

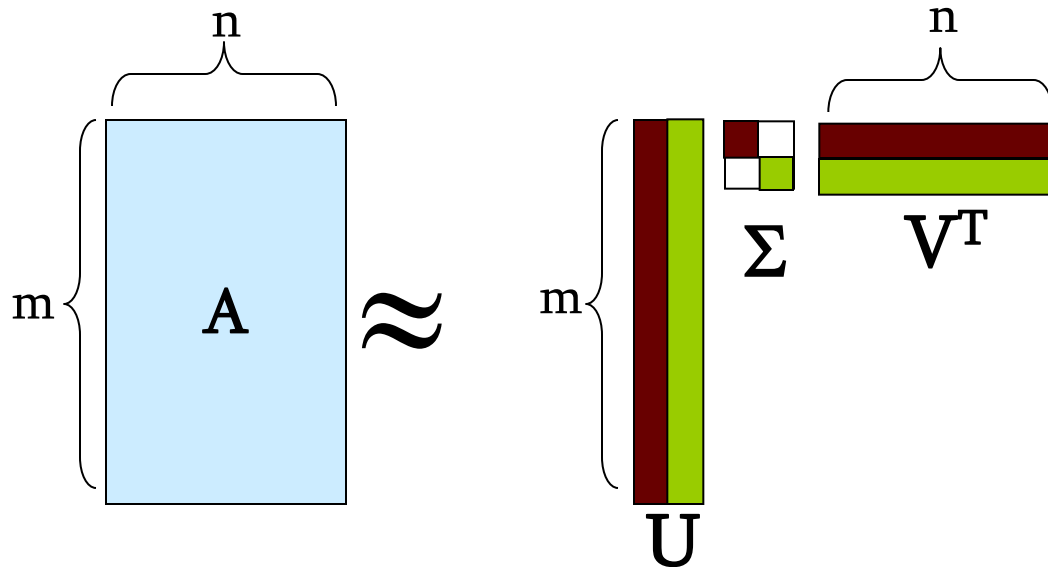
# SVD - Definition

$$\mathbf{A}_{[m \times n]} = \mathbf{U}_{[m \times r]} \mathbf{\Sigma}_{[r \times r]} (\mathbf{V}_{[n \times r]})^T$$

- **A: Input data matrix**
  - $m \times n$  matrix (e.g.,  $m$  documents,  $n$  terms)
- **U: Left singular vectors**
  - $m \times r$  matrix ( $m$  documents,  $r$  concepts)
- **$\Sigma$ : Singular values**
  - $r \times r$  diagonal matrix (strength of each 'concept')  
( $r$  : rank of the matrix **A**)
- **V: Right singular vectors**
  - $n \times r$  matrix ( $n$  terms,  $r$  concepts)

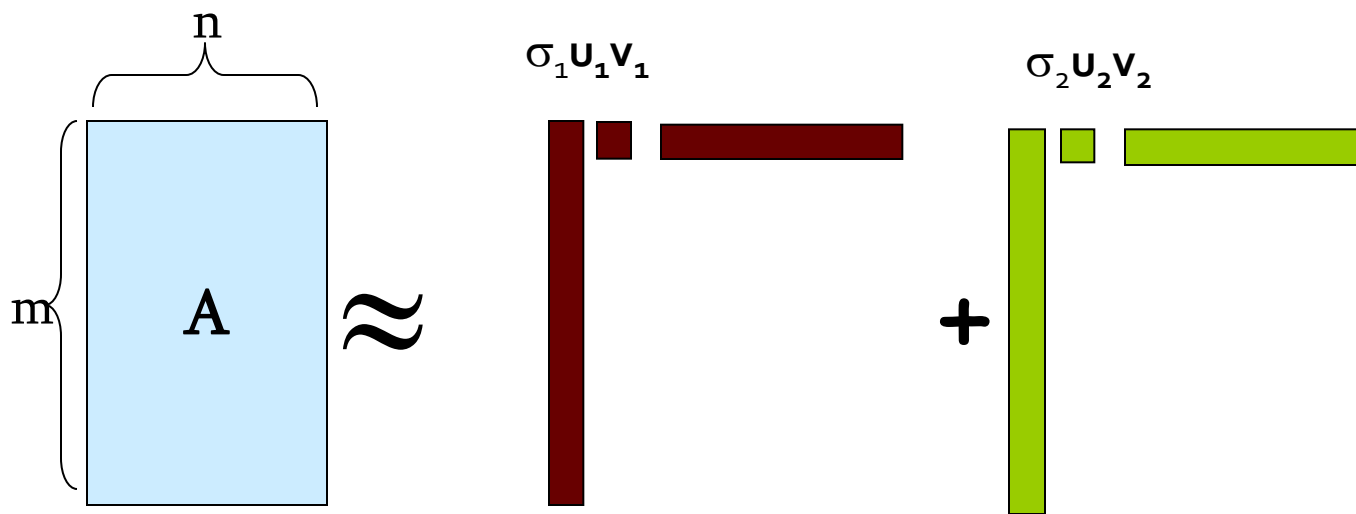
# Singular Value Decomposition

$$\mathbf{A} \approx \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \sum_i \sigma_i \mathbf{u}_i \circ \mathbf{v}_i^T$$



