

# 介电函数实部虚部求解

求：

$$\text{Re } \epsilon(q, \omega) = 1 - \frac{2v(q)}{\hbar} \sum_k n_k \frac{2\omega_{kq}}{\omega^2 - \omega_{kq}^2}$$

sol.

令  $\hbar = 1$ , 并且令

$$\text{Re } \Pi_{MD}^*(q, w) = 2 \sum_k n_k \frac{2\omega_{kq}}{\omega^2 - \omega_{kq}^2}$$

则

$$\text{Re } \epsilon(q, \omega) = 1 - v(q) \text{Re } \Pi_{MD}^*(q, w)$$

$\text{Re } \Pi_{MD}^*(q, w)$  的积分形式为,

$$\text{Re } \Pi_{MD}^*(q, w) = 2 \int \frac{dk}{(2\pi)^3} \theta(k_f - k) \left( \frac{1}{w - \frac{1}{m}(\vec{q} \cdot \vec{k} + \frac{1}{2}q^2)} - \frac{1}{w + \frac{1}{m}(\vec{q} \cdot \vec{k} + \frac{1}{2}q^2)} \right)$$

令

$$v = w \frac{m}{k_F^2}, k \rightarrow kk_F, q \rightarrow qk_F.$$

$$\begin{aligned} \text{Re } \Pi_{MD}^*(q, w) &= 2mk_F \int \frac{dk}{(2\pi)^3} \theta(1 - k) \left( \frac{1}{v - \vec{q} \cdot \vec{k} - \frac{1}{2}q^2} - \frac{1}{v + \vec{q} \cdot \vec{k} + \frac{1}{2}q^2} \right) \\ &= \frac{2mk_F}{(2\pi)^3} 2\pi \int_0^1 dk \int_0^\pi d\theta k^2 \sin \theta \left( \frac{1}{v - qk \cos \theta - \frac{1}{2}q^2} - \frac{1}{v + qk \cos \theta + \frac{1}{2}q^2} \right) \\ &= \frac{2mk_F}{(2\pi)^3} 2\pi \int_0^1 k^2 dk \int_{-1}^1 dx \left( -\frac{1}{-v + qkx + \frac{1}{2}q^2} - \frac{1}{v + qkx + \frac{1}{2}q^2} \right) \\ &= -\frac{2mk_F}{(2\pi)^3} \frac{2\pi}{q} \int_0^1 k^2 dk \int_{-1}^1 dx \left( \frac{1}{kx + \frac{1}{2}q - \frac{v}{q}} + \frac{1}{kx + \frac{1}{2}q + \frac{v}{q}} \right) \end{aligned}$$

令

$$\begin{aligned}
A(a) &= \int_0^1 k^2 dk \int_{-1}^1 dx \frac{1}{kx+a} \\
&= \int_0^1 k dk (\ln|k+a| - \ln|k-a|) \\
&\equiv B(a) - B(-a)
\end{aligned}$$

则有

$$\text{Re } \Pi_{MD}^*(q, w) = -\frac{2mk_F}{(2\pi)^3} \frac{2\pi}{q} \left( A\left(\frac{1}{2}q - \frac{v}{q}\right) + A\left(\frac{1}{2}q + \frac{v}{q}\right) \right)$$

求解 $B(a)$ ,

$$\begin{aligned}
B(a) &= \int_0^1 k \ln|k+a| dk \\
&= \left[ \frac{(k+a)^2}{2} \ln|k+a| - a[(k+a) \ln|k+a| - k] - \frac{1}{2} \frac{(k+a)^2}{2} \right] \Big|_0^1 \\
&= \frac{1-a^2}{2} \ln|1+a| + \frac{a^2}{2} \ln|a| + \frac{a}{2} - \frac{1}{4}
\end{aligned}$$

