Unified Curvature Dynamics of the Orthogonal Compatibility Connection

Abstract

For fixed background metrics (h, g_N), the orthogonal compatibility ansatz on $E = TM \oplus N$ defines an $\mathrm{SO}(n+k)$-connection whose curvature carries tangent, mixed, and normal Gauss–Codazzi–Ricci blocks. The canonical quadratic gauge–natural curvature functional is $S_{\mathrm{YM}} = \frac{1}{2g^2}\int_M \langle \mathcal{F}^\oplus, \mathcal{F}^\oplus \rangle \, dV_h$. Its Euler–Lagrange equation, $D^\oplus * \mathcal{F}^\oplus = 0$, is the Yang–Mills equation of the assembled connection. Tangent, mixed, and normal equations are the block projections of this single system.

1 Introduction

In [1, 2, 3], the orthogonal compatibility connection appears as the integrability object of Gauss–Codazzi–Ricci data. Here (M, h) and (N, g_N) are fixed background geometry, and $(\nabla^\perp, K)$ are dynamical fields through

$$\nabla^\oplus = \nabla^h \oplus \nabla^\perp + \Psi(K).$$

The minimal quadratic curvature action in this class is uniquely determined, and the resulting unified and blockwise equations follow from its variation. The minimal quadratic curvature action in this class is uniquely determined. Its variation yields the unified equation, whose block projections give the tangent, mixed, and normal systems.

2 Configuration Space and Symmetry

2.1 Geometric Data

Consider triples

$$(h, K, \nabla^\perp),$$

where h is a Riemannian metric on M^n,

$$K \in \Gamma(\mathrm{Sym}^2 T^*M \otimes N),$$

and $\nabla^\perp$ is a metric connection on the rank-k normal bundle N. Define $A : N \to \mathrm{End}(TM)$ by

$$\langle A_\nu X, Y \rangle_h = \langle K(X, Y), \nu \rangle_N.$$

Let D^{Hom} denote the induced covariant derivative on $\mathrm{Hom}(TM, N)$ from ∇^h and $\nabla^\perp$. Fix h and g_N throughout the variational principle, and define

$$\mathcal{C}_{h,g_N} := \left\{ (\nabla^\perp, K) \ : \ \nabla^\perp \text{ metric on } (N, g_N), \ K \in \Gamma(\mathrm{Sym}^2 T^*M \otimes N) \right\}.$$

2.2 The Assembled Orthogonal Connection

Set $E = TM \oplus N$ and define

$$\nabla^{\oplus} = \nabla^{h} \oplus \nabla^{\perp} + \Psi, \qquad \Psi = \begin{pmatrix} 0 & -K^{T} \\ K & 0 \end{pmatrix}.$$

2.3 Curvature Decomposition

Its curvature is

$$\mathcal{F}^{\oplus} = (\nabla^{\oplus})^{2} = \begin{pmatrix} \mathcal{G} & -\mathcal{C}^{T} \\ \mathcal{C} & \mathcal{S} \end{pmatrix},$$

with blocks

$$\mathcal{G} = R^{h} - A \wedge K, \qquad \mathcal{C} = D^{\mathrm{Hom}} K, \qquad \mathcal{S} = R^{\perp} - K \wedge A.$$

The block curvature identities are established in [2] in the connection-formalism context of [4]. Here all wedge products are composition wedges of bundle-valued forms.

2.4 Gauge Symmetry

The symmetry group is

$$\mathrm{Diff}(M) \ltimes \Gamma(\mathrm{SO}(N)),$$

and $\nabla^{\oplus}$ transforms as an $\mathrm{SO}(n+k)$-connection.

3 The Unified Curvature Functional

3.1 Quadratic Curvature Functional

Let $g_E := h \oplus g_N$ on $E = TM \oplus N$. This induces a metric on $\mathrm{End}(E)$ and the trace pairing on $\mathfrak{so}(E)$-valued forms. For $\alpha, \beta \in \Omega^{p}(M, \mathfrak{so}(E))$, set

$$\langle \alpha, \beta \rangle \, dV_h := \mathrm{tr}_{g_E}\big(\alpha \wedge *\beta\big),$$

where $*$ is the Hodge operator of (M, h). Define

$$S_{\mathrm{YM}} = \frac{1}{2g^{2}} \int_{M} \langle \mathcal{F}^{\oplus}, \mathcal{F}^{\oplus} \rangle \, dV_h,$$

where the inner product is induced by $h \oplus g_N$.

Definition 3.1 (Admissible Local Functional)**.** *A functional $S[\nabla^{\oplus}; h, g_N]$ is admissible if it is local, invariant under diffeomorphisms and $\Gamma(\mathrm{SO}(N))$, depends only on $(\mathcal{F}^{\oplus}, h, g_N)$, and involves no covariant derivatives of $\mathcal{F}^{\oplus}$.*

This gauge-natural locality class follows the framework of [7].

