Fermion Transport for the Two-site Model

The Hamiltonian is $H = -wc_L^\dagger c_R - wc_R^\dagger c_L$. Then we can define Hermitian Majorana operators according to Eq. (5) in Prosen's paper:

$$w_1 = c_L + c_L^\dagger$$

$$w_2 = i(c_L - c_L^\dagger)$$

$$w_3 = c_R + c_R^\dagger$$

$$w_4 = i(c_R - c_R^\dagger)$$

satisfying $\{w_j, w_k\} = 2\delta_{j,k}$. In terms of these operators, the Hamiltonian becomes:

$$\begin{aligned}
H &= -\frac{w}{4}((w_1 + iw_2)(w_3 - iw_4) + (w_3 + iw_4)(w_1 - iw_2)) \\
&= -\frac{w}{4}(w_1w_3 + iw_2w_3 - iw_1w_4 + w_2w_4 + w_3w_1 + iw_4w_1 - iw_3w_2 + w_4w_2) \\
&= \frac{iw}{4}(w_1w_4 - w_4w_1 + w_3w_2 - w_2w_3) = \frac{iw}{2}(w_1w_4 + w_3w_2)
\end{aligned}$$

This can be written as a matrix

$$H = \begin{bmatrix} 0 & 0 & 0 & \frac{iw}{4} \\ 0 & 0 & -\frac{iw}{4} & 0 \\ 0 & \frac{iw}{4} & 0 & 0 \\ -\frac{iw}{4} & 0 & 0 & 0 \end{bmatrix}$$

which is clearly Hermitian.

Next, we want to represent the bath operators. We let $\epsilon_L = \epsilon_R = 0$, $\mu_L = -\mu_R$, and $\beta_L = \beta_R = \beta$. Therefore the bath operators are $v_L = v_R = \sqrt{\gamma}$ and $u_L = \sqrt{\gamma e^{-\beta\mu_L}}$, $u_R = \sqrt{\gamma e^{-\beta\mu_R}}$. With the notation of our code, we can write $t = e^{-\beta\mu_L}$, and hence $u_L = \sqrt{\gamma t}$, $u_R = \sqrt{\frac{\gamma}{t}}$. The bath operators are

$$L_1 = \frac{1}{\sqrt{2}}u_R^\dagger c_R = \frac{u_R}{2\sqrt{2}}(w_3 - iw_4)$$

$$L_2 = \frac{1}{\sqrt{2}}u_L^\dagger c_L = \frac{u_L}{2\sqrt{2}}(w_1 - iw_2)$$

$$L_3 = \frac{1}{\sqrt{2}} v_R c_R^\dagger = \frac{\sqrt{\gamma}}{2\sqrt{2}}(w_3 + i w_4)$$

$$L_4 = \frac{1}{\sqrt{2}} v_L c_L^\dagger = \frac{\sqrt{\gamma}}{2\sqrt{2}}(w_1 + i w_2)$$

Note that the additional factor of $\frac{1}{\sqrt{2}}$ comes from the discrepancy in the Lindblad equation as defined in Prosen's paper and in the transport paper. The equation in Prosen's paper has an extra factor of 2, for which we compensate above. To rewrite these in the form $L_i = l_i \cdot w$, we use

$$l_1 = \frac{1}{\sqrt{2}}[0, 0, \frac{u_R}{2}, \frac{-i u_R}{2}]$$

$$l_2 = \frac{1}{\sqrt{2}}[\frac{u_L}{2}, \frac{-i u_L}{2}, 0, 0]$$

$$l_3 = \frac{1}{\sqrt{2}}[0, 0, \frac{\sqrt{\gamma}}{2}, \frac{i\sqrt{\gamma}}{2}]$$

$$l_4 = \frac{1}{\sqrt{2}}[\frac{\sqrt{\gamma}}{2}, \frac{i\sqrt{\gamma}}{2}, 0, 0]$$

The complex Hermitian matrix M given by $M_{jk} = \sum_\mu l_{\mu,j} l^*_{\mu,k}$ (Eq. (23)) becomes:

$$M = \begin{bmatrix} \frac{u_L^2+\gamma}{8} & \frac{i(u_L^2-\gamma)}{8} & 0 & 0 \\ \frac{i(-u_L^2+\gamma)}{8} & \frac{u_L^2+\gamma}{8} & 0 & 0 \\ 0 & 0 & \frac{u_R^2+\gamma}{8} & \frac{i(u_R^2-\gamma)}{8} \\ 0 & 0 & \frac{i(-u_R^2+\gamma)}{8} & \frac{u_R^2+\gamma}{8} \end{bmatrix}$$

