On Rallis Inner Product Formula

Chenyan Wu

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In this chapter we study the case of the Rallis inner product formula that relates the pairing of theta functions to the central value of Langlands L-function. We study the Siegel-Weil formula first, as it is a key ingredient in the proof.

We consider the dual reductive pair $H = \mathrm{O}(V)$ and $G = \mathrm{Sp}(2n)$ with n the rank of the symplectic group. We use $\widetilde{G(\mathbb{A})}$ to denote the metaplectic group which is a double cover of $G(\mathbb{A})$. Let V be a vector space over a number field k with the quadractic form Q and let m be the dimension of the vector space and r its Witt index. Set $s_0 = (m-n-1)/2$. Form the Siegel Eisenstein series $E(g, s, f_\Phi)$ and the theta integral $I(g, \Phi)$ for $g \in \widetilde{G(\mathbb{A})}$ and Φ in the Schwartz space $\mathcal{S}_0(V^n(\mathbb{A}))$. The Eisenstein series can be meromorphically continued to the whole s-plane. The theta integral is not necessarily convergent and we will use Ichino's[2] regularized theta integral $I_{\mathrm{REG}}(g, \Phi)$. Please see Sec. 1 for further notations. Roughly speaking, the Siegel-Weil formula gives the relation between the value or the residue at s_0 of the Siegel Eisenstein series and the regularized theta integral.

When the Eisenstein series and the theta integral are both absolutely convergent, Weil[17] proved the formula in great generality. In the case where the groups under consideration are orthogonal group and the metaplectic group, Weil's condition for absolute convergence for the theta integral is that $m - r > n + 1$ or $r = 0$. The Siegel Eisenstein series $E(g, s, f_\Phi)$ is absolutely convergent for $\operatorname{Re} s > (n+1)/2$, so if $m > 2n + 2$, $E(g, s, f_\Phi)$ is absolutely convergent at $s_0 = (m - n - 1)/2$. Then assuming only the absolute convergence of theta integral, i.e., $m - r > n + 1$ or $r = 0$, Kudla and Rallis in [7] and [8] proved that the analytic continued Siegel Eisenstein series is holomorphic at s_0 and showed that the Siegel-Weil formula holds between the value at s_0 of the Siegel Eisenstein series and the theta integral.

In [9] Kudla and Rallis introduced the regularized theta integral to remove the requirement of absolute convergence of the theta integral. The formula then relates the residue of Siegel Eisenstein series at s_0 with the leading term of the regularized theta integral. However they worked under the condition that m is even, in which case the metaplectic group splits. The regularized theta integral is actually associated to V_0, the complementary space of V if $n+1 < m \leq 2n$ and $m - r \leq n+1$ with V isotropic. In the case $m = n+1$ the Eisenstein series is holomorphic at $s_0 = 0$ and the formula relates the value of Eisenstein series at s_0 to the leading term of the regularized theta integral associated to V. Note that in the above summary we excluded the split binary case for clarity.

For m odd Ikeda in [4] proved an analogous formula. However his theta integral does not require regularization since he assumed that the complementary space V_0 of V is anisotropic in the case $n+1 < m < 2n+2$ or that V is anisotropic in the case $m = n+1$. The method for regularizing theta integral was generalized by Ichino[2]. Instead of using differential operator at a real place as in [9], he used a Hecke operator at a finite place and thus did away with the assumption that the ground field k has a real place. In Ichino's notation the Siegel-Weil formula is a relation between the residue at s_0 of the Siegel Eisenstein series and the regularized theta integral $I_{\mathrm{REG}}(g, \Phi)$ itself. He considered the case where $n+1 < m \leq 2n+2$ and $m - r \leq n+1$ with no parity restriction on m. The interesting case $m = n+1$ with m odd, however, is still left open.

The case of Rallis inner product formula we are concerned with involves the orthogonal group $\mathrm{O}(V)$ with V a quadratic space of dimension $2n'+1$ and the symplectic group $\mathrm{Sp}(2n')$ of rank n'. Via the doubling method, to compute the inner product we ultimately need to apply the Siegel-Weil formula with $m = 2n'+1$ and $n = 2n'$. Note that m is odd here. We show that the pairing of theta functions is related to the central value of an L-function.

The idea of proof originates from Kudla and Rallis's paper[9]. We try to show the identity by comparing the Fourier coefficients of the Siegel Eisenstein series and regularized theta integral. By showing that a certain representation is nonsingular (c.f. Section 5) we can find a Schwartz function on $\mathrm{Sym}_n(k_v)$ for some finite place v of k to kill the singular Fourier coefficients of the automorphic form A which is the difference of $E(g, s, \Phi)|_{s=s_0}$ and $2I_{\mathrm{REG}}(g, \Phi)$. The constant 2 is with respect to some normalization of Haar measures. Then via the theory of Fourier-Jacobi coefficients we are able to show the nonsingular Fourier coefficients of A actually vanish. Then by a density argument we show that $A = 0$.

Finally via the new case of Rallis inner product formula we show the relation between nonvanishing of L-value and the nonvanishing of theta lifts.

1 Notations and Preliminaries

Let k be a number field and $\mathbb{A}$ its adele ring. Let U be a vector space of dimension m over k with quadratic form Q. We view the vectors in U as column vectors. The associated bilinear form on U is denoted by $\langle , \rangle_Q$ and it is defined by $\langle x, y\rangle_Q = Q(x+y) - Q(x) - Q(y)$. Thus $Q(x) = \frac{1}{2}\langle x, x\rangle_Q$. Let r denote the Witt index of Q, i.e., the dimension of a maximal isotropic subspace of U. Let $H = O(U)$ denote the orthogonal group of (U, Q) and $G = \mathrm{Sp}(2n)$ the symplectic group of rank n. Let $\widetilde{G(\mathbb{A})}$ be the metaplectic group which is a double cover of $G(\mathbb{A})$ and fix a non-trivial additive character ψ of $\mathbb{A}/k$ and set $\psi_S(\cdot) = \psi(S\cdot)$ for $S \in k$. Locally the multiplication law of $\widetilde{G(k_v)}$ is given by

$$(g_1, \zeta_1)(g_2, \zeta_2) = (g_1 g_2, c(g_1, g_2)\zeta_1\zeta_2).$$

where $\zeta_i \in \{\pm 1\}$ and $c(g_1, g_2)$ is Rao's 2-cocycle on $G(k_v)$ with values in $\{\pm 1\}$. The properties of c can be found in [15, Theorem 5.3]. There the factor $(-1,-1)^{\frac{j(j+1)}{2}}$ should be $(-1,-1)^{\frac{j(j-1)}{2}}$ as pointed out, for example, in [6, Remark 4.6].

Via the Weil representation ω, $\widetilde{G(\mathbb{A})} \times H(\mathbb{A})$ acts on the space of Schwartz functions $\mathcal{S}(U^n(\mathbb{A}))$. Locally it is characterized by the following properties (see e.g., [6, Prop. 4.3]):

$$\begin{aligned}
\omega_v((\begin{pmatrix} A & \\ & {}^{\mathrm{t}}A^{-1} \end{pmatrix}, \zeta))\Phi(X) &= \chi_v(\det A, \zeta)|\det A|_v^{m/2}\Phi(XA), \\
\omega_v((\begin{pmatrix} 1_n & B \\ & 1_n \end{pmatrix}, \zeta))\Phi(X) &= \zeta^m \psi_v(\frac{1}{2}\operatorname{tr}(\langle X, X\rangle_{Q_v} B))\Phi(X), \\
\omega_v((\begin{pmatrix} & -1_n \\ 1_n & \end{pmatrix}, \zeta))\Phi(X) &= \zeta^m \gamma_v(\psi_v \circ Q_v)^{-n}\mathcal{F}\Phi(-X) \\
\omega(h)\Phi(X) &= \Phi(h^{-1}X)
\end{aligned}$$

where $\Phi \in \mathcal{S}(U^n(k_v))$, $X \in U^n(k_v)$, $A \in \mathrm{GL}_n(k_v)$, $B \in \mathrm{Sym}_n(k_v)$, $\zeta \in \{\pm 1\}$ and $h \in H(k_v)$. Here γ_v is the Weil index of the character of second degree

$x \mapsto \psi_v \circ Q_v(x)$ and has values in 8-th roots of unity. The matrix $\langle X, X\rangle_{Q_v}$ has $\langle X_i, X_j\rangle_{Q_v}$ as ij-th entry if we write $X = (X_1, \dots, X_n)$ with X_i column vectors in $U(k_v)$. The Fourier transform of Φ with respect to ψ_v and Q_v is defined to be

$$\mathcal{F}\Phi(X) = \int_{U^n(k_v)} \psi_v(\mathrm{tr}\langle X, Y\rangle_{Q_v})\phi(Y)dY$$

and

$$\chi_v(a,\zeta) = \zeta^m (a, (-1)^{\frac{m(m-1)}{2}} \det\langle\ ,\ \rangle_{Q_v})_{k_v} \cdot \begin{cases} \gamma_v(a, \psi_{v,1/2})^{-1} & \text{if } m \text{ is odd,} \\ 1 & \text{if } m \text{ is even.} \end{cases} \tag{1.1}$$

for $a \in k_v^\times$ and $\zeta \in \{\pm 1\}$. Here $(\ ,\ )_{k_v}$ denote the Hilbert symbol and $\det\langle\ ,\ \rangle_{Q_v}$ is the determinant of the symmetric bilinear form on $U(k_v)$.

