The Strong Disorder Renormalization Group Method

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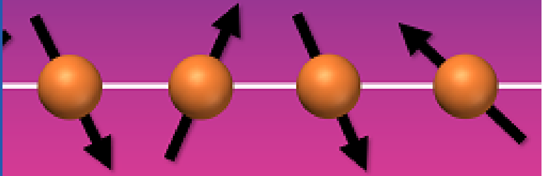


Outline

Disordered Systems

A **disordered (random) system** is governed by a Hamiltonian with **random parameters**.

Strong disorder

- Logarithmic variables are natural
- Fluctuations $\gg$ mean values
- Broad, heavy-tailed distributions

Large-scale behavior

- Disorder may average out
- Or may dominate other fluctuations
- $\rightarrow$ Spin-$\frac{1}{2}$ Heisenberg chain

Strong-Disorder Renormalization Group (SDRG)

SDRG is the principal framework for analyzing the scale evolution of **strongly disordered systems**

Key idea

- Exploits strong randomness
- Tracks full coupling distributions
- Predicts observables at large scales

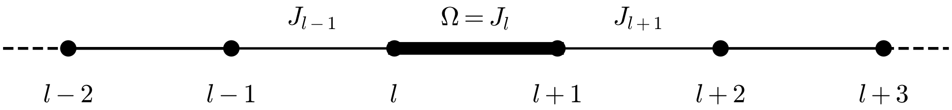
SDRG for spin chains

- Introduced by Ma–Dasgupta (1979)
- Formalized by Fisher (1990s)

Outline

Spin $\frac{1}{2}$ Heisenberg chain and the Ma-Dasgupta decimation rule

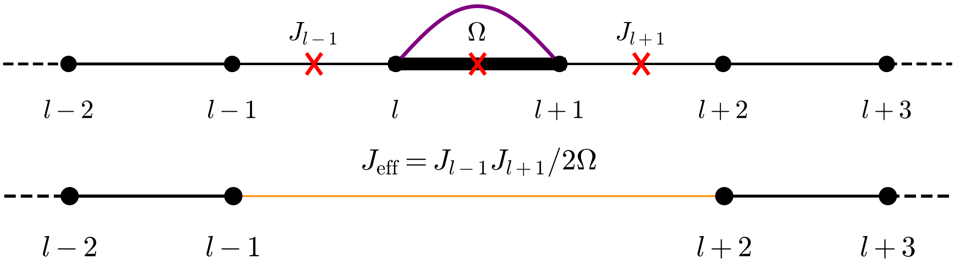
$$H = \sum_i J_i \, \mathbf{S}_i \cdot \mathbf{S}_{i+1}, \qquad J_i > 0 \text{ random (AF chain)}$$



- Identify strongest bond: $\Omega := \max_i J_i = J_l$
- **Strong disorder** $\Rightarrow \Omega \gg J_{l\pm1}$
- $\Rightarrow$ **perturbation theory:** $H_0 = \Omega \, \mathbf{S}_l \cdot \mathbf{S}_{l+1}$

- 0th: **singlet** $|s\rangle$ on $(l, l+1)$, $P = |s\rangle\langle s|$
- 1st: $\langle s|\mathbf{S}_l|s\rangle = \langle s|\mathbf{S}_{l+1}|s\rangle = 0 \Rightarrow$ neighboring bonds $J_{l\pm1}$ suppressed
- 2nd: $H_{\text{eff}}^{(2)} = \frac{J_{l-1} J_{l+1}}{2\Omega} \mathbf{S}_{l-1} \cdot \mathbf{S}_{l+2}$

Spin $\frac{1}{2}$ Heisenberg chain and the Ma-Dasgupta decimation rule

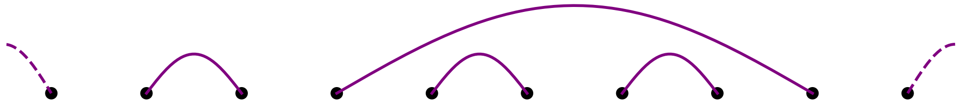


- Ω freezes $l, l+1$ into **singlet**
- remove bonds
- New effective coupling:

$$J_{\text{eff}} = \frac{J_{l-1}J_{l+1}}{2\Omega} \ll J_{l\pm1}$$

- Transformation is **local**

Random-Singlet Phase



- **Iterate** decimation rule: each singlet formed at a different RG step
- Ground state: **non-crossing singlets**

