介电函数实部虚部求解

求:

$$\mathrm{Re}\,\epsilon\,(q,\omega)=1-\frac{2v\,(q)}{\hbar}\sum_k n_k\frac{2\omega_{kq}}{\omega^2-\omega_{kq}^2}$$

sol.

令$\hbar=1$，并且令

$$\mathrm{Re}\,\Pi_{MD}^*(q,w)=2\sum_k n_k\frac{2\omega_{kq}}{\omega^2-\omega_{kq}^2}$$

则

$$\mathrm{Re}\,\epsilon\,(q,\omega)=1-v(q)\mathrm{Re}\,\Pi_{MD}^*(q,w)$$

$\mathrm{Re}\,\Pi_{MD}^*(q,w)$的积分形式为，

$$\mathrm{Re}\,\Pi_{MD}^*(q,w)=2\int\frac{dk}{(2\pi)^3}\theta(k_f-k)\left(\frac{1}{w-\frac{1}{m}(\overrightarrow{q}\cdot\overrightarrow{k}+\frac{1}{2}q^2)}-\frac{1}{w+\frac{1}{m}(\overrightarrow{q}\cdot\overrightarrow{k}+\frac{1}{2}q^2)}\right)$$

令

$$v=w\frac{m}{k_F^2},k\to kk_F,q\to qk_F.$$

$$\begin{aligned}\mathrm{Re}\,\Pi_{MD}^*(q,w)&=2mk_F\int\frac{dk}{(2\pi)^3}\theta(1-k)\left(\frac{1}{v-\overrightarrow{q}\cdot\overrightarrow{k}-\frac{1}{2}q^2}-\frac{1}{v+\overrightarrow{q}\cdot\overrightarrow{k}+\frac{1}{2}q^2}\right)\\&=\frac{2mk_F}{(2\pi)^3}2\pi\int_0^1dk\int_0^\pi d\theta k^2\sin\theta\left(\frac{1}{v-qk\cos\theta-\frac{1}{2}q^2}-\frac{1}{v+qk\cos\theta+\frac{1}{2}q^2}\right)\\&=\frac{2mk_F}{(2\pi)^3}2\pi\int_0^1k^2dk\int_{-1}^1dx(-\frac{1}{-v+qkx+\frac{1}{2}q^2}-\frac{1}{v+qkx+\frac{1}{2}q^2})\\&=-\frac{2mk_F}{(2\pi)^3}\frac{2\pi}{q}\int_0^1k^2dk\int_{-1}^1dx(\frac{1}{kx+\frac{1}{2}q-\frac{v}{q}}+\frac{1}{kx+\frac{1}{2}q+\frac{v}{q}})\end{aligned}$$

令

$$
\begin{aligned}
A(a) &= \int_0^1 k^2 dk \int_{-1}^1 dx \frac{1}{kx+a} \\
&= \int_0^1 kdk(\ln|k+a| - \ln|k-a|) \\
&\equiv B(a) - B(-a)
\end{aligned}
$$

则有

$$
\operatorname{Re}\Pi^*_{MD}(q,w) = -\frac{2mk_F}{(2\pi)^3}\frac{2\pi}{q}\left(A\left(\frac{1}{2}q - \frac{v}{q}\right) + A(\frac{1}{2}q + \frac{v}{q})\right)
$$

求解$B(a)$,

$$
\begin{aligned}
B(a) &= \int_0^1 k \ln|k+a| dk \\
&= \left[\frac{(k+a)^2}{2}\ln|k+a| - a[(k+a)\ln|k+a| - k] - \frac{1}{2}\frac{(k+a)^2}{2}\right]\Bigg|_0^1 \\
&= \frac{1-a^2}{2}\ln|1+a| + \frac{a^2}{2}\ln|a| + \frac{a}{2} - \frac{1}{4}
\end{aligned}
$$