Define the theta function

$$\Theta(g, h; \Phi) = \sum_{u \in U^n(k)} \omega(g, h)\Phi(u)$$

where $g \in \widetilde{G(\mathbb{A})}$, $h \in H(\mathbb{A})$ and $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$ and consider the integral

$$I(g, \Phi) = \int_{H(k)\setminus H(\mathbb{A})} \Theta(g, h; \Phi)dh.$$

It is well-known that this integral is absolutely convergent for all $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$ if either $r = 0$ or $m - r > n + 1$. Thus in the case considered in this paper we will need to regularize the theta integral unless Q is anisotropic.

Let P be the Siegel parabolic subgroup of G, N the unipotent part and $\widetilde{K_G}$ the standard maximal compact subgroup of $\widetilde{G(\mathbb{A})}$. For $g \in \widetilde{G(\mathbb{A})}$ write $g = m(A)nk$ with $A \in \mathrm{GL}_n(\mathbb{A})$, $n \in N$ and $k \in \widetilde{K_G}$. Set $a(g) = \det A$ in any such decomposition of g and it is well-defined. The Siegel-Weil section associate to $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$ is defined to be

$$f_\Phi(g, s) = |a(g)|^{s-s_0}\omega(g)\Phi(0),$$

where $s_0 = (m - n - 1)/2$. Then the Eisenstein series

$$E(g, s, f_\Phi) = \sum_{\gamma \in P(k)\setminus G(k)} f_\Phi(\gamma g, s)$$

is absolutely convergent for $\mathrm{Re}(s) > (n+1)/2$ and has meromorphic continuation to the whole s-plane if Φ is $\widetilde{K_G}$-finite. In the case where $m = n+1$, $E(g, s, f_\Phi)$ is holomorphic at $s = (m-n-1)/2 = 0$[2, Page 216].

Let $\mathcal{S}_0(U^n(\mathbb{A}))$ denote the $\widetilde{K_G}$-finite part of $\mathcal{S}(U^n(\mathbb{A}))$. We will show, under some normalization of Haar measures, the following

Theorem 1.1. *Assume that $m = n+1$ and exclude the split binary case. Then*

$$E(g, s, f_\Phi)|_{s=0} = 2I_{\mathrm{REG}}(g, \Phi)$$

for all $\Phi \in \mathcal{S}_0(U^n(\mathbb{A}))$.

Remark 1.2. The regularized theta integral I_{REG} will be defined in Section 2.

2 Regularization of Theta Integral

The results concerning the regularization of theta integrals summarized in this section are due to Ichino[2, Section 1]. We consider the case where the theta integral is not necessarily absolutely convergent for all $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$ i.e., if Q is isotropic and $m - r \leq n+1$.

Take v a finite place of k and temporarily suppress it from notation. If $2 \nmid q$ then there is a canonical splitting of $\widetilde{G}$ over K_G, the standard maximal compact subgroup of G. Identify K_G with the image of the splitting. Let $\mathcal{H}_G$ and $\mathcal{H}_H$ denote the spherical Hecke algebras of $\widetilde{G}$ and H:

$$\mathcal{H}_G = \{\alpha \in \mathcal{H}(\widetilde{G}//K_G) | \alpha(\epsilon g) = \epsilon^m \alpha(g) \text{ for all } g \in \widetilde{G}\},$$
$$\mathcal{H}_H = \mathcal{H}(H//K_H)$$

where $\epsilon = (\mathbf{1}_{2n}, -1) \in \widetilde{G}$.

Proposition 2.1. *Assume $m \leq n+1$ and $r \neq 0$. Fix $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$ and choose a good place v for Φ. Then there exists a Hecke operator $\alpha \in \mathcal{H}_{G_v}$ satisfying the following conditions:*

1. *$I(g, \omega(\alpha)\Phi)$ is absolutely convergent for all $g \in \widetilde{G(\mathbb{A})}$;*
2. *$\theta(\alpha).\mathbf{1} = c_\alpha.\mathbf{1}$ with $c_\alpha \neq 0$.*

Remark 2.2. For the definition of good place please refer to [2, Page 209]. Here θ is an algebra homomorphism between the Hecke algebras $\mathcal{H}_{G_v}$ and $\mathcal{H}_{H_v}$ such that $\omega_Q(\alpha) = \omega_Q(\theta(\alpha))$ as in [2, Prop 1.1]. The trivial representation of H is denoted by $\mathbf{1}$ here.

Definition 2.3. Define the regularized theta integral by

$$I_{\mathrm{REG}}(g, \Phi) = c_\alpha^{-1} I(g, \omega(\alpha)\Phi).$$

Remark 2.4. Also write $I_{\mathrm{REG}}(g, \Phi) = I(g, \Phi)$ for Q anisotropic. The above definition is independent of the choice of v and α.

Let $\mathcal{S}(U^n(\mathbb{A}))_{\mathrm{abc}}$ denote the subspace of $\mathcal{S}(U^n(\mathbb{A}))$ consisting of Φ such that $I(g, \Phi)$ is absolutely convergent for all g. Then I defines an $H(\mathbb{A})$-invariant map

$$I : \mathcal{S}(U^n(\mathbb{A}))_{\mathrm{abc}} \to \mathcal{A}^\infty(G)$$

where $\mathcal{A}^\infty$ is the space of smooth automorphic forms on $\widetilde{G(\mathbb{A})}$ (left-invariant by $G(k)$) without the $\widetilde{K_G}$-finiteness condition.

Proposition 2.5. [2, Lemma 1.9] *Assume $m \leq n + 1$. Then I_{REG} is the unique $H(\mathbb{A})$-invariant extension of I to $\mathcal{S}(U^n(\mathbb{A}))$.*

3 Siegel Eisenstein Series

Now we define the Siegel Eisenstein series. Let χ be a character of $\widetilde{P(\mathbb{A})}$. Let $I(\chi, s)$ denote the induced representation $\mathrm{Ind}_{\widetilde{P(\mathbb{A})}}^{\widetilde{G(\mathbb{A})}} \chi |\det|^s$. A function $f(g, s)$ on $\widetilde{G(\mathbb{A})} \times \mathbb{C}$ is said to be a holomorphic section of $I(\chi, s)$ if

1. $f(g, s)$ is holomorphic with respect to s for each $g \in \widetilde{G(\mathbb{A})}$,
2. $f(pg, s) = \chi(p)|a(p)|^{s+(n+1)/2} f(g, s)$ for $p \in \widetilde{P(\mathbb{A})}$ and $g \in \widetilde{G(\mathbb{A})}$ and
3. $f(\cdot, s)$ is $\widetilde{K_G}$-finite.

For f a holomorphic section of $I(\chi, s)$ we form the Siegel Eisenstein series

$$E(g, s, f) = \sum_{\gamma \in P(k) \setminus G(k)} f(\gamma g, s).$$

Note that $\widetilde{G(\mathbb{A})}$ splits over $G(k)$.

We will specialize to the case where χ is the character associated to the χ in (1.1):

$$(p, z) \mapsto \chi(a(p), z). \tag{3.1}$$

Still denote this character by χ. For $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$ set

$$f_\Phi(g, s) = |a(g)|^{s - \frac{m-n-1}{2}} \omega(g)\Phi(0).$$

Then f_Φ is a holomorphic section of $I(\chi, s)$. The Eisenstein series $E(g, s, f_\Phi)$ is absolutely convergent for $\mathrm{Re}(s) > (n+1)/2$ and has meromorphic continuation to the whole s-plane. From [2, Page 216] we know that if $m = n+1$, $E(g, s, f_\Phi)$ is holomorphic at $s = 0$.

The following definition will be useful later.

Definition 3.1. A holomorphic section $f \in I(\chi, s)$ is said to be a weak SW section associated to $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$ if $f(g, \frac{m-n-1}{2}) = \omega(g)\Phi(0)$.

Define similarly $I_v(\chi_v, s)$ in the local cases. Fix one place v. For $w \neq v$, fix $\Phi_w \in \mathcal{S}(U^n(k_w))$ and let $f^0_w(g_w, s)$ be the associated holomorphic sections where we suppress the subscript Φ_w. Then if $m = n+1$ we have the map

$$\begin{aligned} &I_v \to \mathcal{A} \\ &f_v \mapsto E(g, s, f_v \otimes (\otimes_{w \neq v} f^0_w))|_{s=0}. \end{aligned} \tag{3.2}$$

Then by [8, Prop. 2.2] this map is $\widetilde{G_v}$-intertwining if v is finite or $(\mathfrak{g}_v, \widetilde{K_v})$-intertwining if v is archimedean.

4 Fourier-Jacobi Coefficients

A key step in the proof of Siegel-Weil formula is the comparison of the B-th Fourier coefficients of the Eisenstein series and the regularized theta integral where B is a nonsingular symmetric matrix. It is easy to show that the B-th Fourier coefficient of the Eisenstein series is a product of Whittaker functions. In the case where m is even by [16] and [5] the Whittaker functions can be analytically continued to the whole complex plane. Also true is the case where $n = 1$ and m arbitrary. However in the case m odd this is not fully known. To work around the problem Ikeda[4] used Fourier-Jacobi coefficients to initiate an induction process. The B-th Fourier coefficients can be calculated from the Fourier-Jacobi coefficients from lower dimensional objects.

