Lindhard函数实部虚部求解

Lindhard函数

$$\Pi^*_{MD}(q,\omega)=2\int\frac{dk}{(2\pi)^3}\theta(|\vec{k}+\vec{q}|-k_F)\theta(k_F-|\vec{k}|)\left(\frac{1}{\omega-\omega_{kq}+i\eta}-\frac{1}{\omega+\omega_{kq}-i\eta}\right)$$

其中,

$$\omega_{kq}=\frac{(\vec{k}+\vec{q})^2}{2m}-\frac{\vec{k}^2}{2m}$$

1 实部

$$\mathrm{Re}\,\Pi^*_{MD}(q,\omega)=2\int\frac{dk}{(2\pi)^3}\theta(|\vec{k}+\vec{q}|-k_F)\theta(k_F-|\vec{k}|)\left(\frac{1}{\omega-\omega_{kq}}-\frac{1}{\omega+\omega_{kq}}\right)$$

$$\begin{aligned}\mathrm{Re}\,\Pi^*_{MD}(q,\omega)&=2\int\frac{dk}{(2\pi)^3}\theta(|\vec{k}+\vec{q}|-k_F)\theta(k_F-|\vec{k}|)\left(\frac{1}{\omega-\omega_{kq}}-\frac{1}{\omega+\omega_{kq}}\right)\\&=2\int\frac{dk}{(2\pi)^3}(1-\theta(k_F-|\vec{k}+\vec{q}|))\theta(k_F-|\vec{k}|)\left(\frac{1}{\omega-\omega_{kq}}-\frac{1}{\omega+\omega_{kq}}\right)\\&=2\int\frac{dk}{(2\pi)^3}\theta(k_F-|\vec{k}|)\left(\frac{1}{\omega-\omega_{kq}}-\frac{1}{\omega+\omega_{kq}}\right)\\&-2\int\frac{dk}{(2\pi)^3}\theta(k_F-|\vec{k}+\vec{q}|)\theta(k_F-|\vec{k}|)\left(\frac{1}{\omega-\omega_{kq}}-\frac{1}{\omega+\omega_{kq}}\right)\end{aligned}$$

对于第二项，做变换$\vec{k}+\vec{q}\rightarrow-\vec{k}$，有$\omega_{kq}\rightarrow-\omega_{kq}$,

$$\begin{aligned}&2\int\frac{dk}{(2\pi)^3}\theta(k_F-|\vec{k}+\vec{q}|)\theta(k_F-|\vec{k}|)\left(\frac{1}{\omega-\omega_{kq}}-\frac{1}{\omega+\omega_{kq}}\right)\\=&2\int\frac{dk}{(2\pi)^3}\theta(k_F-|\vec{k}+\vec{q}|)\theta(k_F-|\vec{k}|)\left(\frac{1}{\omega+\omega_{kq}}-\frac{1}{\omega-\omega_{kq}}\right)\\=&0\end{aligned}$$

$$\begin{aligned}\mathrm{Re}\,\Pi^*_{MD}(q,\omega)=&2\int\frac{dk}{(2\pi)^3}\theta(k_F-|\vec{k}|)\left(\frac{1}{\omega-\omega_{kq}}-\frac{1}{\omega+\omega_{kq}}\right)\\=&2\int\frac{dk}{(2\pi)^3}\theta(k_F-|\vec{k}|)\left(\frac{1}{\omega-\frac{1}{m}(\vec{q}\cdot\vec{k}+\frac{1}{2}q^2)}-\frac{1}{\omega+\frac{1}{m}(\vec{q}\cdot\vec{k}+\frac{1}{2}q^2)}\right)\end{aligned}$$

