Structure Equations for the Orthogonal Compatibility Connection

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Abstract

The orthogonal compatibility connection on $E = TM \oplus N$ packages the Gauss–Weingarten system as an $\mathrm{SO}(n+k)$ connection. Its curvature decomposes into Gauss, Codazzi, and Ricci defect blocks, so flatness is exactly the constraint system. Coupled Bianchi identities and ideal-closure properties are derived in invariant form. The gauge-equivariant constraint operator $\mathcal{D}(h, K, \nabla^\perp) = (\mathcal{G}, \mathcal{C}, \mathcal{S})$ is formulated as a natural PDE operator, and its principal and gauge symbols are computed. For any $n \geq 2$, at frozen background $K = 0$ the Codazzi principal symbol has kernel dimension exactly k at every nonzero covector, and this kernel survives principal-symbol gauge reduction (the δK gauge symbol vanishes at $K = 0$); hence the operator is non-elliptic even after gauge reduction at $K = 0$.

1 Introduction

1.1 Standing Assumptions

Let M^n be a smooth manifold with Riemannian metric h, and let (N, g_N) be a rank-k Riemannian vector bundle equipped with an arbitrary metric connection $\nabla^\perp$. The extrinsic curvature data is a section $K \in \Gamma(\mathrm{Hom}(TM, \mathrm{Hom}(TM, N)))$ with symmetry $K(X, Y) = K(Y, X)$, equivalently $K \in \Gamma(\mathrm{Sym}^2 T^*M \otimes N)$. The shape operator $A : N \to \mathrm{End}(TM)$ is defined by the adjoint relation $\langle A_\nu X, Y\rangle_h = \langle K(X, Y), \nu\rangle_N$. No embedding, immersion, or real-analytic hypothesis is imposed. The bundle $E = TM \oplus N$ is treated as an abstract direct-sum bundle. Assume E is oriented (equivalently, orientations are fixed so that the orthonormal structure group is $\mathrm{SO}(n+k)$; otherwise replace SO by O). The Levi-Civita connection ∇^h is torsion-free. For any connection ∇, the curvature convention is $R(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. Here, "integrable" means locally gauge-trivial as an $\mathrm{SO}(n+k)$-connection, equivalently $\mathcal{F}^\oplus = 0$; no immersion claim is made without additional hypotheses.

The assembled connection is the Gauss–Weingarten system in block form:

$$\nabla^\oplus = \nabla^h \oplus \nabla^\perp + \Psi, \qquad \Psi = \begin{pmatrix} 0 & -K^T \\ K & 0 \end{pmatrix},$$

and its curvature blocks are exactly the Gauss, Codazzi, and Ricci defects. Via the metric $h \oplus g_N$, $\nabla^\oplus$ is equivalently a principal connection $\omega^\oplus$ on the orthonormal frame bundle $P_{\mathrm{SO}(E)}$.

1.2 Conventions and Notation

Composition wedge is used throughout for bundle-valued forms:

$$(\alpha \wedge \beta)(X, Y) = \alpha_X \circ \beta_Y - \alpha_Y \circ \beta_X.$$

Graded commutators are

$$[\alpha, \beta]_{\mathrm{gr}} := \alpha \wedge \beta - (-1)^{pq} \beta \wedge \alpha, \qquad \alpha \in \Omega^p, \ \beta \in \Omega^q.$$

Transpose conventions are metric adjoints with respect to $h \oplus g_N$; in particular K^T denotes the $N \to TM$ adjoint block and agrees with the shape operator block A.

1.3 Main Results and Novelty

The first result is structural: curvature block decomposition, coupled Bianchi identities, and closure of the curvature-generated graded ideal are established in a single invariant algebra on $\Omega^\bullet(M, \mathrm{End}(E))$. The second result is formal-PDE: the full constraint map $\mathcal{D}(h, K, \nabla^\perp)$ is treated as a natural gauge-equivariant operator, with explicit principal symbol and gauge symbol formulas.

At the frozen background $K = 0$ with $(h, \nabla^\perp)$ fixed, the symbol reduces to the Codazzi block and yields an exact kernel dimension count for surfaces. The full-symbol formulation clarifies that non-ellipticity is structural (mixed order and gauge directions), not an artifact of a special coordinate choice. Completeness statements in this paper are always within the gauge-natural operator class (KMS sense), not general PDE-solvability claims.

1.4 Relation to Classical Realization Results

This paper isolates operator structure, compatibility identities, and symbol-level consequences. Local immersion realization under classical fundamental-theorem hypotheses is treated as external geometry and used only as contextual interpretation, not as an input assumption.

Organization. Section 2 defines the assembled connection, uniqueness, and gauge actions on data. Section 3 computes the curvature block decomposition with explicit type conventions. Section 4 states the coupled Bianchi identities and the defect-complex packaging. Sections 5 and 6 establish gauge-natural completeness (as an operator-classification statement) and closure of the curvature-generated differential ideal, including a jet-level consequence. Section 7 defines the full natural constraint operator and computes its linearization, principal symbol, and gauge symbol. Section 8 records symbol consequences, including the frozen-background warm-up. Section 9 gives specializations and consistency checks. Appendix A translates invariant blocks to components, Appendix B records variation formulas, and Appendix C summarizes symbol maps.

2 Foundational Precision

2.1 The Assembled Connection

Define a connection on the direct-sum bundle $E = TM \oplus N$ that couples the intrinsic and extrinsic data.

Definition 2.1 (Orthogonal Compatibility Connection)**.** Let ∇^h be the Levi-Civita connection of (M, h). The **Orthogonal Compatibility Connection** $\nabla^\oplus$ is defined explicitly for sections $Y \in \Gamma(TM)$ and $\nu \in \Gamma(N)$ by:

$$\nabla^\oplus_X (Y \oplus \nu) := \Big(\nabla^h_X Y - A_\nu(X) \Big) \oplus \Big(\nabla^\perp_X \nu + K(X, Y) \Big). \tag{1}$$

Proposition 2.2. *The connection $\nabla^\oplus$ is metric-compatible with respect to the direct sum metric $g_E = h \oplus g_N$.*

Proof. For sections $s_1 = Y_1 \oplus \nu_1, s_2 = Y_2 \oplus \nu_2$:

$$\begin{aligned} X\langle s_1, s_2\rangle_E &= X(\langle Y_1, Y_2\rangle_h + \langle \nu_1, \nu_2\rangle_N) \\ &= \langle \nabla^h_X Y_1, Y_2\rangle_h + \langle Y_1, \nabla^h_X Y_2\rangle_h + \langle \nabla^\perp_X \nu_1, \nu_2\rangle_N + \langle \nu_1, \nabla^\perp_X \nu_2\rangle_N. \end{aligned}$$