最终,

$$A(a) = \frac{1-a^2}{2} \ln \left| \frac{1+a}{1-a} \right| + a$$

$$\begin{aligned}
&\text{Re } \Pi_{MD}^*(q, w) \\
&= -\frac{2mk_F}{(2\pi)^3} \frac{2\pi}{q} \left[ \frac{1 - \left(\frac{q}{2} - \frac{v}{q}\right)^2}{2} \ln \left| \frac{1 + \frac{q}{2} - \frac{v}{q}}{1 - \frac{q}{2} + \frac{v}{q}} \right| + \frac{1 - \left(\frac{q}{2} + \frac{v}{q}\right)^2}{2} \ln \left| \frac{1 + \frac{q}{2} + \frac{v}{q}}{1 - \frac{q}{2} - \frac{v}{q}} \right| + q \right] \\
&= -\frac{mk_F}{2\pi^2} \frac{1}{q} \left[ \frac{1 - \left(\frac{q}{2} - \frac{v}{q}\right)^2}{2} \ln \left| \frac{1 + \frac{q}{2} - \frac{v}{q}}{1 - \frac{q}{2} + \frac{v}{q}} \right| + \frac{1 - \left(\frac{q}{2} + \frac{v}{q}\right)^2}{2} \ln \left| \frac{1 + \frac{q}{2} + \frac{v}{q}}{1 - \frac{q}{2} - \frac{v}{q}} \right| + q \right] \\
&= -\frac{mk_F}{\pi^2} \left[ \frac{1 - \left(\frac{q}{2} - \frac{v}{q}\right)^2}{4q} \ln \left| \frac{1 + \frac{q}{2} - \frac{v}{q}}{1 - \frac{q}{2} + \frac{v}{q}} \right| + \frac{1 - \left(\frac{q}{2} + \frac{v}{q}\right)^2}{4q} \ln \left| \frac{1 + \frac{q}{2} + \frac{v}{q}}{1 - \frac{q}{2} - \frac{v}{q}} \right| + \frac{1}{2} \right] \\
&\quad (\text{无量纲的}q)
\end{aligned}$$

$$v(q) = \frac{4\pi e^2}{q^2}, \quad v(q) \frac{mk_F}{\pi^2} = \frac{\lambda^2}{q^2} (\text{有量纲的}q)$$

将 $v$ 带入, 并且将 $q$ 写成有量纲形式

$$\begin{aligned}\frac{v}{q} - \frac{q}{2} &= \frac{1}{qv_p} \left( w - \frac{q^2}{2m} \right), \\ \frac{v}{q} + \frac{q}{2} &= \frac{1}{qv_p} \left( w + \frac{q^2}{2m} \right) \\ \ln \left| \frac{\frac{v}{q} - \frac{q}{2} - 1}{\frac{v}{q} - \frac{q}{2} + 1} \right| &= \ln \left| \frac{\omega - qv_F - \frac{q^2}{2m}}{\omega + qv_F - \frac{q^2}{2m}} \right| \\ \ln \left| \frac{\frac{v}{q} + \frac{q}{2} + 1}{\frac{v}{q} + \frac{q}{2} - 1} \right| &= \ln \left| \frac{w + qv_F + \frac{q^2}{2m}}{w - qv_F + \frac{q^2}{2m}} \right|\end{aligned}$$

$$\begin{aligned}\text{Re } \Pi_{MD}^*(q, w) \\ &= -\frac{mk_F}{\pi^2} \\ &\left[ \frac{k_F}{4q} \left( \frac{(w + \frac{q^2}{2m})^2}{(qv_p)^2} - 1 \right) \ln \left| \frac{w - qv_F + \frac{q^2}{2m}}{w + qv_F + \frac{q^2}{2m}} \right| - \frac{k_F}{4q} \left( \frac{(w - \frac{q^2}{2m})^2}{(qv_p)^2} - 1 \right) \ln \left| \frac{\omega - qv_F - \frac{q^2}{2m}}{\omega + qv_F - \frac{q^2}{2m}} \right| + \frac{1}{2} \right]\end{aligned}$$

根据上式以及

$$\text{Re } \epsilon(q, \omega) = 1 - v(q) \text{Re } \Pi_{MD}^*(q, w)$$

可以得到书上(4.6.25)。

 求：

$$\text{Im } \epsilon(q, \omega) = \frac{2\pi v(q)}{\hbar} \sum_k n_k (1 - n_{k+q}) \{ \delta(\omega_{kq} - \omega) - \delta(\omega_{kq} + \omega) \}$$

sol.

