Hidden Markov Chains

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Reminder: What a Markov chain is?

- A Markov chain is a stochastic process with a particular characteristic, must be "memory-less" (i.e the probability of future actions are not dependent upon the steps that led up to the present state.). This property is known as the **Markov property**.
- Markov chains are very useful, we can find very often phenomena that satisfy the Markov property. This fact make Markov chains very useful in various application scenarios.
- Additionally, Markov chains allow us modeling sequential processes in a simple but effective way.

Reminder: What a Markov chain is?

A simple example...

- Bag of balls without replacement: **is NOT** a Markov process

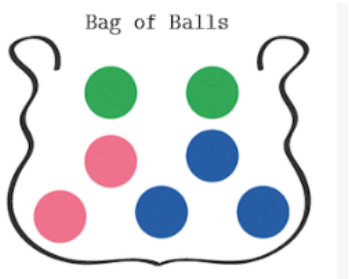


Figure: example: probability of getting a blue ball / without replacement

- Bag of balls with replacement **is** a Markov process

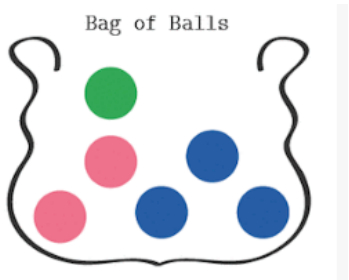


Figure: example: probability of getting a blue ball / with replacement

Markov chains: Definition

Markov Chain: definition

A Markov chain is a sequence $X_{0:T} = (X_0, X_1, \ldots, X_T)$ of random variables taking values in some finite set $\mathcal{X}$ that satisfies the rule of conditional independence:

$$P(X_{t+1} = x_{t+1} \mid X_t = x_t, \ldots, X_0 = x_0) = P(X_{t+1} = x_{t+1} \mid X_t = x_t)$$

For this case, we will study time-homogeneous Markov chains (i.e. the probability of any state transition is independent of time).

$$A_{i,j} := P(X_{t+1} = j \mid X_t = i) \;\; i, j \in \mathcal{X}$$

However, the general case of Markov chains allows for time-inhomogeneous Markov chains (as time goes on, the probability of moving from one state to another may change).

Transition matrices

The movement among states are defined by a **transition matrix**, A_t. Particularly, this matrix contains the information on the probability of transitioning between states.

$$A_{i,j} := P\left(X_{t+1} = j \mid X_t = i\right) \; i, j \in \mathcal{X}$$

Each row of the matrix is a probability vector, and the sum of its entries is 1.

For $x_0 \in \mathcal{X}$, let $\mu_{x0} = P\left(X_0 = x_0\right)$ then, the joint probability mass function of $X_{1:T}$ in terms of $(A_{i,j})_{i,j \in \mathcal{X}}$ and $(\mu_i)_{i \in \mathcal{X}}$ is as follows:

$$\begin{aligned} p\left(x_{0:T}\right) &:= P\left(X_0 = x_0, \ldots, X_T = x_t\right) \\ &= P\left(X_0 = x_0\right) \prod_{t=1}^{T} P\left(X_t = x_t \mid X_{t-1} = x_{t-1}\right) \\ &= \mu_{x0} \prod_{t=1}^{T} A_{x_{t-1}, x_t} \end{aligned}$$

Hidden Markov Chains : Motivation

- A **Hidden Markov Model** is a statistical model which studies a system assumed as a Markov process including hidden (unobservable) states.
- In the simple Markov chain model, the state of the system is directly visible to the observer. On the other hand, HMM assume that the data observed is not the actual state of the model but instead is generated by underlying hidden states.
- Each state has a probability distribution over the possible outputs.
- HMM are often used to model temporal data.

Definition

HMM :Definition

Let $X_{0:T} = (X_0, X_1, \ldots, X_T)$ be a homogeneous Markov chain taking values in $\mathcal{X}$ with and associated transition matrix (A_{ij}). Additionally, there exits another sequence of random variables $Y_{0:T} = (Y_1, \ldots, Y_T)$ taking values in $\mathcal{Y}$, known as the observation space.

We assume that the random variables of sequence $Y_{0:T}$ are independent conditional on the state sequence $X_{0:T}$, which is equivalent to:

$$P\left(Y_1 = y_1, \ldots, Y_T = y_T \mid X_0 = x_0, \ldots, X_T = x_t\right)$$
$$= \prod_{t=1}^{T} P\left(Y_t = y_t \mid X_t = x_t\right)$$

if we are considering an homogeneous HMM, then we have an **emission probability** mass function:

$$g_x(y) := P\left(Y_t = y \mid X_t = x\right)$$

More Definitions

Using the emission probabilities to connect the observation space and the hidden space and introducing the transition matrix that governs the movements among hidden states, we can define the joint probability of the hidden states and observations as follows:

$$P\left(X_{0:T} = x_{0:T}, Y_{0:T} = y_{0:T}\right) = \mu_{x_0} \prod_{t=1}^{T} g_{x_t}\left(y_t\right) A_{x_{t-1}, x_t}$$

Inference in Hidden Markov Models (HMM)

Remark:

For simplicity of exposure **we consider discrete-valued observations** Y_t, however algorithms apply similarly with continuous observation.

Some notations:

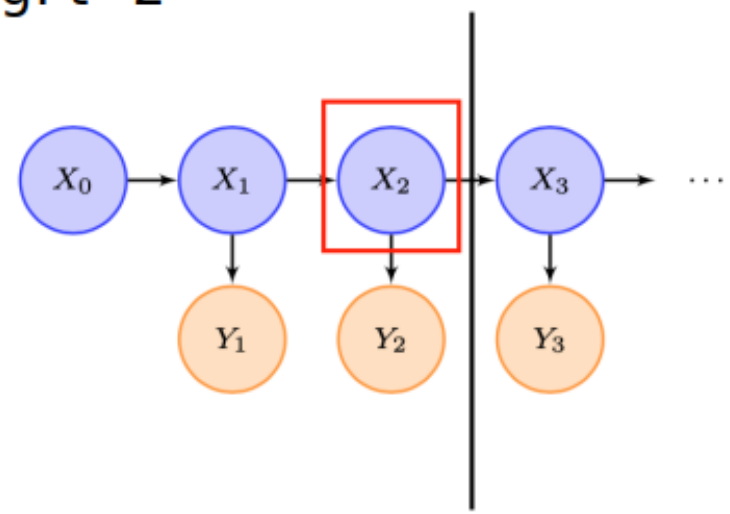
- $p(x_{t+1}|x_t) = P(X_{t+1} = x_{t+1}|X_t = x_t)$
- $p(y_t|x_t) = P(Y_t = y_t|X_t = x_t)$
- $p(x_t|y_{1:t}) = P(X_t = x_t|Y_1 = y_1, \ldots, Y_t = y_t)$
- $p(y_{1:t}) = P(Y_1 = y_1, \ldots, Y_t = y_t)$

What do you mean by inference problems in HMM?

Filtering:
Given measurements up to time t, compute the distribution of X_t,

$$p(x_t|y_{1:t})$$

E.g. t=2

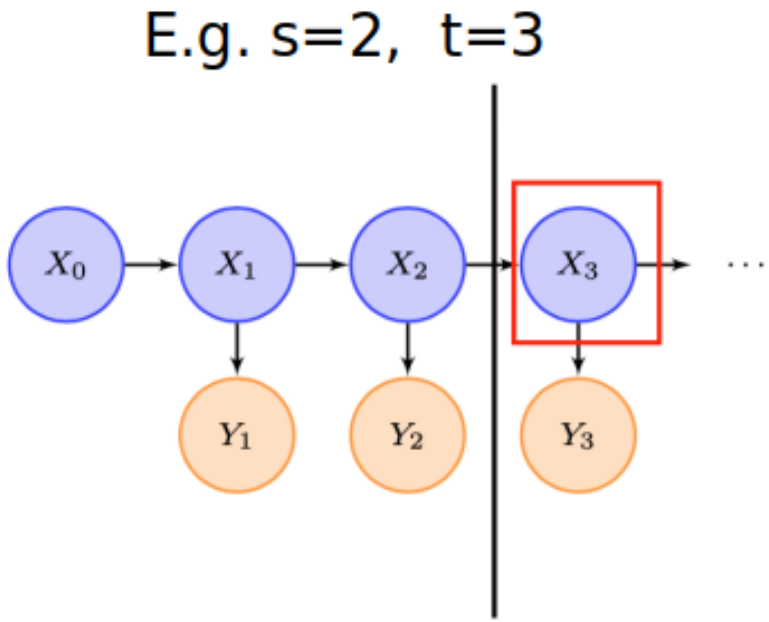


What do you mean by inference problems in HMM?

