nonisogenous simple Abelian varieties. Thus $\operatorname{End}^0 A_{\overline{\mathbb{Q}}} \cong \prod \mathrm{M}_{n_i}(L_i)$ where $L_i = \operatorname{End}^0 B_i$. Then E embeds into $\prod P_i$ with P_i a maximal subfield of $\mathrm{M}_{n_i}(L_i)$. Consider projection to the i-th factor. We have $E \to P_i$. Note that the identity element of E is mapped to the identity element in $\prod P_i$. Hence $E \to P_i$ is not the zero map and so it is injective. Since $[E:\mathbb{Q}] = g$, P_i has degree greater than or equal to g. On the other hand $\sum [P_i:\mathbb{Q}] = 2g$. We are forced to have

$$E \hookrightarrow P_1$$

with $[P_1 : Q] = 2g$ or

$$E \hookrightarrow P_1 \times P_2$$

with $P_1 \cong P_2 \cong E$. Correspondingly we have either $A_{\overline{\mathbb{Q}}} \sim B_1^{n_1}$ or $A_{\overline{\mathbb{Q}}} \sim B_1^{n_1} \times B_2^{n_2}$. In the second case E must be a CM field.

Now we discuss what conditions on A make sure that $A_{\overline{\mathbb{Q}}}$ is isogenous to a power of a simple Abelian variety. Fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$.

Proposition 1.9. *Suppose either that the g embeddings of E into $\mathbb{C}$ coming from the action of E on $\operatorname{Lie}(A_{\mathbb{C}})$ exhaust all possible embeddings of E into $\mathbb{C}$ or that A is defined over a totally real field F. Then $A_{\overline{\mathbb{Q}}} \sim B^n$ for some simple Abelian variety B.*

Proof. Suppose that A is not isogenous to a power of some simple Abelian variety contrary, i.e., $A_{\overline{\mathbb{Q}}} \sim B_1^{n_1} \times B_2^{n_2}$. Let S be the centre of $\operatorname{End}^0 A_{\overline{\mathbb{Q}}}$. Then $S = L_1 \times L_2$ for two number fields L_1 and L_2. Denote by e_i the identity element of L_i. Let A_i be the image of A under e_i or rather he_i for some integer h such that $he_i \in \operatorname{End} A_{\overline{\mathbb{Q}}}$. Then $A \sim A_1 \times A_2$ and E acts on A_i. This gives rise to CM-types (E, Φ_i).

Suppose that the first part of the assumption holds; then $\Phi_1 \sqcup \Phi_2$ gives all possible embeddings of E into $\mathbb{C}$. Thus $\Phi_1 = \iota\Phi_2$ where ι denote the complex conjugation of $\mathbb{C}$. Then $\Phi_1 = \Phi_2\iota_E$ for the complex conjugation ι_E of E. If we change the embedding of E into $\operatorname{End}^0 A_2$ by $\iota_E \in \operatorname{Gal}(E/\mathbb{Q})$ then the action of E on A_2 has also type Φ_1. This means that $A_2 \sim A_1$ and we get a contradiction. Thus A decomposes over $\overline{\mathbb{Q}}$ into a power of a simple Abelian variety.

Now suppose that the second part of the assumption holds; then F is a totally real field. Thus the complex conjugate ${}^{\iota}A$ is isomorphic to A. If ${}^{\iota}A_1 \cong A_1$ then A_1 can be defined over $\mathbb{R}$. This is impossible by Prop. 1.4 since it is CM. Thus ${}^{\iota}A_1 \cong A_2$. Then for some automorphism σ of E, we have $\iota\Phi_1\sigma = \Phi_2$. Thus Φ_1 and Φ_2 are different by an automorphism $\iota_E\sigma$ of E. Again $A_1 \sim A_2$, a contradiction.

Thus A decomposes over $\overline{\mathbb{Q}}$ into a power of a simple Abelian variety. □

1.1.2 Case Non-CM

In this case $E = C$. In particular the centre L of $\operatorname{End}^0(A_{\overline{\mathbb{Q}}})$ is contained in E and hence is a field. Thus $\operatorname{End}^0(A_{\overline{\mathbb{Q}}}) \xleftarrow[\sim]{\theta} \mathrm{M}_n(D)$ for n some positive integer and D some division algebra. Correspondingly $A_{\overline{\mathbb{Q}}} \sim B^n$ with B some simple Abelian variety over $\overline{\mathbb{Q}}$ and $\operatorname{End}^0 B \cong D$. Let $e = [L : \mathbb{Q}]$ and $d = \sqrt{[D : L]}$. Then E is a maximal subfield of $\mathrm{M}_n(D)$. We know that $ed^2 | \frac{2g}{n}$ and that $g = [E : \mathbb{Q}] = nde$. This forces $d|2$. If $d = 1$, $\operatorname{End}^0 B \cong L$. If $d = 2$, $\operatorname{End}^0 B \cong D$ with D a quaternion algebra over L.

Since A is defined over F, we get for all $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/F)$,

$$
{}^\sigma B^n \sim {}^\sigma A_{\overline{\mathbb{Q}}} \cong A_{\overline{\mathbb{Q}}} \sim B^n.
$$

By the uniqueness of decomposition we find ${}^\sigma B \sim B$, for all $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/F)$. Furthermore the canonical isomorphism ${}^\sigma A \cong A$ is L-equivariant, since the endomorphisms in $\theta(L)$ are rational over F. Fix isogeny $A_{\overline{\mathbb{Q}}} \to B^n$. The actions of L on B^n and ${}^\sigma B^n$ are the pullbacks of the actions of L on A and ${}^\sigma A$, so ${}^\sigma B^n \sim B^n$ is L-equivariant. As the L-actions are just diagonal actions, we have ${}^\sigma B^n \sim B$ L-equivariantly.

Let P be a maximal subfield of D. Then B is an Abelian variety of $\mathrm{GL}_2(L)$-type for $d = 1$ or $\mathrm{GL}_2(P)$-type for $d = 2$. Even though we are using different letters L, E and D, they may refer to the same object. In this subsection all Abelian varieties are assumed to be without CM.

First we record a result that follows from the discussion.

Proposition 1.10. *The endomorphism algebra of an Abelian variety of* GL_2*-type has one of the following types: a matrix algebra over some number field or a matrix algebra over some quaternion algebra.*

The following lemma shows that L-equivariance is as strong as D-equivariance.

Lemma 1.11. *Let $B/\overline{\mathbb{Q}}$ be an Abelian variety of* GL_2*-type with $\theta : D \xrightarrow{\sim} \operatorname{End}^0(B)$. Let L be the centre of D. Suppose ${}^\sigma B$ is L-equivariantly isogenous to B for all $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/F)$. Then after modifying θ, we can make B into an F-virtual Abelian variety, i.e., there exist D-equivariant isogenies ${}^\sigma B \to B$ for all $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/F)$.*

Proof. Choose L-equivariant isogenies $\mu_\sigma : {}^\sigma B \to B$. This means that $\mu_\sigma \circ {}^\sigma \phi = \phi \circ \mu_\sigma$ for $\phi \in L$. Thus we have L-algebra isomorphisms:

$$
\begin{aligned}
&D \to D \\
&\phi \mapsto \mu_\sigma \circ {}^\sigma\phi \circ \mu_\sigma^{-1}
\end{aligned}
$$

By Skolem-Noether theorem there exists an element $\psi \in D^\times$ s.t. $\mu_\sigma \circ {}^\sigma\phi \circ \mu_\sigma^{-1} = \psi \circ \phi \circ \psi^{-1}$. Let $\mu'_\sigma = \psi^{-1} \circ \mu_\sigma$. Then μ'_σ gives a D-equivariant isogeny between ${}^\sigma B$ and B. ☐

Proposition 1.12. *An Abelian variety A/F of* GL_2*-type is $\overline{\mathbb{Q}}$-isogenous to a power of a $\overline{\mathbb{Q}}$-simple Abelian variety B. Moreover B is an F-virtual Abelian variety of* GL_2*-type.*

Proof. This follows from the discussion above and the lemma. ☐

1.2 F-virtual Abelian varieties and simple Abelian varieties over F

We consider the converse problem. Given a simple F-virtual Abelian variety B of GL_2-type we want to give an explicit construction of a simple Abelian variety A over F of GL_2-type such that B is a factor of A. We separate into two cases:

1.2.1 Case Non-CM

Theorem 1.13. *Let $A/\overline{\mathbb{Q}}$ be a simple abelian variety of $\mathrm{GL}_2(E)$-type. Suppose that ${}^{\sigma}\!A$ and A are E-equivariantly isogenous for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Then there exists an F-simple Abelian variety B/F of GL_2-type s.t. A is a factor of $B_{\overline{\mathbb{Q}}}$.*

Proof. By Lemma 1.11, the Abelian variety A is actually F-virtual. Find a model of A over a number field K_1 such that K_1/F is Galois and all endomorphisms of A are defined over K_1. We identify ${}^{\sigma}\!A$ and A via the canonical isomorphism for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K_1)$. Let D denote the full endomorphism algebra of A. Choose D-equivariant $\overline{\mathbb{Q}}$-isogenies $\mu_\sigma : {}^{\sigma}\!A \to A$ for representatives σ in $\mathrm{Gal}(\overline{\mathbb{Q}}/F)/\mathrm{Gal}(\overline{\mathbb{Q}}/K_1) \cong \mathrm{Gal}(K_1/F)$. The rest of the μ_σ's for all σ in $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ are determined from these representatives and the canonical isomorphisms. Now let K be a field extension of K_1 such that K is Galois over F and that all μ_σ's for σ in $\mathrm{Gal}(K_1/F)$ are defined over K. Base change the model over K_1 to K and still call this model A. Instead of considering all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$, we only need to consider σ-twists of A for $\sigma \in \mathrm{Gal}(K/F)$.

Define $c(\sigma, \tau) = \mu_\sigma {}^{\sigma}\mu_\tau \mu_{\sigma\tau}^{-1}$. In the quaternion algebra case, note that

$$c(\sigma, \tau).\phi = \mu_\sigma {}^{\sigma}\mu_\tau \mu_{\sigma\tau}^{-1}.\phi = \mu_\sigma {}^{\sigma}\mu_\tau {}^{\sigma\tau}\phi \mu_{\sigma\tau}^{-1} = \mu_\sigma {}^{\sigma}\phi \mu_\tau \mu_{\sigma\tau}^{-1} = \phi \mu_\sigma {}^{\sigma}\mu_\tau \mu_{\sigma\tau}^{-1}$$

for $\phi \in D$. Thus c has values in $L^\times$.

It is easy to check that c is a 2-cocycle on $\mathrm{Gal}(K/F)$ with values in $L^\times$. By inflation we consider the class of c in $H^2(\mathrm{Gal}(\overline{\mathbb{Q}}/F), L^\times)$. It can be shown that $H^2(\mathrm{Gal}(\overline{\mathbb{Q}}/F), \overline{L}^\times)$ with the Galois group acting trivially on $\overline{L}^\times$ is trivial by a theorem of Tate as quoted as Theorem 6.3 in [8]. Hence there exists a locally constant function: $\alpha : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \overline{L}^\times$ s.t. $c(\sigma, \tau) = \frac{\alpha(\sigma)\alpha(\tau)}{\alpha(\sigma\tau)}$. We will use α in the construction of an Abelian variety of GL_2-type.

Let $B = \mathrm{Res}_{K/F} A$ be the restriction of scalars of A from K to F. Then B is defined over F. We let $D \circ \mu_\sigma$ denote the set of isogenies $\{f \circ \mu_\sigma \mid \forall f \in D\}$. Then we have the commutative diagram:

$$\begin{tikzcd}
\mathrm{End}^0 B \arrow[r, "\sim"] \arrow[d, hook] & \prod_\sigma \mathrm{Hom}^0({}^{\sigma}\!A, A) \arrow[r, "\sim"] \arrow[d, hook] & \prod_\sigma D \circ \mu_\sigma \\
\mathrm{End}^0 B_K \arrow[r, "\sim"] & \prod_{\rho,\tau} \mathrm{Hom}^0({}^{\tau}\!A, {}^{\rho}\!A) &
\end{tikzcd}$$

where the right verticle arrow maps $f : {}^{\sigma}\!A \to A$ to ${}^{\tau}\!f : {}^{\tau\sigma}\!A \to {}^{\tau}\!A$, for all τ and the products are running over $\mathrm{Gal}(K/F)$. The multiplication of the ring $\prod_\sigma D \circ \mu_\sigma$ can be described as follows. Let ϕ, $\psi \in D$. Then $(\phi \circ \mu_\sigma).(\psi \circ \mu_\tau) = \phi \circ \mu_\sigma \circ {}^{\sigma}\psi \circ {}^{\sigma}\mu_\tau =$

$\phi \circ \psi \circ \mu_\sigma \circ {}^\sigma\mu_\tau = \phi \circ \psi \circ c(\sigma, \tau)\mu_\sigma\tau$ since μ_σ is D-equivariant. Thus $\prod_\sigma D \circ \mu_\sigma$ can be viewed as $D[\mathrm{Gal}(K/F)]$ twisted by the cocycle c.

