# Zadanie 1. Egzaminacyjne

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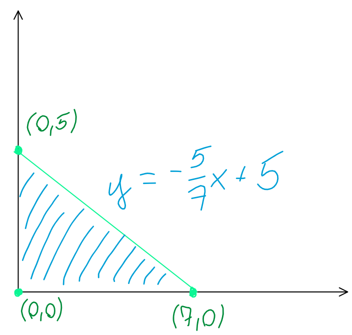
$m = 4 + 1 = 5$

$n = 6 + 1 = 7$

- Gęstość zmiennej losowej $(X, Y)$ to stała $C$.
- Jest opisana na $\Delta$ o wierzchołkach $(0,0)$, $(0,5)$, $(7,0)$.




$$\begin{cases} 0a + b = 5 \\ 7a + b = 0 \end{cases} \Rightarrow \begin{cases} b = 5 \\ 7a = -6 \end{cases}$$

$$7a = -5$$

$$a = -\frac{5}{7}$$

- $\Delta$ ogranicza nam $x \in [0, 7]$ oraz $y \in [0, -\frac{5}{7}x + 5]$
- Chcemy obliczyć gęstość zmiennej losowej $S = 2X + 3Y$.

$\boxed{1.}$

$f(x, y) = C$ ma spełnić warunki:

- $f(x,y) \geq 0$ dla $(x, y) \in R^2 \Longrightarrow C \geq 0$
- $\int_R \int_R f(x, y)\, dy\, dx = 1$

Policzmy $C$ z drugiej kropki:

$$\int_R \int_R f(x, y)\, dy\, dx = 1$$

$$\int_0^7 \int_0^{-\frac{5}{7}x+5} C\, dy\, dx = \int_0^7 C \int_0^{-\frac{5}{7}x+5} 1\, dy\, dx = \int_0^7 C \cdot \left[-\frac{5}{7}x + 5\right] dx = C \int_0^7 \left[-\frac{5}{7}x + 5\right] dx =$$

$$= C \cdot \left[-\frac{5}{7} \int_0^7 x\, dx + 5 \int_0^7 1\, dx\right] = C \cdot \left[-\frac{5}{7} \cdot \left[\frac{x^2}{2}\right]_0^7 + 5 \cdot [x]_0^7\right] =$$

$$= C \cdot \left[-\frac{5}{\cancel{7}_{1}} \cdot \frac{\cancel{49}^{7}}{2} + 5 \cdot 7\right] = C\left[-\frac{35}{2} + 35\right] = \frac{35}{2} C = 1$$

$$\boxed{C = \frac{2}{35}}$$

$\boxed{2.}$

$C = 35$

## 2.

Weźmy $S = 2X + 3Y$ oraz $T = Y$:

$$\begin{cases} S = 2X + 3Y \\ T = Y \end{cases}$$

$$X = \frac{S - 3T}{2} = \frac{S}{2} - \frac{3T}{2}$$

Teraz obliczamy Jakobian:

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$$

## 3.

$$g(s,t) = f(x(s,t), y(s,t)) \cdot |J| = C \cdot |J| = \frac{\cancel{2}^{\,1}}{35} \cdot \frac{1}{\cancel{2}_{\,1}} = \frac{1}{35}$$

$$\int g(s,t)\,dt = \int \frac{1}{35}\,dt = \frac{t}{35} + \underbrace{K}_{\text{jakaś stała}}$$

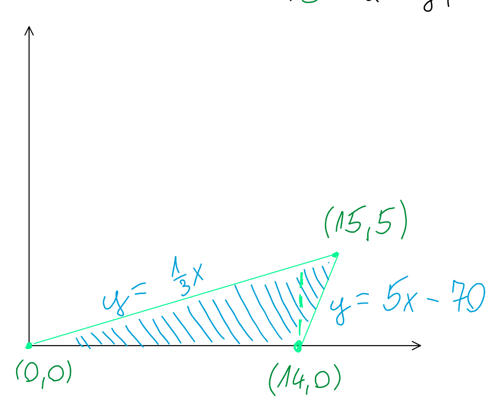
## 4.

