# On the Integrability of Extrinsic Constraint Data

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Given $(h, K, \nabla^\perp)$, one canonically constructs a metric connection on $TM \oplus N$ whose flatness is equivalent to the Gauss–Codazzi–Ricci system. On simply connected domains, flatness integrates to a framed local isometric immersion, unique up to rigid motion and $\nabla^\perp$-parallel $\mathrm{SO}(k)$ gauge. In codimension one, Gauss–Codazzi contractions recover the standard hypersurface constraint identities; under Einstein's equations these become the Hamiltonian and momentum constraints.

Cartan moving-frame structure equations encode the Gauss–Codazzi–Ricci system as a flatness condition for a connection on an orthonormal frame bundle.

We work with triples $(h, K, \nabla^\perp)$, where $h$ is a Riemannian metric on $M$, $K \in \Gamma(\mathrm{Sym}^2 T^*M \otimes N)$, and $\nabla^\perp$ is a metric connection on $N$, modulo diffeomorphisms of $M$ and orthogonal bundle automorphisms of $N$ covering them.

The GCR system is equivalent to flatness of the canonical metric connection on $TM \oplus N$; on simply connected neighborhoods this yields framed local Euclidean realization, recovering the classical fundamental theorem[6,10,11].

In codimension one, the same formalism reduces to Gauss/Codazzi constraints for $(h, K)$ and, in Lorentzian ambient geometry with Einstein equations, to Hamiltonian and momentum constraints.

*a. Main result.* On simply connected $U \subset M$, solutions of the GCR system are in one-to-one correspondence with framed isometric immersions $X : U \to \mathbb{R}^{n+k}$ modulo Euclidean motions, and this correspondence is induced by the flat canonical connection on $TM \oplus N$.

## I. CONSTRAINT DATA

Let $M^n$ be a smooth connected manifold (without boundary). Let $N \to M$ be a real vector bundle of rank $k$ equipped with a positive–definite fiber metric $\langle \cdot, \cdot \rangle_N$ and a metric connection $\nabla^\perp$.

*a. Conventions.* Use

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

For connection 1-forms $\omega_{ij}$, set

$$\mathcal{R}_{ij} := d\omega_{ij} + \omega_{ik} \wedge \omega_{kj}.$$

For an embedded hypersurface with unit normal $n$, set $K_{ij} = \langle \nabla_{e_i} n, e_j \rangle$ (no minus sign). Indices $i, j, k, \ell$ run over tangent indices $1, \ldots, n$, and $\alpha, \beta, \gamma$ over normal indices $1, \ldots, k$.

**Definition I.1** (GCR data)**.** A *GCR triple* on $(M, N)$ is a triple $(h, K, \nabla^\perp)$ where:

- $h$ is a Riemannian metric on $M$ with Levi–Civita connection $\nabla^h$;
- $K \in \Gamma(\mathrm{Sym}^2 T^*M \otimes N)$ is a symmetric $N$–valued $(0, 2)$–tensor;
- $\nabla^\perp$ is a metric connection on $N$.

Choose local $h$–orthonormal tangent frames $\{e_i\}_{i=1}^n$ with dual coframe $\{\theta^i\}$, and local orthonormal frames $\{n_\alpha\}_{\alpha=1}^k$ of $N$, so $K = \sum_\alpha K^{(\alpha)} \otimes n_\alpha$. Let $\omega_{ij}$ denote the Levi–Civita connection 1-forms, and $\omega_{\alpha\beta}$ the normal connection 1-forms. The Levi–Civita connection satisfies the torsion-free structure equation

$$d\theta^i + \omega_{ij} \wedge \theta^j = 0. \tag{1}$$

Define the mixed 1-forms

$$\omega_{i\alpha} := \sum_{j=1}^n K_{ij}^{(\alpha)} \, \theta^j, \qquad \omega_{\alpha i} := -\omega_{i\alpha}, \tag{2}$$

where $K_{ij}^{(\alpha)} := K^{(\alpha)}(e_i, e_j)$.

