Structure Equations for the Orthogonal Compatibility Connection

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Abstract

The orthogonal compatibility connection on $E = TM \oplus N$ packages the Gauss–Weingarten system as an $\mathrm{SO}(n+k)$ connection. Its curvature decomposes into Gauss, Codazzi, and Ricci defect blocks, so flatness is exactly the constraint system. Coupled Bianchi identities and ideal-closure properties are derived in invariant form. For surfaces ($n = 2$), the principal symbol of the frozen-background Codazzi operator has kernel dimension exactly k at every nonzero covector, so the operator is not elliptic; for $k > 2$, this kernel exceeds the frozen gauge symbol at $K = 0$.

1 Introduction

1.1 Standing Assumptions

Let M^n be a smooth manifold with Riemannian metric h, and let (N, g_N) be a rank-k Riemannian vector bundle equipped with an arbitrary metric connection $\nabla^\perp$. The extrinsic curvature data is a section $K \in \Gamma(\mathrm{Hom}(TM, \mathrm{Hom}(TM, N)))$ with symmetry $K(X, Y) = K(Y, X)$, equivalently $K \in \Gamma(\mathrm{Sym}^2 T^*M \otimes N)$. The shape operator $A : N \to \mathrm{End}(TM)$ is defined by the adjoint relation $\langle A_\nu X, Y\rangle_h = \langle K(X, Y), \nu\rangle_N$. No embedding, immersion, or real-analytic hypothesis is imposed. The bundle $E = TM \oplus N$ is treated as an abstract direct-sum bundle. The Levi-Civita connection ∇^h is torsion-free. For any connection ∇, the curvature convention is $R(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. Here, "integrable" means locally gauge-trivial as an $\mathrm{SO}(n + k)$-connection, equivalently $\mathcal{F}^\oplus = 0$; no immersion claim is made without additional hypotheses.

The assembled connection is the Gauss–Weingarten system in block form:

$$\nabla^\oplus = \nabla^h \oplus \nabla^\perp + \Psi, \qquad \Psi = \begin{pmatrix} 0 & -K^T \\ K & 0 \end{pmatrix},$$

and its curvature blocks are exactly the Gauss, Codazzi, and Ricci defects.

Organization. Section 2 defines the assembled connection and its uniqueness. Section 3 computes the curvature block decomposition with explicit type conventions. Section 4 states the coupled Bianchi identities. Sections 5 and 6 establish gauge-natural completeness (as an operator-classification statement) and closure of the curvature-generated differential ideal. Section 7 gives the symbol calculation under frozen-background linearization, and the appendix translates the invariant blocks to component equations.

2 Foundational Precision

2.1 The Assembled Connection

Define a connection on the direct-sum bundle $E = TM \oplus N$ that couples the intrinsic and extrinsic data.

Definition 2.1 (Orthogonal Compatibility Connection)**.** Let ∇^h be the Levi-Civita connection of (M, h). The **Orthogonal Compatibility Connection** $\nabla^\oplus$ is defined explicitly for sections $Y \in \Gamma(TM)$ and $\nu \in \Gamma(N)$ by:

$$\nabla^\oplus_X(Y \oplus \nu) := \Big(\nabla^h_X Y - A_\nu(X)\Big) \oplus \Big(\nabla^\perp_X \nu + K(X, Y)\Big). \tag{1}$$

Proposition 2.2. *The connection $\nabla^\oplus$ is metric-compatible with respect to the direct sum metric $g_E = h \oplus g_N$.*

Proof. For sections $s_1 = Y_1 \oplus \nu_1, s_2 = Y_2 \oplus \nu_2$:

$$\begin{aligned} X\langle s_1, s_2\rangle_E &= X(\langle Y_1, Y_2\rangle_h + \langle \nu_1, \nu_2\rangle_N) \\ &= \langle \nabla^h_X Y_1, Y_2\rangle_h + \langle Y_1, \nabla^h_X Y_2\rangle_h + \langle \nabla^\perp_X \nu_1, \nu_2\rangle_N + \langle \nu_1, \nabla^\perp_X \nu_2\rangle_N. \end{aligned}$$