令

$$\nu = \omega \frac{m}{k_F^2}, k \to k k_F, q \to q k_F.$$

$$\begin{aligned}
\mathrm{Re}\,\Pi^*_{MD}(q,\omega) &= 2mk_F \int \frac{dk}{(2\pi)^3}\theta(1-k)\left(\frac{1}{\nu - \overrightarrow{q}\cdot\overrightarrow{k} - \frac{1}{2}q^2} - \frac{1}{\nu + \overrightarrow{q}\cdot\overrightarrow{k} + \frac{1}{2}q^2}\right) \\
&= \frac{2mk_F}{(2\pi)^3} 2\pi \int_0^1 dk \int_0^\pi d\theta k^2 \sin\theta \left(\frac{1}{\nu - qk\cos\theta - \frac{1}{2}q^2} - \frac{1}{\nu + qk\cos\theta + \frac{1}{2}q^2}\right) \\
&= \frac{2mk_F}{(2\pi)^3} 2\pi \int_0^1 k^2 dk \int_{-1}^1 dx (-\frac{1}{-\nu + qkx + \frac{1}{2}q^2} - \frac{1}{\nu + qkx + \frac{1}{2}q^2}) \\
&= -\frac{2mk_F}{(2\pi)^3} \frac{2\pi}{q} \int_0^1 k^2 dk \int_{-1}^1 dx (\frac{1}{kx + \frac{1}{2}q - \frac{\nu}{q}} + \frac{1}{kx + \frac{1}{2}q + \frac{\nu}{q}})
\end{aligned}$$

令

$$\begin{aligned}
A(a) &= \int_0^1 k^2 dk \int_{-1}^1 dx \frac{1}{kx + a} \\
&= \int_0^1 k dk (\ln|k+a| - \ln|k-a|) \\
&\equiv B(a) - B(-a)
\end{aligned}$$

则有

$$\mathrm{Re}\,\Pi^*_{MD}(q,\omega) = -\frac{2mk_F}{(2\pi)^3}\frac{2\pi}{q}\left(A\left(\frac{1}{2}q - \frac{\nu}{q}\right) + A(\frac{1}{2}q + \frac{\nu}{q})\right)$$

求解$B(a)$,

$$\begin{aligned}
B(a) &= \int_0^1 k \ln|k+a| dk \\
&= \left[\frac{(k+a)^2}{2}\ln|k+a| - a[(k+a)\ln|k+a| - k] - \frac{1}{2}\frac{(k+a)^2}{2}\right]\Bigg|_0^1 \\
&= \frac{1-a^2}{2}\ln|1+a| + \frac{a^2}{2}\ln|a| + \frac{a}{2} - \frac{1}{4}
\end{aligned}$$

最终,

$$A(a) = \frac{1-a^2}{2}\ln|\frac{1+a}{1-a}| + a$$

$$
\begin{aligned}
&\operatorname{Re}\Pi^*_{MD}(q,\omega)\\
&=-\frac{2mk_F}{(2\pi)^3}\frac{2\pi}{q}\left[\frac{1-\left(\frac{q}{2}-\frac{\nu}{q}\right)^2}{2}\ln\left|\frac{1+\frac{q}{2}-\frac{\nu}{q}}{1-\frac{q}{2}+\frac{\nu}{q}}\right|+\frac{1-\left(\frac{q}{2}+\frac{\nu}{q}\right)^2}{2}\ln\left|\frac{1+\frac{q}{2}+\frac{\nu}{q}}{1-\frac{q}{2}-\frac{\nu}{q}}\right|+q\right]\\
&=-\frac{mk_F}{2\pi^2}\frac{1}{q}\left[\frac{1-\left(\frac{q}{2}-\frac{\nu}{q}\right)^2}{2}\ln\left|\frac{1+\frac{q}{2}-\frac{\nu}{q}}{1-\frac{q}{2}+\frac{\nu}{q}}\right|+\frac{1-\left(\frac{q}{2}+\frac{\nu}{q}\right)^2}{2}\ln\left|\frac{1+\frac{q}{2}+\frac{\nu}{q}}{1-\frac{q}{2}-\frac{\nu}{q}}\right|+q\right]\\
&=-\frac{mk_F}{\pi^2}\left[\frac{1-\left(\frac{q}{2}-\frac{\nu}{q}\right)^2}{4q}\ln\left|\frac{1+\frac{q}{2}-\frac{\nu}{q}}{1-\frac{q}{2}+\frac{\nu}{q}}\right|+\frac{1-\left(\frac{q}{2}+\frac{\nu}{q}\right)^2}{4q}\ln\left|\frac{1+\frac{q}{2}+\frac{\nu}{q}}{1-\frac{q}{2}-\frac{\nu}{q}}\right|+\frac{1}{2}\right]
\end{aligned}
$$