The expression $\langle \nabla^\oplus_X s_1, s_2\rangle_E + \langle s_1, \nabla^\oplus_X s_2\rangle_E$ produces the same diagonal terms plus cross terms

$$-\langle A_{\nu_1} X, Y_2\rangle_h + \langle K(X, Y_1), \nu_2\rangle_N - \langle Y_1, A_{\nu_2} X\rangle_h + \langle \nu_1, K(X, Y_2)\rangle_N,$$

which cancel by the adjoint relation between A and K. Therefore $\nabla^\oplus$ is metric-compatible. □

2.2 Uniqueness

$$\nabla_{product} := \nabla^h \oplus \nabla^\perp.$$

Theorem 2.3 (Uniqueness of the Assembled Connection)**.** *$\nabla^\oplus$ is the unique metric connection on E whose restriction to diagonal blocks coincides with $\nabla^h \oplus \nabla^\perp$ and whose off-diagonal block $TM \to N$ is given by K.*

Proof. Equivalently, the off-diagonal condition is $\mathrm{pr}_N(\nabla_X(Y \oplus 0)) = K(X, Y)$ for all $X, Y \in TM$. Any connection can be written as $\nabla = \nabla_{product} + \Psi$, where $\Psi \in \Omega^1(M, \mathrm{End}(E))$. Metric compatibility requires Ψ to be skew-symmetric with respect to g_E. Since the diagonal blocks are fixed, $\Psi_{TM\to TM} = 0$ and $\Psi_{N\to N} = 0$. The off-diagonal block $\Psi_{TM\to N}$ is given by K. By skew-symmetry, the remaining block $\Psi_{N\to TM}$ is uniquely determined to be $-A$. Equivalently, for every X, metric compatibility is

$$\langle \Psi_X u, v\rangle_{h\oplus g_N} + \langle u, \Psi_X v\rangle_{h\oplus g_N} = 0,$$

i.e. Ψ_X is skew-adjoint with respect to $h \oplus g_N$.

$$\Psi_X = \begin{pmatrix} 0 & B_X \\ K_X & 0 \end{pmatrix}, \qquad \Psi_X^* = -\Psi_X$$

with respect to $h \oplus g_N$. Therefore $B_X = -K_X^T$, i.e. $B_X(\nu) = -A_\nu X$. Thus specifying the $TM \to N$ block determines the $N \to TM$ block uniquely. □

2.3 Gauge Actions on Data

Two symmetry classes act on the data triple $(h, K, \nabla^\perp)$:

1. Diffeomorphisms $\varphi \in \mathrm{Diff}(M)$:

$$h \mapsto \varphi^* h, \qquad K \mapsto \varphi^* K, \qquad \nabla^\perp \mapsto \varphi^* \nabla^\perp.$$

2. Orthogonal bundle gauge transformations $g \in \Gamma(\mathrm{SO}(N))$:

$$h \mapsto h, \qquad K \mapsto g \cdot K, \qquad \nabla^\perp \mapsto g \nabla^\perp g^{-1}.$$

Proposition 2.4 (Functoriality of the Assembled Connection)**.** *The assignment $(h, K, \nabla^\perp) \mapsto \nabla^\oplus$ is natural under $\mathrm{Diff}(M) \ltimes \Gamma(\mathrm{SO}(N))$. Consequently, the defect map $(h, K, \nabla^\perp) \mapsto (\mathcal{G}, \mathcal{C}, \mathcal{S})$ is gauge-equivariant.*

Proof. Each block of $\nabla^\oplus = \nabla^h \oplus \nabla^\perp + \Psi$ is functorial: ∇^h under pullback of metrics, $\nabla^\perp$ under pullback and orthogonal conjugation, and Ψ from the tensorial pair (K, K^T). Therefore $\nabla^\oplus$ transforms by pullback/conjugation in the associated principal $\mathrm{SO}(n+k)$ bundle, and curvature transforms covariantly. Hence $(\mathcal{G}, \mathcal{C}, \mathcal{S})$ is equivariant. □

3 Curvature Block Decomposition

The obstruction to integrability is the curvature

$$\mathcal{F}^{\oplus} := (\nabla^{\oplus})^2 \in \Omega^2(M, \mathrm{End}(E)).$$

The convention $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is used for all connections. The invariant defect blocks are defined globally by:

$$\mathcal{G} := (\mathcal{F}^{\oplus})_{TM}, \quad \mathcal{C} := (\mathcal{F}^{\oplus})_{Mix}, \quad \mathcal{S} := (\mathcal{F}^{\oplus})_{N}. \tag{2}$$

Here $(\mathcal{F}^{\oplus})_{Mix} \in \Omega^2(M, \mathrm{Hom}(TM, N))$ denotes the $TM \to N$ block. Thus

$$\mathcal{G} \in \Omega^2(M, \mathrm{End}(TM)), \qquad \mathcal{C} \in \Omega^2(M, \mathrm{Hom}(TM, N)), \qquad \mathcal{S} \in \Omega^2(M, \mathrm{End}(N)).$$

Let $D^{product}$ denote the exterior covariant derivative induced by $\nabla_{product} = \nabla^h \oplus \nabla^{\perp}$, and let

$$\mathcal{F}_{product} := (\nabla_{product})^2 = R^h \oplus R^{\perp} \in \Omega^2(M, \mathrm{End}(E)).$$

The perturbation equation is $\mathcal{F}^{\oplus} = \mathcal{F}_{product} + D^{product}\Psi + \Psi \wedge \Psi$. Here $\wedge$ is the composition wedge product:

$$(\alpha \wedge \beta)(X, Y) = \alpha_X \circ \beta_Y - \alpha_Y \circ \beta_X.$$

This composition wedge is the graded product in $\Omega^{\bullet}(M, \mathrm{End}(E))$; the displayed equation is the degree-1 specialization. Also, view

$$A \in \Omega^1(M, \mathrm{Hom}(N, TM)), \qquad K \in \Omega^1(M, \mathrm{Hom}(TM, N)),$$

via $A_X(\nu) := A_\nu X$ and $K_X(Y) := K(X, Y)$. Then

$$(A \wedge K)(X, Y) := A_X \circ K_Y - A_Y \circ K_X \in \mathrm{End}(TM),$$

$$(K \wedge A)(X, Y) := K_X \circ A_Y - K_Y \circ A_X \in \mathrm{End}(N).$$

Transpose Conventions. For each $X \in TM$, define $K_X \in \mathrm{Hom}(TM, N)$ by $K_X(Y) = K(X, Y)$ and its metric adjoint

$$K_X^T \in \mathrm{Hom}(N, TM), \qquad \langle K_X^T \nu, Y\rangle_h = \langle \nu, K_X Y\rangle_N.$$

Then $K_X^T(\nu) = A_\nu X$, so K^T and A are the same $\mathrm{Hom}(N, TM)$-valued 1-form. Also

$$\mathcal{C} \in \Omega^2(M, \mathrm{Hom}(TM, N)), \qquad \mathcal{C}^T \in \Omega^2(M, \mathrm{Hom}(N, TM)),$$

with $(\mathcal{C}^T)_{X,Y} := (\mathcal{C}_{X,Y})^T$.