Let the number field E embed into $\mathrm{End}^0 B$ via θ into the factor $\mathrm{Hom}^0(A, A)$ of $\mathrm{End}^0 B$. Let $\mathrm{End}^0_E B$ denote the commutant of E in $\mathrm{End}^0 B$. Since the μ_σ's are E-equivariant and the commutant of E in D is E, $\mathrm{End}^0_E B \cong \prod \theta(E) \circ \mu_\sigma$.

Let $L_\alpha(E_\alpha$ resp.) denote the field $L(E$ resp.) adjoined with values of α. Let $D_\alpha = D \otimes_L L_\alpha$. Take

$$\omega : \prod_\sigma D \circ \mu_\sigma \to D_\alpha$$
$$\phi \circ \mu_\sigma \mapsto \phi \otimes \alpha(\sigma).$$

Then ω is a D-algebra homomorphism. If we restrict ω to $\mathrm{End}^0_E B$ we find $\omega|_{\mathrm{End}^0_E B}$ has image E_α. Since $\mathrm{End}^0 B$ is a semisimple $\mathbb{Q}$-algebra, we have $\mathrm{End}^0 B \cong D_\alpha \oplus \ker \omega$. Being the commutant of E in a semisimple algebra $\mathrm{End}^0 B$, $\mathrm{End}^0_E B$ is semisimple and therefore we have $\mathrm{End}^0_E B \cong E_\alpha \oplus \ker \omega|_{\mathrm{End}^0_E B}$. Let $\pi \in \mathrm{End}^0 B$ be a projector to D_α. Let B_α be the image of π. Then B_α has action up to isogeny exactly given by D_α.

Before finishing the proof, we need the following:

Lemma 1.14. *Let R denote $\mathrm{End}^0_E B$. Then the Tate module $V_\ell(B) = V_\ell(B_K)$ is a free $R \otimes \mathbb{Q}_\ell$-module of rank 2.*

Proof. Note $B_K \cong \prod_\sigma {}^\sigma A$. Passing to Tate modules we have $V_\ell(B_K) \cong \oplus_\sigma V_\ell({}^\sigma A)$. We know that $V_\ell(A)$ is a free $E \otimes_\mathbb{Q} \mathbb{Q}_\ell$-module of rank 2. Choose an $E \otimes_\mathbb{Q} \mathbb{Q}_\ell$-basis $\{e_1, e_2\}$ of $V_\ell(A)$. Then ${}^{\sigma^{-1}}\mu_\sigma\{e_1, e_2\}$ gives a basis for the free $E \otimes_\mathbb{Q} \mathbb{Q}_\ell$-module $V_\ell({}^\sigma A)$ of rank 2, since μ_σ's are E-equivariant. Hence $V_\ell(B_K)$ is freely generated over R by $\{e_1, e_2\}$. □

Thus $V_\ell(B_\alpha)$ is a free $E_\alpha \otimes \mathbb{Q}_\ell$-module of rank 2. Therefore $[E_\alpha : \mathbb{Q}] = \dim B_\alpha$. Now we consider 3 cases.

Case $D_\alpha = E_\alpha$. Then B_α is F-simple. We take $B = B_\alpha$ and A is a quotient of B over $\overline{\mathbb{Q}}$.

Case $D_\alpha \neq E_\alpha$ and D_α not split. Then D_α is a quaternion algebra over L_α. Then B_α is F-simple. We take $B = B_\alpha$ and A is a quotient of B over $\overline{\mathbb{Q}}$.

Case $D_\alpha \neq E_\alpha$ and D_α split. Then $D_\alpha \cong L_2(L_\alpha)$. B_α is F-isogenous to $(B)^2$ for some simple Abelian variety B and $\mathrm{End}^0 B \cong L_\alpha$. We check the degree of the endomorphism algebra: $[L_\alpha : \mathbb{Q}] = \frac{1}{2} \dim B_\alpha = \dim B'$. Thus A is a $\overline{\mathbb{Q}}$-quotient of a simple Abelian variety B of $\mathrm{GL}_2(L_\alpha)$-type. □

We record a corollary to the proof of Thm. 1.13. Suppose μ_σ are given isogenies from ${}^\sigma A$ to A. Let $c(\sigma, \tau) = \mu_\sigma {}^\sigma\mu_\tau \mu_{\sigma\tau}^{-1}$ be the associated 2-cocycle. Fix a choice of α that trivializes c. Let E_α denote the field constructed from E by adjoining values of α.

Corollary 1.15. *Let A be a simple F-virtual Abelian variety of $\mathrm{GL}_2(E)$-type. Then there exists an Abelian variety B_α/F that has A as an $\overline{F}$-factor, that has action by*

E_α and that is either simple or isogenous to the square of an F-simple Abelian variety of GL_2*-type.*

1.2.2 Case CM

Next we consider a CM simple Abelian variety $A/\overline{\mathbb{Q}}$. Suppose (A,θ) has type (E,Φ). Then it is known from [9] that the isogeny class of (A,θ) is defined over $E^\sharp$, the reflex field of E, i.e., there exist E-equivariant isogenies $\mu_\sigma : {}^\sigma A \to A$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E^\sharp)$.

Theorem 1.16. *Let A be as above. Then there exists a simple Abelian variety A' over a totally real field F of* GL_2*-type with A a $\overline{\mathbb{Q}}$-quotient.*

Proof. Let $F = E^{\sharp +}$ be the maximal totally real subfield of $E^\sharp$. We will construct a simple Abelian variety over F of GL_2-type with A a $\overline{\mathbb{Q}}$-quotient. First we construct a simple CM Abelian variety over F whose endomorphisms are all defined over F. We proceed as in the proof of Theorem 1.13. Find a model of A over K, a number field Galois over $E^\sharp$ such that all endomorphisms of A is defined over K and that all the μ_σ's are defined over K. Let $B = \mathrm{Res}_{K/E^\sharp} A$. Also define $c(\sigma,\tau) = \mu_\sigma {}^\sigma\mu_\tau \mu_{\sigma\tau}^{-1}$. Then c has values in $E^\times$ and we can trivialize c by $\alpha : \mathrm{Gal}(\overline{Q}/E^\sharp) \to \overline{E}^\times$. Again

$$\mathrm{End}^0 B \cong \prod \mathrm{Hom}^0({}^\sigma A, A) \cong \prod E \circ \mu_\sigma.$$

Similarly we find that $\mathrm{End}^0 B$ can be split into $\ker\omega \oplus E_\alpha$ where E_α is the field by adjoining E with values of α and $\omega : \mathrm{End}^0 B \to E_\alpha, \mu_\sigma \mapsto \alpha(\sigma)$. Hence corresponding to the decomposition of the endomorphism algebra we get an Abelian subvariety B_α of B . Consider $V_\ell(B)$ the Tate module of B. Similarly we find that $V_\ell(B)$ is a free $\prod(E \circ \mu_\sigma) \otimes \mathbb{Q}_\ell$-module of rank 1. Hence $V_\ell(B_\alpha)$ is a free $E_\alpha \otimes \mathbb{Q}_\ell$-module of rank 1. This shows that B_α has CM by E_α. Then we set $A' = \mathrm{Res}_{E^\sharp/F} B_\alpha$. Note that A' is F-simple. Otherwise $A' \sim A_1 \times A_2$ for some Abelian varieties A_1 and A_2 defined over F, so $A'_{E^\sharp} \sim A_{1E^\sharp} \times A_{2E^\sharp}$. Yet $A'_{E^\sharp} \cong B_\alpha \times {}^\iota B_\alpha$ where ι is the nontrivial element in $\mathrm{Gal}(E^\sharp/F)$. Since B_α and ${}^\iota B_\alpha$ are $E^\sharp$-simple, we find $A_{1E^\sharp} \sim B_\alpha$ or $A_{2E^\sharp} \sim B_\alpha$. However Proposition 1.4 says B_α cannot be descended to a totally real field. We get a contradiction. Thus A' is F-simple.

Now $\mathrm{End}^0 A' \cong \mathrm{Hom}^0({}^\iota B_\alpha, B_\alpha) \times \mathrm{End}^0 B_\alpha$ as $\mathbb{Q}$-vector spaces. As $E_\alpha \hookrightarrow \mathrm{End}^0 A'$ we find A' is a simple Abelian variety over F of $\mathrm{GL}_2(E_\alpha)$-type with $\overline{\mathbb{Q}}$-quotient A. $\square$

2 ℓ-adic Representations

In this section F denotes a totally real field. Let A be an Abelian variety over F of $\mathrm{GL}_2(E)$-type. Then the Tate module $V_\ell A$ is free of rank 2 over $E \otimes \mathbb{Q}_\ell$. Let G_F denote the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$. The action of G_F on $V_\ell A$ induces a homomorphism

$$\rho_\ell : G_F \to \mathrm{GL}_2(E \otimes \mathbb{Q}_\ell).$$

For each prime λ of E lying above ℓ if we set $V_\lambda A = V_\ell A \otimes_{E\otimes\mathbb{Q}_\ell} E_\lambda$ then the action of G_F defines a λ-adic reprensentation on $V_\lambda A$:

$$\rho_\lambda : G_F \to \mathrm{GL}_2(E_\lambda)$$

We can also associate Galois representations to an F-virtual Abelian variety. Let B be an F-virtual Abelian variety of $\mathrm{GL}_2(E)$-type defined over $\overline{\mathbb{Q}}$. Define $\rho'_\ell(\sigma).P := \mu_\sigma({}^\sigma P)$ for P in $V_\ell B$ and σ in G_F. This is not an action, since

$$\rho'_\ell(\sigma)\rho'_\ell(\tau)P = \mu_\sigma{}^\sigma\mu_\tau{}^{\sigma\tau}P.$$

The obstruction is given by $c(\sigma,\tau) := \mu_\sigma{}^\sigma\mu_\tau\mu_{\sigma\tau}^{-1}$. This is the same c we constructed in the proof of Thm. 1.13. We know that c is a 2-cocycle on G_F with values in $E^\times$, in fact, in $L^\times$, if we let L denote the centre of the endomorphism algebra of B. Consider the class of c in $H^2(G_F, \overline{E}^\times)$. As is shown by Tate, $H^2(G_F, \overline{E}^\times) = 0$. Thus c can be trivialized:

$$c(\sigma,\tau) = \frac{\alpha(\sigma)\alpha(\tau)}{\alpha(\sigma\tau)}$$

by some locally constant map $\alpha : G_F \to \overline{E}^\times$. Then set

$$\rho_\ell(\sigma)P := \alpha^{-1}(\sigma)\mu_\sigma{}^\sigma P.$$

This gives an action of G_F on $V_\ell B$. We get a homomorphism

$$\rho_\ell : G_F \to \overline{E}^\times \mathrm{GL}_2(E \otimes \mathbb{Q}_\ell).$$

More precisely, ρ_ℓ actually factors through $E_\alpha^\times \mathrm{GL}_2(E \otimes \mathbb{Q}_\ell)$ where E_α denote the subfield of $\overline{\mathbb{Q}}$ generated over E by the values of α. If we fix a choice of α then to simplify notation we write E' for E_α. Let λ' be a prime of E' lying above λ a prime of E which in turn lies above ℓ. Then the representation ρ_ℓ gives rise to:

$$\rho_{\lambda'} : G_F \to E'^\times_{\lambda'} \mathrm{GL}_2(E_\lambda).$$

We can view $\rho_{\lambda'}$ as a representation of G_F on $V_\lambda \otimes_{E_\lambda} E'_{\lambda'}$.

Proposition 2.1. *Let A be a simple F-virtual Abelian variety of* $\mathrm{GL}_2(E)$*-type. Fix a choice of α and let A'/F denote the Abelian variety B_α as in Cor. 1.15. Then the field E' generated over E by the values of α acts up to isogeny on A'. Let λ' be a prime of E'. Associate the representations $\rho_{A,\lambda'}$ and $\rho_{A',\lambda'}$ to A and A' respectively. Then $\rho_{A,\lambda'} \cong \rho_{A',\lambda'}$ as $E'_{\lambda'}[G_F]$-module.*

Proof. Note that μ_σ's correspond to $\alpha(\sigma)$'s in the endomorphism algebra of A'. The proposition follows from the definiton of the λ'-adic representations. $\square$

Because of this proposition we will focus on the Galois representations associated to Abelian varieties defined over F. We record some properties of the λ-adic representations.

Definition 2.2. Let E be a totally real field or CM field. A polarized Abelian variety of $\mathrm{GL}_2(E)$-type is an Abelian variety of $\mathrm{GL}_2(E)$-type with a polarization that is compatible with the canonical involution on E.

Remark 2.3. The canonical involution is just identity if E is totally real.

Proposition 2.4. *If A is a polarized simple F-virtual Abelian variety of* $\mathrm{GL}_2(E)$*-type then the associated Abelian variety A'/F can also be endowed with a polarization compatible with E'.*

Proof. Let L' be the centre of the endomorphism algebra of A'. Then by construction E' is actually the composite of L' and E and hence is totally real or CM as L' is totally real by Prop. 2.12 and Prop. 2.14 (whose proofs depends solely on the analysis on polarized Abelian varieties over F and do not depend on this proposition) and E either CM or totally real. □

Let A be a polarized Abelian variety over F of $\mathrm{GL}_2(E)$-type. Let V_ℓ denote $V_\ell A$. Then $V_\ell = \oplus_{\lambda|\ell} V_\lambda$ corresponding to the decomposition of $E \otimes \mathbb{Q}_\ell = \oplus E_\lambda$ where λ's are primes of E lying above ℓ. We get the λ-adic representations ρ_λ of G_F on V_λ. The set of ρ_λ's for all λ forms a family of strictly compatible system of E-rational representations[7].