最终,

$$
A(a) = \frac{1-a^2}{2}\ln|\frac{1+a}{1-a}| + a
$$

$$
\begin{aligned}
&\operatorname{Re}\Pi^*_{MD}(q,w) \\
&= -\frac{2mk_F}{(2\pi)^3}\frac{2\pi}{q}\left[\frac{1-\left(\frac{q}{2}-\frac{v}{q}\right)^2}{2}\ln|\frac{1+\frac{q}{2}-\frac{v}{q}}{1-\frac{q}{2}+\frac{v}{q}}| + \frac{1-\left(\frac{q}{2}+\frac{v}{q}\right)^2}{2}\ln\left|\frac{1+\frac{q}{2}+\frac{v}{q}}{1-\frac{q}{2}-\frac{v}{q}}\right| + q\right] \\
&= -\frac{mk_F}{2\pi^2}\frac{1}{q}\left[\frac{1-\left(\frac{q}{2}-\frac{v}{q}\right)^2}{2}\ln|\frac{1+\frac{q}{2}-\frac{v}{q}}{1-\frac{q}{2}+\frac{v}{q}}| + \frac{1-\left(\frac{q}{2}+\frac{v}{q}\right)^2}{2}\ln\left|\frac{1+\frac{q}{2}+\frac{v}{q}}{1-\frac{q}{2}-\frac{v}{q}}\right| + q\right] \\
&= -\frac{mk_F}{\pi^2}\left[\frac{1-\left(\frac{q}{2}-\frac{v}{q}\right)^2}{4q}\ln|\frac{1+\frac{q}{2}-\frac{v}{q}}{1-\frac{q}{2}+\frac{v}{q}}| + \frac{1-\left(\frac{q}{2}+\frac{v}{q}\right)^2}{4q}\ln\left|\frac{1+\frac{q}{2}+\frac{v}{q}}{1-\frac{q}{2}-\frac{v}{q}}\right| + \frac{1}{2}\right]
\end{aligned}
$$

（无量纲的q）

$$
v(q) = \frac{4\pi e^2}{q^2}, \quad v(q)\frac{mk_F}{\pi^2} = \frac{\lambda^2}{q^2}(\text{有量纲的q})
$$

将v带入，并且将q写成有量纲形式

$$\frac{v}{q}-\frac{q}{2}=\frac{1}{qv_p}(w-\frac{q^2}{2m}),$$
$$\frac{v}{q}+\frac{q}{2}=\frac{1}{qv_p}(w+\frac{q^2}{2m})$$
$$\ln\left|\frac{\frac{v}{q}-\frac{q}{2}-1}{\frac{v}{q}-\frac{q}{2}+1}\right|=\ln\left|\frac{\omega-qv_F-\frac{q^2}{2m}}{\omega+qv_F-\frac{q^2}{2m}}\right|$$
$$\ln|\frac{\frac{v}{q}+\frac{q}{2}+1}{\frac{v}{q}+\frac{q}{2}-1}|=\ln|\frac{w+qv_F+\frac{q^2}{2m}}{w-qv_F+\frac{q^2}{2m}}|$$

$$\begin{aligned}&\mathrm{Re}\,\Pi^*_{MD}(q,w)\\&=-\frac{mk_F}{\pi^2}\\&\left[\frac{k_F}{4q}(\frac{(w+\frac{q^2}{2m})^2}{(qv_p)^2}-1)\ln|\frac{w-qv_F+\frac{q^2}{2m}}{w+qv_F+\frac{q^2}{2m}}|-\frac{k_F}{4q}(\frac{(w-\frac{q^2}{2m})^2}{(qv_p)^2}-1)\ln|\frac{\omega-qv_F-\frac{q^2}{2m}}{\omega+qv_F-\frac{q^2}{2m}}|+\frac{1}{2}\right]\end{aligned}$$

根据上式以及

$$\mathrm{Re}\,\epsilon\,(q,\omega)=1-v(q)\mathrm{Re}\,\Pi^*_{MD}(q,w)$$

可以得到书上(4.6.25)。

求:

$$\mathrm{Im}\varepsilon(q,\omega)=\frac{2\pi v(q)}{\hbar}\sum_k n_k(1-n_{k+q})\left\{\delta(w_{kq}-\omega)-\delta(\omega_{kq}+\omega)\right\}$$

sol.
令

$$I=\sum_k n_k(1-n_{k+q})\left\{\delta(\omega_{kq}-\omega)-\delta(\omega_{kq}+\omega)\right\}$$

其中,

$$1-n_{k+q}=\theta(|\overrightarrow{k}+\overrightarrow{q}|-k_F)$$
$$n_k=\theta(k_F-|\overrightarrow{k}|)$$
$$\omega_{kq}=\frac{k}{2m}[(\overrightarrow{k}+\overrightarrow{q})^2-k^2]=\frac{k}{2m}(\overrightarrow{q}^2+2\overrightarrow{k}\cdot\overrightarrow{q})$$

$$\text{因为}\omega\geq 0,\quad\text{而}\theta(k_F-|\overrightarrow{k}|)\theta(|\overrightarrow{k}+\overrightarrow{q}|-k_F)\text{保证}\omega_{kq}>0\text{,所以}\delta(\omega_{kq}+\omega)=0.$$