Prediction:

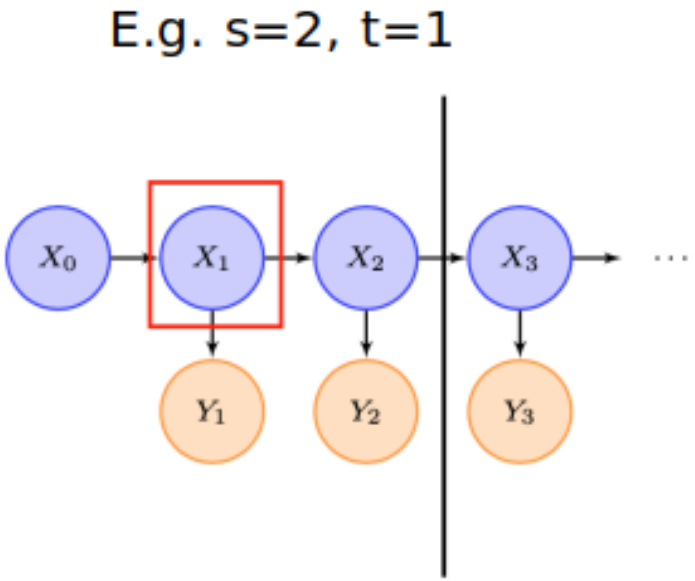
Given measurements up to time $s < t$, compute the distribution of X_t,

$$p(x_t|y_{1:s})$$



What do you mean by inference problems in HMM?

Smoothing:
Given measurements up to time $s > t$, compute the distribution of X_t, $p(x_t|y_{1:s})$.

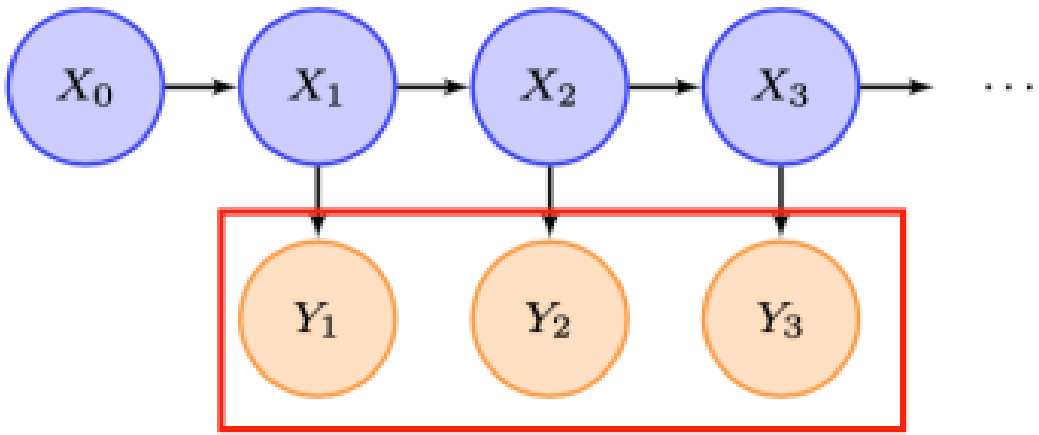


What do you mean by inference problems in HMM?

Likelihood:

Given observations $y_1, \ldots, y_T$ compute its likelyhood model,

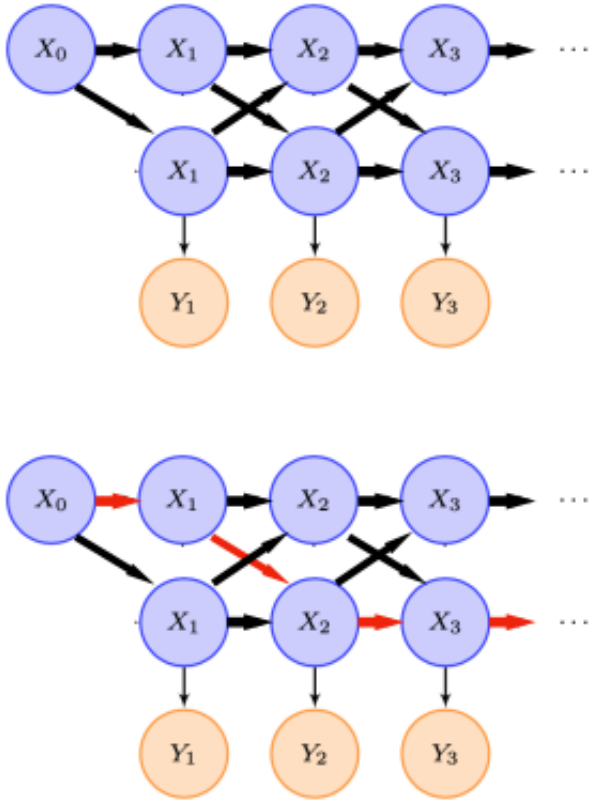
$$p(y_{1:T})$$



What do you mean by inference problems in HMM?

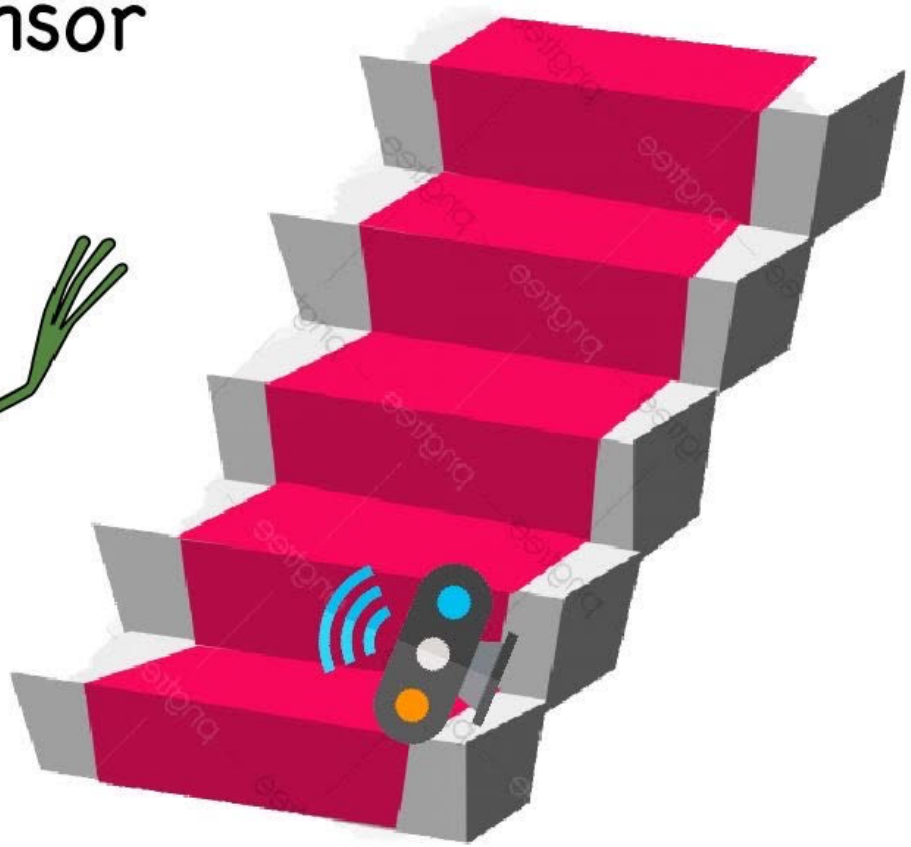
Decoding:
Find the most likely state history $\mathcal{X}$ given the observation history:

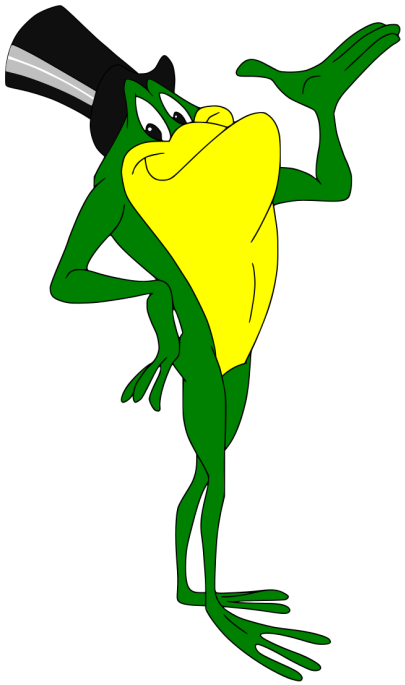
$$arg \max_{x_{0:t}} p(x_{0:T}|y_{1:T})$$



Example: Hidden Mark-frog Chains

Movement sensor





- Our frog is jumping on a ladder with $K = 6$ levels.
- From position X_t level at which the frog is at time t is **not observed**.
- Frog's detector at the lowest level of the ladder sends a signal Y_t

$$Y_t = 1 \; non - detection$$

$$Y_t = 2 \; detection$$

- Observations

$$y_{1:14} = (1, 1, 1, 1, 2, 2, 1, 1, 1, 1, 2, 2, 1, 2)$$

Transition matrix $p = 0.4$

$$A_{1,2} = 1 - p, A_{K,1} = \frac{1-p}{2}$$

$$A_{i,i+1} = \frac{1-p}{2} \text{ for } i = 1, \ldots, K-1$$

$$A_{i,i} = p \text{ for } i = 1, \ldots, K$$

$$A_{i,i-1} = \frac{1-p}{2} \text{ for } i = 2, \ldots, K$$

Probability of detection

$$B_{k,2} := P(Y_t = 2 | X_t = k)$$

$$= \begin{cases} 0.9 & \text{if } k = 1 \\ 0.5 & \text{if } k = 1 \\ 0.1 & \text{if } k = 2 \\ 0 & \textit{otherwise} \end{cases}$$

Problems: From frog's observations of position at each time t, infer 1) filtering and 2) smoothing and 3) MAP estimate of pmf's.

Where can the frog jump? - Transition probabilities (Matrix A)

		Level to				
	1	2	3	4	5	6
Level from 1	0.4	0.6				
2	0.3	0.4	0.3			
3		0.3	0.4	0.3		
4			0.3	0.4	0.3	
5				0.3	0.4	0.3
6	0.3				0.3	0.4

Table: Hidden Mark-frog chain example - Part II

Where is the frog at the beginning?

1	2	3	4	5	6
1/6	1/6	1/6	1/6	1/6	1/6

Table: Hidden Mark-frog chain example - Part I

What is the probability of detecting the frog? - Emission probabilities (Matrix B)

Movement sensor

At Level k, probabilitiy of being:

At Level	Detected	Not detected
1	0.9	0.1
2	0.5	0.5
3	0.1	0.9
4	0	1
5	0	1
6	0	1

Table: Hidden Mark-frog chain example - Part III

Using filtering algorithm

Forward Algorithm initialization equation

$$\alpha_1(i) = \pi_i \cdot b_i(O)$$

Forward Algorithm recursion equation

$$\alpha_{t+1}(j) = \sum_{i=1}^{N} \alpha_t(i)\alpha_{ij}b_{ij}(O)$$

Termination

$$P(O|\lambda) = \sum_{i=1}^{N} \alpha_T(i)$$

Formal Filtering

We are interested in the conditional probability mass function $p\left(x_t \mid y_{1:t}\right)$ of the state X_t given the data observed up to time t .

$$p(x_t|y_{1:t}) = \frac{p(x_t, y_{1:t})}{\sum_{x'_t \in \mathcal{X}} p(x'_t, y_{1:t})}$$

Formal Filtering

We will derive a recursion for:

$$\begin{aligned} p(x_t, y_{t:T}) &= \sum_{x_{t-1} \in \mathcal{X}} p(x_t, x_{t-1}, y_t, y_{1:t-1}) \\ &= \sum_{x_t \in \mathcal{X}} p(y_t|x_t, x_{t-1}, y_{1:t-1}) p(x_t|x_{t-1}, y_{1:t-1}) p(x_{t-1}, y_{1:t-1}) \\ &= p(y_t|x_t) \sum_{x_t \in \mathcal{X}} p(x_t|x_{t-1}) p(x_{t-1}, y_{1:t-1}) \end{aligned}$$

If we define $\alpha(t) := p(x_t, y_{1:t})$, then we have a forward recursion for $t = 1, \ldots, T$, $x_t \in \mathcal{X}$:

$$\alpha_t(x_t) = p(y_t|x_t) \sum_{x_t \in \mathcal{X}} p(x_t|x_{t-1}) \alpha_{t-1}(x_{t-1}); \ \alpha_0(x_0) = 0$$

Formal Filtering and likelihood

So, we compute a filtering using:
For $t = 1, \ldots, T$, $x \in X$:

$$\alpha_t(x_t) = p(y_t \mid x_t) \sum_{x_{t-1} \in \mathcal{X}} p(x_t \mid x_{t-1}) \alpha_{t-1}(x_{t-1})$$

The filtering pmf is obtained by normalizing $\alpha_t(x_t)$ as:

$$p(x_t | y_{1:t}) = \frac{p(x_t, y_{1:t})}{p(y_{1:t})} = \frac{\alpha_t(x_t)}{\sum_{x \in \mathcal{X}} \alpha_t(x)}$$

Formal Filtering and likelihood

The likelihood term $p\left(y_{1:t-1}\right)$ can be computed from the α-recursion:

$$p(y_{1:T}) = \sum_{x \in \mathcal{X}} \alpha_T(x)$$

Predict

$$p(x_t|y_{1:t-1}) = \sum_{x_{t-1} \in \mathcal{X}} p(x_t|x_{t-1}) p(x_{t-1}|y_{1:t-1})$$

Update

$$p(x_t|y_{1:t}) = \frac{g_{x_t}(y_t) p(x_t|y_{1:t})}{\sum_{x'_t \in \mathcal{X}} g_{x'_t}(y_t) p(x'_t|y_{1:t})}$$