We generalize the calculation done in [2] and in [4]. First we introduce some subgroups of G, describe Weil representation realized on some other space and then define the Fourier-Jacobi coefficients. The exposition closely follows that in [2]. Put

$$\begin{aligned}
V &= \left\{ v(x,y,z) = \left(\begin{array}{cc|cc} 1 & x & z & y \\ & 1_{n-1} & {}^{\mathrm{t}}y & \\ \hline & & 1 & \\ & & -{}^{\mathrm{t}}x & 1_{n-1} \end{array}\right) \middle| x, y \in k^{n-1}, z \in k \right\}, \\
Z &= \{v(0,0,z) \in V\}, \\
W &= \{v(x,y,0) \in V\}, \\
L &= \{v(x,0,0) \in V\}, \\
G_1 &= \left\{ \left(\begin{array}{cc|cc} 1 & & & \\ & a & & b \\ \hline & & 1 & \\ & c & & d \end{array}\right) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{n-1} \right\}, \\
N_1 &= \left\{ \left(\begin{array}{cc|cc} 1 & & & \\ & 1_{n-1} & & n_1 \\ \hline & & 1 & \\ & & & 1_{n-1} \end{array}\right) \middle| n_1 \in \mathrm{Sym}_{n-1} \right\}.
\end{aligned}$$

Then $V = W \oplus Z$ is a Heisenberg group with centre Z and the symplectic form on W is set to be $\langle v(x_1,y_1,0), v(x_2,y_2,0)\rangle_W = 2(x_1{}^{\mathrm{t}}y_2 - y_1{}^{\mathrm{t}}x_2)$. Here the coefficient 2 is added to facilitate later computation. We set $\langle x, y\rangle = 2x{}^{\mathrm{t}}y$, for x and y row vectors of length $n-1$. Sometimes we identify L with row vectors of length $n-1$. The Schrödinger representation ω of $V(\mathbb{A})$ with central character ψ can be realized on the Schwartz space $\mathcal{S}(L(\mathbb{A}))$:

$$\omega(v(x,y,z))\phi(t) = \phi(t+x)\psi(z + \langle t, y\rangle + \frac{1}{2}\langle x, y\rangle)$$

for $\phi \in \mathcal{S}(L(\mathbb{A}))$. By the Stone-von Neumann theorem, ω is irreducible and unique up to isomorphism. Moreover the Schrödinger representation ω of $V(\mathbb{A})$ naturally extends to the Weil representation ω of $V(\mathbb{A}) \rtimes \widetilde{G_1(\mathbb{A})}$ on $\mathcal{S}(L(\mathbb{A}))$. Let $\widetilde{K_{G_1}}$ denote the standard maximal compact subgroup of $\widetilde{G_1(\mathbb{A})}$ and $\mathcal{S}_0(L(\mathbb{A}))$ the $\widetilde{K_{G_1}}$-finite vectors in $\mathcal{S}(L(\mathbb{A}))$.

For each $\phi \in \mathcal{S}(L(\mathbb{A}))$ define the theta function

$$\vartheta(vg_1, \phi) = \sum_{t \in L(k)} \omega(vg_1)\phi(t)$$

for $v \in V(\mathbb{A})$ and $g_1 \in \widetilde{G_1(\mathbb{A})}$. Suppose that A is an automorphic form on $\widetilde{G(\mathbb{A})}$. Then define a function on $G_1(k) \setminus \widetilde{G_1(\mathbb{A})}$ by

$$\mathrm{FJ}^{\phi}(g_1; A) = \int_{V(k) \setminus V(\mathbb{A})} A(vg_1)\overline{\vartheta(vg_1, \phi)}dv.$$

For $\beta \in \mathrm{Sym}_{n-1}(k)$, let $\mathrm{FJ}^{\phi}_{\beta}(g_1; A)$ be the β-th Fourier coefficient of $\mathrm{FJ}^{\phi}(g_1; A)$.

Suppose that the bilinear form $\langle \ , \ \rangle_Q$ is equal to $\langle \ , \ \rangle_S + \langle \ , \ \rangle_{Q_1}$ where S and Q_1 are quadratic forms of dimension 1 and $n-1$ respectively. Decompose accordingly $U = k \oplus U_1$. Note that $\langle x, y \rangle_S = 2Sxy$. Let $H_1 = \mathrm{O}(U_1)$. With this setup we will use the character ψ_S in the Schrödinger model instead of ψ.

Lemma 4.1. [2, Lemma 4.1] *Let $S \in k^{\times}$ and $\beta \in \mathrm{Sym}_{n-1}(k)$. Let A be an automorphic form on $\widetilde{G(\mathbb{A})}$, and assume that $\mathrm{FJ}^{\phi}_{\beta}(g_1; \rho(f)A) = 0$ for all $\phi \in \mathcal{S}_0(L(\mathbb{A}))$ and all $f \in \mathcal{H}(\widetilde{G(\mathbb{A})})$. Then $A_B = 0$ for*

$$B = \begin{pmatrix} S & \\ & \beta. \end{pmatrix}$$

Proof. For $n \in N$ we set $b(n)$ to be the upper-right block of n and set $b_1(n)$

to be the lower-right block of size $(n-1)\times(n-1)$ of $b(n)$. We compute

$$\begin{aligned}
&\mathrm{FJ}^{\phi}_{\beta}(g_1, A)\\
&= \int_{N_1(k)\,\backslash\, N_1(\mathbb{A})} \int_{V(k)\,\backslash\, V(\mathbb{A})} A(vn_1g_1)\overline{\vartheta(vn_1g_1,\phi)}\psi(-\operatorname{tr}(b_1(n_1)\beta))dvdn_1\\
&= \int_{L(k)\,\backslash\, L(\mathbb{A})} \int_{N(k)\,\backslash\, N(\mathbb{A})} A(nxg_1)\overline{\vartheta(nxg_1)}\psi(-\operatorname{tr}(b_1(n)\beta))dndx\\
&= \int_{L(k)\,\backslash\, L(\mathbb{A})} \int_{N(k)\,\backslash\, N(\mathbb{A})} \sum_{t\in L(k)} A(nxg_1)\overline{\omega(tnxg_1)\phi(0)}\psi(-\operatorname{tr}(b_1(n)\beta))dndx\\
&= \int_{L(k)\,\backslash\, L(\mathbb{A})} \sum_{t\in L(k)} \int_{N(k)\,\backslash\, N(\mathbb{A})} A(ntxg_1)\overline{\omega(ntxg_1)\phi(0)}\psi(-\operatorname{tr}(b_1(n)\beta))dndx\\
&= \int_{L(\mathbb{A})} \int_{N(k)\,\backslash\, N(\mathbb{A})} A(nxg_1)\overline{\omega(nxg_1)\phi(0)}\psi(-\operatorname{tr}(b_1(n)\beta))dndx\\
&= \int_{L(\mathbb{A})} \int_{N(k)\,\backslash\, N(\mathbb{A})} A(nxg_1)\overline{\omega(g_1)\phi(x)\psi_S(z)}\psi(-\operatorname{tr}(b_1(n)\beta))dndx\\
&= \int_{L(\mathbb{A})} \int_{N(k)\,\backslash\, N(\mathbb{A})} A(nxg_1)\overline{\omega(g_1)\phi(x)}\psi(-\operatorname{tr}(b(n)B))dndx\\
&= \int_{L(\mathbb{A})} A_B(xg_1)\overline{\omega(g_1)\phi(x)}dx.
\end{aligned}$$

Since $\mathrm{FJ}^{\phi}_{\beta}(g_1, A) = 0$ for all $g_1 \in \widetilde{G_1(\mathbb{A})}$ we conclude that $A_B(g_1) = 0$ for all $g_1 \in \widetilde{G_1(\mathbb{A})}$. Then we apply a sequence of $f_i \in \mathcal{H}(\widetilde{G(\mathbb{A})})$ that converges to the Dirac delta at $g \in \widetilde{G(\mathbb{A})}$ to conclude that $A_B(g) = 0$ for all $g \in \widetilde{G(\mathbb{A})}$. $\square$

4.1 Fourier-Jacobi coefficients of the regularized theta integrals

Now we consider the Fourier-Jacobi coefficients of the regularized theta integrals

$$\mathrm{FJ}^{\phi}(g_1; I_{\mathrm{REG}}(\Phi)) = c_\alpha^{-1} \int_{V(k)\,\backslash\, V(\mathbb{A})} \int_{H(k)\,\backslash\, H(\mathbb{A})} \Theta(vg_1, h; \omega(\alpha)\Phi)\overline{\vartheta(vg_1,\phi)}dhdv.$$

Put

$$\Psi(\Phi,\phi;u) = \int_{L(\mathbb{A})} \Phi\begin{pmatrix} 1 & x\\ 0 & u\end{pmatrix}\overline{\phi(x)}dx$$

for $u \in U_1^{n-1}(\mathbb{A})$. Then the map

$$\begin{aligned} \mathcal{S}(U^n(\mathbb{A})) \otimes \mathcal{S}(L(\mathbb{A})) &\to \mathcal{S}(U_1^{n-1}(\mathbb{A})) \\ \Phi \otimes \phi &\mapsto \Psi(\Phi, \phi) \end{aligned}$$

is $\widetilde{G_1(\mathbb{A})}$-intertwining, i.e.,

$$\omega(g_1)\Psi(\Phi, \phi) = \Psi(\omega(g_1)\Phi, \omega(g_1)\phi)$$

for $g_1 \in \widetilde{G_1(\mathbb{A})}$. Notice that on $\mathcal{S}(U^n(\mathbb{A}))$ and $\mathcal{S}(U_1^{n-1}(\mathbb{A}))$ the Weil representations are associated with the character ψ and on $\mathcal{S}(L(\mathbb{A}))$ the Weil representation is associated with the character ψ_S. Then we have:

Proposition 4.2. *Suppose that* $\beta \in \mathrm{Sym}_{n-1}(k)$ *with* $\det(\beta) \neq 0$. *Then*

$$\mathrm{FJ}_\beta^\phi(g_1; I_{\mathrm{REG}}(\Phi))$$

is equal to the absolutely convergent integral

$$\int_{H_1(\mathbb{A}) \setminus H(\mathbb{A})} I_{\mathrm{REG},\beta}(g_1, \Psi(\omega(h)\Phi, \phi)) dh.$$

Proof. We need to compute the following integral.

$$\begin{aligned} &\mathrm{FJ}_\beta^\phi(g_1; I_{\mathrm{REG}}(\Phi)) \\ =&c_\alpha^{-1} \int_{N_1(k) \setminus N_1(\mathbb{A})} \int_{V(k) \setminus V(\mathbb{A})} \int_{H(k) \setminus H(\mathbb{A})} \theta(vn_1g_1, h_0, \omega(\alpha)\Phi)\overline{\vartheta(vn_1g_1, \phi)} \\ &\times \psi(-\operatorname{tr} b_1(n_1)\beta) dh_0 dv dn_1. \end{aligned} \tag{4.1}$$

First we consider

$$\int_{V(k) \setminus V(\mathbb{A})} \theta(vg_1, h_0, \Phi)\overline{\vartheta(vg_1, \phi)} dv. \tag{4.2}$$

Suppose $v = v(x,0,0)v(0,y,z)$. Then

$$
\begin{aligned}
&\theta(vg_1, h_0, \Phi)\\
&= \sum_{t\in U^n(k)} \omega(vg_1,h_0)\Phi(t)\\
&= \sum_{t\in U^n(k)} \omega(v(0,y,z)g_1,h_0)\Phi(t\begin{pmatrix}1 & x\\ & 1\end{pmatrix})\\
&= \sum_{t\in U^n(k)} \omega(g_1,h_0)\Phi(t\begin{pmatrix}1 & x\\ & 1\end{pmatrix})\psi(\frac{1}{2}\operatorname{tr}\left(\left\langle t\begin{pmatrix}1 & x\\ & 1\end{pmatrix}, t\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\right\rangle_Q \begin{pmatrix} z & y\\ {}^{\mathrm{t}}y & \end{pmatrix}\right))\\
&= \sum_{t=\left(\begin{smallmatrix} t_1 & t_2\\ t_3 & t_4\end{smallmatrix}\right)} \omega(g_1,h_0)\Phi(t\begin{pmatrix}1 & x\\ & 1\end{pmatrix})\psi(\frac{1}{2}\left\langle\begin{pmatrix}t_1\\ t_3\end{pmatrix}, \begin{pmatrix}t_1\\ t_3\end{pmatrix}\right\rangle_Q (z+2x{}^{\mathrm{t}}y))\\
&\quad\times \psi(\left\langle\begin{pmatrix}t_1\\ t_3\end{pmatrix}, \begin{pmatrix}t_2\\ t_4\end{pmatrix}\right\rangle_Q {}^{\mathrm{t}}y)
\end{aligned}
$$

where $t_1 \in k$, $t_2 \in k^{n-1}$, $t_3 \in U_1(k)$ and $t_4 \in U_1^{n-1}(k)$. Also we expand

$$
\begin{aligned}
&\vartheta(vg_1,\phi)\\
&= \sum_{t\in L(k)} \omega(g_1)\phi(t+x)\psi_S(z+\langle x,y\rangle + \langle t,y\rangle)\\
&= \sum_{t\in L(k)} \omega(g_1)\phi(t+x)\psi_S(z+2x{}^{\mathrm{t}}y + \langle t,y\rangle).
\end{aligned}
$$

Thus if we integrate against z the integral (4.2) vanishes unless

$$\left\langle\begin{pmatrix}t_1\\ t_3\end{pmatrix}, \begin{pmatrix}t_1\\ t_3\end{pmatrix}\right\rangle_Q = 2S.$$

By Witt's theorem there exists some $h \in H(k)$ such that

$$\begin{pmatrix}t_1\\ t_3\end{pmatrix} = h^{-1}\begin{pmatrix}1\\ 0\end{pmatrix}.$$

Note that the stabilizer of $\left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)$ in $H(k)$ is $H_1(k)$. After changing $\left(\begin{smallmatrix}t_2\\ t_4\end{smallmatrix}\right)$ to

$h^{-1}\left(\begin{smallmatrix} t_2 \\ t_4 \end{smallmatrix}\right)$ we find that (4.2) is equal to

$$\int_{W(k)\setminus W(\mathbb{A})} \sum_{h\in H_1(k)\setminus H(k)} \sum_{t_2,t_4} \sum_{t\in L(k)} \omega(g_1)\omega(\alpha)\Phi(h_0^{-1}h^{-1}\begin{pmatrix}1 & t_2\\ 0 & t_4\end{pmatrix}\begin{pmatrix}1 & x\\ & 1\end{pmatrix})$$

$$\times \overline{\omega(g_1)\phi(t+x)}\psi(\mathrm{tr}\left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}t_2\\t_4\end{pmatrix}\right\rangle_Q {}^{\mathrm{t}}y)\psi_S(-\langle t,y\rangle))dxdy$$

$$= \int_{W(k)\setminus W(\mathbb{A})} \sum_{h\in H_1(k)\setminus H(k)} \sum_{t_2,t_4} \sum_{t\in L(k)} \omega(g_1)\Phi(h_0^{-1}h^{-1}\begin{pmatrix}1 & t_2\\ 0 & t_4\end{pmatrix}\begin{pmatrix}1 & x\\ & 1\end{pmatrix})$$

$$\times \overline{\omega(g_1)\phi(t+x)}\psi(2St_2{}^{\mathrm{t}}y)\psi_S(-\langle t,y\rangle))dxdy.$$

Now the integration against y vanishes unless $t=t_2$ and we get

$$= \int_{L(\mathbb{A})} \sum_{h\in H_1(k)\setminus H(k)} \sum_{t_4} \omega(g_1)\Phi(h_0^{-1}h^{-1}\begin{pmatrix}1 & t\\ 0 & t_4\end{pmatrix}\begin{pmatrix}1 & x\\ & 1\end{pmatrix})\overline{\omega(g_1)\phi(t+x)}dx$$

$$= \sum_{h\in H_1(k)\setminus H(k)} \sum_{t\in U_1^{n-1}(k)} \omega(g_1)\Psi(t,\Phi,\phi).$$

Then we consider the integration over $N_1(k)\setminus N_1(\mathbb{A})$ in (4.1). This will kill those terms such that $\langle t,t\rangle_{Q_1}\neq\beta$. Thus (4.1) is equal to

$$c_\alpha^{-1}\int_{H(k)\setminus H(\mathbb{A})} \sum_{h\in H_1(k)\setminus H(k)} \sum_{\substack{t\in U_1^{n-1}(k),\\ \langle t,t\rangle_{Q_1}=\beta}} \omega(g_1)\Psi(t,\omega(hh_0)\omega(\alpha)\Phi,\phi)dh_0.$$

We assume that $2Q_1$ represents β, since otherwise the lemma obviously holds. As $\mathrm{rk}\,\beta=n-1$, $\{t\in U_1^{n-1}(k)|\langle t,t\rangle_{Q_1}=\beta\}$ is a single $H_1(k)$-orbit. Fix a representative t_0 of this orbit. Since the stabilizer of t_0 in $H_1(k)$ is of order $\kappa=2$, $\mathrm{FJ}_\beta^\phi(g_1;I_{\mathrm{REG}}(\Phi))$ is equal to

$$\kappa^{-1}c_\alpha^{-1}\int_{H(k)\setminus H(\mathbb{A})} \sum_{h\in H(k)} \omega(g_1)\Psi(\omega(hh_0)\omega(\alpha)\Phi,\phi;t_0)dh_0.$$

The convergence Lemma in [2] holds also for $m=n+1$ which is recorded here as Lemma 4.3. Thus we can exchange the orders of integration in

$\mathrm{FJ}^{\phi}_{\beta}(g_1; I_{\mathrm{REG}}(\Phi))$ and continue the computation to get

$$\begin{aligned}
&\kappa^{-1}c_\alpha^{-1}\int_{H(\mathbb{A}}\omega(g_1)\Psi(\omega(h)\omega(\alpha)\Phi,\phi;t_0)dh\\
=&\kappa^{-1}c_\alpha^{-1}\sum_{\gamma\in H_1(k)}\int_{H_1(k)\,\backslash\, H(\mathbb{A}}\omega(g_1)\Psi(\gamma t_0,\omega(h)\omega(\alpha)\Phi,\phi)dh\\
=&c_\alpha^{-1}\int_{H_1(k)\,\backslash\, H(\mathbb{A})}\sum_{\langle t,t\rangle_{Q_1}=\beta}\omega(g_1)\Psi(\omega(\gamma h)\omega(\alpha)\Phi,\phi;t)dh\\
=&c_\alpha^{-1}\int_{H_1(\mathbb{A})\,\backslash\, H(\mathbb{A})}\int_{H_1(k)\,\backslash\, H_1(\mathbb{A})}\sum_{\langle t,t\rangle_{Q_1}=\beta}\omega(g_1,h_1)\Psi(\omega(h)\omega(\alpha)\Phi,\phi;t)dh_1dh\\
=&\int_{H_1(\mathbb{A})\,\backslash\, H(\mathbb{A})}I_{\mathrm{REG},\beta}(g_1,\Psi(\omega(h)\Phi,\phi))dh.
\end{aligned}$$