The expression $\langle \nabla^\oplus_X s_1, s_2\rangle_E + \langle s_1, \nabla^\oplus_X s_2\rangle_E$ produces the same diagonal terms plus cross terms

$$-\langle A_{\nu_1} X, Y_2\rangle_h + \langle K(X, Y_1), \nu_2\rangle_N - \langle Y_1, A_{\nu_2} X\rangle_h + \langle \nu_1, K(X, Y_2)\rangle_N,$$

which cancel by the adjoint relation between A and K. Therefore $\nabla^\oplus$ is metric-compatible. □

2.2 Uniqueness

$$\nabla_{product} := \nabla^h \oplus \nabla^\perp.$$

Theorem 2.3 (Uniqueness of the Assembled Connection)**.** *$\nabla^\oplus$ is the unique metric connection on E whose restriction to diagonal blocks coincides with $\nabla^h \oplus \nabla^\perp$ and whose off-diagonal block $TM \to N$ is given by K.*

Proof. Equivalently, the off-diagonal condition is $\mathrm{pr}_N(\nabla_X(Y \oplus 0)) = K(X, Y)$ for all $X, Y \in TM$. Any connection can be written as $\nabla = \nabla_{product} + \Psi$, where $\Psi \in \Omega^1(M, \mathrm{End}(E))$. Metric compatibility requires Ψ to be skew-symmetric with respect to g_E. Since the diagonal blocks are fixed, $\Psi_{TM\to TM} = 0$ and $\Psi_{N\to N} = 0$. The off-diagonal block $\Psi_{TM\to N}$ is given by K. By skew-symmetry, the remaining block $\Psi_{N\to TM}$ is uniquely determined to be $-A$.

$$\Psi_X = \begin{pmatrix} 0 & B_X \\ K_X & 0 \end{pmatrix}, \qquad \Psi^*_X = -\Psi_X$$

with respect to $h \oplus g_N$. Therefore $B_X = -K_X^T$, i.e. $B_X(\nu) = -A_\nu X$. Thus specifying the $TM \to N$ block determines the $N \to TM$ block uniquely. □

3 Curvature Block Decomposition

The obstruction to integrability is the curvature

$$\mathcal{F}^{\oplus} := (\nabla^{\oplus})^2 \in \Omega^2(M, \mathrm{End}(E)).$$

The convention $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is used for all connections. The invariant defect blocks are defined globally by:

$$\mathcal{G} := (\mathcal{F}^{\oplus})_{TM}, \quad \mathcal{C} := (\mathcal{F}^{\oplus})_{Mix}, \quad \mathcal{S} := (\mathcal{F}^{\oplus})_{N}. \tag{2}$$

Here $(\mathcal{F}^{\oplus})_{Mix} \in \Omega^2(M, \mathrm{Hom}(TM, N))$ denotes the $TM \to N$ block. Thus

$$\mathcal{G} \in \Omega^2(M, \mathrm{End}(TM)), \qquad \mathcal{C} \in \Omega^2(M, \mathrm{Hom}(TM, N)), \qquad \mathcal{S} \in \Omega^2(M, \mathrm{End}(N)).$$

Let $D^{product}$ denote the exterior covariant derivative induced by $\nabla_{product} = \nabla^h \oplus \nabla^\perp$, and let

$$\mathcal{F}_{product} := (\nabla_{product})^2 = R^h \oplus R^\perp \in \Omega^2(M, \mathrm{End}(E)).$$

The perturbation equation is $\mathcal{F}^{\oplus} = \mathcal{F}_{product} + D^{product}\Psi + \Psi \wedge \Psi$. Here $\wedge$ is the composition wedge product:

$$(\alpha \wedge \beta)(X,Y) = \alpha_X \circ \beta_Y - \alpha_Y \circ \beta_X.$$

This composition wedge is the graded product in $\Omega^\bullet(M, \mathrm{End}(E))$; the displayed equation is the degree-1 specialization. Also, view

$$A \in \Omega^1(M, \mathrm{Hom}(N, TM)), \qquad K \in \Omega^1(M, \mathrm{Hom}(TM, N)),$$

via $A_X(\nu) := A_\nu X$ and $K_X(Y) := K(X,Y)$. Then

$$(A \wedge K)(X,Y) := A_X \circ K_Y - A_Y \circ K_X \in \mathrm{End}(TM),$$

$$(K \wedge A)(X,Y) := K_X \circ A_Y - K_Y \circ A_X \in \mathrm{End}(N).$$

Transpose Conventions. For each $X \in TM$, define $K_X \in \mathrm{Hom}(TM, N)$ by $K_X(Y) = K(X,Y)$ and its metric adjoint

$$K_X^T \in \mathrm{Hom}(N, TM), \qquad \langle K_X^T \nu, Y\rangle_h = \langle \nu, K_X Y\rangle_N.$$