The induced connection on $\mathrm{Hom}(TM, N)$ is

$$\nabla_X^{\mathrm{Hom}} T = \nabla_X^{\perp} \circ T - T \circ \nabla_X^h, \qquad T \in \Gamma(\mathrm{Hom}(TM, N)),$$

and the induced exterior covariant derivative on $\mathrm{Hom}(TM, N)$-valued 1-forms is

$$(D^{\mathrm{Hom}} K)(X, Y) := \nabla_X^{\mathrm{Hom}} K_Y - \nabla_Y^{\mathrm{Hom}} K_X - K_{[X,Y]}.$$

Theorem 3.1 (Curvature Equations)**.** *The curvature 2-form decomposes into:*

$$\mathcal{G} = R^h - A \wedge K \tag{3}$$
$$\mathcal{C} = D^{\mathrm{Hom}} K \tag{4}$$
$$\mathcal{S} = R^{\perp} - K \wedge A \tag{5}$$

with D^{Hom} as defined above.

Proof. Write $\nabla^{\oplus} = \nabla_{product} + \Psi$ with $\nabla_{product} = \nabla^h \oplus \nabla^{\perp}$ and

$$\Psi = \begin{pmatrix} 0 & -K^T \\ K & 0 \end{pmatrix}.$$

Also, $\Psi \wedge \Psi$ has only diagonal blocks: since Ψ_X maps $TM \to N$ and $N \to TM$, the composition $\Psi_X \circ \Psi_Y$ preserves TM and N, so mixed blocks of $\Psi \wedge \Psi$ vanish. With curvature convention $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, this yields the Gauss block sign $R^h - A \wedge K$.

$$\begin{aligned}(\mathcal{F}^{\oplus})_{TM} &= R^h + (\Psi \wedge \Psi)_{TM} = R^h - A \wedge K, \\ (\mathcal{F}^{\oplus})_{N} &= R^{\perp} + (\Psi \wedge \Psi)_{N} = R^{\perp} - K \wedge A, \\ (\mathcal{F}^{\oplus})_{Mix} &= (D^{product}\Psi)_{Mix}.\end{aligned}$$

Since $\nabla_{product} = \nabla^h \oplus \nabla^{\perp}$ preserves TM and N separately, the projections are parallel: $[\nabla^{product}, P_{TM}] = [\nabla^{product}, P_N] = 0$. Together with $P_{TM}\Psi P_{TM} = 0 = P_N \Psi P_N$, this implies $P_{TM}(D^{product}\Psi)P_{TM} = 0$ and $P_N(D^{product}\Psi)P_N = 0$. Here P_{TM}, P_N are the bundle projections associated to $E = TM \oplus N$. Explicitly, for $Z \in TM$,

$$((\Psi \wedge \Psi)_{TM}(X,Y))Z = \Psi_X(\Psi_Y Z) - \Psi_Y(\Psi_X Z) = -A_{K(Y,Z)}X + A_{K(X,Z)}Y = -(A \wedge K)(X,Y)Z.$$

Also,

$$((\Psi \wedge \Psi)(X,Y))_{TM \to N} = 0$$

because each composition $\Psi_X \circ \Psi_Y$ sends $TM \to TM$ and $N \to N$, so there is no $TM \to N$ component. For the mixed block, define $K_X \in \mathrm{Hom}(TM, N)$ by $K_X(Y) = K(X,Y)$ and use

$$(D^{product}\Psi)(X,Y) = \nabla_X^{product}\Psi_Y - \nabla_Y^{product}\Psi_X - \Psi_{[X,Y]}.$$

Projecting to $\mathrm{Hom}(TM, N)$ and evaluating on $Z \in \Gamma(TM)$ gives

$$\begin{aligned}((D^{product}\Psi)_{Mix}(X,Y))Z &= (\nabla_X^{\mathrm{Hom}} K_Y)Z - (\nabla_Y^{\mathrm{Hom}} K_X)Z - K_{[X,Y]}Z \\ &= (D^{\mathrm{Hom}}K)(X,Y)Z,\end{aligned}$$

Therefore $\mathcal{C} = D^{\mathrm{Hom}}K$, proving the stated block equations. □

Proposition 3.2 (Flatness Equivalent to Gauss–Codazzi–Ricci)**.**

$$\mathcal{F}^{\oplus} = 0 \iff \mathcal{G} = 0,\ \mathcal{C} = 0,\ \mathcal{S} = 0.$$

Equivalently, flatness of $\nabla^{\oplus}$ is exactly the Gauss, Codazzi, and Ricci system.

Proof. By block decomposition,

$$\mathcal{F}^{\oplus} = \begin{pmatrix} \mathcal{G} & -\mathcal{C}^T \\ \mathcal{C} & \mathcal{S} \end{pmatrix}.$$

Hence $\mathcal{F}^{\oplus} = 0$ iff each block vanishes. □

4 Coupled Bianchi Identities

Let D^T and $D^\perp$ denote the exterior covariant derivatives on $\mathrm{End}(TM)$ and $\mathrm{End}(N)$-valued forms, respectively; D^T is induced by ∇^h on TM, and $D^\perp$ is induced by $\nabla^\perp$ on N. The operator $D^{product}$ acts blockwise as D^T on $\mathrm{End}(TM)$, as $D^\perp$ on $\mathrm{End}(N)$, and as D^{Hom} on $\mathrm{Hom}(TM, N)$. All wedges in this section are composition wedges, and all brackets $[\cdot,\cdot]$ are graded commutators in $\Omega^\bullet(M, \mathrm{End}(E))$.

$$[\alpha, \beta]_{\mathrm{gr}} := \alpha \wedge \beta - (-1)^{pq} \beta \wedge \alpha, \qquad \alpha \in \Omega^p(M, \mathrm{End}(E)),\ \beta \in \Omega^q(M, \mathrm{End}(E)).$$

The exterior covariant derivative $D^\oplus$ is a graded derivation:

$$D^\oplus(\alpha \wedge \beta) = D^\oplus \alpha \wedge \beta + (-1)^p \alpha \wedge D^\oplus \beta, \qquad \alpha \in \Omega^p(M, \mathrm{End}(E)).$$