□

Lemma 4.3. *1. Let $t\in U^n(k)$. If $\operatorname{rk} t = n$ then $\int_{H(\mathbb{A})}\omega(h)\Phi(t)dh$ is absolutely convergent for any $\Phi\in\mathcal{S}(U^n(\mathbb{A}))$.*

2. Let $t_1\in U_1^{n-1}(k)$. If $\operatorname{rk} t_1 = n-1$ then

$$\int_{H(\mathbb{A})}\Psi(\omega(h)\Phi,\phi;t_1)dh=\int_{H(\mathbb{A})}\int_{L(\mathbb{A})}\omega(h)\Phi\begin{pmatrix}1 & x\\ 0 & t_1\end{pmatrix}\overline{\phi(x)}dxdh$$

is absolutely convergent for any $\Phi\in\mathcal{S}(U^n(\mathbb{A}))$ and $\phi\in\mathcal{S}(L(\mathbb{A}))$.

Proof. The argument in [9, pp. 59-60] also includes the case $m=n+1$ and it proves (1). For (2) consider the function on $U^n(\mathbb{A})$

$$\varphi(u)=\int_{L(\mathbb{A})}\Phi(u\begin{pmatrix}1 & x\\ & 1_{n-1}\end{pmatrix})\overline{\phi(x)}dx.$$

This integral is absolutely convergent and defines a smooth function on $U^n(\mathbb{A})$. Furthermore $\varphi\in\mathcal{S}(U^n(\mathbb{A}))$. Then we apply (1) to get (2). □

4.2 Fourier-Jacobi coefficients of the Siegel Eisenstein series

Now we compute the Fourier-Jacobi coefficients of the Siegel Eisenstein series

$$\mathrm{FJ}^{\phi}(g_1,E(f,s))=\int_{V(k)\,\backslash\, V(\mathbb{A})}E(vg_1,f,s)\overline{\vartheta(vg_1,\phi)}dv.$$

Let χ_1 be the character associated to ψ and Q_1 defined similarly as in (1.1).

Proposition 4.4. *For $\phi \in \mathcal{S}_0(\widetilde{G_1(\mathbb{A})})$ we have*

$$\mathrm{FJ}^{\phi}(g_1, E(f,s)) = \sum_{\gamma \in P_1(k) \setminus G_1(k)} R(\gamma g_1, f, s, \phi)$$

where

$$R(g_1, f, s, \phi) = \int_{V(\mathbb{A})} f(w_n v w_{n-1} h) \overline{\omega(v w_{n-1} h)\phi(0)} dv.$$

is a holomorphic section of $Ind_{\widetilde{P_1(\mathbb{A})}}^{\widetilde{G_1(\mathbb{A})}}(\chi_1, s)$ for $\mathrm{Re}\, s >> 0$. Furthermore $R(g_1, f, s, \phi)$ is absolutely convergent for $\mathrm{Re}\, s > -(n-3)/2$ and can be analytically continued to the domain $\mathrm{Re}\, s > -(n-2)/2$.

Proof. This was proved in [3, Theorem 3.2 and Theorem 3.3]. □

Now we will relate $R(g_1, f_\Phi, s, \phi)$ to $\Psi(g_1, \Phi, \phi)$. First we need a lemma.

Lemma 4.5. *Let $n = 1$ and $S \in k^\times$. Assume $m \geq 5$ or $(m, r) = (4, 0)$, $(4, 1)$, $(3, 0)$ or $(2, 0)$. Let $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $s_0 = \frac{m}{2} - 1$. Then*

$$\int_{\mathbb{A}} f_\Phi(w \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, s) \overline{\psi_S(z)} dz \tag{4.3}$$

can be meromorphically continued to the whole s-plane and is holomorphic at $s = s_0$. Its value at $s = s_0$ is 0 if Q does not represent S. If $Q = \left(\begin{smallmatrix} S & \\ & Q_1 \end{smallmatrix}\right)$ then its value at $s = s_0$ is equal to the absolutely convergent integral

$$\kappa \int_{H_1(\mathbb{A}) \setminus H(\mathbb{A})} \Phi(h^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}) dh$$

where $\kappa = 2$ for $(m, r) = (2, 0)$ and $\kappa = 1$ otherwise.

Proof. The cases excluded are those where the Eisenstein series has a pole at $s = s_0$ or when the theta integral is not absolutely convergent. Then (4.3) is the S-th Fourier coefficient of $E(g, s, f)$ and

$$\int_{H_1(\mathbb{A}) \setminus H(\mathbb{A})} \Phi(h^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}) dh$$

is the S-th Fourier coefficient of $I(g, s)$. Thus the lemma follows from the known Siegel-Weil formula for $n = 1$. Please see [4] for details. □

Proposition 4.6. *Assume $m = n+1$. Also assume that $m \geq 5$ or $(m,r) = (4,0)$, $(4,1)$, $(3,0)$ or $(2,0)$. Let $\phi \in \mathcal{S}_0(L(\mathbb{A}))$ and $f_\Phi(s)$ be a holomorphic section of $I(\chi,s)$ associated to $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$. If Q does not represent S then $R(g_1, f_\Phi, s, \phi) = 0$. If*

$$Q = \begin{pmatrix} S & \\ & Q_1 \end{pmatrix}$$

then

$$\mathrm{FJ}^\phi(g_1; E(s, f_\Phi))|_{s=0} = \int_{H_1(\mathbb{A}) \setminus H(\mathbb{A})} E(g_1, f_{\Psi(\omega_Q(h)\Phi, \phi)}(s)) dh.$$

Proof. First we simplify $R(g_1, f_\Phi, s, \phi)$. We will suppress the subscript Φ. Suppose $v = v(x,0,0)v(0,y,z)$. Then $R(g_1, f, s, \phi)$ is equal to

$$\begin{aligned}
&\int_{V(\mathbb{A})} f(w_n v(0,y,z) w_{n-1} g_1, s) \overline{\omega(w_{n-1}g_1)\phi(x)\psi_S(z + \langle x, y\rangle)} dv \\
= &\int_{V(\mathbb{A})} \int_{L(\mathbb{A})} f(w_n v(0,y,z) w_{n-1} g_1, s) \overline{\omega(g_1)\phi(t)\psi_S(\langle -x, t\rangle)\psi_S(z + \langle x, y\rangle)} dt dv.
\end{aligned}$$

Integration against x vanishes unless $y = t$. Thus we continue

$$= \int_{\mathbb{A}} \int_{L(\mathbb{A})} f(w_n v(0,y,z) w_{n-1} g_1, s) \overline{\omega(g_1)\phi(y)\psi_S(z)} dy dz.$$

Embed $\mathrm{Sp}(2)$ into $G = \mathrm{Sp}(2n)$ by

$$g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & & b & \\ & 1_{n-1} & & 0_{n-1} \\ c & & d & \\ & 0_{n-1} & & 1_{n-1} \end{pmatrix}$$

and denote this embedding by ι. Also denote the lift $\widetilde{\mathrm{Sp}(2)} \to \widetilde{\mathrm{Sp}(2n)}$ by ι. Then as a function of $g_0 \in \widetilde{\mathrm{Sp}(2)}$,

$$f(\iota(g_0) w_{n-1} \left(\begin{array}{c|cc} 1_n & 0 & y \\ & {}^t y & 0_{n-1} \\ \hline 0_n & \multicolumn{2}{c}{1_n} \end{array}\right) w_{n-1} g_1)$$

is a weak SW section associated to

$$u \mapsto \omega(w_{n-1} \left(\begin{array}{c|cc} 1_n & 0 & y \\ & {}^{\mathrm{t}}y & 0_{n-1} \\ \hline 0_n & \multicolumn{2}{c}{1_n} \end{array}\right) w_{n-1} g_1))\Phi(u, 0),$$

a Schwartz function in $\mathcal{S}(U(\mathbb{A}))$. Then by lemma 4.5, if Q does not represent S then $R(g_1, f, 0, \phi) = 0$. If $Q = \left(\begin{smallmatrix} S & \\ & Q_1 \end{smallmatrix}\right)$ then by Lemma 4.5 $R(g_1, f, 0, \phi)$ is equal to

$$\int_{H_1(\mathbb{A}) \setminus H(\mathbb{A})} \int_{L(\mathbb{A})} \omega(w_{n-1} \left(\begin{array}{c|cc} 1_n & 0 & y \\ & {}^{\mathrm{t}}y & 0_{n-1} \\ \hline 0_n & \multicolumn{2}{c}{1_n} \end{array}\right) w_{n-1} g_1)\Phi(h^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})\overline{\omega(g_1)\phi(-y)} dy dh.$$

Note the part

$$\omega(w_{n-1} \left(\begin{array}{c|cc} 1_n & 0 & y \\ & {}^{\mathrm{t}}y & 0_{n-1} \\ \hline 0_n & \multicolumn{2}{c}{1_n} \end{array}\right) w_{n-1} g_1)\Phi(h^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = \omega(g_1)\Phi(h^{-1} \begin{pmatrix} 1 & -y \\ 0 & 0 \end{pmatrix}).$$

Thus we find

$$R(g_1, f, 0, \phi) = \int_{H_1(\mathbb{A}) \setminus H(\mathbb{A})} \omega(g_1)\Psi(0, \omega(h)\Phi, \phi) dh.$$

The calculation relies on the Siegel-Weil formula in the case $n = 1$ and $m - r > 2$ or $r = 0$ arbitrary. Thus we have to exclude certain cases where the Eisenstein series may have a pole at $s_0 = (m-2)/2$. $\square$

Remark 4.7. The cases not covered above are $(m, r) = (4, 2)$, $(3, 1)$ and $(2, 1)$. The anisotropic cases and the m even cases of the Siegel-Weil formula were dealt with in [9]. Thus we cannot go down only when we reach the $(m, r) = (3, 1)$ case.