Outline

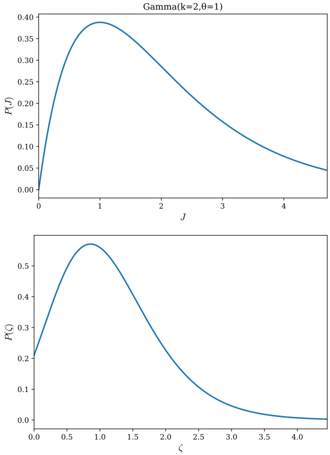
RG flow setup

- $P(J, \Omega)$ distribution of **surviving + effective** couplings at **energy scale Ω**
- **Goal:** track $P(J, \Omega)$ as $\Omega_0 \to \Omega$
- **Dimensionless variables**:

$$\zeta_i = \ln \frac{\Omega}{J_i}, \quad \Gamma = \ln \frac{\Omega_0}{\Omega}, \quad \zeta_i = -\Gamma + c$$

- Advantages of ζ:
  - Domain fixed: $[0, \infty)$
  - Maximum bond $\to \zeta_l = 0$
- Decimation step: $\Omega \searrow$, $\zeta_i \searrow$, $\Gamma \nearrow$

Initial pdf for J and ζ:

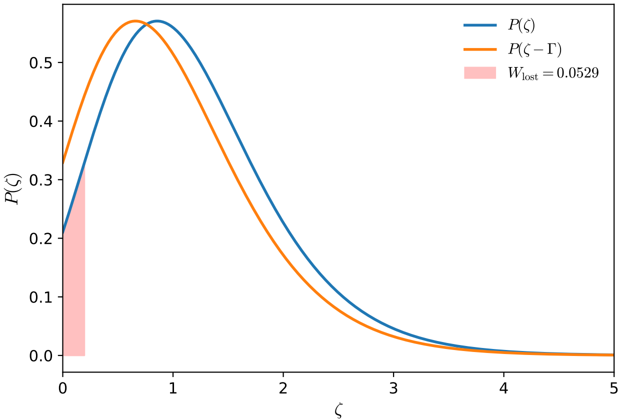


Decimation rule in RG flow framework

- Consider **infinitesimal change** in the scale $\boldsymbol{d\Gamma}$
- $\zeta_i \to \zeta_i - d\Gamma \;\Rightarrow\; P(\zeta)$ shifts to the left:

$$dP = -\frac{\partial P}{\partial \zeta} d\zeta = \frac{\partial P}{\partial \zeta} d\Gamma$$

- Leftmost weight removed: **strongest bonds decimation**



Decimation rule in RG flow framework

Effective bond formula is additive in ζ:

$$J_{\text{eff}} = \frac{J_{l-1} J_{l+1}}{2\Omega} \Rightarrow \quad \zeta_{\text{eff}} = \zeta_{i-1} + \zeta_{i+1} + \ln 2$$

Distribution of new effective bond:

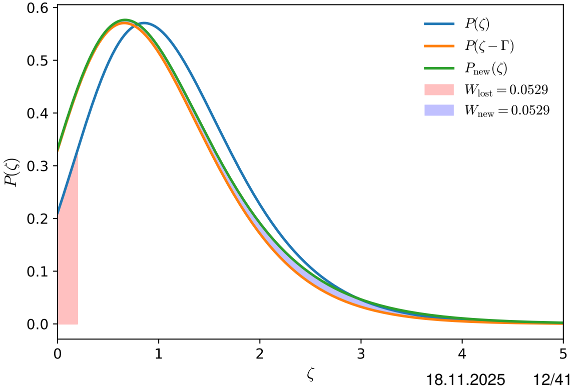
$$P_{\text{eff}}(\zeta) = \int_0^{\zeta} d\zeta' \, P(\zeta') P(\zeta - \zeta')$$

- **Convolution** over $[0, \zeta]$
- Neglect $\ln 2$ because by strong disorder $\zeta_{i\pm1} \gg \ln 2$
- $\int_0^{\infty} P_{\text{eff}}(\zeta) \, d\zeta = 1$

Master equation

$$\frac{\mathrm{d}P(\zeta)}{\mathrm{d}\Gamma} = \frac{\partial P(\zeta)}{\partial \zeta} + P(0)\int_0^{\zeta} d\zeta'\, P(\zeta')P(\zeta-\zeta')$$

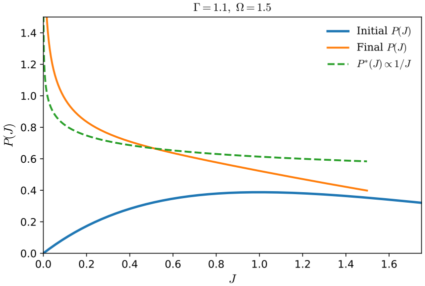
- **Master equation** of SDRG flow
- Weight of new bonds is $\mathrm{d}\Gamma P(0)$
- Total **normalization preserved**

