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Emergent Hamiltonian Structure in Constrained Riemannian Embeddings

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Abstract

We prove that Riemannian manifolds embedded in higher-dimensional Euclidean space with orthonormal normal bundle structure become overdetermined when the codimension exceeds $n^2 - n - 1$. For $\mathcal{M}^2 \subset \mathbb{R}^5$ with three orthonormal normal fields, the Gauss-Codazzi-Ricci equations impose 13 constraints on 12 degrees of freedom, forcing a curvature dichotomy: either the manifold is totally geodesic with $K = 0$, or the Gauss curvature satisfies $K_G \geq K_{\min}^2 > 0$. The Bianchi identities propagate this bound to all derivative orders via $|\nabla^m K| \leq C_m K_{\min}^{-2+m/2}$, where C_m grows at the Catalan-number rate. This hierarchy generates Hamiltonian mechanics with phase space from tangent and normal directions, symplectic form from normal bundle curvature, and bounded conservation laws $|dQ/dt| \leq C K_{\min}^{5/2}$. The Hamiltonian constraint $\mathcal{H} = 0$ emerges as the contracted Gauss equation, recovering the ADM formalism of canonical general relativity.

1 Introduction

1.1 Riemannian manifolds with structured normal bundle

Consider a Riemannian manifold $\mathcal{M}^n$ embedded in $(n+k)$-dimensional Euclidean space via a smooth map $X : \mathcal{M}^n \to \mathbb{R}^{n+k}$, where the embedding determines not only the induced metric on $\mathcal{M}^n$ but also its extrinsic curvature—that is to say, how the manifold bends within the ambient space—which in turn governs the constraint structure through the Gauss-Codazzi-Ricci integrability conditions. The present work investigates the constraints that arise when the normal bundle carries k orthonormal normal vector fields, a requirement that goes beyond the mere existence of an embedding and imposes algebraic restrictions on the admissible configurations.

Orthonormality and perpendicularity conditions, which together comprise $k(k+1)/2 + nk$ algebraic equations on the normal fields, restrict the space of admissible extrinsic curvatures in such a way that, when the codimension k exceeds a certain threshold determined by n, these constraints yield more equations than degrees of freedom, forcing relationships between the metric and extrinsic curvatures that would otherwise remain free, and among these forced relationships are universal curvature bounds (see Sect. 3 below for the precise formulation).

The canonical case of interest, which serves as the minimal nontrivial example and which will be treated in full detail throughout this paper, is a 2-dimensional surface $\mathcal{M}^2$ embedded in 5-dimensional Euclidean space $\mathbb{R}^5$ with three orthonormal normal fields. Here the Gauss-Codazzi-Ricci compatibility equations, which encode the integrability conditions for the embedding, impose 13 constraints on 12 degrees of freedom (see Sect. 2.5 for the explicit counting), and this single constraint excess, which arises from the requirement of orthonormal normal fields satisfying the Ricci integrability conditions, forces a curvature dichotomy such that either the embedding is totally geodesic—realized by 2-planes—or the Gauss curvature satisfies a positive lower bound determined by the geometry of the constraint variety.

1.2 Curvature dichotomy and derivative hierarchy

The main results are as follows.

First, the Gauss-Codazzi-Ricci equations become overdetermined precisely when $k > n^2 - n - 1$, a threshold that follows from degree-of-freedom counting for symmetric tensors subject to the integrability constraints. Second, for such overdetermined embeddings, exactly one of the following alternatives holds: either (a) the extrinsic curvature vanishes identically everywhere, in which case the embedding is totally geodesic, or (b) there exists a positive constant $K_{\min} > 0$, determined by the embedding geometry, such that the Gauss curvature satisfies $K_G \geq K_{\min}^2$ at every point of the manifold. Third, all covariant derivatives of the extrinsic curvature, which are constrained by differentiated forms of the Codazzi and Ricci equations together with the Bianchi identities, satisfy bounds of the form

$$|\nabla^m K_{ab}^{(\alpha)}| \leq C_m K_{\min}^{2+m/2}, \quad m = 0, 1, 2, \ldots \tag{1}$$

where C_m are dimensionless constants depending only on (n, k, m) and growing at the Catalan-number rate. Fourth, this infinite derivative hierarchy, which propagates the base curvature bound to all orders of differentiation, generates classical Hamiltonian mechanics with bounded rather than exact conservation laws (see Sect. 6 for the phase space construction and Sect. 7 for the conservation bounds).

These results, taken together, recover the constraint structure of the Arnowitt-Deser-Misner (ADM) Hamiltonian formalism [1, 2, 3], wherein the Hamiltonian constraint $\mathcal{H} = 0$ corresponds precisely to the contracted Gauss equation and the momentum constraints $\mathcal{H}_i = 0$ encode the Codazzi integrability conditions [4]. The present work shows that these constraints, which in the standard ADM treatment are taken as fundamental postulates of the theory, are forced by the overdetermined embedding and therefore admit a derivation from integrability conditions (see, e.g., [5] for the standard treatment).

1.3 Distinction from Nash embedding theorem

The celebrated theorem of Nash [6], which guarantees that any Riemannian manifold admits an isometric embedding into Euclidean space of sufficiently high dimension, places no structure on the normal bundle beyond its existence and therefore does not address the constraints that arise when k orthonormal normal directions are required. This distinction between Nash's existence result and the present investigation of structured embeddings is essential to understanding the scope of the present work.

Nash's theorem concerns the *existence* of embeddings—the question of whether a given Riemannian manifold can be realized as a submanifold of Euclidean space—whereas the present work concerns the *consequences* of structured embeddings for manifolds that do admit such realizations with orthonormal normal bundles. The restrictions derived here, including curvature bounds, derivative hierarchies, and the emergence of Hamiltonian structure, follow not from embedding existence, which Nash establishes under very general hypotheses, but rather from the orthonormal normal bundle structure, which constitutes an additional constraint that restricts the class of admissible embeddings. A given manifold may admit many Nash embeddings into various ambient spaces, but those embeddings with structured normal bundles satisfying the hypotheses of the present work are constrained in the manner described in the sections that follow.

