

# Supplementary: Conservation Laws from Geometry

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Overdetermined embeddings constrain curvature and all geometric quantities constructed from it. Physical observables, including energy, momentum, and angular momentum, correspond to geometric quantities on the embedded manifold; hierarchy bounds translate directly into constraints on time evolution. Bounded conservation laws emerge from this structure, reducing to exact conservation in the  $K_{\min} \rightarrow 0$  limit.

## 0.1 Bounded Conservation Laws Theorem

**Theorem (Bounded Conservation Laws).** For any quantity  $Q$  constructed from curvature and its derivatives up to order  $\ell$ , all time derivatives satisfy:

$$\left| \frac{d^m Q}{dt^m} \right| \leq \tilde{C}_{m,\ell} K_{\min}^{2+m/2}, \quad m = 1, 2, 3, \dots$$

where  $\tilde{C}_{m,\ell}$  are dimensionless constants depending on  $m$ ,  $\ell$ , and the geometric configuration.

### Proof.

Consider a physical quantity  $Q$  constructed from curvature and derivatives:

$$Q = Q(K_{ab}^{(\alpha)}, \nabla K_{ab}^{(\alpha)}, \nabla^2 K_{ab}^{(\alpha)}, \dots, \nabla^\ell K_{ab}^{(\alpha)}).$$

The time derivative of  $Q$  involves spatial and temporal derivatives of the arguments. On a time-dependent embedding where coordinates evolve as  $\varphi^a = \varphi^a(t)$ :

$$\frac{dQ}{dt} = \sum_{|j| \leq \ell} \frac{\partial Q}{\partial (\nabla^j K)} \cdot \frac{d}{dt} (\nabla^j K).$$

The time derivative of covariant derivatives relates to spatial derivatives

through coordinate evolution:

$$\frac{d}{dt}(\nabla^j K) = \sum_{p=0}^{j+1} c_{j,p} \nabla^p K \cdot \dot{\varphi}^a,$$

where  $c_{j,p}$  are combinatorial coefficients and  $\dot{\varphi}^a = d\varphi^a/dt$  represents coordinate velocity.

From the hierarchy bounds, each term satisfies:

$$|\nabla^p K| \leq C_p K_{\min}^{2+p/2}.$$

Substituting and using dimensional analysis with  $|\dot{\varphi}| \sim K_{\min}^{1/2}$  (set by geometric time scale):

$$\left| \frac{dQ}{dt} \right| \lesssim \sum_{p=0}^{\ell+1} K_{\min}^{2+p/2} \cdot K_{\min}^{1/2} \sim K_{\min}^{5/2},$$

matching the  $m = 1$  bound with  $\tilde{C}_{1,\ell} \sim (\ell + 2)$ .

Higher time derivatives follow by induction, each adding a factor  $K_{\min}^{1/2}$  from the time evolution operator, yielding the general bound.

## 0.2 Physical Examples

### 0.2.1 Energy Conservation

Energy density on the embedded manifold is geometrically defined through the extrinsic curvature:

$$E \sim \int_V K_{ab}^{(\alpha)} K^{ab(\alpha)} \sqrt{h} d^n \varphi \sim K_{\min}^2 V,$$

where  $V$  is the spatial volume. The time derivative satisfies:

$$\left| \frac{dE}{dt} \right| \leq C_1 K_{\min}^{5/2} V.$$

For cosmological embeddings with  $K_{\min} \sim H_0 \sim 10^{-18} \text{ s}^{-1}$  and  $V \sim (10^{26} \text{ m})^3$ :

$$\left| \frac{dE}{dt} \right| \lesssim 10^{-9} \text{ J/s}.$$

This bound is extraordinarily tight, effectively indistinguishable from exact conservation in laboratory or astrophysical measurements. The difference is fundamental: energy flows between the observable manifold and the embedding space at rate bounded by  $K_{\min}^{5/2}$ . Exact conservation holds in the full embedding space, not the observable manifold alone.

### 0.2.2 Momentum Conservation

Spatial momentum components arise from mixed space-time extrinsic curvature. Each component satisfies:

$$\left| \frac{dp^i}{dt} \right| \leq C_1 K_{\min}^{5/2} V, \quad i = 1, 2, 3.$$

The bound applies independently to each spatial direction, preserving rotational symmetry of the geometric constraints.

### 0.2.3 Angular Momentum Conservation

Angular momentum  $L^{ij} = \int (x^i p^j - x^j p^i) dV$  combines position and momentum. The antisymmetric structure ensures:

$$\left| \frac{dL^{ij}}{dt} \right| \leq C_1 K_{\min}^{5/2} V \ell,$$

where  $\ell$  is a characteristic length scale. For isolated systems where  $\ell \sim K_{\min}^{-1}$ , this gives  $|dL/dt| \lesssim K_{\min}^{3/2} V$ .

## 0.3 Noether Limit and Emergence of Exact Conservation

**Corollary (Noether Limit).** As  $K_{\min} \rightarrow 0$ , bounded conservation laws reduce to exact conservation:

$$\lim_{K_{\min} \rightarrow 0} \left| \frac{dQ}{dt} \right| = 0.$$

Physically, the  $K_{\min} \rightarrow 0$  limit decouples the manifold from the embedding space. Energy exchange between manifold and embedding vanishes, and the manifold becomes a closed system. This limit recovers standard Noether structure where continuous symmetries of the action yield exact conservation laws within the manifold.

Bounded conservation is thus a generalization that:

- Reduces to exact conservation when geometric constraints vanish ( $K_{\min} \rightarrow 0$ )
- Provides quantitative bounds in physically realized embeddings ( $K_{\min} \sim H_0$ )
- Unifies conservation laws and geometric structure without invoking symmetry postulates

## 0.4 Universal Geometric Conservation Principle

**Corollary (Universality).** Every geometric quantity in an overdetermined embedding with  $k \leq n^2 - n - 1$  satisfies bounded conservation laws.

This universality replaces symmetry postulates as the foundation for conservation. Gauge symmetries arise from redundancy in choosing orthonormal normal vectors: U(1) from single-vector phase freedom, SU(2) and SU(3) from rotations among multiple normals. The embedding constraints, not the action, determine conservation structure.

## 0.5 Observational Consequences

Bounded conservation differs observationally from exact conservation only at scales where  $K_{\min}^{5/2}V$  becomes measurable. For terrestrial experiments with  $V \sim (1 \text{ m})^3$  and  $K_{\min} \sim H_0$ :

$$\left| \frac{dE}{dt} \right| \lesssim 10^{-54} \text{ W},$$

far below any conceivable measurement precision. Even for galactic scales with  $V \sim (10^{21} \text{ m})^3$ , the bound remains  $\lesssim 10^{-27} \text{ W}$ , undetectable with current technology.

At cosmological scales or in extreme environments (early universe, black hole horizons), the bounds become relevant. Testing requires:

- Precision cosmology measuring total energy evolution over Hubble time
- Black hole thermodynamics tracking information and energy transfer
- Early universe nucleosynthesis sensitive to small energy nonconservation

These observational frontiers distinguish bounded conservation laws from exact conservation, providing falsifiable tests.

## 0.6 Falsification

Bounded conservation predicts violations at the geometric scale  $K_{\min}^{5/2}$ , while exact conservation predicts zero violations. Detection of systematic conservation violations exceeding geometric bounds would falsify overdetermined embedding. Current observations are consistent with bounded conservation at cosmological scales.