Let δ_λ denote $\det \rho_\lambda$ and χ_ℓ the ℓ-adic cyclotomic character.

Lemma 2.5. *There exists a character of finite order $\varepsilon : G_F \to E^*$ such that $\delta_\lambda = \varepsilon\chi_\ell$. Furthermore ε is unramified at primes which are primes of good reduction for A.*

Remark 2.6. ε is trivial if E is totally real, as can be seen from Prop. 2.9.

Proof. Since the δ_λ's arise from an Abelian variety, they are of the Hodge-Tate type. They are associated with an E-valued Grossencharacter of type A_0 of F. Thus they have to be of the form $\delta_\lambda = \varepsilon\chi_\ell^n$ for some E-valued character of finite order.

By the criterion of Néron-Ogg-Shafarevich ρ_λ is unramified at primes of F which are primes of good reduction for A and which do not divide ℓ. Then δ_λ is also unramified at those primes. Since χ_ℓ is unramified at primes not dividing ℓ, ε is unramified at primes of F which are primes of good reduction for A and which do not divide ℓ. Let ℓ vary and we find that ε is unramified at primes of F which are primes of good reduction for A.

Now consider the representation of G_F on $\det_{\mathbb{Q}_\ell} V_\ell$. It is known to be given by $\chi_\ell^{\dim A}$ which is equal to $\chi_\ell^{[E:Q]}$ since A is of $\mathrm{GL}_2(E)$-type. On the other hand it is also equal to $\prod_{\lambda|\ell} \mathbf{N}_{E_\lambda/\mathbb{Q}_\ell} \delta_\lambda = \mathbf{N}_{E/\mathbb{Q}}\, \varepsilon.\chi_\ell^{n[E:\mathbb{Q}]}$. Since χ_ℓ has infinite order and ε has finite order we are forced to have $n = 1$ and $\mathbf{N}_{E/\mathbb{Q}}\, \varepsilon = 1$. □

Lemma 2.7. *The character δ_λ is odd, i.e., δ_λ sends all complex conjugations to -1.*

Proof. For each embedding of fields $\overline{\mathbb{Q}} \to \mathbb{C}$ we have a comparison isomorphism $V_\lambda \cong H_1(A(\mathbb{C}), \mathbb{Q}) \otimes_E E_\lambda$. Via this isomorphism complex conjugation acts as $F_\infty \otimes 1$ where F_∞ comes from the action of complex conjugation on $A(\mathbb{C})$ by transport of structure. We need to show that $\det F_\infty$ is -1 where det is taken with respect to the E-linear action of F_∞ on $H_1(A(\mathbb{C}), \mathbb{Q})$.

Note that $H_1(A(\mathbb{C}), \mathbb{Q})$ is of dimension 2 over E. Since F_∞ is an involution on $H_1(A(\mathbb{C}), \mathbb{Q})$ its determinant is 1 if and only if F_∞ acts as a scalar. Since F_∞

permutes $H_{0,1}$ and $H_{1,0}$ in the Hodge decomposition of $H_1(A(\mathbb{C}), \mathbb{Q})$ it obviously does not act as a scalar. Thus $\det F_\infty$ is -1. $\square$

Proposition 2.8. *For each λ, ρ_λ is an absolutely irreducible 2-dimensional representation of G_F over E_λ and $\mathrm{End}_{E_\lambda[G_F]} V_\lambda = E_\lambda$.*

Proof. By Faltings's results V_ℓ is a semisimple G_F-module and $\mathrm{End}_{E\otimes\mathbb{Q}_\ell[G_F]} V_\ell = E\otimes\mathbb{Q}$. Thus corresponding to the decomposition of $V_\ell = \oplus_{\lambda|\ell} V_\lambda$ we have $\mathrm{End}_{E_\lambda[G_F]} V_\lambda = E_\lambda$. This shows V_λ is simple over E_λ. Hence ρ_λ is absolutely irreducible. $\square$

For prime v of F at which A has good reduction let $a_v = \mathrm{tr}_{E_\lambda}(\mathrm{Frob}_v |_{V_\lambda})$ if $v \nmid \ell$ for ℓ lying below λ. Let ι be the canonical involution on E if E is CM and identity if E is totally real.

Proposition 2.9. *For each place v of good reduction $a_v = \iota(a_v)\varepsilon(\mathrm{Frob}_v)$.*

Proof. For each embedding σ of E into $\overline{\mathbb{Q}}_\ell$ denote by $V_\sigma := V_\ell \otimes_{E\otimes\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ where the algebra homomorphism $E \otimes \mathbb{Q}_\ell \to \overline{\mathbb{Q}}_\ell$ is induced by σ.

Fix a polarization of A that is compatible with ι. We have a Weil pairing:

$$\langle\ ,\ \rangle : V_\ell \times V_\ell \to \mathbb{Q}_\ell(1)$$

such that $\langle ex, y\rangle = \langle x, \iota(e)y\rangle$ for e in E. This pairing is G_F-equivariant, i.e.,$\langle {}^\tau x, {}^\tau y\rangle = {}^\tau\langle x, y\rangle$ for all $\tau \in G_F$.

After extending scalar to $\overline{\mathbb{Q}}_\ell$ we get pairings between the spaces V_σ and $V_{\sigma\iota}$. Thus we have isomorphism of $\overline{\mathbb{Q}}_\ell[G_F]$-modules

$$V_{\sigma\iota} \cong \mathrm{Hom}(V_\sigma, \overline{\mathbb{Q}}_\ell(1)).$$

On $\det V_\sigma$, G_F acts by ${}^\sigma\varepsilon\chi_\ell$. As V_σ is 2-dimensional over $\overline{\mathbb{Q}}_\ell$

$$\mathrm{Hom}(V_\sigma, \overline{\mathbb{Q}}_\ell({}^\sigma\varepsilon\chi_\ell)) \cong V_\sigma$$

as G_F-modules. Thus we find that $V_{\sigma\iota}({}^\sigma\varepsilon) \cong V_\sigma$. As $\mathrm{tr}\,\mathrm{Frob}_v$ is $\sigma(a_v)$ on V_σ and is $\sigma\iota(a_v){}^\sigma\varepsilon(\mathrm{Frob}_v)$ on $V_{\sigma\iota}({}^\sigma\varepsilon)$ for v place of good reduction and v prime to ℓ we get

$$\sigma(a_v) = \sigma\iota(a_v){}^\sigma\varepsilon(\mathrm{Frob}_v).$$

It follows $a_v = \iota(a_v)\varepsilon(\mathrm{Frob}_v)$. $\square$

Corollary 2.10. *ε is trivial when E is totally real.*

Proposition 2.11. *Let S be a finite set containing all the places of bad reduction. Then E is generated over $\mathbb{Q}$ by the a_v's with $v \notin S$.*

Proof. Again we consider the V_σ's corresponding to embeddings of E into $\overline{\mathbb{Q}}_\ell$ as in the proof of Prop. 2. As V_ℓ is semisimple, so is $V_\ell \otimes \overline{\mathbb{Q}}_\ell$. Since $V_\ell \otimes \overline{\mathbb{Q}}_\ell = \oplus_\sigma V_\sigma$ we get $\mathrm{End}_{E\otimes\overline{\mathbb{Q}}_\ell[G_F]}(\oplus_\sigma V_\sigma) = E \otimes \overline{\mathbb{Q}}_\ell = \prod_\sigma \overline{\mathbb{Q}}_\ell$. This shows that the V_σ's are simple and pairwise nonisomorphic. Thus their traces are pairwise distinct. Since the trace of Frob_v acting on V_σ is $\sigma(a_v)$ for v place of good reduction, by Cebotarev Density theorem the embeddings σ are pairwise distinct when restricted to the set of a_v's for $v \notin S$. Thus E is generated over $\mathbb{Q}$ by the set of a_v's for $v \notin S$. $\square$

Proposition 2.12. *Let L be the subfield of E generated by $a_v^2/\varepsilon(\mathrm{Frob}_v)$ for $v \notin S$. Then L is a totally real field and E/L is Abelian.*

Proof. We check

$$\iota(\frac{a_v^2}{\varepsilon(\mathrm{Frob}_v)}) = \frac{\iota(a_v)^2}{\iota(\varepsilon(\mathrm{Frob}_v))} = a_v^2\varepsilon^{-2}(\mathrm{Frob}_v)\varepsilon(\mathrm{Frob}_v) = \frac{a_v^2}{\varepsilon(\mathrm{Frob}_v)}.$$

Thus L is totally real.

Since E is contained in the extension of L obtained by adjoining the square root of all of the $a_v^2/\varepsilon(\mathrm{Frob}_v)$'s and all roots of unity, E is an Abelian extension of L. $\square$

Now we consider the reductions of ρ_λ. Replace A by an isogenous Abelian variety so that $\mathcal{O}_E$ actually acts on A. Consider the action of G_F on the λ-torsion points $A[\lambda]$ of A and we get a 2-dimensional representation $\bar{\rho}_\lambda$ of G_F over $\mathbb{F}_\lambda$, the residue field at λ.

Lemma 2.13. *For almost all λ, the representation $\bar{\rho}_\lambda$ is absolutely irreducible.*

Proof. A result of Faltings implies that for almost all λ's $A[\lambda]$ is a semisimple $\mathbb{F}_\lambda[G_F]$-module whose commutant is $\mathbb{F}_\lambda$. The lemma follows immediately. $\square$

Proposition 2.14. *Let A/F be a simple Abelian variety of $\mathrm{GL}_2(E)$-type. Let D denote the endomorphism algebra of $A_{\overline{\mathbb{Q}}}$ and L its centre. Then L is generated over $\mathbb{Q}$ by $a_v^2/\epsilon(\mathrm{Frob}_v)$.*

Proof. Let ℓ be a prime that splits completely in E. Then all embeddings of E into $\overline{\mathbb{Q}_\ell}$ actually factors through $\mathbb{Q}_\ell$. Suppose the isogenies in D are defined over a number field K. Fix an embedding of K into $\overline{\mathbb{Q}}$. Let H denote the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ which is an open subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Shrink H if necessary so that H is contained in the kernel of ϵ. By a result of Faltings's, $D \otimes \mathbb{Q}_\ell \cong \mathrm{End}_{\mathbb{Q}_\ell[H]} V_\ell$. The centre of $D \otimes \mathbb{Q}_\ell$ is $L \otimes \mathbb{Q}_\ell$. By our choice of ℓ, the Tate module V_ℓ decomposes as $\oplus_\sigma V_\sigma$ where σ runs over all embedding of E into $\mathbb{Q}_\ell$ and where $V_\sigma := V_\ell \otimes_{E\otimes\mathbb{Q}_\ell} \mathbb{Q}_\ell$. Note that $\mathbb{Q}_\ell$ is viewed as an $E \otimes \mathbb{Q}_\ell$-module via σ. Each V_σ is a simple $\mathbb{Q}_\ell[H]$-module. Thus $\mathrm{End}_{\mathbb{Q}_\ell[H]} V_\sigma = \mathbb{Q}_\ell$.

For each prime w of K prime to ℓ and not a prime of bad reduction of A, we have a trace of Frobenius at w associated to the λ-adic representations. Denote it by b_w and it is in E. Then $\mathrm{tr}\,\mathrm{Frob}_w |V_\sigma = \sigma(b_w)$. The $\mathbb{Q}_\ell[H]$-modules V_σ and V_τ are isomorphic if and only if $\sigma(b_w) = \tau(b_w)$ for all w. If we let L' denote the field generated over $\mathbb{Q}$ by the b_w's, then the $\mathbb{Q}_\ell[H]$-modules V_σ and V_τ are isomorphic if and only if $\sigma|_{L'} = \tau|_{L'}$. Thus the centre of $D \otimes \mathbb{Q}_\ell$ is isomorphic to $L' \otimes \mathbb{Q}_\ell$. We have $L \otimes \mathbb{Q}_\ell = L' \otimes \mathbb{Q}_\ell$ with equality taken inside $E \otimes \mathbb{Q}_\ell$. Thus $L = L'$.

Now suppose that σ and τ agree on L so that $V_\sigma \cong V_\tau$ as $\mathbb{Q}_\ell[H]$-modules. There is a character $\varphi : G_F \to \mathbb{Q}_\ell^\times$ such that $V_\sigma \cong V_\tau \otimes \varphi$ as $\mathbb{Q}_\ell[G_F]$-modules. Taking traces we get $\sigma(a_v) = \varphi(\mathrm{Frob}_v)\tau(a_v)$ for all primes v of F which are prime to ℓ and not primes of bad reduction of A. Taking determinants we get ${}^\sigma\epsilon = \varphi^2\,{}^\tau\epsilon$. Thus σ and τ agree on a_v^2/ϵ for all good v prime to ℓ. This shows that a_v^2/ϵ is in L.