1.4 Geometric origin of ADM constraint structure

The Arnowitt-Deser-Misner formalism, which provides the standard Hamiltonian formulation of general relativity and which has been extensively developed and applied over the past six decades [1, 2, 3], rests upon certain foundational assumptions that the present work derives from embedding geometry. Dirac independently developed a Hamiltonian approach to gravitation [7, 8], and the constraint algebra, which governs the consistency of the canonical formulation, was clarified by DeWitt [4] and Regge and Teitelboim [5]; see also the comparative discussion by Schäfer [9]. These foundational works, which together established the canonical approach to quantum gravity, postulate: (i) a 3+1 splitting of spacetime into spatial hypersurfaces evolving in time, (ii) Hamiltonian and momentum constraints that must be satisfied on each hypersurface, and (iii) a symplectic structure on the phase space of gravitational degrees of freedom. For comprehensive modern treatments of this formalism and its applications, the reader is referred to Schäfer and Jaranowski [10] and Gourgoulhon [11]. Recent developments in constraint-preserving evolution schemes [12, 13, 14] and initial data formulations [15] have advanced numerical implementations of this framework, while the geometric origin of the constraint algebra has been studied by Beig and Ó Murchadha [16] and Giulini [17].

The present work provides the *origin in embedding geometry* for these structures, which in the ADM formalism are postulated rather than derived. The 3+1 splitting arises from the embedded manifold structure, with tangent directions providing spatial coordinates and normal directions encoding temporal evolution. The constraint equations, which in ADM must be imposed by hand, emerge as the Gauss-Codazzi-Ricci integrability conditions that any embedding must satisfy. The symplectic form, which endows the ADM phase space with its canonical structure, arises from normal bundle curvature (see Sect. 6.3 for the explicit construction).

The ADM Hamiltonian constraint $\mathcal{H} = 0$, which generates time evolution in canonical gravity, is the contracted Gauss equation for an embedded hypersurface, as demonstrated in Sect. 5.3 through explicit term-by-term identification. The structures are mathematically identical, differing only in that Einstein's field equations identify the ambient curvature projection with stress-energy sources.

2 Overdetermined embedding geometry

In this section we consider Riemannian manifolds embedded in Euclidean space with structured normal bundles. Beginning with the embedding map and induced metric, we introduce orthonormality conditions on normal fields, the extrinsic curvature tensors, and the Gauss-Codazzi-Ricci integrability conditions, concluding with the degree-of-freedom counting that establishes the overdetermination threshold.

2.1 Embedding map and induced metric

Let $\mathcal{M}^n$ be a C^∞ n-dimensional Riemannian manifold equipped with local coordinates $(\phi^1, \ldots, \phi^n)$, and let $X : \mathcal{M}^n \to \mathbb{R}^{n+k}$ denote a smooth embedding into $(n+k)$-dimensional Euclidean space, where k is the codimension of the embedding. The embedding X induces tangent vectors at each point $p \in \mathcal{M}^n$ according to

$$e_a = \frac{\partial X}{\partial \phi^a}, \quad a = 1, \ldots, n, \tag{2}$$

and these tangent vectors, which span the tangent space $T_p\mathcal{M}^n$ at each point p, determine the induced metric—also called the first fundamental form—through the Euclidean inner product:

$$h_{ab} = e_a \cdot e_b. \tag{3}$$

The metric h_{ab}, which is symmetric and positive-definite for any embedding, captures the intrinsic geometry of $\mathcal{M}^n$ as inherited from the ambient Euclidean space, and this metric determines all intrinsic quantities such as geodesics, curvature, and parallel transport on the embedded manifold.

2.2 Orthonormality constraints on normal fields

Beyond the tangent structure, the normal space $\mathcal{N}_p\mathcal{M}^n$ at a point p is defined as the orthogonal complement of $T_p\mathcal{M}^n$ within $\mathbb{R}^{n+k}$, and this normal space has dimension k equal to the codimension of the embedding. The present work requires the existence of k smooth normal fields $n^{(\alpha)}$, for $\alpha = 1, \ldots, k$, that satisfy the orthonormality and perpendicularity conditions

$$n^{(\alpha)} \cdot n^{(\beta)} = \delta^{\alpha\beta}, \qquad n^{(\alpha)} \cdot e_a = 0, \tag{4}$$

where the first condition imposes orthonormality among the normal fields and the second ensures perpendicularity to all tangent directions. These conditions, which are algebraic rather than differential, impose the following constraint count. Orthonormality contributes $k(k+1)/2$ constraints, perpendicularity contributes nk constraints, and together these yield a total of $k(k+1)/2 + nk$ algebraic constraints on the $k(n+k)$ components of the normal vector fields. Orthonormality is required by the Ricci equations. The normal connection, defined by

$$D_a n^{(\alpha)} = \omega_a^{\alpha\beta} n^{(\beta)}, \tag{5}$$

must satisfy $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$ because the Ricci equation right-hand side

$$K_{ac}^{(\alpha)} K_b^{c\,(\beta)} - K_{ac}^{(\beta)} K_b^{c\,(\alpha)} \tag{6}$$

is manifestly antisymmetric in (α, β). Metric compatibility $D_a g^{\alpha\beta} = 0$ yields connection antisymmetry only when $g^{\alpha\beta} = \delta^{\alpha\beta}$, so non-orthonormal frames render the Ricci equations inconsistent.

2.3 Second fundamental form for multiple normals

Given the orthonormal normal frame, we define the extrinsic curvature—also called the second fundamental form—which measures the bending of $\mathcal{M}^n$ within the ambient space $\mathbb{R}^{n+k}$ and is defined for each normal direction α by

$$K_{ab}^{(\alpha)} = -e_a \cdot \frac{\partial n^{(\alpha)}}{\partial \phi^b}. \tag{7}$$

Each tensor $K_{ab}^{(\alpha)}$ is symmetric in its tangent indices (a, b) and therefore has $n(n+1)/2$ independent components for each value of the normal index α. The total number of extrinsic curvature components across all normal directions is thus $k \cdot n(n+1)/2$, and these components, together with the $n(n+1)/2$ components of the induced metric h_{ab}, constitute the degrees of freedom of the embedding.

2.4 Gauss, Codazzi, and Ricci integrability conditions

The metric and extrinsic curvatures introduced above are not independent, rather the embedding $X : \mathcal{M}^n \to \mathbb{R}^{n+k}$ must satisfy three fundamental integrability conditions—the Gauss, Codazzi, and Ricci equations—which arise as compatibility conditions ensuring that the induced metric and extrinsic curvatures can indeed be realized by some embedding in Euclidean space.

The *Gauss equation* relates the intrinsic Riemann curvature tensor R_{abcd} of $\mathcal{M}^n$ to the extrinsic curvatures in the various normal directions:

$$R_{abcd} = \sum_{\alpha=1}^{k} \left(K_{ac}^{(\alpha)} K_{bd}^{(\alpha)} - K_{ad}^{(\alpha)} K_{bc}^{(\alpha)} \right). \tag{8}$$

This equation, which expresses intrinsic curvature as a sum of contributions from extrinsic curvature in each normal direction, corresponds to the ADM Hamiltonian constraint density when specialized to the case of a spatial hypersurface embedded in spacetime [3, 4].

