BAYESIAN NETWORKS

CS221 Fall 2019
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What are we covering today?

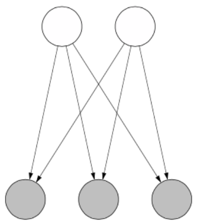
INTRODUCTION TO BAYESIAN NETWORKS

PROBABILISTIC INFERENCE COOKBOOK

SAMPLE PROBLEMS

CONDITIONAL INDEPENDENCE

Bayesian Networks



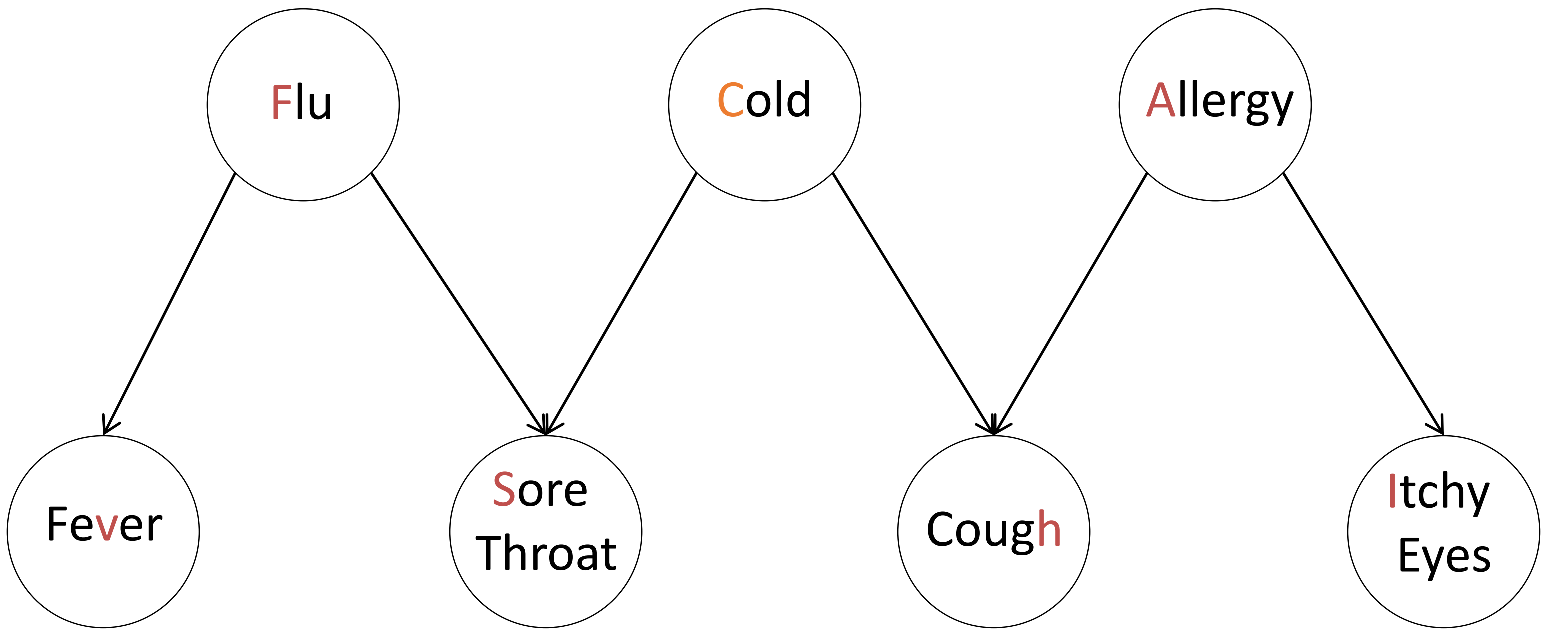
Definition: Bayesian network

Let $X = (X_1, \dots, X_n)$ be random variables.

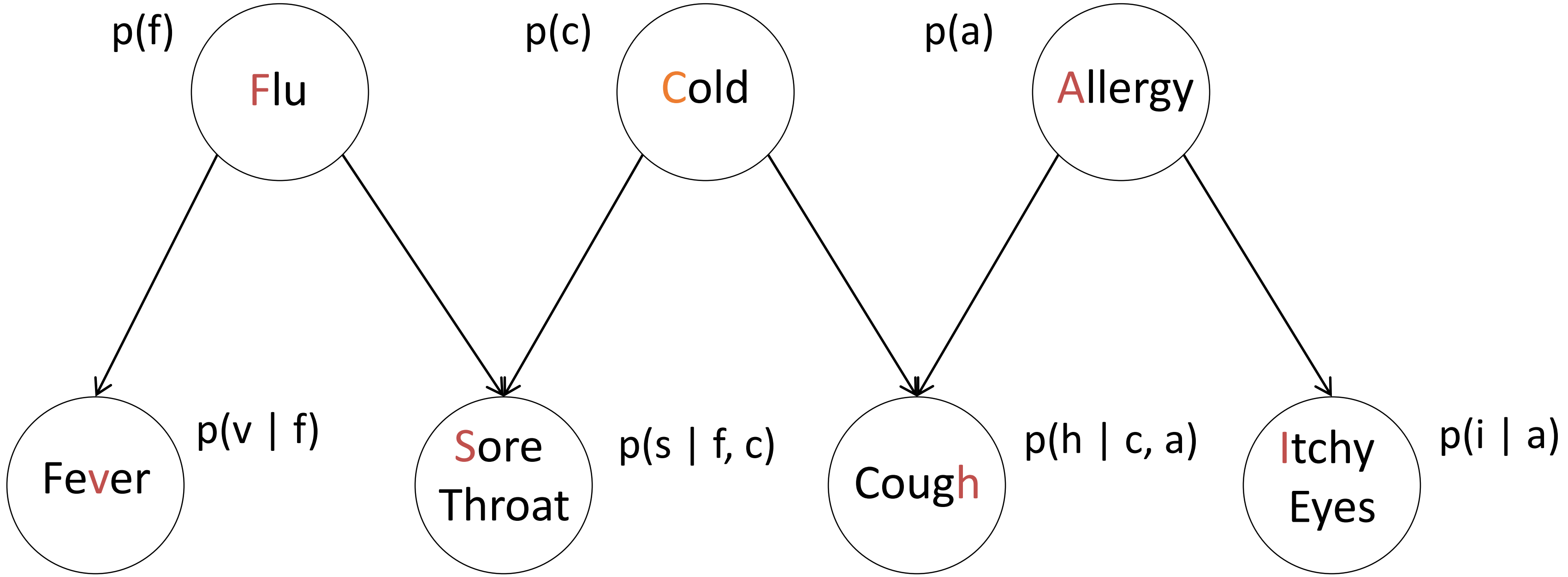
A **Bayesian network** is a directed acyclic graph (DAG) that specifies a joint distribution over X as a product of local conditional distributions, one for each node:

$$\mathbb{P}(X_1 = x_1, \dots, X_n = x_n) = \prod_{i=1}^{n} p(x_i \mid x_{\text{Parents}(i)})$$

Bayesian Networks



A Bayesian network represents a joint probability distribution.



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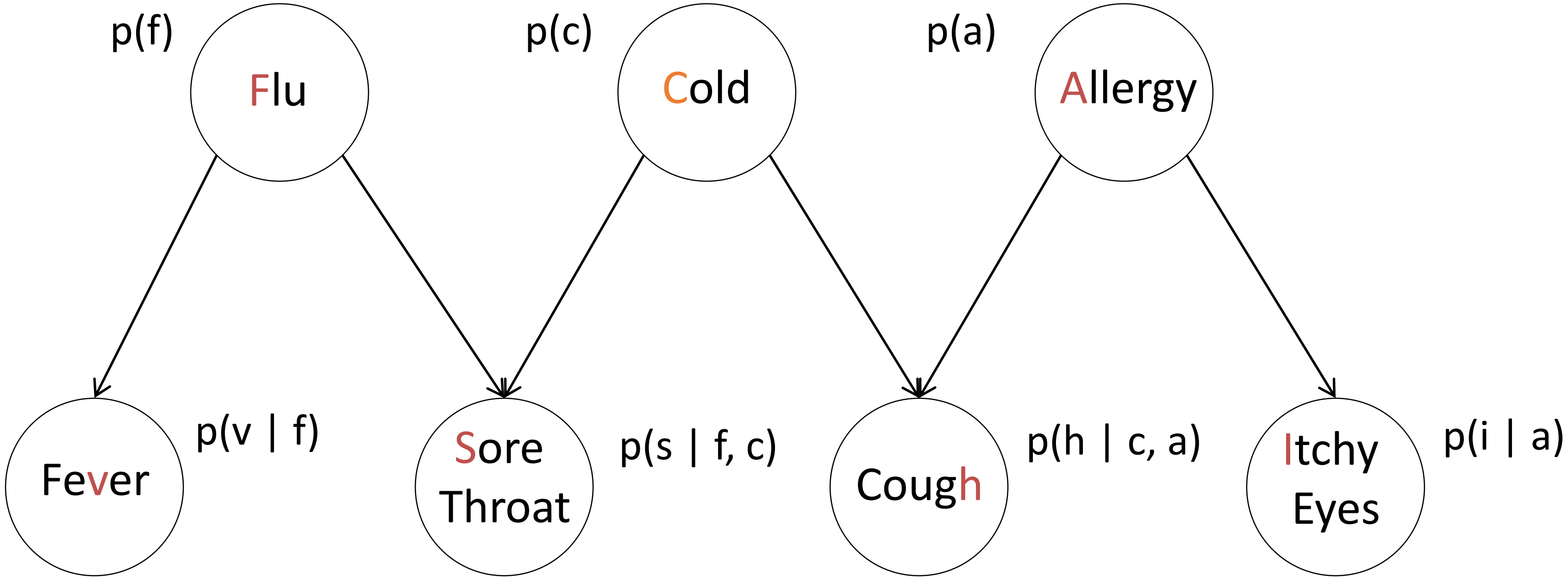


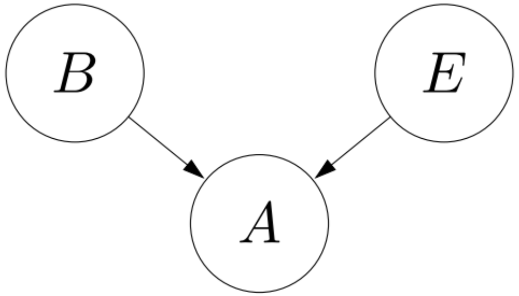
$$P(F=f, C=c, A=a, V=v, S=s, H=h, I=i) = p(f)\, p(c)\, p(a)\, p(v \mid f)\, p(s \mid f, c)\, p(h \mid c, a)\, p(i \mid a)$$

Probabilistic Inference Cookbook

Given a query $P(Q \mid E = e)$:

1. Remove (marginalize) variables not ancestors of Q or E.
2. Convert Bayesian network to factor graph.
3. Condition (shade nodes / disconnect) on $E = e$.
4. Remove (marginalize) nodes disconnected from Q.
5. Run probabilistic inference algorithm (manual, variable elimination, Gibbs sampling, particle filtering).

Example: alarm



b	$p(b)$
1	ϵ
0	$1-\epsilon$

e	$p(e)$
1	ϵ
0	$1-\epsilon$

b	e	a	$p(a \mid b, e)$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

[whiteboard]

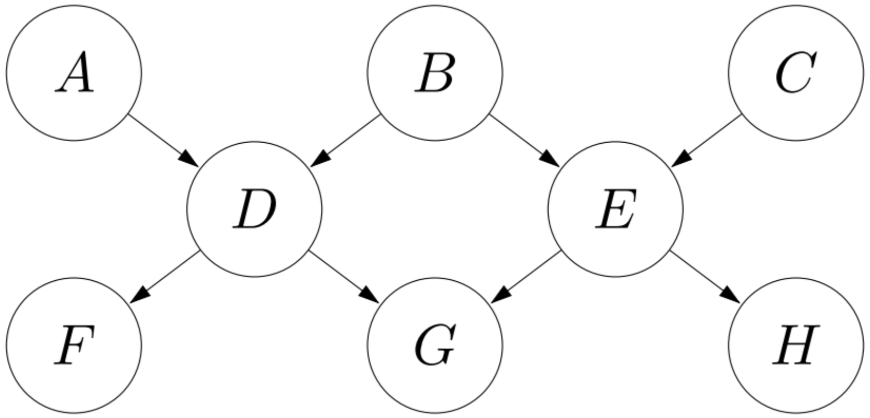
Query: $\mathbb{P}(B)$

- Marginalize out A, E

Query: $\mathbb{P}(B \mid A = 1)$

- Condition on $A = 1$

Example: A-H (section)



[whiteboard]

Query: $\mathbb{P}(C \mid B = b)$

- Marginalize out everything else, note $C \perp\!\!\!\perp B$

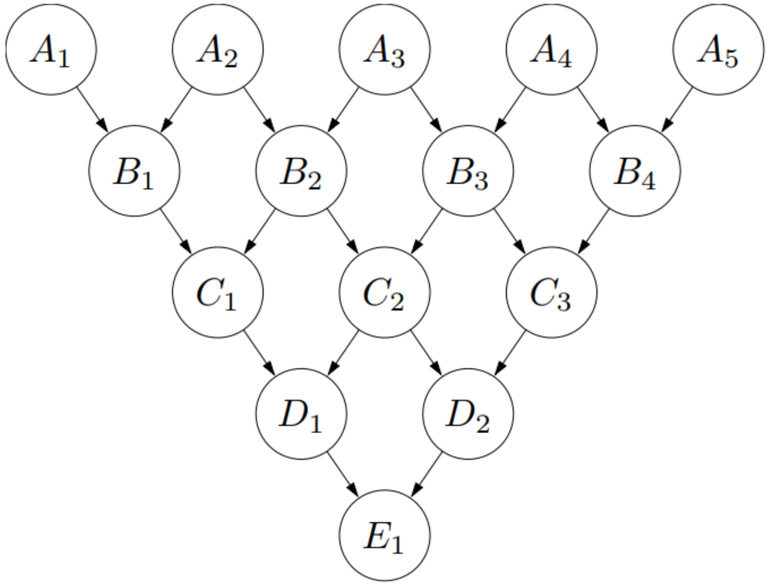
Query: $\mathbb{P}(C, H \mid E = e)$

- Marginalize out A, D, F, G, note $C \perp\!\!\!\perp H \mid E$

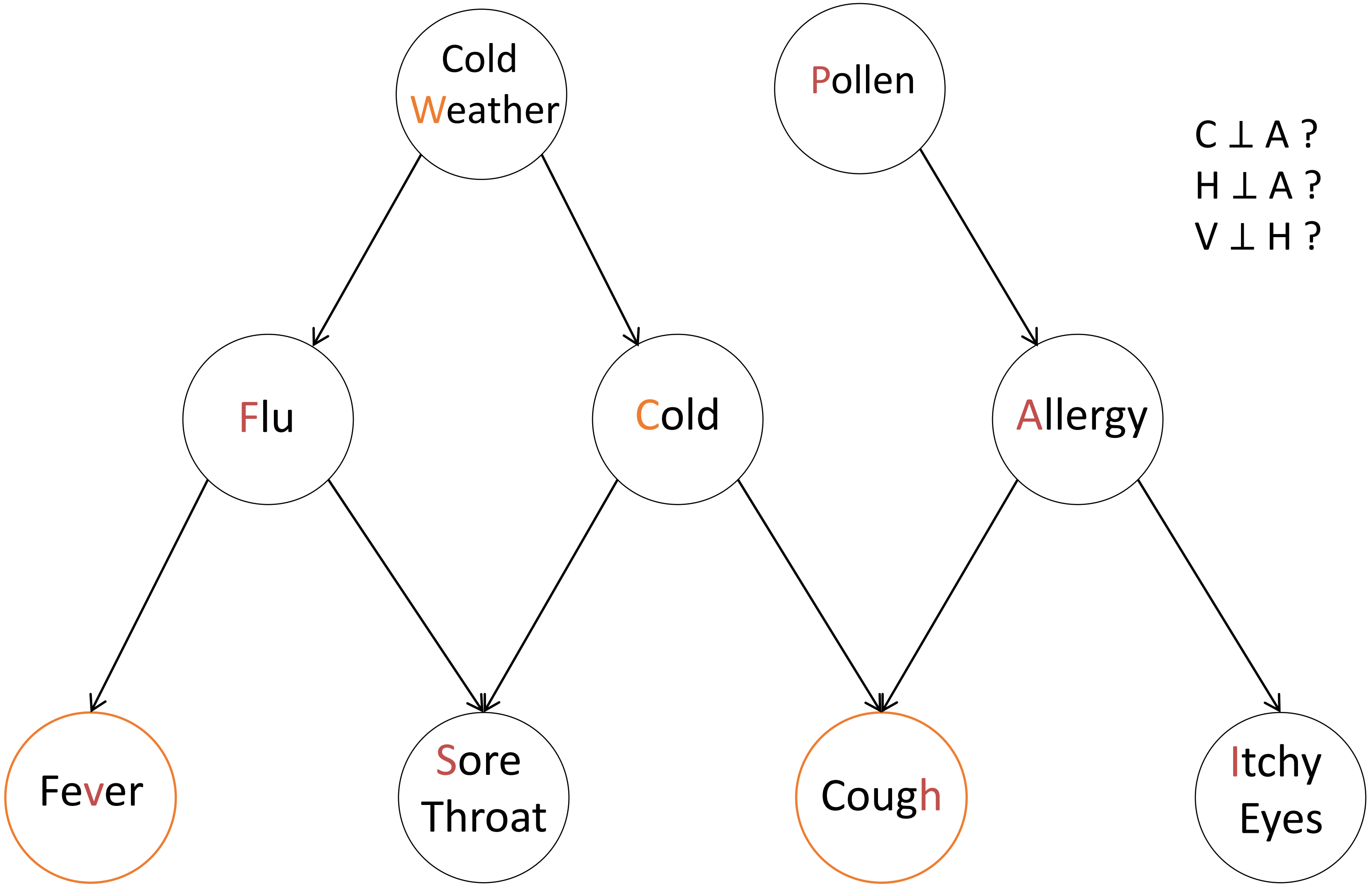
Bayesian Lights
(Fall 2016 Midterm)

This holiday season, you decide to put your knowledge of Bayesian networks to good use. You decide to create Bayesian Lights™, an arrangement of lights that turn on and off randomly according to the joint distribution of a Bayesian network.