求和转化为积分，

$$\sum_k \to \frac{1}{8\pi^3}\int d^3k = \frac{1}{8\pi^3}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^{\infty} dk k^2 \sin\theta$$

所以I写成，

$$I = \frac{1}{8\pi^3}\int d^3k \theta(k_F - \left|\overrightarrow{k}\right|)\theta(\left|\overrightarrow{k}+\overrightarrow{q}\right| - k_F)\delta(w_{kq} - \omega)$$

使用费米波矢k_F作为波矢单位，$k \to kk_F, q \to qk_F$,所以有

$$\sum_k \to \frac{1}{8\pi^3}\int d^3k = \frac{k_F^3}{8\pi^3}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^{\infty} dk k^2 \sin\theta$$

$$\omega_{kq} = \frac{\hbar k_F^2}{2m}(q^2 + 2\overrightarrow{k}\cdot\overrightarrow{q})$$

并且定义

$$\nu = \omega\frac{m}{\hbar k_F^2}$$

根据δ函数性质，

$$\delta(ax) = |a|^{-1}\delta(x)$$

$$\delta(\omega_{kq} - \omega) = \frac{m}{\pi k_F^2}\delta(\nu - \overrightarrow{q}\cdot\overrightarrow{k} - \frac{q^2}{2})$$

最终得到I，

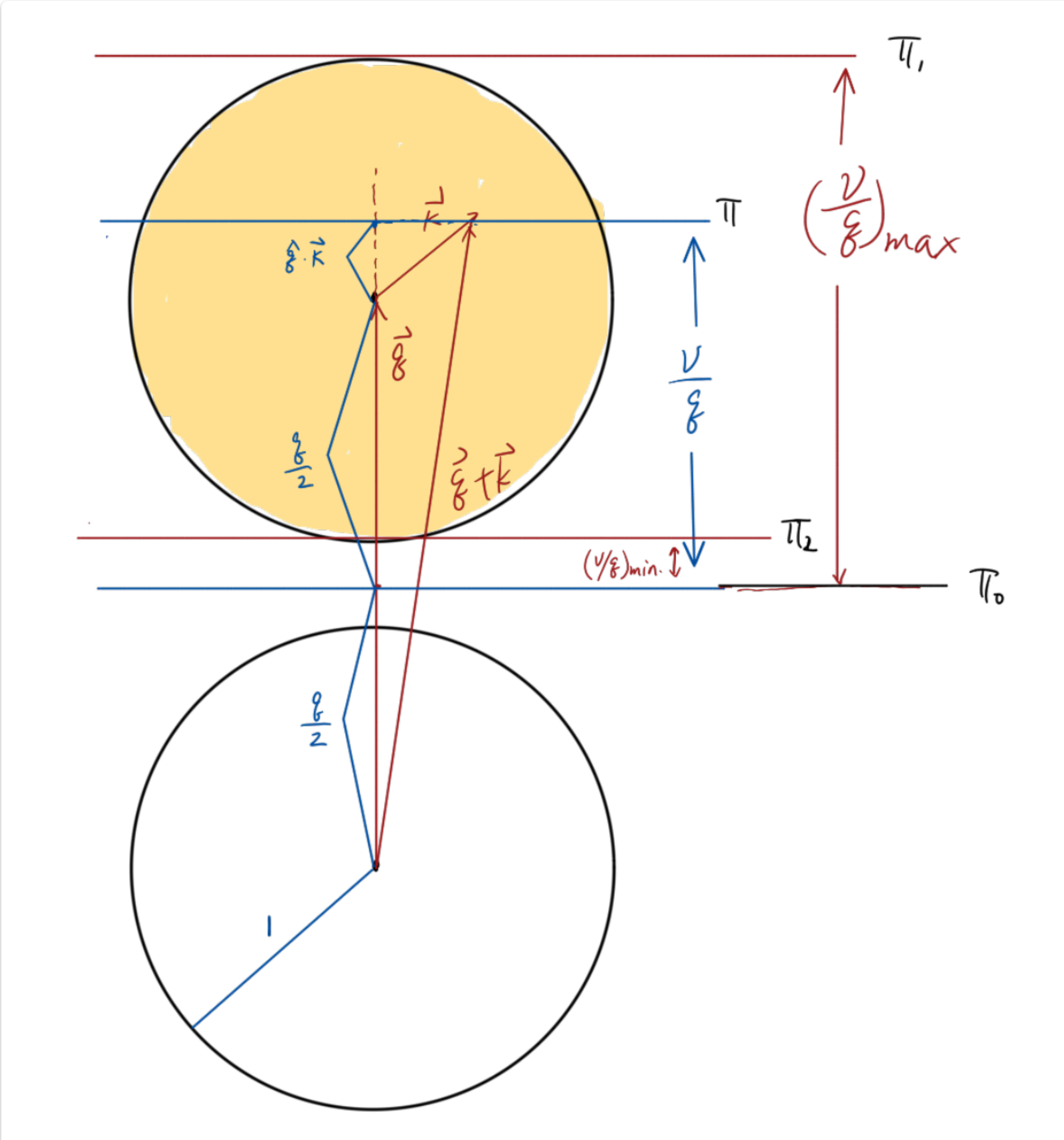
$$I = \frac{mk_F}{4\pi^2\hbar}\int\limits_0^{\pi} d\theta \int\limits_0^{\infty} dk k^2 \sin\theta \cdot \theta(1-k)\theta(|\overrightarrow{k}+\overrightarrow{q}| - 1)\delta\left(v - \overrightarrow{q}\cdot\overrightarrow{k} - \frac{q^2}{2}\right)$$