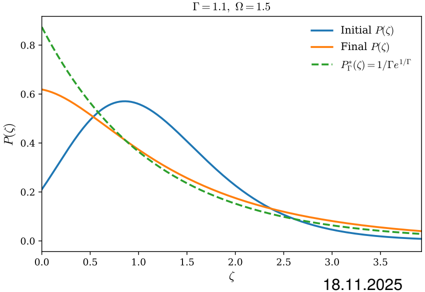


Infinite-Disorder Fixed Point

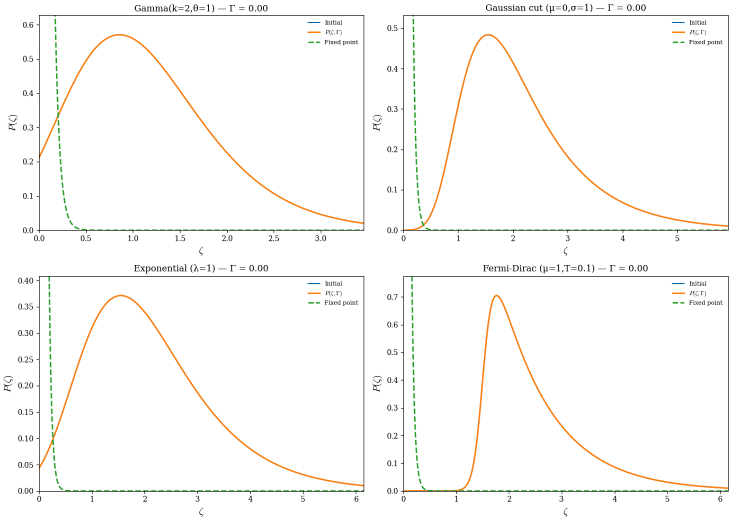
$$P^*(\zeta,\Gamma) = \frac{1}{\Gamma}\, e^{-\zeta/\Gamma}$$
$$P^*(J,\Omega) = \frac{1}{\Omega\Gamma}\left(\frac{\Omega}{J}\right)^{1-\frac{1}{\Gamma}}$$

- **infinite disorder:** $\sigma(\zeta) = \Gamma \xrightarrow{\Gamma\to\infty} \infty$
- $P(J) \sim 1/J$
- Asymptotically exact solution

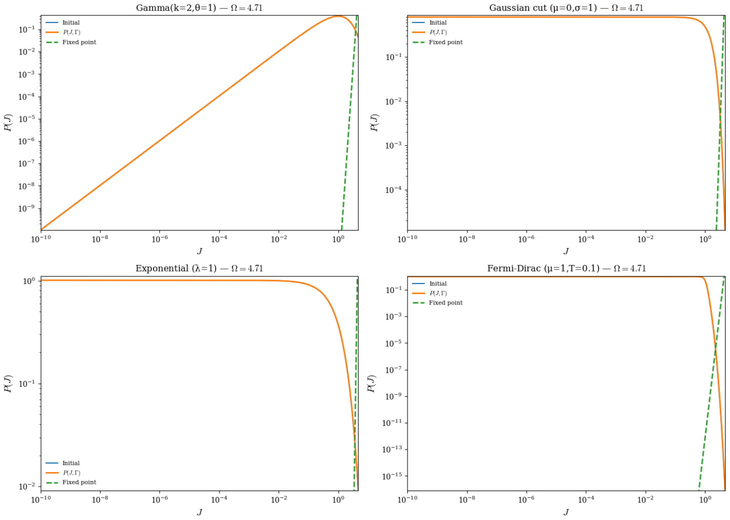




Flow solutions

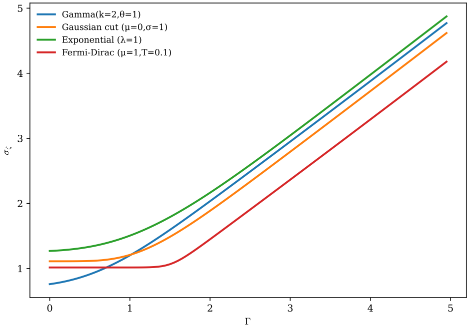


Flow solutions

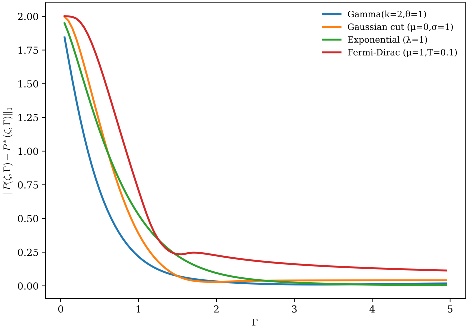


Flow solutions

- **Infinite disorder:** $\sigma(\zeta) = \Gamma \xrightarrow{\Gamma\to\infty} \infty$
- **Asymptotic exactness:** $\|P(\zeta,\Gamma) - P^*(\zeta,\Gamma)\| \xrightarrow{\Gamma\to\infty} 0$
- This fixed point is a **global attractor**