*b. Covariant derivative of $K$.* Write $\nabla K$ for the induced covariant derivative on $\mathrm{Sym}^2 T^*M \otimes N$:

$$\begin{aligned}(\nabla_X K)(Y, Z) = \nabla^\perp_X\big(K(Y, Z)\big) - K\big(\nabla^h_X Y, Z\big) \\ - K\big(Y, \nabla^h_X Z\big).\end{aligned}$$

**Definition I.2** (Gauss–Codazzi–Ricci equations)**.** The triple $(h, K, \nabla^\perp)$ satisfies the *Gauss–Codazzi–Ricci (GCR) equations* if, in any local adapted orthonormal frames, the following hold:

$$\text{(Gauss)} \quad R_{ijk\ell} = \sum_{\alpha=1}^k \Big( K_{ik}^{(\alpha)} K_{j\ell}^{(\alpha)} - K_{i\ell}^{(\alpha)} K_{jk}^{(\alpha)} \Big). \tag{3}$$

$$\text{(Codazzi)} \quad (\nabla_{e_i} K)_{jk}^{(\alpha)} = (\nabla_{e_j} K)_{ik}^{(\alpha)}. \tag{4}$$

$$\text{(Ricci)} \quad R^\perp_{\alpha\beta ij} = \sum_{m=1}^n \Big( K_{im}^{(\alpha)} K_{jm}^{(\beta)} - K_{im}^{(\beta)} K_{jm}^{(\alpha)} \Big). \tag{5}$$

Here $R_{ijk\ell}$ are the components of the Riemann curvature tensor of $(M, h)$, and $R^\perp_{\alpha\beta ij}$ are the components of the curvature of $\nabla^\perp$.

**Definition I.3** (Gauge equivalence)**.** Two triples $(h, K, \nabla^\perp)$ and $(\tilde{h}, \tilde{K}, \tilde{\nabla}^\perp)$ are *equivalent* if there exists a diffeomorphism $\varphi \in \mathrm{Diff}(M)$ and an orthogonal bundle isomorphism $g : N \to N$ covering $\varphi$ such that

$$\tilde{h} = \varphi^* h, \qquad \tilde{K} = g \cdot (\varphi^* K), \qquad \tilde{\nabla}^\perp = g \circ (\varphi^* \nabla^\perp) \circ g^{-1}.$$

**Remark I.4** (Ambient signatures)**.** The construction extends to pseudo-Riemannian ambient signatures by replacing $\mathrm{SO}(n+k)$ with $\mathrm{SO}(p,q)$ and adjusting sign conventions.

## II. FLATNESS

### A. The Orthogonal Compatibility Connection

Consider the orthogonal sum bundle $E = TM \oplus N$ equipped with the direct sum metric. An adapted orthonormal frame for $E$ is a tuple $(e_1, \ldots, e_n, n_1, \ldots, n_k)$ where $\{e_i\}$ frames $TM$ and $\{n_\alpha\}$ frames $N$.

Given a GCR triple $(h, K, \nabla^\perp)$, define a metric connection $\nabla^\oplus$ on $E = TM \oplus N$ by the Gauss–Weingarten equations: for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(N)$,

$$\nabla^\oplus_X(Y, 0) = (\nabla^h_X Y,\ K(X,Y)), \tag{6}$$
$$\nabla^\oplus_X(0, \xi) = (-A_\xi X,\ \nabla^\perp_X \xi), \tag{7}$$

where the shape operator $A_\xi : TM \to TM$ is defined by

$$h(A_\xi X, Y) = \langle K(X,Y), \xi \rangle_N.$$

Metric compatibility follows from the defining adjoint relation between $A_\xi$ and $K$.

In an adapted orthonormal frame, the connection 1-form matrix of $\nabla^\oplus$ is given block-wise by:

$$\Omega = \begin{pmatrix} \omega_{ij} & \omega_{i\alpha} \\ \omega_{\alpha i} & \omega_{\alpha\beta} \end{pmatrix}, \qquad \omega_{\alpha i} = -\omega_{i\alpha}. \tag{8}$$

The diagonal blocks are the Levi–Civita and normal connection forms, and the mixed block is $\omega_{i\alpha} = \sum_j K^{(\alpha)}_{ij} \theta^j$. The curvature 2-form is $\mathcal{F} = d\Omega + \Omega \wedge \Omega$.