Theorem 4.1 (Universal Bianchi Identities)**.** *The defect tensors satisfy the following coupled identities:*

$$D^T \mathcal{G} + [K^T, \mathcal{C}] = 0 \tag{6}$$

$$D^{\mathrm{Hom}} \mathcal{C} + K \wedge \mathcal{S} - \mathcal{G} \wedge K = 0 \tag{7}$$

$$D^\perp \mathcal{S} + [K, \mathcal{C}^T] = 0 \tag{8}$$

where $[K^T, \mathcal{C}] = K^T \wedge \mathcal{C} - \mathcal{C}^T \wedge K$. *The first identity lies in* $\Omega^3(M, \mathrm{End}(TM))$*, the second in* $\Omega^3(M, \mathrm{Hom}(TM, N))$*, and the third in* $\Omega^3(M, \mathrm{End}(N))$*.*

Proof. Write the connection perturbation and curvature in block form:

$$\Psi = \begin{pmatrix} 0 & -K^T \\ K & 0 \end{pmatrix}, \qquad \mathcal{F}^\oplus = \begin{pmatrix} \mathcal{G} & -\mathcal{C}^T \\ \mathcal{C} & \mathcal{S} \end{pmatrix}.$$

Using $D^\oplus = D^{product} + [\Psi, \cdot]_{\mathrm{gr}}$, the Bianchi identity is

$$0 = D^\oplus \mathcal{F}^\oplus = D^{product} \mathcal{F}^\oplus + [\Psi, \mathcal{F}^\oplus]_{\mathrm{gr}}.$$

Since $\deg \Psi = 1$ and $\deg \mathcal{F}^\oplus = 2$ (so $1 \cdot 2$ is even), $[\Psi, \mathcal{F}^\oplus]_{\mathrm{gr}} = \Psi \wedge \mathcal{F}^\oplus - \mathcal{F}^\oplus \wedge \Psi$. A direct block computation gives

$$[\Psi, \mathcal{F}^\oplus]_{\mathrm{gr}} = \begin{pmatrix} K^T \wedge \mathcal{C} - \mathcal{C}^T \wedge K & K^T \wedge \mathcal{S} - \mathcal{G} \wedge K^T \\ \mathcal{S} \wedge K - K \wedge \mathcal{G} & K \wedge \mathcal{C}^T - \mathcal{C} \wedge K^T \end{pmatrix}.$$

For example, the (TM, TM) block is

$$([\Psi, \mathcal{F}^\oplus]_{\mathrm{gr}})_{TM} = (\Psi \wedge \mathcal{F}^\oplus)_{TM} - (\mathcal{F}^\oplus \wedge \Psi)_{TM} = [(-K^T) \wedge \mathcal{C}] - [(-\mathcal{C}^T) \wedge K] = K^T \wedge \mathcal{C} - \mathcal{C}^T \wedge K.$$

Equating (TM, TM), (Mix), and (N, N) blocks in $D^{product} \mathcal{F}^\oplus + [\Psi, \mathcal{F}^\oplus]_{\mathrm{gr}} = 0$ yields the three displayed identities. □

Remark 4.2 (Consistency Check)**.** If $K = 0$ (totally geodesic), then $K^T = A = 0$, hence $\mathcal{C} = 0$, and by the curvature equations $\mathcal{G} = R^h$, $\mathcal{S} = R^\perp$. The coupled identities reduce to $D^T R^h = 0$ and $D^\perp R^\perp = 0$, which are the standard Bianchi identities for the intrinsic connections.

4.1 Defect-Complex Viewpoint

The three coupled identities are exactly the block components of one graded equation:

$$D^{\oplus}\mathcal{F}^{\oplus} = 0 \quad \text{in} \quad \Omega^3(M, \mathrm{End}(E)).$$

This packages the Gauss, Codazzi, and Ricci differential compatibilities into a single off-shell identity in the non-commutative graded algebra of $\mathrm{End}(E)$-valued forms.

5 Gauge-Natural Completeness

Completeness of $\mathcal{F}^{\oplus}$ as an obstruction is formulated below.

Theorem 5.1 (Completeness of the Obstruction)**.** *Let $\omega^{\oplus}$ be the principal $\mathrm{SO}(n+k)$-connection induced by $\nabla^{\oplus}$ on $P_{\mathrm{SO}(E)}$, and let P be any finite-order (r) classical gauge-natural differential operator (in the sense of KMS) constructed from $\omega^{\oplus}$ and its finite jet prolongations, mapping to a tensor bundle. If P vanishes whenever $\mathcal{F}^{\oplus} = 0$, then there exists finite s and a bundle map Φ such that*

$$P = \Phi\big(j^s\mathcal{F}^{\oplus}\big).$$

Moreover s depends on the order r of P (write $s = s(r)$).

Proof. The connection $\nabla^{\oplus}$ determines uniquely a principal connection $\omega^{\oplus}$ on $P_{\mathrm{SO}(E)}$. A classical gauge-natural operator means a local finite-order functorial differential operator on associated bundles, equivariant under principal automorphisms. By the KMS regularity theorem (cf. Theorem 19.7 in [1]), order-r operators depend on finite jets of $\omega^{\oplus}$, hence on finite jets of curvature and covariant derivatives up to an order $s = s(r)$. Here "flat data" means exactly $\mathcal{F}^{\oplus} = 0$. Therefore any such operator vanishing on $\mathcal{F}^{\oplus} = 0$ factors through $j^{s(r)}\mathcal{F}^{\oplus}$. This is a classification of local invariants; it does not imply formal integrability, involutivity, or PDE solvability for immersion equations. □

5.1 Corollary Class

Any local gauge-natural scalar or tensor invariant built from $\omega^{\oplus}$ and vanishing on flat data factors through finite jets of $\mathcal{F}^{\oplus}$. This isolates curvature jets as the complete local generating source for such invariants in the KMS sense.

6 Hierarchy Closure

Theorem 6.1 ($D^{\oplus}$-Stable Graded Two-Sided Ideal)**.** *Let $\mathcal{I}$ be the graded two-sided ideal in $\Omega^{\bullet}(M, \mathrm{End}(E))$ generated by $\mathcal{F}^{\oplus}$:*

$$\mathcal{I} = \Big\{ \sum_i \alpha_i \wedge \mathcal{F}^{\oplus} \wedge \beta_i : \alpha_i, \beta_i \in \Omega^{\bullet}(M, \mathrm{End}(E)) \Big\}.$$

Then $\mathcal{I}$ is stable under $D^{\oplus}$.

