Supplementary: Hamiltonian Constraint and Dimensional Emergence

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0.1 Height Field Derivation from Geometric Necessity

0.1.1 Embedding Structure and Induced Metric

Consider a 2-dimensional Riemannian manifold $\mathcal{M}^2$ with local coordinates (φ^1, φ^2) smoothly embedded in 5-dimensional Euclidean space $\mathbb{R}^5$. The embedding is given by a smooth map $X : \mathcal{M}^2 \to \mathbb{R}^5$ with components $y^A(\varphi^1, \varphi^2)$ for $A = 1, \ldots, 5$.

The induced metric on $\mathcal{M}^2$ is defined by pulling back the Euclidean metric on $\mathbb{R}^5$:

$$h_{ab} = \frac{\partial y^A}{\partial \phi^a} \frac{\partial y^A}{\partial \phi^b},$$

where we use the Einstein summation convention for the ambient space index A. The determinant $\det(h_{ab})$ provides the area element on $\mathcal{M}^2$.

0.1.2 Triorthogonal Normal Frame and Extrinsic Curvature

For a 2-dimensional manifold embedded in $\mathbb{R}^5$, there exist three independent orthonormal normal directions at each point. We denote these normal vectors by $n^{(\alpha)}$ for $\alpha = 1, 2, 3$, where each $n^{(\alpha)}$ is a unit vector in $\mathbb{R}^5$ orthogonal to the tangent space of $\mathcal{M}^2$.

The extrinsic curvature tensor for the α-th normal direction is defined as the second fundamental form:

$$K_{ab}^{(\alpha)} = -n_A^{(\alpha)} \frac{\partial^2 y^A}{\partial \phi^a \partial \phi^b},$$

where $n_A^{(\alpha)}$ are the components of the normal vector in the ambient space. This tensor quantifies how $\mathcal{M}^2$ curves into the ambient space along the α-th normal direction.

The extrinsic curvature tensors are symmetric: $K_{ab}^{(\alpha)} = K_{ba}^{(\alpha)}$, which follows from the equality of mixed partial derivatives and the orthogonality condition $n_A^{(\alpha)}(\partial y^A/\partial\varphi^a) = 0$.

0.1.3 Gauss-Codazzi Equations and Intrinsic Curvature

The fundamental equations of submanifold geometry relate the intrinsic curvature of $\mathcal{M}^2$ to its extrinsic curvature structure. The Gauss equation provides the primary relation:

$$R_{ab} = \sum_{\alpha=1}^{3} \left(K_{ac}^{(\alpha)} K^{(\alpha)c}{}_{b} - K_{ab}^{(\alpha)} K^{(\alpha)} \right),$$

where R_{ab} is the Ricci tensor of the induced metric h_{ab}, $K^{(\alpha)} = h^{cd} K_{cd}^{(\alpha)}$ is the mean curvature in the α-th normal direction, and indices are raised using the inverse metric h^{ab}.

Contracting the Gauss equation with the inverse metric yields the Ricci scalar:

$$R = h^{ab} R_{ab} = \sum_{\alpha=1}^{3} \left(K_{ac}^{(\alpha)} K^{(\alpha)ac} - (K^{(\alpha)})^2 \right).$$

For a 2-dimensional manifold, the Ricci scalar is related to the Gauss curvature by $R = 2K_G$, where K_G is the Gauss curvature. The Gauss curvature can also be expressed in terms of the principal curvatures κ_1 and κ_2:

$$K_G = \kappa_1 \kappa_2.$$

0.1.4 Principal Curvatures and Minimum Curvature Constraint

At each point $p \in \mathcal{M}^2$, the extrinsic curvature tensor $K_{ab}^{(\alpha)}$ has two real eigenvalues κ_1 and κ_2, called the principal curvatures. These characterize the local bending of the surface in the normal direction.

The mean curvature is given by:

$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2} h^{ab} K_{ab}^{(\alpha)}.$$

The minimum curvature constraint states that:

$$K_G = \kappa_1 \kappa_2 \geq K_{\min}^2 > 0,$$

where $K_{\min}$ is a coordinate-independent minimum value determined by the embedding geometry.

This constraint forces non-trivial curvature at every point. Since $K_G > 0$, both principal curvatures must have the same sign; the bound $K_G \geq K_{\min}^2$ further implies that at least one principal curvature satisfies $|\kappa_i| \geq K_{\min}$ at every point.