Width grows linearly with Γ



Convergence to the fixed-point distribution

Outline

Ensembles

Consider a random Hamiltonian $H = \boldsymbol{J} \cdot \boldsymbol{K} := \sum_{i=1}^{n} J_i K_i$

- $\boldsymbol{J}$: random vector, J_i i.i.d. on $D \subset [0, \infty)$
- $\boldsymbol{K}$: deterministic vector, K_i Hermitian

It naturally induces an ensemble

$$\mathcal{M}(H) = \left\{ \Big(\mathbb{P}(\boldsymbol{J} = \boldsymbol{j}), H(\boldsymbol{j}) \Big) : \ \boldsymbol{j} \in D^n \right\}. \tag{1}$$

Given an observable O, one defines the *ensemble average*

$$\langle O \rangle_{\mathcal{M}} := \sum_{\boldsymbol{j} \in D^n} O(H(\boldsymbol{j}))\, \mathbb{P}(\boldsymbol{J} = \boldsymbol{j}), \tag{2}$$

and the *typical value*

$$O_{\mathcal{M}}^{\mathrm{typ}} := \exp\left(\langle \ln(O) \rangle_{\mathcal{M}}\right), \tag{3}$$

Example: correlation length

Example: correlation length $C_{ij} := \langle \boldsymbol{S}_i \cdot \boldsymbol{S}_j \rangle$

$$\langle C_{ij} \rangle_{\mathcal{M}} := \sum_{\boldsymbol{j} \in D^n} \langle \mathrm{gs} | \, \boldsymbol{S}_i \cdot \boldsymbol{S}_j \, | \mathrm{gs} \rangle_{H(\boldsymbol{j})} \, \mathbb{P}(\boldsymbol{J} = \boldsymbol{j}) \tag{4}$$

$$C^{\mathrm{typ}}_{ij,\mathcal{M}} := \exp\left(\sum_{\boldsymbol{j} \in D^n} \ln\Big(\langle \mathrm{gs} | \, \boldsymbol{S}_i \cdot \boldsymbol{S}_j \, | \mathrm{gs} \rangle_{H(\boldsymbol{j})} \Big) \, \mathbb{P}(\boldsymbol{J} = \boldsymbol{j}) \right) \tag{5}$$

Collection of samples

For numerical implementations, we do not have access to the full ensemble. We only have a collection of N samples

$$\mathcal{C} := \{H(\boldsymbol{J} = \boldsymbol{j}_1), \ldots, H(\boldsymbol{J} = \boldsymbol{j}_N)\}\,, \tag{6}$$

- $\boldsymbol{j}_1, \ldots, \boldsymbol{j}_N \in D^n$ are drawn in an i.i.d. manner according to the distribution of $\boldsymbol{J}$

We define *perceived ensemble average*

$$\langle O \rangle_{\mathcal{C}} := \frac{1}{N} \sum_{i=1}^{N} O(H(\boldsymbol{j}_i)) \tag{7}$$

and *perceived typical value* is defined by

$$O_{\mathcal{C}}^{\mathrm{typ}} := \exp\left(\langle \ln(O) \rangle_{\mathcal{C}}\right) = \exp\left(\frac{1}{N} \sum_{i=1}^{N} \ln(O(H(\boldsymbol{j}_i)))\right), \tag{8}$$

By the law of large numbers, they converge in the large N limit to $\langle O \rangle_{\mathcal{M}}$ and $O_{\mathcal{M}}^{\mathrm{typ}}$.

Outline

Initial setup

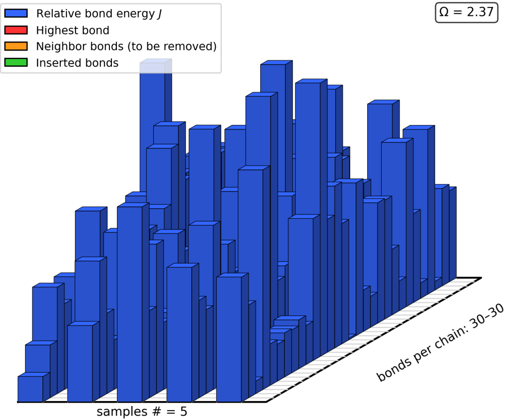


Figure: Initial $\mathcal{C}^{(0)}$ drawn from $\mathcal{N}(0,1)$.