Then $K_X^T(\nu) = A_\nu X$, so K^T and A are the same $\mathrm{Hom}(N, TM)$-valued 1-form. Also

$$\mathcal{C} \in \Omega^2(M, \mathrm{Hom}(TM, N)), \qquad \mathcal{C}^T \in \Omega^2(M, \mathrm{Hom}(N, TM)),$$

with $(\mathcal{C}^T)_{X,Y} := (\mathcal{C}_{X,Y})^T$.

The induced connection on $\mathrm{Hom}(TM, N)$ is

$$\nabla_X^{\mathrm{Hom}} T = \nabla_X^\perp \circ T - T \circ \nabla_X^h, \qquad T \in \Gamma(\mathrm{Hom}(TM, N)),$$

and the induced exterior covariant derivative on $\mathrm{Hom}(TM, N)$-valued 1-forms is

$$(D^{\mathrm{Hom}} K)(X,Y) := \nabla_X^{\mathrm{Hom}} K_Y - \nabla_Y^{\mathrm{Hom}} K_X - K_{[X,Y]}.$$

Theorem 3.1 (Curvature Equations)**.** *The curvature 2-form decomposes into:*

$$\mathcal{G} = R^h - A \wedge K \tag{3}$$

$$\mathcal{C} = D^{\mathrm{Hom}} K \tag{4}$$

$$\mathcal{S} = R^{\perp} - K \wedge A \tag{5}$$

with D^{Hom} as defined above.

Proof. Write $\nabla^{\oplus} = \nabla_{product} + \Psi$ with $\nabla_{product} = \nabla^h \oplus \nabla^{\perp}$ and

$$\Psi = \begin{pmatrix} 0 & -K^T \\ K & 0 \end{pmatrix}.$$

Also, $\Psi \wedge \Psi$ has only diagonal blocks: since Ψ_X maps $TM \to N$ and $N \to TM$, the composition $\Psi_X \circ \Psi_Y$ preserves TM and N, so mixed blocks of $\Psi \wedge \Psi$ vanish.

$$\begin{aligned} (\mathcal{F}^{\oplus})_{TM} &= R^h + (\Psi \wedge \Psi)_{TM} = R^h - A \wedge K, \\ (\mathcal{F}^{\oplus})_{N} &= R^{\perp} + (\Psi \wedge \Psi)_{N} = R^{\perp} - K \wedge A, \\ (\mathcal{F}^{\oplus})_{Mix} &= (D^{product}\Psi)_{Mix}. \end{aligned}$$

Indeed, with P_{TM}, P_N the splitting projections, $P_{TM}\Psi P_{TM} = 0 = P_N \Psi P_N$ and $[\nabla^{product}, P_{TM}] = [\nabla^{product}, P_N] = 0$, so $P_{TM}(D^{product}\Psi)P_{TM} = 0$ and $P_N(D^{product}\Psi)P_N = 0$. Explicitly, for $Z \in TM$,

$$((\Psi \wedge \Psi)_{TM}(X,Y))Z = \Psi_X(\Psi_Y Z) - \Psi_Y(\Psi_X Z) = -A_{K(Y,Z)}X + A_{K(X,Z)}Y = -(A \wedge K)(X,Y)Z.$$

Also,

$$((\Psi \wedge \Psi)(X,Y))_{TM \to N} = 0$$

because each composition $\Psi_X \circ \Psi_Y$ sends $TM \to TM$ and $N \to N$, so there is no $TM \to N$ component. For the mixed block, define $K_X \in \mathrm{Hom}(TM, N)$ by $K_X(Y) = K(X,Y)$ and use

$$(D^{product}\Psi)(X,Y) = \nabla_X^{product}\Psi_Y - \nabla_Y^{product}\Psi_X - \Psi_{[X,Y]}.$$