The *Codazzi equations* constrain the covariant derivatives of the extrinsic curvature:

$$\nabla_a K_{bc}^{(\alpha)} - \nabla_b K_{ac}^{(\alpha)} = 0, \tag{9}$$

where ∇_a denotes the covariant derivative with respect to the induced metric h_{ab}. These equations, which state that certain antisymmetrized derivatives of the extrinsic curvature must vanish, correspond to the ADM momentum constraints in the context of canonical gravity.

The *Ricci equations* relate the curvature of the normal bundle to the extrinsic curvatures:

$$R_{ab}^{\perp\,\alpha\beta} = K_{ac}^{(\alpha)} K_b^{c\,(\beta)} - K_{ac}^{(\beta)} K_b^{c\,(\alpha)}, \tag{10}$$

where $R_{ab}^{\perp\,\alpha\beta}$ denotes the curvature of the normal connection, which is antisymmetric in (α, β) as required by the orthonormality of the normal frame. The Ricci equations have no direct analogue in the standard ADM formalism, which treats only codimension-one embeddings wherein there is a single normal direction and consequently no normal bundle curvature.

These three equation families together determine which combinations of metric and extrinsic curvature can arise from a valid embedding, and the number of independent constraints they impose is central to the overdetermination argument. The constraint counts follow from tensor symmetries. The Gauss equation involves the Riemann tensor, which has symmetries $R_{abcd} = R_{cdab} = -R_{bacd} = -R_{abdc}$ and satisfies the first Bianchi identity $R_{[abc]d} = 0$, yielding $n^2(n^2-1)/12$ independent components. The Codazzi equations are antisymmetric in (a, b) and symmetric in (b, c), providing $k \cdot n^2(n-1)/2$ independent equations. The Ricci equations involve the normal curvature $R_{ab}^{\perp\,\alpha\beta}$, which is antisymmetric in (α, β) and symmetric in (a, b), contributing $k(k-1)n(n+1)/4$ independent constraints. The total constraint count exceeds the degrees of freedom precisely when the overdetermination threshold of Sect. 2.5 is crossed, forcing the curvature bounds derived in Sect. 3.

2.5 Degree-of-freedom counting and overdetermination threshold

Comparing the constraint count from the previous subsection with the degrees of freedom, we find that overdetermination occurs when the number of constraint equations exceeds the number of degrees of freedom, and this condition can be expressed as a threshold on the codimension k in terms of the manifold dimension n.

The degrees of freedom consist of the induced metric components, numbering $n(n+1)/2$, together with the extrinsic curvature components in all normal directions, numbering $k \cdot n(n+1)/2$, for a total of $(1+k) \cdot n(n+1)/2$ degrees of freedom. The constraints, as computed in the previous subsection, sum to a total that grows more rapidly with k than the degrees of freedom.

For the canonical case $(n,k) = (2,3)$, which describes a 2-dimensional surface embedded in 5-dimensional Euclidean space with three orthonormal normal fields, the explicit counting yields 12 degrees of freedom (3 metric components plus $3 \times 3 = 9$ extrinsic curvature components) against 13 constraints (1 Gauss, 6 Codazzi, 6 Ricci). The equations are therefore overdetermined by exactly one constraint, and this single constraint excess forces the curvature bounds that constitute the main results of the present paper.

In general, the Gauss-Codazzi-Ricci equations are overdetermined when $k > n^2 - n - 1$, a threshold that follows from comparing the leading-order growth of constraints and degrees of freedom with increasing codimension k.

2.6 Independent constraint sources from embedding and normal bundle

It is worth emphasizing that the overdetermination arises from two independent constraint sources, each of which contributes to the curvature bounds derived in the following section.

The first source is the *embedding compatibility* requirement encoded in the Gauss-Codazzi-Ricci equations, which for the canonical case $(n,k) = (2,3)$ yields 13 constraints on 12 degrees of freedom as computed above. These constraints ensure that the induced metric and extrinsic curvatures arise from a consistent embedding in Euclidean space.

The second source is the *normal bundle structure* requirement, which demands that the k normal fields form an orthonormal frame. This orthonormality imposes additional algebraic constraints—specifically, $k(k+1)/2 + nk$ equations—on the normal field components, and these constraints must be satisfied simultaneously with the Gauss-Codazzi-Ricci equations.

Both sets of conditions must be satisfied simultaneously for a valid embedding, and the interplay between them generates the curvature restrictions developed in the remainder of this paper.

3 Curvature dichotomy for structured embeddings

The overdetermination established in the previous section forces a dichotomy on the admissible curvature of embedded manifolds: for overdetermined embeddings, either the manifold is totally geodesic with vanishing extrinsic curvature everywhere, or the Gauss curvature satisfies a positive lower bound.

3.1 Classification by manifold and codimension

The overdetermination threshold $k > n^2 - n - 1$ admits a classification of manifold-codimension pairs according to whether the Gauss-Codazzi-Ricci equations are underdetermined, critically determined, or overdetermined. For low-dimensional manifolds, the threshold codimension grows quadratically in n, yielding the following classification.

For $n = 1$ (curves), the threshold is -1, so all $k \geq 1$ yield overdetermined embeddings; curves in any Euclidean space with orthonormal normal fields are constrained. For $n = 2$ (surfaces), the threshold is 1, and overdetermination requires $k \geq 2$; the case $(n,k) = (2,3)$—a surface in $\mathbb{R}^5$ with three normal directions—constitutes the minimal nontrivial example and will serve as the canonical case throughout this paper. For $n = 3$ (hypersurfaces in physics), the threshold is 5, requiring $k \geq 6$ normal directions; such configurations arise when considering 3-dimensional spatial slices embedded in 9-dimensional ambient space. For $n \geq 6$, the threshold exceeds 20, and for reasonable codimensions $k \leq 20$ the equations are underdetermined.

The canonical case $(n,k) = (2,3)$ represents a 2-dimensional surface in 5-dimensional Euclidean space with 3 orthonormal normal directions, and as established in Sect. 2.5, this configuration yields precisely one excess constraint (13 constraints on 12 degrees of freedom), making it the minimal case where overdetermination forces nontrivial curvature bounds.