Figure 1 shows the Bayesian network corresponding to the lights. Each light is associated with a variable which takes on values in $\{0, 1\}$ (off: 0, on: 1). For example, A_1 is the light in the upper-left corner and E_1 is the light at the very bottom. A light in the top row is on with probability α. The status of a light in subsequent rows is governed by the two parent lights directly above it, and it is on with probability β when the two parent lights above it have different statuses (on-off or off-on), and off with probability β when the two parent lights above it have the same status. In other words, each light is the result of applying a noisy XOR function.

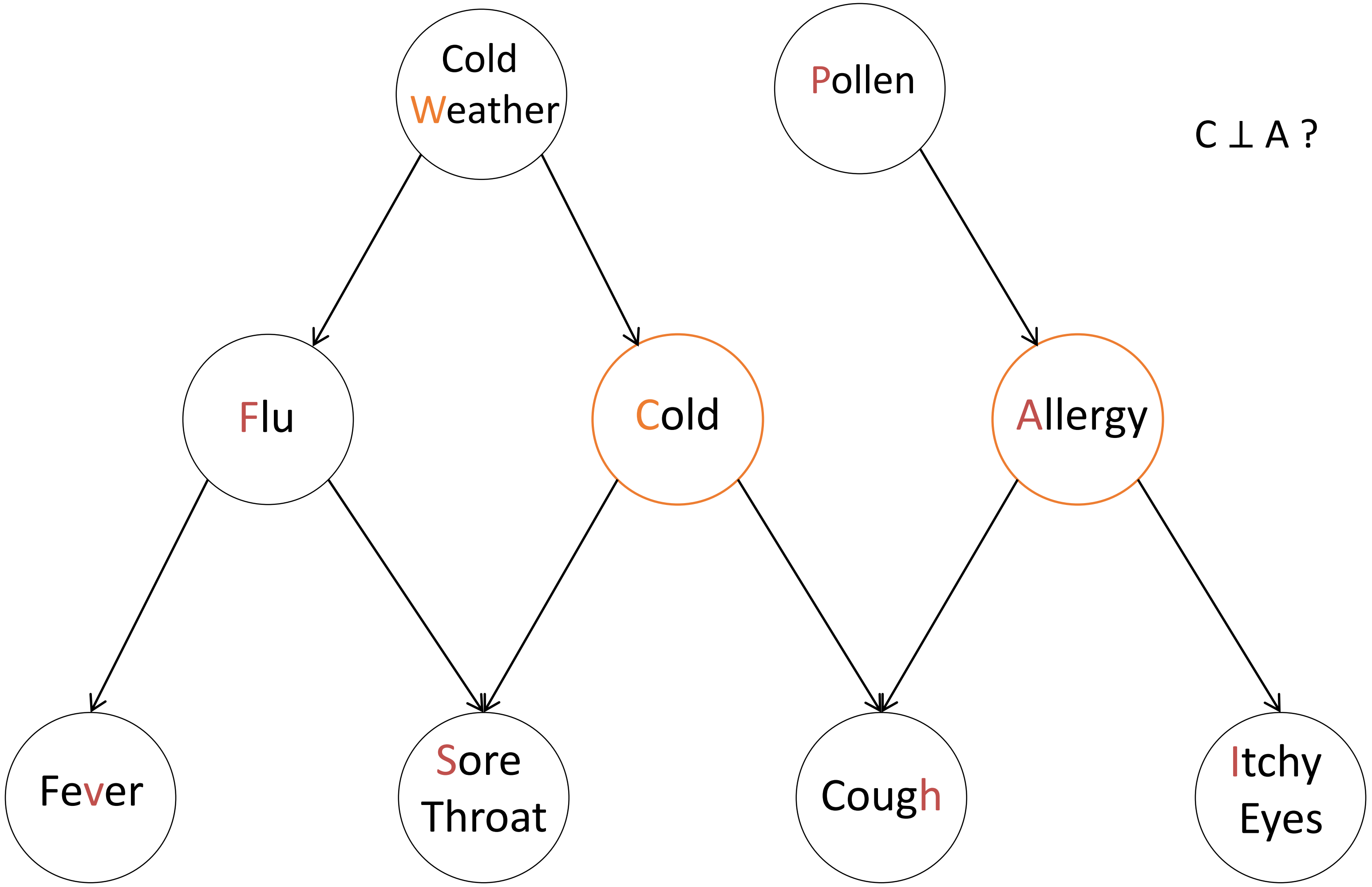


Conditional Independence



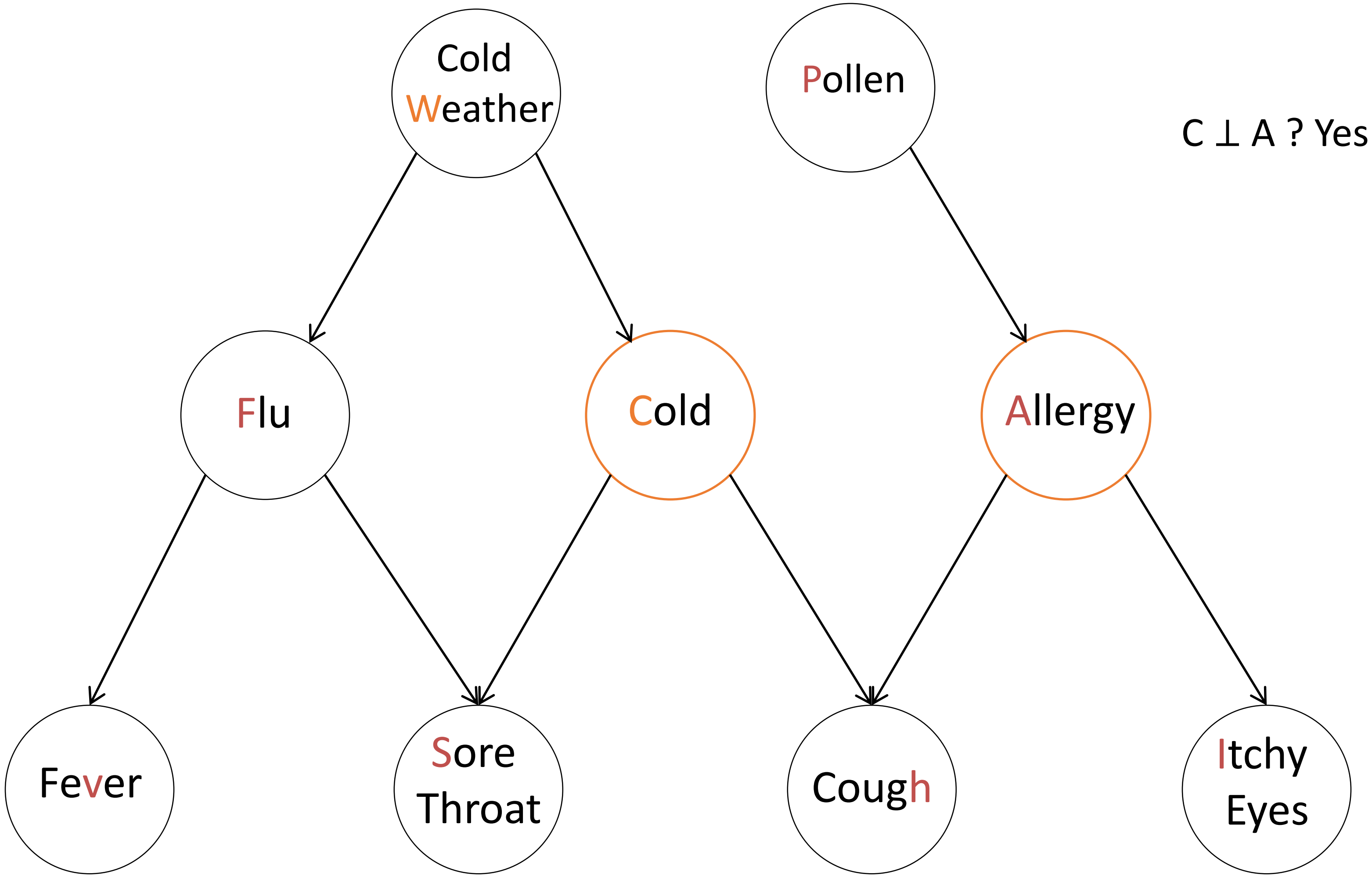
$C \perp A$?
$H \perp A$?
$V \perp H$?

Conditional Independence



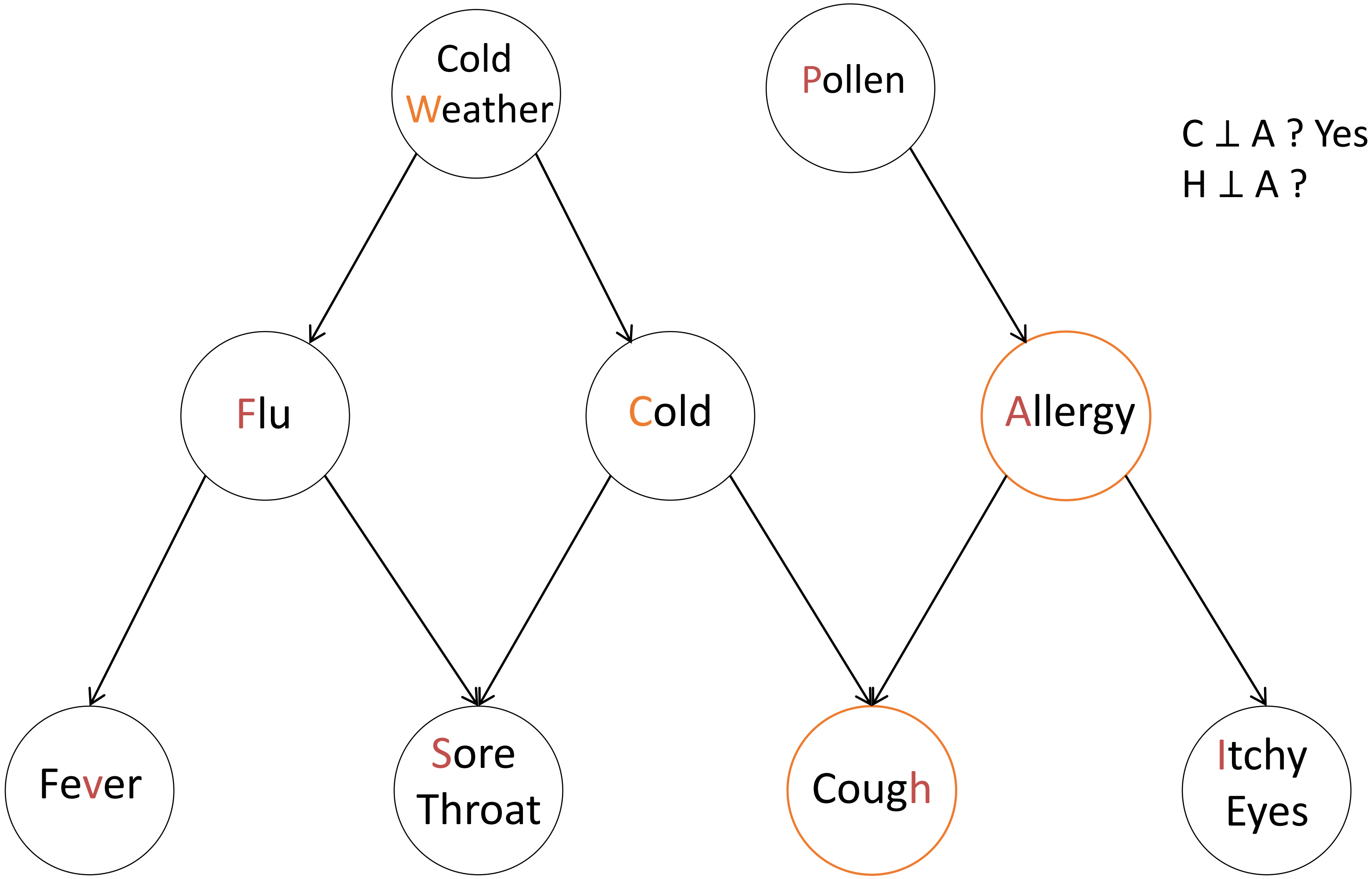
C ⊥ A ?

Conditional Independence



C ⊥ A ? Yes

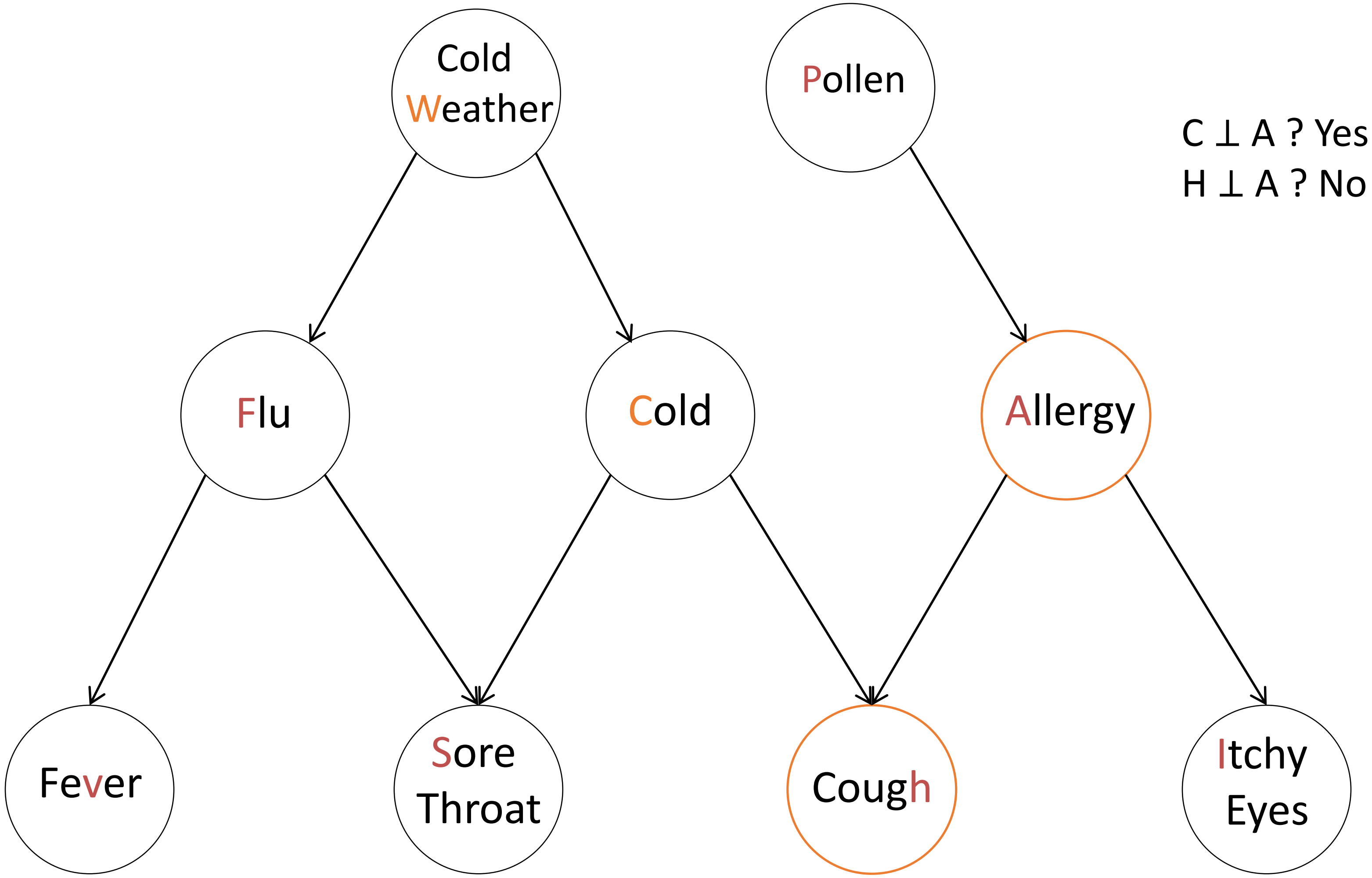
Conditional Independence



C ⊥ A ? Yes

H ⊥ A ?