Przekształćmy punkty według nowych równań:

A: $(0,0) \longrightarrow (0,0)$

B: $(0,5) \longrightarrow (S = 2x + 3y,\ T = y) = (15, 5)$

C: $(7,0) \longrightarrow (S = 2x + 3y,\ T = y) = (14, 0)$




$$\begin{cases} 15a + b = 5 \\ 0a + b = 0 \end{cases} \Rightarrow \begin{cases} 15a = 5 \\ b = 0 \end{cases} \Rightarrow \begin{cases} a = \frac{1}{3} \\ b = 0 \end{cases}$$

$$\begin{cases} 15a + b = 5 \\ 14a + b = 0 \end{cases} \Rightarrow \begin{cases} 15a - 14a = 5 \\ b = -14a \end{cases} \Rightarrow \begin{cases} a = 5 \\ b = -14 \cdot 5 = -70 \end{cases}$$

1) $s \in [0, 14]$, $t \in [0, \frac{1}{35}]$

2) $s \in [14, 15]$, $t \in [5s - 70, \frac{1}{3}s]$

# $\boxed{5.}$

Policzmy teraz gęstość zmiennej $S = 2X + 3Y$, dla ustalonego $s$.
Mamy 2 przedziały, więc mamy 2 funkcje:

- dla $s \in [0, 14]$, $t \in [0, \frac{1}{3}s]$

$$f_S(s) = \int_0^{\frac{1}{3}s} g(s,t)\,dt = \frac{1}{35} \cdot t \Big|_0^{\frac{1}{3}s} = \boxed{\frac{1}{105}s}$$

- dla $s \in [14, 15]$, $t \in [5s - 70, \frac{1}{3}s]$

$$f_S(s) = \int_{5s-70}^{\frac{1}{3}s} g(s,t)\,dt = \frac{1}{35} \cdot t \Big|_{5s-70}^{\frac{1}{3}s} = \frac{1}{35} \cdot \left(\frac{1}{3}s - 5s + 70\right) = \frac{1}{105}s - \frac{1}{7}s + 2 =$$

$$= \frac{1}{105}s - \frac{15}{105}s + 2 = \boxed{-\frac{14}{105}s + 2}$$

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Sprawdźmy jeszcze warunki na bycie gęstością:

- $\forall_{s \in [0, 15]} f_S(s) \geq 0$
  - dla $s \in [0, 14]$

    $$f_S(s) = \frac{1}{105}s \geq 0 \text{ dla każdego } s \in [0, 14]$$

  - dla $s \in [14, 15]$

    $$f_S(s) = -\frac{14}{105}s + 2$$

    dla $s = 14$: $-\frac{14^2}{105} + 2 = -\frac{196}{105} + \frac{210}{105} = \frac{14}{105} \geq 0$

    dla $s = 15$: $\frac{-14 \cdot 15}{105} + 2 = \frac{-210}{105} + \frac{210}{105} = 0 \geq 0$

- $\int_0^{15} f_S(s)\,ds = 1$

$$\int_0^{15} f_S(s)\,ds = \int_0^{14} f_S(s)\,ds + \int_{14}^{15} f_S(s)\,ds = \int_0^{14} \frac{1}{105}s\,ds + \int_{14}^{15} -\frac{14}{105}s + 2\,ds =$$

$$= \frac{1}{105}\int_0^{14} s\,ds - \frac{14}{105}\int_{14}^{15} s\,ds + 2\int_{14}^{15} 1\,ds = \frac{1}{105} \cdot \left[\frac{s^2}{2}\Big|_0^{14}\right] - \frac{14}{105} \cdot \left[\frac{s^2}{2}\Big|_{14}^{15}\right] + 2 \cdot \left[s\Big|_{14}^{15}\right] =$$

$$= \frac{1}{105} \cdot \frac{196}{2} - \frac{14}{105} \cdot \left( \frac{225}{2} - \frac{196}{2} \right) + 2 \cdot 1 = \frac{196}{210} - \frac{14}{105} \cdot \frac{29}{2} + 2 =$$

$$= \frac{196}{210} - \frac{406}{210} + 2 = -\frac{210}{210} + 2 = -1 + 2 = \boxed{1}$$