Proof. By Bianchi, $D^{\oplus}\mathcal{F}^{\oplus} = 0$. Since $D^{\oplus}$ is a graded derivation for composition wedge, for homogeneous α, β one has

$$D^{\oplus}(\alpha \wedge \mathcal{F}^{\oplus} \wedge \beta) = (D^{\oplus}\alpha) \wedge \mathcal{F}^{\oplus} \wedge \beta + (-1)^{|\alpha|}\alpha \wedge (D^{\oplus}\mathcal{F}^{\oplus}) \wedge \beta + (-1)^{|\alpha|+2}\alpha \wedge \mathcal{F}^{\oplus} \wedge D^{\oplus}\beta.$$

The middle term vanishes, and the remaining terms are again in the two-sided ideal generated by $\mathcal{F}^{\oplus}$. Hence $D^{\oplus}\mathcal{I} \subseteq \mathcal{I}$. No commutativity assumption is used: the argument only uses graded derivation and two-sided ideal structure in the non-commutative algebra $\Omega^{\bullet}(M, \mathrm{End}(E))$. $\square$

Remark 6.2. For associated bundles (here $\mathrm{End}(E)$-valued forms), the curvature representation identity $(D^{\oplus})^2\eta = [\mathcal{F}^{\oplus}, \eta]$ also holds.

6.1 Jet-Level Closure Statement

The $D^{\oplus}$-stability of the curvature-generated ideal implies that all iterated covariant derivatives $(D^{\oplus})^r\mathcal{F}^{\oplus}$ remain in the same graded two-sided ideal. In prolongation language, universal differential consequences generated from curvature remain algebraically controlled by the curvature ideal itself. This is an algebraic closure statement and does not assert involutivity or formal integrability of the PDE system.

7 Structural Interpretation

Proposition 7.1 (Structural Interpretation of Compatibility)**.** *Let $E = TM \oplus N$ with metric $h \oplus g_N$, let ∇^h be the torsion-free Levi-Civita connection on TM, let $\nabla^{\perp}$ be a metric connection on N, and let $K \in \Gamma(\mathrm{Sym}^2 T^*M \otimes N)$ define the mixed block via the canonical inclusion $TM \hookrightarrow E$. Then $\nabla^{\oplus}$ induces a principal $\mathrm{SO}(n+k)$-connection $\omega^{\oplus}$ on $P_{\mathrm{SO}(E)}$ such that:*

1. *local flatness/integrability of the Gauss–Weingarten system is equivalent to $\mathcal{F}^{\oplus} = 0$;*
2. *$\mathcal{F}^{\oplus}$ is complete as a local structure function within the gauge-natural class, i.e. every local gauge-natural obstruction factors through finite jets of $\mathcal{F}^{\oplus}$.*

Proof. Item (1) is exactly the Gauss–Codazzi–Ricci flatness criterion encoded by the block equations above. Flatness of $\omega^{\oplus}$ implies local principal-bundle trivialization with local parallel adapted orthonormal frames on simply connected neighborhoods. This is an integrability statement for the abstract $\mathrm{SO}(n+k)$-connection on E, not yet an immersion statement. Under additional classical hypotheses of the fundamental theorem of submanifolds (compatible first/second fundamental data on a simply connected local domain), this specializes to local immersion realization into Euclidean space [3, 2]. No global embedding claim is made. Item (2) is the gauge-natural completeness theorem in the previous section, applied to $\omega^{\oplus}$. $\square$

8 The Constraint Operator as a Natural PDE

8.1 Definition of the Constraint Operator

Definition 8.1 (Nonlinear Constraint Operator)**.**

$$\mathcal{D} : \ (h, K, \nabla^{\perp}) \longmapsto (\mathcal{G}, \mathcal{C}, \mathcal{S}),$$

with

$$\mathcal{G} = R^h - A \wedge K, \qquad \mathcal{C} = D^{\mathrm{Hom}} K, \qquad \mathcal{S} = R^{\perp} - K \wedge A.$$

Thus $\mathcal{D} = 0$ is exactly the Gauss–Codazzi–Ricci system in invariant block form.

Definition 8.2 (Linearization and Frozen Operator)**.** At background $(h, K, \nabla^\perp)$, write

$$\mathbf{L}_{(h,K,\nabla^\perp)} := D\mathcal{D}\big|_{(h,K,\nabla^\perp)}.$$

At frozen background $(h, 0, \nabla^\perp)$, the restricted Codazzi operator is

$$L : \Gamma(\mathrm{Sym}^2 T^*M \otimes N) \to \Omega^2(M, \mathrm{Hom}(TM, N)), \qquad L(K) := D^{\mathrm{Hom}} K.$$

Definition 8.3 (Block Principal Symbol and Reduced Symbol)**.** At $x \in M$, define

$$U_x := \mathrm{Sym}^2 T_x^*M \oplus (\mathrm{Sym}^2 T_x^*M \otimes N_x) \oplus (T_x^*M \otimes \mathfrak{so}(N_x)),$$

$$V_x := \Lambda^2 T_x^*M \otimes \mathrm{End}(T_xM) \oplus \Lambda^2 T_x^*M \otimes \mathrm{Hom}(T_xM, N_x) \oplus \Lambda^2 T_x^*M \otimes \mathrm{End}(N_x).$$

For $\xi \neq 0$, $\sigma_\xi(\mathbf{L}_{(h,K,\nabla^\perp)}) : U_x \to V_x$ is defined blockwise by highest-order terms in each variable block. For

$$\sigma_\xi(\delta_{\mathrm{gauge}}) : \mathfrak{g}_x \to U_x, \qquad \mathfrak{g}_x := T_xM \oplus \mathfrak{so}(N_x),$$

the reduced symbol is

$$\overline{\sigma}_\xi(\mathbf{L}_{(h,K,\nabla^\perp)}) : U_x / \operatorname{im} \sigma_\xi(\delta_{\mathrm{gauge}}) \to V_x.$$

Definition 8.4 (Reduced Ellipticity Convention)**.** All ellipticity statements in this paper mean injectivity of $\overline{\sigma}_\xi(\mathbf{L}_{(h,K,\nabla^\perp)})$ for every $\xi \neq 0$.