Similarly, the complex antisymmetric matrix A defined in Eq. (27) is given by

$$A = \begin{bmatrix} 0 & \frac{i(u_L^2+\gamma)}{4} & \frac{i(-u_L^2+\gamma)}{4} & \frac{u_L^2-\gamma}{4} & 0 & 0 & \frac{w}{2} & 0 \\ -\frac{i(u_L^2+\gamma)}{4} & 0 & \frac{u_L^2-\gamma}{4} & \frac{i(u_L^2-\gamma)}{4} & 0 & 0 & 0 & \frac{w}{2} \\ \frac{i(u_L^2-\gamma)}{4} & \frac{-u_L^2+\gamma}{4} & 0 & \frac{i(u_L^2+\gamma)}{4} & -\frac{w}{2} & 0 & 0 & 0 \\ \frac{-u_L^2+\gamma}{4} & \frac{i(-u_L^2+\gamma)}{4} & -\frac{i(u_L^2+\gamma)}{4} & 0 & 0 & -\frac{w}{2} & 0 & 0 \\ 0 & 0 & \frac{w}{2} & 0 & 0 & \frac{i(u_R^2+\gamma)}{4} & \frac{i(-u_R^2+\gamma)}{4} & \frac{u_R^2-\gamma}{4} \\ 0 & 0 & 0 & \frac{w}{2} & -\frac{i(u_R^2+\gamma)}{4} & 0 & \frac{u_R^2-\gamma}{4} & \frac{i(u_R^2-\gamma)}{4} \\ -\frac{w}{2} & 0 & 0 & 0 & \frac{i(u_R^2-\gamma)}{4} & \frac{-u_R^2+\gamma}{4} & 0 & \frac{i(u_R^2+\gamma)}{4} \\ 0 & -\frac{w}{2} & 0 & 0 & \frac{-u_R^2+\gamma}{4} & \frac{i(-u_R^2+\gamma)}{4} & -\frac{i(u_R^2+\gamma)}{4} & 0 \end{bmatrix}$$

Using the fact that $u_L^2 = \gamma t$ and $u_R^2 = \dfrac{\gamma}{t}$, we can rewrite the above matrix

$$A = \begin{bmatrix} 0 & \frac{i\gamma(t+1)}{4} & \frac{i\gamma(-t+1)}{4} & \frac{\gamma(t-1)}{4} & 0 & 0 & \frac{w}{2} & 0 \\ -\frac{i\gamma(t+1)}{4} & 0 & \frac{\gamma(t-1)}{4} & \frac{i\gamma(t-1)}{4} & 0 & 0 & 0 & \frac{w}{2} \\ \frac{\gamma(t-1)}{4} & \frac{\gamma(-t+1)}{4} & 0 & \frac{i\gamma(t+1)}{4} & -\frac{w}{2} & 0 & 0 & 0 \\ \frac{\gamma(-t+1)}{4} & \frac{i\gamma(-t+1)}{4} & -\frac{i\gamma(t+1)}{4} & 0 & 0 & -\frac{w}{2} & 0 & 0 \\ 0 & 0 & \frac{w}{2} & 0 & 0 & \frac{i\gamma(1+t)}{4t} & \frac{i\gamma(-1+t)}{4t} & \frac{\gamma(1-t)}{4t} \\ 0 & 0 & 0 & \frac{w}{2} & -\frac{i\gamma(1+t)}{4t} & 0 & \frac{\gamma(1-t)}{4t} & \frac{i\gamma(1-t)}{4t} \\ -\frac{w}{2} & 0 & 0 & 0 & \frac{i\gamma(1-t)}{4t} & \frac{\gamma(-1+t)}{4t} & 0 & \frac{i\gamma(1+t)}{4t} \\ 0 & -\frac{w}{2} & 0 & 0 & \frac{\gamma(-1+t)}{4t} & \frac{i\gamma(-1+t)}{4t} & -\frac{i\gamma(1+t)}{4t} & 0 \end{bmatrix}$$

Since A is an antisymmetric matrix of even dimension, its eigenvalues come in pairs $(\beta, -\beta)$. Arrange the eigenvalues as $\beta_1, -\beta_1, ..., \beta_4, -\beta_4$, $Re(\beta_1) \geq ... \geq Re(\beta_4)$, and let $v_1, ..., v_8$ be the corresponding eigenvectors. We would like to normalize the eigenvectors such that they satisfy Eq. (30), $v_r \cdot v_s = J_{rs}$, where

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The eigenvectors computed by MATLAB are such that $v_j \cdot v_j = 0$ and $v_{2j-1} \cdot v_{2j} = 1$. However, because the spectrum $\{\beta_j\}$ is degenerate, we are not guaranteed that $v_{2j} \cdot v_k = 0$ for all $k \neq 2j - 1$. Therefore, we must consider a linear combination of v_js.