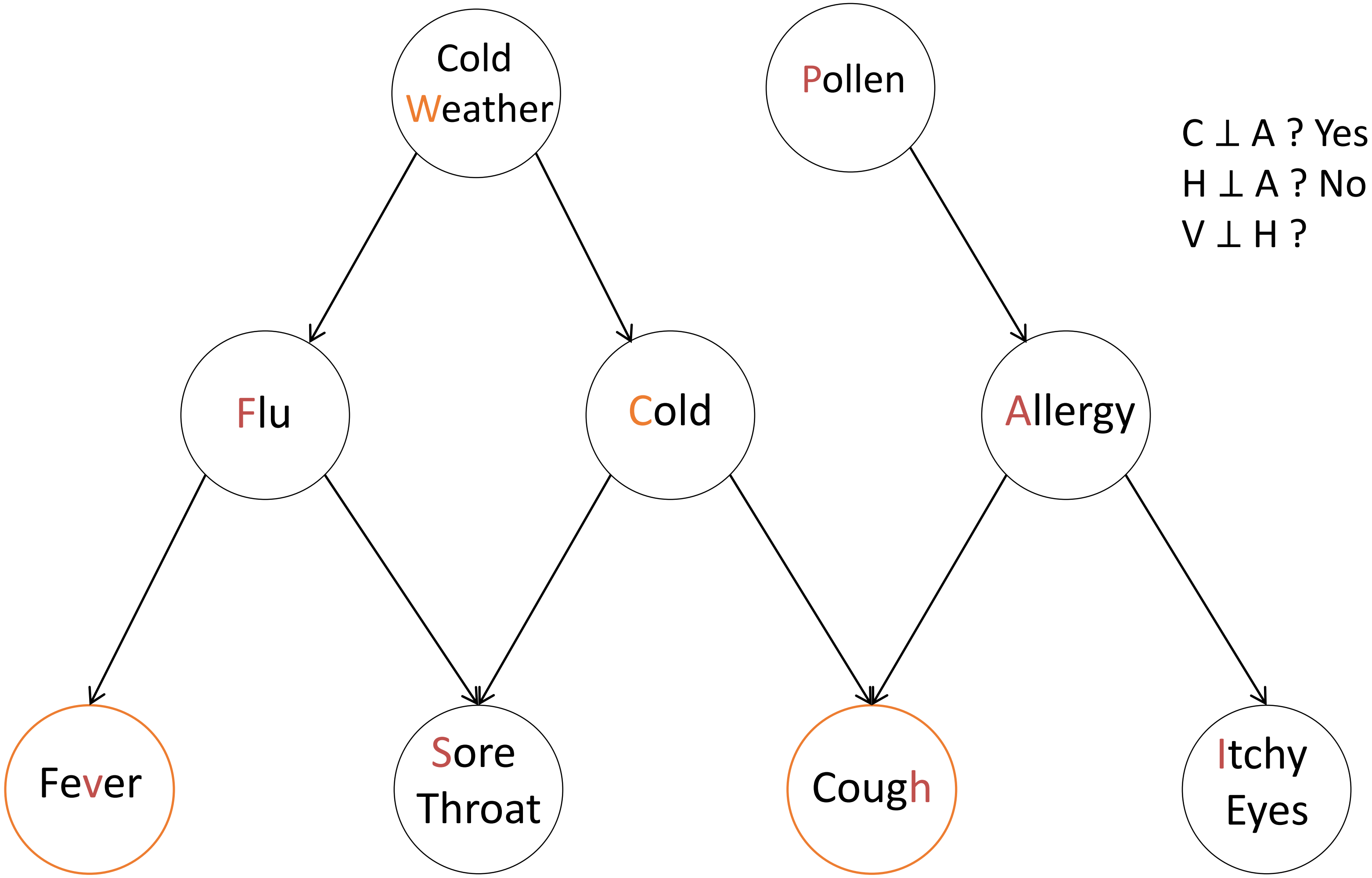
Conditional Independence



C ⊥ A ? Yes

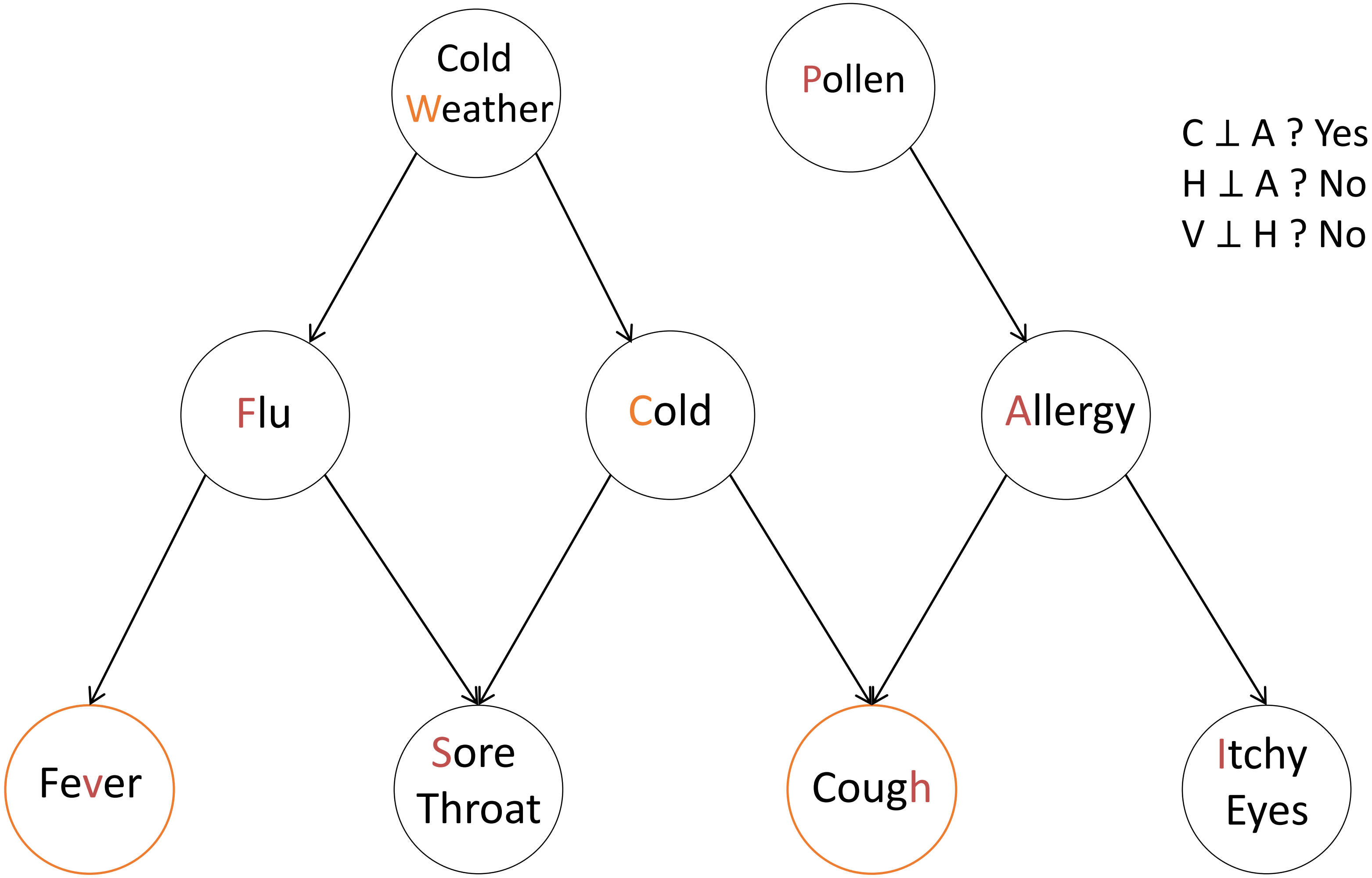
H ⊥ A ? No

Conditional Independence



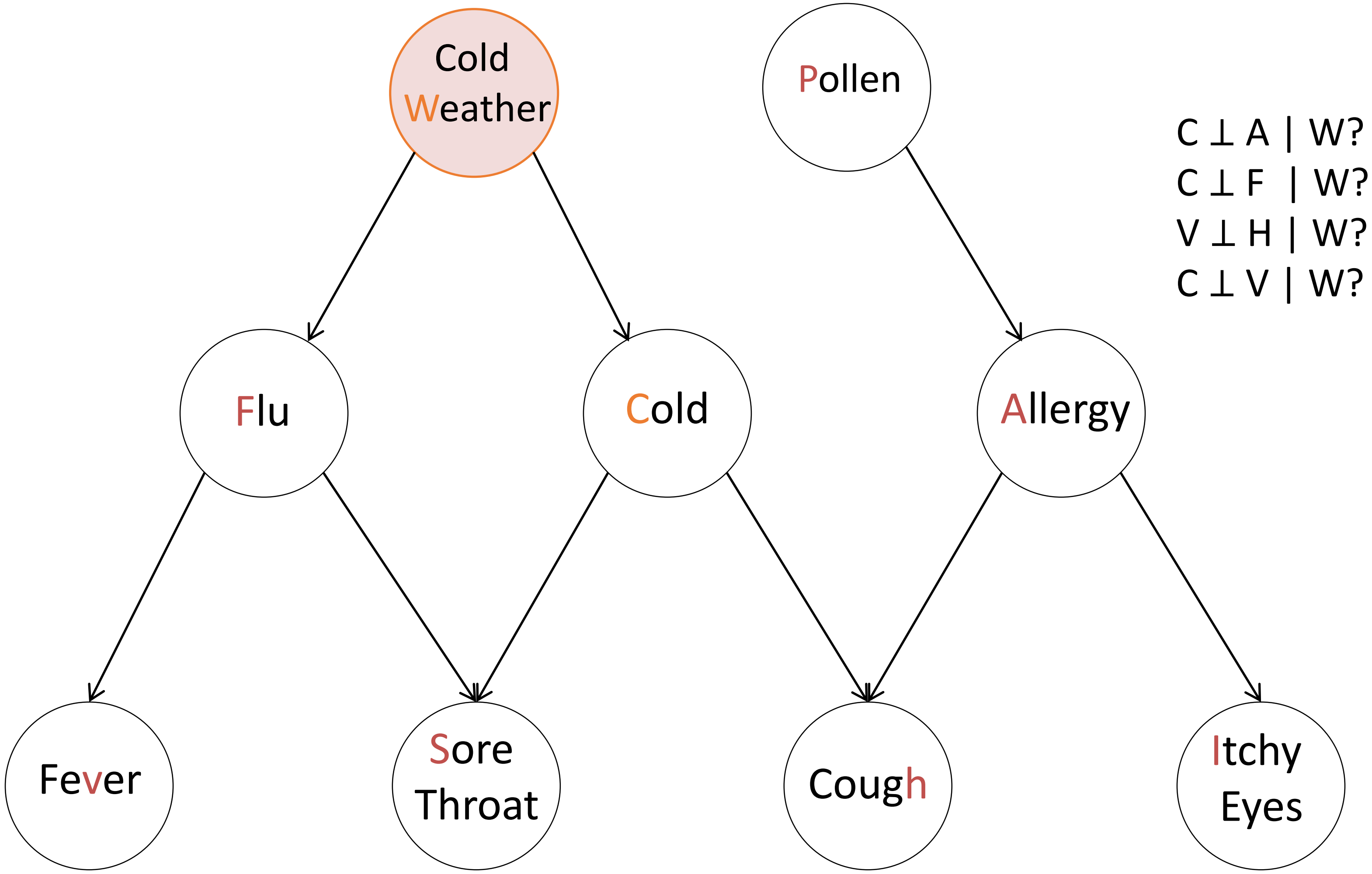
C ⊥ A ? Yes
H ⊥ A ? No
V ⊥ H ?

Conditional Independence



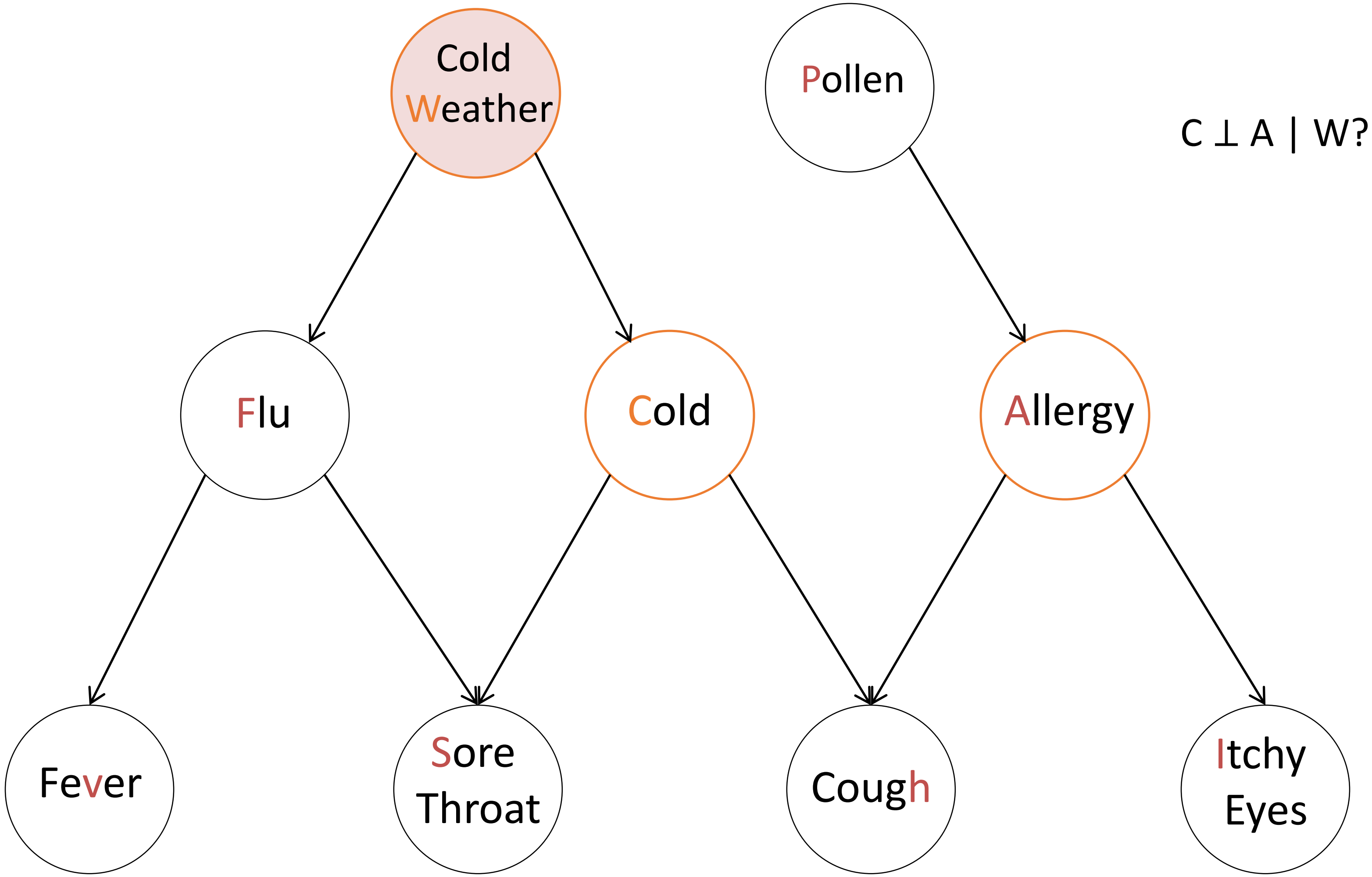
C ⊥ A ? Yes
H ⊥ A ? No
V ⊥ H ? No

Conditional Independence



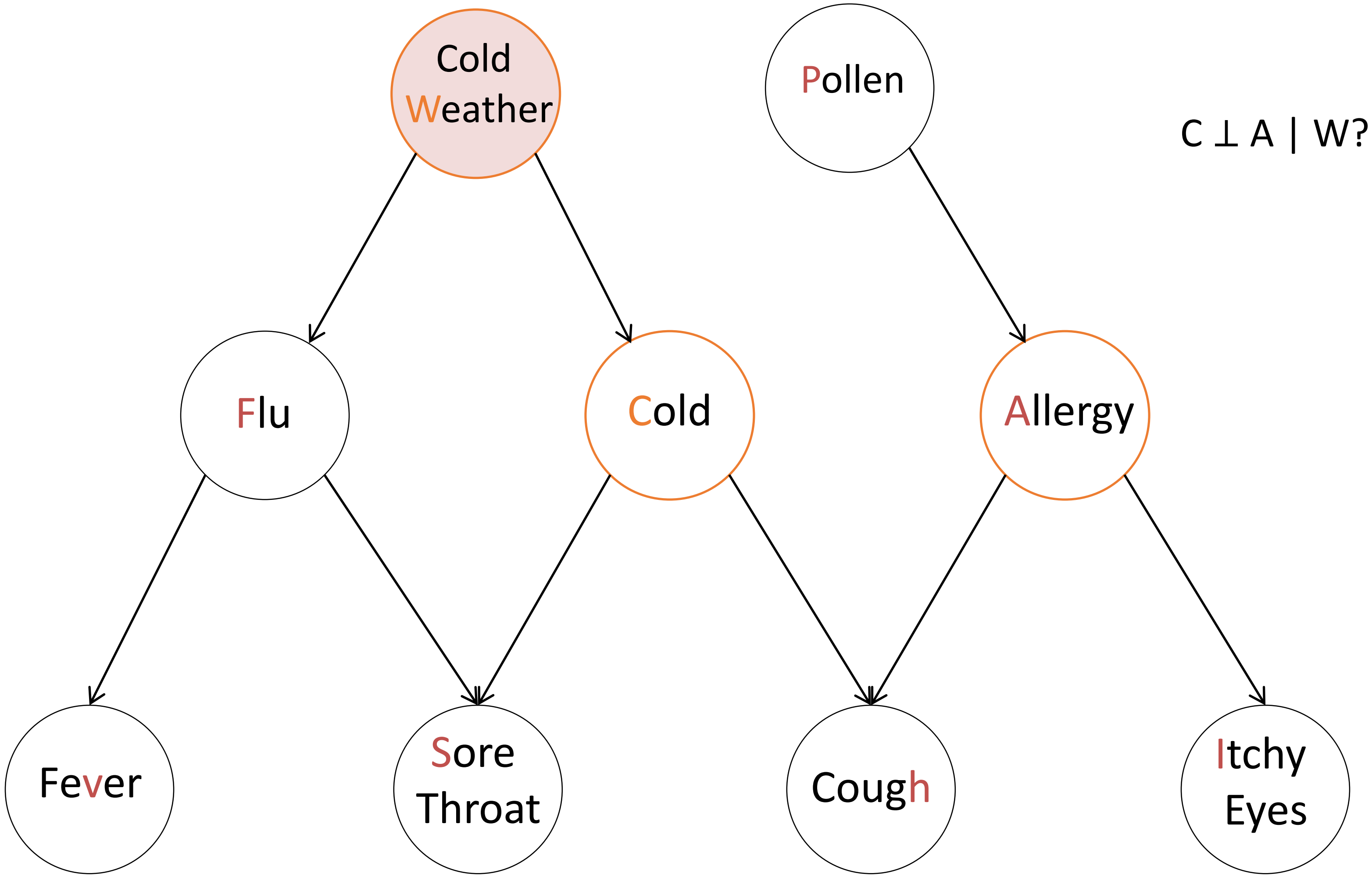
$C \perp A \mid W$?
$C \perp F \mid W$?
$V \perp H \mid W$?
$C \perp V \mid W$?