Projecting to $\mathrm{Hom}(TM, N)$ and evaluating on $Z \in \Gamma(TM)$ gives

$$\begin{aligned} ((D^{product}\Psi)_{Mix}(X,Y))Z &= (\nabla_X^{\mathrm{Hom}} K_Y)Z - (\nabla_Y^{\mathrm{Hom}} K_X)Z - K_{[X,Y]}Z \\ &= (D^{\mathrm{Hom}} K)(X,Y)Z, \end{aligned}$$

Therefore $\mathcal{C} = D^{\mathrm{Hom}} K$, proving the stated block equations. □

Proposition 3.2 (Flatness Equivalent to Gauss–Codazzi–Ricci)**.**

$$\mathcal{F}^{\oplus} = 0 \quad \Longleftrightarrow \quad \mathcal{G} = 0,\ \mathcal{C} = 0,\ \mathcal{S} = 0.$$

Equivalently, flatness of $\nabla^{\oplus}$ is exactly the Gauss, Codazzi, and Ricci system.

Proof. By block decomposition,

$$\mathcal{F}^{\oplus} = \begin{pmatrix} \mathcal{G} & -\mathcal{C}^T \\ \mathcal{C} & \mathcal{S} \end{pmatrix}.$$

Hence $\mathcal{F}^{\oplus} = 0$ iff each block vanishes. □

4 Coupled Bianchi Identities

Let D^T and $D^\perp$ denote the exterior covariant derivatives on $\mathrm{End}(TM)$ and $\mathrm{End}(N)$-valued forms, respectively; D^T is induced by ∇^h on TM, and $D^\perp$ is induced by $\nabla^\perp$ on N. The operator $D^{product}$ acts blockwise as D^T on $\mathrm{End}(TM)$, as $D^\perp$ on $\mathrm{End}(N)$, and as D^{Hom} on $\mathrm{Hom}(TM, N)$. All wedges in this section are composition wedges, and all brackets $[\cdot,\cdot]$ are graded commutators in $\Omega^\bullet(M, \mathrm{End}(E))$.

$$[\alpha, \beta]_{\mathrm{gr}} := \alpha \wedge \beta - (-1)^{pq} \beta \wedge \alpha, \qquad \alpha \in \Omega^p(M, \mathrm{End}(E)),\ \beta \in \Omega^q(M, \mathrm{End}(E)).$$

The exterior covariant derivative $D^\oplus$ is a graded derivation:

$$D^\oplus(\alpha \wedge \beta) = D^\oplus \alpha \wedge \beta + (-1)^p \alpha \wedge D^\oplus \beta, \qquad \alpha \in \Omega^p(M, \mathrm{End}(E)).$$

Theorem 4.1 (Universal Bianchi Identities)**.** *The defect tensors satisfy the following coupled identities:*

$$D^T \mathcal{G} + [K^T, \mathcal{C}] = 0 \tag{6}$$
$$D^{\mathrm{Hom}} \mathcal{C} + K \wedge \mathcal{S} - \mathcal{G} \wedge K = 0 \tag{7}$$
$$D^\perp \mathcal{S} + [K, \mathcal{C}^T] = 0 \tag{8}$$

where $[K^T, \mathcal{C}] = K^T \wedge \mathcal{C} - \mathcal{C}^T \wedge K$*. The first identity lies in* $\Omega^3(M, \mathrm{End}(TM))$*, the second in* $\Omega^3(M, \mathrm{Hom}(TM, N))$*, and the third in* $\Omega^3(M, \mathrm{End}(N))$*.*

Proof. Write the connection perturbation and curvature in block form:

$$\Psi = \begin{pmatrix} 0 & -K^T \\ K & 0 \end{pmatrix}, \qquad \mathcal{F}^\oplus = \begin{pmatrix} \mathcal{G} & -\mathcal{C}^T \\ \mathcal{C} & \mathcal{S} \end{pmatrix}.$$

Using $D^\oplus = D^{product} + [\Psi, \cdot]_{\mathrm{gr}}$, the Bianchi identity is

$$0 = D^\oplus \mathcal{F}^\oplus = D^{product} \mathcal{F}^\oplus + [\Psi, \mathcal{F}^\oplus]_{\mathrm{gr}}.$$