When we have multiple eigenvectors with the same eigenvalues, we put them next to each other in blocks $\beta_1, -\beta_1, \beta_1, -\beta_1, ...$ Let the corresponding eigenvectors in this block be $a_1, b_1, a_2, b_2, ...,$ such that a_j is the eigenvector for the j^{th} positive eigenvalue β_1 and b_j is the eigenvector for the j^{th} negative eigenvalue $-\beta_1$. We are given that $a_i \cdot a_j = 0$ and $b_i \cdot b_j = 0$. Define linear combinations $c_i = \sum_j U_{ij} b_j$. Note that c_i is also an eigenvector with eigenvalue $-\beta_1$. We want c_js to satisfy $a_i \cdot c_j = \delta_{i,j}$. Hence

$$\sum_k U_{jk} a_i \cdot b_k = \delta_{i,j}$$

Define matrix T:

$$T_{ik} = a_i \cdot b_k = \sum_\alpha a_i^\alpha b_k^\alpha$$

$$\sum_k (U)^T_{kj} T_{ik} = (TU^T)_{ij} = \delta_{i,j}$$

Therefore $TU^T = I$ and $U = (T^{-1})^T$. Now we can use the matrix U to compute c_j as shown above. As a result of this procedure, we are left with eigenvectors $a_1, c_1, a_2, c_2, ...$

For this particular matrix A, the eigenvalues are

$$\beta_1 = \beta_2 = \frac{\gamma(t+1)^2 + \sqrt{\gamma^2(t^2-1)^2 - 16t^2w^2}}{8t}$$

$$\beta_3 = \beta_4 = \frac{\gamma(t+1)^2 - \sqrt{\gamma^2(t^2-1)^2 - 16t^2w^2}}{8t}$$

Notice that for $\gamma \gg w$, the above expressions reduce to

$$\beta_1 = \beta_2 \approx \frac{\gamma(t+1)}{4}$$

$$\beta_3 = \beta_4 \approx \frac{\gamma(t+1)}{4t}$$

After applying these changes to all blocks of equal eigenvalues (in this case 2 blocks with 4 entries each), we introduce the matrix V whose j^{th} row is the j^{th} eigenvector. This matrix is used to compute quadratic observables as described in Eq. (47) of Prosen's paper. Our observable is the current (the factor of $\frac{1}{2}$ is there to match the analytical formula)

$$\begin{aligned}
I &= < \frac{i}{2}(c_L c_R^\dagger - c_R c_L^\dagger) > \\
&= < -\frac{1}{8i}((w_1 - iw_2)(w_3 + iw_4) - (w_3 - iw_4)(w_1 + iw_2)) > \\
&= < -\frac{1}{8i}(w_1w_3 - iw_2w_3 + iw_1w_4 + w_2w_4 - w_3w_1 + iw_4w_1 - iw_3w_2 - w_4w_2) > \\
&= -\frac{1}{4i}(< w_1w_3 > + < w_2w_4 >)
\end{aligned}$$

where the expectation values are given by Eq. (47)

$$< w_1w_3 >= \frac{1}{2}\sum_{m=1}^{2n}(V_{2m,1}V_{2m-1,5} - V_{2m,2}V_{2m-1,6} - iV_{2m,2}V_{2m-1,5} - iV_{2m,1}V_{2m-1,6})$$

$$< w_2w_4 >= \frac{1}{2}\sum_{m=1}^{2n}(V_{2m,3}V_{2m-1,7} - V_{2m,4}V_{2m-1,8} - iV_{2m,4}V_{2m-1,7} - iV_{2m,3}V_{2m-1,8})$$

On the other hand, the current can be computed analytically, and is given by

$$I = -\frac{2w}{\gamma(1+r_0)}\frac{\delta r}{\frac{4w^2}{\gamma^2} + (1+r_0)^2 - \delta r^2}$$

where $r_0 + \delta r = e^{-\beta\mu_L} = t$ and $r_0 - \delta r = e^{-\beta\mu_R} = \frac{1}{t}$. Expressing the above relation in terms of t, after a little algebra, we get

$$I = -\frac{2w}{(1+t)} \frac{(t-1)}{\frac{4w^2}{\gamma} + \frac{\gamma(1+t)^2}{t}}$$

As of right now, the two methods agree on their value of I only in the case $\gamma \gg w$ (e.g. $\gamma = 1$, $w = 0.1$).