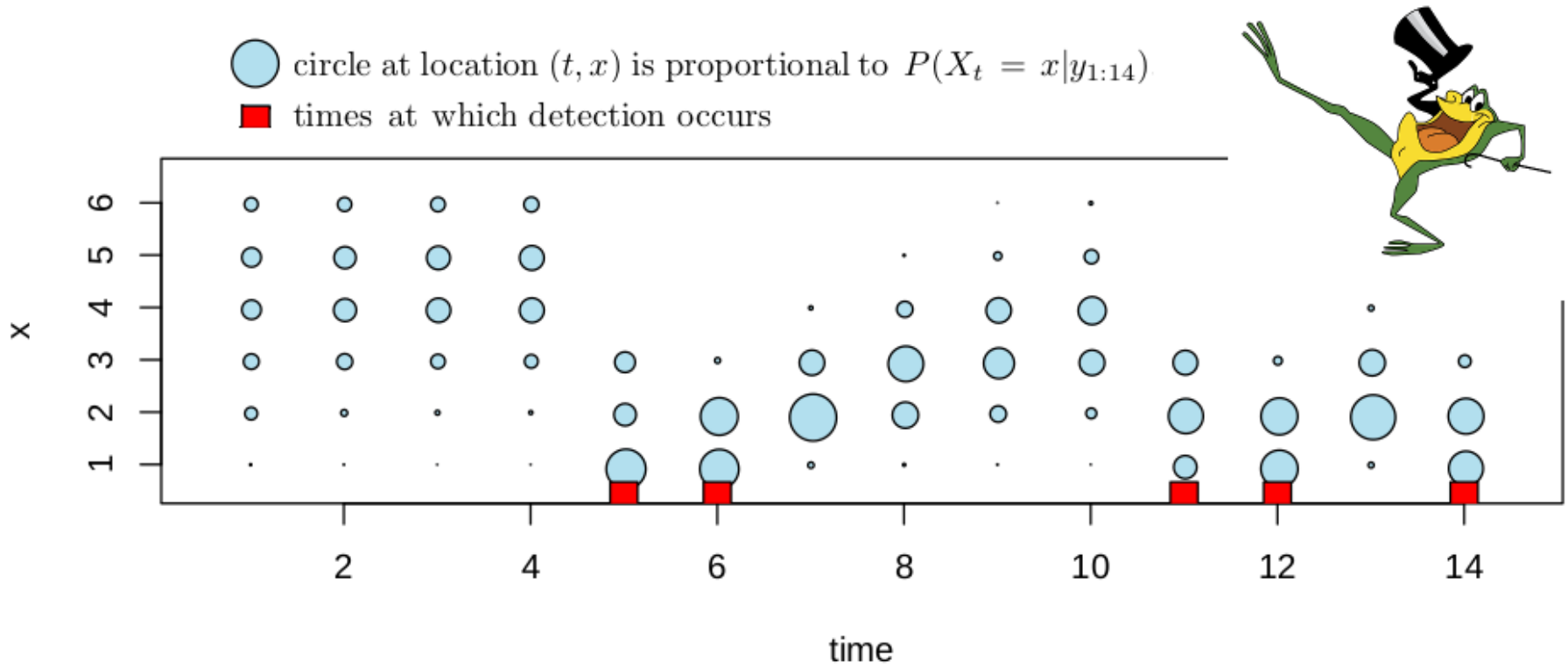
Considerations in Filtering

- The proposed recursion may suffer from numerical underflow/overflow, as α_t may become very small or very large for large t.
- The proposed recursion may suffer from numerical underflow/overflow, as α_t may become very small or very large for large t.
- To avoid this, we can normalize α_t, or propagate the filtering *pmf* $p(x_t \mid y_{1:t})$ instead of α_t, using the following two-step predict-update recursion:

$$p(x_t \mid y_{1:t-1}) = \sum_{x_{t-1} \in \mathcal{X}} p(x_t \mid x_{t-1})\, p(x_{t-1} \mid y_{1:t-1}) \qquad \textit{Predict}$$

$$p(x_t \mid y_{1:t}) = \frac{g_{x_t}(y_t)\, p(x_t \mid y_{1:t-1})}{\sum_{x'_t \in \mathcal{X}} g_{x'_t}(y_t)\, p(x'_t \mid y_{1:t-1})} \qquad \textit{Update}$$

Example: Hidden Mark-frog Chains - Filtering pmf



Smoothing

General problem of smoothing:
Given: observations up to time s (i.e. $y_{1:s}$ known),
Problem: compute $p(x_t|y_{1:s})$ distribution of X_t, for $t < s$.

A particular case is when we know all history of observations:

Smoothing with all history of observations

Given: observations up to time T (i.e. $y_{1:T}$ known),
Problem: compute $p(x_t|y_{1:T})$ distribution of X_t, for $t < T$.

This case can be solved using a recursive algorithm **Forward-backward Smoothing**.

Forward-backward Smoothing I

Ideas:

- $p(x_t|y_{1:T})$ can be split as:

$$p(x_t|y_{1:T}) = \frac{p(x_t, y_{1:t}) \cdot p(y_{t+1:T}|x_t)}{p(y_{1:T})}$$

- $p(y_{1:T})$ normalization constant (calculation explained later).
- Algorithm 1 Forward α-recursion can estimate $\alpha_t(x_t) = p(x_t, y_{1:t})$.
- We can approach HMM structure to generate a recursion rule for

$$\beta_t(x_t) := p(y_{t+1:T}|x_t)$$

Forward-backward Smoothing II

$$
\begin{aligned}
p(y_{t:T}|x_{t-1}) &= \sum_{x_t \in \mathcal{X}} p(y_{t:T}, x_t|x_{t-1}) \\
&= \sum_{x_t \in \mathcal{X}} p(y_t|y_{t+1:T}, x_t, x_{t-1}) p(y_{t+1:T}, x_t|x_{t-1}) \\
&= \sum_{x_t \in \mathcal{X}} p(y_t|x_t) p(y_{t+1:T}|x_t, x_{t-1}) p(x_t|x_{t-1}) \\
&= \sum_{x_t \in \mathcal{X}} p(y_t|x_t) p(y_{t+1:T}|x_t) p(x_t|x_{t-1})
\end{aligned}
$$

We have a backward recursion for $t = T, \ldots, 2$

$$
\beta_{t-1}(x_{t-1}) = \sum_{x_t \in \mathcal{X}} p(y_t|x_t) p(x_t|x_{t-1}) \beta_t(x_t), \;\; \beta_T(x_T) = 1
$$

Forward-backward Smoothing III

If $\mathcal{X} = \{1, \ldots, K\}$:

Algorithm 2 Backward β-recursion

- For $i = 1, \ldots, K$, set $\beta_T(i) = 1$
- For $t = 1, \ldots, T$
  - For $i = 1, \ldots, K$, set

$$\beta_{t-1}(i) = \sum_{j=1}^{K} g_j(y_t) A_{i,j} \beta_t(j)$$

Figure: Backward β-recursion Algorithm

Forward-backward Smoothing IV

Backward and Forward Algorithm for smoothing:

- Algorithm 1 Forward α-recursion to get $\alpha_t(x_t) = p(x_t, y_{1:t})$.
- Algorithm 2 Backward β-recursion to get $\beta_t(x_t) = p(y_{t+1:T}|x_t)$.
- Smoothing probability mass function is estimated as:

$$p(x_t|y_{1:T}) = \frac{p(x_t, y_{1:T})}{p(y_{1:T})} = \frac{\alpha_t(x_t)\beta_t(x_t)}{\sum_{x_t \in \mathcal{X}} \alpha_t(x_t)\beta_t(x_t)}$$

Notes: 1) Algorithms 1 y 2 can be run independently, 2) Backward and Forward Algorithm is from order $\mathcal{O}(T \cdot |\mathcal{X}|^2)$