We define the global *perceived renormalization parameter*

$$\Omega_{\mathcal{C}} := \max_{1\leq i\leq N} \max_{1\leq k\leq n} j_{i,k}. \tag{9}$$

Start with

- an initial collection of samples $\mathcal{C}^{(0)}$.
- initial cutoff $\Omega^{(0)} := \Omega_{\mathcal{C}^{(0)}}$
- shell width $d\Omega > 0$

SDRG steps

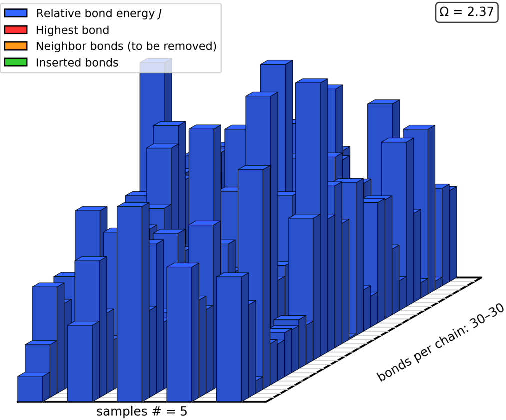


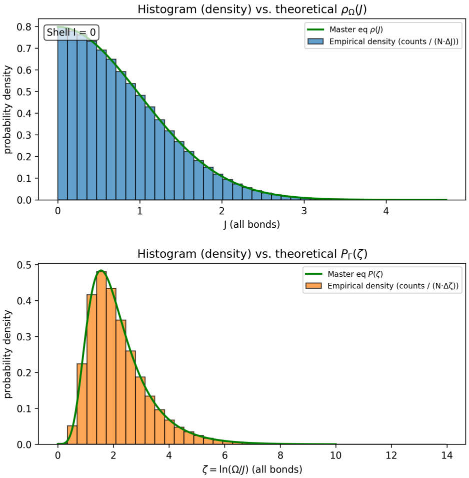
Figure: Initial $\mathcal{C}^{(0)}$ drawn from $\mathcal{N}(0, 1)$.

For $l > 0$, we define $\Omega^{(l+1)} := \Omega^{(l)} - d\Omega$. At the l-th step, we iterate the following cycle for each sample $H(\boldsymbol{j}_k) \in \mathcal{C}^{(l)}$:

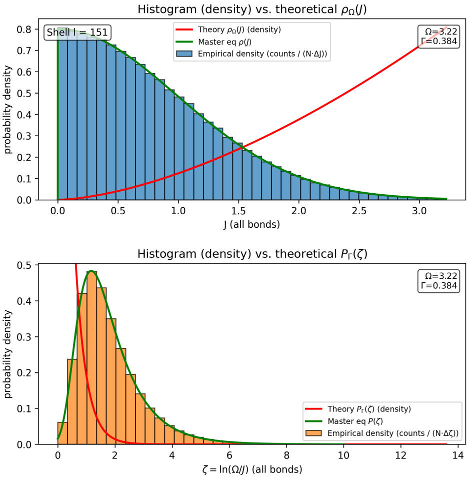
- Let h be the strongest bond of $H(\boldsymbol{j}_k)$.
- If $h \in [\Omega^{(l+1)}, \Omega^{(l)}]$
  1. Apply decimation rules
  2. Obtain a modified Hamiltonian $H'(\boldsymbol{j}'_k)$
  3. Redefine $H(\boldsymbol{j}_k) \longleftarrow H'(\boldsymbol{j}'_k)$
  4. Redefine h
  5. Repeat
- Else
  1. Pass the (modified) $H(\boldsymbol{j}_k)$ to a new set $\mathcal{C}^{(l+1)}$
  2. Move on to the next sample in $\mathcal{C}^{(l)}$.