On the other hand, suppose $\sigma(a_v^2/\epsilon) = \tau(a_v^2/\epsilon)$ for all good v prime to ℓ. This implies by Chebotarev density theorem that $\mathrm{tr}^2/\det$ are the same for the representations of G_F on V_σ and V_τ. Since ϵ is trivial on H we get $\mathrm{tr}(h|V_\sigma) = \pm\,\mathrm{tr}(h|V_\tau)$. If we choose H small enough we will have $\mathrm{tr}(h|V_\sigma) = \mathrm{tr}(h|V_\tau)$. Then $V_\sigma \cong V_\tau$ as $\mathbb{Q}_\ell[H]$-modules. Thus L is contained in the field generated over $\mathbb{Q}$ by the a_v^2/ϵ's. $\square$

Corollary 2.15. *Let A be a polarized simple F-virtual Abelian variety of $\mathrm{GL}_2(E)$-type. Let L be the centre of $\mathrm{End}^0(A)$. Then L is generated over $\mathbb{Q}$ by $a_v^2/\epsilon(\mathrm{Frob}_v)$. In particular L is totally real.*

Proof. By 1.13 there exists a simple Abelian variety B/F of GL_2-type such that A is a $\overline{\mathbb{Q}}$-factor. The centre of the endomorphism algebra of A is just the centre of the endormorphism algebra of B by Prop 1.12. The corollary follows from the proposition. That L is totally real is established in Proposition 2.12. $\square$

Proposition 2.16. *Suppose that 3 is totally split in E and suppose for $\lambda|3$ the representation $\bar{\rho}_\lambda$ is irreducible. Then $\bar{\rho}_\lambda$ is modular.*

Proof. In this case we have $\bar{\rho}_\lambda : G_F \to \mathrm{GL}_2(\mathbb{F}_3)$. We have already shown that $\bar{\rho}_\lambda$ is odd. Then the proposition follows from the theorem of Langlands and Tunnel. $\square$

We record a modularity result first.

Theorem 2.17 (Shepherd-Barron, Taylor)**.** *Let ℓ be 3 or 5 and let F be a field of characteristic different from ℓ. Suppose $\rho : G_F \to \mathrm{GL}_2(\mathbb{F}_\ell)$ is a representation such that $\det(\rho) = \chi_\ell$. Then there is an elliptic curve C defined over F such that $\rho \cong \bar{\rho}_{C,\ell}$.*

Then from our analysis we get:

Corollary 2.18. *Let ℓ be 3 or 5. Suppose that E is totally real and that ℓ is totally split in E. Then for $\lambda|\ell$ there exists an elliptic curve C/F such that $\bar{\rho}_{A,\lambda} \cong \bar{\rho}_{C,\ell}$.*

Proof. Since E is totally real, $\det(\rho) = \chi_\ell$. Then all conditions in the quoted theorem is satisfied. $\square$

3 Moduli of F-virtual Abelian Varieties of GL_2-type

Now we consider the moduli space of F-virtual Abelian Varieties of GL_2-type. Roughly speaking, in the construction we will produce trees whose vertices are such Abelian varieties and whose edges represent isogenies. Then via graph theoretic properties of the trees, we can locate a nice Abelian variety that is isogenous to a given F-virtual Abelian Varieties of GL_2-type and that is represented by F-points on certain quotients of Shimura varieties. We exclude the CM case and consider the non-quaternion and quaternion cases separately.

3.1 Case $\mathrm{End}^0(A) \cong E$

Let $\mathcal{A}$ be the category of Abelian varieties over $\overline{\mathbb{Q}}$. Let $\mathcal{A}_0$ be the category of Abelian varieties over $\overline{\mathbb{Q}}$ up to isogeny. We consider the subcategory $\mathcal{B}$ of $\mathcal{A}$ defined as follows. The objects are pairs (A, ι) where A is an Abelian variety over $\overline{\mathbb{Q}}$ of dimension $[E : \mathbb{Q}]$ and $\iota : \mathcal{O}_E \to \mathrm{End}(A)$ is a ring isomorphism. The morphisms $\mathrm{Mor}_{\mathcal{B}}((A_1, \iota_1), (A_2, \iota_2))$ are those homomorphisms in $\mathrm{Hom}(A_1, A_2)$ that respect $\mathcal{O}_E$-action and is denoted by $\mathrm{Hom}_{\mathcal{O}_E}(A_1, A_2)$. Also define the categories $\mathcal{B}_\lambda$, where λ is a prime of $\mathcal{O}_E$, as follows. The objects are the same as in $\mathcal{B}$. The morphisms $\mathrm{Mor}_{\mathcal{B}_\lambda}((A_1, \iota_1), (A_2, \iota_2))$ are $\mathrm{Hom}_{\mathcal{O}_E}(A_1, A_2) \otimes_{\mathcal{O}_E} \mathcal{O}_{E,(\lambda)}$, where $\mathcal{O}_{E,(\lambda)}$ denotes the ring $\mathcal{O}_E$ localized at λ (but not completed). If no confusion arises the ι's will be omitted to simplify notation. Most often we will work in the category $\mathcal{B}$.

Let f be a morphism in $\mathrm{Mor}_{\mathcal{B}_\lambda}(A, B)$. Then f can be written as $g \otimes (1/s)$ with g in $\mathrm{Hom}(A, B)$ and s in $\mathcal{O}_E \setminus \lambda$. Define the kernel of f, $\ker f$ to be the λ part of the kernel of g in the usual sense, $(\ker g)_\lambda$. Note here kernel is not in the sense of category theory. Suppose that we write f in another way by $g' \otimes (1/s')$. Then consider $f = gs' \otimes (1/ss')$ and we find $(\ker gs')_\lambda = (\ker g)_\lambda$ as s' is prime to λ and also $(\ker g's)_\lambda = (\ker g')_\lambda$. Since $g's = gs'$ we find our kernel well-defined.

We construct a graph associated to each $\mathcal{B}_\lambda$. Let the vertices be the isomorphism classes of $\mathcal{B}_\lambda$ modulo the relation $\approx$, where $A \approx B$ if there exists a fractional ideal $\mathfrak{A}$ of E such that

$$A \otimes_{\mathcal{O}_E} \mathfrak{A} \cong B$$

Denote the vertex associated to A by $[A]$ and connect $[A]$ and $[B]$ if there exists $f \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A, B)$ such that

$$\ker f \cong \mathcal{O}_E/\lambda^{r+1} \oplus \mathcal{O}_E/\lambda^r$$

for some r. If we change A to $A/A[\lambda] = A \otimes_{\mathcal{O}_E} \lambda^{-1}$ then the kernel becomes $\mathcal{O}_E/\lambda^r \oplus \mathcal{O}_E/\lambda^{r-1}$ and as we quotient out more we will get to $\mathcal{O}_E/\lambda$. Obviously there is an $f' \in \mathrm{Mor}_{\mathcal{B}_\lambda}(B, A)$ such that $\ker f' \cong \mathcal{O}_E/\lambda^{s+1} \oplus \mathcal{O}_E/\lambda^s$ for some s.

Lemma 3.1. *Suppose that two vertices $[A]$ and $[B]$ can be connected by a path of length n. Then there exists $f \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A, B)$ such that $\ker f \cong \mathcal{O}_E/\lambda^n$ for some representatives A and B.*

Proof. For $n = 1$ the lemma is true. For $n = 2$ suppose we have $[A]$ connected to $[A_1]$ and then $[A_1]$ to $[B]$ and suppose we have $f \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A, A_1)$ and $g \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A_1, B)$ with $\ker f \cong \mathcal{O}_E/\lambda$ and $\ker g \cong \mathcal{O}_E/\lambda$. Then if $\ker g \circ f \cong \mathcal{O}_E/\lambda \oplus \mathcal{O}_E/\lambda$ we find $B \approx A$ and thus $[A] = [B]$. Then $[A]$ and $[A_1]$ are connected by two edges. However by the construction there is at most one edge between two vertices. Thus we must have $\ker g \circ f \cong \mathcal{O}_E/\lambda^2$.

Now suppose the lemma holds for all paths of length $n - 1$. Suppose that $[A]$ and $[B]$ are connected via $[A_1], \cdots, [A_{n-1}]$ and that we have morphisms $A \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-2}} A_{n-1} \xrightarrow{f_{n-1}} B$ where all kernels are isomorphic to $\mathcal{O}_E/\lambda$. Then $f = f_{n-2} \circ \cdots \circ f_0$ and $g = f_{n-1} \circ \cdots \circ f_1$ have kernel isomorphic to $\mathcal{O}_E/\lambda^{n-1}$. If $g \circ f_0$ has kernel isomorphic to $\mathcal{O}_E/\lambda^{n-1} \oplus \mathcal{O}_E/\lambda$ then f must have kernel isomorphic to $\mathcal{O}_E/\lambda^{n-2} \oplus \mathcal{O}_E/\lambda$ and we get a contradiction. Thus $g \circ f_0$ has kernel isomorphic to $\mathcal{O}_E/\lambda^n$. This concludes the proof. □

Proposition 3.2. *Each connected component of the the graph is a tree.*

Proof. We need to show that there is no loop. Suppose there is one. By the previous lemma we have a morphism $f \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A, A')$ for some A' in the same $\approx$-equivalence class as A such that $\ker f \cong \mathcal{O}_E/\lambda^n$. This means we have a morphism $f \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A, A)$ with kernel isomorphic to $\mathcal{O}_E/\lambda^{n+r} \oplus \mathcal{O}_E/\lambda^r$ for some r. This is impossible as any isogeny in $\mathrm{End}_{\mathcal{O}_E}(A)$ has kernel isomorphic to $(\mathcal{O}_E/\lambda^s)^2$ for some s. Thus there is no loop. □

Definition 3.3. For an Abelian variety A and for each λ the tree that contains the vertex associated to A is called the λ-local tree associated to A.

Obviously the Galois group G_F acts on the graph. If a connect component has a vertex coming from an F-virtual Abelian variety, then automatically all vertices in this component come from F-virtual Abelian varieties. Also G_F preserves this component and we get another characterization of F-virtual Abelian varieties.

Lemma 3.4. *The G_F-orbit of an F-virtual Abelian variety is contained in the same λ-local tree.*

Now let A_0 be an F-virtual Abelian variety in $\mathcal{B}$. If considered as an object in $\mathcal{B}_\lambda$, it is mapped to a vertex in its λ-local tree. Also for all $\sigma \in G_F$ the vertex $[{}^\sigma A_0]$ is in the same λ-local tree. A priori, the Galois orbit of the vertex associated to A_0 is hard to describe. However for some special vertex in the tree the Galois orbit is essentially contained in the Atkin-Lehner orbit which we will describe below.

Definition 3.5. For a finite subset S of vertices of a tree the centre is the central edge or central vertex on any one of the longest paths connecting two vertices in S.

Remark 3.6. There are possibly more than one longest path, but they give the same centre. Thus the centre is well-defined.

Definition 3.7. The centre associated to A_0 in each local tree is defined to be the centre of the image of the set $\{{}^\sigma A_0 : \sigma \in G_F\}$ in the tree.

Obviously we have:

Proposition 3.8. *The associated centre of A_0 is fixed under the action of G_F. (Note if the centre is an edge it can be flipped.) Furthermore the vertices in the image of $\{{}^\sigma A_0 : \sigma \in G_F\}$ are at the same distance to the nearer vertex of the centre.*

Proposition 3.9. *The set of central edges associated to an F-virtual Abelian variety A of $\mathrm{GL}_2(E)$-type such that $\mathrm{End}^0 A \cong E$ is an E-linear isogeny invariant.*

Proof. Suppose in the λ-local tree the centre associated to A_0 is an edge. Then there is an element in G_F that exchanges the two vertices connected by the edge. Otherwise all Galois conjugates of A_0 will be on one side of the edge, contrary to the fact that this edge is central.

Once we have a fixed edge which is flipped under Galois action there can be no fixed vertices or other fixed edges in the tree. We take an Abelian variety B_0 which is E-linearly isogenous to A_0. Then the centre associated to B_0 is also fixed under Galois action and hence must be an edge. Furthermore it must coincide with the central edge associated to A_0. Thus central edges are E-linear isogeny invariants. $\square$

Remark 3.10. Central vertices are not necessarily isogeny invariants. For example we can just take an Abelian variety A over F such that $\mathrm{End}^0 A = \mathcal{O}_E$ and take $B = A/C$ where C is a subgroup of A isomorphic to $\mathcal{O}_E/\lambda$. Then obviously the central vertex for A is $[A]$ and for B it is $[B]$ and they are not the same by construction.

Let Σ be the set of primes where the centre is an edge.

Lemma 3.11. *The set Σ is a finite set.*

Proof. The Abelian varieties ${}^{\sigma}A_0$ for σ in G_F end up in the same equivalence class as long as λ does not devide the degree of the isogenies μ_σ's between the Galois conjugates. Thus there are only finitely many λ's where the associated centre can be an edge. $\square$

Remark 3.12. Also we note that for almost all λ's, $[A_0]$ is just its own associated central vertex.

For each central edge we choose one of the vertices and for central vertices we just use the central vertices. Then these vertices determine an Abelian variety up to $\approx$. Recall that $A \approx B$ if $A \otimes_{\mathcal{O}_E} \mathfrak{A}$ is isomorphic to B for some fractional ideal $\mathfrak{A}$.