Lemma 3.2 (No Nontrivial Linear Invariant)**.** *For $\mathfrak{so}(n+k)$-valued curvature forms,*

$$\mathrm{tr}(\mathcal{F}^{\oplus}) = 0.$$

Hence no nontrivial gauge-invariant local action linear in $\mathcal{F}^{\oplus}$ exists.

Proof. At each point, $\mathcal{F}^{\oplus}$ takes values in skew endomorphisms of (E, g_E). Skew endomorphisms have zero trace. $\square$

Theorem 3.3 (Quadratic Minimality at Fixed Canonical Pairing)**.** *Assume $n + k \geq 3$. Fix the canonical* Ad*-invariant pairing on $\mathfrak{so}(E)$ induced by $g_E = h \oplus g_N$. Among admissible local functionals polynomial in $\mathcal{F}^{\oplus}$, the action*

$$\int_M \mathrm{tr}(\mathcal{F}^{\oplus} \wedge *\mathcal{F}^{\oplus})$$

is the unique nontrivial quadratic invariant, up to an overall scale and addition of Chern–Weil topological densities (in even dimensions).

Proof. Any quadratic invariant has the form

$$\mathcal{L}_2 = B_{ab}\, F^a \wedge *F^b,$$

where B is an Ad-invariant symmetric bilinear form on $\mathfrak{so}(n+k)$ and $\mathcal{F}^{\oplus} = F^a T_a$. Ad-invariance is

$$f_{ia}{}^{m} B_{mk} + f_{ik}{}^{m} B_{am} = 0$$

for structure constants $f_{ij}{}^{k}$. For $n+k \geq 5$, $\mathfrak{so}(n+k)$ is simple; for $n+k = 3$, $\mathfrak{so}(3)$ is also simple; for $n + k = 4$, $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ is semisimple. By standard Lie algebra structure theory (e.g. [8, Ch. I–II]), invariant symmetric bilinear forms on a simple Lie algebra are one-dimensional. Hence in the simple cases such forms are proportional to the Killing form, and the metric-trace pairing is proportional to this form [8]. In the $n+k = 4$ case, the invariant family is two-parameter, and fixing the canonical trace pairing from g_E selects the contraction used in S_{YM}. The linear term vanishes by the previous lemma. Admissibility excludes derivative terms such as $|D^{\oplus}\mathcal{F}^{\oplus}|^2$. Among admissible Lagrangians involving no derivatives of curvature, the lowest nontrivial homogeneous degree under scaling $F \mapsto \lambda F$ is therefore quadratic. Chern–Weil topological densities contribute only boundary terms and hence do not alter the Euler–Lagrange equation for compactly supported variations [9]. Therefore the stated quadratic action is unique in this class up to scale and topological addition. $\square$

3.2 Block Expansion

Orthogonality of blocks implies

$$|\mathcal{F}^{\oplus}|^2 = |\mathcal{G}|^2 + 2|\mathcal{C}|^2 + |\mathcal{S}|^2.$$

The factor 2 comes from equal contributions of the two off-diagonal blocks $\mathcal{C}$ and $-\mathcal{C}^T$ under the trace pairing. Hence

$$S_{\mathrm{YM}} = \frac{1}{2g^2} \int_M \Big(|R^h - A \wedge K|^2 + 2|D^{\mathrm{Hom}} K|^2 + |R^{\perp} - K \wedge A|^2 \Big)\, dV_h.$$

3.3 Interpretation of Sectors

The three projected equations correspond to the tangent, mixed, and normal curvature blocks of the $\mathrm{SO}(n+k)$ Yang–Mills equation.

4 Euler–Lagrange Equations

4.1 Unified Field Equation

Assume either M is compact without boundary or all variations are compactly supported. Then boundary terms vanish under covariant integration by parts. Let $(\nabla^\perp, K) \in \mathcal{C}_{h,g_N}$. Variations of the assembled connection are compactly supported and induced from $(\delta\nabla^\perp, \delta K)$. Since h and g_N are fixed, $\delta\mathcal{F}^\oplus = D^\oplus(\delta\nabla^\oplus)$. Covariant integration by parts yields [6, 4]

$$\delta S_{\mathrm{YM}} = \frac{1}{g^2}\int_M \langle \delta\nabla^\oplus, D^\oplus * \mathcal{F}^\oplus\rangle \, dV_h.$$

Hence the Euler–Lagrange equation is

$$D^\oplus * \mathcal{F}^\oplus = 0.$$

This is a second-order system in the dynamical fields $(\nabla^\perp, K)$ for fixed (h, g_N).