Lemma 8.5 (Invariance of the Block Symbol Construction)**.** *The block principal symbol and reduced symbol above are coordinate-independent and depend only on* (x, ξ)*.*

Proof. Each block is the principal part of a natural tensorial differential operator block; under coordinate change and orthonormal bundle gauge change, principal parts transform by the corresponding representation action. Therefore $\sigma_\xi(\mathbf{L}_{(h,K,\nabla^\perp)})$ is intrinsically defined. The quotient by $\operatorname{im} \sigma_\xi(\delta_{\mathrm{gauge}})$ is intrinsic because $\sigma_\xi(\delta_{\mathrm{gauge}})$ is itself natural. □

Proposition 8.6 (Gauge Equivariance)**.** *The operator* $\mathcal{D}$ *is natural and equivariant under* $\mathrm{Diff}(M) \ltimes \Gamma(\mathrm{SO}(N))$*.*

Proof. Equivariance follows from functoriality of $\nabla^\oplus$ and covariance of curvature under pullback/conjugation, then projection to the three bundle-valued blocks. □

8.2 Linearization of $\mathcal{D}$

Let $(\delta h, \delta K, a)$ be a variation of $(h, K, \nabla^\perp)$, where $a \in \Omega^1(M, \mathrm{End}(N))$ is the connection variation. The linearization decomposes blockwise:

$$\delta\mathcal{D} = (\delta\mathcal{G}, \delta\mathcal{C}, \delta\mathcal{S}).$$

Principal-order terms are:

$$\delta\mathcal{G} = \delta R^h + (\text{terms of order} \leq 1 \text{ in } \delta h, \text{ order } 0 \text{ in } \delta K, a),$$

$$\delta\mathcal{C} = D^{\mathrm{Hom}} \delta K + (\text{terms of order} \leq 1 \text{ in } \delta h, \text{ order } 0 \text{ in } a, \delta K),$$

$$\delta\mathcal{S} = \delta R^\perp + (\text{terms of order } 0 \text{ in } \delta h, \delta K, a).$$

Hence the mixed-order structure is intrinsic: second order in h, first order in K and $\nabla^\perp$. Since $\mathcal{D}$ is not pure-order, principal symbols are taken blockwise by highest-order contributions in each variable block.

8.3 Principal Symbol of the Full Operator

For $\xi \neq 0$, the symbol is computed from highest-order terms:

$$\sigma_\xi(\delta\mathcal{D})(\delta h, \delta K, a) = \Big(\mathcal{R}_\xi(\delta h),\ \xi \wedge \delta K + \mathcal{B}_\xi(\delta h; K),\ \xi \wedge a\Big),$$

where:

1. $\mathcal{R}_\xi$ is the Riemann-curvature symbol in the metric variation, e.g.

$$(\mathcal{R}_\xi(\delta h))_{ijkl} = \frac{1}{2}(\xi_k\xi_i\delta h_{jl} + \xi_l\xi_j\delta h_{ik} - \xi_k\xi_j\delta h_{il} - \xi_l\xi_i\delta h_{jk}) .$$

   This block takes values in $\Lambda^2 T_x^* M \otimes \Lambda^2 T_x^* M$ with the standard Riemann symmetries.

2. $\xi \wedge \delta K$ is the Codazzi principal symbol, defined by

$$((\xi \wedge \delta K)(X, Y))Z := \xi(X)\,\delta K(Y, Z) - \xi(Y)\,\delta K(X, Z).$$

3. $\xi \wedge a$ is the curvature symbol of the normal connection variation, defined by

$$((\xi \wedge a)(X, Y)) := \xi(X)\,a(Y) - \xi(Y)\,a(X) \in \mathrm{End}(N).$$

4. $\mathcal{B}_\xi(\delta h; K)$ is first order in δh and vanishes at $K = 0$.

8.4 Gauge Symbol and Reduced Symbol

The infinitesimal gauge action generated by $(V, \phi) \in \Gamma(TM) \oplus \Gamma(\mathfrak{so}(N))$ is

$$\delta_{(V,\phi)} h = \mathcal{L}_V h, \qquad \delta_{(V,\phi)} K = \mathcal{L}_V K + \phi \cdot K, \qquad \delta_{(V,\phi)} \nabla^\perp = \mathcal{L}_V \nabla^\perp + d^{\nabla^\perp}\phi.$$

Its principal symbol at $\xi \neq 0$ is

$$\sigma_\xi(\delta_{\text{gauge}})(V, \phi) = \big(\xi \odot V^\flat,\ \mathcal{K}_\xi(V, \phi; K),\ \xi \otimes \phi\big),$$

where $\xi \odot V^\flat := \frac{1}{2}(\xi \otimes V^\flat + V^\flat \otimes \xi)$ is the symmetric product (principal symbol of $\mathcal{L}_V h$), and $\mathcal{K}_\xi(V, \phi; K)$ is linear in K and vanishes at $K = 0$. The reduced symbol is the quotient

$$\overline{\sigma}_\xi(\mathbf{L}_{(h,K,\nabla^\perp)}) := \sigma_\xi(\mathbf{L}_{(h,K,\nabla^\perp)}) \bmod \operatorname{im}\sigma_\xi(\delta_{\text{gauge}}),$$

which identifies genuine underdetermination directions as $\ker\sigma_\xi(\mathbf{L}_{(h,K,\nabla^\perp)})/\operatorname{im}\sigma_\xi(\delta_{\text{gauge}})$. The reduced characteristic set is defined by failure of injectivity of $\overline{\sigma}_\xi(\mathbf{L}_{(h,K,\nabla^\perp)})$.

Lemma 8.7 (Principal Gauge Symbol Computation)**.** *In a local frame at x, the principal blocks are*

$$\sigma_\xi(\mathcal{L}_V h) = \xi \odot V^\flat, \qquad \sigma_\xi(d^{\nabla^\perp}\phi) = \xi \otimes \phi,$$

and the K-block principal symbol is K-linear, hence vanishes at $K = 0$.

Proof. Using $\mathcal{L}_V h_{ij} = \nabla_i V_j + \nabla_j V_i$, principal order gives $\xi_i V_j + \xi_j V_i = \xi \odot V^\flat$. For the normal gauge block, $(d^{\nabla^\perp}\phi)_i = \nabla_i^\perp \phi + \cdots$, so principal order is $\xi_i\phi$, i.e. $\xi \otimes \phi$. For K, $\delta_{(V,\phi)} K = \mathcal{L}_V K + \phi \cdot K$, whose principal part is linear in coefficients of K; therefore it vanishes at $K = 0$. □

Lemma 8.8 (Quotient-Kernel Criterion)**.** *For linear maps $\sigma : U_x \to V_x$ and $g : \mathfrak{g}_x \to U_x$ with induced $\overline{\sigma}$ on $U_x/\operatorname{im} g$,*

$$\ker\overline{\sigma} \cong (\ker\sigma + \operatorname{im} g)/\operatorname{im} g.$$

Proof. This is the standard kernel description for induced maps on quotient spaces. □