# SVD

$$\mathbf{A} \approx \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \sum_i \sigma_i \mathbf{u}_i \circ \mathbf{v}_i^T$$



$\sigma_i$  ... scalar  
 $\mathbf{u}_i$  ... vector  
 $\mathbf{v}_i$  ... vector

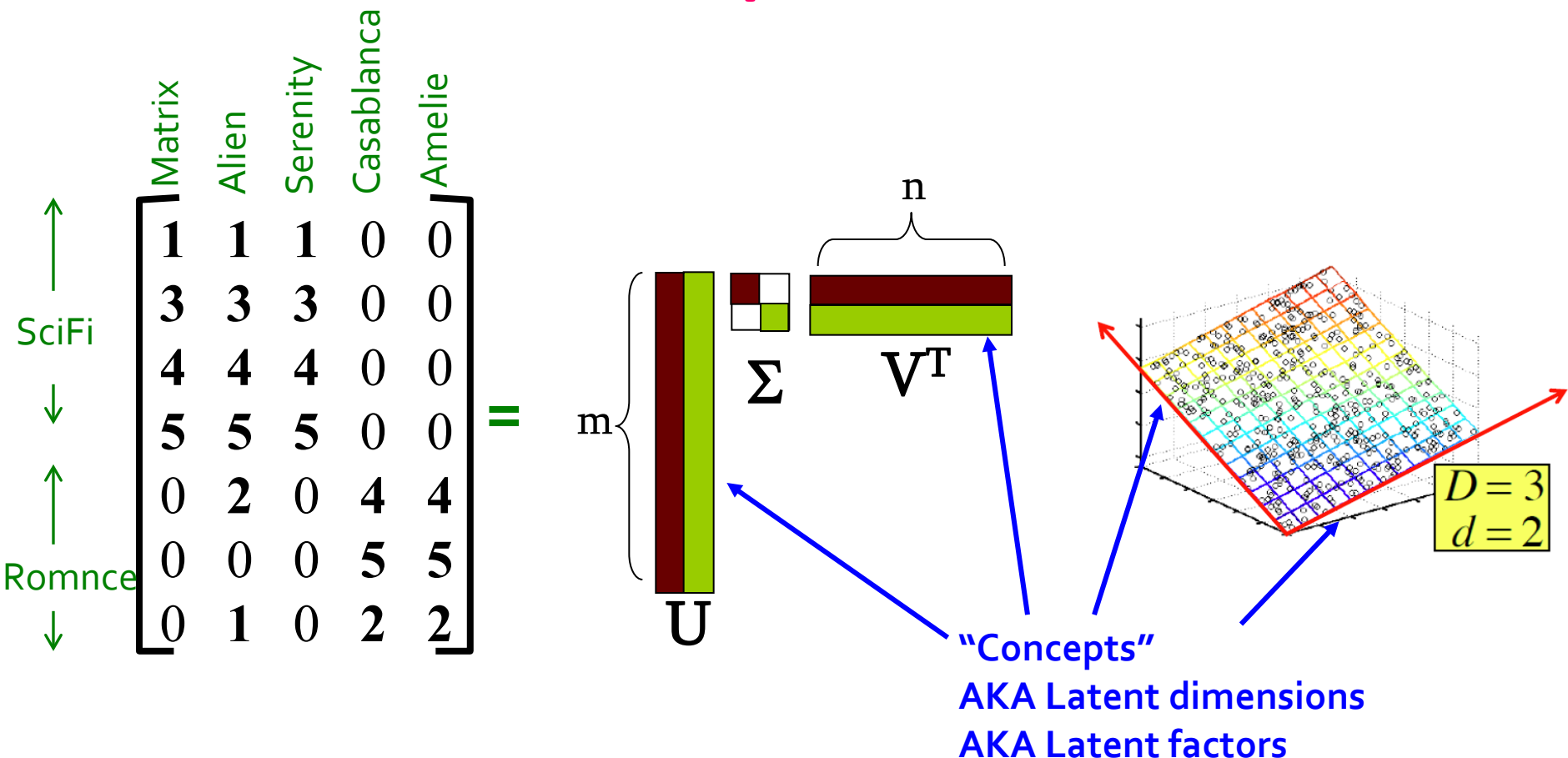
# SVD - Properties

It is **always** possible to decompose a real matrix  $A$  into  $A = U \Sigma V^T$ , where