令

$$I = \sum_k n_k (1 - n_{k+q}) \{ \delta(\omega_{kq} - \omega) - \delta(\omega_{kq} + \omega) \}$$

其中,

$$\begin{aligned}1 - n_{k+q} &= \theta(|\vec{k} + \vec{q}| - k_F) \\ n_k &= \theta(k_F - |\vec{k}|) \\ \omega_{kq} &= \frac{k}{2m} [(\vec{k} + \vec{q})^2 - k^2] = \frac{k}{2m} (\vec{q}^2 + 2\vec{k} \cdot \vec{q})\end{aligned}$$

因为 $\omega \geq 0$ , 而 $\theta(k_F - |\vec{k}|)\theta(|\vec{k} + \vec{q}| - k_F)$ 保证 $\omega_{kq} > 0$ , 所以 $\delta(\omega_{kq} + \omega) = 0$ .

求和转化为积分,

$$\sum_k \rightarrow \frac{1}{8\pi^3} \int d^3k = \frac{1}{8\pi^3} \int_0^{2\pi} d\varphi \int_0^\pi d\theta \int_0^\infty dk k^2 \sin \theta$$

所以 $I$ 写成,

$$I = \frac{1}{8\pi^3} \int d^3k \theta(k_F - |\vec{k}|) \theta(|\vec{k} + \vec{q}| - k_F) \delta(\omega_{kq} - \omega)$$

使用费米波矢 $k_F$ 作为波矢单位,  $k \rightarrow k k_F, q \rightarrow q k_F$ , 所以有

$$\sum_k \rightarrow \frac{1}{8\pi^3} \int d^3k = \frac{k_F^3}{8\pi^3} \int_0^{2\pi} d\varphi \int_0^\pi d\theta \int_0^\infty dk k^2 \sin \theta$$

$$\omega_{kq} = \frac{\hbar k_F^2}{2m} (q^2 + 2\vec{k} \cdot \vec{q})$$

并且定义

$$\nu = \omega \frac{m}{\hbar k_F^2}$$

根据 $\delta$ 函数性质,

$$\delta(ax) = |a|^{-1} \delta(x)$$

$$\delta(\omega_{kq} - \omega) = \frac{m}{\pi k_F^2} \delta\left(\nu - \vec{q} \cdot \vec{k} - \frac{q^2}{2}\right)$$

最终得到 $I$ ,

$$I = \frac{m k_F}{4\pi^2 \hbar} \int_0^\pi d\theta \int_0^\infty dk k^2 \sin \theta \cdot \theta(1 - k) \theta(|\vec{k} + \vec{q}| - 1) \delta\left(\nu - \vec{q} \cdot \vec{k} - \frac{q^2}{2}\right)$$

$k$ 需要同时满足3个条件,

$$\begin{cases} k < 1, |\vec{k} + \vec{q}| > 1, \\ \nu - \vec{q} \cdot \vec{k} - \frac{q^2}{2} = 0 \Leftrightarrow \frac{q}{2} + \hat{q} \cdot \vec{k} = \frac{\nu}{q}. \end{cases}$$

其几何含义是平面 $\frac{q}{2} + \hat{q} \cdot \vec{k} = \frac{\nu}{q}$ 需要与 $k < 1, |\vec{k} + \vec{q}| > 1$ 确定的三维空间相交, 这确定了 $\nu$ 的取值范围。

The diagram illustrates the geometry of a particle in a magnetic field, showing the relationship between the particle's position, velocity, and the magnetic field lines. It consists of two parts: a top part showing a particle in a magnetic field and a bottom part showing a geometric construction.

**Top Part:** A yellow circle represents a particle. A vertical dashed line passes through its center. A horizontal blue line represents the magnetic field line, labeled  $\pi$ . A horizontal red line is labeled  $\pi_1$ . A horizontal red line is labeled  $\pi_2$ . A horizontal red line is labeled  $\pi_0$ . A vertical blue double-headed arrow between  $\pi$  and  $\pi_2$  is labeled  $\frac{v}{\xi}$ . A vertical red double-headed arrow between  $\pi_1$  and  $\pi_0$  is labeled  $(\frac{v}{\xi})_{\max}$ . A vertical red double-headed arrow between  $\pi_2$  and  $\pi_0$  is labeled  $(\frac{v}{\xi})_{\min}$ . A vector  $\vec{g}$  points from the center of the circle to the  $\pi$  line. A vector  $\vec{k}$  points from the center of the circle to the  $\pi$  line. A vector  $\vec{g} + \vec{k}$  points from the center of the circle to the  $\pi$  line. A vector  $\vec{g} \cdot \vec{k}$  points from the center of the circle to the  $\pi$  line. A vector  $\frac{g}{2}$  points from the center of the circle to the  $\pi$  line.