（无量纲的q）

2 虚部

$$\operatorname{Im}\Pi^*_{MD}(q,\omega)=-2\pi\frac{1}{8\pi^3}\int d^3k\theta(k_F-|\overrightarrow{k}|)\theta(|\overrightarrow{k}+\overrightarrow{q}|-k_F)(\delta(\omega_{kq}-\omega)+\delta(\omega_{kq}+\omega))$$

$$\text{因为}\omega\geq 0,\quad\text{而}\theta(k_F-|\overrightarrow{k}|)\theta(|\overrightarrow{k}+\overrightarrow{q}|-k_F)\text{保证}\omega_{kq}>0\text{，所以}\delta(\omega_{kq}+\omega)=0.$$

所以，

$$\operatorname{Im}\Pi^*_{MD}(q,\omega)=-2\pi\frac{1}{8\pi^3}\int d^3k\theta\left(k_F-\left|\overrightarrow{k}\right|\right)\theta\left(\left|\overrightarrow{k}+\overrightarrow{q}\right|-k_F\right)\delta(\omega_{kq}-\omega)$$

使用费米波矢k_F作为波矢单位，$k\to kk_F, q\to qk_F$,
并且定义

$$\nu=\omega\frac{m}{k_F^2}$$

根据δ函数性质，

$$\delta(ax)=|a|^{-1}\delta(x)$$

$$\delta(\omega_{kq}-\omega)=\frac{m}{\pi k_F^2}\delta(\nu-\overrightarrow{q}\cdot\overrightarrow{k}-\frac{q^2}{2})$$

最终得到$\operatorname{Im}\Pi^*_{MD}(q,\omega)$,

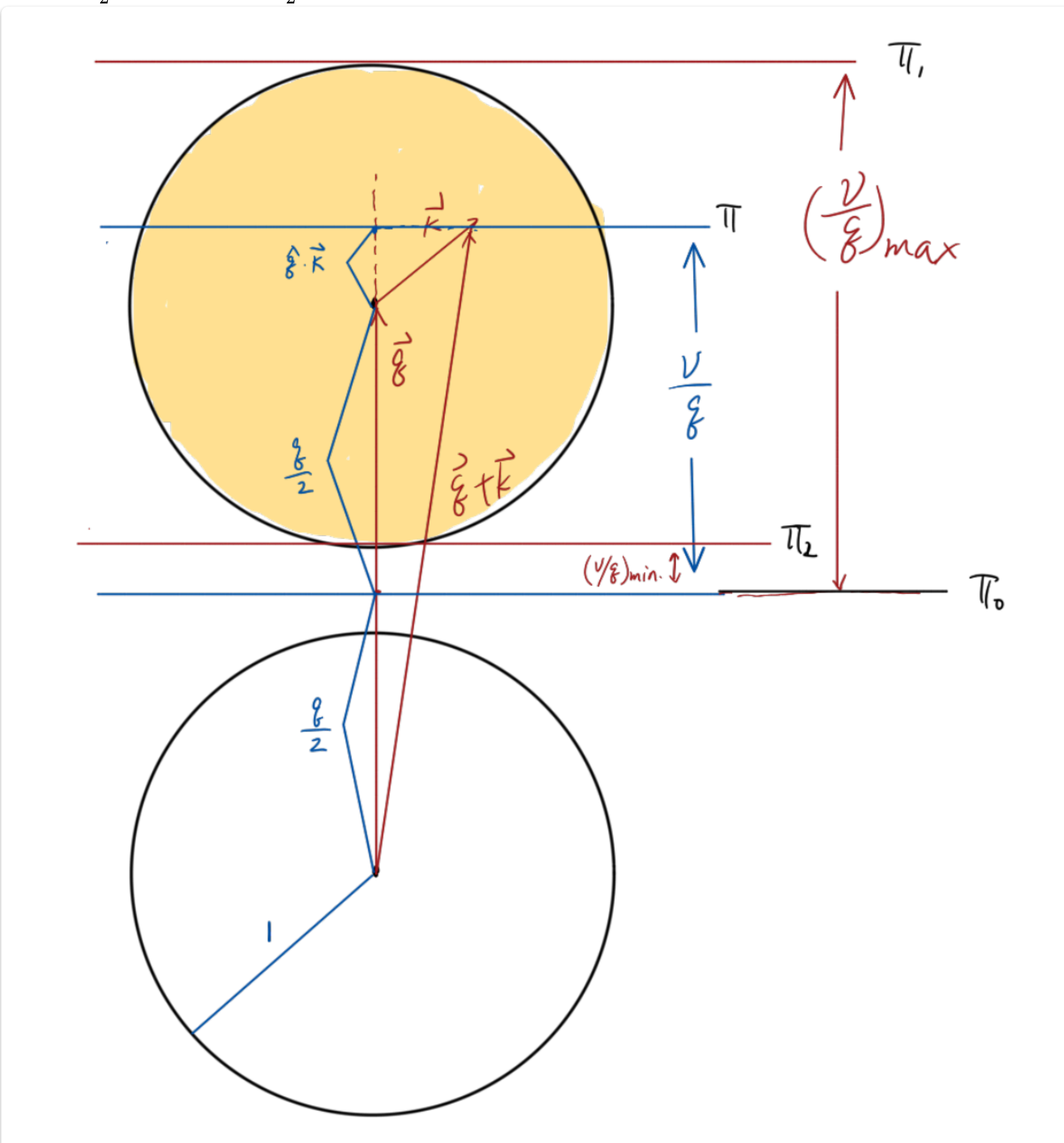
$$\operatorname{Im}\Pi^*_{MD}(q,w)=-2\pi\frac{mk_F}{4\pi^2}\int_0^\pi d\theta\int_0^\infty dkk^2\sin\theta\cdot\theta(1-k)\theta(|\overrightarrow{k}+\overrightarrow{q}|-1)\delta\left(\nu-\overrightarrow{q}\cdot\overrightarrow{k}-\frac{q^2}{2}\right)$$