k需要同时满足3个条件，

$$\begin{cases} k < 1, \left|\overrightarrow{k}+\overrightarrow{q}\right| > 1, \\ v - \overrightarrow{q}\cdot\overrightarrow{k} - \dfrac{q^2}{2} = 0 \Leftrightarrow \dfrac{q}{2} + \hat{q}\cdot\overrightarrow{k} = \dfrac{v}{q}. \end{cases}$$

其几何含义是平面$\frac{q}{2} + \hat{q}\cdot\overrightarrow{k} = \frac{v}{q}$需要与$k < 1, \left|\overrightarrow{k}+\overrightarrow{q}\right| > 1$确定的三维空间相交，这确定了$\nu$的取值范围。

1. $q > 2,\ \ \frac{1}{2}q^2 + q \geq \nu \geq \frac{1}{2}q^2 - q$



令$t = \cos\theta$,

$$I = \frac{mk_F}{4\pi^2\hbar}\int_{\frac{\nu}{q}-\frac{q}{2}}^{1} k^2 dk \int_{-1}^{1} dt \frac{1}{qk}\delta\left(\frac{\nu}{qk} - \frac{1}{2}\frac{q}{k} - t\right)$$

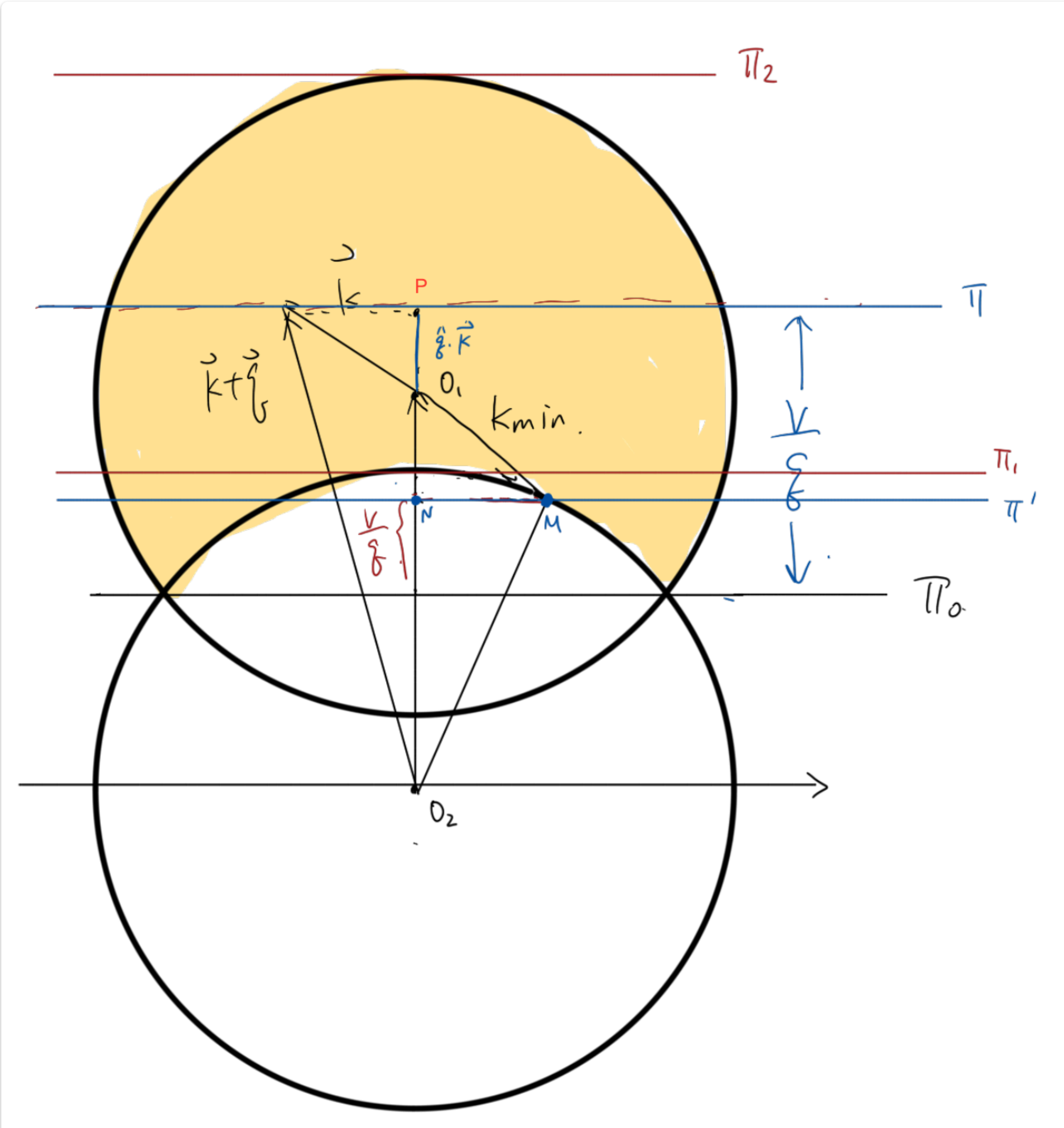
$$I = \frac{mk_F}{4\pi^2\hbar}\frac{1}{q}\frac{1}{2}[1-(\frac{\nu}{q}-\frac{q}{2})^2]$$

带入$\nu = \omega\frac{m}{\hbar k_F^2}, qk_F \to q$,

定义费米速度$v_F = \frac{\hbar k_F}{m}$, $\quad \lambda^2 = \frac{4me^2k_F}{\pi\hbar^2}$,

$$\mathrm{Im}\varepsilon(q,\omega)=\frac{\pi}{4}\frac{\lambda^2}{q^2}\frac{k_F}{q}[1-\frac{(\omega-\frac{\hbar q^2}{2m})^2}{(qv_F)^2}]$$

2. $1<q<2$,



需要分为两种情况，

- 平面π在π_1,π_2之间时$(q+\frac{1}{2}q^2\geq\nu\geq q-\frac{1}{2}q^2)$，$k$最小值为$O_1P$，此时$\mathrm{Im}\varepsilon(q,\omega)$表达式同$q>2$情况。
- 平面π在π_0,π_1之间时$(0\leq\nu\leq q-\frac{1}{2}q^2)$，$k$的下限只能到$O_1M$。