5 Some Representation Theory

Now we want to study irreducible submodules of the induced representations and show that it is nonsingular in the sense of Howe[1]. In Section 6 we will interpret the difference $A(g, \Phi) = E(g, s, f_\Phi)|_{s=0} - 2I(g, \Phi)$ as an element in an irreducible nonsingular submodules. This forces the B-th Fourier coefficients of A to vanish if B is not of full rank.

Fix v a finite place of k and suppress it from notation. Thus k is a nonarchimedean local field for the present. We consider the various groups over k. Let χ be a quasicharacter of $\widetilde{P}$ trivial on N and form the normalised induced representation $I(\chi) = \mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}}(\chi)$. Define maximal parabolic subgroups of GL_n:

$$Q_r = \left\{ \begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \middle| a \in \mathrm{GL}_{n-r}, b \in \mathrm{GL}_r \right\}.$$

Here r is not related to the Witt index of Q.

Lemma 5.1. *The Jacquet module of $I(\chi)_N$ has an $\widetilde{M}$-stable filtration*

$$I(\chi)_N = I^0 \supset I^1 \supset \cdots \supset I^n \supset I^{n+1} = 0$$

with successive quotients

$$Z^r(\chi) = I^r/I^{r+1} \cong \mathrm{Ind}_{\widetilde{Q_r}}^{\widetilde{\mathrm{GL}_n}}(\xi_r)$$

where ξ_r is the quasicharacter of Q_r given by

$$\xi_r\left(\left(\begin{pmatrix} a & * \\ 0 & b \end{pmatrix}, \zeta\right)\right) = \chi\left(\left(m\left(\begin{pmatrix} a & \\ & {}^t b^{-1} \end{pmatrix}\right), \zeta\right)\right) |\det a|^{\frac{n+1-r}{2}} |\det b|^{\frac{r+1}{2}}.$$

Proof. We follow the proof in [10]. Choose double coset decomposition representatives w_r for $\widetilde{P} \backslash \widetilde{G} / \widetilde{P}$: for $0 \leq r \leq n$, let

$$w_r = \left(\begin{pmatrix} 1_{n-r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_r \\ 0 & 0 & 1_{n-r} & 0 \\ 0 & -1_r & 0 & 0 \end{pmatrix}, 1 \right).$$

Then the relative Bruhat decomposition holds $\widetilde{G} = \amalg_{j=0}^n \widetilde{P} w_j \widetilde{P}$. Let $J^0 = I(\chi)$ and for $1 \leq r \leq n+1$, set $J^r = \left\{ f \in J^0 \middle| f = 0 \text{ on } \widetilde{P} w_{r-1} \widetilde{P} \right\}$. Alternatively set $J^{n+1} = 0$ and $J^r = \left\{ f \in I(\chi) \middle| \mathrm{supp}(f) \subset \amalg_{j=r}^n \widetilde{P} w_j \widetilde{P} \right\}$. Also define

$$N_r = \left\{ n \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \middle| a \in \mathrm{Sym}_r(k) \right\}.$$

We check that we have a $\widetilde{P}$-intertwining map

$$
\begin{aligned}
&J^r \to \operatorname{Ind}_{\widetilde{Q_r}}^{\widetilde{\mathrm{GL}_n}}(\xi_r),\\
&\Phi \mapsto \left\{\Psi : (m(a),\zeta) \mapsto \int_{N_r} \Phi(w_r n(m(a),\zeta))dn\right\}.
\end{aligned}
$$

Obviously the map factors through J^{r+1} if the above is well-defined. Notice that the properties of Rao's 2-cocycle[15, Theorem 5.3] implies that $w_r n(m,\zeta) = (m',\zeta) w_r n'$ for some other elements $m' \in M$ and $n' \in N$. Standard computation then shows that Ψ is in the space of $\operatorname{Ind}_{\widetilde{Q_r}}^{\widetilde{\mathrm{GL}_n}}(\xi_r)$.

Gustafson checked in Sp_n case that the integral converges and that the map is surjective and that the kernel of the map $J^r/J^{r+1} \to \operatorname{Ind}_{\widetilde{Q_r}}^{\widetilde{\mathrm{GL}_n}} \xi_r$ is $J^r/J^{r+1}(N)$. By exactness of Jacquet functor we get an $\widetilde{M}$-module isomorphism of J_N^r/J_N^{r+1} with the space $\operatorname{Ind}_{\widetilde{Q_r}}^{\widetilde{\mathrm{GL}_n}} \xi_r$. Setting $I^r = J_N^r$ for each r finishes the proof. □

We are interested in the case where $\chi(m(a),\zeta)$ is the one in (3.1).

Lemma 5.2. *Suppose that $\pi \subset I(\chi)$ is a $\widetilde{G}$-submodule. Then*

$$\dim \operatorname{Hom}_{\widetilde{G}}(\pi, I(\chi)) \le 2.$$

In particular, $I(\chi)$ has at most two irreducible submodules.

Proof. The centre $\widetilde{Z}$ of $\widetilde{\mathrm{GL}_n}$ consists of elements of the form

$$(aI_n, \zeta).$$

Also note that we can view χ as a character on $\widetilde{\mathrm{GL}_n}$. Given π

$$\operatorname{Hom}_{\widetilde{G}}(\pi, I(\chi)) = \operatorname{Hom}_{\widetilde{\mathrm{GL}_n}}(\pi_N, \chi|\ |^{(n+1)/2}).$$

Now we consider the generalized eigenspaces of π_N and of $I(\chi)_N$ with respect to the action of $\widetilde{Z}$, where the eigencharacter of interest is

$$\mu(aI_n,\zeta) = \chi(aI_n,\zeta)|a|^{n(n+1)/2}.$$

On the other hand the central characters of the successive quotients $Z_r(\chi)$ of $I(\chi)_N$ are

$$(aI_n,\zeta) \mapsto \chi\left(\begin{pmatrix} aI_{n-r} & \\ & aI_r \end{pmatrix}, \zeta\right) |a|^{\frac{n^2+n-2nr-2r}{2}}.$$

If $r=0$ then one of these coincides with μ. If $r=n$ and since $\chi(a^n,\zeta)=\chi(a^{-n},\zeta)$ we get one more solution. Thus we get the bound

$$\dim \mathrm{Hom}_{\widetilde{G}}(\pi, I(\chi)) \leq 2.$$

$\square$

Let $R_n(U)$ denote the image of the map

$$\begin{aligned} \mathcal{S}(U^n) &\to I(\chi) \\ \Phi &\mapsto \omega(g)\Phi(0). \end{aligned}$$

This map induces an isomorphism $\mathcal{S}(U^n)_H \cong R_n(U)$ by [13]. Let U' be the quadratic space with the same dimension and determinant with U but with opposition Hasse invariant.

Lemma 5.3. *The G-modules $R_n(U)$ and $R_n(U')$ are irreducible. Furthermore $I(\chi) \cong R_n(U) \oplus R_n(U')$,*

Proof. We have an intertwining operator

$$\begin{aligned} M : I(\chi) &\to I(\chi) \\ f &\mapsto (g \mapsto \int_N f(w_n n g) dn). \end{aligned}$$

Thus $I(\chi)$ is unitarizable and hence completely reducible. Also by [8, Prop. 3.4] we know that $R_n(U)$ and $R_n(U')$ are inequivalent and by [8, Lemmas 3.5 and 3.6] it cannot happen that one is contained in the other. These combined with Lemma 5.2 force $I(\chi) \cong R_n(U) \oplus R_n(U')$ with $R_n(U)$ and $R_n(U')$ irreducible. $\square$

Lemma 5.4. *Assume $m=n+1$. Then $R_n(U)$ is a nonsingular representation of $\widetilde{G}$ in the sense of Howe[1].*

Proof. This follows from [8, Prop 3.2(ii)]. $\square$

6 Proof of Siegel-Weil Formula

Combining the results above we are ready to show the Siegel-Weil formula. Note the assumption that $m=n+1$. We will focus on the cases where metaplectic double cover of $\mathrm{Sp}(2n)$ must be considered so in the proofs we only deal with the cases where m is odd. For the cases where m is even please refer to [9]. Set $A(g,\Phi)=E(g,s,f_\Phi)|_{s=0}-2I(g,\Phi)$.