### B. Flatness and the GCR equations

**Proposition II.1** (GCR $\iff$ flatness)**.** *The assembled connection $\nabla^\oplus$ is flat (i.e., $\mathcal{F} = 0$) if and only if the triple $(h, K, \nabla^\perp)$ satisfies the Gauss–Codazzi–Ricci equations.*

*Proof.* In adapted orthonormal frames, the curvature 2-form has block components

$$\mathcal{F}_{ij} = d\omega_{ij} + \omega_{ik} \wedge \omega_{kj} + \omega_{i\alpha} \wedge \omega_{\alpha j}, \tag{9}$$
$$\mathcal{F}_{i\alpha} = d\omega_{i\alpha} + \omega_{ik} \wedge \omega_{k\alpha} + \omega_{i\beta} \wedge \omega_{\beta\alpha}, \tag{10}$$
$$\mathcal{F}_{\alpha\beta} = d\omega_{\alpha\beta} + \omega_{\alpha i} \wedge \omega_{i\beta} + \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta}. \tag{11}$$

*$(i,j)$-block (Gauss):* Substitute $\omega_{i\alpha} = K^{(\alpha)}_{im} \theta^m$ into (9) and equate coefficients of $\theta^k \wedge \theta^\ell$; this yields (3).

*$(i,\alpha)$-block (Codazzi):* The component $\mathcal{F}_{i\alpha}$ is the covariant exterior derivative $D\omega_{i\alpha}$ with respect to the Levi–Civita and normal connections. In components,

$$\mathcal{F}_{i\alpha}(e_j, e_k) = (\nabla_{e_j} K)^{(\alpha)}_{ik} - (\nabla_{e_k} K)^{(\alpha)}_{ij},$$

so $\mathcal{F}_{i\alpha} = 0$ is equivalent to the Codazzi symmetry (4).

*$(\alpha,\beta)$-block (Ricci):* Substitute $\omega_{i\alpha} = K^{(\alpha)}_{im} \theta^m$ into (11) and equate coefficients; this yields (5). □

## III. INTEGRATION

Flatness integrates to a local realization on simply connected domains.

**Theorem III.1** (Local realization from GCR data)**.** *Let $(h, K, \nabla^\perp)$ satisfy the Gauss–Codazzi–Ricci equations on an open set $U \subset M$. Assume $U$ is simply connected. Then there exists a smooth map*

$$X : U \to \mathbb{R}^{n+k}$$

*and smooth fields $(E_1, \ldots, E_n, N_1, \ldots, N_k)$ along $X$ forming an orthonormal frame of $\mathbb{R}^{n+k}$ at each point, such that:*

1. *$X$ is a local isometric immersion: $dX(e_i) = E_i$ and $\langle dX(\cdot), dX(\cdot) \rangle = h$;*
2. *the second fundamental form and normal connection induced by the frame $(N_\alpha)$ coincide with $(K, \nabla^\perp)$ on $U$.*

*Proof.* Fix an adapted orthonormal frame $\mathbf{u}$ of $E|_U$. In this trivialization, $\nabla^\oplus$ is represented by the $\mathfrak{so}(n+k)$-valued 1-form $\Omega$ defined in (8). By Proposition II.1, $d\Omega + \Omega \wedge \Omega = 0$. By standard integration for flat principal connections on simply connected domains (e.g.[9]), there exists a smooth map

$$F : U \to \mathrm{SO}(n+k)$$

satisfying $F^{-1} dF = \Omega$.

