

# Fermion Transport for the Two-site Model

The Hamiltonian is  $H = -wc_L^\dagger c_R - wc_R^\dagger c_L$ . Then we can define Hermitian Majorana operators according to Eq. (5) in Prosen's paper:

$$w_1 = c_L + c_L^\dagger$$

$$w_2 = i(c_L - c_L^\dagger)$$

$$w_3 = c_R + c_R^\dagger$$

$$w_4 = i(c_R - c_R^\dagger)$$

satisfying  $\{w_j, w_k\} = \delta_{j,k}$ . In terms of these operators, the Hamiltonian becomes:

$$\begin{aligned} H &= -\frac{w}{4}((w_1 + iw_2)(w_3 - iw_4) + (w_3 + iw_4)(w_1 - iw_2)) \\ &= -\frac{w}{4}(w_1w_3 + iw_2w_3 - iw_1w_4 + w_2w_4 + w_3w_1 + iw_4w_1 - iw_3w_2 + w_4w_2) \\ &= \frac{iw}{4}(w_1w_4 - w_4w_1 + w_3w_2 - w_2w_3) = \frac{iw}{2}(w_1w_4 + w_3w_2) \end{aligned}$$

This can be written as a matrix

$$H = \begin{bmatrix} 0 & 0 & 0 & \frac{iw}{4} \\ 0 & 0 & -\frac{iw}{4} & 0 \\ 0 & \frac{iw}{4} & 0 & 0 \\ -\frac{iw}{4} & 0 & 0 & 0 \end{bmatrix}$$

which is clearly Hermitian.

Next, we want to represent the bath operators. We let  $\epsilon_L = \epsilon_R = 0$ ,  $\mu_L = -\mu_R$ , and  $\beta_L = \beta_R = \beta$ . Therefore the bath operators are  $v_L = v_R = \sqrt{\gamma}$  and  $u_L = \sqrt{\gamma e^{-\beta\mu_L}}$ ,  $u_R = \sqrt{\gamma e^{-\beta\mu_R}}$ . With the notation of our code, we can write  $t = e^{-\beta\mu_L}$ , and hence  $u_L = \sqrt{\gamma t}$ ,  $u_R = \sqrt{\frac{\gamma}{t}}$ . The bath operators are

$$L_1 = u_R c_R = \frac{u_R}{2}(w_3 - iw_4)$$

$$L_2 = u_L c_L = \frac{u_L}{2}(w_1 - iw_2)$$

$$L_3 = v_R c_R^\dagger = \frac{\sqrt{\gamma}}{2}(w_3 + iw_4)$$

$$L_4 = v_L c_L^\dagger = \frac{\sqrt{\gamma}}{2}(w_1 + iw_2)$$

To rewrite these in the form  $L_i = l_i \cdot w$ , we use

$$l_1 = [0, 0, \frac{u_R}{2}, \frac{-iu_R}{2}]$$

$$l_2 = [\frac{u_L}{2}, \frac{-iu_L}{2}, 0, 0]$$

$$l_3 = [0, 0, \frac{\sqrt{\gamma}}{2}, \frac{i\sqrt{\gamma}}{2}]$$

$$l_4 = [\frac{\sqrt{\gamma}}{2}, \frac{i\sqrt{\gamma}}{2}, 0, 0]$$

The complex Hermitian matrix  $M$  given by  $M_{jk} = \sum_\mu l_{\mu,j} l_{\mu,k}^*$  (Eq. (23)) becomes:

$$M = \begin{bmatrix} \frac{u_L^2 + \gamma}{4} & \frac{i(u_L^2 - \gamma)}{4} & 0 & 0 \\ \frac{i(-u_L^2 + \gamma)}{4} & \frac{u_L^2 + \gamma}{4} & 0 & 0 \\ 0 & 0 & \frac{u_R^2 + \gamma}{4} & \frac{i(u_R^2 - \gamma)}{4} \\ 0 & 0 & \frac{i(-u_R^2 + \gamma)}{4} & \frac{u_R^2 + \gamma}{4} \end{bmatrix}$$

