

On the Integrability of Extrinsic Constraint Data

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Given (h, K, ∇^\perp) , one canonically constructs a metric connection on $TM \oplus N$ whose flatness is equivalent to the Gauss–Codazzi–Ricci system. On simply connected domains, flatness integrates to a framed local isometric immersion, unique up to rigid motion and ∇^\perp -parallel $SO(k)$ gauge. In codimension one, Gauss–Codazzi contractions recover the standard hypersurface constraint identities; under Einstein’s equations these become the Hamiltonian and momentum constraints.

Cartan moving-frame structure equations encode the Gauss–Codazzi–Ricci system as a flatness condition for a connection on an orthonormal frame bundle.

We work with triples (h, K, ∇^\perp) , where h is a Riemannian metric on M , $K \in \Gamma(\text{Sym}^2 T^*M \otimes N)$, and ∇^\perp is a metric connection on N , modulo diffeomorphisms of M and orthogonal bundle automorphisms of N covering them.

The GCR system is equivalent to flatness of the canonical metric connection on $TM \oplus N$; on simply connected neighborhoods this yields framed local Euclidean realization, recovering the classical fundamental theorem^{6,10,11}.

In codimension one, the same formalism reduces to Gauss/Codazzi constraints for (h, K) and, in Lorentzian ambient geometry with Einstein equations, to Hamiltonian and momentum constraints.

a. Main result. On simply connected $U \subset M$, solutions of the GCR system are in one-to-one correspondence with framed isometric immersions $X : U \rightarrow \mathbb{R}^{n+k}$ modulo Euclidean motions, and this correspondence is induced by the flat canonical connection on $TM \oplus N$.

I. CONSTRAINT DATA

Let M^n be a smooth connected manifold (without boundary). Let $N \rightarrow M$ be a real vector bundle of rank k equipped with a positive-definite fiber metric $\langle \cdot, \cdot \rangle_N$ and a metric connection ∇^\perp .

a. Conventions. Use

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

For connection 1-forms ω_{ij} , set

$$\mathcal{R}_{ij} := d\omega_{ij} + \omega_{ik} \wedge \omega_{kj}.$$

For an embedded hypersurface with unit normal n , set $K_{ij} = \langle \nabla_{e_i} n, e_j \rangle$ (no minus sign). Indices i, j, k, ℓ run over tangent indices $1, \dots, n$, and α, β, γ over normal indices $1, \dots, k$.

Definition I.1 (GCR data). A *GCR triple* on (M, N) is a triple (h, K, ∇^\perp) where:

- h is a Riemannian metric on M with Levi–Civita connection ∇^h ;

- $K \in \Gamma(\text{Sym}^2 T^*M \otimes N)$ is a symmetric N -valued $(0, 2)$ -tensor;

- ∇^\perp is a metric connection on N .

Choose local h -orthonormal tangent frames $\{e_i\}_{i=1}^n$ with dual coframe $\{\theta^i\}$, and local orthonormal frames $\{n_\alpha\}_{\alpha=1}^k$ of N , so $K = \sum_\alpha K^{(\alpha)} \otimes n_\alpha$. Let ω_{ij} denote the Levi–Civita connection 1-forms, and $\omega_{\alpha\beta}$ the normal connection 1-forms. The Levi–Civita connection satisfies the torsion-free structure equation

$$d\theta^i + \omega_{ij} \wedge \theta^j = 0. \quad (1)$$

Define the mixed 1-forms

$$\omega_{i\alpha} := \sum_{j=1}^n K_{ij}^{(\alpha)} \theta^j, \quad \omega_{\alpha i} := -\omega_{i\alpha}, \quad (2)$$

where $K_{ij}^{(\alpha)} := K^{(\alpha)}(e_i, e_j)$.

b. Covariant derivative of K . Write ∇K for the induced covariant derivative on $\text{Sym}^2 T^*M \otimes N$:

$$(\nabla_X K)(Y, Z) = \nabla_X^\perp(K(Y, Z)) - K(\nabla_X^h Y, Z) - K(Y, \nabla_X^h Z).$$

Definition I.2 (Gauss–Codazzi–Ricci equations). The triple (h, K, ∇^\perp) satisfies the *Gauss–Codazzi–Ricci (GCR) equations* if, in any local adapted orthonormal frames, the following hold:

$$(\text{Gauss}) \quad R_{ijk\ell} = \sum_{\alpha=1}^k \left(K_{ik}^{(\alpha)} K_{j\ell}^{(\alpha)} - K_{i\ell}^{(\alpha)} K_{jk}^{(\alpha)} \right). \quad (3)$$

$$(\text{Codazzi}) \quad (\nabla_{e_i} K)_{jk}^{(\alpha)} = (\nabla_{e_j} K)_{ik}^{(\alpha)}. \quad (4)$$

$$(\text{Ricci}) \quad R_{\alpha\beta ij}^\perp = \sum_{m=1}^n \left(K_{im}^{(\alpha)} K_{jm}^{(\beta)} - K_{im}^{(\beta)} K_{jm}^{(\alpha)} \right). \quad (5)$$

Here $R_{ijk\ell}$ are the components of the Riemann curvature tensor of (M, h) , and $R_{\alpha\beta ij}^\perp$ are the components of the curvature of ∇^\perp .