Since $\deg \Psi = 1$ and $\deg \mathcal{F}^\oplus = 2$ (so $1 \cdot 2$ is even), $[\Psi, \mathcal{F}^\oplus]_{\mathrm{gr}} = \Psi \wedge \mathcal{F}^\oplus - \mathcal{F}^\oplus \wedge \Psi$. A direct block computation gives

$$[\Psi, \mathcal{F}^\oplus]_{\mathrm{gr}} = \begin{pmatrix} K^T \wedge \mathcal{C} - \mathcal{C}^T \wedge K & K^T \wedge \mathcal{S} - \mathcal{G} \wedge K^T \\ \mathcal{S} \wedge K - K \wedge \mathcal{G} & K \wedge \mathcal{C}^T - \mathcal{C} \wedge K^T \end{pmatrix}.$$

For example, the (TM, TM) block is

$$([\Psi, \mathcal{F}^\oplus]_{\mathrm{gr}})_{TM} = (\Psi \wedge \mathcal{F}^\oplus)_{TM} - (\mathcal{F}^\oplus \wedge \Psi)_{TM} = [(-K^T) \wedge \mathcal{C}] - [(-\mathcal{C}^T) \wedge K] = K^T \wedge \mathcal{C} - \mathcal{C}^T \wedge K.$$

Equating (TM, TM), (Mix), and (N, N) blocks in $D^{product} \mathcal{F}^\oplus + [\Psi, \mathcal{F}^\oplus]_{\mathrm{gr}} = 0$ yields the three displayed identities. □

Remark 4.2 (Consistency Check)**.** If $K = 0$ (totally geodesic), then $K^T = A = 0$, hence $\mathcal{C} = 0$, and by the curvature equations $\mathcal{G} = R^h$, $\mathcal{S} = R^\perp$. The coupled identities reduce to $D^T R^h = 0$ and $D^\perp R^\perp = 0$, which are the standard Bianchi identities for the intrinsic connections.

5 Gauge-Natural Completeness

Completeness of $\mathcal{F}^{\oplus}$ as an obstruction is formulated below.

Theorem 5.1 (Completeness of the Obstruction)**.** *Let $\omega^{\oplus}$ be the principal $\mathrm{SO}(n+k)$-connection induced by $\nabla^{\oplus}$ on $P_{\mathrm{SO}(E)}$, and let P be any classical gauge-natural differential operator (in the sense of KMS) constructed from $\omega^{\oplus}$ and its finite jet prolongations, mapping to a tensor bundle. If P vanishes whenever $\mathcal{F}^{\oplus}=0$, then P factors through finite-order jets of $\mathcal{F}^{\oplus}$.*

Proof. The connection $\nabla^{\oplus}$ determines uniquely a principal connection $\omega^{\oplus}$ on $P_{\mathrm{SO}(E)}$. A classical gauge-natural operator means a local finite-order functorial differential operator on associated bundles, equivariant under principal automorphisms. By the KMS regularity theorem [1], such operators depend on finite jets of $\omega^{\oplus}$, hence universally on finite jets of curvature and covariant derivatives. Here "flat data" means exactly $\mathcal{F}^{\oplus}=0$. Therefore any such operator vanishing on $\mathcal{F}^{\oplus}=0$ factors through finite jets of $\mathcal{F}^{\oplus}$. This is a classification of local invariants; it does not imply formal integrability, involutivity, or PDE solvability for immersion equations. □

6 Hierarchy Closure

Theorem 6.1 ($D^{\oplus}$-Stable Graded Two-Sided Ideal)**.** *Let $\mathcal{I}$ be the graded two-sided ideal in $\Omega^{\bullet}(M,\mathrm{End}(E))$ generated by $\mathcal{F}^{\oplus}$:*

$$\mathcal{I}=\Big\{\sum_i \alpha_i\wedge\mathcal{F}^{\oplus}\wedge\beta_i : \alpha_i,\beta_i\in\Omega^{\bullet}(M,\mathrm{End}(E))\Big\}.$$

Then $\mathcal{I}$ is stable under $D^{\oplus}$.