Conditional Independence



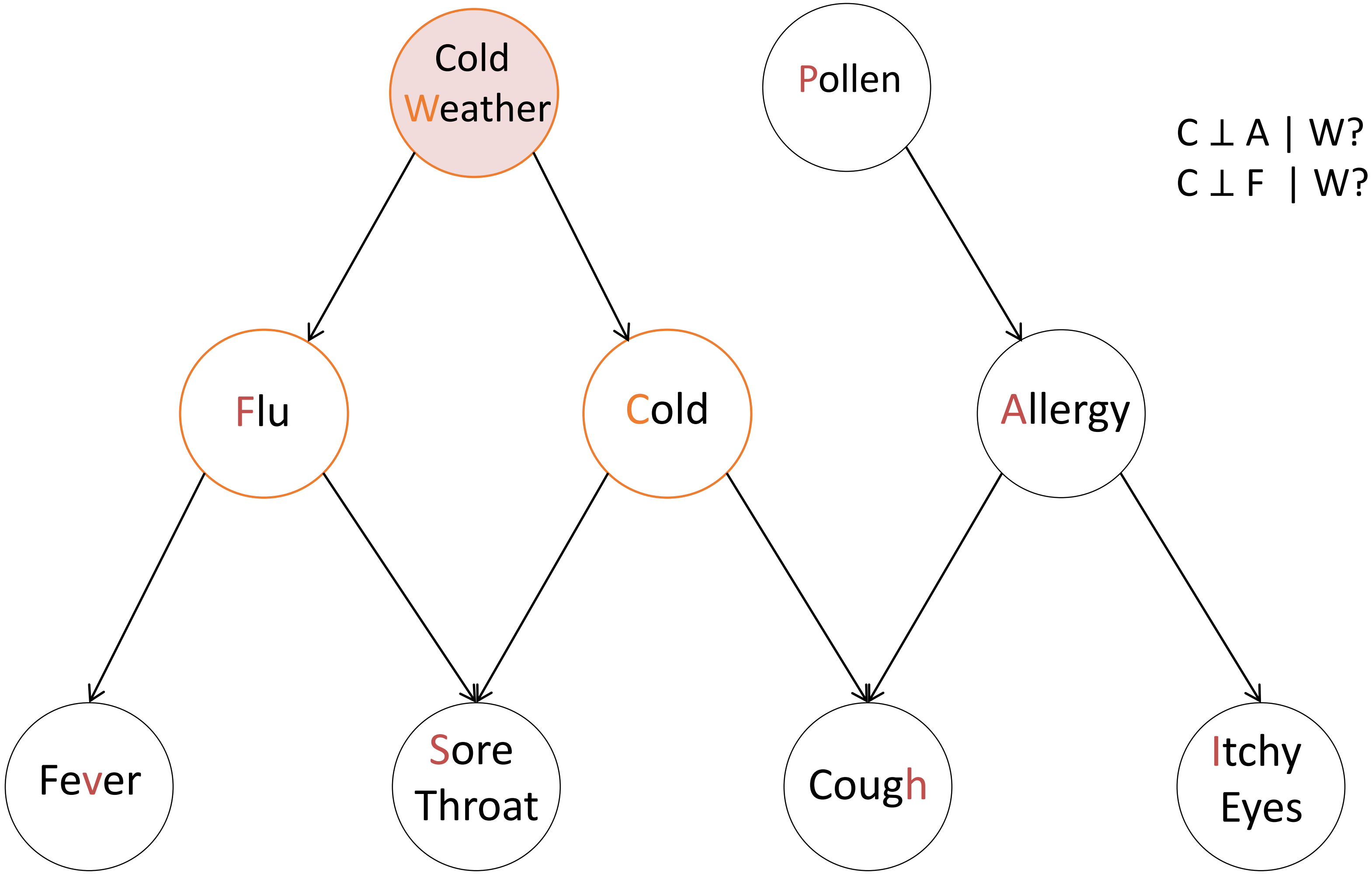
C ⊥ A | W?

Conditional Independence



C ⊥ A | W? Yes

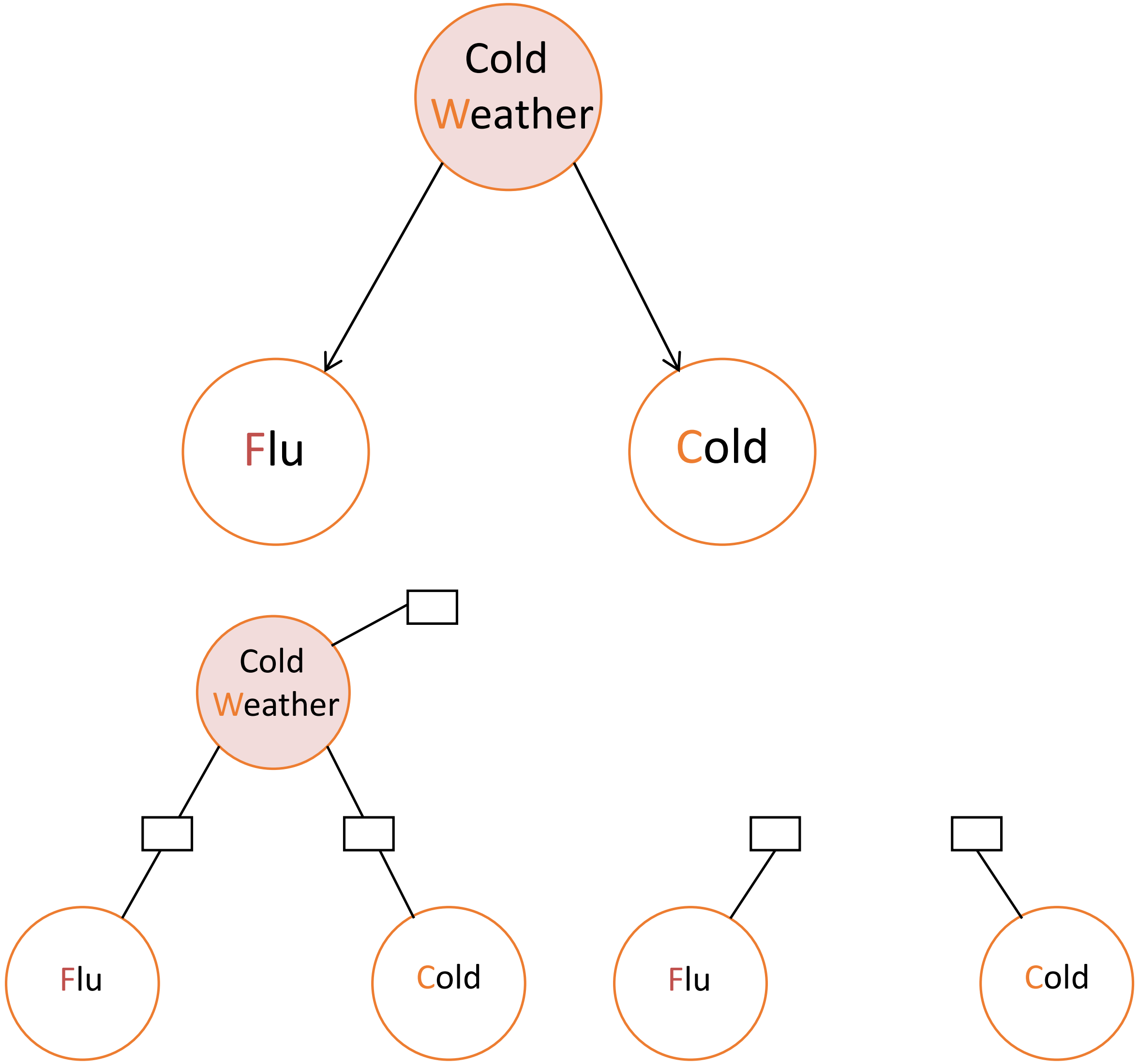
Conditional Independence



$C \perp A \mid W$? Yes

$C \perp F \mid W$?

Conditional Independence



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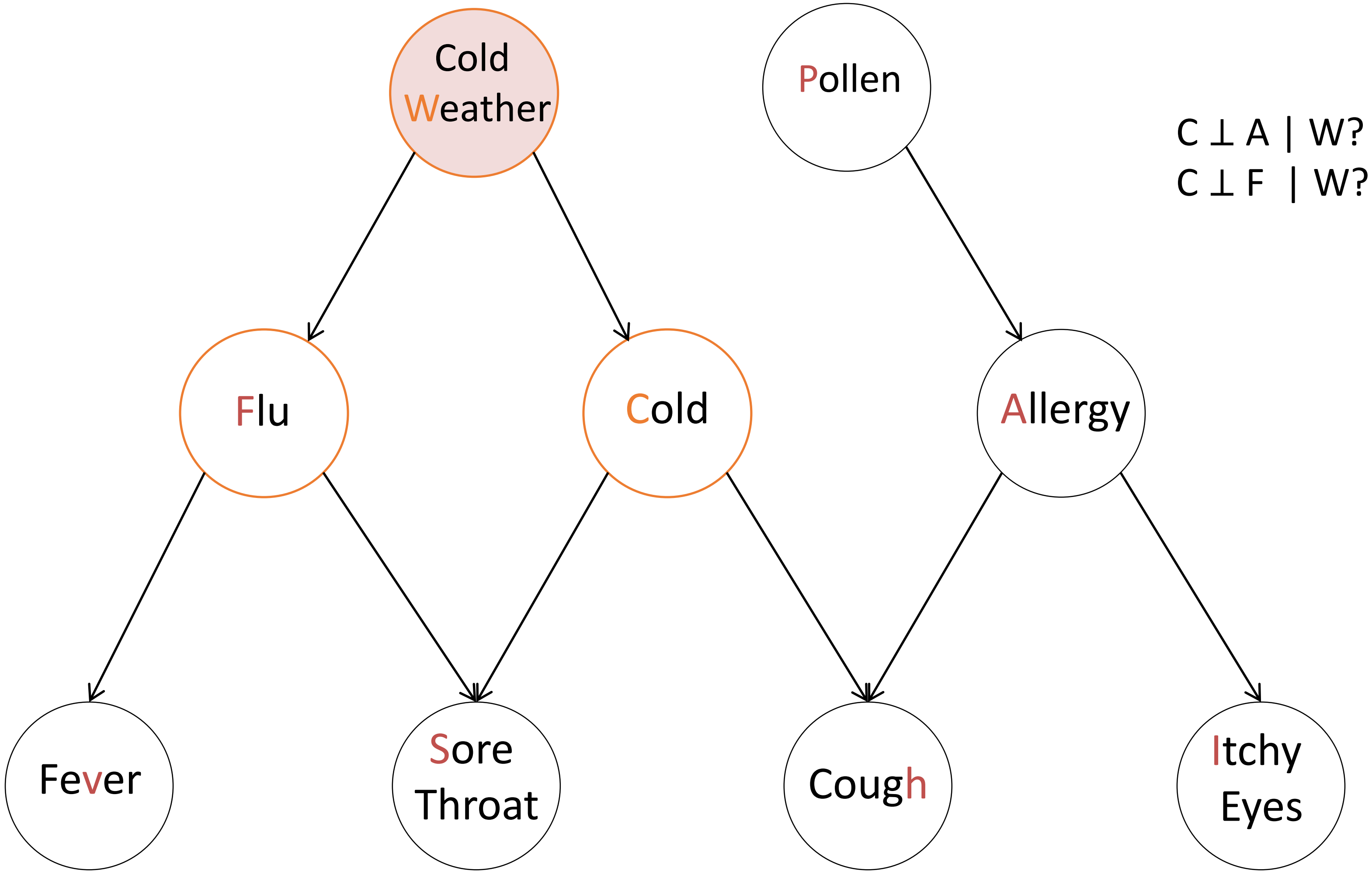
$C \perp F \mid W$?

