Fermion Transport for the Two-site Model

The Hamiltonian is $H = -wc_L^\dagger c_R - wc_R^\dagger c_L$. Then we can define Hermitian Majorana operators according to Eq. (5) in Prosen's paper:

$$w_1 = c_L + c_L^\dagger$$

$$w_2 = i(c_L - c_L^\dagger)$$

$$w_3 = c_R + c_R^\dagger$$

$$w_4 = i(c_R - c_R^\dagger)$$

satisfying $\{w_j, w_k\} = \delta_{j,k}$. In terms of these operators, the Hamiltonian becomes:

$$\begin{aligned} H &= -\frac{w}{4}((w_1 + iw_2)(w_3 - iw_4) + (w_3 + iw_4)(w_1 - iw_2)) \\ &= -\frac{w}{4}(w_1w_3 + iw_2w_3 - iw_1w_4 + w_2w_4 + w_3w_1 + iw_4w_1 - iw_3w_2 + w_4w_2) \\ &= \frac{iw}{4}(w_1w_4 - w_4w_1 + w_3w_2 - w_2w_3) = \frac{iw}{2}(w_1w_4 + w_3w_2) \end{aligned}$$

This can be written as a matrix

$$H = \begin{bmatrix} 0 & 0 & 0 & \frac{iw}{4} \\ 0 & 0 & -\frac{iw}{4} & 0 \\ 0 & \frac{iw}{4} & 0 & 0 \\ -\frac{iw}{4} & 0 & 0 & 0 \end{bmatrix}$$

which is clearly Hermitian.

Next, we want to represent the bath operators. We let $\epsilon_L = \epsilon_R = 0$, $\mu_L = -\mu_R$, and $\beta_L = \beta_R = \beta$. Therefore the bath operators are $v_L = v_R = \sqrt{\gamma}$ and $u_L = \sqrt{\gamma e^{-\beta\mu_L}}$, $u_R = \sqrt{\gamma e^{-\beta\mu_R}}$. With the notation of our code, we can write $t = e^{-\beta\mu_L}$, and hence $u_L = \sqrt{\gamma t}$, $u_R = \sqrt{\frac{\gamma}{t}}$. The bath operators are

$$L_1 = u_R^\dagger c_R = \frac{u_R}{2}(w_3 - iw_4)$$

$$L_2 = u_L^\dagger c_L = \frac{u_L}{2}(w_1 - iw_2)$$

$$L_3 = v_R c_R^\dagger = \frac{\sqrt{\gamma}}{2}(w_3 + i w_4)$$

$$L_4 = v_L c_L^\dagger = \frac{\sqrt{\gamma}}{2}(w_1 + i w_2)$$

To rewrite these in the form $L_i = l_i \cdot w$, we use

$$l_1 = [0, 0, \frac{u_R}{2}, \frac{-iu_R}{2}]$$

$$l_2 = [\frac{u_L}{2}, \frac{-iu_L}{2}, 0, 0]$$

$$l_3 = [0, 0, \frac{\sqrt{\gamma}}{2}, \frac{i\sqrt{\gamma}}{2}]$$

$$l_4 = [\frac{\sqrt{\gamma}}{2}, \frac{i\sqrt{\gamma}}{2}, 0, 0]$$

The complex Hermitian matrix M given by $M_{jk} = \sum_\mu l_{\mu,j} l^*_{\mu,k}$ (Eq. (23)) becomes:

$$M = \begin{bmatrix} \frac{u_L^2+\gamma}{4} & \frac{i(u_L^2-\gamma)}{4} & 0 & 0 \\ \frac{i(-u_L^2+\gamma)}{4} & \frac{u_L^2+\gamma}{4} & 0 & 0 \\ 0 & 0 & \frac{u_R^2+\gamma}{4} & \frac{i(u_R^2-\gamma)}{4} \\ 0 & 0 & \frac{i(-u_R^2+\gamma)}{4} & \frac{u_R^2+\gamma}{4} \end{bmatrix}$$