9 Symbol Consequences and Non-Ellipticity

9.1 Frozen-Background Linearization (Warm-Up)

Linearize at $K = 0$ with h and $\nabla^\perp$ fixed. Then $\delta R^h = 0$, $\delta R^\perp = 0$, and

$$\delta(A \wedge K)\big|_{K=0} = 0, \qquad \delta(K \wedge A)\big|_{K=0} = 0.$$

Only the first-order Codazzi block remains:

$$L : \Gamma(\mathrm{Sym}^2 T^*M \otimes N) \to \Omega^2(\mathrm{Hom}(TM, N)), \qquad L(K) := D^{\mathrm{Hom}} K.$$

Theorem 9.1 (Frozen-Background Kernel Dimension)**.** *Let M be a surface ($n = 2$) and N have rank k. For any non-zero covector $\xi \in T^*M$, the kernel of $\sigma_\xi(L)$ has dimension exactly k.*

Proof. The symbol is

$$(\sigma_\xi(L)\,\delta K)^\alpha_{ij|\ell} = \xi_i\,\delta K^\alpha_{j\ell} - \xi_j\,\delta K^\alpha_{i\ell}.$$

Domain dimension is $3k$. For $(i, j) = (1, 2)$ and $\ell \in \{1, 2\}$:

$$\xi_1 \delta K^\alpha_{2\ell} - \xi_2 \delta K^\alpha_{1\ell} = 0.$$

By linear change of basis in $T^*_x M$, it suffices to choose $\xi = dx^1$. Then $\delta K^\alpha_{2\ell} = 0$, so $\delta K^\alpha_{21} = \delta K^\alpha_{22} = 0$, and symmetry $\delta K^\alpha_{ij} = \delta K^\alpha_{ji}$ gives $\delta K^\alpha_{12} = 0$. Only δK^α_{11} is free. Hence $\dim\ker\sigma_\xi(L) = k$, independent of nonzero ξ. Therefore L is not elliptic. □

Proposition 9.2 (Frozen Gauge Symbol Does Not Remove Codazzi Kernel)**.** *Fix $(h, \nabla^\perp)$ and linearize at $K = 0$. For $\xi \in T^*_x M \setminus \{0\}$, the principal symbol of the infinitesimal gauge action satisfies*

$$\sigma_\xi(\delta_{\mathrm{gauge}})(V, \phi) = \big(\xi \odot V^\flat,\ 0,\ \xi \otimes \phi\big).$$

In particular, the δK-component of $\operatorname{im}\sigma_\xi(\delta_{\mathrm{gauge}})$ vanishes.

Proof. At $K = 0$, one has

$$\delta_{(V,\phi)} K = \mathcal{L}_V K + \phi \cdot K = 0.$$

Hence the principal gauge symbol has no δK-component. The remaining principal blocks are $\xi \odot V^\flat$ on the metric and $\xi \otimes \phi$ on the normal connection. □

Theorem 9.3 (Reduced Frozen Symbol Has k-Dimensional Kernel)**.** *Fix $(h, \nabla^\perp)$ and linearize at $K = 0$. Let $\overline{\sigma}_\xi(L)$ denote the Codazzi principal symbol induced on the quotient by $\operatorname{im}\sigma_\xi(\delta_{\mathrm{gauge}})$. Then for every $\xi \neq 0$,*

$$\dim\ker\overline{\sigma}_\xi(L) = k.$$

Consequently, the compatibility operator is not elliptic at $K = 0$ even after principal-symbol gauge reduction.

Proof. The previous theorem gives $\dim\ker\sigma_\xi(L) = k$. By the preceding proposition, $\operatorname{im}\sigma_\xi(\delta_{\mathrm{gauge}})$ has zero δK-component at $K = 0$, so quotienting does not remove any δK kernel direction. Therefore $\ker\overline{\sigma}_\xi(L) = \ker\sigma_\xi(L)$ and has dimension k. □

9.2 Non-Ellipticity Regimes and Rank Structure

The full operator has mixed order and persistent symbol kernels:

1. metric directions enter at order 2 through $\mathcal{R}_\xi$;
2. K- and $\nabla^\perp$-directions enter at order 1;
3. gauge directions survive at symbol level through $\xi \odot V^\flat$ and $\xi \otimes \phi$.

Consequently, ellipticity fails generically before gauge reduction; after gauge reduction, injectivity still fails in the frozen background for the surface codimension-k family above.

Remark 9.4 (Representative Frozen Codazzi Ranks (Illustrative))**.** For normalized nonzero covector $\xi = e^1$ and dimensions $n = 2, 3, 4, 5$, the frozen Codazzi symbol per normal direction has:

n	$\dim \mathrm{Dom}$	$\dim \mathrm{Codom}$	$\mathrm{rank}(\sigma_\xi)$	$\dim \ker(\sigma_\xi)$
2	3	2	2	1
3	6	9	5	1
4	10	24	9	1
5	15	50	14	1

This table is illustrative only and consistent with the all-dimension theorem above. Values are obtained by symbolic matrix-rank computation for the map

$$(\sigma_\xi \delta K)_{ij|\ell} = \xi_i \delta K_{j\ell} - \xi_j \delta K_{i\ell},$$

first per normal direction and then multiplied by codimension k.

Theorem 9.5 (All-Dimension Frozen Codazzi Kernel)**.** *Let $n \geq 2$ and N have rank k. For nonzero covector ξ, consider*

$$\sigma_\xi(L) : \ \mathrm{Sym}^2(T_x^*M) \otimes N_x \to \Lambda^2(T_x^*M) \otimes T_x^*M \otimes N_x,$$

given by $(\sigma_\xi \delta K)^\alpha_{ij|\ell} = \xi_i \delta K^\alpha_{j\ell} - \xi_j \delta K^\alpha_{i\ell}$. Then $\dim \ker \sigma_\xi(L) = k$.

Proof. By linear invariance, choose $\xi = e^1$. Then

$$(\sigma_\xi \delta K)^\alpha_{ij|\ell} = \delta_{i1} \delta K^\alpha_{j\ell} - \delta_{j1} \delta K^\alpha_{i\ell}.$$

For $i = 1 < j$, all equations force $\delta K^\alpha_{j\ell} = 0$ for $j > 1$ and all ℓ. By symmetry in (j, ℓ), every component with at least one index > 1 vanishes. The only free component per normal direction is δK^α_{11}. Hence per normal direction nullity is 1, and total nullity is k. Equivalently, invariantly

$$\ker \sigma_\xi(L) \cong \mathrm{span}\{\xi \otimes \xi\} \otimes N_x.$$

□

Theorem 9.6 (Reduced Non-Ellipticity at Frozen Background)**.** *Fix $(h, \nabla^\perp)$ and background $K = 0$. For every nonzero $\xi \in T_x^*M$,*

$$\dim \ker \overline{\sigma}_\xi(\mathbf{L}_{(h,0,\nabla^\perp)}) \geq k.$$

Hence $\mathbf{L}_{(h,0,\nabla^\perp)}$ is not reduced-elliptic.