Similarly, the complex antisymmetric matrix  $A$  defined in Eq. (27) is given by

$$A = \begin{bmatrix} 0 & \frac{i(u_L^2 + \gamma)}{2} & \frac{i(-u_L^2 + \gamma)}{2} & \frac{u_L^2 - \gamma}{2} & 0 & 0 & \frac{w}{2} & 0 \\ -\frac{i(u_L^2 + \gamma)}{2} & 0 & \frac{u_L^2 - \gamma}{2} & \frac{i(u_L^2 - \gamma)}{2} & 0 & 0 & 0 & \frac{w}{2} \\ \frac{i(u_L^2 - \gamma)}{2} & \frac{-u_L^2 + \gamma}{2} & 0 & \frac{i(u_L^2 + \gamma)}{2} & -\frac{w}{2} & 0 & 0 & 0 \\ \frac{-u_L^2 + \gamma}{2} & \frac{i(-u_L^2 + \gamma)}{2} & -\frac{i(u_L^2 + \gamma)}{2} & 0 & 0 & -\frac{w}{2} & 0 & 0 \\ 0 & 0 & \frac{w}{2} & 0 & 0 & \frac{i(u_R^2 + \gamma)}{2} & \frac{i(-u_R^2 + \gamma)}{2} & \frac{u_R^2 - \gamma}{2} \\ 0 & 0 & 0 & \frac{w}{2} & -\frac{i(u_R^2 + \gamma)}{2} & 0 & \frac{u_R^2 - \gamma}{2} & \frac{i(u_R^2 - \gamma)}{2} \\ -\frac{w}{2} & 0 & 0 & 0 & \frac{i(u_R^2 - \gamma)}{2} & \frac{-u_R^2 + \gamma}{2} & 0 & \frac{i(u_R^2 + \gamma)}{2} \\ 0 & -\frac{w}{2} & 0 & 0 & \frac{-u_R^2 + \gamma}{2} & \frac{i(-u_R^2 + \gamma)}{2} & -\frac{i(u_R^2 + \gamma)}{2} & 0 \end{bmatrix}$$

Using the fact that  $u_L^2 = \gamma t$  and  $u_R^2 = \frac{\gamma}{t}$ , we can rewrite the above matrix

$$A = \begin{bmatrix} 0 & \frac{i\gamma(t+1)}{2} & \frac{i\gamma(-t+1)}{2} & \frac{\gamma(t-1)}{2} & 0 & 0 & \frac{w}{2} & 0 \\ -\frac{i\gamma(t+1)}{2} & 0 & \frac{\gamma(t-1)}{2} & \frac{i\gamma(-t+1)}{2} & 0 & 0 & 0 & \frac{w}{2} \\ \frac{\gamma(t-1)}{2} & \frac{\gamma(-t+1)}{2} & 0 & \frac{i\gamma(t+1)}{2} & -\frac{w}{2} & 0 & 0 & 0 \\ \frac{i\gamma(-t+1)}{2} & \frac{i\gamma(t+1)}{2} & -\frac{i\gamma(t+1)}{2} & 0 & 0 & -\frac{w}{2} & 0 & 0 \\ 0 & 0 & \frac{w}{2} & 0 & 0 & \frac{i\gamma(1+t)}{2t} & \frac{i\gamma(-1+t)}{2t} & \frac{\gamma(1-t)}{2t} \\ 0 & 0 & 0 & \frac{w}{2} & -\frac{i\gamma(1+t)}{2t} & 0 & \frac{\gamma(1-t)}{2t} & \frac{i\gamma(1-t)}{2t} \\ -\frac{w}{2} & 0 & 0 & 0 & \frac{i\gamma(1-t)}{2t} & \frac{\gamma(-1+t)}{2t} & 0 & \frac{i\gamma(1+t)}{2t} \\ 0 & -\frac{w}{2} & 0 & 0 & \frac{\gamma(-1+t)}{2t} & \frac{i\gamma(-1+t)}{2t} & -\frac{i\gamma(1+t)}{2t} & 0 \end{bmatrix}$$

Since  $A$  is an antisymmetric matrix of even dimension, its eigenvalues come in pairs  $(\beta, -\beta)$ . Arrange the eigenvalues as  $\beta_1, -\beta_1, \dots, \beta_4, -\beta_4$ ,  $Re(\beta_1) \geq \dots \geq Re(\beta_4)$ , and let  $v_1, \dots, v_8$  be the corresponding eigenvectors. We would like to normalize the eigenvectors such that they satisfy Eq. (30),  $v_r \cdot v_s = J_{rs}$ , where

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The eigenvectors computed by MATLAB are such that  $v_j \cdot v_j = 0$  and  $v_{2j-1} \cdot v_{2j} = 1$ . However, because the spectrum  $\{\beta_j\}$  is degenerate, we are not guaranteed that  $v_{2j} \cdot v_k = 0$  for all  $k \neq 2j - 1$ . Therefore, we must consider a linear combination of  $v_j$ s.