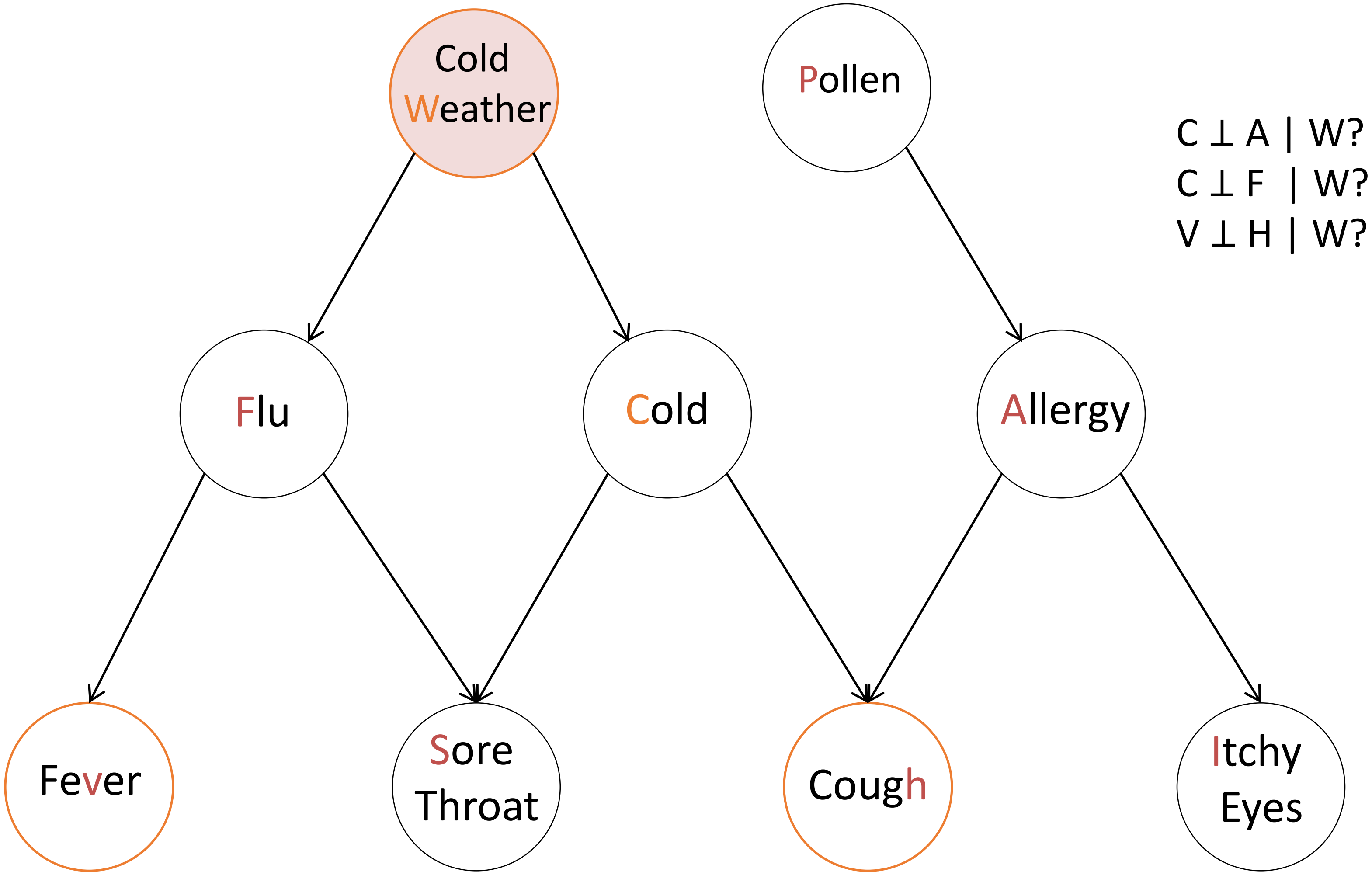
Conditional Independence



C ⊥ A | W? Yes

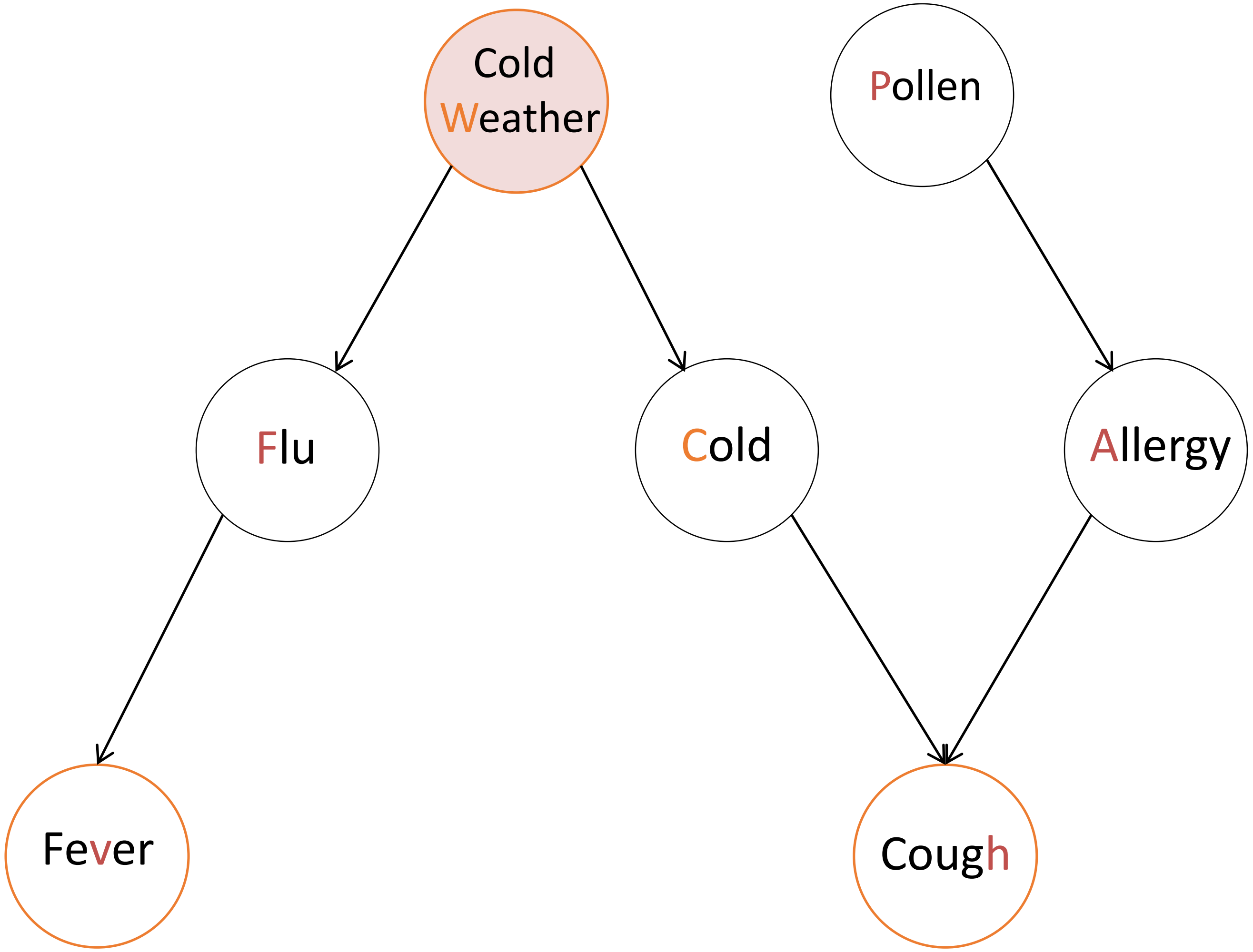
C ⊥ F | W? Yes

Conditional Independence



$C \perp A \mid W$? Yes
$C \perp F \mid W$? Yes
$V \perp H \mid W$?

Conditional Independence



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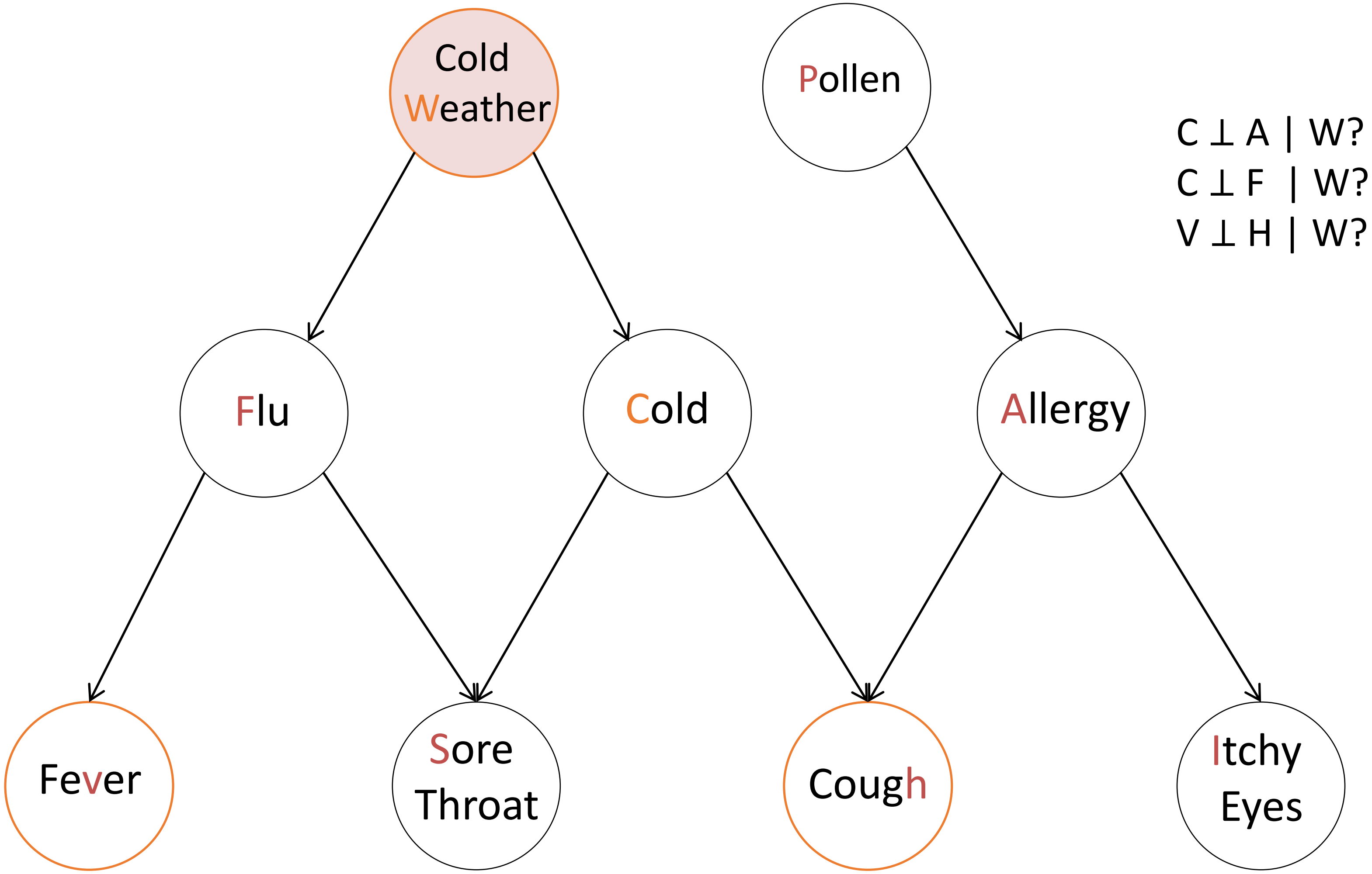
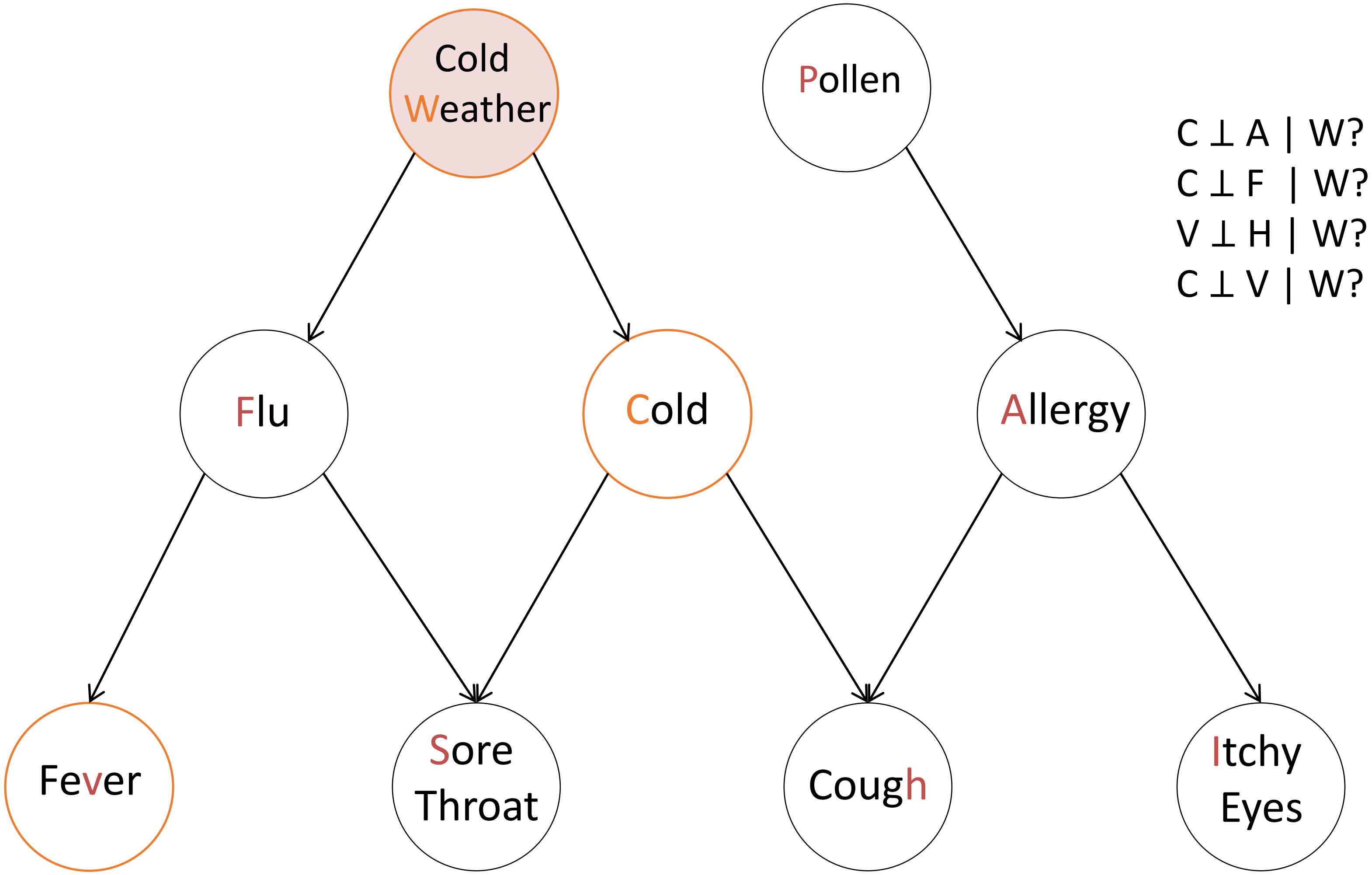
Conditional Independence



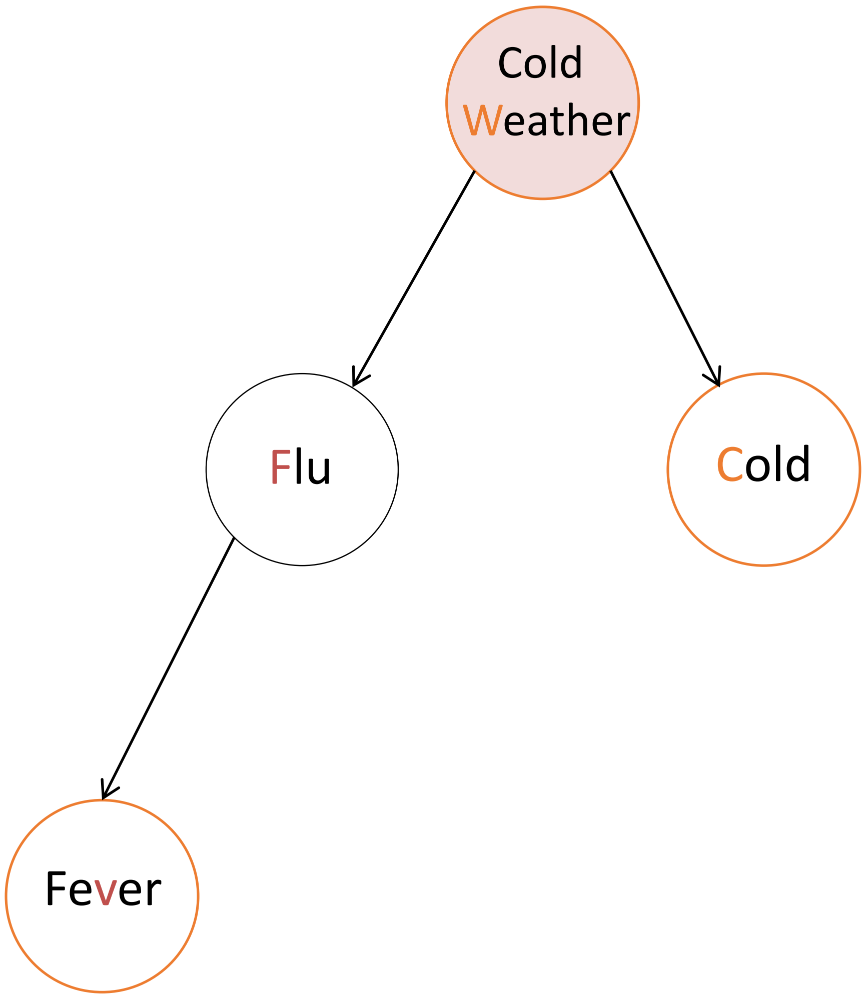
C ⊥ A | W? Yes
C ⊥ F | W? Yes
V ⊥ H | W? Yes

Conditional Independence



C ⊥ A | W? Yes
C ⊥ F | W? Yes
V ⊥ H | W? Yes
C ⊥ V | W?

Conditional Independence



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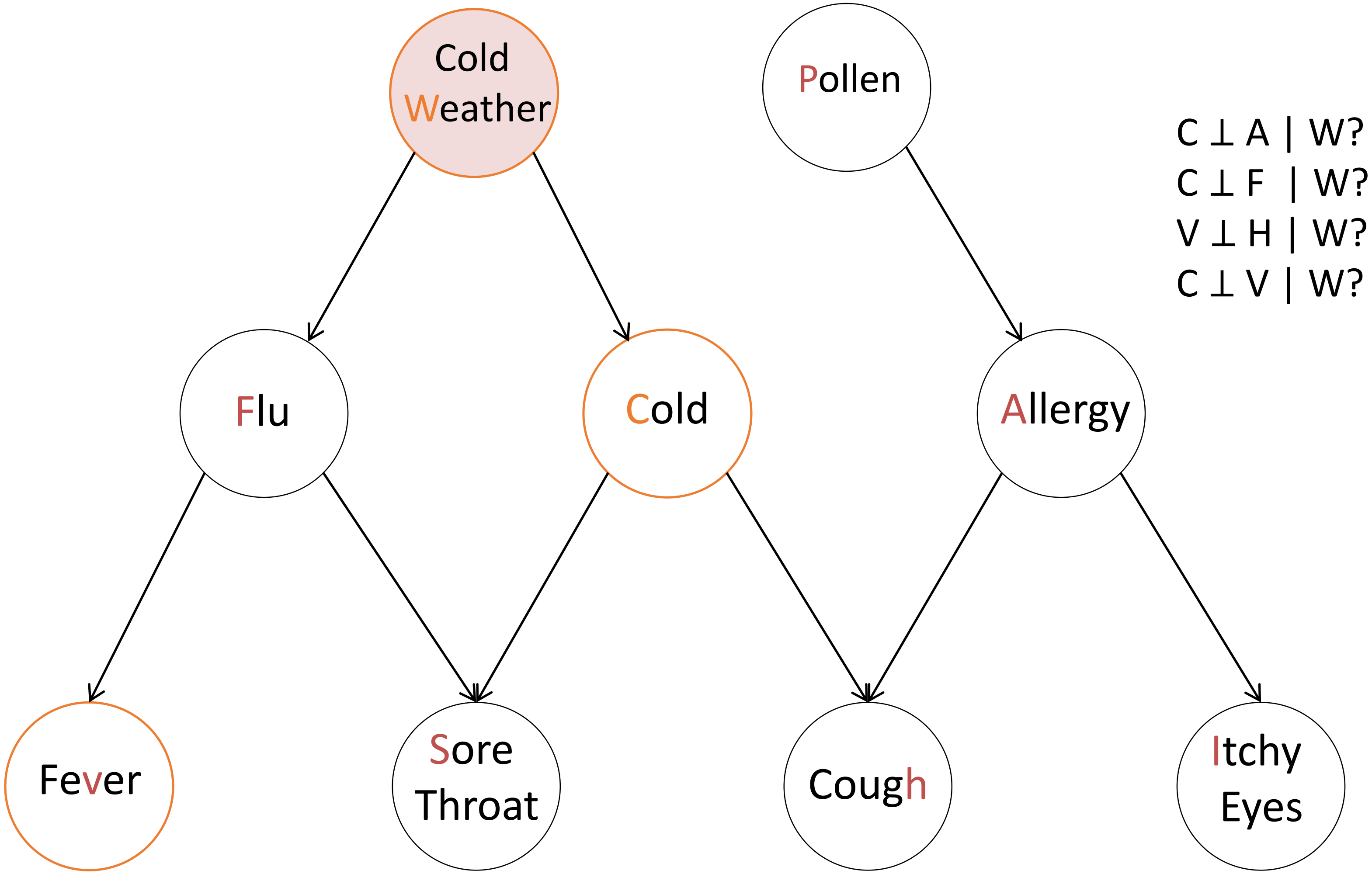
$V \perp H \mid W$? Yes

$C \perp V \mid W$?

