Supplementary: Cosmological Constant Problem

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0.1 The 10^{123} Problem

Quantum field theory predicts vacuum energy density $\rho_{\text{vac}} \sim M_P^4 \sim 10^{76}$ GeV^4 from zero-point fluctuations. The observed dark energy density is $\rho_\Lambda \sim 10^{-47}$ GeV^4, a ratio of 10^{123}. This is the cosmological constant problem.

Standard approaches (supersymmetry, anthropic selection, quintessence) either fail to cancel the vacuum energy or provide no predictive mechanism. The problem persists because QFT and general relativity are treated as independent theories with no geometric constraint relating them.

Overdetermined embedding resolves this by showing that QFT and gravity are jointly constrained by embedding geometry, which bounds Λ to cosmological scales.

0.2 Height Field and Dimensional Emergence

The minimum curvature constraint $K_G \geq K_{\text{min}}^2$ prevents flatness, forcing the manifold to curve into the ambient space. This creates a height field $x^3(\varphi^1, \varphi^2)$ measuring displacement in the normal direction.

The height field satisfies a Poisson equation:

$$\frac{\partial^2 x^3}{\partial(\phi^1)^2} + \frac{\partial^2 x^3}{\partial(\phi^2)^2} = K(\phi^1, \phi^2),$$

where $K = 2H^2 - K_G$ with $H = (\kappa_1 + \kappa_2)/2$ the mean curvature.

When $x^3 \neq 0$, the 2-dimensional surface extends into a 3-dimensional spatial manifold: $\mathcal{M}^2 \to \mathcal{M}^3$. The volume element becomes:

$$dV = \sqrt{\det(h_{ab})}\, |x^3(\phi^1, \phi^2)|\, d\phi^1 d\phi^2.$$

0.3 Hamiltonian Constraint from Volume Element

The volume element determines the measure on configuration space $\Phi = \mathrm{Riem}(\mathcal{M}^3) \times \mathrm{Sym}(\mathcal{M}^3)^3 \times \Omega^1(\mathcal{M}^3)$. Variation of the action $S = \int L \, d\mu$ with respect to the lapse function yields the Hamiltonian constraint:

$$H = \sqrt{\det(h)} \left[\pi_{ij}\pi^{ij} - \frac{1}{2}(\pi^i_i)^2 - R \right] = 0,$$

where π_{ij} are momenta conjugate to h_{ij} and R is the Ricci scalar.

This coincides with the ADM constraint:

$$H_{\mathrm{ADM}} = R + K^2 - K_{ij}K^{ij} - 16\pi G\rho = 0,$$

through the relation $\pi_{ij} = \sqrt{\det(h)}(K_{ij} - h_{ij}K)$.

0.4 Time as Configuration Space Parameter

The Hamiltonian constraint $H = 0$ is algebraic, containing no time derivatives. Time t functions as a parameter labeling configurations in timeless configuration space $\Phi = (h_{ij}, K_{ij}, \rho)$ rather than a structural dimension.

Under reparameterization $t \to t' = f(t)$, the lapse transforms as $N' = N/f'(t)$ while (h_{ij}, K_{ij}, R) remain invariant. Time is gauge, not structure.

0.5 Cosmological Constant from Einstein Equations

The Einstein field equations with cosmological constant are:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

In vacuum ($T_{\mu\nu} = 0$), the cosmological constant contributes to spacetime curvature:

$$R = 4\Lambda.$$

If spacetime is an overdetermined embedding, curvature is bounded: $|R| \lesssim K^2_{\min}$. This immediately implies:

$$|\Lambda| \lesssim K^2_{\min}.$$

In timeless configuration space, fundamental geometric scales coincide. The minimum curvature sets length scale $L_{\min} = 1/K_{\min}$; the Hubble radius $R_H = c/H_0$ characterizes cosmological structure. Therefore:

$$K_{\min} \sim \frac{H_0}{c}.$$

From the Gauss equation for 2D manifolds:

$$R_{\text{min}} = 2K_{\text{min}}^2.$$

The effective cosmological constant:

$$\Lambda_{\text{eff}} = \frac{3}{2}K_{\text{min}}^2 = \frac{3}{2}\left(\frac{H_0}{c}\right)^2 \sim H_0^2.$$

0.6 Numerical Comparison with Planck 2018

Using $H_0 = 67.4$ km/s/Mpc $= 2.184 \times 10^{-18}$ s^{-1} and $c = 2.998 \times 10^8$ m/s:

$$K_{\text{min}} = 7.29 \times 10^{-27} \text{ m}^{-1},$$

$$\Lambda_{\text{predicted}} = 7.96 \times 10^{-53} \text{ m}^{-2}.$$

The observed value $\Lambda_{\text{observed}} = 1.09 \times 10^{-52}$ m^{-2} gives:

$$\frac{\Lambda_{\text{observed}}}{\Lambda_{\text{predicted}}} = 1.37.$$

Agreement within a factor of 1.37 demonstrates the cosmological constant emerges from geometry without fine-tuning.

0.7 UV Regulator Mechanism

The resolution does not require QFT vacuum energy to vanish. Vacuum fluctuations exist and carry energy, yet this energy does not curve spacetime arbitrarily; the embedding geometry bounds the effective contribution to Λ.

High-energy vacuum fluctuations are geometrically suppressed. The embedding acts as a UV regulator: contributions above the curvature scale $K_{\text{min}}^{-1/2}$ cannot propagate to macroscopic curvature. Embedding structure regularizes other divergences through the same mechanism.

The 10^{123} discrepancy is resolved: QFT and gravity are not independent theories but are jointly constrained by the embedding geometry, which bounds Λ to cosmological scales.

0.8 Falsification Condition

If dark energy measurements reveal $\Lambda > K_{\text{min}}^2$ (e.g., through an equation of state $w < -1$ that grows with time), overdetermined embedding is falsified. The geometric bound $|\Lambda| \leq K_{\text{min}}^2 \sim H_0^2$ is a firm prediction that can be tested by precision cosmology.