- $U, \Sigma, V$ : **unique**
- $U, V$ : **column orthonormal**
  - $U^T U = I; V^T V = I$  ( $I$ : identity matrix)
  - (Columns are orthogonal unit vectors)
- $\Sigma$ : **diagonal**
  - Entries (**singular values**) are **positive**, and sorted in decreasing order ( $\sigma_1 \geq \sigma_2 \geq \dots \geq 0$ )

# SVD – Example: Users-to-Movies

## ■ $A = U \Sigma V^T$ - example: Users to Movies



# SVD – Example: Users-to-Movies

## ■ $A = U \Sigma V^T$ - example: Users to Movies

$$\begin{array}{c}
 \begin{array}{c} \uparrow \\ \text{SciFi} \\ \downarrow \\ \uparrow \\ \text{Romnce} \\ \downarrow \end{array}
 \begin{array}{c} \text{Matrix} \\ \text{Alien} \\ \text{Serenity} \\ \text{Casablanca} \\ \text{Amelie} \end{array}
 \begin{bmatrix}
 1 & 1 & 1 & 0 & 0 \\
 3 & 3 & 3 & 0 & 0 \\
 4 & 4 & 4 & 0 & 0 \\
 5 & 5 & 5 & 0 & 0 \\
 0 & 2 & 0 & 4 & 4 \\
 0 & 0 & 0 & 5 & 5 \\
 0 & 1 & 0 & 2 & 2
 \end{bmatrix}
 =
 \begin{bmatrix}
 0.13 & 0.02 & -0.01 \\
 0.41 & 0.07 & -0.03 \\
 0.55 & 0.09 & -0.04 \\
 0.68 & 0.11 & -0.05 \\
 0.15 & -0.59 & 0.65 \\
 0.07 & -0.73 & -0.67 \\
 0.07 & -0.29 & 0.32
 \end{bmatrix}
 \times
 \begin{bmatrix}
 12.4 & 0 & 0 \\
 0 & 9.5 & 0 \\
 0 & 0 & 1.3
 \end{bmatrix}
 \times
 \begin{bmatrix}
 0.56 & 0.59 & 0.56 & 0.09 & 0.09 \\
 0.12 & -0.02 & 0.12 & -0.69 & -0.69 \\
 0.40 & -0.80 & 0.40 & 0.09 & 0.09
 \end{bmatrix}
 \end{array}$$

# SVD – Example: Users-to-Movies

## ■ $A = U \Sigma V^T$ - example: Users to Movies

Matrix    Alien    Serenity    Casablanca    Amelie

SciFi ↑    ↓ Romnce

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0.13 & 0.02 & -0.01 \\ 0.41 & 0.07 & -0.03 \\ 0.55 & 0.09 & -0.04 \\ 0.68 & 0.11 & -0.05 \\ 0.15 & -0.59 & 0.65 \\ 0.07 & -0.73 & -0.67 \\ 0.07 & -0.29 & 0.32 \end{bmatrix} \times \begin{bmatrix} 12.4 & 0 & 0 \\ 0 & 9.5 & 0 \\ 0 & 0 & 1.3 \end{bmatrix} \times \begin{bmatrix} 0.56 & 0.59 & 0.56 & 0.09 & 0.09 \\ 0.12 & -0.02 & 0.12 & -0.69 & -0.69 \\ 0.40 & -0.80 & 0.40 & 0.09 & 0.09 \end{bmatrix}$$

SciFi-concept    Romance-concept

# SVD – Example: Users-to-Movies

■  $A = U \Sigma V^T$  - example:

$U$  is “user-to-concept” similarity matrix

	Matrix	Alien	Serenity	Casablanca	Amelie		SciFi-concept	Romance-concept	
	1	1	1	0	0	=	0.13	0.02	-0.01
	3	3	3	0	0		0.41	0.07	-0.03
	4	4	4	0	0		0.55	0.09	-0.04
	5	5	5	0	0		0.68	0.11	-0.05
	0	2	0	4	4		0.15	-0.59	0.65
	0	0	0	5	5		0.07	-0.73	-0.67
	0	1	0	2	2		0.07	-0.29	0.32
SciFi	↑								
	↓								
	↑								
Romnce	↓								