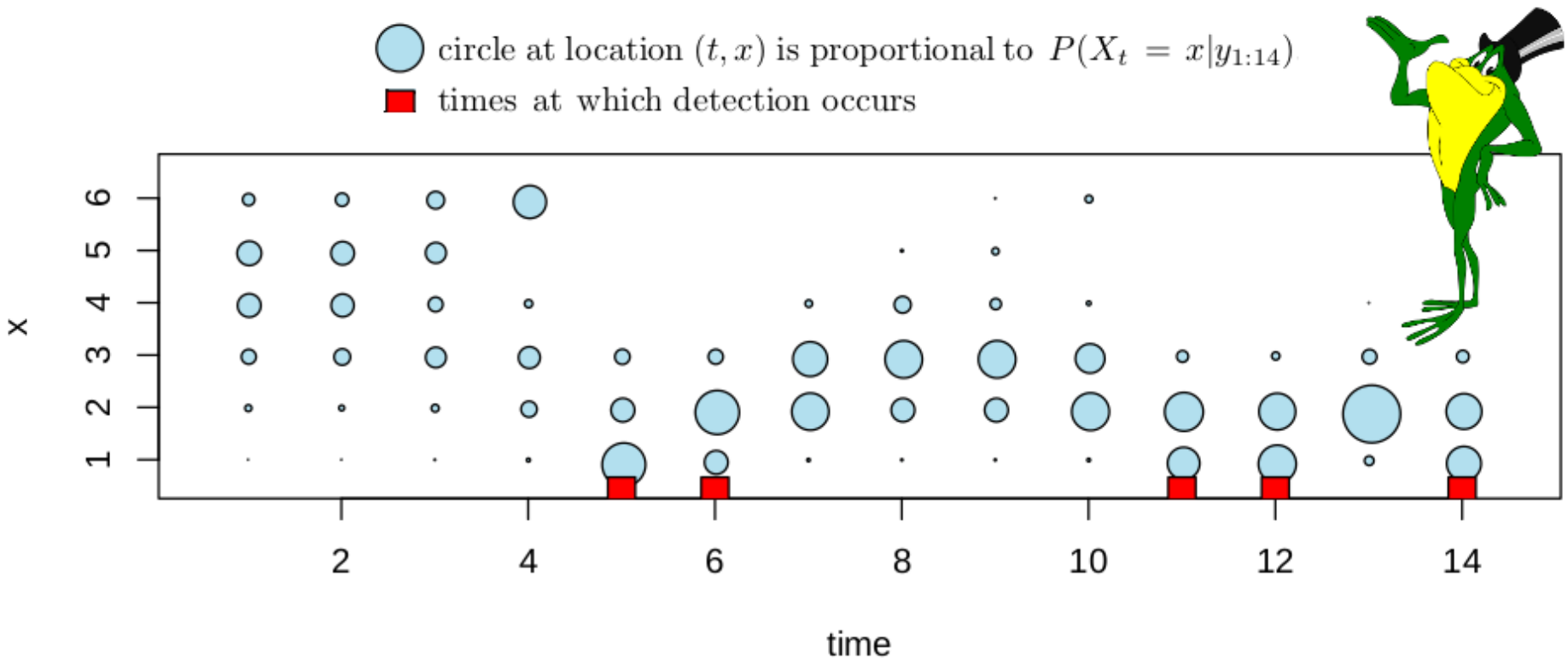
Likelihood

Given: all observations (i.e. $y_{1:T}$ known),
Problem: compute $p(y_{1:T})$ likelihood function of observations.

Ideas:

- Notice
$$p(y_{1:T}) = \sum_{x_t \in \mathcal{X}} p(x_t, y_{1:t}) \cdot p(y_{t+1:T}|x_t)$$
- This implies
$$p(y_{1:T}) = \sum_{x_t \in \mathcal{X}} \alpha_t(x_t)\beta_t(x_t)$$
- $p(y_{1:T})$ can be estimated using algorithm 1 Forward α-recursion and algorithm 2 Backward β-recursion.

Example: Hidden Mark-frog Chains - Smoothing pmf



Most likely state path/Decoding

Given: history of observations (i.e. $y_{1:T}$ known),
Problem: Find the most likely state history $x_{0:T}$.

For fixed $y_{1:T}$, solve MAP problem with **conditional distribution**:

$$\hat{x}_0 = argmax_{x_{0:T}} \; p(x_{0:T}|y_{1:T})$$

Or equivalently with **joint distribution**:

$$\hat{x}_0 = argmax_{x_{0:T}} \; p(x_{0:T}, y_{1:T})$$

Note: unfeasible if number of different state paths $|\mathcal{X}|^{T+1}$ is large from optimization point of view!!.

Most likely state path - Viterbi algorithm

However, we can use **Viterbi algorithm** to do MAP estimation.
Ideas:

- Explote structure of HMM do estimations based on backward-forward or forward-backward recursions.
- Compute messages from time t to $t-1$ $(m_{t-1}(x_{t-1}) \leftarrow m_t(x_t))$
- Use messages to estimate feasible points $t-1$ to t $(\hat{x}_{t-1} \rightarrow \hat{x}_t)$

Figure: Scheme of Vitterbi algorithm recursions

What do you mean by messages $m_t(x_t)$? Part I

Factorizing $p(x_{0:T}, y_{1:T})$:

$$p(x_{0:T}, y_{1:T}) = p(x_0) \prod_{t=1}^{T} p(x_t|x_{t-1}) p(y_t|x_t)$$

We can split our optimization problem in different optimization problems:

$$\Rightarrow \max_{x_0:x_T} p(x_{0:T}, y_{1:T}) =$$

$$\max_{x_0:x_{T-1}} \left\{ \left\{ p(x_0) \prod_{t=1}^{T-1} p(x_t|x_{t-1}) p(y_t|x_t) \right\} \max_{x_T} p(x_T|x_{T-1}) p(y_T|x_T) \right\}$$

$$= \max_{x_0:x_{T-1}} \left\{ \left\{ p(x_0) \prod_{t=1}^{T-1} p(x_t|x_{t-1}) p(y_t|x_t) \right\} m_{T-1}(x_{T-1}) \right\}$$

What do you mean by messages $m_t(x_t)$? Part II

Continuing this process we can define a set of messages based on iterative optimization problems:

$$m_{t-1}(x_{t-1}) = \max_{x_{t:T}} \left\{ \prod_{k=t}^{T} p(x_k|x_{k-1}) p(y_k|x_k) \right\} \text{ for } t = T-1, \ldots, 1$$

$$m_T(x_T) = 1$$

This satisfies the following recursion:

$$m_{t-1}(x_{t-1}) = \max_{x_t} p(y_t|x_t) p(x_t|x_{t-1}) m_t(x_t)$$

What do you mean by messages $m_t(x_t)$? Part III

Definitions of messages is good for our problem because:

$$p(x_0)m_0(x_0) = \max_{x_{1:T}} p(x_{0:T}, y_{1:T})$$

$$\Rightarrow \hat{x_0} = arg\max_{x_0}\left(\max_{x_{1:T}} p(x_{0:T}, y_{1:T})\right) = arg\max_{x_0} m_0(x_0)p(x_0)$$

And also

$$\hat{x_t} = arg\max_{x_t} p(\hat{x}_{0:t-1}, x_t, x_{t+1:T}, y_{1:T})$$

$$= arg\max_{x_t} p(\hat{x}_{t-1}, x_t, x_{t+1:T}, y_{1:T})$$

$$= arg\max_{x_t} (m_t(x_t)p(y_t|x_t)p(x_t|\hat{x}_{t-1}))$$

Most likely state path - Viterbi algorithm

Algorithm 3 Viterbi algorithm for maximum a posteriori estimation

- For $i = 1, \ldots, K$, set $m_T(i) = 1$.
- For $t = T, \ldots, 1$
  - For $i = 1, \ldots, K$, let