**Bottom Part:** A circle represents a particle. A vertical blue line passes through its center. A horizontal blue line is labeled  $\pi$ . A horizontal red line is labeled  $\pi_1$ . A horizontal red line is labeled  $\pi_2$ . A horizontal red line is labeled  $\pi_0$ . A vertical blue double-headed arrow between  $\pi$  and  $\pi_2$  is labeled  $\frac{v}{\xi}$ . A vertical red double-headed arrow between  $\pi_1$  and  $\pi_0$  is labeled  $(\frac{v}{\xi})_{\max}$ . A vertical red double-headed arrow between  $\pi_2$  and  $\pi_0$  is labeled  $(\frac{v}{\xi})_{\min}$ . A vector  $\vec{g}$  points from the center of the circle to the  $\pi$  line. A vector  $\vec{k}$  points from the center of the circle to the  $\pi$  line. A vector  $\vec{g} + \vec{k}$  points from the center of the circle to the  $\pi$  line. A vector  $\vec{g} \cdot \vec{k}$  points from the center of the circle to the  $\pi$  line. A vector  $\frac{g}{2}$  points from the center of the circle to the  $\pi$  line.

$$I = \frac{mk_F}{4\pi^2\hbar} \int_{\frac{\nu}{q} - \frac{q}{2}}^1 k^2 dk \int_{-1}^1 dt \frac{1}{qk} \delta\left(\frac{\nu}{qk} - \frac{1}{2} \frac{q}{k} - t\right)$$

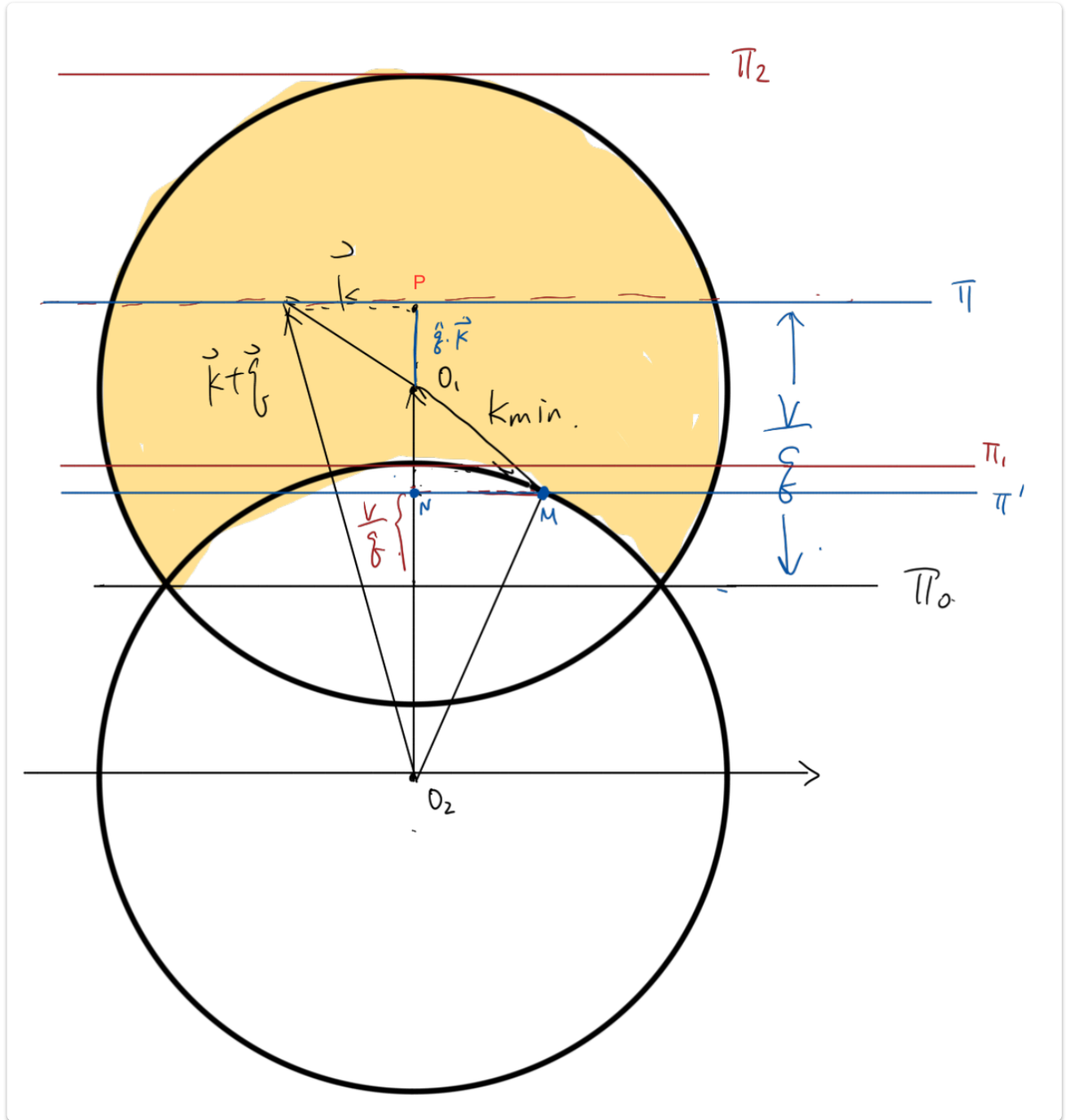
$$I = \frac{mk_F}{4\pi^2\hbar} \frac{1}{q} \frac{1}{2} \left[1 - \left(\frac{\nu}{q} - \frac{q}{2}\right)^2\right]$$

