Lindhard函数实部虚部求解

∠ Lindhard函数

$$\Pi_{MD}^*(q,\omega) = 2\int rac{dk}{(2\pi)^3} heta(|\overrightarrow{k}+\overrightarrow{q}|-k_F) heta(k_F-|\overrightarrow{k}|) \left(rac{1}{\omega-\omega_{kq}+i\eta}-rac{1}{\omega+\omega_{kq}-i\eta}
ight)$$

其中,

$$\omega_{kq} = rac{(\overrightarrow{k} + \overrightarrow{q})^2}{2m} - rac{\overrightarrow{k}^2}{2m}$$

1实部

$$\operatorname{Re} \Pi_{MD}^{*}(q,\omega) = 2 \int \frac{dk}{(2\pi)^{3}} \theta(|\overrightarrow{k}+\overrightarrow{q}|-k_{F}) \theta(k_{F}-|\overrightarrow{k}|) \left(\frac{1}{\omega-\omega_{kq}} - \frac{1}{\omega+\omega_{kq}}\right)$$

$$\operatorname{Re} \Pi_{MD}^{*}(q,\omega) = 2 \int \frac{dk}{(2\pi)^{3}} \theta(|\overrightarrow{k}+\overrightarrow{q}|-k_{F}) \theta(k_{F}-|\overrightarrow{k}|) \left(\frac{1}{\omega-\omega_{kq}} - \frac{1}{\omega+\omega_{kq}}\right)$$

$$= 2 \int \frac{dk}{(2\pi)^{3}} (1-\theta(k_{F}-|\overrightarrow{k}+\overrightarrow{q}|)) \theta(k_{F}-|\overrightarrow{k}|) \left(\frac{1}{\omega-\omega_{kq}} - \frac{1}{\omega+\omega_{kq}}\right)$$

$$= 2 \int \frac{dk}{(2\pi)^{3}} \theta(k_{F}-|\overrightarrow{k}|) \left(\frac{1}{\omega-\omega_{kq}} - \frac{1}{\omega+\omega_{kq}}\right)$$

$$-2 \int \frac{dk}{(2\pi)^{3}} \theta(k_{F}-|\overrightarrow{k}+\overrightarrow{q}|) \theta(k_{F}-|\overrightarrow{k}|) \left(\frac{1}{\omega-\omega_{kq}} - \frac{1}{\omega+\omega_{kq}}\right)$$

对于第二项,做变换 $\overrightarrow{k}+\overrightarrow{q}
ightarrow -\overrightarrow{k}$,有 $\omega_{kq}
ightarrow -\omega_{kq}$,

$$egin{aligned} &2\intrac{dk}{(2\pi)^3} heta(k_F-|\overrightarrow{k}+\overrightarrow{q|}) heta(k_F-|\overrightarrow{k}|)\left(rac{1}{\omega-\omega_{kq}}-rac{1}{\omega+\omega_{kq}}
ight)\ =&2\intrac{dk}{(2\pi)^3} heta(k_F-|\overrightarrow{k}+\overrightarrow{q|}) heta(k_F-|\overrightarrow{k}|)\left(rac{1}{\omega+\omega_{kq}}-rac{1}{\omega-\omega_{kq}}
ight)\ =&0 \end{aligned}$$

$$egin{aligned} \operatorname{Re} \Pi_{MD}^*(q,\omega) = & 2 \int rac{dk}{(2\pi)^3} heta(k_F - |\overrightarrow{k}|) \left(rac{1}{\omega - \omega_{kq}} - rac{1}{\omega + \omega_{kq}}
ight) \ = & 2 \int rac{dk}{(2\pi)^3} heta(k_F - |\overrightarrow{k}|) \left(rac{1}{\omega - rac{1}{m}(\overrightarrow{q} \cdot \overrightarrow{k} + rac{1}{2}q^2)} - rac{1}{\omega + rac{1}{m}(\overrightarrow{q} \cdot \overrightarrow{k} + rac{1}{2}q^2)}
ight) \end{aligned}$$