$$m_{t-1}(i) = \max_{j=1,\ldots,K} g_j(y_t) A_{i,j} m_t(j)$$

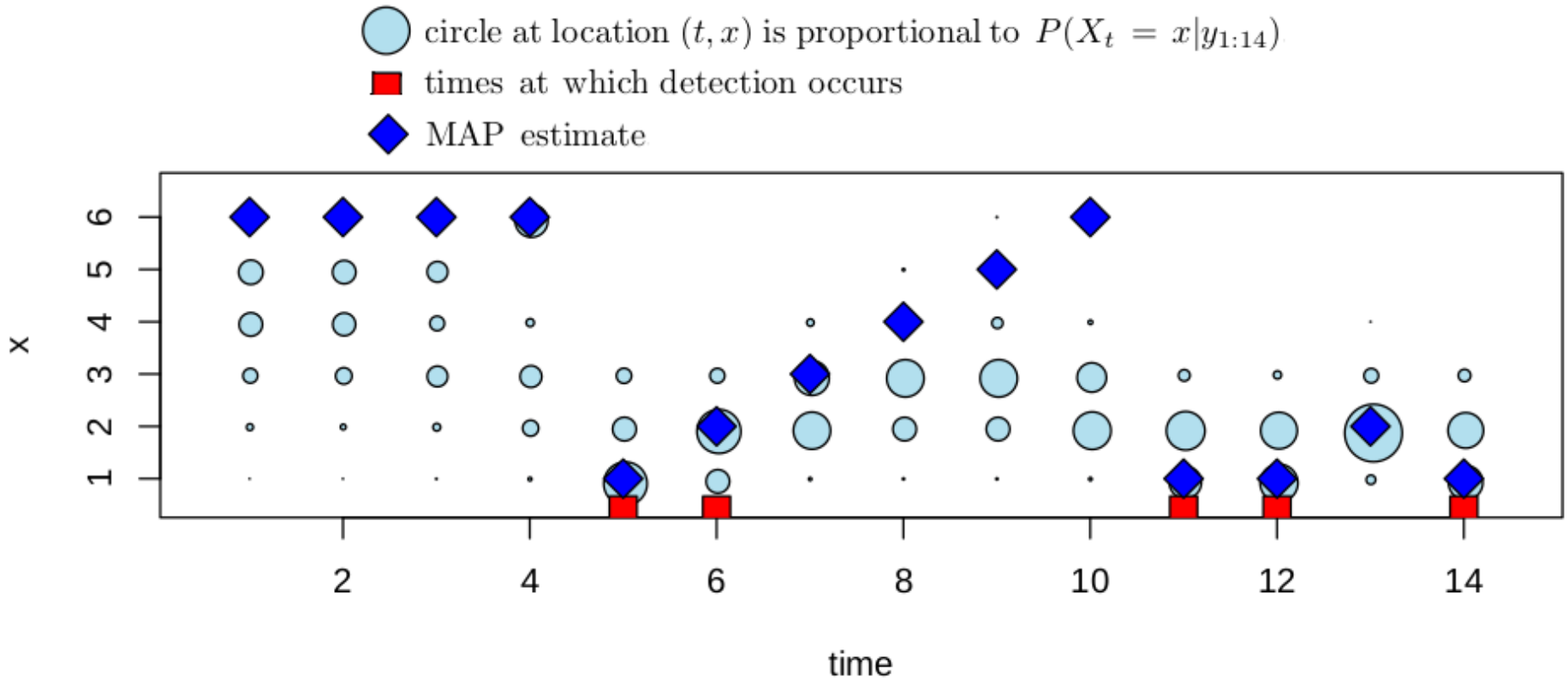
- Set $\hat{x}_0 = \arg\max_{i=1\ldots,K} \; m_0(i)\mu(i)$
- For $t = 1, \ldots, T$
  - Set

$$\hat{x}_t = \arg\max_{i=1\ldots,K} \; m_t(i) g_i(y_t) A_{\hat{x}_{t-1},i}.$$

Figure: Vitterbi algorithm

Notes: 1) Viterbi algorithm has order $\mathcal{O}(T|\mathcal{X}|^2)$, 2) In practice logarithms are computed to assure numerical stability.

Example: Hidden Mark-frog Chains - MAP pmf



Note: Image shows Smoothing pmf over time t and MAP estimate

Continuous-state Hidden Markov Models

In many problems often hidden parameters of interest are continuous.

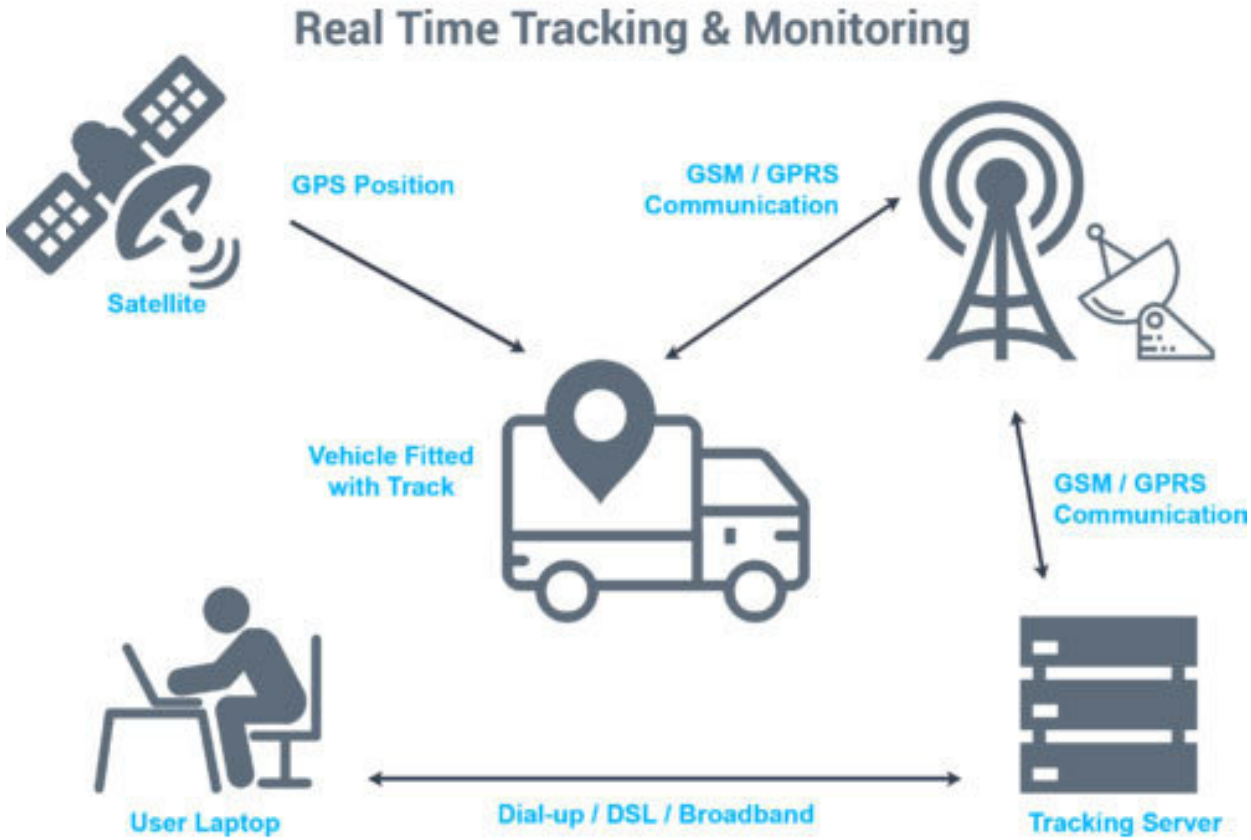


Figure: GPS object tracking; hidden parameters: position + velocity

Continuous-state Hidden Markov Models

- Ideas developed for discrete case can be generalized to aboard problems with hidden continuous parameters of interest.
- We call these **Continuous-state Hidden Markov Models** (CS-HMM), also named as state-space models or dynamical systems.
- Let's focus on a particular subset of CS-HMM: **linear Gaussian state-space model** (LGSSM).

Linear Gaussian state-space model (LGSSM)

Hidden states: $(X_0, \ldots, X_T)$ continuous r.v. taking values in $\mathbb{R}^{d_x}$
Observations: $(Y_1, \ldots, Y_T)$ continuous r.v. taking values in $\mathbb{R}^{d_y}$

$(X_0, \ldots, X_T, Y_1, \ldots, Y_T)$ is LGSSM if has two components such as:

- **state model:** X_t is a linear transformation of X_{t-1} plus a linear combination of Gaussian noise.

- **observation model:** Y_t is a linear transformation of X_t plus Gaussian noise.