$$I = \frac{mk_F}{4\pi^2\hbar} \frac{1}{q} \frac{1}{2} \left[ 1 - \left( \frac{\nu}{q} - \frac{q}{2} \right)^2 \right]$$

定义费米速度  $v_F = \frac{\hbar k_F}{m}$ ,  $\lambda^2 = \frac{4me^2 k_F}{\pi \hbar^2}$ ,

$$\text{Im}\varepsilon(q, \omega) = \frac{\pi}{4} \frac{\lambda^2}{q^2} \frac{k_F}{q} \left[ 1 - \frac{(\omega - \frac{\hbar q^2}{2m})^2}{(qv_F)^2} \right]$$

2.  $1 < q < 2$ ,



需要分为两种情况，

- 平面 $\pi$ 在 $\pi_1, \pi_2$ 之间时( $q + \frac{1}{2}q^2 \geq \nu \geq q - \frac{1}{2}q^2$ )， $k$ 最小值为 $O_1P$ ，此时 $\text{Im}\varepsilon(q, \omega)$ 表达式同 $q > 2$ 情况。
- 平面 $\pi$ 在 $\pi_0, \pi_1$ 之间时( $0 \leq \nu \leq q - \frac{1}{2}q^2$ )， $k$ 的下限只能到 $O_1M$ 。