Consider the Hilbert modular variety $Y_0(\mathfrak{n})$ classifying Abelian varieties with real multiplication E with level structure $K_0(\mathfrak{n})$ where $\mathfrak{n}$ is the product of the prime in Σ.

$$Y_0(\mathfrak{n}) := \mathrm{GL}_2(E) \setminus \mathcal{H}^n \times \mathrm{GL}_2(\widehat{E})/K_0(\mathfrak{n})$$

First we have $\mathrm{Pic}\,\mathcal{O}_E$ acting on $Y_0(\mathfrak{n})$ which is defined by:

$$\rho(\mathfrak{m}) : (A \to B) \mapsto (A \otimes_{\mathcal{O}_E} \mathfrak{m} \to B \otimes_{\mathcal{O}_E} \mathfrak{m}).$$

for $\mathfrak{m} \in \mathrm{Pic}(\mathcal{O}_E)$. This corresponds to $\approx$.

On the quotient $\mathrm{Pic}(\mathcal{O}_E) \setminus Y_0(\mathfrak{n})$ we have the Atkin-Lehner operators w_λ for $\lambda|\mathfrak{n}$ defined as follows. Suppose we are given a point on $\mathrm{Pic}(\mathcal{O}_E) \setminus Y_0(\mathfrak{n})$ represented by $(A \xrightarrow{f} B)$ and let C denote the kernel of f. Then w_λ sends this point to

$$(A/C[\lambda] \to B/f(A[\lambda])).$$

This operation of w_{λ_0} when viewed on the λ-tree sends one vertex on the central edge to the other one if $\lambda = \lambda_0$ or does nothing if $\lambda \neq \lambda_0$.

Definition 3.13. Let W be the group generated by w_λ's and let $\widetilde{W}$ be the group $W \ltimes \mathrm{Pic}(\mathcal{O}_E)$ acting on $Y_0(\mathfrak{n})$.

Thus we have:

Proposition 3.14. *The G_F-orbit of any Abelian variety coming the central vertices is contained in the $\widetilde{W}$-orbit.*

Denote $\widetilde{W} \setminus Y_0(\mathfrak{n})$ by $Y_0^+(\mathfrak{n})$. Then we have associated to A_0 a point in $Y_0^+(\mathfrak{n})(\overline{\mathbb{Q}})$. As this point is fixed by G_F this is actually an F-point.

On the other hand take an F-rational point of $Y_0^+(\mathfrak{n})$ and we get a set of Abelian varieties in $Y_0(\mathfrak{n})$ that lie above it. They are all isogenous. Take any one of them, say A. Then its Galois conjugates are still in the set and they are E-linearly isogenous to A by construction. We get an F-virtual Abelian variety of $\mathrm{GL}_2(E)$-type. However we cannot rule out the possibility that it may have larger endomorphism algebra than E.

We have shown

Theorem 3.15. *Every F-point on the Hilbert modular variety $Y_0^+(\mathfrak{n})$ gives an F-virtual Abelian variety of $\mathrm{GL}_2(E)$-type. Conversely for any F-virtual Abelian variety A of $\mathrm{GL}_2(E)$-type there is an isogenous F-virtual Abelian variety of $\mathrm{GL}_2(E)$-type A' which corresponds to an F-rational point on a Hilbert modular variety of the form $Y_0^+(\mathfrak{n})$ where $\mathfrak{n}$ is given as in the tree construction above.*

3.2 Case $\mathrm{End}^0 A = D$

We cannot simply follow Case $\mathrm{End}^0 A = E$, since we will get a graph with loop in that way. We will construct local trees in a slightly different way. Otherwise everything is the same as in Case $\mathrm{End}^0 A = E$. Let L denote the centre of D and $\mathcal{O}_D$ a maximal order of D containing $\mathcal{O}_E$.

Let $\mathcal{B}$ be the subcategory of $\mathcal{A}$ defined as above except for the requirements that $\iota : \mathcal{O}_D \to \mathrm{End}(A)$ is a ring isomorphism and that morphisms should respect $\mathcal{O}_D$-action. For the definition of $\mathcal{B}_\lambda$ we divide into 2 cases.

If D does not split at λ, let $\mathfrak{Q}$ be the prime of $\mathcal{O}_D$ lying above λ. We have $\mathfrak{Q}^2 = \lambda$. Then define $\mathrm{Mor}_{\mathcal{B}_\lambda}(A_1, A_2)$ to be $\mathrm{Hom}_{\mathcal{O}_D}(A_1, A_2) \otimes_{\mathcal{O}_D} \mathcal{O}_{D,(\mathfrak{Q})}$. Define the kernel of $f \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A_1, A_2)$ to be the λ-part of the usual kernel of g in some decomposition of $f = g \otimes (s^{-1})$. The equivalence relation $\approx$ on objects of $\mathcal{B}_\lambda$ is given as follows: $A \approx B$ if and only if there exists some $f \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A, B)$ such that

$$\ker f \cong \mathcal{O}_D/\lambda^r \mathcal{O}_D$$

for some r. Connect $[A]$ and $[B]$ if there exists $f \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A, B)$ such that

$$\ker f \cong \mathcal{O}_D/\mathfrak{Q} \oplus \mathcal{O}_D/\lambda^r \mathcal{O}_D$$

for some r.

If D splits at λ, then define $\mathrm{Mor}_{\mathcal{B}_\lambda}(A_1, A_2)$ to be $\mathrm{Hom}_{\mathcal{O}_D}(A_1, A_2) \otimes_{\mathcal{O}_L} \mathcal{O}_{L,(\lambda)}$. Still define the kernel of $f \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A_1, A_2)$ to be the λ-part of the usual kernel of

g in some decomposition of $f = g \otimes (s^{-1})$. The equivalence relation $\approx$ on objects of $\mathcal{B}_\lambda$ is given as follows: $A \approx B$ if and only if there exists some $f \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A, B)$ such that

$$\ker f \cong \mathrm{M}_2(\mathcal{O}_L/\lambda^r \mathcal{O}_L)$$

for some r. Connect $[A]$ and $[B]$ if there exists $f \in \mathrm{Mor}_{\mathcal{B}_\lambda}(A_1, A_2)$ such that

$$\ker f \cong (\mathcal{O}_L/\lambda)^2 \oplus \mathrm{M}_2(\mathcal{O}_L/\lambda^r \mathcal{O}_L)$$

for some r.

Now we follow what we have done in the previous subsection. Similarly we get

Proposition 3.16. *Each connected component of the graph is a tree.*

Proposition 3.17. *The set of central edges associated to an F-virtual Abelian variety A of $\mathrm{GL}_2(E)$-type such that $\mathrm{End}^0(A) \cong D$ is a D-linear isogeny invariant.*

Let Σ denote the set of primes at which the associated centre is an edge. Note that Σ is again a finite set.

Proposition 3.18.

Now we consider the Shimura variety S_Σ parametrizing the quadruples $(A, *, \iota, C)$ where the level structure C is isomorphic to

$$(\oplus_{\substack{\lambda\in\Sigma \\ \lambda\nmid\mathrm{disc}(D)}} (\mathcal{O}_L/\lambda)^2) \oplus (\oplus_{\substack{\lambda\in\Sigma \\ \lambda\mid\mathrm{disc}(D)}} \mathcal{O}_{D_\lambda}/\mathfrak{Q}_\lambda).$$

Let $\widetilde{W}$ be the group acting on S_Σ generated by

$$w_\lambda : (A, *, \iota, C) \mapsto (A/C[\lambda], *', \iota', C + A[\lambda]/C[\lambda]) \quad \text{if } \lambda \in \Sigma \text{ and } \lambda \nmid \mathrm{disc}(D);$$
$$w_\lambda : (A, *, \iota, C) \mapsto (A/C[\mathfrak{Q}_\lambda], *', \iota', C/C[\mathfrak{Q}_\lambda] + (A/C[\mathfrak{Q}_\lambda])[\mathfrak{Q}_\lambda]) \quad \text{if } \lambda \in \Sigma \text{ and } \lambda \mid \mathrm{disc}(D);$$
$$w'_{\mathfrak{J}} : (A, *, \iota, C) \mapsto (A/A[\mathfrak{J}], *', \iota', C) \quad \text{if } \mathfrak{J} \text{ prime to } \Sigma.$$

Also we have

Lemma 3.19. *The group $\widetilde{W}$ is a finitely generated Abelian group.*

Denote $\widetilde{W} \setminus S_\Sigma$ by S_Σ^+. Similarly we have that the F-points of S_Σ^+ give isogeny classes of F-virtual Abelian varieties of $\mathrm{GL}_2(E)$-type.

Theorem 3.20. *Every F-point on the Shimura variety S_Σ^+ gives an F-virtual Abelian variety of $\mathrm{GL}_2(E)$-type. Conversely for any F-virtual Abelian variety A of $\mathrm{GL}_2(E)$-type s.t. $\mathrm{End}^0 A = D$ there is an isogenous Abelian variety A' of $\mathrm{GL}_2(E)$-type which corresponds to an F-rational point on S_Σ^+, a quotient of Shimura variety of PEL-type, where Σ is given above.*

4 Field of Definition

Now we consider the field of definition of F-virtual Abelian varieties of GL_2-type. Even though they are not necessarily defined over F we will show that in their isogenous class there exist ones that can be defined over some polyquadratic field extension of F. Here we follow essentially [5] except in the quaternion algebra case.

Proposition 4.1. *Let A be an F-virtual Abelian variety. Let c be the associated 2-cocycle on G_F. If c is trivial in $H^2(G_{F'}, L^\times)$, then A is isogenous to an Abelian variety defined over F'.*

Proof. Suppose $c(\sigma,\tau) = \frac{\alpha(\sigma)\alpha(\tau)}{\alpha(\sigma\tau)}$ where σ and τ are in $G_{F'}$. Let $\nu_\sigma = \mu_\sigma \alpha^{-1}(\sigma)$. Let K be Galois over F' such that A is defined over K, that all the isogenies involved are defined over K as we did in the proof of Thm 1.13 and α is constant on G_K . Let $B = \mathrm{Res}_{K/F'} A$. Then the assignment $\sigma \mapsto \nu_\sigma$ is a D-linear algebra isomorphism between $D[G_{K/F'}]$ and $\mathrm{End}^0 B$. Consider the D-algebra homomorphism $D[G_{K/F'}] \to D$ where σ is sent to 1. Since $D[G_{K/F'}]$ is semisimple, D is a direct summand of $D[G_{K/F'}]$. Since $e = \frac{1}{[K:F']} \sum_{\sigma \in G_{K/F'}} \sigma$ is an idempotent of $D[G_{K/F'}]$ which splits off D, then if we let $A' = Ne(B)$ where N is chosen so that Ne is an endomorphism of B, then A' is defined over F' and is isogenous to A. $\square$

Proposition 4.2. *The two cocycle c is of 2-torsion in $H^2(G_F, L^\times)$ and c is trivial in $H^2(G_{F'}, L^\times)$ where F' can be taken as a polyquadratic extension.*

Proof. First consider the case when D is a quaternion algebra. Let Λ be the lattice in $\mathbb{C}^g$ that corresponds to A. Then $\Lambda \otimes \mathbb{Q}$ is isomorphic to D as left D-modules. Fix such an isomorphism. For ${}^\sigma A$ we get an induced isomorphism between ${}^\sigma\Lambda \otimes \mathbb{Q}$ and D as left D-modules. Then a D-linear isogeny f between ${}^\sigma A$ and ${}^\tau A$ induces a D-module isomorphism $D \to D$ which is just right multiplication by an element in $D^\times$. Let $d(f)$ denote the reduced norm of that element from D to L. Obviously $d(f) = d({}^\sigma f)$, for $\sigma \in G_F$. Furthermore $d(f \circ g) = d(f)d(g)$. Thus $d(c(\sigma,\tau)) = c^2(\sigma,\tau) = \frac{d(\mu_\sigma)d(\mu_\tau)}{d(\mu_{\sigma\tau})}$, which shows that c^2 is a coboundary.

Second consider the case when $D = E = L$. Then the lattice Λ corresponding to A is isomorphic to E^2 as E-vector space after tensoring with $\mathbb{Q}$. Fix such an isomorphism and correspondingly isomorphisms between ${}^\sigma\Lambda \otimes \mathbb{Q}$ and E^2. An E-linear isogeny f between ${}^\sigma A$ and ${}^\tau A$ then induces linear transformation. Let $d(f)$ denote the determinant of the linear transformation. Still d is multiplicative and $d(f) = d({}^\sigma f)$. Thus $d(c(\sigma,\tau)) = (c(\sigma,\tau))^2 = \frac{d(\mu_\sigma)d(\mu_\tau)}{d(\mu_{\sigma\tau})}$, which shows that c is 2-torsion in $H^2(G_F, L^\times)$.

Now all that is left is to show that c is trivial after a polyquadratic extension of F.