Similarly, the complex antisymmetric matrix A defined in Eq. (27) is given by

$$A = \begin{bmatrix} 0 & \frac{i(u_L^2+\gamma)}{2} & \frac{i(-u_L^2+\gamma)}{2} & \frac{u_L^2-\gamma}{2} & 0 & 0 & \frac{w}{2} & 0 \\ -\frac{i(u_L^2+\gamma)}{2} & 0 & \frac{u_L^2-\gamma}{2} & \frac{i(u_L^2-\gamma)}{2} & 0 & 0 & 0 & \frac{w}{2} \\ \frac{i(u_L^2-\gamma)}{2} & \frac{-u_L^2+\gamma}{2} & 0 & \frac{i(u_L^2+\gamma)}{2} & -\frac{w}{2} & 0 & 0 & 0 \\ \frac{-u_L^2+\gamma}{2} & \frac{i(-u_L^2+\gamma)}{2} & -\frac{i(u_L^2+\gamma)}{2} & 0 & 0 & -\frac{w}{2} & 0 & 0 \\ 0 & 0 & \frac{w}{2} & 0 & 0 & \frac{i(u_R^2+\gamma)}{2} & \frac{i(-u_R^2+\gamma)}{2} & \frac{u_R^2-\gamma}{2} \\ 0 & 0 & 0 & \frac{w}{2} & -\frac{i(u_R^2+\gamma)}{2} & 0 & \frac{u_R^2-\gamma}{2} & \frac{i(u_R^2-\gamma)}{2} \\ -\frac{w}{2} & 0 & 0 & 0 & \frac{i(u_R^2-\gamma)}{2} & \frac{-u_R^2+\gamma}{2} & 0 & \frac{i(u_R^2+\gamma)}{2} \\ 0 & -\frac{w}{2} & 0 & 0 & \frac{-u_R^2+\gamma}{2} & \frac{i(-u_R^2+\gamma)}{2} & -\frac{i(u_R^2+\gamma)}{2} & 0 \end{bmatrix}$$

Using the fact that $u_L^2 = \gamma t$ and $u_R^2 = \frac{\gamma}{t}$, we can rewrite the above matrix

$$A = \begin{bmatrix} 0 & \frac{i\gamma(t+1)}{2} & \frac{i\gamma(-t+1)}{2} & \frac{\gamma(t-1)}{2} & 0 & 0 & \frac{w}{2} & 0 \\ -\frac{i\gamma(t+1)}{2} & 0 & \frac{\gamma(t-1)}{2} & \frac{i\gamma(t-1)}{2} & 0 & 0 & 0 & \frac{w}{2} \\ \frac{\gamma(t-1)}{2} & \frac{\gamma(-t+1)}{2} & 0 & \frac{i\gamma(t+1)}{2} & -\frac{w}{2} & 0 & 0 & 0 \\ \frac{\gamma(-t+1)}{2} & \frac{i\gamma(-t+1)}{2} & -\frac{i\gamma(t+1)}{2} & 0 & 0 & -\frac{w}{2} & 0 & 0 \\ 0 & 0 & \frac{w}{2} & 0 & 0 & \frac{i\gamma(1+t)}{2t} & \frac{i\gamma(-1+t)}{2t} & \frac{\gamma(1-t)}{2t} \\ 0 & 0 & 0 & \frac{w}{2} & -\frac{i\gamma(1+t)}{2t} & 0 & \frac{\gamma(1-t)}{2t} & \frac{i\gamma(1-t)}{2t} \\ -\frac{w}{2} & 0 & 0 & 0 & \frac{i\gamma(1-t)}{2t} & \frac{\gamma(-1+t)}{2t} & 0 & \frac{i\gamma(1+t)}{2t} \\ 0 & -\frac{w}{2} & 0 & 0 & \frac{\gamma(-1+t)}{2t} & \frac{i\gamma(-1+t)}{2t} & -\frac{i\gamma(1+t)}{2t} & 0 \end{bmatrix}$$

Since A is an antisymmetric matrix of even dimension, its eigenvalues come in pairs $(\beta, -\beta)$. Arrange the eigenvalues as $\beta_1, -\beta_1, ..., \beta_4, -\beta_4$, $Re(\beta_1) \geq ... \geq Re(\beta_4)$, and let $v_1, ..., v_8$ be the corresponding eigenvectors. We would like to normalize the eigenvectors such that they satisfy Eq. (30), $v_r \cdot v_s = J_{rs}$, where

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The eigenvectors computed by MATLAB are such that $v_j \cdot v_j = 0$ and $v_{2j-1} \cdot v_{2j} = 1$. However, because the spectrum $\{\beta_j\}$ is degenerate, we are not guaranteed that $v_{2j} \cdot v_k = 0$ for all $k \neq 2j - 1$. Therefore, we must consider a linear combination of v_js.