Conditional Independence



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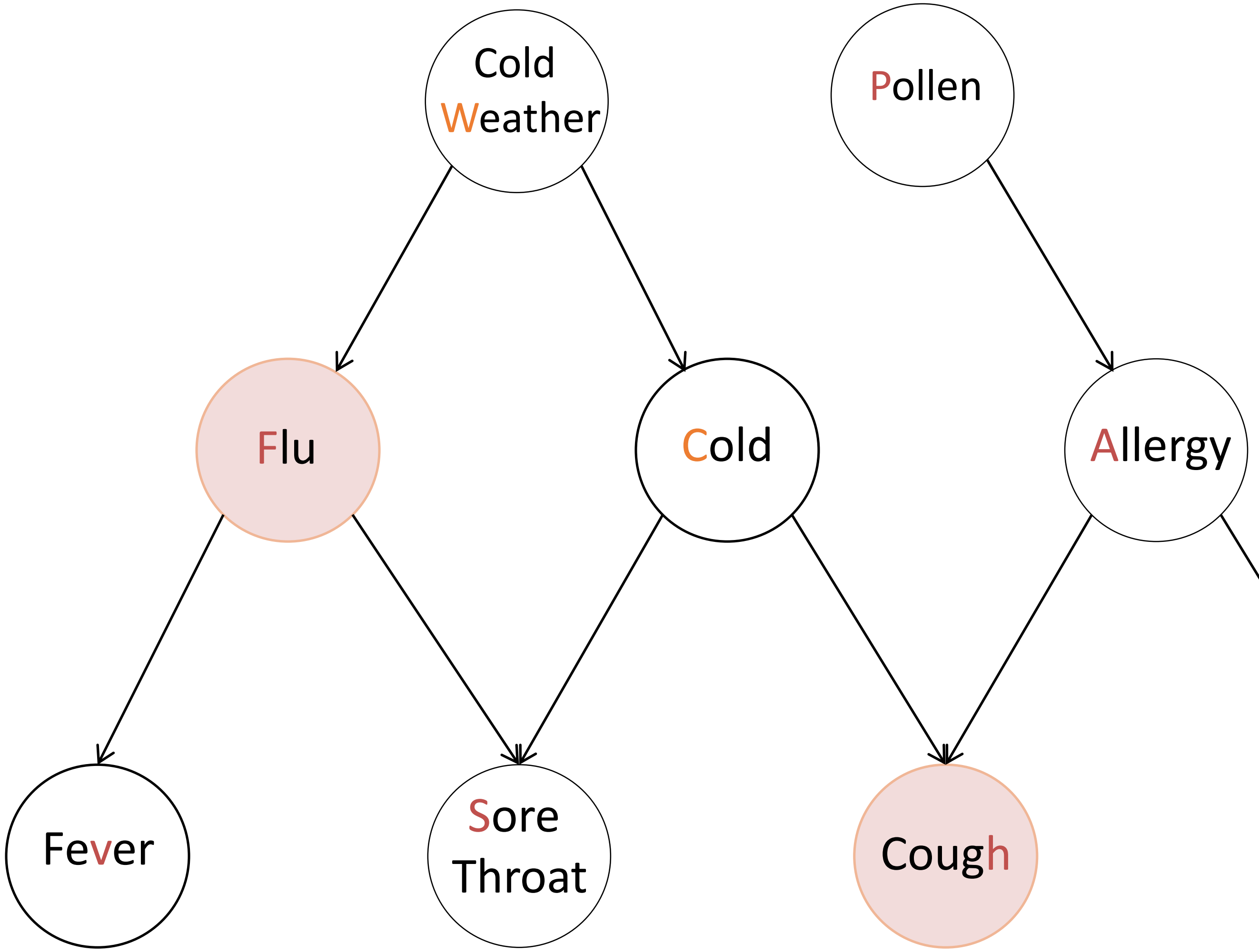
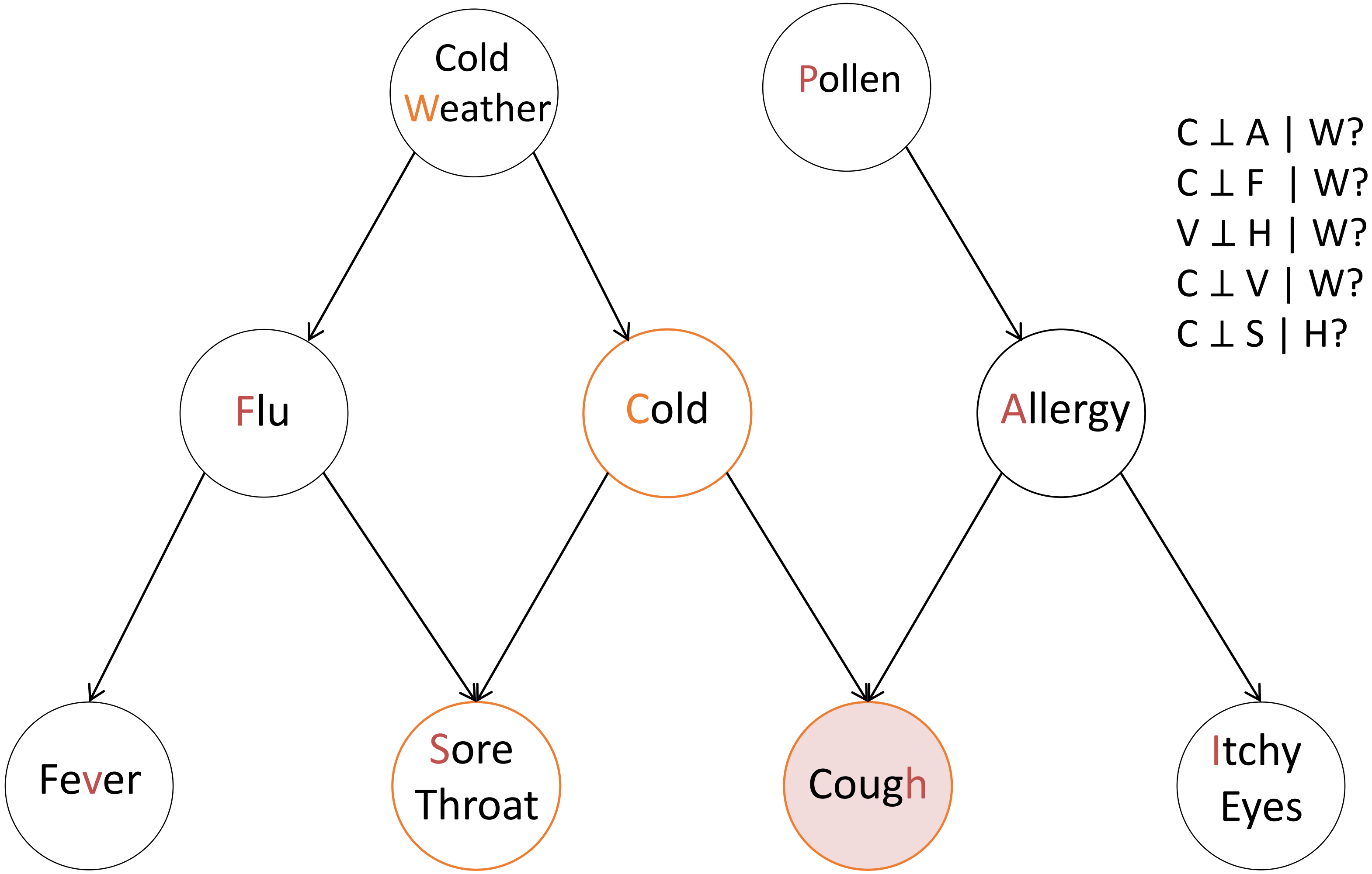
Conditional Independence



$C \perp A \mid W$? Yes
$C \perp F \mid W$? Yes
$V \perp H \mid W$? Yes
$C \perp V \mid W$? Yes
$C \perp S \mid H$?
$C \perp A \mid H$?
$C \perp P \mid H$?
$C \perp V \mid F$?

Conditional Independence



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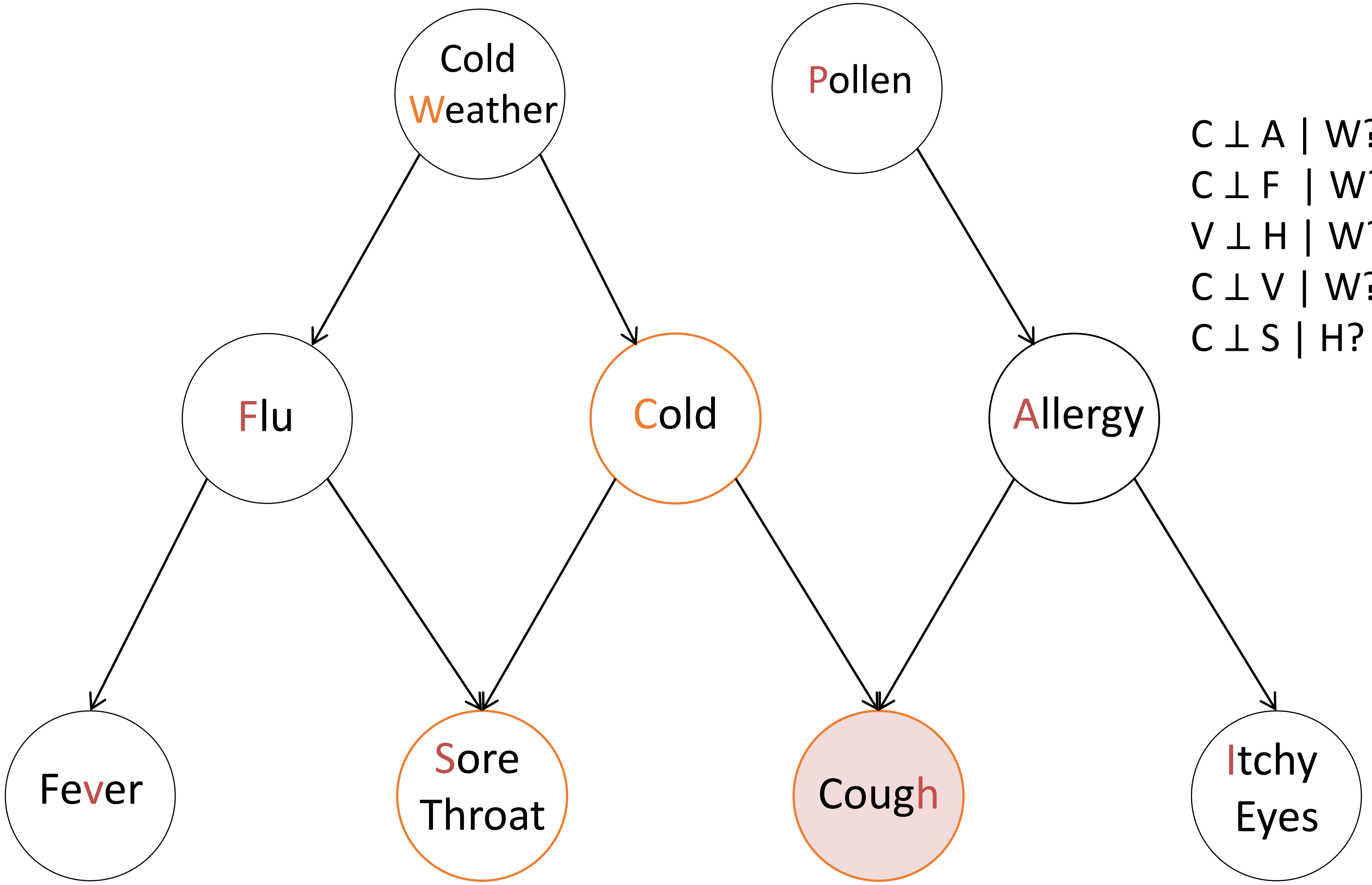
$C \perp V \mid W$? Yes
$C \perp S \mid H$?