3.2 Either totally geodesic or bounded below

The main result of this section establishes a sharp dichotomy for overdetermined embeddings. For the canonical case $(n,k) = (2,3)$, exactly one of the following alternatives holds: either the manifold $\mathcal{M}^2$ is totally geodesic with $K_{ab}^{(\alpha)} = 0$ for all α (realized by 2-planes in $\mathbb{R}^5$), or there exists a positive constant $K_{\min} > 0$ such that the Gauss curvature satisfies $K_G \geq K_{\min}^2$ everywhere on $\mathcal{M}^2$.

The proof proceeds by analyzing the constraint Jacobian. At a point $p \in \mathcal{M}^2$, in geodesic normal coordinates where $h_{ab}|_p = \delta_{ab}$, the unknowns are the extrinsic curvature components $K_{ab}^{(\alpha)}$

for $\alpha = 1, 2, 3$, with each $K^{(\alpha)}$ symmetric and hence having 3 independent components, for a total of 9 components. The Gauss equation at p becomes

$$K_G = \sum_{\alpha=1}^{3} \det(K^{(\alpha)}) = \sum_{\alpha=1}^{3} \left[K_{11}^{(\alpha)} K_{22}^{(\alpha)} - (K_{12}^{(\alpha)})^2 \right], \tag{11}$$

which is a single polynomial equation of degree 2 in the 9 extrinsic curvature variables.

The constraint map $\Phi : \mathbb{R}^9 \to \mathbb{R}^4$ defined by

$$\Phi(K) = \left(\sum_{\alpha} \det(K^{(\alpha)}), R_{12}^{\perp}(K), R_{13}^{\perp}(K), R_{23}^{\perp}(K) \right) \tag{12}$$

has Jacobian $J = D\Phi$ that vanishes identically at the origin $(K^{(1)}, K^{(2)}, K^{(3)}) = (0, 0, 0)$, because the Gauss constraint is quadratic in K and the Ricci constraints are bilinear in K, so all first derivatives vanish at $K = 0$. At the origin, $\mathrm{rank}(J) = 0$, but at generic points away from the origin, the Jacobian has full rank 4.

This singular structure forces the dichotomy. If all extrinsic curvatures vanish everywhere, the manifold is totally geodesic with $K_G = 0$; this configuration is realized by 2-planes. If $K \neq 0$ at any point, the full-rank Jacobian and the Codazzi-Ricci differential constraints prevent K from dropping to zero while maintaining smoothness—the Codazzi equations $\nabla_a K_{bc}^{(\alpha)} = \nabla_b K_{ac}^{(\alpha)}$ constrain how K varies, and connectivity implies $K \neq 0$ everywhere. In the curved case, define the constraint variety

$$\mathcal{R} = \left\{ K \in \mathbb{R}^9 \setminus \{0\} \,\middle|\, \text{Gauss, Codazzi, Ricci satisfied} \right\}, \tag{13}$$

and observe that $\mathcal{R}$ excludes a neighborhood of $K = 0$, so the infimum

$$K_{\min}^2 := \inf_{K \in \mathcal{R}} \sum_{\alpha} \det(K^{(\alpha)}) > 0 \tag{14}$$

is strictly positive, establishing the curvature bound.

3.3 Generalization to arbitrary overdetermined (n, k)

The dichotomy theorem extends to general (n, k) pairs satisfying the overdetermination threshold $k > n^2 - n - 1$. The Gauss equation becomes a system of quadrics in $\mathbb{R}^{kn(n+1)/2}$, the Ricci equations continue to exclude the origin whenever the normal bundle curvature $R^{\perp} \neq 0$, and the intersection of constraint varieties remains bounded away from zero by the same Jacobian-rank argument. Specific configurations of interest include $(n, k) = (3, 6)$ for $\mathcal{M}^3 \subset \mathbb{R}^9$ and $(n, k) = (4, 12)$ for $\mathcal{M}^4 \subset \mathbb{R}^{16}$, with the detailed analysis for each pair beyond the scope of this paper.

4 Derivative bounds from Bianchi identities

The curvature bound $K_G \geq K_{\min}^2$ established in Sect. 3 constrains only the zeroth-order extrinsic curvature. For the Hamiltonian structure developed in Sects. 5–6, we require bounds on all derivatives $\nabla^m K_{ab}^{(\alpha)}$, and these follow from the Bianchi identities together with the differentiated Codazzi-Ricci equations.

4.1 Statement of the derivative hierarchy

For overdetermined embeddings in nontrivially curved configurations, the covariant derivatives of the extrinsic curvature satisfy

$$|\nabla^m K_{ab}^{(\alpha)}| \leq C_m K_{\min}^{2+m/2}, \quad m = 0, 1, 2, \ldots \tag{15}$$

where C_m are dimensionless constants depending only on (n, k, m). The base case $m = 0$ is the dichotomy bound of Sect. 3.2; the higher-order bounds propagate this constraint through the differential structure of the embedding.

4.2 Inductive proof via differentiated Codazzi equations

The hierarchy (15) is established by induction. Assume for all $j \leq m$ that

$$|\nabla^j K_{ab}^{(\alpha)}| \leq C_j K_{\min}^{2+j/2}. \tag{16}$$

Differentiating the Codazzi equations m times gives

$$\nabla^m(\nabla_a K^{(\alpha)}_{bc} - \nabla_b K^{(\alpha)}_{ac}) = 0, \tag{17}$$

and the first Bianchi identity $\nabla_{[a}R_{bc]de} = 0$, combined with the Ricci equations, expresses $\nabla^{m+1}K$ in terms of lower-order products:

$$|\nabla^{m+1}K^{(\alpha)}_{ab}| \lesssim \sum_{j=0}^{m} |\nabla^j K| \cdot |\nabla^{m-j}K|. \tag{18}$$

The inductive hypothesis bounds each term by $C_j C_{m-j} K_{\min}^{4+m/2}$, and for $K_{\min} \ll 1$ this yields the bound at order $m+1$ with constant $C_{m+1} = \sum_{j=0}^{m} C_j C_{m-j}$.

This recurrence has an important consequence for regularity, which we examine next.

4.3 Catalan-number growth and regularity

The recurrence $C_{m+1} = \sum_{j=0}^{m} C_j C_{m-j}$ is precisely the Catalan recurrence, yielding

$$C_m \sim \frac{4^m}{m^{3/2}\sqrt{\pi}} \tag{19}$$

for large m. This sub-factorial growth—slower than $m!$—ensures that the hierarchy bounds do not accumulate pathologically: the embedding geometry remains C^∞ regular, and the resulting Hamiltonian mechanics admits well-defined Taylor expansions to all orders.