4.2 Normal Bundle Equation

The normal projection of $D^\oplus * \mathcal{F}^\oplus = 0$ is

$$D^\perp * \mathcal{S} + [K, *\mathcal{C}^T] = 0.$$

4.3 Extrinsic Curvature Equation

The mixed projection of $D^\oplus * \mathcal{F}^\oplus = 0$ is

$$D^{\mathrm{Hom}} * \mathcal{C} + (A \wedge *\mathcal{G}) + (*\mathcal{S} \wedge A) = 0.$$

Proposition 4.1 (Blockwise Decomposition)**.** *Relative to $E = TM \oplus N$, the unified equation*

$$D^\oplus * \mathcal{F}^\oplus = 0$$

is equivalent to the coupled projected system

$$\Pi_T(D^\oplus * \mathcal{F}^\oplus) = 0, \qquad \Pi_{\mathrm{mix}}(D^\oplus * \mathcal{F}^\oplus) = 0, \qquad \Pi_N(D^\oplus * \mathcal{F}^\oplus) = 0.$$

In particular, the normal and mixed projections yield the displayed normal-bundle and K equations.

Proof. Set

$$\Psi = \begin{pmatrix} 0 & -K^T \\ K & 0 \end{pmatrix}, \qquad *\mathcal{F}^\oplus = \begin{pmatrix} *\mathcal{G} & -(*\mathcal{C})^T \\ *\mathcal{C} & *\mathcal{S} \end{pmatrix},$$

and use

$$D^\oplus(*\mathcal{F}^\oplus) = D^{\mathrm{prod}}(*\mathcal{F}^\oplus) + [\Psi, *\mathcal{F}^\oplus].$$

Here D^{prod} is the block-diagonal covariant derivative induced by $\nabla^h \oplus \nabla^\perp$. Block commutator algebra yields

$$[\Psi, *\mathcal{F}^\oplus]_{TT} = -A(*\mathcal{C}) + (*\mathcal{C})^T K,$$
$$[\Psi, *\mathcal{F}^\oplus]_{TN} = -A(*\mathcal{S}) + (*\mathcal{G})A,$$
$$[\Psi, *\mathcal{F}^\oplus]_{NN} = (*\mathcal{C})A - K(*\mathcal{C})^T.$$

Hence the three block equations are exactly the projected equations

$$\Pi_T(D^\oplus * \mathcal{F}^\oplus) = 0, \quad \Pi_{\mathrm{mix}}(D^\oplus * \mathcal{F}^\oplus) = 0, \quad \Pi_N(D^\oplus * \mathcal{F}^\oplus) = 0.$$

Conversely, reconstruction of a block matrix from its projected blocks is exact, so vanishing of all three projections implies $D^\oplus * \mathcal{F}^\oplus = 0$. □

5 Vacuum Structure

5.1 Flat Vacuum

The flat configuration

$$\mathcal{F}^{\oplus} = 0$$

corresponds to totally geodesic compatibility data and satisfies the field equation.

6 Effective Low-Energy Regime

6.1 Small-K Expansion

In the regime of small K,

$$S_{\mathrm{YM}} = \frac{1}{2g^2}\int_M \Big(|R^h|^2 + |R^\perp|^2 + 2|D^{\mathrm{Hom}}K|^2\Big)\, dV_h + \text{terms quadratic in } K \text{ coupled to } R^h \text{ and } R^\perp.$$

7 Structural Unification Statement

Theorem 7.1 (Unified Curvature Dynamics)**.** *Under the admissibility hypotheses of Section 3, all dynamical sectors arise as block projections of the single equation*

$$D^{\oplus} * \mathcal{F}^{\oplus} = 0.$$

Proof. The unified Euler–Lagrange equation is $D^{\oplus} * \mathcal{F}^{\oplus} = 0$. By the blockwise decomposition proposition, this is equivalent to the coupled tangent, mixed, and normal projected equations. Hence all sector equations are projections of one connection equation. □

References

[1] C. Nacode, *On the Integrability of Extrinsic Constraint Data*, Zenodo, 2026. `https://doi.org/10.5281/zenodo.18689081`

[2] C. Nacode, *On the Structure of the Orthogonal Compatibility Connection*, Zenodo, 2026. `https://doi.org/10.5281/zenodo.18720045`

[3] C. Nacode, *Orthogonal Structure of Noether Identities*, Zenodo, 2026. `https://doi.org/10.5281/zenodo.18722210`

[4] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol. I*, Wiley, 1963.

[5] R. Utiyama, Invariant theoretical interpretation of interaction, *Phys. Rev.* **101** (1956), 1597–1607.

[6] D. S. Freed and K. K. Uhlenbeck, *Instantons and Four-Manifolds*, Springer, 1984.

[7] I. Kolář, P. W. Michor, and J. Slovák, *Natural Operations in Differential Geometry*, Springer, 1993.

[8] A. W. Knapp, *Lie Groups Beyond an Introduction*, 2nd ed., Birkhäuser, 2002.

[9] J. W. Milnor and J. D. Stasheff, *Characteristic Classes*, Princeton University Press, 1974.