Linear Gaussian state-space model (LGSSM)

Definition (LGSSM - State model)

$$X_t = F_t X_{t-1} + G_t V_t \text{ for } t = 1, \ldots, T \text{ State model} \tag{1}$$

- $X_t \in \mathbb{R}^{d_x}$ hidden state at time t,
- $F_t \in \mathbb{R}^{d_x \times d_x}$ transition state matrix at time t,
- $G_t \in \mathbb{R}^{d_x \times d_v}$ noise transfer matrix,
- $V_t \in \mathbb{R}^{d_v}$ state noise matrix, $V_t \sim \mathcal{N}(0, Q_t)$.

Linear Gaussian state-space model (LGSSM)

Definition (Observation model)

$$Y_t = H_t X_t + W_t \text{ for } t = 1, \ldots, T \qquad \textit{Observation model} \qquad (2)$$

- $Y_t \in \mathbb{R}^{d_y}$ observation at time t,
- $H_t \in \mathbb{R}^{d_x \times d_x}$ observation matrix at time t,
- $W_t \in \mathbb{R}^{d_v}$ observation noise, $W_t \sim \mathcal{N}(0, R_t)$.

Definition (Other conditions)

- $X_0 \sim \mathcal{N}(\mu_0, \Sigma_0)$,
- $V_t \sim \mathcal{N}(0, Q_t)$, $W_t \sim \mathcal{N}(0, R_t)$, *for* $t = 1, \ldots T$,
- $X_0, V_1, \ldots V_T, W_1, \ldots W_T$ *are independent.*

LGSSM and parametrization of joint pdf

As in discrete case, joint pdf of hidden variables and observation can be described as:

$$p(X_{0:T}, Y_{1:T}) = \prod_{t=1}^{T} p(y_t|x_t) \cdot p(x_{t-1}|x_t) \tag{3}$$

Using **LGSSM structure** and **properties of multivariate normal distributions** it can be shown that if $G_t Q_T G_T \in \mathbb{R}^{d_x \times d_x}$ has full rank then:

$$p(y_t|x_t) = \mathcal{N}(H_t x_t, R_t) \tag{4}$$

$$p(x_t|x_{t-1}) = \mathcal{N}(F_t x_{t-1}, G_t Q_T G_T) \tag{5}$$

Inference in dynamic LGSSM

Let's consider the following inference problem:
P1: Given **observations up to time** t find $Y_1 = y_1, \ldots, Y_t = y_t$, find joint pdf $p(x_t|y_{1:t})$.

P2: Given **all observations** $Y_1 = y_1, \ldots, Y_T = y_T$, find joint pdf $p(x_t|y_{1:T})$

Both can be solve by **sequentially computation of means and covariance matrices of conditionally distributions**: 1) P1 is called **Kallman filter**, and in 2) P2 is named **Kallman smoother**.

Inference LGSSM - Kallman filter

P1: Determine $p(x_t|y_{1:t})$ of the hidden state X_t given t observations $y_{1:t}$.

$$\mu_{t|t-1} := E[X_t|Y_{1:t} = y_{1:t-1}] \tag{6}$$

$$\mu_{t|t} := E[X_t|Y_{1:t} = y_{1:t}] \tag{7}$$

$$\Sigma_{t|t-1} := E[(X_t - \mu_{t|t-1})(X_t - \mu_{t|t-1})^T|Y_{1:t} = y_{1:t-1}] \tag{8}$$

$$\Sigma_{t|t} := E[(X_t - \mu_{t|t})(X_t - \mu_{t|t})^T|Y_{1:t} = y_{1:t}] \tag{9}$$

Kallman filter ideas - Prediction

Structure of LGSSM implies $p(x_t|y_{1:t-1}) = \mathcal{N}(\mu_{t|t-1}, \Sigma_{t|t-1})$ and the existence of recursive rules:

Prediction $(\mu_{t-1|t-1} \to \mu_{t|t-1})$ and $(\Sigma_{t-1|t-1} \to \Sigma_{t|t-1})$:

- $\mu_{t|t-1} = F_t \mu_{t-1|t-1}$,
- $\Sigma_{t|t-1} = F_t \Sigma_{t-1|t-1} F_t^T + G_t Q_t G_t^T$

Kallman filter ideas - Update/correction

Again, structure of LGSSM implies $p(x_t|y_{1:t}) = \mathcal{N}(\mu_{t|t}, \Sigma_{t|t})$ and following recursion rule:

Update/correction $(\mu_{t|t-1} \to \mu_{t|t})$ and $(\Sigma_{t|t-1} \to \Sigma_{t|t})$:

- $\mu_{t|t} = \mu_{t|t-1} + K_t \nu_t$,
- $\Sigma_{t|t-1} = (I - K_t H_t)\Sigma_{t-1|t-1}$

Where $\nu_t = y_t - \hat{y}_t$, $\hat{y}_t = E[Y_t | Y_{1:t-1} = y_{1:t-1}] = H_t \mu_{t|t-1}$
$K_t = \Sigma_{t|t-1} H^T S_t^{-1}$,
$S_t = E[(Y_t - \hat{y}_t)(Y_t - \hat{y}_t)^T | Y_{1:t-1} = y_{1:t-1}] = H_t \Sigma_{t|t-1} H_t^T + R_t$
K_t is called **Kallman gain**

Kallman filter ideas

Recursive strategy of Kallman filter

$p(x_t|y_{1:t})$ can be determined estimating parameters for $s \in \mathbb{N}$ of:

- $p(x_s|y_{1:s-1}) = \mathcal{N}(\mu_{s|s-1}, \Sigma_{s|s-1})$
- $p(x_s|y_{1:s}) = \mathcal{N}(\mu_{s|s}, \Sigma_{s|s})$

as follows:

$$(\mu_0, \Sigma_0) \xrightarrow{\text{Predict.}} (\mu_{1|0}, \Sigma_{1|0}) \xrightarrow{\text{Update}} \dots$$

$$\dots \xrightarrow{\text{Predict.}} (\mu_{t-1|t-1}, \Sigma_{t-1|t-1}) \xrightarrow{\text{Update}}$$

$$(\mu_{t|t-1}, \Sigma_{t|t-1}) \xrightarrow{\text{Predict.}} (\mu_{t|t}, \Sigma_{t|t}) \to \dots$$

Inference LGSSM - Kallman Smoother

P2: Determine $p(x_t|y_{1:T})$ of the hidden state X_t given all the observations $y_{1:T}$.

$$\mu_{t|T} := E[X_t|Y_{1:T} = y_{1:T}] \tag{10}$$

$$\Sigma_{t|T} := E[(X_t - \mu_{t|T})(X_t - \mu_{t|T})^T|Y_{1:T} = t_{1:T}] \tag{11}$$

Kallman Smoother ideas

Structure of LGSSM implies $p(x_t|y_{1:T}) = \mathcal{N}(\mu_{t|T}, \Sigma_{t|T})$ and the existence of recursive rules:

Backward recursion:

- $\mu_{t|T} = \mu_{t|t} + J_t(\mu_{t+1|T} - \mu_{t+1|t})$,
- $\Sigma_{t|T} = \Sigma_{t|t} + J_t(\Sigma_{t+1|T} - \Sigma_{t+1|t})J_t^T$

Where $J_t = \Sigma_{t|t}F_{t+1}^T\Sigma_{t+1|t}^{-1}$, is called **Backwards Kallman gain**.

Recursive strategy of Kallman Smoother

$p(x_t|y_{1:T})$ can be determined estimating parameters as follows:

- Compute $(\mu_{t|t}, \Sigma_{t|t})$ and $(\mu_{t+1|t}, \Sigma_{t+1|t})$ for $t, t+1 \leq T$ using Kallman filter.
- Use backward recursion until obtain $(\mu_{t|T}, \Sigma_{t|T})$.