$$u = \omega rac{m}{k_F^2}, k o k k_F, q o q k_F.$$

$$\operatorname{Re} \Pi_{MD}^{*}(q,\omega) = 2mk_{F} \int \frac{dk}{(2\pi)^{3}} \theta(1-k) \left(\frac{1}{\nu - \overrightarrow{q} \cdot \overrightarrow{k} - \frac{1}{2}q^{2}} - \frac{1}{\nu + \overrightarrow{q} \cdot \overrightarrow{k} + \frac{1}{2}q^{2}} \right) \\
= \frac{2mk_{F}}{(2\pi)^{3}} 2\pi \int_{0}^{1} dk \int_{0}^{\pi} d\theta k^{2} \sin \theta \left(\frac{1}{\nu - qk \cos \theta - \frac{1}{2}q^{2}} - \frac{1}{\nu + qk \cos \theta + \frac{1}{2}q^{2}} \right) \\
= \frac{2mk_{F}}{(2\pi)^{3}} 2\pi \int_{0}^{1} k^{2} dk \int_{-1}^{1} dx \left(-\frac{1}{-\nu + qkx + \frac{1}{2}q^{2}} - \frac{1}{\nu + qkx + \frac{1}{2}q^{2}} \right) \\
= -\frac{2mk_{F}}{(2\pi)^{3}} \frac{2\pi}{q} \int_{0}^{1} k^{2} dk \int_{-1}^{1} dx \left(\frac{1}{kx + \frac{1}{2}q - \frac{\nu}{q}} + \frac{1}{kx + \frac{1}{2}q + \frac{\nu}{q}} \right)$$

令

$$egin{aligned} A(a) &= \int_0^1 k^2 dk \int_{-1}^1 dx rac{1}{kx+a} \ &= \int_0^1 k dk (\ln|k+a| - \ln|k-a|) \ &\equiv B(a) - B(-a) \end{aligned}$$

则有

$$ext{Re}\,\Pi^*_{MD}(q,\omega) = -rac{2mk_F}{(2\pi)^3}rac{2\pi}{q}igg(A\left(rac{1}{2}q-rac{
u}{q}
ight) + A(rac{1}{2}q+rac{
u}{q})igg)$$

求解B(a),

$$egin{align} B(a) &= \int_0^1 k \ln |k+a| dk \ &= \left[rac{(k+a)^2}{2} \ln |k+a| - a[(k+a) \ln |k+a| - k] - rac{1}{2} rac{(k+a)^2}{2}
ight]
ight|_0^1 \ &= rac{1-a^2}{2} \ln |1+a| + rac{a^2}{2} \ln |a| + rac{a}{2} - rac{1}{4} \end{split}$$

最终,

$$A(a) = rac{1-a^2}{2} {\ln |rac{1+a}{1-a}|} + a$$

$$\begin{split} & = -\frac{2mk_F}{(2\pi)^3} \frac{2\pi}{q} \left[\frac{1 - \left(\frac{q}{2} - \frac{\nu}{q}\right)^2}{2} \ln |\frac{1 + \frac{q}{2} - \frac{\nu}{q}}{1 - \frac{q}{2} + \frac{\nu}{q}}| + \frac{1 - \left(\frac{q}{2} + \frac{\nu}{q}\right)^2}{2} \ln \left|\frac{1 + \frac{q}{2} + \frac{\nu}{q}}{1 - \frac{q}{2} - \frac{\nu}{q}}\right| + q \right] \\ & = -\frac{mk_F}{2\pi^2} \frac{1}{q} \left[\frac{1 - \left(\frac{q}{2} - \frac{\nu}{q}\right)^2}{2} \ln |\frac{1 + \frac{q}{2} - \frac{\nu}{q}}{1 - \frac{q}{2} + \frac{\nu}{q}}| + \frac{1 - \left(\frac{q}{2} + \frac{\nu}{q}\right)^2}{2} \ln \left|\frac{1 + \frac{q}{2} + \frac{\nu}{q}}{1 - \frac{q}{2} - \frac{\nu}{q}}\right| + q \right] \\ & = -\frac{mk_F}{\pi^2} \left[\frac{1 - \left(\frac{q}{2} - \frac{\nu}{q}\right)^2}{4q} \ln \left|\frac{1 + \frac{q}{2} - \frac{\nu}{q}}{1 - \frac{q}{2} + \frac{\nu}{q}}\right| + \frac{1 - \left(\frac{q}{2} + \frac{\nu}{q}\right)^2}{4q} \ln \left|\frac{1 + \frac{q}{2} + \frac{\nu}{q}}{1 - \frac{q}{2} - \frac{\nu}{q}}\right| + \frac{1}{2} \right] \end{split}$$
 (无量纲的q)

2虚部

$$\operatorname{Im}\Pi_{MD}^*(q,\omega) = -2\pirac{1}{8\pi^3}\int d^3k heta(k_F-|\overrightarrow{k}|) heta(|\overrightarrow{k}+\overrightarrow{q}|-k_F)(\delta(\omega_{kq}-\omega)+\delta(\omega_{kq}+\omega))$$

因为
$$\omega \geq 0$$
, 而 $heta(k_F - |\overrightarrow{k}|) heta(|\overrightarrow{k} + \overrightarrow{q}| - k_F)$ 保证 $\omega_{kq} > 0$,所以 $\delta(\omega_{kq} + \omega) = 0$.