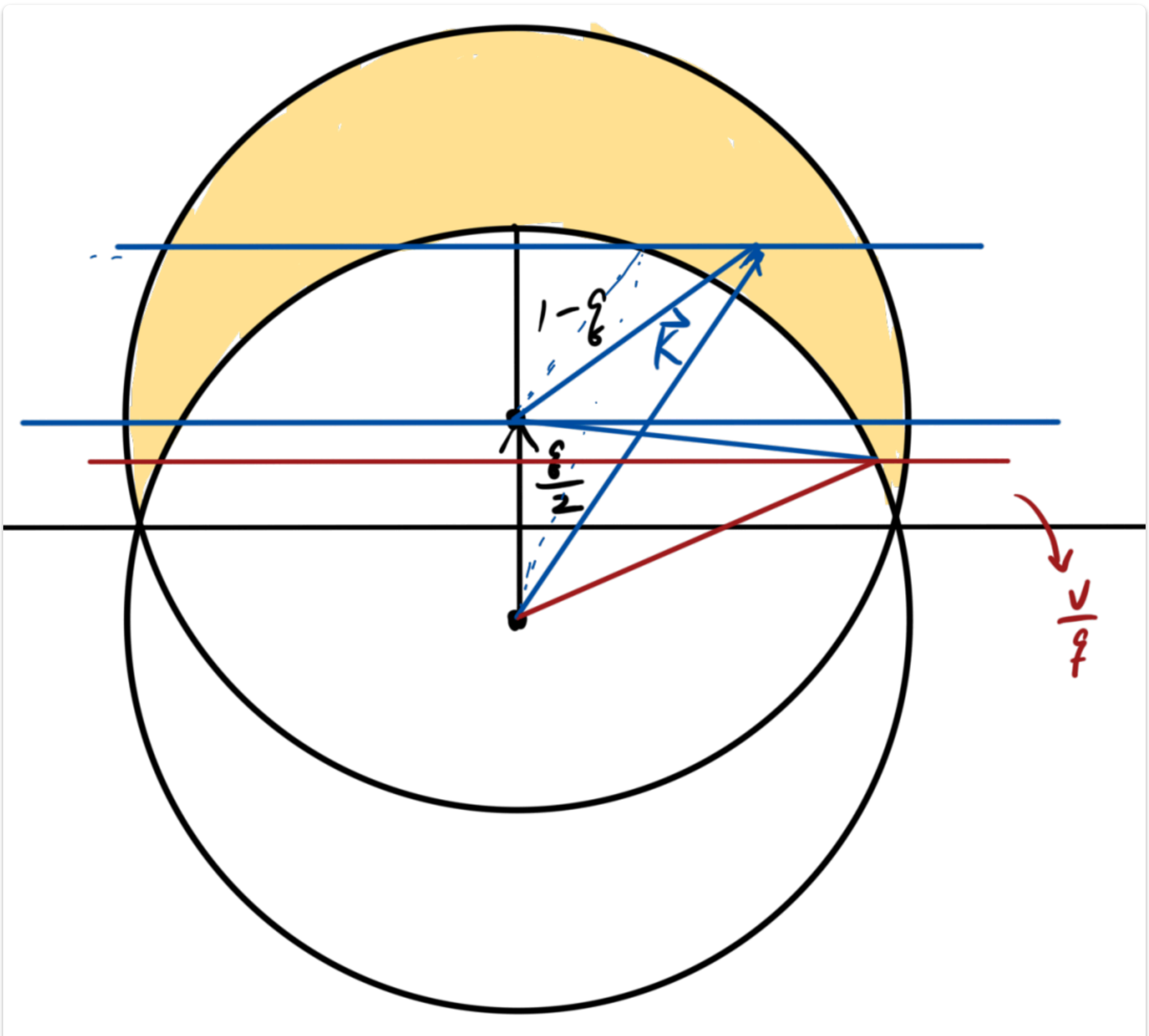
$$
\begin{aligned}
O_1M^2 &= MN^2 + O_1N^2 \\
&= O_2M^2 - O_2N^2 + O_1N^2 \\
&= \left(\frac{1}{2}q + \frac{\nu}{q}\right)^2 + \left(\frac{1}{2}q - \frac{\nu}{q}\right)^2 \\
&= 1 - 2\nu.
\end{aligned}
$$

所以，

$$
I = \frac{mk_F}{4\pi^2 k}\int_{\sqrt{1-2\nu}}^{1} k^2 dk \int_{-1}^{1} dt \frac{1}{qk}\delta\left(\frac{\nu}{qk} - \frac{1}{2}\frac{q}{k} - t\right) = \frac{mk_F}{4\pi^2\hbar}\frac{\nu}{q}
$$

$$
\mathrm{Im}\varepsilon(q,\omega) = \frac{\pi}{2}\frac{\lambda^2}{q^2}\frac{\omega}{qv_F}
$$

3. $q < 1$，$\mathrm{Im}\varepsilon(q,\omega)$表达式同2。



综上，

$$
\mathrm{Im}\varepsilon(q,\omega) = \begin{cases} \frac{\pi}{2}\frac{\lambda^2}{q^2}\frac{\omega}{qv_F}, 0 \le \omega \le qv_F - \hbar\frac{q^2}{2m} \\ \frac{\pi}{4}\frac{\lambda^2}{q^2}\frac{k_F}{q}\left[1 - \frac{\left(\omega - \frac{\hbar q^2}{2m}\right)^2}{(qv_F)^2}\right], \qquad -\hbar\frac{q^2}{2m} \le \omega - qv_F \le \hbar\frac{q^2}{2m} \\ 0, \qquad \omega \ge qv_F + \hbar\frac{q^2}{2m} \end{cases}
$$