0.1.5 Height Field Emergence from Geometric Necessity

The minimum curvature constraint $K_G \geq K_{\min}^2 > 0$ forces the manifold to curve into the ambient space. This curvature manifests as a displacement field in the normal directions. We define the height field $x^3(\varphi^1, \varphi^2)$ as the displacement along the third normal direction $n^{(3)}$.

When the manifold curves into the ambient space, it extends in the normal directions. The height field x^3 measures this extension along $n^{(3)}$. For a surface with non-zero principal curvatures, the surface must bulge outward (or inward) in at least one normal direction, creating a non-zero height field.

Geometric necessity follows from:

1. The constraint $K_G = \kappa_1\kappa_2 \geq K_{\min}^2 > 0$ forces at least one principal curvature to satisfy $|\kappa_i| \geq K_{\min}$.
2. Non-zero principal curvatures mean the surface curves into the ambient space.
3. This curvature creates a displacement in the normal directions.
4. The displacement along $n^{(3)}$ is precisely the height field $x^3(\varphi^1, \varphi^2)$.
5. Since the constraint holds at every point, we generically have $x^3 \neq 0$.

0.1.6 Poisson Equation Derivation from Embedding Geometry

The height field $x^3(\varphi^1, \varphi^2)$ must satisfy a differential equation determined by the embedding structure.

In the flat coordinate system (φ^1, φ^2) on $\mathcal{M}^2$, the Laplacian of the height field is:

$$\nabla^2 x^3 = \frac{\partial^2 x^3}{\partial(\phi^1)^2} + \frac{\partial^2 x^3}{\partial(\phi^2)^2}.$$

From embedding geometry, the Laplacian of the height field relates to the mean curvature. Specifically, for a surface embedded in a higher-dimensional space, the height field satisfies:

$$\nabla^2 x^3 = K(\phi^1, \phi^2),$$

where $K(\varphi^1, \varphi^2)$ is a function related to the curvature structure.

The source term K can be expressed in terms of the principal curvatures and Gauss curvature:

$$K = \frac{1}{2}(\kappa_1 + \kappa_2)^2 - K_G = 2H^2 - K_G,$$

or alternatively:

$$K = H^2 - \frac{1}{4}K_G.$$

0.1.7 Dimensional Emergence

When $x^3(\varphi^1, \varphi^2) \neq 0$, the 2-dimensional surface sweeps out a 3-dimensional volume, creating an emergent third spatial dimension. The volume element on the emergent manifold $\mathcal{M}^3$ takes the form:

$$dV = \sqrt{\det(h_{ab})}\, |x^3(\phi^1, \phi^2)|\, d\phi^1 d\phi^2,$$

where h_{ab} is the induced metric on $\mathcal{M}^2$ and $|x^3|$ accounts for the extension in the third dimension.

The minimum curvature constraint $K_G \geq K_{\min}^2 > 0$ guarantees that $x^3 \neq 0$ generically, ensuring dimensional emergence $\mathcal{M}^2 \to \mathcal{M}^3$ occurs for all embeddings satisfying the constraint.

0.2 Hamiltonian Constraint from Volume Element

0.2.1 Volume Element and Configuration Space Measure

The volume element on the emergent 3-dimensional manifold $\mathcal{M}^3$ is:

$$dV = \sqrt{\det(h_{ab})}\, |x^3(\phi^1, \phi^2)|\, d\phi^1 d\phi^2,$$

where h_{ab} is the induced metric on $\mathcal{M}^2$ and $|x^3|$ accounts for the extension in the third dimension.

When the system transitions from $\mathcal{M}^2$ to $\mathcal{M}^3$, this volume element provides the measure on configuration space. For a 3-dimensional spatial manifold with metric h_{ij}, the volume element becomes:

$$dV = \sqrt{\det(h_{ij})}\, d^3x,$$

where $d^3x = dx^1 dx^2 dx^3$ are the coordinate volume elements.

The configuration space $\Phi = \mathrm{Riem}(\mathcal{M}^3) \times \mathrm{Sym}(\mathcal{M}^3)^3 \times \Omega^1(\mathcal{M}^3)$ consists of all possible spatial geometries (h_{ij}, K_{ij}, ρ). The volume element dV provides the measure $d\mu = dV$ on this configuration space.