$\times$

12.4	0	0
0	9.5	0
0	0	1.3

$\times$

0.56	0.59	0.56	0.09	0.09
0.12	-0.02	0.12	-0.69	-0.69
0.40	-0.80	0.40	0.09	0.09



# SVD – Example: Users-to-Movies

## ■ $A = U \Sigma V^T$ - example:

Matrix    Alien    Serenity    Casablanca    Amelie

SciFi ↑    1    1    1    0    0

↓    3    3    3    0    0

4    4    4    0    0

5    5    5    0    0

Romnce ↑    0    2    0    4    4

↓    0    0    0    5    5

0    1    0    2    2

SciFi-concept ↓

0.13    0.02    -0.01

0.41    0.07    -0.03

0.55    0.09    -0.04

0.68    0.11    -0.05

0.15    -0.59    0.65

0.07    -0.73    -0.67

0.07    -0.29    0.32

"strength" of the SciFi-concept

12.4    0    0

0    9.5    0

0    0    1.3

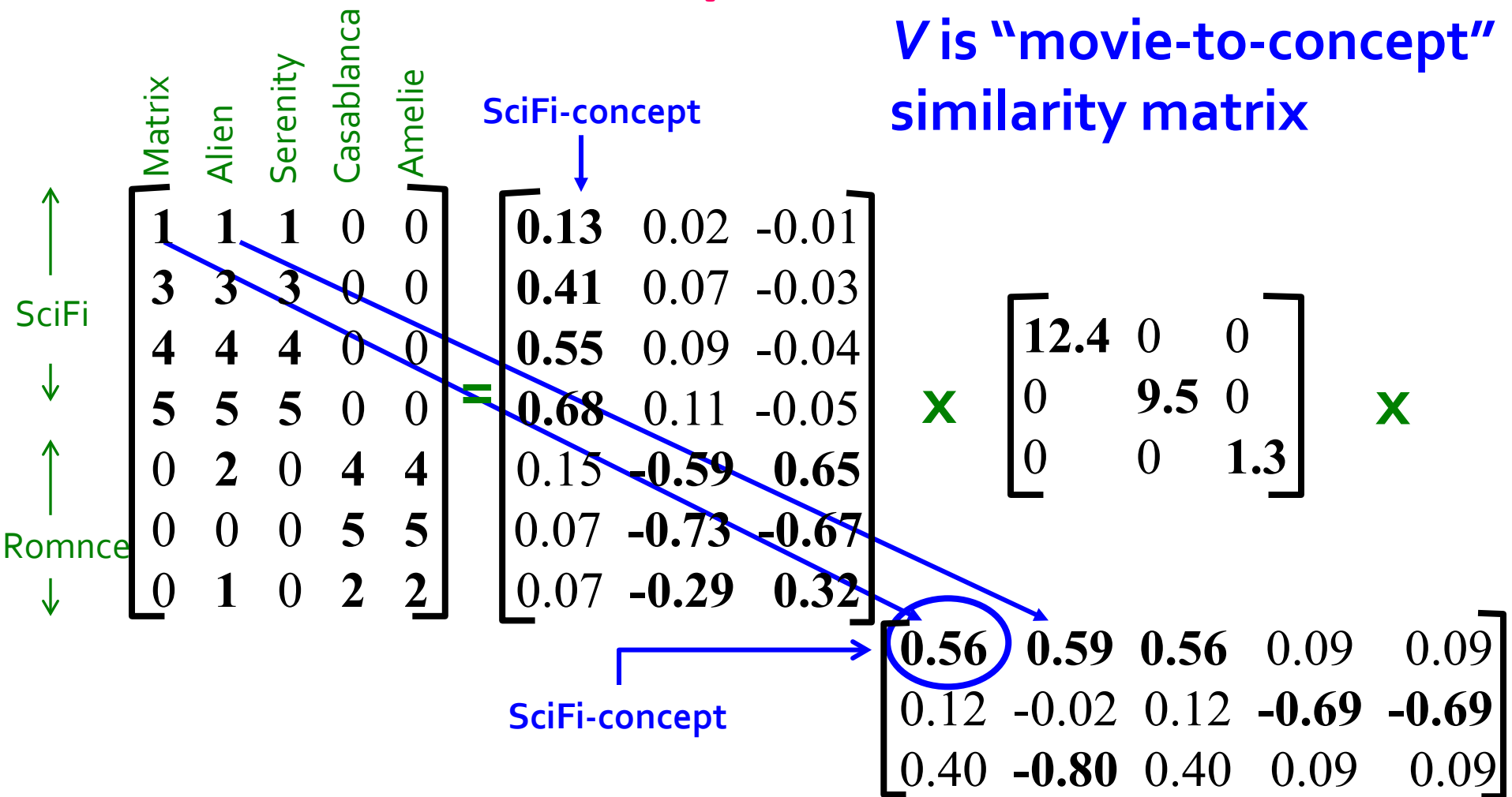
0.56    0.59    0.56    0.09    0.09

0.12    -0.02    0.12    -0.69    -0.69

0.40    -0.80    0.40    0.09    0.09

# SVD – Example: Users-to-Movies

## ■ $A = U \Sigma V^T$ - example:



# SVD - Interpretation #1

‘**movies**’, ‘**users**’ and ‘**concepts**’:

- $U$ : user-to-concept similarity matrix
- $V$ : movie-to-concept similarity matrix
- $\Sigma$ : its diagonal elements:  
‘strength’ of each concept