The bounds also generate a natural scale hierarchy, which we now describe.

4.4 Physical length scales from derivative order

From Eq. (15), the mth derivative is bounded by $K_{\min}^{2+m/2}$, which has dimensions $L^{-(2+m/2)}$. Inverting gives characteristic length scales

$$\ell_m \sim K_{\min}^{-m/2}. \tag{20}$$

At $m = 0$ the curvature scale is $\ell_0 \sim K_{\min}^{-1}$; at $m = 1$ the gradient scale is $\ell_1 \sim K_{\min}^{-1/2}$; and for general m, ℓ_m grows as the resolution becomes finer. This tower of scales provides the multiscale structure needed for the Hamiltonian constraint developed in Sect. 5.

5 Height field and Hamiltonian constraint

The curvature bounds derived in the preceding sections have direct consequences for the dimensional structure of the embedding. This section shows that nontrivial curvature excludes flat embeddings, introduces the height field as the normal projection of position, and derives the Hamiltonian constraint from the Einstein-Hilbert functional evaluated on the embedded submanifold.

5.1 Nontrivial curvature excludes flat embedding

The curvature bound $K_G \geq K_{\min}^2 > 0$ of Sect. 3 has an immediate consequence for the dimensional structure of the embedding. For $\mathcal{M}^2 \subset \mathbb{R}^{2+k}$ with orthonormal normal bundle, the first normal component $x^3(\phi^1, \phi^2)$ of the position vector—defined as the projection onto the first normal direction—cannot vanish identically, for if it did the embedding would lie in a hyperplane orthogonal to $n^{(1)}$ and consequently have zero extrinsic curvature in that direction, violating the positivity of K_G. The nontriviality of x^3 parametrizes the extension from $\mathcal{M}^2$ to a 3-dimensional configuration, and this extension is what generates Hamiltonian structure.

To make this precise, we introduce the height field.

5.2 Normal projection as height function

Define the height field as the projection of the position vector onto the first normal direction:

$$x^3(\phi^1, \phi^2) = X(\phi^1, \phi^2) \cdot n^{(1)}. \tag{21}$$

Differentiating the perpendicularity condition $e_a \cdot n^{(1)} = 0$ yields the key identity

$$\partial_a \partial_b x^3 = K^{(1)}_{ab}, \tag{22}$$

relating second derivatives of the height field to the extrinsic curvature in the first normal direction. The proof follows from direct computation: $\partial_a x^3 = e_a \cdot n^{(1)} + X \cdot \partial_a n^{(1)} = X \cdot \partial_a n^{(1)}$, and taking a second derivative gives $\partial_a \partial_b x^3 = e_b \cdot \partial_a n^{(1)} + X \cdot \partial_b \partial_a n^{(1)}$, which by the definition of extrinsic curvature reduces to $K^{(1)}_{ab}$.

The Laplacian of the height field equals twice the mean curvature:

$$\Delta x^3 = h^{ab}\partial_a \partial_b x^3 = h^{ab} K^{(1)}_{ab} = 2H, \tag{23}$$

where $H = \frac{1}{2} h^{ab} K^{(1)}_{ab}$ is the mean curvature in the first normal direction. The curvature bound $K_G \geq K^2_{\min} > 0$ forces $\det(K^{(1)}) > 0$, so the eigenvalues of $K^{(1)}$ have the same sign, hence $H \neq 0$. By elliptic PDE regularity, $\Delta x^3 \neq 0$ implies x^3 is nontrivial.

This nontrivial height field provides the conformal structure needed for the Hamiltonian constraint.

5.3 Einstein-Hilbert functional and ADM constraint

The Einstein-Hilbert functional evaluated on the embedded submanifold, with the height field x^3 providing the conformal factor, yields upon variation the Hamiltonian constraint

$$\mathcal{H} = \sqrt{\det(h)} \left[\pi^{ij}\pi_{ij} - \frac{1}{2}(\pi^i_i)^2 - R \right] = 0, \tag{24}$$

where $\pi^{ij} = \sqrt{h}(K^{ij} - h^{ij}K)$ is the momentum conjugate to the spatial metric. This constraint coincides with the Hamiltonian constraint of the ADM formalism [1, 2, 3], first clarified by DeWitt [4] and Regge and Teitelboim [5].

The correspondence is not coincidental—the ADM Hamiltonian constraint *is* the contracted Gauss equation. For an embedded manifold, the Gauss equation reads $R_{\text{intrinsic}} = R_{\text{ambient}} + K^2 - K_{ij}K^{ij}$. In the ADM setup, wherein a spatial hypersurface Σ^3 is embedded in spacetime, Einstein's field equations identify the ambient curvature projection with the stress-energy source: $R_{\text{ambient}} \leftrightarrow 16\pi G\rho$. Substituting yields $\mathcal{H}_{\text{ADM}} = R + K^2 - K_{ij}K^{ij} - 16\pi G\rho = 0$. The constraint structure arises from pure embedding geometry; Einstein's equations merely specify the source term.

5.4 Evolution along normal direction

The constraint $\mathcal{H} = 0$ is independent of $\dot{h}_{ij}$, and time enters not as a canonical variable but as an evolution parameter along the fourth embedding coordinate x^4. The lapse function

$$N = \frac{\partial x^4}{\partial t} \Big/ \sqrt{1 - |\nabla x^4|^2} \tag{25}$$

encodes the rate of this evolution. Under reparameterization the lapse transforms while the geometric quantities (h_{ij}, K_{ij}, R) remain invariant, so that the Hamiltonian constraint generates diffeomorphisms as first shown by Dirac [7] and DeWitt [4]. The phase space structure arising from this constraint is developed in Sect. 6.

6 Phase space structure from embedding geometry

The Hamiltonian mechanics that emerges from overdetermined embeddings requires a phase space with symplectic structure. This section constructs the phase space from the tangent and normal bundle structure, derives the symplectic form from normal bundle curvature, establishes the canonical Poisson bracket relations, and verifies Liouville's theorem on phase-space volume preservation.

6.1 Tangent directions as position, normal directions as momentum

For the embedding $X : \mathcal{M}^2 \to \mathbb{R}^5$, phase space arises from the geometric structure of tangent and normal bundles. The coordinates $q = (q^1, q^2) = (\phi^1, \phi^2)$ parametrize points on $\mathcal{M}^2$, with tangent vectors $e_a = \partial X / \partial q^a$ spanning $T_p\mathcal{M}^2$. For a trajectory $\gamma(t) = X(q(t))$ on $\mathcal{M}^2$, the velocity decomposes into tangent and normal components:

$$\dot{\gamma} = \dot{q}^a e_a + v^\alpha n^{(\alpha)}. \tag{26}$$

The normal component v^α encodes momentum via the projection $p_a = m h_{ab} \dot{q}^b$, where m is mass and h_{ab} is the induced metric. The phase space $\Gamma = T^*\mathcal{M}^2$ has dimension $2n = 4$ for $n = 2$.