Outline

Sampled distribution: Gaussian

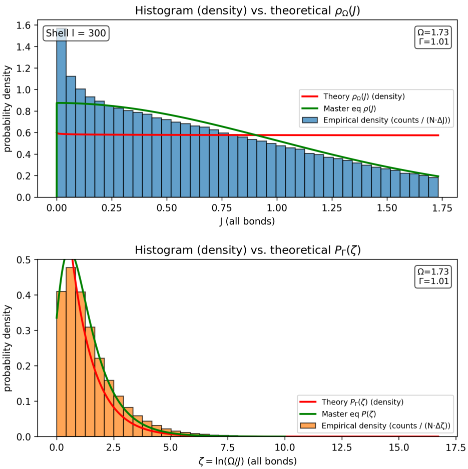


(a) Initial distribution

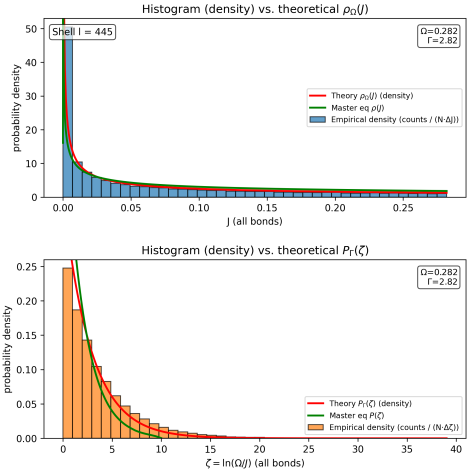


(b) At shell $l = 151$

Sampled distribution: Gaussian



(a) At shell $l = 300$



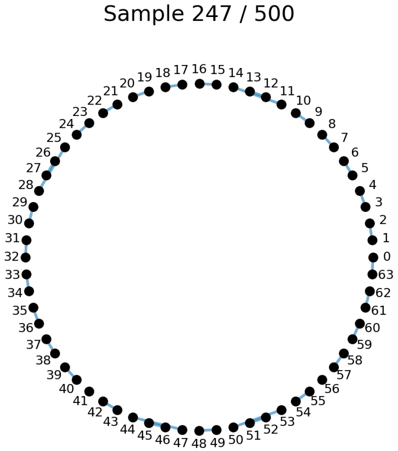
(b) At shell $l = 445$

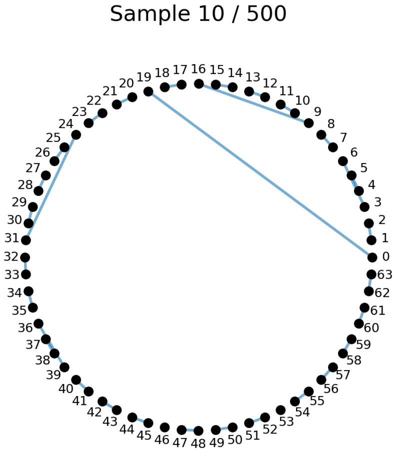
Outline

Ground state

After all bonds are decimated, we associate each sample $H(\boldsymbol{j}_k)$ with a unique ground state $|\mathrm{gs}\rangle_{H(\boldsymbol{j}_k)}$.

- For the Heisenberg, we get this ground state by keeping track of the frozen singlets.



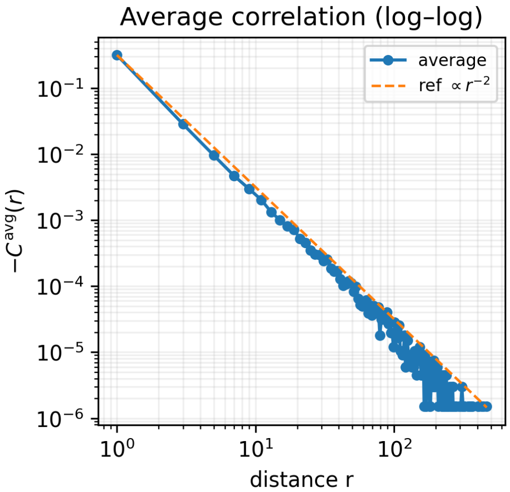


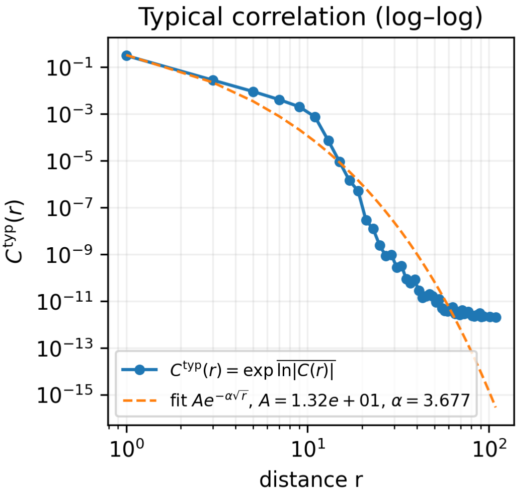
Correlation length

For any ground state $|\mathrm{gs}\rangle_{H(\boldsymbol{j}_k)}$, if a singlet part $|ij\rangle_{H(\boldsymbol{j}_k)}$ exists, it contributes $-\frac{3}{4}$ to C_{ij}, otherwise, it contributes nothing.