Proof. By Bianchi, $D^{\oplus}\mathcal{F}^{\oplus}=0$. Since $D^{\oplus}$ is a graded derivation for composition wedge, for homogeneous α,β one has

$$D^{\oplus}(\alpha\wedge\mathcal{F}^{\oplus}\wedge\beta)=(D^{\oplus}\alpha)\wedge\mathcal{F}^{\oplus}\wedge\beta+(-1)^{|\alpha|}\alpha\wedge(D^{\oplus}\mathcal{F}^{\oplus})\wedge\beta+(-1)^{|\alpha|+2}\alpha\wedge\mathcal{F}^{\oplus}\wedge D^{\oplus}\beta.$$

The middle term vanishes, and the remaining terms are again in the two-sided ideal generated by $\mathcal{F}^{\oplus}$. Hence $D^{\oplus}\mathcal{I}\subseteq\mathcal{I}$. No commutativity assumption is used: the argument only uses graded derivation and two-sided ideal structure in the non-commutative algebra $\Omega^{\bullet}(M,\mathrm{End}(E))$. □

Remark 6.2. For associated bundles (here $\mathrm{End}(E)$-valued forms), the curvature representation identity $(D^{\oplus})^2\eta=[\mathcal{F}^{\oplus},\eta]$ also holds.

7 Structural Interpretation

Proposition 7.1 (Structural Interpretation of Compatibility)**.** *Let $E=TM\oplus N$ with metric $h\oplus g_N$, let ∇^h be the torsion-free Levi-Civita connection on TM, let $\nabla^{\perp}$ be a metric connection on N, and let $K\in\Gamma(\mathrm{Sym}^2T^*M\otimes N)$ define the mixed block via the canonical inclusion $TM\hookrightarrow E$. Then $\nabla^{\oplus}$ induces a principal $\mathrm{SO}(n+k)$-connection $\omega^{\oplus}$ on $P_{\mathrm{SO}(E)}$ such that:*

1. *local flatness/integrability of the Gauss–Weingarten system is equivalent to $\mathcal{F}^{\oplus}=0$;*
2. *$\mathcal{F}^{\oplus}$ is the complete local structure function, i.e. every local gauge-natural obstruction factors through finite jets of $\mathcal{F}^{\oplus}$.*

Proof. Item (1) is exactly the Gauss–Codazzi–Ricci flatness criterion encoded by the block equations above. Flatness of $\omega^{\oplus}$ implies local principal-bundle trivialization with local parallel adapted orthonormal frames on simply connected neighborhoods. This is an integrability statement for the abstract $\mathrm{SO}(n+k)$-connection on E, not yet an immersion statement. Under additional classical hypotheses of the fundamental theorem of submanifolds (compatible first/second fundamental data on a simply connected local domain), this specializes to local immersion realization into Euclidean space [3, 2]. No global embedding claim is made. Item (2) is the gauge-natural completeness theorem in the previous section, applied to $\omega^{\oplus}$. □

8 Linearization and Symbol Analysis

The linearization of the compatibility operator is analyzed:

$$\mathcal{D}_{h,\nabla^{\perp}} : \Gamma(\mathrm{Sym}^2 T^*M \otimes N) \to \Omega^2(\mathrm{End}(TM)) \oplus \Omega^2(\mathrm{Hom}(TM, N)) \oplus \Omega^2(\mathrm{End}(N)).$$

Linearizing at $K = 0$ with h and $\nabla^{\perp}$ fixed, only the Codazzi block contributes at principal level. The principal symbol convention is

$$\sigma_{\xi}(L)(\delta K) = \text{coefficient of highest-order derivatives in } L(e^{i\lambda\phi}\delta K) \text{ at } \xi = d\phi,$$

equivalently the standard coordinate-symbol definition. Since h is frozen, $\delta R^h = 0$, and the Gauss/Ricci blocks are quadratic in K (via $-A \wedge K$ and $-K \wedge A$), so their first variations vanish at $K = 0$, e.g.

$$\delta(A \wedge K)\big|_{K=0} = 0, \qquad \delta(K \wedge A)\big|_{K=0} = 0.$$

The principal symbol is exactly that of the first-order Codazzi block. Define the restricted first-order operator

$$L : \Gamma(\mathrm{Sym}^2 T^*M \otimes N) \to \Omega^2(\mathrm{Hom}(TM, N)), \qquad L(K) := D^{\mathrm{Hom}} K.$$

Thus L is first-order, and the principal symbol of the full defect map, restricted to δK-variations with frozen $(h, \nabla^{\perp})$, is exactly $\sigma(L)$.

