

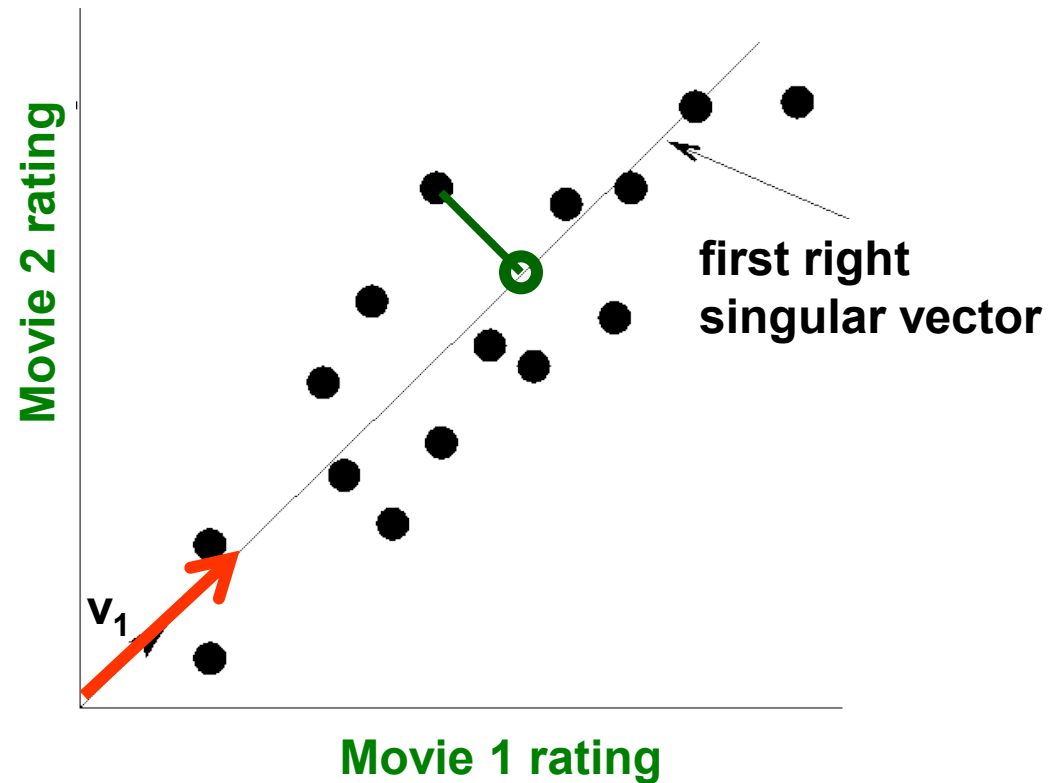
Dimensionality Reduction with SVD

Mining of Massive Datasets
Leskovec, Rajaraman, and Ullman
Stanford University



SVD – Dimensionality Reduction

- SVD gives ‘best’ axis to project on:
 - ‘best’ = min sum of squares of projection errors
- In other words, **minimum reconstruction error**



SVD – Dimensionality Reduction

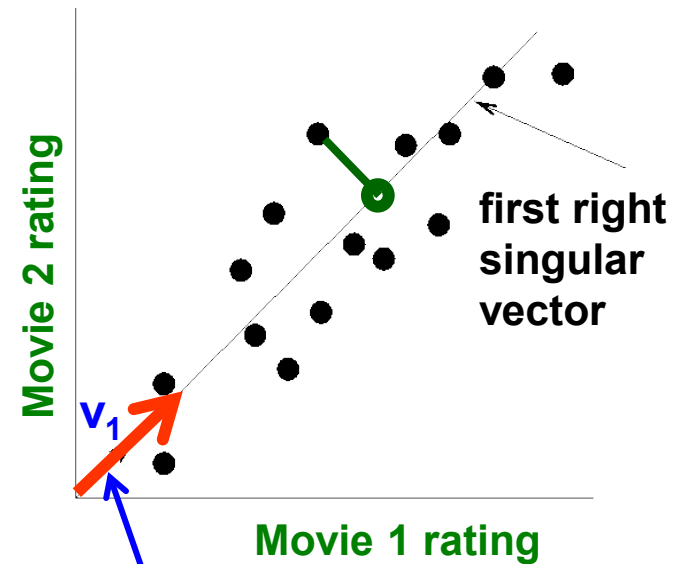
■ $A = U \Sigma V^T$ - example:

- V : “movie-to-concept” matrix
- U : “user-to-concept” matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0.13 & 0.02 & -0.01 \\ 0.41 & 0.07 & -0.03 \\ 0.55 & 0.09 & -0.04 \\ 0.68 & 0.11 & -0.05 \\ 0.15 & -0.59 & 0.65 \\ 0.07 & -0.73 & -0.67 \\ 0.07 & -0.29 & 0.32 \end{bmatrix} \times$$

$$\begin{bmatrix} 12.4 & 0 & 0 \\ 0 & 9.5 & 0 \\ 0 & 0 & 1.3 \end{bmatrix} \times$$

$$\begin{bmatrix} 0.56 & 0.59 & 0.56 & 0.09 & 0.09 \\ 0.12 & -0.02 & 0.12 & -0.69 & -0.69 \\ 0.40 & -0.80 & 0.40 & 0.09 & 0.09 \end{bmatrix}$$

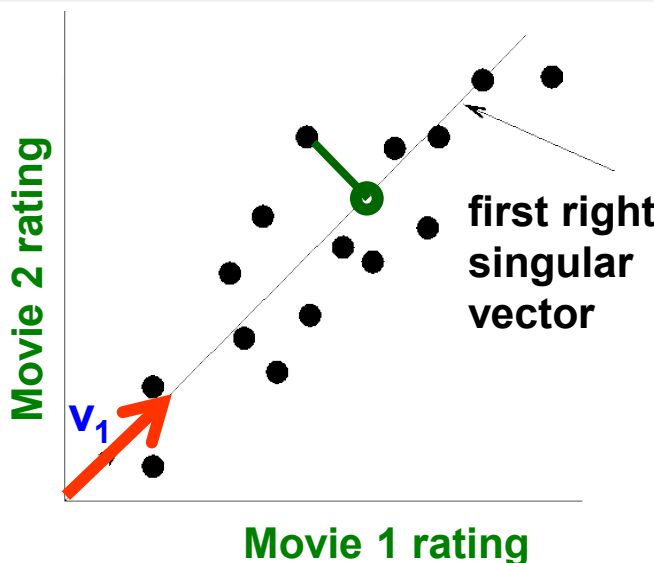


SVD – Dimensionality Reduction

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variance ('spread') on the v_1 axis



Movie 2 rating
 Movie 1 rating
 first right singular vector
 v_1

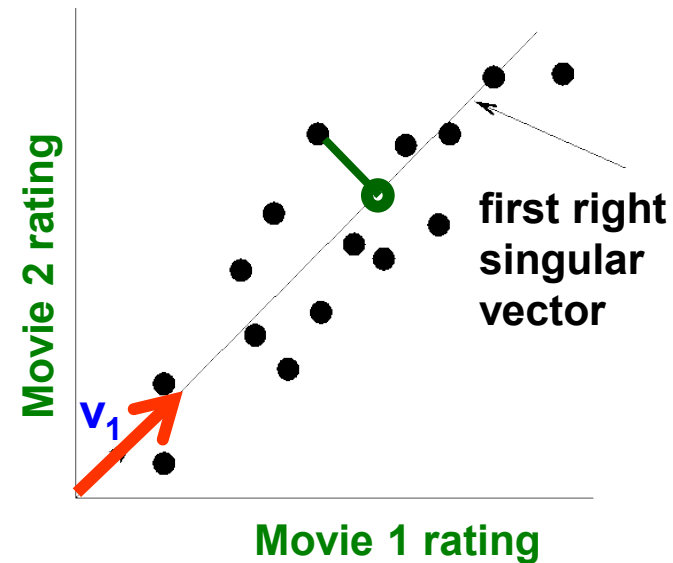
SVD – Dimensionality Reduction

$A = U \Sigma V^T$ - example:

- **$U \Sigma$:** Gives the coordinates of the points in the projection axis

1	1	1	0	0
3	3	3	0	0
4	4	4	0	0
5	5	5	0	0
0	2	0	4	4
0	0	0	5	5
0	1	0	2	2

**Projection of users
on the “Sci-Fi” axis
($(U \Sigma)^T$):**



1.61	0.19	-0.01
5.08	0.66	-0.03
6.82	0.85	-0.05
8.43	1.04	-0.06
1.86	-5.60	0.84
0.86	-6.93	-0.87
0.86	-2.75	0.41

SVD – Dimensionality Reduction

More details

- **Q:** How exactly is dim. reduction done?

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The diagram illustrates the SVD decomposition of a matrix. The first matrix is a 7x5 matrix. The second matrix is a 7x3 matrix, with its third column crossed out by a red 'X'. The third matrix is a 3x3 diagonal matrix, with its third diagonal element (1.3) crossed out by a red 'X'. The fourth matrix is a 3x5 matrix, with its third row crossed out by a red 'X'. This visualizes the process of dimensionality reduction by setting the smallest singular values to zero.

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Frobenius norm:

$$\|M\|_F = \sqrt{\sum_{ij} M_{ij}^2}$$

$$\|A-B\|_F = \sqrt{\sum_{ij} (A_{ij}-B_{ij})^2}$$

is “small”