Proof. Restrict to the δK-subspace and use the frozen Codazzi block L. By the all-dimension kernel theorem, $\dim \ker \sigma_\xi(L) = k$. By the frozen gauge-symbol proposition, the gauge image has zero δK-component at $K = 0$, so these kernel directions survive in the quotient. Therefore the reduced kernel of the full linearization has dimension at least k. □

9.3 Minimal Prolongation Identities from Bianchi

Theorem 9.7 (First Universal Compatibility Conditions)**.** *For the system $\mathcal{D} = 0$, the only universal off-shell identities obtained functorially from $\nabla^{\oplus}$ are the coupled Bianchi identities from Section 4.*

Proof. The system is equivalent to $\mathcal{F}^{\oplus} = 0$. Applying $D^{\oplus}$ gives $D^{\oplus}\mathcal{F}^{\oplus} = 0$, whose block projections are precisely the coupled identities for $(\mathcal{G}, \mathcal{C}, \mathcal{S})$. Section 6 shows closure under repeated covariant differentiation in the curvature-generated ideal, so these identities generate the first prolongation-level universal consequences. □

Remark 9.8. Here "universal" means gauge-natural jet-level identities holding for all $(h, K, \nabla^{\perp})$, not identities tied to specific solution subclasses.

10 Examples and Specializations

10.1 Codimension-One Reduction

If $\operatorname{rank}(N) = 1$, then $\mathrm{SO}(1)$ is trivial, so every metric normal connection is flat and $R^{\perp} \equiv 0$. The system reduces to the classical Gauss–Codazzi constraints with the same curvature convention.

10.2 Consistency Cases

Three basic checks are immediate:

1. $K = 0$: $\mathcal{C} = 0$, $\mathcal{G} = R^h$, $\mathcal{S} = R^{\perp}$.
2. $R^{\perp} = 0$: the Ricci defect is purely quadratic, $\mathcal{S} = -K \wedge A$.
3. Space-form tangent geometry ($R^h = \kappa\, h \wedge h$): Gauss defect is $\kappa\, h \wedge h - A \wedge K$.

In each case, Section 4 reduces to standard Bianchi-type identities for the surviving blocks.

A Component Translation

Restricting to an orthonormal frame e_i and normal frame n_α, the invariant blocks correspond to the classical Gauss-Codazzi-Ricci equations: here $\nabla^{h,\perp}$ denotes the induced connection on $\mathrm{Sym}^2 T^*M \otimes N$ (Levi-Civita on tangent indices and $\nabla^{\perp}$ on the normal index). The second fundamental form components satisfy $K^{\alpha}_{ij} = K^{\alpha}_{ji}$. Define

$$\mathcal{G}_{ijkl} := \langle \mathcal{G}(e_i, e_j)e_k, e_l \rangle_h, \qquad \mathcal{C}^{\alpha}_{ijk} := \langle \mathcal{C}(e_i, e_j)e_k, n_\alpha \rangle_N, \qquad \mathcal{S}_{\alpha\beta ij} := \langle \mathcal{S}(e_i, e_j)n_\alpha, n_\beta \rangle_N.$$

1. **Gauss:** $\mathcal{G}_{ijkl} = R^h_{ijkl} - \sum_\alpha (K^\alpha_{ik} K^\alpha_{jl} - K^\alpha_{il} K^\alpha_{jk})$.
2. **Codazzi:** $\mathcal{C}^\alpha_{ijk} = \nabla^{h,\perp}_k K^\alpha_{ij} - \nabla^{h,\perp}_j K^\alpha_{ik}$.
3. **Ricci:** $\mathcal{S}_{\alpha\beta ij} = R^{\perp}_{\alpha\beta ij} - \sum_k (K^\alpha_{ik} K^\beta_{jk} - K^\beta_{ik} K^\alpha_{jk})$.

Remark A.1. With the curvature convention fixed above, these Gauss/Ricci sign conventions agree with standard references [3, 2].

B Variation Formulas

For metric variation δh on (M, h), the Levi-Civita variation has principal part

$$\delta\Gamma^i_{jk} = \frac{1}{2}h^{i\ell}\big(\nabla_j \delta h_{k\ell} + \nabla_k \delta h_{j\ell} - \nabla_\ell \delta h_{jk}\big),$$

and curvature variation has principal part

$$(\delta R^h)^i{}_{jkl} = \nabla_k(\delta\Gamma^i_{jl}) - \nabla_l(\delta\Gamma^i_{jk}) + \text{l.o.t.}$$

For normal connection variation $a = \delta\nabla^\perp \in \Omega^1(M, \mathrm{End}(N))$:

$$\delta R^\perp = D^\perp a + \text{l.o.t.}$$

For mixed block variation:

$$\delta(D^{\mathrm{Hom}} K) = D^{\mathrm{Hom}}\delta K + (\delta\nabla^\perp)\cdot K - K\cdot(\delta\nabla^h) + \text{l.o.t.}$$

These formulas justify the principal symbols stated in Sections 7 and 8.

C Symbol Tables

Block	δh	δK	$a = \delta\nabla^\perp$
$\sigma_\xi(\delta\mathcal{G})$	$\mathcal{R}_\xi(\delta h)$	0	0
$\sigma_\xi(\delta\mathcal{C})$	$\mathcal{B}_\xi(\delta h; K)$	$\xi \wedge \delta K$	0
$\sigma_\xi(\delta\mathcal{S})$	0	0	$\xi \wedge a$

Gauge Generator	Field Variation	Principal Symbol
$V \in \Gamma(TM)$	$\delta h = \mathcal{L}_V h$	$\xi \odot V^\flat$
$\phi \in \Gamma(\mathfrak{so}(N))$	$\delta\nabla^\perp = d^{\nabla^\perp}\phi + \cdots$	$\xi \otimes \phi$

At $K = 0$, $\mathcal{B}_\xi(\delta h; K) = 0$ and the K-gauge symbol contribution vanishes, recovering the frozen-background symbol used in Section 8.1.

References

[1] I. Kolář, P. W. Michor, and J. Slovák, *Natural Operations in Differential Geometry*, Springer, 1993.

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[3] M. Spivak, *A Comprehensive Introduction to Differential Geometry, Vol. IV*, Publish or Perish, 3rd edition, 1999.