Consider the split short exact sequence of G_F-modules

$$0 \to \mu_2 \to L^\times \to P \to 0$$

where $P \cong L^\times/\mu_2$. We get that $H^2(G_F, L^\times) \cong H^2(G_F, \mu_2) \times H^2(G_F, P)$. Since $H^2(G_F, \mu_2)$ corresponds to those quaternion algebra elements in $Br(F)$ it can be killed by a quadratic extension of F. To kill the 2-torsion elements in $H^2(G_F, P)$, consider the short exact sequence:

$$0 \to P \xrightarrow{\times 2} P \to P/2P \to 0$$

where group multiplication is written additively. We get a long exact sequence:

$$\mathrm{Hom}(G_F, P) \to \mathrm{Hom}(G_F, P/2P) \to H^2(G_F, P) \xrightarrow{\times 2} H^2(G_F, P) \to \cdots$$

Since P is torsion-free $\mathrm{Hom}(G_F, P)$ is trivial, we find $\mathrm{Hom}(G_F, P/2P) \cong H^2(G_F, P)[2]$. After a polyquadratic extension of F to F' we can make $\mathrm{Hom}(G_{F'}, P/2P)$ as well as $H^2(G_F, \mu_2)$ trivial. Thus c is trivial in $H^2(G_{F'}, L^\times)$. $\square$

5 Classification of Hilbert Modular Surfaces

We will focus on the case where E is a real quadratic field with narrow class number 1 and study the Hilbert modular surfaces $Y_0^+(\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal of E. We have shown in Section 3 that the F-points of $Y_0^+(\mathfrak{p})$ represent F-virtual Abelian varieties. Suppose $E = \mathbb{Q}(\sqrt{D})$ where D is the discriminant. Because of the narrow class number 1 assumption, necessarily D is either a prime congruent to 1 modulo 4 or $D = 8$. We fix an embedding of E into $\mathbb{R}$. The conjugate of an element a in E is denoted by a^c. Since the class group of E is trivial, the group $\widetilde{W}$ in Definition 3.13 is just W, a group of order 2. The group $W\Gamma_0(\mathfrak{p})$ is in fact the normalizer of $\Gamma_0(\mathfrak{p})$ in $\mathrm{PGL}_2^+(E)$ and $Y_0^+(\mathfrak{p})$ is isomorphic to $W\Gamma_0(\mathfrak{p}) \setminus \mathcal{H}^2$ where

$$\Gamma_0(\mathfrak{p}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}) : c \equiv 0 \pmod{\mathfrak{p}} \right\}$$

In our case $\mathrm{PSL}_2(\mathcal{O}_E) = \mathrm{PGL}_2^+(\mathcal{O}_E)$. Let $X_0^+(\mathfrak{p})$ denote the minimal desingularity of $W\Gamma_0(\mathfrak{p}) \setminus \mathcal{H}^2 \cup \mathbb{P}^1(E)$. The classification done in [10] does not cover our case. We will follow the line of [10] and show that most surfaces in question are of general type and will give examples of surfaces not of general type. Our method relies on the estimation of Chern numbers. To do so we must study the singularities on the surfaces.

5.1 Cusp Singularities

Obviously for $\Gamma_0(\mathfrak{p}) \setminus \mathcal{H}^2 \cup \mathbb{P}^1(E)$ there are two inequivalent cusps 0 and ∞. They are identified via the Atkin-Lehner operator $w_\mathfrak{p} = \left(\begin{smallmatrix} 0 & 1 \\ -\varpi_\mathfrak{p} & 0 \end{smallmatrix}\right)$, where the prime ideal $\mathfrak{p}$ is equal to $(\varpi_\mathfrak{p})$ and $\varpi_\mathfrak{p}$ is chosen to be totally positive. This is possible as we assumed that the narrow class number of E is 1. The isotropy group of the unique inequivalent cusp ∞ in $W\Gamma_0(\mathfrak{p})$ is equal to that in $\mathrm{PSL}_2(\mathcal{O}_E)$, as $W\Gamma_0(\mathfrak{p})$ contains all those elements in $\mathrm{PSL}_2(\mathcal{O}_E)$ that are of the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$. Thus the type of the cusp singularity is the same as that for $\mathrm{PSL}_2(\mathcal{O})$ and the isotropy group is equal to

$$\begin{aligned}
&\left\{\begin{pmatrix}\epsilon & \mu \\ 0 & \epsilon^{-1}\end{pmatrix} \in \mathrm{PSL}_2(E) : \epsilon \in \mathcal{O}_E^\times, \mu \in \mathcal{O}_E\right\} \\
\cong &\left\{\begin{pmatrix}\epsilon & \mu \\ 0 & 1\end{pmatrix} \in \mathrm{PGL}_2^+(E) : \epsilon \in \mathcal{O}_E^{\times+}, \mu \in \mathcal{O}_E\right\} \\
\cong &\ \mathcal{O}_E \rtimes \mathcal{O}_E^{\times+}.
\end{aligned} \tag{5.1}$$

By [10, Chapter II] we have the minimal resolution of singularity resulting from toroidal embedding and the exceptional divisor consists of a cycle of rational curves.

5.2 Elliptic Fixed Points

Now consider the inequivalent elliptic fixed points of $W\Gamma_0(\mathfrak{p})$ on $\mathcal{H}^2$. More generally we consider the elliptic fixed points of $\mathrm{PGL}_2^+(E)$. Suppose $z = (z_1, z_2)$ is fixed by $\alpha = (\alpha_1, \alpha_2)$ in the image of $\mathrm{PGL}_2^+(E)$ in $\mathrm{PGL}_2^+(\mathbb{R})^2$. Then

$$\begin{pmatrix}a_j & b_j \\ c_j & d_j\end{pmatrix}.z_j = z_j$$

for $j = 1$ or 2 where $\alpha_j = \left(\begin{smallmatrix}a_j & b_j \\ c_j & d_j\end{smallmatrix}\right)$. Solving the equation we get

$$z_j = \frac{a_j - d_j}{2c_j} + \frac{1}{2|c_j|}\sqrt{\mathrm{tr}(\alpha_j)^2 - 4\det(\alpha_j)}.$$

Tranform z_j to 0 via möbius transformation $\zeta_j \mapsto \frac{\zeta_j - z_j}{\zeta_j - \bar{z_j}}$ then the isotropy group of (z_1, z_2) acts as rotation around 0 on each factor. For α_j the rotation factor is $r_j = e^{2i\theta_j}$ where

$$\cos\theta_j = \frac{\mathrm{tr}(\alpha_j)}{2\sqrt{\det(\alpha_j)}}, \quad c_j \sin\theta_j > 0. \tag{5.2}$$

The isotropy group of an elliptic point is cyclic.

Definition 5.1. We say that the quotient singularity is of type $(n; a, b)$ if the rotation factor associated to a generator of the isotropy group acts as $(w_1, w_2) \mapsto (\zeta_n^a w_1, \zeta_n^b w_2)$ where ζ_n is a primitive n-th root of 1.

Remark 5.2. We require that a and b are coprime to n respectively and we can make a equal to 1 by changing the primitive n-th root of 1. The method of resolution of singularity in [10, Section 6, Chapter II] depends only on the type.

Definition 5.3. Let $a_2(\Gamma)$ denote the number of Γ-inequivalent elliptic points of type $(2; 1, 1)$. Let $a_n^+(\Gamma)$ denote the number of Γ-inequivalent elliptic points of type $(n; 1, 1)$. Let $a_n^-(\Gamma)$ denote the number of Γ-inequivalent elliptic points of type $(n; 1, -1)$.

From the expression for θ_j we get:

Lemma 5.4. *Assume $D > 12$. Then the elliptic elements of $\Gamma_0(\mathfrak{p})$ can only be of order 2 or 3.*

5.3 Estimation of Chern Numbers

Now we estimate the Chern numbers of $X_0^+(\mathfrak{p})$. We will use the following criterion (Prop. 5.5) found on page 171 of [10] to show that most of our surfaces are of general type. Let χ denote the Euler characteristic and c_i be the i-th Chern class. The Chern class of a surface S, $c_i(S)$, is the Chern class of the tangent bundle.

Proposition 5.5. *Let S be a nonsingular algebraic surface with vanishing irregularity. If $\chi > 1$ and $c_1^2 > 0$, then S is of general type.*

Definition 5.6. Let S be a normal surface with isolated singular points and let S' be its desingularization. Suppose p is a singular point on S and the irreducible curves $C_1, \dots, C_m$ on S' form the resolution of p. Then the local Chern cycle of p is defined to be the unique divisor $Z = \sum_{i=1}^m a_i C_i$ with rational numbers a_i such that the adjunction formula holds:

$$ZC_i - C_iC_i = 2 - 2p_a(C_i).$$

Remark 5.7. For quotient singularity of type $(n; 1, 1)$, the exceptional divisor consists of one rational curve S_0 with $S_0^2 = -n$ and the local Chern cycle is $(1-2/n)S_0$. For quotient singularity of type $(n; 1, -1)$, the exceptional divisor consists of $n-1$ rational curves $S_1, \dots, S_{n-1}$ with $S_i^2 = -2$, $S_{i-1}.S_i = 1$ and the rest of the intersection numbers involving these rational curves are 0. The local Chern cycle is 0. For cusp singularity, the exceptional divisor consists of several rational curves $S_0, \dots, S_m$ such that $S_{i-1}.S_i = 1$, $S_0.S_m = 1$, $S_i^2 \leq -2$ and the rest of the intersection numbers involving these rational curves are 0. The local Chern cycle is $\sum_i S_i$.

We will make frequent comparison to the surface associated to the full Hilbert modular group $\mathrm{PSL}_2(\mathcal{O})$. As is computed on page 64 of [10] we have the following with a slight change of notation:

Theorem 5.8. *Let $\Gamma \subset \mathrm{PGL}_2^+(\mathbb{R})^2$ be commensurable with $\mathrm{PSL}_2(\mathcal{O})$ and let X_Γ be the minimal desingularization of $\overline{\Gamma \setminus \mathcal{H}^2}$. Then*

$$c_1^2(X_\Gamma) = 2\,\mathrm{vol}(\Gamma \setminus \mathcal{H}^2) + c + \sum a(\Gamma; n; a, b)c(n; a, b), \tag{5.3}$$

$$c_2(X_\Gamma) = \mathrm{vol}(\Gamma \setminus \mathcal{H}^2) + l + \sum a(\Gamma; n; a, b)(l(n; a, b) + \frac{n-1}{n}) \tag{5.4}$$

where

$a(\Gamma; n; a, b) =$ *#quotient singularity of $\Gamma \setminus \mathcal{H}^2$ of type $(n; a, b)$,*

$c =$ *sum of the self-intersection number of the local Chern cycles of cusp singularities,*

$c(n; a, b) =$ *self-intersection number of the local Chern cycle of a quotient singularity of type $(n; a, b)$,*

$l =$ *#curves in the resolution of cusps,*

$l(n; a, b) =$ *#curves in the resolution of a quotient singularity of type $(n; a, b)$.*

Lemma 5.9.
$$\mathrm{vol}(\mathrm{PSL}_2(\mathcal{O}_E)\setminus\mathcal{H}^2) = 2\zeta_E(-1). \tag{5.5}$$

Now we will estimate the chern numbers under the assumption that $D > 12$. This ensures that we only have $(2;1,1)$ or $(3;1,\pm1)$ points for $\Gamma_0(\mathfrak{p})$ and hence only $(2;1,1)$, $(3;1,\pm1)$, $(4;1,\pm1)$ and $(6;1,\pm1)$ points for $W\Gamma_0(\mathfrak{p})$. From [10, II. 6] as summarized in Remark 5.7, we know how the elliptic singularities are resolved and can compute the self-intersection number of local chern cycles. Thus after we plug in the values equation (5.3) reads

$$\begin{aligned} c_1^2(X(W\Gamma_0(\mathfrak{p}))) =& \frac{1}{2}[\mathrm{PSL}_2(\mathcal{O}):\Gamma_0(\mathfrak{p})]4\zeta_E(-1) + c - \frac{1}{3}a_3^+ - a_4^+ - \frac{8}{3}a_6^+;\\ c_2(X(W\Gamma_0(\mathfrak{p}))) =& \frac{1}{2}[\mathrm{PSL}_2(\mathcal{O}):\Gamma_0(\mathfrak{p})]2\zeta_E(-1) + l + (1+\frac{1}{2})a_2 + (1+\frac{2}{3})a_3^+ + (2+\frac{2}{3})a_3^-\\ &+ (1+\frac{3}{4})a_4^+ + (3+\frac{3}{4})a_4^- + (1+\frac{5}{6})a_6^+ + (5+\frac{5}{6})a_6^-. \end{aligned} \tag{5.6}$$

We quote some results in [10, Section VII.5].

Lemma 5.10. *For all D a fundamental discriminant $\zeta_E(-1) > \frac{D^{3/2}}{360}$.*

This is equation (1) in [10, Section VII.5].

As a_2, $a_3^\pm$, $a_4^\pm$, $a_6^\pm$ and l are non-negative, $c_2(W\Gamma_0(\mathfrak{p})) > (\mathbf{N}\,\mathfrak{p}+1)\frac{D^{3/2}}{360}$. Thus if

$$(\mathbf{N}\,\mathfrak{p}+1)\frac{D^{3/2}}{360} > 12 \tag{5.7}$$

then $c_2(W\Gamma_0(\mathfrak{p})) > 12$.