k需要同时满足3个条件，

$$\begin{cases} k<1, \left|\vec{k}+\vec{q}\right|>1, \\ \nu-\vec{q}\cdot\vec{k}-\dfrac{q^2}{2}=0 \Leftrightarrow \dfrac{q}{2}+\hat{q}\cdot\vec{k}=\dfrac{\nu}{q}. \end{cases}$$

其几何含义是平面$\frac{q}{2}+\hat{q}\cdot\vec{k}=\frac{\nu}{q}$需要与$k<1, \left|\vec{k}+\vec{q}\right|>1$确定的三维空间相交，这确定了$\nu$的取值范围。

1. $q>2$，$\frac{1}{2}q^2+q\geq\nu\geq\frac{1}{2}q^2-q$

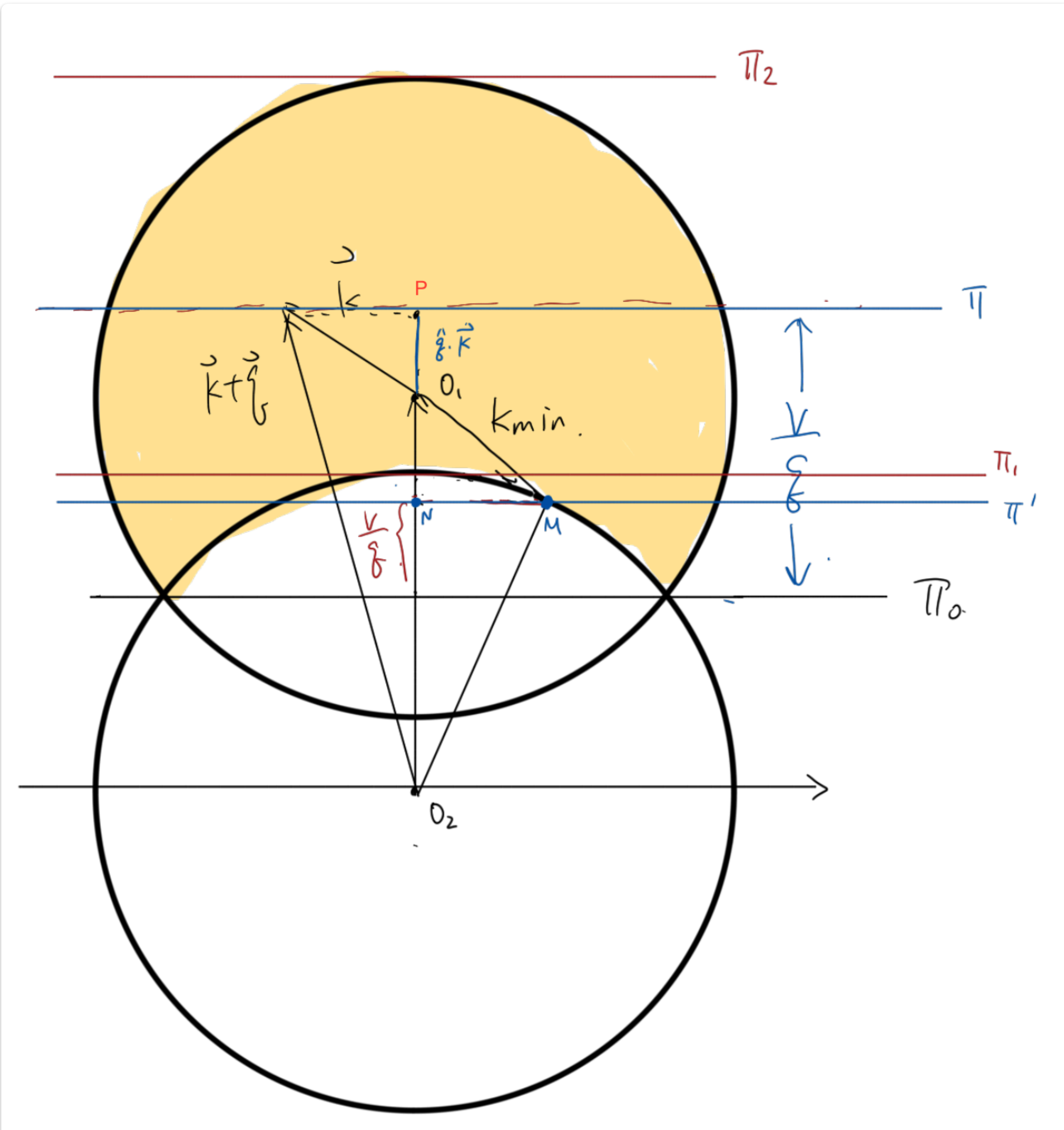


令$t=\cos\theta$,

$$\operatorname{Im}\Pi^*_{MD}(q,\omega) = -2\pi\frac{mk_F}{4\pi^2}\int_{\frac{\nu}{q}-\frac{q}{2}}^{1}k^2dk\int_{-1}^{1}dt\frac{1}{qk}\delta\left(\frac{\nu}{qk}-\frac{1}{2}\frac{q}{k}-t\right)$$

$$\operatorname{Im}\Pi^*_{MD}(q,\omega) = -\frac{mk_F}{4\pi q}[1-(\frac{\nu}{q}-\frac{q}{2})^2]$$