Proposition 6.1. *Assume $m \geq 3$ or $m = 2$ and V anisotropic. Then for $B \in \mathrm{Sym}_n(k)$ with rank n, the Fourier coefficients $A_B = 0$.*

Proof. Without loss of generality suppose $B = \begin{pmatrix} S & \\ & \beta \end{pmatrix}$ for some $S \in k^\times$ and some nonsingular $\beta \in \mathrm{Sym}_{n-1}(k)$. First we prove the anisotropic case. The base case $m = 2$ and $n = 1$ was proved in [14, Chapter 4]. Now for $m = n+1$, if Q does not represent S then by Prop. 4.2 and Prop. 4.6 we obviously have $A_B = 0$. If Q represents S then we can just assume that $Q = \begin{pmatrix} S & \\ & Q_1 \end{pmatrix}$. Note that Q_1 is still anistropic. Again by Prop. 4.2 and Prop. 4.6 and the induction hypothesis we conclude that A_B=0.

Secondly we assume Q to be isotropic and $m \geq 4$, so Q represents S. We can just assume that $Q = \begin{pmatrix} S & \\ & Q_1 \end{pmatrix}$.

From Section 4 we get by Prop. 4.2 and Prop. 4.6 and the m even case $\mathrm{FJ}^{\phi}_{\beta}(A) = 0$ for all $\phi \in \mathcal{S}(L(\mathbb{A}))$ if the rank of β is $n-1$. Then by Lemma 4.1, A_B vanishes for $B \in \mathrm{Sym}^n(k)$ such that $\det B \neq 0$.

Finally assume that Q is isotropic and $m = 3$. By the expression for $E_B(g, s, f_\Phi)$ in Remark 4.1 of [14] we know that $E_B(g, s, f_\Phi)$ is analytic at $s = 0$. Thus Prop 4.2 of [14] holds: $E_B(g, 0, f_\Phi) = cI_B(g, \Phi)$ where c does not depend on Φ or B. Now we consider objects in dimension $m = 4$ and $n = 3$. Here $E_B(g, 0, f_\Phi) = 2I_B(g, \Phi)$. By Prop. 4.2 and Prop. 4.6 and the independence of c on Φ we conclude that $c = 2$ and this finishes the proof of the lemma. $\square$

Remark 6.2. For the split binary case please refer to [9] and note that the Eisenstein series vanishes at 0, so the Siegel-Weil formula takes a different form.

In the above proof we the argument dealing with the case $(m, r) = (3, 1)$ can also be used to prove other cases. We use two methods for the record.

Proof of Theorem 1.1. Fix a finite place v of k and fix for each place w not equal to v a $\Phi^0_w \in \mathcal{S}_0(U^n(\mathbb{A}))$. Consider the map A_v which sends $\Phi_v \in \mathcal{S}_0(U^n_v)$ to $A(g, \Phi_v \otimes (\otimes_{w \neq v} \Phi^0_w))$. By invariant distribution theorem $R_n(U_v) \cong \mathcal{S}(U^n_v)_{H_v}$. Thus A_v is a $\widetilde{G_v}$-intertwining operator

$$\mathcal{S}(U^n_v) \to \mathcal{A}$$

which actually factors through $R_n(U_v)$. As $R_n(U_v)$ is nonsingular in the sense of [1] by Lemma 5.4 take $f \in \mathcal{S}(\mathrm{Sym}^n(k_v))$ such that its Fourier transform is supported on nonsingular symmetric matrices. Then for all $g \in G(\mathbb{A})$ with $g_v = 1$ and all $B \in \mathrm{Sym}_n(k)$ we have

$$
\begin{aligned}
(\rho(f).A(\Phi))_B(g) &= \int_{\mathrm{Sym}_n(k)\setminus \mathrm{Sym}_n(\mathbb{A})} \int_{\mathrm{Sym}_n(k_v)} f(c)A(\Phi)(ngn(c))\psi(-\operatorname{tr}(Bb))dcdb \\
&= \int_{\mathrm{Sym}_n(k)\setminus \mathrm{Sym}_n(\mathbb{A})} \int_{\mathrm{Sym}_n(k_v)} f(c)A(\Phi)(nn(c)g)\psi(-\operatorname{tr}(Bb))dcdb \\
&= \int_{\mathrm{Sym}_n(k)\setminus \mathrm{Sym}_n(\mathbb{A})} \int_{\mathrm{Sym}_n(k_v)} f(c)A(\Phi)(ng)\psi(-\operatorname{tr}(B(b-c)))dcdb \\
&= \hat{f}(-B)A(\Phi)_B(g).
\end{aligned}
$$

The above is always 0, since $\hat{f}(B) = 0$ if $\operatorname{rk} B < n$ and $A(\Phi)_B \equiv 0$ if $\operatorname{rk} B = n$. Thus $\rho(f)A(\Phi) = 0$ as $G(k)\prod_{w\neq v} G_w$ is dense in $G(\mathbb{A})$. Since f does not act by zero and $R_n(U_v)$ is irreducible we find that in fact $A(\Phi) = 0$ and this concludes the proof. □

7 Inner Product Formula

We will apply Theorem 1.1 to show a case of Rallis Inner Product formula via the doubling method. We will also deduce the location of poles of Langlands L-function from information on the theta lifting.

Let G_2 denote the symplectic group of rank $2n$, P_2 its Siegel parabolic and $\widetilde{G_2(\mathbb{A})}$ the metaplectic group. Let $H = \mathrm{O}(U, Q)$ with (U, Q) a quadratic space of dimension $2n+1$. Let π be a genuine irreducible cuspidal automorphic representation of $\widetilde{G(\mathbb{A})}$. For $f \in \pi$ and $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$ define

$$
\Theta(h; f, \Phi) = \int_{G(k)\setminus \widetilde{G(\mathbb{A})}} f(g)\Theta(g, h; \Phi)dg.
$$

Consider the mapping

$$
\begin{aligned}
\iota_0 : \widetilde{G(\mathbb{A})} \times \widetilde{G(\mathbb{A})} &\to \widetilde{G_2(\mathbb{A})} \\
((g_1, \zeta_1), (g_2, \zeta_2)) &\mapsto (\begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix}, \zeta_1\zeta_2)
\end{aligned}
$$

if $g_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$. For $g \in G$ set

$$\check{g} = \begin{pmatrix} 1_n & \\ & -1_n \end{pmatrix} g \begin{pmatrix} 1_n & \\ & -1_n \end{pmatrix}.$$

Then let $\iota((g_1, \zeta_1), (g_2, \zeta_2)) = \iota_0((g_1, \zeta_1), (\check{g}_2, \zeta_2))$. In fact we are just mapping $((g_1, \zeta_1), (g_2, \zeta_2))$ to

$$(\begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & -b_2 \\ c_1 & & d_1 & \\ & -c_2 & & d_2 \end{pmatrix}, \zeta_1\zeta_2).$$

With this we find $\Theta(\iota(g_1, g_2), h; \Phi) = \Theta(g_1, h; \Phi_1)\overline{\Theta(g_2, h; \Phi_2)}$ if we set $\Phi = \Phi_1 \otimes \overline{\Phi_2}$ for $\Phi_i \in \mathcal{S}(U^n(\mathbb{A}))$. Suppose the inner product

$$\begin{aligned} &\langle \Theta(f_1, \Phi_1), \Theta(f_2, \Phi_2) \rangle \\ &= \int_{H(k) \setminus H(\mathbb{A})} \int_{(G(k) \times G(k) \setminus (\widetilde{G(\mathbb{A})} \times \widetilde{G(\mathbb{A})})} f_1(g_1)\Theta(g_1, h; \Phi_1)\overline{f_2(g_2)\Theta(g_2, h; \Phi_2)} dg_1 dg_2 dh \end{aligned}$$

is absolutely convergent. Then it is equal to

$$\begin{aligned} &\int_{(G(k) \times G(k) \setminus (\widetilde{G(\mathbb{A})} \times \widetilde{G(\mathbb{A})})} f_1(g_1)\overline{f_2(g_2)} \left(\int_{H(k) \setminus H(\mathbb{A})} \Theta(g_1, h; \Phi_1)\overline{\Theta(g_2, h; \Phi_2)} dh \right) dg_1 dg_2 \\ &= \int_{(G(k) \times G(k) \setminus (\widetilde{G(\mathbb{A})} \times \widetilde{G(\mathbb{A})})} f_1(g_1)\overline{f_2(g_2)} \left(\int_{H(k) \setminus H(\mathbb{A})} \Theta(\iota(g_1, g_2), h; \Phi) \right) dg_1 dg_2 \\ &= \int_{(G(k) \times G(k) \setminus (\widetilde{G(\mathbb{A})} \times \widetilde{G(\mathbb{A})})} f_1(g_1)\overline{f_2(g_2)} I(\iota(g_1, g_2); \Phi) dg_1 dg_2. \end{aligned}$$

Thus we define the regularized inner product by

$$\begin{aligned} &\langle \Theta(f_1, \Phi_1), \Theta(f_2, \Phi_2) \rangle_{\mathrm{REG}} \\ &\qquad = \int_{(G(k) \times G(k) \setminus (\widetilde{G(\mathbb{A})} \times \widetilde{G(\mathbb{A})})} f_1(g_1)\overline{f_2(g_2)} I_{\mathrm{REG}}(\iota(g_1, g_2); \Phi) dg_1 dg_2. \end{aligned} \tag{7.1}$$

We could apply the Siegle-Weil formula now, but then we would not be able to use the basic identity in [12] directly. Thus we will follow Li[11] to continue the computation. Now consider G to be the group of isometry of the