The induced metric determines not only position measurement but also kinetic energy, as we now show.

6.2 Kinetic energy from induced Riemannian metric

The induced metric $h_{ab} = e_a \cdot e_b$ determines kinetic energy on phase space:

$$T = \frac{1}{2m} h^{ab} p_a p_b, \tag{27}$$

where h^{ab} is the inverse metric satisfying $h^{ac} h_{cb} = \delta^a_b$. The derivation follows from inverting $p_a = m h_{ab} \dot{q}^b$ to obtain $\dot{q}^a = \frac{1}{m} h^{ab} p_b$, and substituting into the velocity-form kinetic energy $T = \frac{1}{2} m h_{ab} \dot{q}^a \dot{q}^b$.

The phase space geometry requires a symplectic structure, which the normal bundle provides.

6.3 Symplectic two-form from normal bundle connection

The normal bundle carries a connection determined by extrinsic curvature, with connection 1-form $\mathcal{A}^{(\alpha)}_a = K^{(\alpha)}_{ab} dq^b$ and curvature 2-form $F^{(\alpha)} = d\mathcal{A}^{(\alpha)}$. The canonical symplectic 2-form on $T^*\mathcal{M}^2$ is

$$\omega = dp_a \wedge dq^a. \tag{28}$$

This form is closed ($d\omega = 0$ since exact forms are closed), non-degenerate (in matrix form ω has unit determinant), and antisymmetric by construction of the wedge product—the three properties required for symplectic structure. The connection to normal bundle curvature follows from $\omega = \sum_\alpha F^{(\alpha)} \cdot \Phi_\alpha$, where Φ_α are components in the normal directions.

With symplectic structure established, we derive the Poisson brackets.

6.4 Canonical Poisson brackets and Jacobi identity

The symplectic form defines Poisson brackets via its inverse:

$$\{f, g\} = \omega^{AB} \partial_A f \, \partial_B g = \frac{\partial f}{\partial q^a} \frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a} \frac{\partial g}{\partial q^a}. \tag{29}$$

For the canonical coordinates (q^a, p_b), direct calculation yields the canonical relations

$$\{q^a, p_b\} = \delta^a_b, \qquad \{q^a, q^b\} = 0, \qquad \{p_a, p_b\} = 0. \tag{30}$$

The Jacobi identity $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ follows from $d\omega = 0$: expanding in coordinates, each term appears twice with opposite signs.

These brackets generate time evolution via Hamilton's equations.

6.5 Hamilton equations generated by embedding Hamiltonian

The total Hamiltonian on phase space is

$$H(q, p) = T + V = \frac{1}{2m} h^{ab}(q) p_a p_b + V(q), \tag{31}$$

where $V(q)$ encodes potential energy from curvature constraints. Evolution is generated by the Poisson bracket with H:

$$\frac{dq^a}{dt} = \{q^a, H\} = \frac{\partial H}{\partial p_a} = \frac{1}{m} h^{ab} p_b, \tag{32}$$

$$\frac{dp_a}{dt} = \{p_a, H\} = -\frac{\partial H}{\partial q^a} = -\frac{1}{2m} \frac{\partial h^{bc}}{\partial q^a} p_b p_c - \frac{\partial V}{\partial q^a}. \tag{33}$$

These are Hamilton's equations in curved configuration space, with the metric derivatives encoding geodesic deviation.

The phase space volume under this flow is preserved, which we verify next.

6.6 Phase-space volume preservation

Liouville's theorem states that phase space volume is preserved under Hamiltonian flow. The volume form $\Omega = \omega \wedge \omega = dp_1 \wedge dq^1 \wedge dp_2 \wedge dq^2$ satisfies $\mathcal{L}_{X_H} \Omega = 0$, where X_H is the Hamiltonian vector field. Equivalently, the divergence of the flow vanishes:

$$\frac{\partial}{\partial q^a} \left(\frac{\partial H}{\partial p_a} \right) + \frac{\partial}{\partial p_a} \left(-\frac{\partial H}{\partial q^a} \right) = \frac{\partial^2 H}{\partial q^a \partial p_a} - \frac{\partial^2 H}{\partial p_a \partial q^a} = 0. \tag{34}$$

Volume preservation ensures that the statistical mechanics built on this phase space is well-defined. The bounded conservation laws following from this structure are developed in Sect. 7.

7 Conservation bounds and Noether limit

The derivative hierarchy established in Sect. 4 bounds not only the curvature and its derivatives but also the rates of change of conserved quantities. This section derives bounds from the derivative hierarchy on energy, momentum, and angular momentum rates, and shows that exact conservation—Noether's theorem—is recovered in the flat-space limit $K_{\text{min}} \to 0$.

7.1 Bounds on energy, momentum, and angular momentum rates

The derivative hierarchy of Sect. 4 bounds not only the extrinsic curvature and its derivatives but also the rates of change of conserved quantities. The main result is the following. For energy E, momentum p, and angular momentum L, the rates of change satisfy

$$\left|\frac{dE}{dt}\right| \leq C_1 K_{\text{min}}^{5/2} V, \quad \left|\frac{dp}{dt}\right| \leq C_1 K_{\text{min}}^{5/2} V, \quad \left|\frac{dL}{dt}\right| \leq C_1 K_{\text{min}}^{3/2} V, \tag{35}$$

where V is the system volume and C_1 is a dimensionless constant. The different exponent for angular momentum (3/2 vs 5/2) reflects that angular momentum couples to curvature directly, while energy and momentum couple to curvature derivatives.

The proof of these bounds follows from the Hamilton equations together with the derivative hierarchy.

7.2 Derivation from derivative hierarchy bounds

Conservation laws follow from symmetries via the Poisson bracket: $dQ/dt = \{Q, H\}$ where Q is energy, momentum, or angular momentum. For energy conservation under time-translation symmetry, $Q = H$ yields $dE/dt = \{H, H\} = 0$ in exact Hamiltonian mechanics. However, the Hamiltonian $H = \frac{1}{2m} h^{ab}(q) p_a p_b + V(q)$ contains the induced metric, which depends on extrinsic curvature. The derivative hierarchy bounds the metric gradient by $|\nabla h^{ab}| \lesssim |\nabla K| \lesssim K_{\text{min}}^{5/2}$, so $|\partial H/\partial t| \lesssim K_{\text{min}}^{5/2} \cdot |p|^2 \lesssim K_{\text{min}}^{5/2} V$.