2. $1<q<2$,



需要分为两种情况，

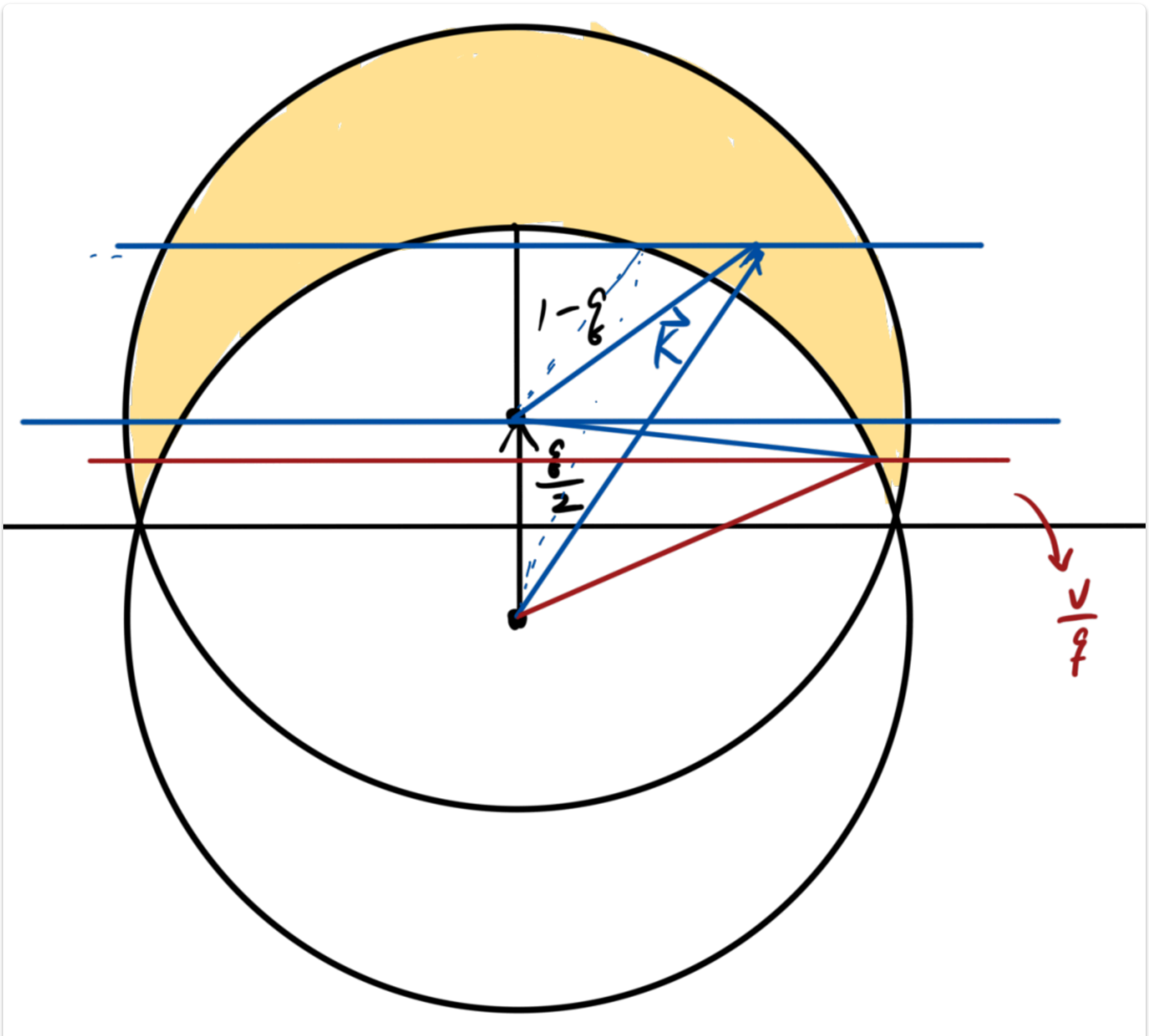
- 平面π在π_1,π_2之间时($q+\frac{1}{2}q^2\geq\nu\geq q-\frac{1}{2}q^2$)，$k$最小值为$O_1P$，此时$\operatorname{Im}\Pi^*_{MD}(q,w)$表达式同$q>2$情况。
- 平面π在π_0,π_1之间时($0\leq\nu\leq q-\frac{1}{2}q^2$)，$k$的下限只能到$O_1M$。

$$
\begin{aligned}
O_1M^2 &= MN^2 + O_1N^2 \\
&= O_2M^2 - O_2N^2 + O_1N^2 \\
&= \left(\frac{1}{2}q + \frac{\nu}{q}\right)^2 + \left(\frac{1}{2}q - \frac{\nu}{q}\right)^2 \\
&= 1 - 2\nu.
\end{aligned}
$$

所以，

$$
\operatorname{Im}\Pi^*_{MD}(q,\omega) = -2\pi\frac{mk_F}{4\pi^2}\int_{\sqrt{1-2\nu}}^{1} k^2 dk \int_{-1}^{1} dt \frac{1}{qk}\delta\left(\frac{\nu}{qk} - \frac{1}{2}\frac{q}{k} - t\right) = -\frac{mk_F}{4\pi}\frac{2\nu}{q}
$$

3. $q < 1$，$\operatorname{Im}\Pi^*_{MD}(q,\omega)$表达式同2。



综上，

$$\mathrm{Im}\,\Pi^*_{MD}(q,\omega) = \begin{cases} -\dfrac{mk_F}{4\pi q}[1-(\dfrac{\nu}{q}-\dfrac{q}{2})^2], & (q-\dfrac{1}{2}q^2 \leq \nu \leq q+\dfrac{1}{2}q^2) \\ -\dfrac{mk_F}{4\pi}\dfrac{2\nu}{q}, & (0 \leq \nu \leq q-\dfrac{1}{2}q^2) \\ 0, & (\nu \geq q+\dfrac{1}{2}q^2) \end{cases}$$