Choose a parallel frame of the trivial bundle $U \times \mathbb{R}^{n+k}$ induced by the gauge $F$. Writing the columns of $F$ as $\mathbb{R}^{n+k}$-valued functions gives an orthonormal frame

$$(E_1, \ldots, E_n, N_1, \ldots, N_k)$$

satisfying

$$dE_i = \omega_{ij} E_j + \omega_{i\alpha} N_\alpha, \tag{12}$$
$$dN_\alpha = -\omega_{i\alpha} E_i + \omega_{\alpha\beta} N_\beta. \tag{13}$$

Define an $\mathbb{R}^{n+k}$-valued 1-form on $U$ by

$$\eta := \sum_{i=1}^n E_i\, \theta^i.$$

Using (12) and the torsion-free structure equation (1), one computes

$$\begin{aligned} d\eta &= \sum_i dE_i \wedge \theta^i + \sum_i E_i\, d\theta^i \\ &= \sum_{i,\alpha} \omega_{i\alpha} N_\alpha \wedge \theta^i + \sum_{i,j} (\omega_{ij} E_j \wedge \theta^i - E_i\, \omega_{ij} \wedge \theta^j). \end{aligned}$$

Since

$$\sum_{i,\alpha} \omega_{i\alpha} N_\alpha \wedge \theta^i = \sum_{i,j,\alpha} K_{ij}^{(\alpha)} N_\alpha\, \theta^j \wedge \theta^i = 0,$$

because $K_{ij}^{(\alpha)} = K_{ji}^{(\alpha)}$ while $\theta^j \wedge \theta^i$ is antisymmetric, and since $\omega_{ij} = -\omega_{ji}$, one gets $d\eta = 0$. Hence $\eta$ is closed. Since $U$ is simply connected, there exists $X : U \to \mathbb{R}^{n+k}$ with $dX = \eta$. Then $dX(e_i) = E_i$ and $\langle E_i, E_j \rangle = \delta_{ij}$ implies $X$ is an isometric immersion.

Differentiating $dX(e_i) = E_i$ and projecting the derivative of $E_i$ onto the normal span of $\{N_\alpha\}$ using (12) yields the second fundamental form coefficients $K_{ij}^{(\alpha)}$ (via (2)), while (13) gives the normal connection forms $\omega_{\alpha\beta}$, i.e. $\nabla^\perp$. □

**Theorem III.2** (Local uniqueness)**.** *Under the hypotheses of Theorem III.1, let $(X, F)$ and $(\tilde{X}, \tilde{F})$ be two realizations constructed from the same connection form $\Omega$ (i.e., using the same adapted frame). Then there exists a constant $A \in \mathrm{SO}(n+k)$ and a constant vector $b \in \mathbb{R}^{n+k}$ such that $\tilde{X} = AX + b$ and $\tilde{F} = AF$.*

*Equivalently, uniqueness holds modulo rigid motions once a gauge (trivialization) of $E|_U$ is fixed.*

*Proof.* If $F^{-1}dF = \Omega = \tilde{F}^{-1}d\tilde{F}$ on a simply connected domain, then $d(\tilde{F}F^{-1}) = 0$, hence $\tilde{F} = AF$ for constant $A \in \mathrm{SO}(n+k)$. Since $d\tilde{X} = \sum_i \tilde{E}_i \theta^i = \sum_i (AE_i)\theta^i = A\, dX$, it follows that $d(\tilde{X} - AX) = 0$, so $\tilde{X} = AX + b$ for constant $b$. □

**Remark III.3** (Gauge freedom)**.** Changing adapted normal frame corresponds to $g : U \to \mathrm{SO}(k)$. If $(h, K, \nabla^\perp)$ is fixed, residual normal gauge freedom is exactly $\nabla^\perp$-parallel $\mathrm{SO}(k)$-automorphisms.

## IV. LORENTZIAN HYPERSURFACES

The codimension one reduction isolates the standard contractions in Lorentzian hypersurface geometry.

### A. Codimension one reduction

Assume $\mathrm{rank}(N) = 1$ and fix a unit normal field $n$ (locally). Then $\nabla^\perp$ is trivial and $K$ is a symmetric $(0,2)$-tensor on $M$, so Ricci is identically satisfied and only Gauss/Codazzi remain.