$$\begin{aligned}
\langle C_r \rangle_{\mathcal{C}} &:= \frac{1}{M} \sum_{H(\boldsymbol{j}_k) \in \mathcal{C}} \sum_{|i-j|=r} \langle \mathrm{gs} | \boldsymbol{S}_i \cdot \boldsymbol{S}_j | \mathrm{gs} \rangle_{H(\boldsymbol{j}_k)} \\
&:= -\frac{3}{4} \frac{1}{M} \sum_{H(\boldsymbol{j}_k) \in \mathcal{C}} \sum_{|i-j|=r} \langle ij | \mathrm{gs} \rangle_{H(\boldsymbol{j}_k)} \\
&\sim \frac{1}{r^2} \\
C_{r,\mathcal{C}}^{\mathrm{typ}} &:= -\frac{3}{4} \exp\left(\frac{1}{M} \sum_{H(\boldsymbol{j}_k) \in \mathcal{C}} \ln \left[\sum_{|i-j|=r} \langle ij | \mathrm{gs} \rangle_{H(\boldsymbol{j}_k)} \right] \right) \\
&\sim \exp\left(-\alpha \sqrt{r}\right)
\end{aligned} \tag{10}$$

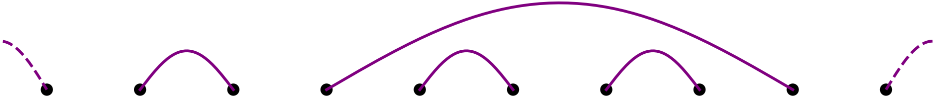
Correlation length





Outline

Revisiting the Random-Singlet Groundstate
Groundstate Properties



- Our groundstate consists of **disconnected** singlets with random distances l.
- We expect singlets with very large l to be very rare and have very low coupling strengths as they appear later in the decimation process.
- Low energy excitations are given by turning a random singlet into a triplet, this contributes an energy $\Delta E = J_{ij}$.

Length Distribution in the Random-Singlet Groundstate
Scaling relation at IDFP

Dasgupta-Ma Decimation changes number of available bonds:

$$dN/d\Gamma = -2P^*(0,\Gamma)N \implies N(\Gamma) \sim N_0/\Gamma^2$$

The average length at a decimation step Γ is thus:

$$L(\Gamma) = l_0 N_0/N(\Gamma) \sim \Gamma^2 \implies L(\Omega) \sim l_0 \ln(\Omega_0/\Omega)$$

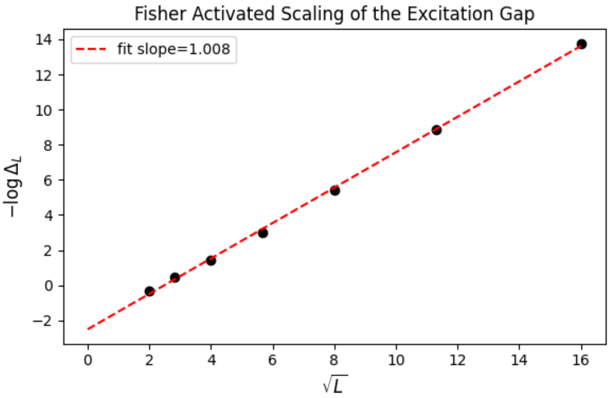
- Using the shape of the probability distribution at IDFP $P^*(\zeta,\Gamma) = P^*(\zeta,\Gamma) = \frac{1}{\Gamma}\, e^{-\zeta/\Gamma}$ which implies $P^*(0,\Gamma) = 1/\Gamma$.
- The total system size is given by $L_{tot} = l_0 N_0$ and is a conserved quantity.

Appearance and Numerical verification of the typical Energy Gap
Difference of physical behavior

- We expect the typical energy gap appearing in the random-singlet groundstate to be:

$$\ln(\Delta_L) \sim -\sqrt{L_{tot}}$$

- This contrasts the usual behavior of the spin 1/2 Heisenberg chain with uniform coupling, where we expect the lowest energy excitations to be spinons which form an excitation spectrum without gap.