When we have multiple eigenvectors with the same eigenvalues, we put them next to each other in blocks  $\beta_1, -\beta_1, \beta_1, -\beta_1, \dots$ . Let the corresponding eigenvectors in this block be  $a_1, b_1, a_2, b_2, \dots$ , such that  $a_j$  is the eigenvector for the  $j^{th}$  positive eigenvalue  $\beta_1$  and  $b_j$  is the eigenvector for the  $j^{th}$  negative eigenvalue  $-\beta_1$ . We are given that  $a_i \cdot a_j = 0$  and  $b_i \cdot b_j = 0$ . Define linear combinations  $c_i = \sum_j U_{ij} b_j$ . Note that  $c_i$  is also an eigenvector with eigenvalue  $-\beta_1$ . We want  $c_j$ s to satisfy  $a_i \cdot c_j = \delta_{i,j}$ . Hence

$$\sum_k U_{jk} a_i \cdot b_k = \delta_{i,j}$$

Define matrix  $T$ :

$$T_{ik} = a_i \cdot b_k = \sum_{\alpha} a_i^{\alpha} b_k^{\alpha}$$

$$\sum_k (U)_{kj}^T T_{ik} = (TU^T)_{ij} = \delta_{i,j}$$

Therefore  $TU^T = I$  and  $U = (T^{-1})^T$ . Now we can use the matrix  $U$  to compute  $c_j$  as shown above. As a result of this procedure, we are left with eigenvectors  $a_1, c_1, a_2, c_2, \dots$

For this particular matrix  $A$ , the eigenvalues are

$$\beta_1 = \beta_2 = \frac{\gamma(t+1)^2 + \sqrt{\gamma^2(t^2-1)^2 - 4t^2w^2}}{4t}$$

$$\beta_3 = \beta_4 = \frac{\gamma(t+1)^2 - \sqrt{\gamma^2(t^2-1)^2 - 4t^2w^2}}{4t}$$

Notice that for  $\gamma \gg w$ , the above expressions reduce to

$$\beta_1 = \beta_2 \approx \frac{\gamma(t+1)}{2}$$

$$\beta_3 = \beta_4 \approx \frac{\gamma(t+1)}{2t}$$

After applying these changes to all blocks of equal eigenvalues (in this case 2 blocks with 4 entries each), we introduce the matrix  $V$  whose  $j^{th}$  row is the  $j^{th}$  eigenvector. This matrix is used to compute quadratic observables as described in Eq. (47) of Prosen's paper. Our observable is the current

$$\begin{aligned} I &= \langle -i(c_L c_R^\dagger - c_R c_L^\dagger) \rangle \\ &= \langle \frac{1}{4i}((w_1 - iw_2)(w_3 + iw_4) - (w_3 - iw_4)(w_1 + iw_2)) \rangle \\ &= \langle \frac{1}{4i}(w_1 w_3 - iw_2 w_3 + iw_1 w_4 + w_2 w_4 - w_3 w_1 + iw_4 w_1 - iw_3 w_2 - w_4 w_2) \rangle \\ &= \frac{1}{2i}(\langle w_1 w_3 \rangle + \langle w_2 w_4 \rangle) \end{aligned}$$

where the expectation values are given by Eq. (47)

$$\langle w_1 w_3 \rangle = \frac{1}{2} \sum_{m=1}^{2n} (V_{2m,1} V_{2m-1,5} - V_{2m,2} V_{2m-1,6} - iV_{2m,2} V_{2m-1,5} - iV_{2m,1} V_{2m-1,6})$$

$$\langle w_2 w_4 \rangle = \frac{1}{2} \sum_{m=1}^{2n} (V_{2m,3} V_{2m-1,7} - V_{2m,4} V_{2m-1,8} - iV_{2m,4} V_{2m-1,7} - iV_{2m,3} V_{2m-1,8})$$

On the other hand, the current can be computed analytically, and is given by

$$I = -\frac{2w}{\gamma(1+r_0)} \frac{\delta r}{\frac{4w^2}{\gamma^2} + (1+r_0)^2 - \delta r^2}$$

where  $r_0 + \delta r = e^{-\beta\mu_L} = t$  and  $r_0 - \delta r = e^{-\beta\mu_R} = \frac{1}{t}$ . Expressing the above relation in terms of  $t$ , after a little algebra, we get

$$I = -\frac{2w}{(1+t)} \frac{(t-1)}{\frac{4w^2}{\gamma} + \frac{\gamma(1+t)^2}{t}}$$

As of right now, the two methods agree on their value of  $I$  only in the case  $\gamma \gg w$  (e.g.  $\gamma = 1, w = 0.1$ ).