### B. Lorentzian ambient geometry and constraint equations

Let $(\mathcal{M}^{n+1}, g)$ be a Lorentzian manifold with signature $(-, +, \dots, +)$, and let $M^n \hookrightarrow \mathcal{M}$ be a *spacelike* hypersurface with induced Riemannian metric $h$ and second fundamental form $K$ (defined by the conventions of Section I). Let $n^\mu$ be the future-pointing unit normal and $h^\mu{}_\nu = \delta^\mu{}_\nu + n^\mu n_\nu$ the tangential projector.

Standard contractions of the Gauss and Codazzi equations yield the geometric identities:

$$ {}^{(n)}R + K^2 - K_{ij}K^{ij} = 2\, G_{\mu\nu} n^\mu n^\nu, \tag{14}$$

$$ \nabla_j \left( K^j{}_i - K\, \delta^j{}_i \right) = -\, G_{\mu\nu} n^\mu h^\nu{}_i, \tag{15}$$

where ${}^{(n)}R$ is the scalar curvature of $(M, h)$, $K = K^i{}_i$ is the trace, and $G_{\mu\nu}$ is the Einstein tensor of $(\mathcal{M}, g)$.

**Remark IV.1** (Sign conventions)**.** The sign of the extrinsic term $K^2 - K_{ij}K^{ij}$ in (14) depends on the chosen signatures for the Riemann tensor, the normal $n$, and the definition of $K$. With our conventions fixed in Section I, this contraction matches the standard form found in[2,7].

Equations (14)–(15) are geometric Gauss–Codazzi contractions. Under Einstein's equations ($G_{\mu\nu} = 8\pi T_{\mu\nu}$), the right-hand sides become energy and momentum densities:

$$2G_{\mu\nu} n^\mu n^\nu = 16\pi\rho, \qquad -G_{\mu\nu} n^\mu h^\nu_i = 8\pi j_i.$$

In vacuum ($G_{\mu\nu} = 0$) they vanish, yielding the familiar Hamiltonian and momentum constraints of the 3+1 decomposition.

**Remark IV.2** (Canonical momentum in ADM)**.** In a 3+1 split, the momentum conjugate to $h_{ab}$ is

$$\pi^{ab} = \sqrt{h}\, (K^{ab} - K h^{ab}),$$

and the constraints are $\mathcal{H}(h, \pi) = 0$ and $\mathcal{H}_a(h, \pi) = 0$.

## DATA AVAILABILITY

The data that support the findings of this study are available within the article.


[1] R. Arnowitt, S. Deser, and C. W. Misner, *Republication of: The Dynamics of General Relativity*, Gen. Relativ. Gravit. **40** (2008), 1997–2027. (Originally in *Gravitation: an introduction to current research*, ed. L. Witten, Wiley, 1962.) See also arXiv:gr-qc/0405109.

[2] E. Gourgoulhon, 3+1 *Formalism and Bases of Numerical Relativity*, arXiv:gr-qc/0703035 (2007).

[3] R. Bartnik and J. Isenberg, *The Constraint Equations*, arXiv:gr-qc/0405092 (2004).

[4] C. Teitelboim, *How commutators of constraints reflect the spacetime structure*, Ann. Phys. **79** (1973), 542–557.

[5] P. A. M. Dirac, *The theory of gravitation in Hamiltonian form*, Proc. Roy. Soc. Lond. A **246** (1958), 333–343.

[6] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol. II*, Interscience/Wiley, 1969.

[7]R. M. Wald, *General Relativity*, University of Chicago Press, 1984.
[8]T. A. Ivey and J. M. Landsberg, *Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Systems*, Graduate Studies in Mathematics, Vol. 61, AMS, 2003.
[9]R. W. Sharpe, *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program*, Graduate Texts in Mathematics 166, Springer, 1997.
[10]M. Spivak, *A Comprehensive Introduction to Differential Geometry, Vol. IV*, Publish or Perish, 3rd ed.
[11]M. P. do Carmo, *Riemannian Geometry*, Birkhäuser, 1992.