Conditional Independence



C ⊥ A | W? Yes

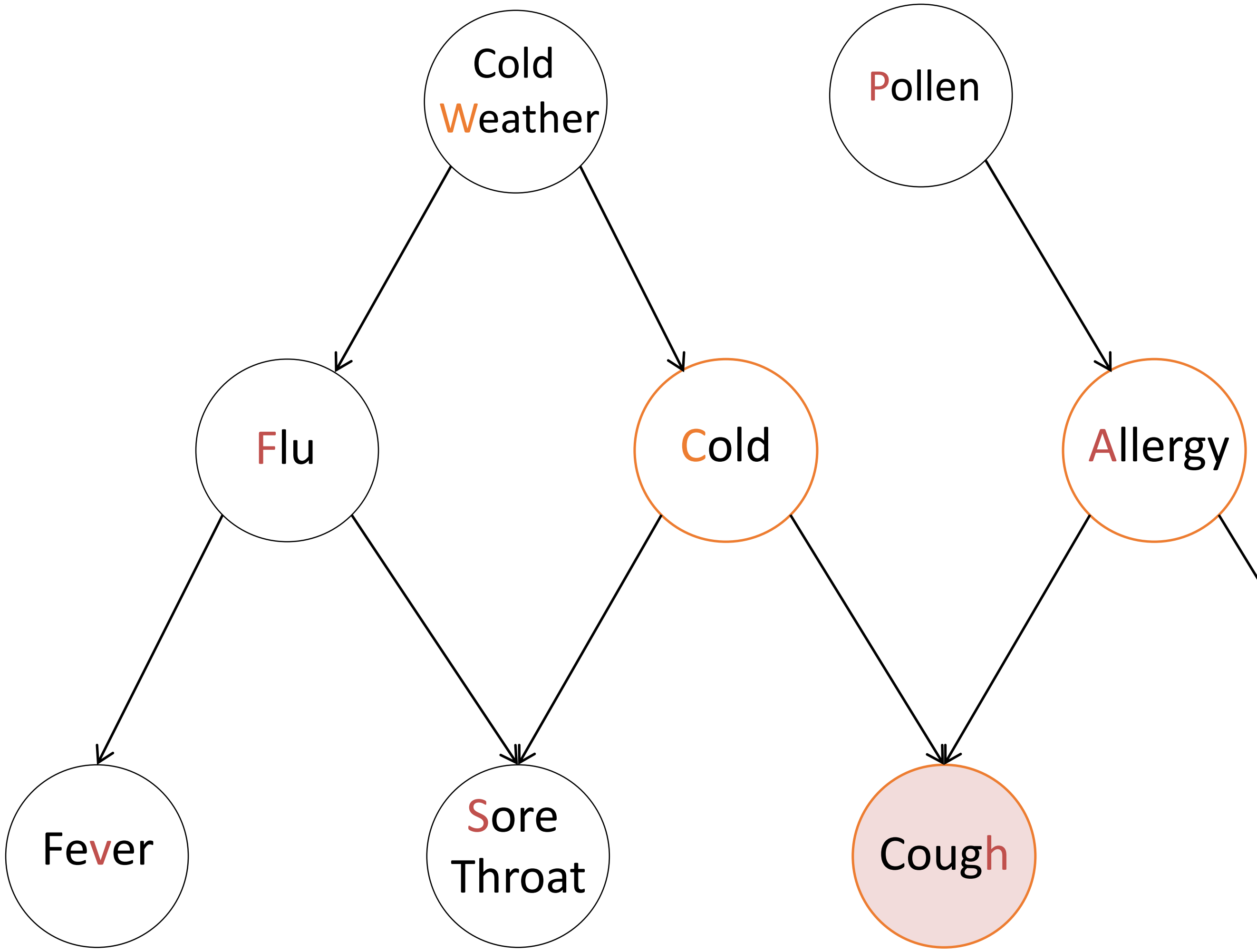
C ⊥ F | W? Yes

V ⊥ H | W? Yes

C ⊥ V | W? Yes

C ⊥ S | H? No

Conditional Independence



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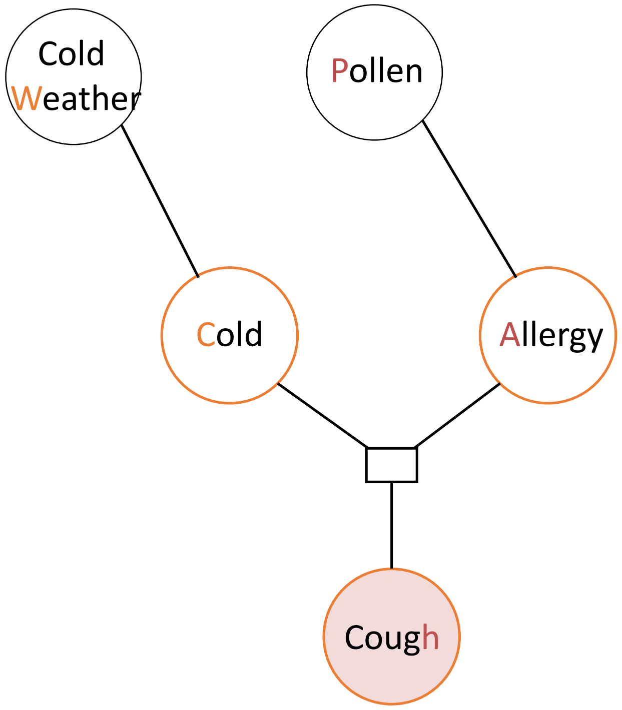
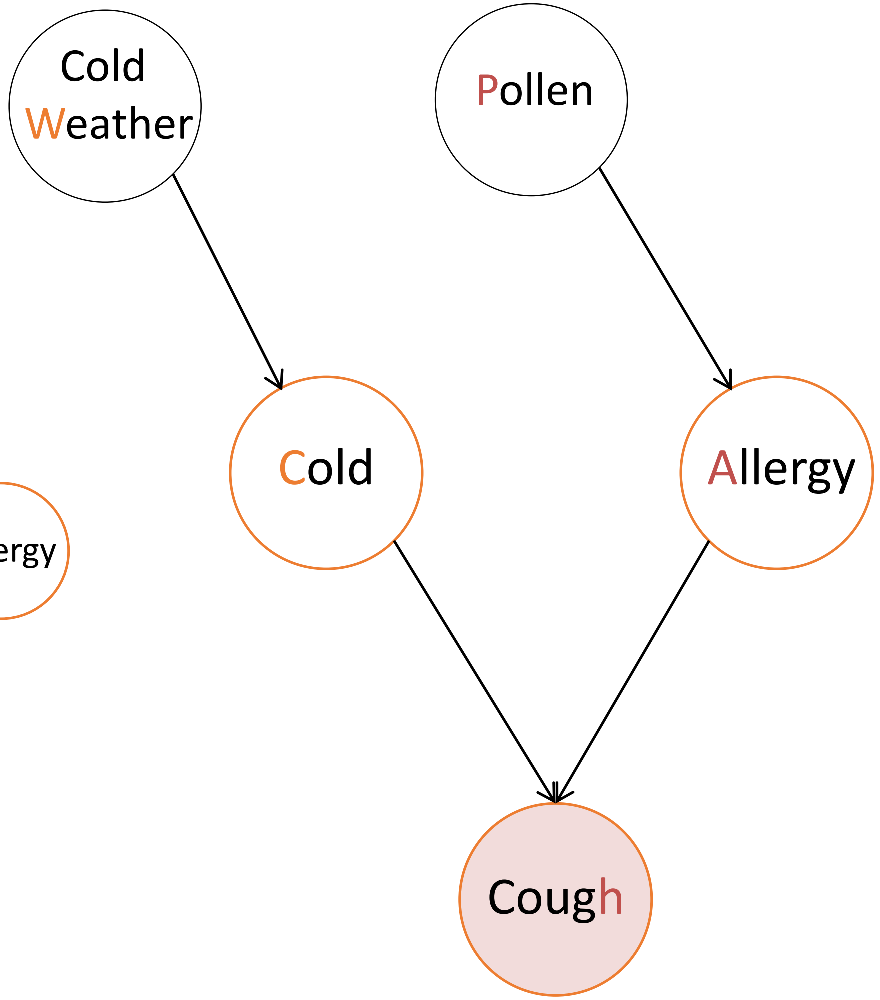
$C \perp F \mid W$? Yes
$V \perp H \mid W$? Yes
$C \perp V \mid W$? Yes
$C \perp S \mid H$? No
$C \perp A \mid H$?

Conditional Independence



Explaining Away!



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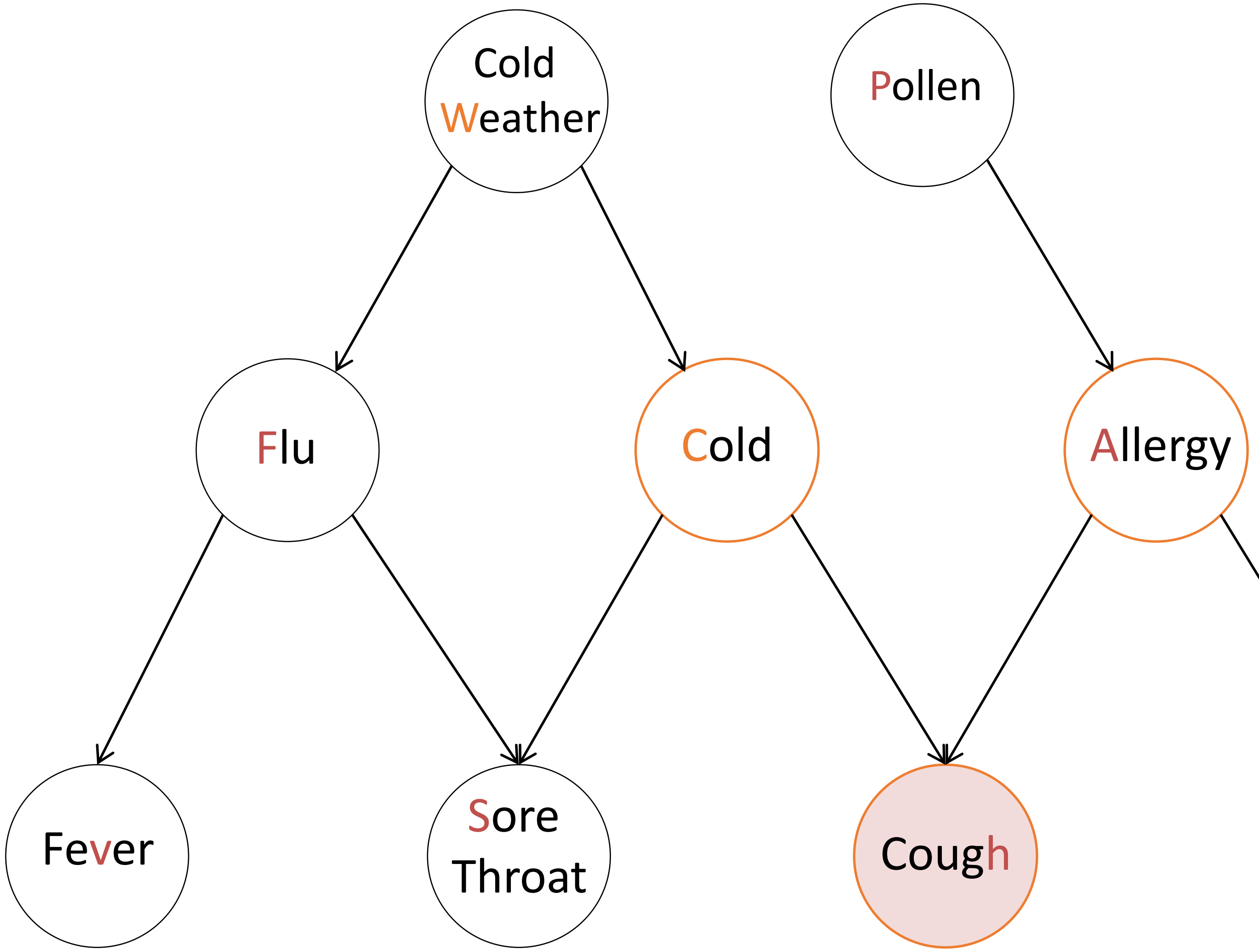
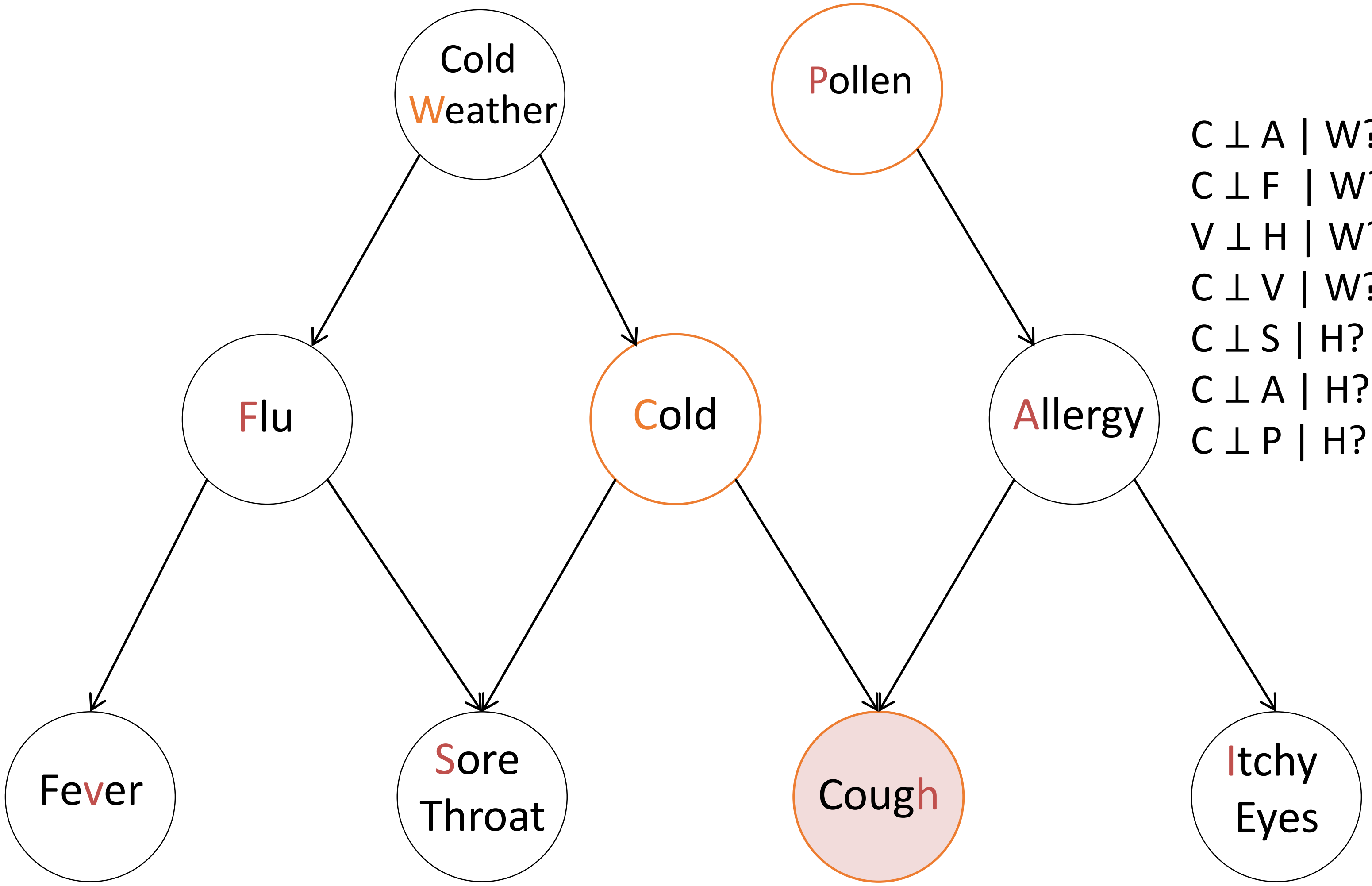
$V \perp H \mid W$? Yes
$C \perp V \mid W$? Yes
$C \perp S \mid H$? No
$C \perp A \mid H$?

Conditional Independence



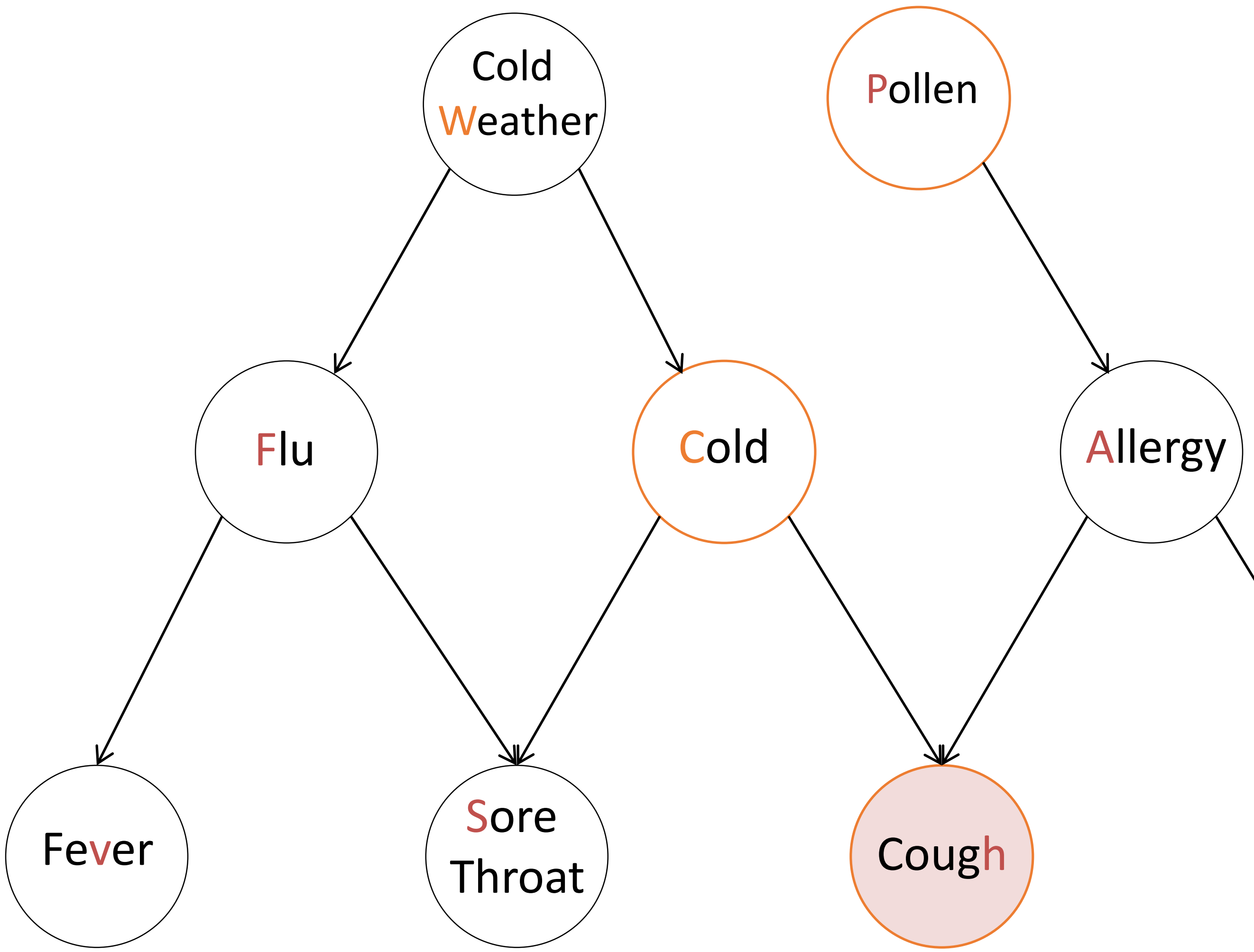
C ⊥ A | W? Yes
C ⊥ F | W? Yes
V ⊥ H | W? Yes
C ⊥ V | W? Yes
C ⊥ S | H? No
C ⊥ A | H? No