所以,

$$\dim\Pi_{MD}^*(q,\omega) = -2\pirac{1}{8\pi^3}\int d^3k heta(k_F-\left|\overrightarrow{k}
ight|) heta(\left|\overrightarrow{k}+\overrightarrow{q}
ight|-k_F)\delta(\omega_{kq}-\omega)$$

使用费米波矢 k_F 作为波矢单位, $k \to kk_F, q \to qk_F,$ 并且定义

$$u = \omega rac{m}{k_F^2}$$

根据 δ 函数性质,

$$\delta(ax) = |a|^{-1}\delta(x) \ \delta(\omega_{kq} - \omega) = rac{m}{\pi k_F^2}\delta(
u - \overrightarrow{q} \cdot \overrightarrow{k} - rac{q^2}{2})$$

最终得到 $\operatorname{Im}\Pi_{MD}^*(q,\omega)$,

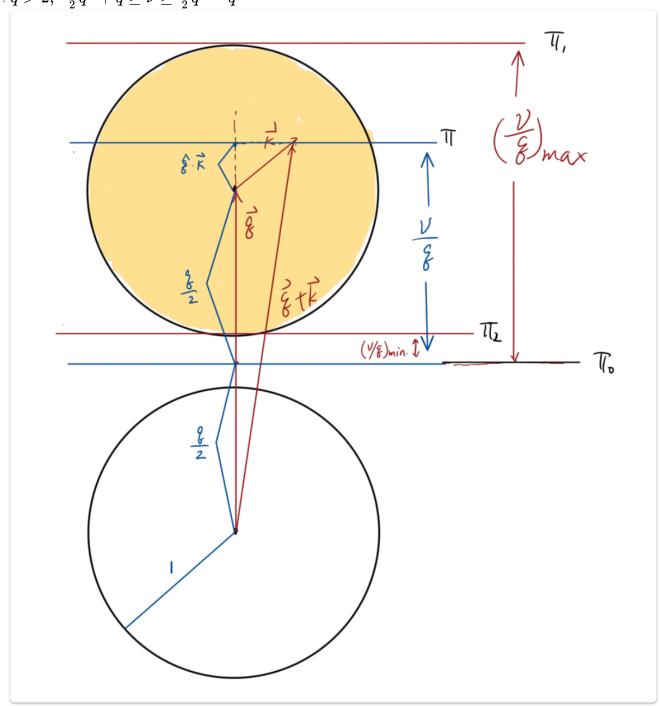
$$\operatorname{Im}\Pi_{MD}^*(q,w) = -2\pirac{mk_F}{4\pi^2}\int\limits_0^\pi d heta\int\limits_0^\infty dk k^2\sin heta\cdot heta(1-k) heta(|\overrightarrow{k}+\overrightarrow{q}|-1)\delta\left(
u-\overrightarrow{q}\cdot\overrightarrow{k}-rac{q^2}{2}
ight)$$

k需要同时满足3个条件,

$$egin{cases} k < 1, |\overrightarrow{k}+\overrightarrow{q}| > 1, \
u - \overrightarrow{q} \cdot \overrightarrow{k} - rac{q^2}{2} = 0 \Leftrightarrow rac{q}{2} + \hat{q} \cdot \overrightarrow{k} = rac{
u}{q}. \end{cases}$$

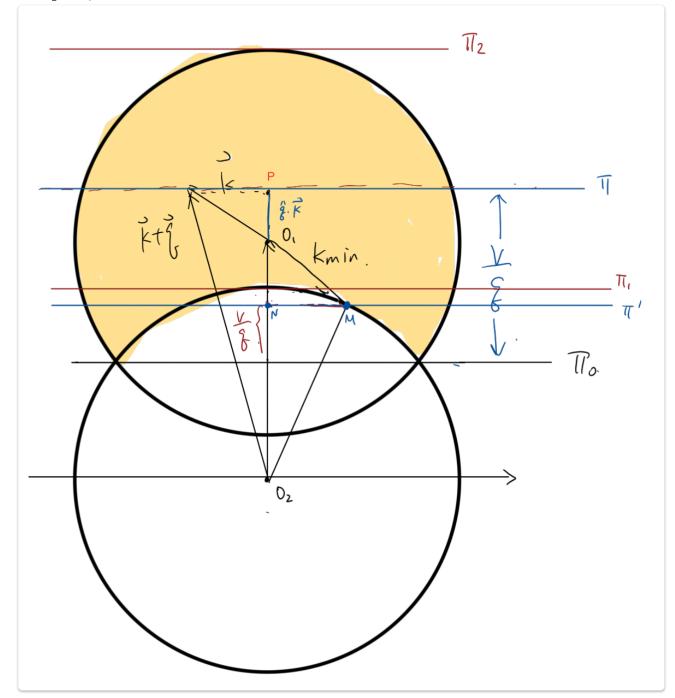
其几何含义是平面 $\frac{q}{2}+\hat{q}\cdot\overrightarrow{k}=\frac{\nu}{q}$ 需要与k<1, $|\overrightarrow{k}+\overrightarrow{q}|>1$ 确定的三维空间相交,这确定了 ν 的取值范围。