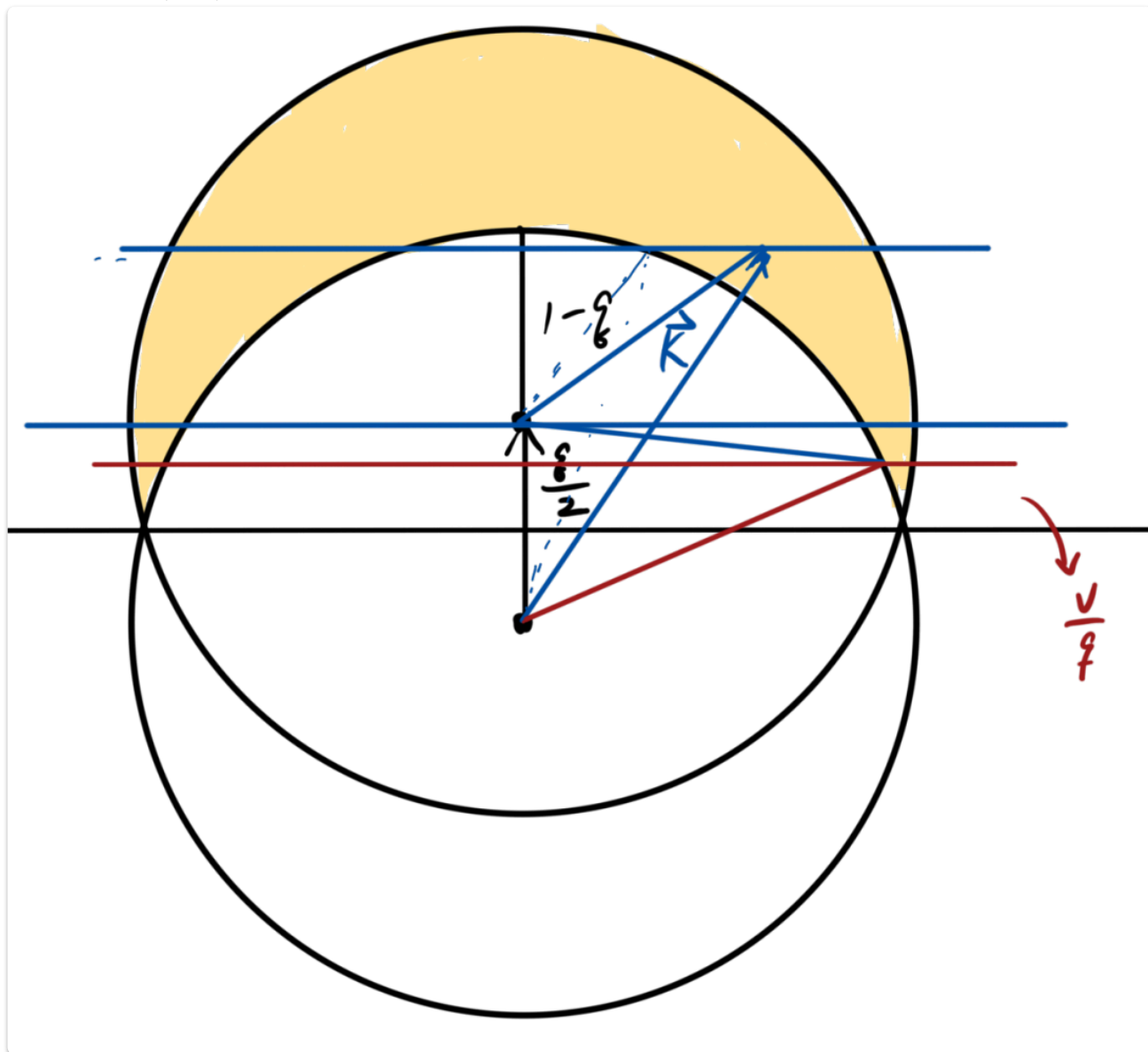
$$\begin{aligned}
 O_1 M^2 &= MN^2 + O_1 N^2 \\
 &= O_2 M^2 - O_2 N^2 + O_1 N^2 \\
 &= \left( \frac{1}{2} q + \frac{\nu}{q} \right)^2 + \left( \frac{1}{2} q - \frac{\nu}{q} \right)^2 \\
 &= 1 - 2\nu.
 \end{aligned}$$

所以,

$$I = \frac{mk_F}{4\pi^2 k} \int_{\sqrt{1-2\nu}}^1 k^2 dk \int_{-1}^1 dt \frac{1}{qk} \delta \left( \frac{\nu}{qk} - \frac{1}{2} \frac{q}{k} - t \right) = \frac{mk_F}{4\pi^2 \hbar} \frac{\nu}{q}$$

$$\text{Im}\varepsilon(q, \omega) = \frac{\pi}{2} \frac{\lambda^2}{q^2} \frac{\omega}{qv_F}$$

3.  $q < 1$ ,  $\text{Im}\varepsilon(q, \omega)$ 表达式同2。



综上,

$$\mathrm{Im}\varepsilon(q, \omega) = \begin{cases} \frac{\pi}{2} \frac{\lambda^2}{q^2} \frac{\omega}{qv_F}, & 0 \leq \omega \leq qv_F - \hbar \frac{q^2}{2m} \\ \frac{\pi}{4} \frac{\lambda^2}{q^2} \frac{k_F}{q} \left[ 1 - \frac{\left( \omega - \frac{\hbar q^2}{2m} \right)^2}{(qv_F)^2} \right], & -\hbar \frac{q^2}{2m} \leq \omega - qv_F \leq \hbar \frac{q^2}{2m} \\ 0, & \omega \geq qv_F + \hbar \frac{q^2}{2m} \end{cases}$$