Typical vs. Average correlations at the IDFP

How rare events dominate averages

At IDFP we get probability of finding a singlet of length r:

$$N(\Gamma) \sim N_0/\Gamma^2 \implies (n_L)^2 \sim 1/L^2$$

The typical correlation enters through perturbation and is much smaller:

$$\delta \langle S_i S_{i+l} \rangle \sim J_R^{eff}/\Omega = e^{-\zeta_{eff}} \sim e^{-a\Gamma} \sim e^{-a\sqrt{l}}$$

- Using the shape of the probability distribution at IDFP $P^*(\zeta,\Gamma) = P^*(\zeta,\Gamma) = \frac{1}{\Gamma}\, e^{-\zeta/\Gamma}$ which implies that the typical value of ζ at cutoff Γ is of the order of Gamma.
- Perturbative corrections to an observable $\hat{O}$ are $\sum_{n\neq 0} \frac{\langle 0|\hat{O}|n\rangle\langle n|\hat{V}|0\rangle}{\Delta E_n}$.

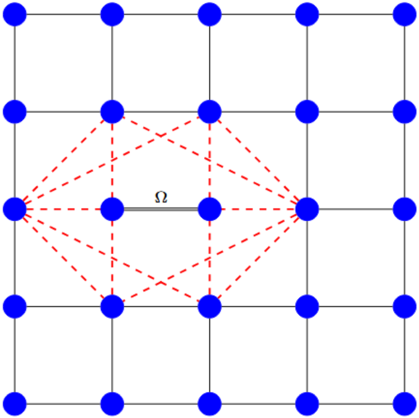
Typical vs. average correlations at IDFP

Correlation Type	Comes from	Shape
Average	Rare singlet pairs contribute O(1) terms that dominate average	goes as $\sim 1/L^2$
Typical	Effective coupling with typical value at a given length scale contributes perturbative correction	goes as $e^{-a\sqrt{L}}$

Outline

Issues with higher dimensionalities

- The second order perturbations generated by fixing a singlet connect not just nearest neighbours.
- The Lattice coordination number is not preserved.
- Thus we are unable to write an analytic equation for the probability distribution.



Connectivity diagram after singlet decimation

Failure of renormalization for naive SDRG implementation in the Spin 1 Heisenberg chain

$$\mathcal{H} = \sum_i J_i \, \mathbf{S}_i \cdot \mathbf{S}_{i+1}, \qquad J_i > 0 \quad \textbf{but} \quad S = 1$$

- We now have three eigenstates: $S^z \left| e_i \right\rangle = e_i \left| e_i \right\rangle$ with $e_i \in \{0, 1, -1\}$
- coupling between two sites now has three energy levels:

$$\begin{aligned} \text{singlet:} \quad & e_o = -2J \\ \text{triplet:} \quad & e_o = -J \\ \text{quintuplet:} \quad & e_o = J \end{aligned}$$

- Naive implementation for the SDRG flow $\implies$ projecting on singlet state between sites i and j
- However the new perturbative coupling generated by projecting onto a singlet is:

$$\tilde{J}_{lm} = \frac{4}{3} \frac{J_{li} J_{jm}}{J_{ij}}$$

- $\implies$ even with strong disorder assumption flow might not converge

Modern Extensions

Current research regarding long range interacting 1D spin chains

$$J_{ij} = J_0 |r_i - r_j|^{-\alpha}$$

- The initial random distribution is that of distances. The couplings follow from the power law.
- The decimation procedure does not add new bonds. Due to the long range nature of the model, only existing bonds get renormalized.

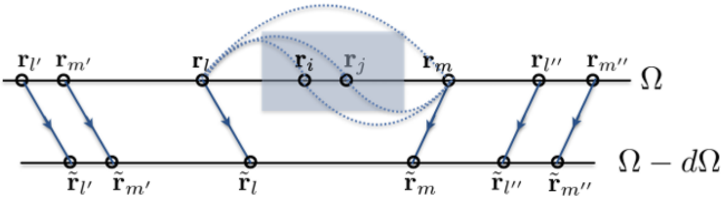


Figure: S. Kettemann (2025) arXiv:2501.07298

In conclusion

- When applicable Strong-disorder renormalization group (SDRG) provides a controlled real-space RG scheme for disordered quantum systems when disorder generates a strong hierarchy of energy scales.
- In the 1D random spin-$\frac{1}{2}$ Heisenberg chain, SDRG is asymptotically exact and yields an infinite-randomness fixed point (IDFP) with a random-singlet ground state.
- At IDFP, energy and length are coupled by activated scaling, $\Gamma = \ln(\Omega_0/\Omega) \sim \sqrt{\ell}$, allowing direct access to typical low-energy observables.
- A hallmark of infinite randomness is the strong separation between typical and disorder-averaged observables, with rare events dominating averages.