Conditional Independence



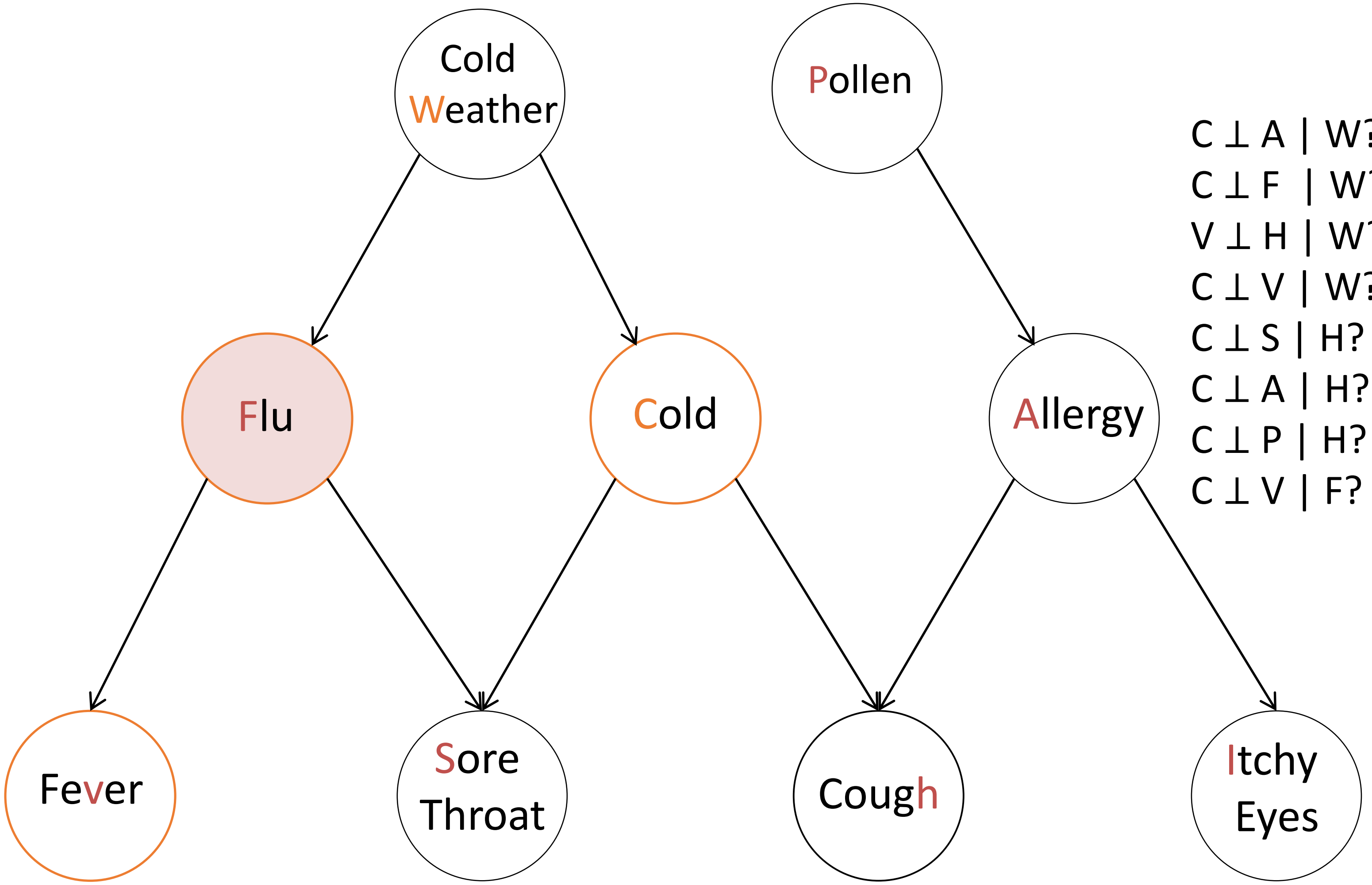
C ⊥ A | W? Yes
C ⊥ F | W? Yes
V ⊥ H | W? Yes
C ⊥ V | W? Yes
C ⊥ S | H? No
C ⊥ A | H? No
C ⊥ P | H?

Conditional Independence



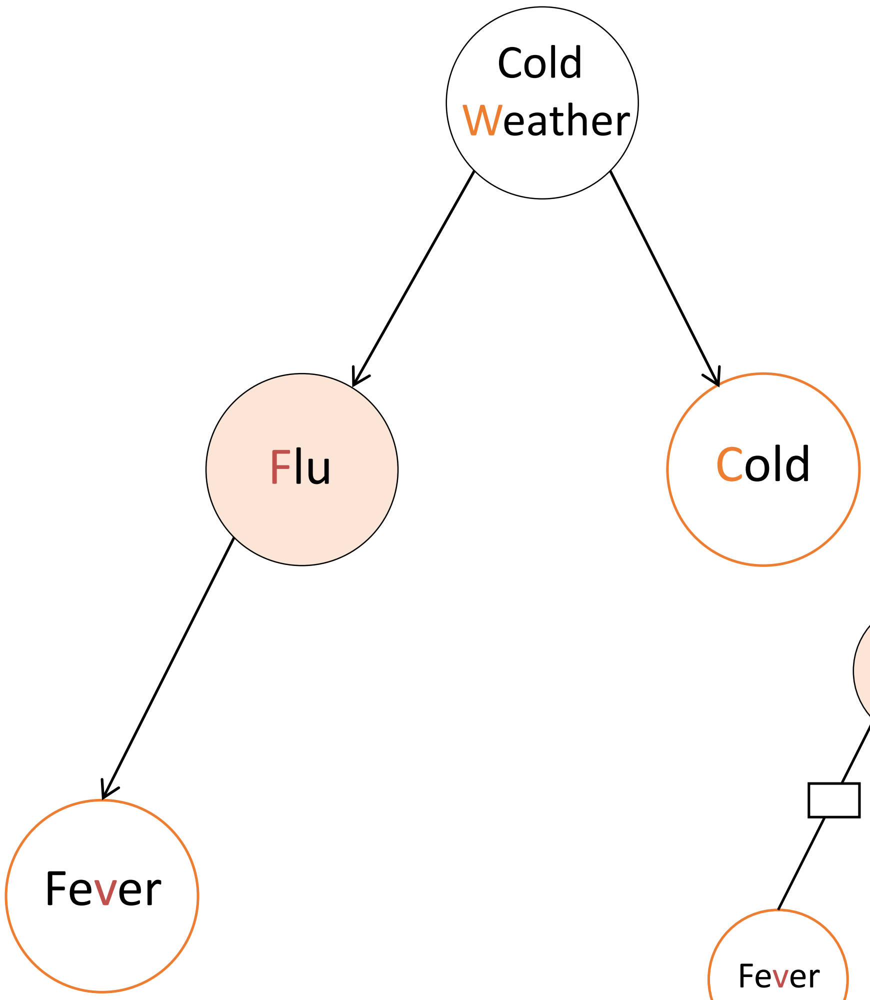
C ⊥ A | W? Yes
C ⊥ F | W? Yes
V ⊥ H | W? Yes
C ⊥ V | W? Yes
C ⊥ S | H? No
C ⊥ A | H? No
C ⊥ P | H? No

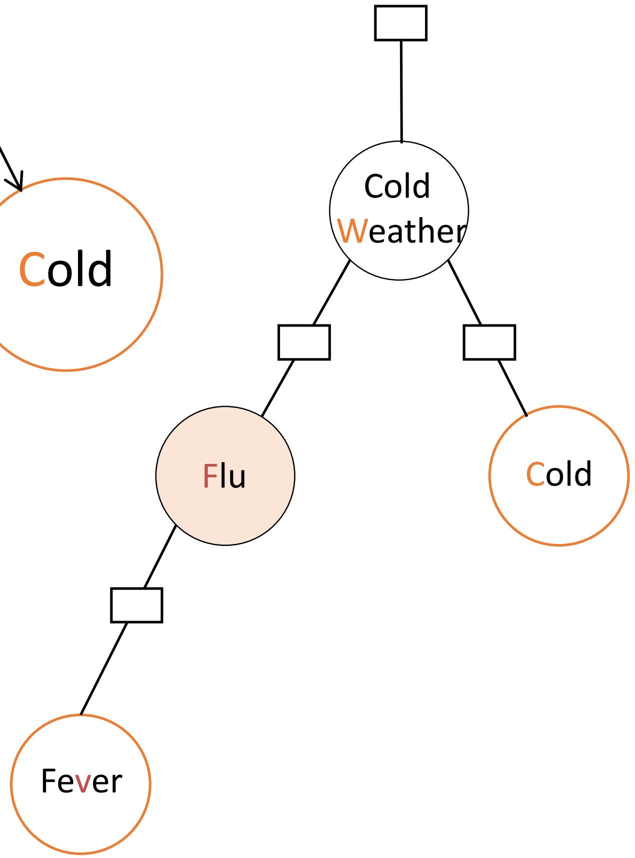
Conditional Independence



$C \perp A \mid W$? Yes
$C \perp F \mid W$? Yes
$V \perp H \mid W$? Yes
$C \perp V \mid W$? Yes
$C \perp S \mid H$? No
$C \perp A \mid H$? No
$C \perp P \mid H$? No
$C \perp V \mid F$?

Conditional Independence





C ⊥ A | W? Yes

C ⊥ F | W? Yes

V ⊥ H | W? Yes

C ⊥ V | W? Yes

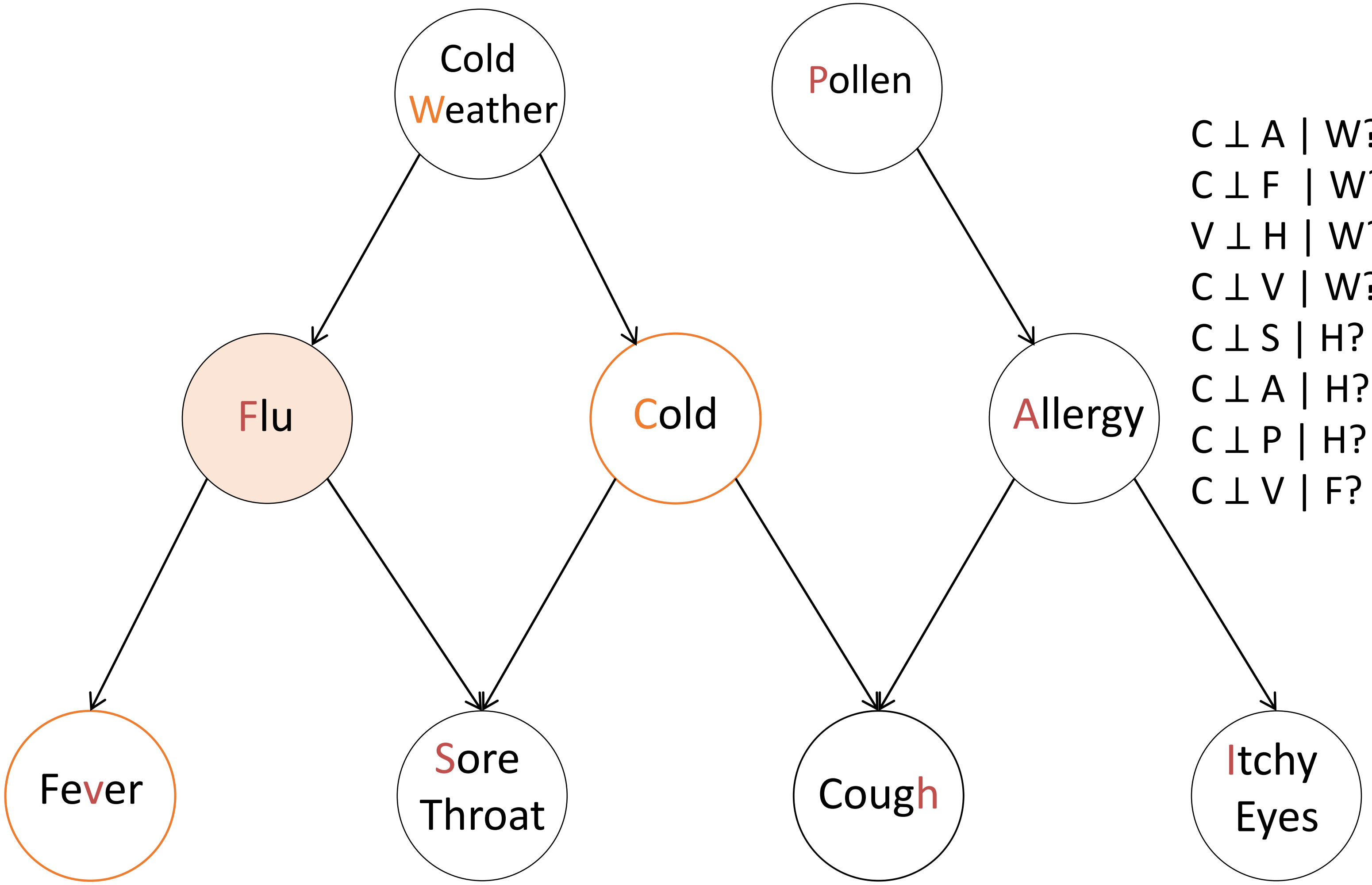
C ⊥ S | H? No

C ⊥ A | H? No

C ⊥ P | H? No

C ⊥ V | F?

Conditional Independence



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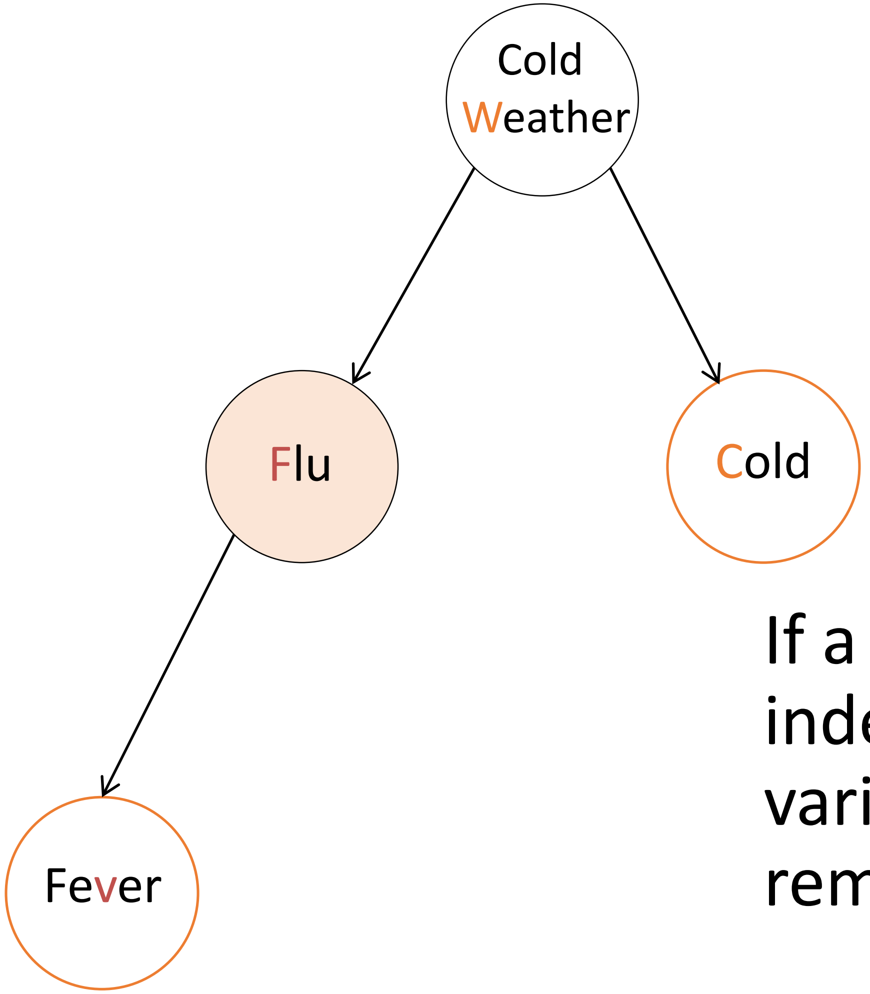
$C \perp F \mid W$? Yes
$V \perp H \mid W$? Yes
$C \perp V \mid W$? Yes
$C \perp S \mid H$? No
$C \perp A \mid H$? No
$C \perp P \mid H$? No
$C \perp V \mid F$? Yes

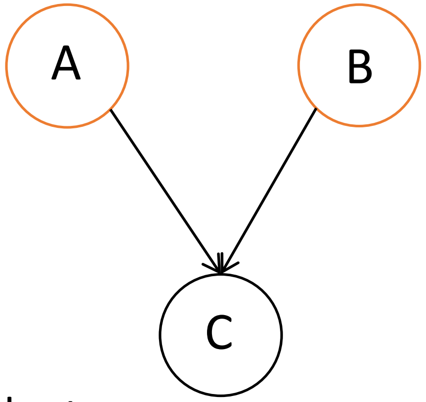
Conditional Independence



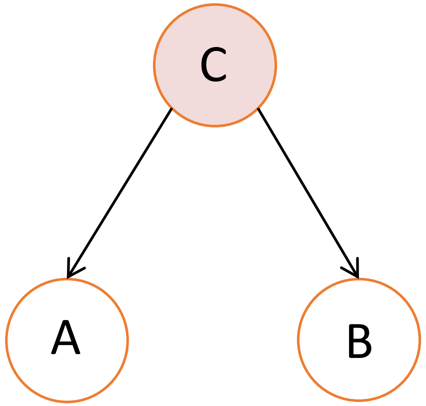
P(C=c | F=f) = ?

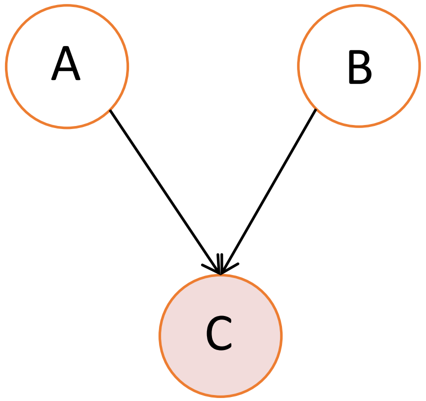
If a variable (Fever) is independent of the Query variable Q (Cold), we can remove (marginalize) it.

Patterns



Independent





Dependent