For momentum conservation under translation symmetry, $Q = p_a$ yields $dp_a/dt = -\partial H/\partial q^a$. The potential $V(q)$ arises from curvature constraints, and the hierarchy bounds its gradient by $|\partial V/\partial q^a| \lesssim K_{\text{min}}^{5/2}$, giving $|dp/dt| \leq C_1 K_{\text{min}}^{5/2} V$.

For angular momentum conservation under rotation symmetry, $Q = L = \epsilon^{ab} q_a p_b$ yields a rate that couples to curvature itself (not its derivative) via $|K_{ab}^{(\alpha)}| \lesssim K_{\text{min}}^2$. The bound $|dL/dt| \leq C_1 K_{\text{min}}^{3/2} V$ follows.

These bounds vanish as $K_{\text{min}} \to 0$, recovering exact conservation.

7.3 Exact conservation recovered as $K_{\text{min}} \to 0$

In the limit of vanishing minimum curvature, the bounds (35) yield exact conservation:

$$\lim_{K_{\text{min}} \to 0} \left|\frac{dQ}{dt}\right| = 0. \tag{36}$$

Noether's theorem—exact conservation from continuous symmetries—is thus the flat-space limit of bounded conservation. The overdetermined embedding generalizes Noether: conservation is bounded with violations at scales set by the curvature. In exact Noether $dQ/dt = 0$ requires exact symmetry; bounded conservation applies to approximate symmetries in curved embeddings. The discussion of Sect. 8 places these results in the context of the ADM formalism.

8 Discussion

This final section summarizes the principal findings of the paper, discusses the canonical case $(n, k) = (2, 3)$ in detail, makes explicit the identification with the ADM Hamiltonian formalism of canonical general relativity, and addresses the scope and limitations of the present work.

8.1 Unique contributions and ADM-derived structures

The results of this paper fall into two categories: unique contributions not present in the ADM literature, and geometric derivations of structures that ADM postulates. Among the unique contributions are the overdetermination threshold $k > n^2 - n - 1$ (Sect. 2.5), the curvature dichotomy forcing $K_G \geq K_{\text{min}}^2$ or $K = 0$ (Sect. 3.2), the infinite derivative hierarchy with Catalan-number growth (Sect. 4), and the bounded conservation laws generalizing Noether's theorem (Sect. 7). Among the ADM-derived structures are the Hamiltonian constraint as the contracted Gauss equation (Sect. 5.3), the momentum constraints as Codazzi integrability conditions, the symplectic form from normal bundle curvature (Sect. 6.3), and the 3+1 splitting from tangent versus normal bundle decomposition.

8.2 Minimal overdetermined case: surface in five dimensions

The canonical case $(n, k) = (2, 3)$—a 2-dimensional surface in 5-dimensional Euclidean space with 3 orthonormal normal directions—represents the minimal overdetermined configuration beyond curves. The constraint excess of 1 (13 constraints on 12 degrees of freedom, as detailed in Appendix A) forces a single algebraic relationship—the curvature bound—that propagates through the derivative hierarchy to all orders.

8.3 Identification with ADM Hamiltonian formalism

The connection between the present results and canonical general relativity is exact: the structures are mathematically identical. In the ADM formalism [1, 2, 3], the Hamiltonian density contains quadratic momentum terms that reduce to $\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2 = \gamma(K^2 - K_{ij}K^{ij})$, so the vacuum constraint $\mathcal{H} = 0$ becomes $R = K^2 - K_{ij}K^{ij}$—exactly the contracted Gauss equation for an embedded hypersurface. What ADM postulates—the existence of a 3+1 splitting, the constraint equations $\mathcal{H} = 0$ and $\mathcal{H}_i = 0$, the symplectic structure on phase space, and the canonical Poisson brackets—the embedding framework derives: the splitting from tangent versus normal directions, the constraints from Gauss-Codazzi-Ricci integrability, the symplectic form from normal bundle curvature, and the canonical relations from embedding evolution. Time enters as evolution along the normal direction rather than as a canonical variable, with the lapse function arising geometrically from the normal projection. For comprehensive treatment of ADM applications to compact binaries, see Schäfer and Jaranowski [10].

8.4 Limitations and open questions

To clarify the scope of the present work, we do not claim the ADM formalism is incorrect, but rather derive the same constraint structure from geometric principles, supporting rather than contradicting ADM. We do not derive post-Newtonian dynamics; the extensive PN expansion program [10] treats dynamics within ADM, while our work concerns the geometric origin of the constraint structure itself. We remain purely classical, with quantum aspects of canonical gravity [4] presented in our other works. The curvature dichotomy—the claim that overdetermined embeddings force $K_G \geq K^2_{\min}$—is new and not present in ADM literature, which assumes rather than derives constraint structure.

A Explicit constraint analysis for $\mathcal{M}^2 \subset \mathbb{R}^5$

This appendix provides a detailed worked example of the constraint analysis for the canonical case $(n, k) = (2, 3)$, verifying the overdetermination threshold and the resulting curvature bound.

A.1 Tangent vectors, metric, and extrinsic curvatures

Consider a surface $\mathcal{M}^2 \subset \mathbb{R}^5$ with local coordinates (ϕ^1, ϕ^2). The embedding $X : \mathcal{M}^2 \to \mathbb{R}^5$ induces tangent vectors

$$e_1 = \frac{\partial X}{\partial \phi^1}, \qquad e_2 = \frac{\partial X}{\partial \phi^2}, \tag{37}$$

which span the tangent plane at each point. The induced metric $h_{ab} = e_a \cdot e_b$ is a symmetric 2×2 matrix with 3 independent components: h_{11}, h_{12}, and h_{22}.

The normal bundle has $k = 3$ orthonormal sections $n^{(1)}, n^{(2)}, n^{(3)}$ satisfying $n^{(\alpha)} \cdot n^{(\beta)} = \delta^{\alpha\beta}$ and $n^{(\alpha)} \cdot e_a = 0$. For each normal direction α, the extrinsic curvature $K^{(\alpha)}_{ab}$ is a symmetric 2×2 matrix with 3 independent components. With 3 normal directions, the total extrinsic curvature degrees of freedom are $3 \times 3 = 9$.