When we have multiple eigenvectors with the same eigenvalues, we put them next to each other in blocks $\beta_1, -\beta_1, \beta_1, -\beta_1, ...$ Let the corresponding eigenvectors in this block be $a_1, b_1, a_2, b_2, ...$, such that a_j is the eigenvector for the j^{th} positive eigenvalue β_1 and b_j is the eigenvector for the j^{th} negative eigenvalue $-\beta_1$. We are given that $a_i \cdot a_j = 0$ and $b_i \cdot b_j = 0$. Define linear combinations $c_i = \sum_j U_{ij} b_j$. Note that c_i is also an eigenvector with eigenvalue $-\beta_1$. We want c_js to satisfy $a_i \cdot c_j = \delta_{i,j}$. Hence

$$\sum_k U_{jk} a_i \cdot b_k = \delta_{i,j}$$

Define matrix T:

$$T_{ik} = a_i \cdot b_k = \sum_\alpha a_i^\alpha b_k^\alpha$$

$$\sum_k (U)^T_{kj} T_{ik} = (TU^T)_{ij} = \delta_{i,j}$$

Therefore $TU^T = I$ and $U = (T^{-1})^T$. Now we can use the matrix U to compute c_j as shown above. As a result of this procedure, we are left with eigenvectors $a_1, c_1, a_2, c_2, ...$

For this particular matrix A, the eigenvalues are

$$\beta_1 = \beta_2 = \frac{\gamma(t+1)^2 + \sqrt{\gamma^2(t^2-1)^2 - 4t^2w^2}}{4t}$$

$$\beta_3 = \beta_4 = \frac{\gamma(t+1)^2 - \sqrt{\gamma^2(t^2-1)^2 - 4t^2w^2}}{4t}$$

Notice that for $\gamma \gg w$, the above expressions reduce to

$$\beta_1 = \beta_2 \approx \frac{\gamma(t+1)}{2}$$

$$\beta_3 = \beta_4 \approx \frac{\gamma(t+1)}{2t}$$

After applying these changes to all blocks of equal eigenvalues (in this case 2 blocks with 4 entries each), we introduce the matrix V whose j^{th} row is the j^{th} eigenvector. This matrix is used to compute quadratic observables as described in Eq. (47) of Prosen's paper. Our observable is the current

$$\begin{aligned}
I &=< -i(c_L c_R^\dagger - c_R c_L^\dagger) > \\
&=< \frac{1}{4i}((w_1 - iw_2)(w_3 + iw_4) - (w_3 - iw_4)(w_1 + iw_2)) > \\
&=< \frac{1}{4i}(w_1w_3 - iw_2w_3 + iw_1w_4 + w_2w_4 - w_3w_1 + iw_4w_1 - iw_3w_2 - w_4w_2) > \\
&= \frac{1}{2i}(< w_1w_3 > + < w_2w_4 >)
\end{aligned}$$

where the expectation values are given by Eq. (47)

$$< w_1w_3 >= \frac{1}{2}\sum_{m=1}^{2n}(V_{2m,1}V_{2m-1,5} - V_{2m,2}V_{2m-1,6} - iV_{2m,2}V_{2m-1,5} - iV_{2m,1}V_{2m-1,6})$$

$$< w_2w_4 >= \frac{1}{2}\sum_{m=1}^{2n}(V_{2m,3}V_{2m-1,7} - V_{2m,4}V_{2m-1,8} - iV_{2m,4}V_{2m-1,7} - iV_{2m,3}V_{2m-1,8})$$

On the other hand, the current can be computed analytically, and is given by

$$I = -\frac{2w}{\gamma(1+r_0)}\frac{\delta r}{\frac{4w^2}{\gamma^2} + (1+r_0)^2 - \delta r^2}$$

where $r_0 + \delta r = e^{-\beta\mu_L} = t$ and $r_0 - \delta r = e^{-\beta\mu_R} = \frac{1}{t}$. Expressing the above relation in terms of t, after a little algebra, we get

$$I = -\frac{2w}{(1+t)} \frac{(t-1)}{\frac{4w^2}{\gamma} + \frac{\gamma(1+t)^2}{t}}$$

As of right now, the two methods agree on their value of I only in the case $\gamma \gg w$ (e.g. $\gamma = 1$, $w = 0.1$).