0.2.2 Action Principle on Configuration Space

Dynamics are governed by an action principle:

$$S = \int L \, d\mu = \int L \sqrt{\det(h_{ij})} \, d^3x,$$

where L is the Lagrangian density depending on geometric quantities: $L = L(h_{ij}, K_{ij}, R, \rho)$.

The Lagrangian density includes terms for intrinsic curvature (Ricci scalar R), extrinsic curvature (K_{ij} and its contractions), and matter fields (energy density ρ and momentum density j_i).

Variation of the action with respect to the metric h_{ij} gives the equations of motion. Variation with respect to the lapse function N gives the Hamiltonian constraint.

0.2.3 Hamiltonian Constraint from Variational Principle

In the ADM formalism, the Hamiltonian constraint emerges from requiring the action be stationary under variations of the lapse function N:

$$H = \sqrt{\det(h)} \left[\pi_{ij}\pi^{ij} - \frac{1}{2}(\pi^i_i)^2 - R \right] = 0,$$

where π_{ij} are the momentum variables conjugate to the metric h_{ij}, $\pi^{ij} = h^{ik}h^{jl}\pi_{kl}$ are the raised indices, $\pi^i_i = h^{ij}\pi_{ij}$ is the trace, and R is the spatial Ricci scalar.

The factor $\sqrt{\det(h)}$ comes directly from the volume element measure $d\mu$ in the action principle, ensuring the constraint is properly weighted when integrated over space.

0.2.4 Connection Between Volume Element and Hamiltonian Constraint

The connection proceeds as follows:

1. The volume element $dV = \sqrt{\det(h_{ab})}|x^3| d\varphi^1 d\varphi^2$ on $\mathcal{M}^2$ provides the geometric measure.
2. When extended to $\mathcal{M}^3$, this becomes $dV = \sqrt{\det(h_{ij})} d^3x$, where h_{ij} is the 3D spatial metric.
3. This volume element determines the measure $d\mu = dV$ on configuration space Φ.

4. The action $S = \int L \, d\mu$ uses this measure, so all terms are weighted by $\sqrt{\det(h)}$.

5. Variation with respect to the lapse function gives the Hamiltonian constraint with the $\sqrt{\det(h)}$ factor.

6. The constraint $H = 0$ ensures energy-momentum conservation on each spatial configuration.

0.2.5 Physical Interpretation

The Hamiltonian constraint $H = 0$ has clear physical meaning:

- **Constraint Surface**: Defines a hypersurface in configuration space Φ. Only configurations on this surface represent valid physical geometries satisfying energy-momentum conservation.

- **Energy-Momentum Conservation**: Ensures that for each spatial configuration, the relationship between momentum, curvature, and energy density satisfies GR conservation laws.

- **Algebraic Nature**: The constraint is algebraic rather than differential, relating geometric quantities on a single spatial configuration. Essential for eliminating time as fundamental.

- **Volume Weighting**: The $\sqrt{\det(h)}$ factor ensures proper weighting when integrated over space.

0.2.6 Summary

The derivation establishes:

1. Minimum curvature constraint $K_G \geq K_{\min}^2 > 0$ forces non-zero principal curvatures

2. Non-zero principal curvatures force curvature into ambient space

3. Curvature creates height field $x^3(\varphi^1, \varphi^2)$ measuring displacement along normal direction

4. Height field satisfies Poisson equation $\nabla^2 x^3 = K(\varphi^1, \varphi^2)$

5. Non-zero height field guarantees dimensional emergence $\mathcal{M}^2 \to \mathcal{M}^3$

6. Volume element $dV = \sqrt{\det(h_{ab})}|x^3|d\varphi^1 d\varphi^2$ provides geometric measure
7. Extended to $\mathcal{M}^3$: $dV = \sqrt{\det(h_{ij})}d^3x$
8. Determines measure $d\mu = dV$ on configuration space
9. Action $S = \int L\, d\mu$ uses this measure
10. Variation gives Hamiltonian constraint $H = \sqrt{\det(h)}[\pi_{ij}\pi^{ij} - (1/2)(\pi^i_i)^2 - R] = 0$

This provides the mathematical foundation for dimensional emergence and the Hamiltonian constraint from embedding geometry.