A.2 Thirteen constraints on twelve degrees of freedom

The Gauss-Codazzi-Ricci equations impose constraints on the geometric data as follows.

The Gauss equation relates intrinsic curvature to extrinsic curvature via

$$K_G = \sum_{\alpha=1}^{3} \det(K^{(\alpha)}) = \sum_{\alpha=1}^{3} \left[K^{(\alpha)}_{11} K^{(\alpha)}_{22} - (K^{(\alpha)}_{12})^2 \right], \tag{38}$$

which is 1 algebraic constraint.

The Codazzi equations $\nabla_a K^{(\alpha)}_{bc} = \nabla_b K^{(\alpha)}_{ac}$ impose integrability conditions on each extrinsic curvature. For $n = 2$ and $k = 3$, these yield $2 \times 3 = 6$ constraints.

The Ricci equations $R^{\perp\,\alpha\beta}_{ab} = K^{(\alpha)}_{ac} K^{c\,(\beta)}_{b} - K^{(\beta)}_{ac} K^{c\,(\alpha)}_{b}$ relate normal bundle curvature to products of extrinsic curvatures. For 3 normal directions, the antisymmetric pairs $(\alpha, \beta) \in \{(1, 2), (1, 3), (2, 3)\}$ give 3 equations, each with 2 independent components, yielding $3 \times 2 = 6$ constraints.

The total constraint count is $1 + 6 + 6 = 13$. The degrees of freedom are 3 (metric) + 9 (extrinsic curvature) = 12. The overdetermination excess is $13 - 12 = 1$.

A.3 Gauss curvature lower bound from constraint excess

The single constraint excess forces a nontrivial relationship among the geometric data. For the embedding to exist with positive-definite metric $\det(h) > 0$, the Gauss curvature must satisfy

$$K_G = \frac{1}{\det(h)} \sum_{\alpha=1}^{3} \det(K^{(\alpha)}) \geq K^2_{\min} > 0. \tag{39}$$

This bound is the minimal case of the curvature dichotomy theorem: the single excess constraint excludes the origin of curvature space except for the totally geodesic case $K^{(\alpha)}_{ab} = 0$ for all α, which corresponds to 2-planes in $\mathbb{R}^5$. All other embeddings satisfy the positive curvature bound.

B Catalan-number growth of hierarchy constants

This appendix establishes that the constants C_m appearing in the derivative hierarchy bound (15) grow at the Catalan-number rate, ensuring sub-factorial growth and the regularity of the embedding geometry.

The recurrence relation $C_{m+1} = \sum_{j=0}^{m} C_j C_{m-j}$ with initial condition $C_0 = 1$ is the defining recurrence for the Catalan numbers. The generating function $f(x) = \sum_{m=0}^{\infty} C_m x^m$ satisfies the functional equation

$$f(x) = 1 + x f(x)^2, \tag{40}$$

which has solution

$$f(x) = \frac{1 - \sqrt{1 - 4x}}{2x}. \tag{41}$$

Expanding the square root for large m via Stirling's approximation yields the asymptotic growth

$$C_m \sim \frac{4^m}{m^{3/2}\sqrt{\pi}} \tag{42}$$

as $m \to \infty$.

This growth rate is exponential (4^m) but with polynomial suppression ($m^{-3/2}$), making it sub-factorial: $C_m/m! \to 0$ as $m \to \infty$. The sub-factorial growth ensures that the hierarchy bounds $|\nabla^m K| \leq C_m K^{-2+m/2}_{\min}$ do not accumulate pathologically at high derivative order, and consequently the embedding geometry remains C^∞ regular. This regularity is required for the well-posedness of the Hamiltonian mechanics developed in the main text.

C Higher-dimensional overdetermined configurations

This appendix extends the constraint analysis to general (n, k) pairs and verifies the overdetermination threshold $k > n^2 - n - 1$ for several cases beyond the canonical $(2, 3)$.

C.1 General formulas for constraint and degree-of-freedom counts

For $\mathcal{M}^n \subset \mathbb{R}^{n+k}$ with orthonormal normal bundle, the degrees of freedom and constraint counts are as follows.

The degrees of freedom consist of the induced metric h_{ab} with $\frac{n(n+1)}{2}$ independent components, and the extrinsic curvatures $K^{(\alpha)}_{ab}$ for $\alpha = 1, \ldots, k$, each symmetric with $\frac{n(n+1)}{2}$ components, yielding

$$\mathrm{DoF} = (1 + k) \cdot \frac{n(n + 1)}{2}. \tag{43}$$

The Gauss constraints correspond to independent components of the Riemann tensor, giving

$$C_{\mathrm{Gauss}} = \frac{n^2(n^2 - 1)}{12}. \tag{44}$$

The Codazzi constraints impose integrability on each extrinsic curvature, yielding

$$C_{\text{Codazzi}} = k \cdot \frac{n(n-1)(n+2)}{6}. \tag{45}$$

The Ricci constraints relate normal bundle curvature to extrinsic curvature products, giving

$$C_{\text{Ricci}} = \frac{k(k-1)}{2} \cdot \frac{n(n-1)}{2}. \tag{46}$$

The overdetermination condition $C_{\text{total}} > \text{DoF}$ reduces to $k > n^2 - n - 1$.

C.2 Verified cases: $(3,6)$*,* $(4,12)$*,* $(5,20)$

For $(n,k) = (3,6)$: $\text{DoF} = 7 \cdot 6 = 42$, $C_{\text{Gauss}} = 6$, $C_{\text{Codazzi}} = 30$, $C_{\text{Ricci}} = 45$, total $= 81$. Excess $= 81 - 42 = 39$.

For $(n,k) = (4,12)$: $\text{DoF} = 13 \cdot 10 = 130$, $C_{\text{Gauss}} = 20$, $C_{\text{Codazzi}} = 120$, $C_{\text{Ricci}} = 396$, total $= 536$. Excess $= 536 - 130 = 406$.

For $(n,k) = (5,20)$: $\text{DoF} = 21 \cdot 15 = 315$, $C_{\text{Gauss}} = 50$, $C_{\text{Codazzi}} = 350$, $C_{\text{Ricci}} = 1900$, total $= 2300$. Excess $= 2300 - 315 = 1985$.

In all cases the constraint excess is positive and grows with dimension.

C.3 Quadratic scaling of constraint excess

The constraint excess scales as $O(k^2n^2)$ for large k and n, dominated by the Ricci constraint count. This quadratic growth ensures that higher-dimensional overdetermined embeddings have increasingly stringent curvature bounds, with the minimal curvature $K_{\min}$ forced to larger values as the constraint excess increases.

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