Now we estimate $c_1^2(X(W\Gamma_0(\mathfrak{p})))$. Let n denote the index of $\Gamma_0(\mathfrak{p})$ in $\mathrm{PSL}_2(\mathcal{O}_E)$, which is equal to $\mathbf{N}\,\mathfrak{p}+1$. First the self-intersection number of the local Chern cycle at the cusp, c, is equal to that for $\mathrm{PSL}_2(\mathcal{O})$ as the isotropy group for the unique cusp in $W\Gamma_0(\mathfrak{p})$ is the same as that in $\mathrm{PSL}_2(\mathcal{O})$.

Lemma 5.11. *The local Chern cycle*

$$c = \frac{1}{2}\sum_{x^2<D, x^2\equiv D \pmod 4}\ \sum_{a>0, a|\frac{D-x^2}{4}} 1. \tag{5.8}$$

and if $D > 500$,

$$c \geq -\frac{1}{2}D^{1/2}\left(\frac{3}{2\pi^2}\log^2 D + 1.05\log D\right). \tag{5.9}$$

This is [10, Lemma VII.5.3].

Definition 5.12. Let $h(D)$ denote the class number of the quadratic field $\mathbb{Q}(\sqrt{D})$ where D is a fundamental discriminant.

Lemma 5.13. *If $D > 0$ is a fundamental discriminant then $h(-D) \leq \frac{\sqrt{D}}{\pi}\log D$.*

This is [10, Lemma VII.5.2].

Lemma 5.14. *If $D > 12$ and $\mathrm{Cl}^+(\mathbb{Q}(\sqrt{D})) = 1$, then*

$$\begin{aligned} a_2(\mathrm{PSL}_2(\mathcal{O})) =& h(-4D) \\ a_3^+(\mathrm{PSL}_2(\mathcal{O})) =& \frac{1}{2}h(-3D) \end{aligned} \tag{5.10}$$

Combining the above two lemmas we get:

Lemma 5.15.

$$\begin{aligned} a_2(\mathrm{PSL}_2(\mathcal{O})) &\le \frac{\sqrt{4D}}{\pi}\log 4D \\ a_3^+(\mathrm{PSL}_2(\mathcal{O})) &\le \frac{1}{2\pi}\sqrt{3D}\log(3D). \end{aligned} \tag{5.11}$$

Lemma 5.16. *If $D > 12$ and $\mathrm{Cl}^+(\mathbb{Q}(\sqrt{D})) = 1$ then*

$$\begin{aligned} a_2(\Gamma_0(\mathfrak{p})) &\le \frac{3\sqrt{4D}}{\pi}\log 4D \\ a_3^+(\Gamma_0(\mathfrak{p})) &\le \frac{3}{2\pi}\sqrt{3D}\log(3D). \end{aligned} \tag{5.12}$$

Proof. Let z be an elliptic point of $\mathrm{PSL}_2(\mathcal{O}_E)$ with isotropy group generated by $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We have coset decomposition of $\mathrm{SL}_2(\mathcal{O}_E) = \cup_\alpha \Gamma_0(\mathfrak{p})\delta_\alpha \cup \Gamma_0(\mathfrak{p})\delta_\infty$, where $\delta_\alpha = \left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)$ with $\alpha \in \mathcal{O}_E$ running through a set of representatives of $\mathcal{O}_E/\mathfrak{p}$ and $\delta_\infty = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. For each δ_α we need to check if $\delta_\alpha g \delta_\alpha^{-1}$ is in $\Gamma_0(\mathfrak{p})$, i.e., if $c + (a-d)\alpha - b\alpha^2$ is in $\mathfrak{p}$. In $\mathbb{F}_\mathfrak{p}$, the equation $c + (a-d)\alpha - b\alpha^2 = 0$ has at most two solutions unless $c, a-d, b \in \mathfrak{p}$. This cannot happen if g is elliptic. Indeed from $ad - bc = 1$ we get $a^2 \equiv 1 \pmod{\mathfrak{p}}$ and thus $a \equiv \pm 1 \pmod{\mathfrak{p}}$. Replace g by $-g$ if necessary we suppose that $a \equiv 1 \pmod{\mathfrak{p}}$. Suppose $a + d = t$ with $t = 0$ or ± 1 and suppose $a = 1 + v$ with $v \in \mathfrak{p}$. Then

$$\begin{aligned} 1 &= ad - bc \\ &= a(t-a) - bc \\ &\equiv a(t-a) \pmod{\mathfrak{p}^2} \\ &\equiv t - 1 + (t-2)v \pmod{\mathfrak{p}^2}. \end{aligned} \tag{5.13}$$

We find that $(t-2)(1+v) \pmod{\mathfrak{p}^2}$. By assumption $D > 12$ is a prime so (2) or (3) cannot ramify. We always have that $1 + v \equiv 0 \pmod{\mathfrak{p}}$, which is impossible.

Thus the number of elliptic points of any type increases to at most threefold that for $\mathrm{PSL}_2(\mathcal{O}_E)$. □

Lemma 5.17. *Suppose $D > 12$ and $\mathrm{Cl}^+(\mathbb{Q}(\sqrt{D})) = 1$. Then $a_6^+(W\Gamma_0(\mathfrak{p})) = 0$ unless (3) is inert in $\mathcal{O}_E$ and $\mathfrak{p} = (3)$ and*

$$2a_3^+(W\Gamma_0(\mathfrak{p})) + a_6^+(W\Gamma_0(\mathfrak{p})) \le \frac{2}{3\pi}\sqrt{3D}\log(3D); \tag{5.14}$$

and $a_4^+(W\Gamma_0(\mathfrak{p}))=0$ *unless* (2) *is inert and* $\mathfrak{p}=(2)$ *and*

$$a_4^+(W\Gamma_0(\mathfrak{p}))\leq\frac{3\sqrt{4D}}{\pi}\log 4D. \tag{5.15}$$

Proof. We check the rotation factor

$$\cos\theta_j=\frac{\mathrm{tr}(\alpha_j)}{2\sqrt{\det(\alpha_j)}} \tag{5.16}$$

associated to an elliptic element α. In order to have a point with isotropy group of order 4 in $W\Gamma_0(\mathfrak{p})$ we must have $\cos\theta_j=\pm\frac{\sqrt{2}}{2}$. As $D>12$, we need $\det(\alpha_j)=\varpi_{\mathfrak{p}}$ in $2\mathcal{O}_E^2$ and also $\mathfrak{p}=(2)$. In order to have a point with isotropy group of order 6 in $W\Gamma_0(\mathfrak{p})$ we must have $\cos\theta_j=\pm\frac{\sqrt{3}}{2}$. As $D>12$, we need $\det(\alpha_j)=\varpi_{\mathfrak{p}}$ in $3\mathcal{O}_E^2$ and also $\mathfrak{p}=(3)$.

The Atkin-Lehner operator exchanges some of the $\Gamma_0(\mathfrak{p})$-inequivalent $(3;1,1)$-points which result in $(3;1,1)$-points and fixes the rest of the points which result in $(6;1,1)$-points. All $(3;1,1)$- and $(6;1,1)$-points for $W\Gamma_0(\mathfrak{p})$ arise in this way. We have

$$2a_3^+(W\Gamma_0(\mathfrak{p}))+a_6^+(\Gamma_0(\mathfrak{p}))=a_3^+(\Gamma_0(\mathfrak{p})). \tag{5.17}$$

The Atkin-Lehner operator exchanges some of the $\Gamma_0(\mathfrak{p})$-inequivalent $(2;1,1)$-points which result in $(2;1,1)$-points and fixes the rest of the points which result in $(4;1,1)$-points. All $(4;1,1)$-points for $W\Gamma_0(\mathfrak{p})$ arise in this way, but we may get $(2;1,1)$-points not arising in this way. We have

$$a_4^+(\Gamma_0(\mathfrak{p}))\leq a_2^+(\Gamma_0(\mathfrak{p})). \tag{5.18}$$

Combining with Lemma 5.16 we prove this lemma. □

Lemma 5.18. *Suppose* $D>12$ *and* $\mathrm{Cl}^+(\mathbb{Q}(\sqrt{D}))=1$*. Then*

$$\frac{1}{3}a_3^+(W\Gamma_0(\mathfrak{p})+8a_6^+(W\Gamma_0(\mathfrak{p}))\leq\frac{8}{3}a_3^+(\Gamma_0(\mathfrak{p})) \tag{5.19}$$

if (3) *is inert and* $\mathfrak{p}=(3)$*. If* $\mathfrak{p}\neq(3)$ *then*

$$\frac{1}{3}a_3^+(W\Gamma_0(\mathfrak{p}))=\frac{1}{6}a_3^+(\Gamma_0(\mathfrak{p})). \tag{5.20}$$

Combining all these inequalities we find

Lemma 5.19. *Suppose* $D>500$ *and* $\mathrm{Cl}^+(\mathbb{Q}(\sqrt{D}))=1$*. Then*

$$\begin{aligned}c_1^2(X_0^+(\mathfrak{p}))>{}&\frac{nD^{3/2}}{180}-\frac{1}{2}D^{1/2}(\frac{3}{2\pi^2}\log^2 D+1.05\log D)\\&-\begin{cases}\frac{1}{4\pi}\sqrt{3D}\log(3D) & \textit{if } \mathfrak{p}\neq(3)\\ \frac{4}{\pi}\sqrt{3D}\log(3D) & \textit{if } \mathfrak{p}=(3)\end{cases}\\&-\begin{cases}0 & \textit{if } \mathfrak{p}\neq(2)\\ \frac{3}{\pi}\sqrt{4D}\log(4D) & \textit{if } \mathfrak{p}=(2).\end{cases}\end{aligned} \tag{5.21}$$

Now it is easy to estimate for what values of D and n we have $c_1^2(W\Gamma_0(\mathfrak{p})) > 0$. For small D we just compute c precisely by using Equation 5.8 instead of using the estimates.

Theorem 5.20. *Suppose $D > 12$, $D \equiv 1 \pmod 4$ and $\mathrm{Cl}^+(\mathbb{Q}(\sqrt{D})) = 1$. Then the Hilbert modular surface $X_0^+(\mathfrak{p})$ is of general type if D or $n = \mathbf{N}\,\mathfrak{p} + 1$ is suffiently large or more precisely if the following conditions on D and n are satisfied:*

$D \geq 853$ *or* $D = 193,\ 241,\ 313,\ 337,\ 409,\ 433,\ 457,\ 521,\ 569,\ 593,\ 601,\ 617,\ 641,\ 673,\ 769,\ 809$	n *can be arbitrary*
$D = 157,\ 181,\ 277,\ 349,\ 373,\ 397,\ 421,\ 509,\ 541,\ 557,\ 613,\ 653,\ 661,\ 677,\ 701,\ 709,\ 757,\ 773,\ 797,\ 821,\ 829$	$n \neq 5$, *i.e.*, $\mathfrak{p} \neq (2)$
$D = 137, 233, 281, 353, 449,$	$n \neq 10$, *i.e.*, $\mathfrak{p} \neq (3)$
$D = 149, 173, 197, 269, 293, 317, 389, 461$	$n \neq 5, 10$, *i.e.*, $\mathfrak{p} \neq (2), (3)$
$D = 113$	$n \geq 8$ *and* $n \neq 10$
$D = 109$	$n \geq 6$
$D = 101$	$n \geq 6$ *and* $n \neq 10$
$D = 97$	$n \geq 5$
$D = 89$	$n \geq 6$
$D = 73$	$n \geq 7$
$D = 61$	$n \geq 10$
$D = 53$	$n \geq 12$
$D = 41$	$n \geq 17$
$D = 37$	$n \geq 20$
$D = 29$	$n \geq 28$
$D = 17$	$n \geq 62$
$D = 13$	$n \geq 93$

Proof. First we note that the condition for (2) to split is that $D \equiv 1 \pmod 8$ and for (3) to split is that $D \equiv 1 \pmod 3$. We check for what values of D and n the inequality 5.7 is satisfied and the right hand side of the inequaltiy in Inequality 5.21 is greater than 0 by using a computer program. Then we have $c_1^2(X_0^+(\mathfrak{p})) > 0$ and $c_2(X_0^+(\mathfrak{p})) > 12$ and thus $\chi(X_0^+(\mathfrak{p})) > 1$. The surface is of general type by Prop. 5.5. $\square$

Remark 5.21. There are some values that, a priori, n cannot achieve.

5.4 Examples

The first two examples give rational surfaces and the third example is neither a rational surface nor a surface of general type.

5.4.1 $D = 5$

Consider the Hilbert modular surface $\mathrm{PSL}_2(\mathcal{O}_E) \setminus \mathcal{H}^2$ where $E = \mathbb{Q}(\sqrt{5})$. The cusp resolution is a nodal curve. We will focus on quotient resolutions to study the configurations of rational curves on the surface. Following the method in [4], we can locate all the $\mathrm{PSL}_2(\mathcal{O}_E)$-inequivalent elliptic points.

$\mathfrak{p} = (2)$ The $\Gamma_0(\mathfrak{p})$-inequivalent elliptic points and their types are summarized in the following table. As the coordinates of the points themselves are not important we only list a generator of the isotropy groups.