A example for LGSSM

Let's consider a toy example for a random walk given by:

$$X_t = X_{t-1} + V_t$$

$$Y_t = X_t + W_t$$

Where

- X_t: trigonometric function plus gaussian noise, for $t = 1, \ldots, 50$.
- $X_0 = \mathcal{N}(0, 1)$, $V_t \sim \mathcal{N}(0, Q)$, $W_t \sim \mathcal{N}(0, R)$, with $Q = 0.02$ and $R = 0.2$

Filtering

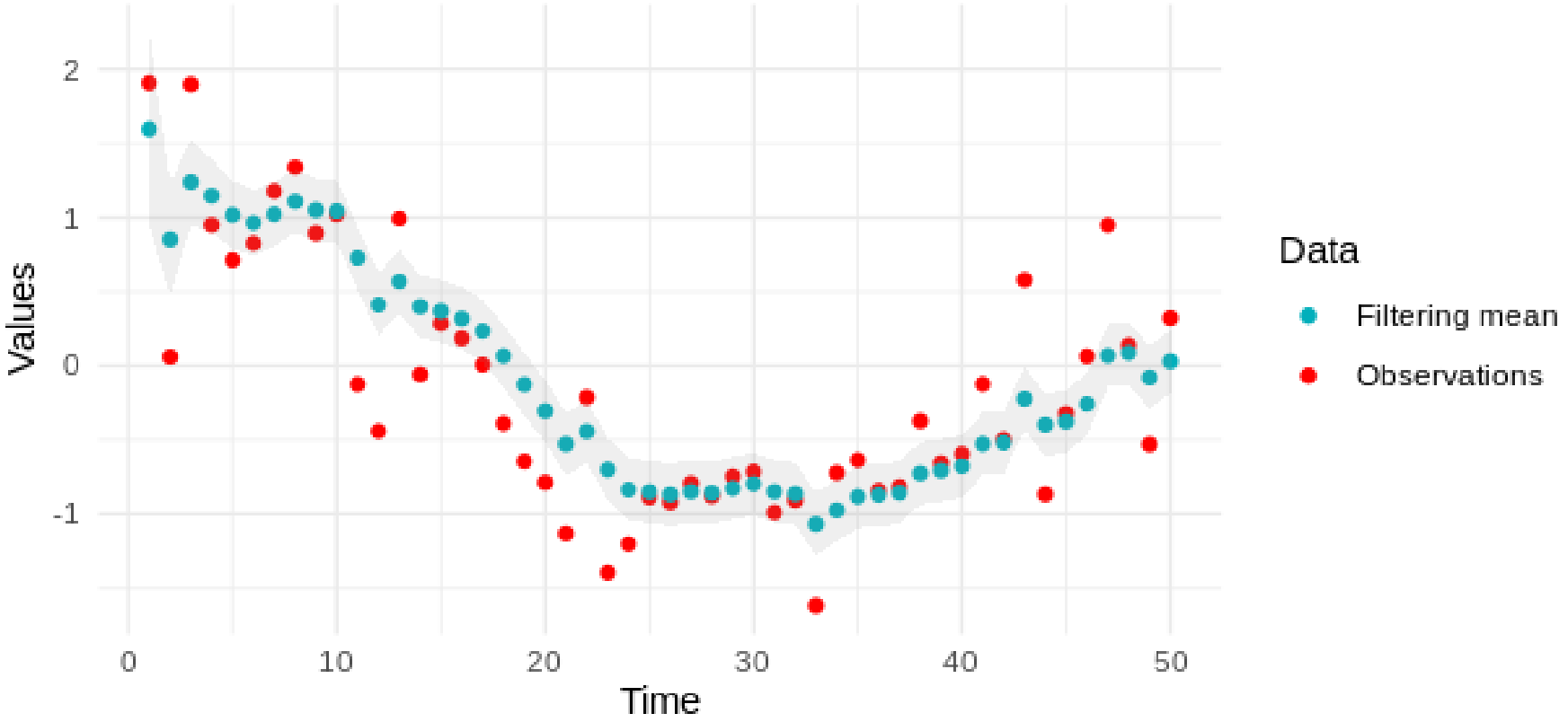


Figure: Observations and filtering mean and 99% credible intervals over time.

Smoothing

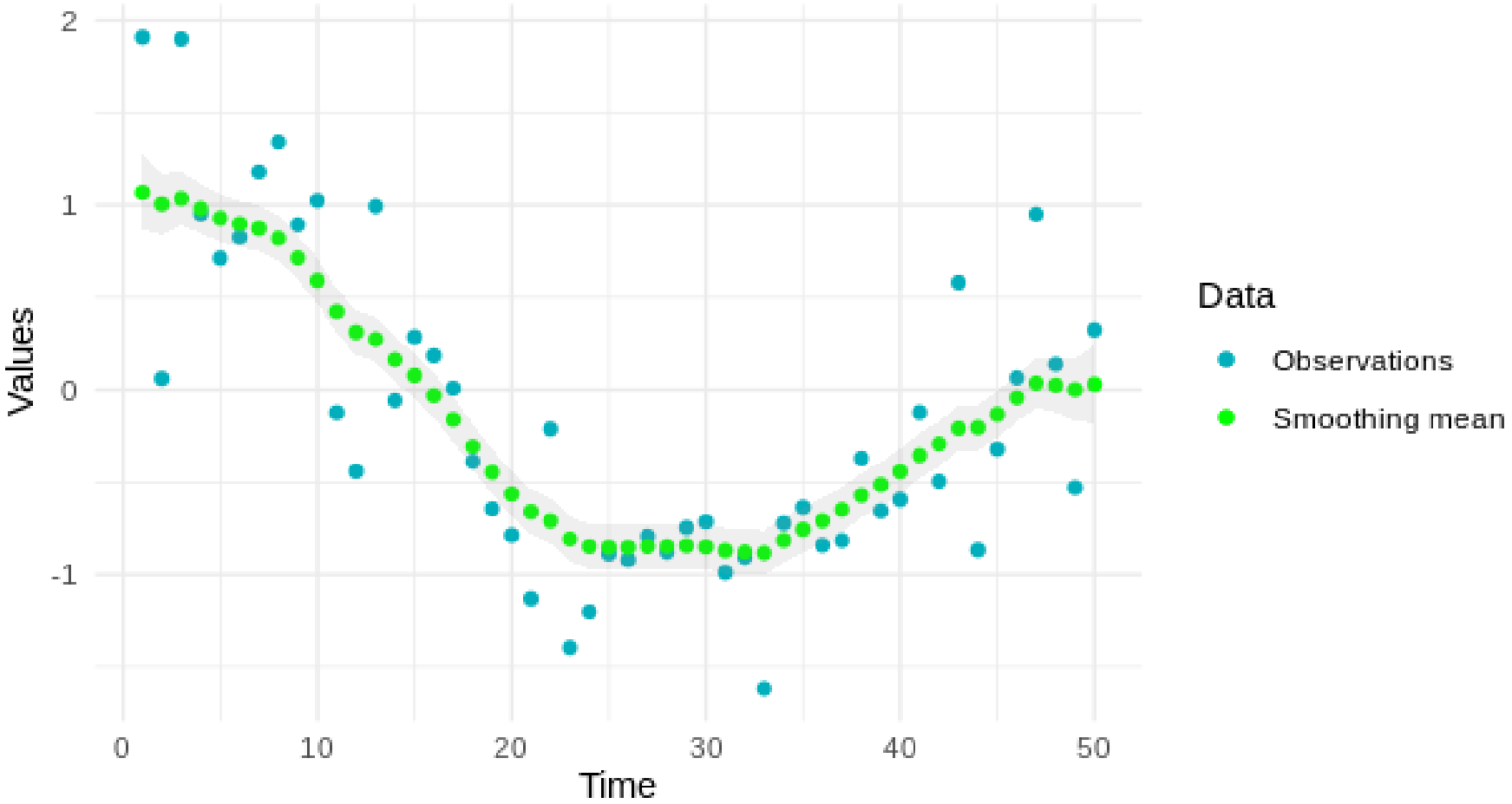


Figure: Observations and smoothing mean and 99% credible intervals over time