$2n$-dimensional space V with symplectic form $(\ ,\ )$ and suppose $V = X \oplus Y$ with X and Y maximal isotropic subspaces. Then the Weil representation ω considered up till now is in fact realised on $\mathcal{S}(U \otimes X(\mathbb{A}))$. Let $V_2 = V \oplus V$ be endowed with the split form $(\ ,\ ) - (\ ,\ )$. The space $U \otimes V_2$ has two complete polarizations $U \otimes V_2 = (U \otimes (X \oplus X)) \oplus (U \otimes (Y \oplus Y))$ and $U \otimes V_2 = (U \otimes V^d) \oplus (U \otimes V_d)$, where $V^d = \{(v, v) | v \in V\}$ and $V_d = \{(v, -v) | v \in V\}$. There is an isometry

$$\delta : \mathcal{S}((U \otimes (X \oplus X))(\mathbb{A})) \to \mathcal{S}((U \otimes V_d)(\mathbb{A}))$$

intertwining the action of $\widetilde{G_2(\mathbb{A})}$. We have as in [11, Eq. (13)]

$$\delta(\Phi_1 \otimes \overline{\Phi_2})(0) = \langle \Phi_1, \Phi_2 \rangle.$$

Then (7.1) is equal to

$$\int_{(G(k)\times G(k)\,\backslash(\widetilde{G(\mathbb{A})}\times\widetilde{G(\mathbb{A})})} f_1(g_1)\overline{f_2(g_2)} I_{\mathrm{REG}}(\iota(g_1, g_2); \delta\Phi) dg_1 dg_2.$$

Note here the theta function is associated to the Weil representation realised on $\mathcal{S}(U \otimes V_d)(\mathbb{A}))$. Now we apply the regularized Siegel-Weil formula and get

$$2^{-1} \int_{(G(k)\times G(k)\,\backslash(\widetilde{G(\mathbb{A})}\times\widetilde{G(\mathbb{A})})} f_1(g_1)\overline{f_2(g_2)} E(\iota(g_1, g_2), s, F_{\delta\Phi})|_{s=0} dg_1 dg_2$$

where to avoid conflict of notation we use $F_{\delta\Phi}$ to denote the Siegel-Weil section associated to $\delta\Phi$.

Set the zeta function to be

$$Z(f_1, f_2, s, F) = 2^{-1} \int_{(G(k)\times G(k)\,\backslash(\widetilde{G(\mathbb{A})}\times\widetilde{G(\mathbb{A})})} f_1(g_1)\overline{f_2(g_2)} E(\iota(g_1, g_2), s, F) dg_1 dg_2 \tag{7.2}$$

and we will deduce some of its properties.

By the basic identity in [12] generalized to the metaplectic case and by [11, Eq. (25)] $Z(f_1, f_2, s, F)$ is equal to

$$\begin{aligned} &2^{-1} \int_{\widetilde{G(\mathbb{A})}} F(\iota(g, 1), s) \int_{G(k)\,\backslash\,\widetilde{G(\mathbb{A})}} f_1(g_2 g)\overline{f_2(g)} dg_2 dg \\ &= 2^{-1} \int_{\widetilde{G(\mathbb{A})}} F(\iota(g, 1), s).\langle \pi(g) f_1, f_2 \rangle dg \\ &= \int_{G(\mathbb{A})} F(\iota(g, 1), s) \langle \pi(g) f_1, f_2 \rangle dg. \end{aligned}$$

The last equation holds since we are dealing with genuine representations.

Suppose F and f_i are factorizable. Then the above factorizes into a product of local zeta integrals

$$Z(f_{1,v}, f_{2,v}, s, F_v) = \int_{G_v} F_v(\iota(g_v, 1), s)\langle \pi_v(g_v) f_{1,v}, f_{2,v}\rangle dg_v.$$

Let S be a finite set of places of k containing all the archimedean places, even places, outside which π_v is an unramified principal series representation, f_i spherical and normalized, F normalized spherical Siegel-Weil section and ψ_v unramified. Notice $\pi_v \otimes \chi_v$ can be viewed as a representation of G_v rather than $\widetilde{G_v}$. Then by [11, Prop. 4.6] the local integral $Z(f_{1,v}, f_{2,v}, s, F_v)$ is equal to

$$\frac{L(s + \frac{1}{2}, \pi_v \otimes \chi_v)}{\widetilde{d}_{G_{2,v}}(s)}$$

where $L(s + \frac{1}{2}, \pi_v \otimes \chi_v)$ is the Langlands L-function associated to $\pi \otimes \chi$ and

$$\widetilde{d}_{G_{2,v}}(s) = \zeta_v(s + \frac{1}{2}) \cdot \prod_{i=1}^{n} \zeta_v(2s + 2i).$$

Note here we normalize the Haar measure on $\widetilde{G_v}$ so that K_{G_v} has volume 1.

Proposition 7.1. *The poles of $L^S(s, \pi \otimes \chi)$ in $\mathrm{Re}(s) > 1/2$ are simple and are contained in the set*

$$\{1, \frac{3}{2}, \frac{5}{2}, \cdots, n + \frac{1}{2}\}.$$

Proof. By [9, Prop. 7.2.1] we deduce that the poles of $L^S(s + \frac{1}{2}, \pi \otimes \chi)$ are contained in the set of poles of $\widetilde{d}^S_{G_2}(s) E(s, \iota(g_1, g_2), F)$. The poles of the Eisenstein series in $\mathrm{Re}(s) > 0$ are simple and are contained in $\{1, 2, \ldots, n\}$, c.f. [2, Page 216]. From this we get the proposition. $\square$

Our result combined with that of Ichino's[2] gives an analogue of Kudla and Rallis's [9, Thm. 7.2.5]. Let $m_0 = 4n + 2 - m$ be the dimension of the complementary space U_0 of U.

Theorem 7.2. *1. The poles of $L^S(s, \pi \otimes \chi)$ in the half plane $\mathrm{Re}\, s > 1/2$ are simple and are contained in the set*

$$\left\{1, \frac{3}{2}, \frac{5}{2}, \ldots, \left[\frac{n+1}{2}\right] + \frac{1}{2}\right\}.$$

2. *If $4n+2 > m > 2n+1$ then suppose $L^S(s,\pi\otimes\chi)$ has a pole at $s = n+1-(m_0/2)$. If $m = 2n+1$ then suppose $L^S(s,\pi\otimes\chi)$ does not vanish at $s = n+1-(m_0/2) = 1/2$. Then there exists a quadratic space U_0 over k with dimension m_0 and $\chi_{U_0} = \chi$ such that $\Theta_{U_0}(\pi) \neq 0$ where $\Theta_{U_0}(\pi)$ denotes the space of automorphic forms $\Theta(f,\Phi)$ on $\mathrm{O}_{U_0}(\mathbb{A})$ for $f \in \pi$ and $\Phi \in \mathcal{S}(U_0(\mathbb{A})^n)$.*

Proof. Consider the residue of $L^S(s,\pi\otimes\chi)$ at $s_0+\frac{1}{2}$ with $s_0 \in \{1,2,\ldots,n\}$. Then it vanishes if the residue of $Z(f_1,f_2,s,F_{\delta\Phi})$ vanishes at s_0. Note that for some choice of Schwartz function Φ, F is the normalized spherical standard Siegel-Weil section. We apply the Siegel-Weil formula of Ichino's[2] and ours and get

$$\mathrm{Res}_{s=s_0} Z(f_1,f_2,s,F_{\delta\Phi}) = \int_{(G(k)\times G(k)\backslash(\widetilde{G(\mathbb{A})}\times\widetilde{G(\mathbb{A})})} f_1(g_1)\overline{f_2(g_2)} I_{\mathrm{REG}}(\iota(g_1,g_2),\Phi)dg_1dg_2$$

or

$$Z(f_1,f_2,0,F_{\delta\Phi}) = \int_{(G(k)\times G(k)\backslash(\widetilde{G(\mathbb{A})}\times\widetilde{G(\mathbb{A})})} f_1(g_1)\overline{f_2(g_2)} I_{\mathrm{REG}}(\iota(g_1,g_2),\Phi)dg_1dg_2$$

which is exactly the regularized pairing of theta liftings $\Theta(f_1,\Phi_1)$ and $\Theta(f_2,\Phi_2)$. Then if the residue of $L^S(s,\pi\otimes\chi)$ at $s_0+\frac{1}{2}$ does not vanish or $L^S(s,\pi\otimes\chi)$ does not vanish at $\frac{1}{2}$ then the space of theta lifting does not vanish and we prove 2).

On the other hand the space of theta lifting vanishes if $m_0 < n$ by [9, Lemma 7.2.6]. This means $s_0 > (n+1)/2$, so $L^S(s,\pi\otimes\chi)$ can only have poles for $s \leq \frac{n+2}{2}$ and we prove 1). $\square$

Finally we set s to 0 in the zeta function and get the Rallis inner product formula:

Theorem 7.3. *Suppose $m = 2n+1$. Then*

$$\langle\Theta(f_1,\Phi_1),\Theta(f_2,\Phi_2)\rangle_{\mathrm{REG}} = \frac{L^S(\frac{1}{2},\pi\otimes\chi)}{\widetilde{d}^S_{G_2}(0)}\cdot\langle\pi(\Xi_S)f_1,f_2\rangle$$

where

$$\Xi_S(g) = \langle\omega_S(g)\Phi_{1,S},\Phi_{2,S}\rangle.$$

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