Theorem 8.1 (Algebraic Kernel Dimension)**.** *Let M be a surface ($n = 2$) and N have rank k. For any non-zero covector $\xi \in T^*M$, the kernel of the principal symbol map $\sigma_{\xi}(\delta K)$ has dimension exactly k.*

Proof. This is a dimension count in the mixed block (Codazzi). The symbol is

$$(\sigma_{\xi}(L)\, \delta K)^{\alpha}_{ij|\ell} = \xi_i\, \delta K^{\alpha}_{j\ell} - \xi_j\, \delta K^{\alpha}_{i\ell}.$$

Domain dimension (δK): components in $\mathrm{Sym}^2(\mathbb{R}^2) \otimes \mathbb{R}^k$. Dimension $3 \times k = 3k$. For $(i, j) = (1, 2)$ and $\ell \in \{1, 2\}$, $\alpha \in \{1, \ldots, k\}$, the symbol equation is $\xi_1 \delta K^{\alpha}_{2\ell} - \xi_2 \delta K^{\alpha}_{1\ell} = 0$. By linear invariance under change of basis in $T^*_x M$, it suffices to choose $\xi = dx^1$ for $\xi \neq 0$. Then $0 = \delta K^{\alpha}_{2\ell}$ for $\ell \in \{1, 2\}$ and $\alpha \in \{1, \ldots, k\}$, so $\delta K^{\alpha}_{21} = \delta K^{\alpha}_{22} = 0$; by symmetry $\delta K^{\alpha}_{21} = \delta K^{\alpha}_{12}$, hence $\delta K^{\alpha}_{12} = 0$. Unknowns per α: $\delta K^{\alpha}_{11}, \delta K^{\alpha}_{12}, \delta K^{\alpha}_{22}$. Constraints per α: $\delta K^{\alpha}_{21} (= \delta K^{\alpha}_{12}) = 0$ and $\delta K^{\alpha}_{22} = 0$. Hence only δK^{α}_{11} is free, and $\dim \ker \sigma_{\xi} = k$. Therefore $\dim \ker \sigma_{\xi}$ is independent of nonzero ξ. Since $\sigma_{\xi}(L)$ is not injective for all $\xi \neq 0$, the linearized operator is not elliptic. □

Remark 8.2. Within the restricted problem where only K varies and $(h, \nabla^{\perp})$ are frozen, the diffeomorphism symbol on K vanishes at $K = 0$. Thus, for $k > 2$, the k kernel dimensions exceed the (trivial) gauge kernel, implying genuine underdetermination.

A Component Translation

Restricting to an orthonormal frame e_i and normal frame n_α, the invariant blocks correspond to the classical Gauss-Codazzi-Ricci equations: here $\nabla^{h,\perp}$ denotes the induced connection on $\mathrm{Sym}^2 T^*M \otimes N$ (Levi-Civita on tangent indices and $\nabla^\perp$ on the normal index). Define

$$\mathcal{G}_{ijkl} := \langle \mathcal{G}(e_i, e_j)e_k, e_l \rangle_h, \qquad \mathcal{C}^\alpha_{ijk} := \langle \mathcal{C}(e_i, e_j)e_k, n_\alpha \rangle_N, \qquad \mathcal{S}_{\alpha\beta ij} := \langle \mathcal{S}(e_i, e_j)n_\alpha, n_\beta \rangle_N.$$

1. **Gauss:** $\mathcal{G}_{ijkl} = R^h_{ijkl} - \sum_\alpha (K^\alpha_{ik}K^\alpha_{jl} - K^\alpha_{il}K^\alpha_{jk})$.
2. **Codazzi:** $\mathcal{C}^\alpha_{ijk} = \nabla^{h,\perp}_k K^\alpha_{ij} - \nabla^{h,\perp}_j K^\alpha_{ik}$.
3. **Ricci:** $\mathcal{S}_{\alpha\beta ij} = R^\perp_{\alpha\beta ij} - \sum_k (K^\alpha_{ik}K^\beta_{jk} - K^\beta_{ik}K^\alpha_{jk})$.

Remark A.1. With the curvature convention fixed above, these Gauss/Ricci sign conventions agree with standard references [3, 2].

References

[1] I. Kolář, P. W. Michor, and J. Slovák, *Natural Operations in Differential Geometry*, Springer, 1993.

[2] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol. II*, Wiley-Interscience, 1969.

[3] M. Spivak, *A Comprehensive Introduction to Differential Geometry, Vol. IV*, Publish or Perish, 3rd edition, 1999.