Type	Generator of Isotropy Group
$(2;1,1)$	$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$
$(2;1,1)$	$\begin{pmatrix} -1 & \frac{-1+\sqrt{5}}{2} \\ -1-\sqrt{5} & 1 \end{pmatrix}$
$(3;1,1)$	$\begin{pmatrix} \frac{1+\sqrt{5}}{2} & -1 \\ 2 & \frac{1-\sqrt{5}}{2} \end{pmatrix}$
$(3;1,1)$	$\begin{pmatrix} \frac{3+\sqrt{5}}{2} & -1 \\ 3+\sqrt{5} & -\frac{1+\sqrt{5}}{2} \end{pmatrix}$
$(3;1,-1)$	$\begin{pmatrix} \frac{1-\sqrt{5}}{2} & \frac{-1+\sqrt{5}}{2} \\ -1-\sqrt{5} & \frac{1+\sqrt{5}}{2} \end{pmatrix}$
$(3;1,-1)$	$\begin{pmatrix} -\frac{1+\sqrt{5}}{2} & \frac{-1+\sqrt{5}}{2} \\ -4-2\sqrt{5} & \frac{3+\sqrt{5}}{2} \end{pmatrix}$

The Atkin-Lehner operator fixes the two $(2;1,1)$-points respectively and exchanges the two $(3;1,1)$- (resp. $(3;1,-1)$-) points. We get one $(4;1,1)$-, one $(4;1,-1)$-, one $(3;1,1)$-, one $(3;1,-1)$- and possibly some new $(2;1,1)$-points. We consider the $(4;1,-1)$-point represented by $(\frac{1+i}{1+\sqrt{5}}, \frac{1-i}{1-\sqrt{5}})$ and the $(3;1,-1)$-point represented by $(\frac{\sqrt{5}+i\sqrt{3}}{2(1+\sqrt{5})}, \frac{\sqrt{5}+i\sqrt{3}}{-2(1-\sqrt{5})})$. Consider the curve F_B on $W\Gamma_0(\mathfrak{p}) \setminus \mathcal{H}^2$ defined as the image of the curve

$$\widetilde{F_B} = \left\{ (z_1, z_2) : \begin{pmatrix} z_2 & 1 \end{pmatrix} B \begin{pmatrix} z_1 \\ 1 \end{pmatrix} = 0 \right\} \tag{5.22}$$

and let F'_B denote the strict transform of F_B in $X_0^+(\mathfrak{p})$, the desingularity of $W\Gamma_0(\mathfrak{p}) \setminus \mathcal{H}^2 \cup \mathbb{P}^1(E)$. Let $B = \left(\begin{smallmatrix} 0 & \frac{(1-\sqrt{5})\sqrt{5}}{2} \\ \frac{(1+\sqrt{5})\sqrt{5}}{2} & 0 \end{smallmatrix} \right)$. Then the two elliptic points noted above lie on F_B. The stabilizer Γ_B of B in $\Gamma_0(\mathfrak{p})$ is a degree 2 extension of the group

$$\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(\mathfrak{p}) : a, d \in \mathbb{Z}, c \in (1+\sqrt{5})\sqrt{5}\mathbb{Z}, b \in \frac{(1-\sqrt{5})\sqrt{5}}{2}\mathbb{Z} \right\} \tag{5.23}$$

generated by $\left(\begin{smallmatrix} \sqrt{5} & \sqrt{5}-1 \\ 1+\sqrt{5} & \sqrt{5} \end{smallmatrix} \right)$. The stabilizer of B in $W\Gamma_0(\mathfrak{p})$ is a degree 2 extension of Γ_B by $\left(\begin{smallmatrix} 2 & \frac{-(1-\sqrt{5})\sqrt{5}}{2} \\ -(1+\sqrt{5})\sqrt{5} & -4 \end{smallmatrix} \right)$. We find that the image of $\widetilde{F_B}$ in $W\Gamma_0(\mathfrak{p}) \setminus \mathcal{H}^2$ is a quotient of $\Gamma_0(10\mathbb{Z}) \setminus \mathcal{H}$ and thus F'_B is a rational curve in $X_0^+(\mathfrak{p})$. Consider the intersection of F'_B with the local Chern cycles. Following the method in [10, V.2] we find that the intersection number of F'_B with the cusp resolution is 2. Thus we have

$$c_1(X_0^+(\mathfrak{p})).F'_B = -2 \cdot \frac{1}{6} \cdot \frac{18}{4} + 2 + \frac{1}{3} \cdot n_3 + \frac{1}{2} \cdot n_4 \tag{5.24}$$

where n_3 is the number of $(3;1,1)$-points that F'_B passes through and n_4 is the number of $(4;1,1)$-points that F'_B passes through. As intersection number is an integer, we are force to have $n_3 = 0$ and $n_4 = 1$ and thus $c_1(X_0^+(\mathfrak{p})).F'_B = 1$. By Adjunction formula, $F'^2_B = -1$. We get a linear configuration of rational curves with self-intersection numbers -2, -1, -2, where the (-2)-curves come from desingularity of the $(3;1,-1)$- and the $(4;1,-1)$-points mentioned above. After blowing down F_B we acquire two intersecting (-1)-curves and this shows that the surface $X_0^+(\mathfrak{p})$ is a rational surface.

5.4.2 $D = 13$

Consider the Hilbert modular surface $\mathrm{PSL}_2(\mathcal{O}_E) \setminus \mathcal{H}^2$ where $E = \mathbb{Q}(\sqrt{13})$. The cusp resolution is of type $(5;2,2)$ and the rational curves are labelled as S_0, S_1 and S_2. Following the method in [4], we can locate all the $\mathrm{PSL}_2(\mathcal{O}_E)$-inequivalent elliptic points.

It is easy to find the $\Gamma_0(\mathfrak{p})$-inequivalent elliptic points from right coset decomposition $\mathrm{PSL}_2(\mathcal{O}_E) = \cup_\alpha \Gamma_0(\mathfrak{p}) g_\alpha \cup \Gamma_0(\mathfrak{p}) g_\infty$ where $g_\alpha = \left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)$ and $g_\infty = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ with α running over a set of representatives of $\mathcal{O}_E/\mathfrak{p}$.

$D = 13$, $\mathfrak{p} = (4+\sqrt{13})$ Suppose $\mathfrak{p} = (4+\sqrt{13})$. Then we list one generator of isotropy group for each $\Gamma_0(\mathfrak{p})$-inequivalent elliptic point:

Type	Generator of Isotropy Group
$(3;1,1)$	$\begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix}$
$(3;1,1)$	$\begin{pmatrix} (-1+\sqrt{13})/2 & -2 \\ (5-\sqrt{13})/2 & (3-\sqrt{13})/2 \end{pmatrix}$
$(3;1,-1)$	$\begin{pmatrix} 2 & (-1+\sqrt{13})/2 \\ -(1+\sqrt{13})/2 & -1 \end{pmatrix}$
$(3;1,-1)$	$\begin{pmatrix} (5+\sqrt{13})/2 & (3+\sqrt{13})/2 \\ -(1+\sqrt{13}) & -(3+\sqrt{13})/2 \end{pmatrix}$

Since there cannot be any elliptic points with isotropy group of 6 for $W\Gamma_0(\mathfrak{p})$ acting on $\mathcal{H}^2$. The Atkin-Lehner operator exchanges the two $(3;1,1)$-points (resp. $(3;1,-1)$-points).

Now consider the curve F_B on $W\Gamma_0(\mathfrak{p}) \setminus \mathcal{H}^2$ defined as in (5.22) and set $B = \left(\begin{smallmatrix} 0 & 4-\sqrt{13} \\ -4-\sqrt{13} & 0 \end{smallmatrix}\right)$. Still let F_B denote the closure of F_B in $W\Gamma_0(\mathfrak{p}) \setminus \mathcal{H}^2 \cup \mathbb{P}^1(E)$ and let F'_B denote the strict transform of F_B in $X_0^+(\mathfrak{p})$, the desingularity of $W\Gamma_0(\mathfrak{p}) \setminus \mathcal{H}^2 \cup \mathbb{P}^1(E)$.

The elements of $W\Gamma_0(\mathfrak{p})$ that stabilize $\widetilde{F_B}$ are of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, d \in \mathbb{Z}$, $c \in (4+\sqrt{13})\mathbb{Z}$ and $b \in (4-\sqrt{13})\mathbb{Z}$ and determinant 1. Thus we find that $F'_B \cong \Gamma_0(3\mathbb{Z}) \setminus \mathcal{H}$ which is of genus 0. Furthermore $c_1(X_0^+(\mathfrak{p})).F'_B = 2\,\mathrm{vol}(F'_B) + \sum Z_x.F'_B$

where Z_x is the local Chern cycle at x a singular point. We compute that

$$c_1(X_0^+(\mathfrak{p})).F'_B = -2 \cdot \frac{1}{6} \cdot 4 + 2 + \frac{1}{3} \cdot n \tag{5.25}$$

with n the number of $(3;1,1)$-points that F'_B passes through. As there is just one $(3;1,1)$-point, we are forced to have $n = 1$ and thus $c_1(X_0^+(\mathfrak{p})).F'_B = 1$. By Adjunction formula $F_B'^2 = -1$. We also find that F'_B intersects with the cusp resolution: $F'_B.S_1 = F'_B.S_2 = 1$. Note that S_1 and S_2 have self-intersection number -2. After blowing down F'_B we get two intersecting (-1)-curves. Again by an algebraic geometry criterion, the surface $W\Gamma_0((4+\sqrt{13})) \setminus \mathcal{H}^2$ is a rational surface.

$D = 13$, $\mathfrak{p} = (2)$ Suppose $\mathfrak{p} = (2)$. We have two inequivalent $(2;1,1)$-points namely $((i+1)/2, (i+1)/2)$ and $((i+1)/(3+\sqrt{13}), (i-1)/(-3+\sqrt{13}))$, four inequivalent $(3;1,1)$-points and four inequivalent $(3;1,-1)$-points. The Atkin-Lehner operator fixes the $(2;1,1)$-points and exchanges $(3;1,1)$-points (resp. $(3;1,-1)$-points). It is easy to check that we get one $(4;1,1)$-, one $(4;1,-1)$-, two $(3;1,1)$- and two $(3;1,-1)$-points and some new $(2;1,1)$-points. We compute that $c_1(X_0^+(\mathfrak{p}))^2 = 2 \cdot 2 \cdot \frac{1}{6} \cdot \frac{5}{2} - \frac{1}{3} \cdot 2 - 1 = -3$ and $c_2(X_0^+(\mathfrak{p})) = 2 \cdot \frac{1}{6} \cdot \frac{5}{2} + (1+\frac{1}{2})a_2 + (1+\frac{2}{3})2 + (2+\frac{2}{3})2 + (1+\frac{3}{4})1 + (3+\frac{3}{4})1 = 18 + \frac{3}{2}a_2$. The Euler characteristic $\chi(X_0^+(\mathfrak{p})) = (c_1(X_0^+(\mathfrak{p}))^2 + c_2(X_0^+(\mathfrak{p})))/12 = (15+\frac{3}{2}a_2)/12 \geq 2$. Thus $X_0^+(\mathfrak{p})$ cannot be a rational surface.

Consider the curve F'_B with

$$B = \begin{pmatrix} 0 & 4-\sqrt{13} \\ -4-\sqrt{13} & 0. \end{pmatrix} \tag{5.26}$$

The stabilizer Γ_B of $\widetilde{F_B}$ in $\Gamma_0(\mathfrak{p})$ consists of elements of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, d \in \mathbb{Z}$, $c \in (2(4+\sqrt{13}))\mathbb{Z}$ and $b \in (4-\sqrt{13})\mathbb{Z}$ and determinant 1. The stabilizer of $\widetilde{F_B}$ in $W\Gamma_0(\mathfrak{p})$ is a degree 2 extension of Γ_B generated by $\left(\begin{smallmatrix} 2 & 4-\sqrt{13} \\ -2(4+\sqrt{13}) & -2 \end{smallmatrix}\right)$. Thus we find that $\Gamma_0(6\mathbb{Z}) \setminus \mathcal{H}$ is a degree 2 cover of F'_B. Thus F'_B is of genus 0. We compute that

$$c_1(X_0^+(\mathfrak{p})).F'_B = -2 \cdot \frac{1}{6} \cdot \frac{12}{2} + 2 + \frac{1}{3} \cdot n_3 + \frac{1}{2} \cdot n_4 \tag{5.27}$$

where n_3 is the number of $(3;1,1)$-points that F'_B passes through and n_4 is the number of $(4;1,1)$-points that F'_B passes through. As there are two $(3;1,1)$-points and one $(4;1,1)$-point on $X_0^+(\mathfrak{p})$ we are forced to have that $n_3 = 0$ and $n_4 = 0$. Thus $c_1(X_0^+(\mathfrak{p})) = 0$. By Adjunction formula, $F_B'^2 = -2$. We have the configuration of (-2)-curves consisting of F'_B, S_1 and S_2 such that $F'_B.S_1 = F'_B.S_2 = S_1.S_2 = 1$. This cannot occur on a surface of general type by [10, Prop. VII.2.7]. Hence in this example we find a surface which is birationally equivalent to a K3, an Enrique surface or an honestly elliptic surface.

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