1. q > 2, $\frac{1}{2}q^2 + q \ge \nu \ge \frac{1}{2}q^2 - q$



$$egin{align} \operatorname{Im}\Pi_{MD}^*(q,\omega) &= -2\pirac{mk_F}{4\pi^2}\int_{rac{
u}{q}-rac{q}{2}}^1k^2dk\int_{-1}^1dtrac{1}{qk}\delta\left(rac{
u}{qk}-rac{1}{2}rac{q}{k}-t
ight) \ & \operatorname{Im}\Pi_{MD}^*(q,\omega) = -rac{mk_F}{4\pi q}[1-(rac{
u}{q}-rac{q}{2})^2] \end{aligned}$$

2.1 < q < 2,



需要分为两种情况,

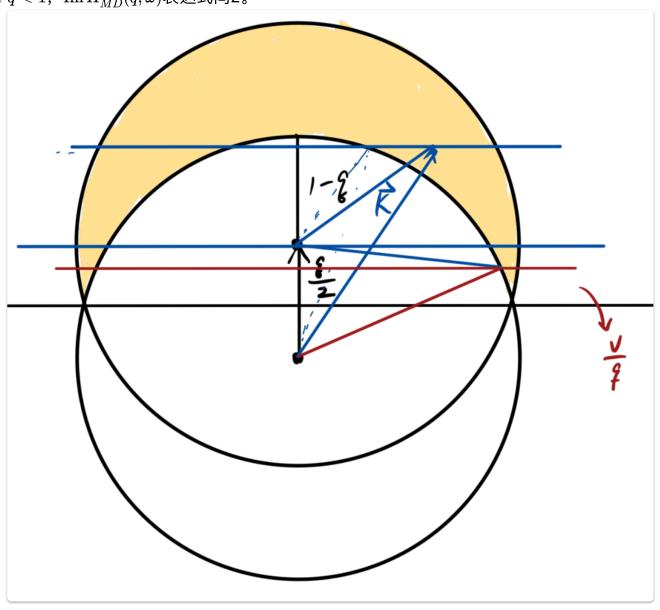
- 平面 π 在 π_1,π_2 之间时 $(q+\frac{1}{2}q^2\geq \nu\geq q-\frac{1}{2}q^2)$,k最小值为 O_1P ,此时 $\operatorname{Im}\Pi^*_{MD}(q,w)$ 表达式同q>2情况。
- 平面 π 在 π_0, π_1 之间时 $(0 \le \nu \le q \frac{1}{2}q^2)$,k的下限只能到 O_1M 。

$$egin{aligned} O_1 M^2 &= M N^2 + O_1 N^2 \ &= O_2 M^2 - O_2 N^2 + O_1 N^2 \ &= \left(rac{1}{2} q + rac{
u}{q}
ight)^2 + \left(rac{1}{2} q - rac{
u}{q}
ight)^2 \ &= 1 - 2
u. \end{aligned}$$

所以,

$${
m Im}\,\Pi_{MD}^*(q,\omega) = -2\pirac{mk_F}{4\pi^2}\int_{\sqrt{1-2
u}}^1 k^2 dk \int_{-1}^1 dtrac{1}{qk}\delta\left(rac{
u}{qk}-rac{1}{2}rac{q}{k}-t
ight) = -rac{mk_F}{4\pi}rac{2
u}{q}$$

3. q<1, $\operatorname{Im}\Pi_{MD}^*(q,\omega)$ 表达式同2。



综上,

$$\operatorname{Im}\Pi_{MD}^*(q,\omega) = egin{cases} -rac{mk_F}{4\pi q}[1-(rac{
u}{q}-rac{q}{2})^2], & (q-rac{1}{2}q^2 \leq
u \leq q+rac{1}{2}q^2) \ -rac{mk_F}{4\pi}rac{2
u}{q}, & (0 \leq
u \leq q-rac{1}{2}q^2) \ 0, & (
u \geq q+rac{1}{2}q^2) \end{cases}$$