Definition I.3 (Gauge equivalence). Two triples (h, K, ∇^\perp) and $(\tilde{h}, \tilde{K}, \tilde{\nabla}^\perp)$ are *equivalent* if there exists a diffeomorphism $\varphi \in \text{Diff}(M)$ and an orthogonal bundle isomorphism $g : N \rightarrow N$ covering φ such that

$$\tilde{h} = \varphi^* h, \quad \tilde{K} = g \cdot (\varphi^* K), \quad \tilde{\nabla}^\perp = g \circ (\varphi^* \nabla^\perp) \circ g^{-1}.$$

Remark I.4 (Ambient signatures). The construction extends to pseudo-Riemannian ambient signatures by replacing $\text{SO}(n+k)$ with $\text{SO}(p, q)$ and adjusting sign conventions.

II. FLATNESS

A. The Orthogonal Compatibility Connection

Consider the orthogonal sum bundle $E = TM \oplus N$ equipped with the direct sum metric. An adapted orthonormal frame for E is a tuple $(e_1, \dots, e_n, n_1, \dots, n_k)$ where $\{e_i\}$ frames TM and $\{n_\alpha\}$ frames N .

Given a GCR triple (h, K, ∇^\perp) , define a metric connection ∇^\oplus on $E = TM \oplus N$ by the Gauss–Weingarten equations: for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(N)$,

$$\nabla_X^\oplus(Y, 0) = (\nabla_X^h Y, K(X, Y)), \quad (6)$$

$$\nabla_X^\oplus(0, \xi) = (-A_\xi X, \nabla_X^\perp \xi), \quad (7)$$

where the shape operator $A_\xi : TM \rightarrow TM$ is defined by

$$h(A_\xi X, Y) = \langle K(X, Y), \xi \rangle_N.$$

Metric compatibility follows from the defining adjoint relation between A_ξ and K .

In an adapted orthonormal frame, the connection 1-form matrix of ∇^\oplus is given block-wise by:

$$\Omega = \begin{pmatrix} \omega_{ij} & \omega_{i\alpha} \\ \omega_{\alpha i} & \omega_{\alpha\beta} \end{pmatrix}, \quad \omega_{\alpha i} = -\omega_{i\alpha}. \quad (8)$$

The diagonal blocks are the Levi–Civita and normal connection forms, and the mixed block is $\omega_{i\alpha} = \sum_j K_{ij}^{(\alpha)} \theta^j$. The curvature 2-form is $\mathcal{F} = d\Omega + \Omega \wedge \Omega$.

B. Flatness and the GCR equations

Proposition II.1 (GCR \iff flatness). *The assembled connection ∇^\oplus is flat (i.e., $\mathcal{F} = 0$) if and only if the triple (h, K, ∇^\perp) satisfies the Gauss–Codazzi–Ricci equations.*

Proof. In adapted orthonormal frames, the curvature 2-form has block components

$$\mathcal{F}_{ij} = d\omega_{ij} + \omega_{ik} \wedge \omega_{kj} + \omega_{i\alpha} \wedge \omega_{\alpha j}, \quad (9)$$

$$\mathcal{F}_{i\alpha} = d\omega_{i\alpha} + \omega_{ik} \wedge \omega_{k\alpha} + \omega_{i\beta} \wedge \omega_{\beta\alpha}, \quad (10)$$

$$\mathcal{F}_{\alpha\beta} = d\omega_{\alpha\beta} + \omega_{\alpha i} \wedge \omega_{i\beta} + \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta}. \quad (11)$$

(i, j)-block (Gauss): Substitute $\omega_{i\alpha} = K_{im}^{(\alpha)} \theta^m$ into (9) and equate coefficients of $\theta^k \wedge \theta^\ell$; this yields (3).

(i, α)-block (Codazzi): The component $\mathcal{F}_{i\alpha}$ is the covariant exterior derivative $D\omega_{i\alpha}$ with respect to the Levi–Civita and normal connections. In components,

$$\mathcal{F}_{i\alpha}(e_j, e_k) = (\nabla_{e_j} K)_{ik}^{(\alpha)} - (\nabla_{e_k} K)_{ij}^{(\alpha)},$$

so $\mathcal{F}_{i\alpha} = 0$ is equivalent to the Codazzi symmetry (4).

(α, β)-block (Ricci): Substitute $\omega_{i\alpha} = K_{im}^{(\alpha)} \theta^m$ into (11) and equate coefficients; this yields (5). \square

III. INTEGRATION

Flatness integrates to a local realization on simply connected domains.

Theorem III.1 (Local realization from GCR data). *Let (h, K, ∇^\perp) satisfy the Gauss–Codazzi–Ricci equations on an open set $U \subset M$. Assume U is simply connected. Then there exists a smooth map*

$$X : U \rightarrow \mathbb{R}^{n+k}$$

and smooth fields $(E_1, \dots, E_n, N_1, \dots, N_k)$ along X forming an orthonormal frame of \mathbb{R}^{n+k} at each point, such that:

1. *X is a local isometric immersion: $dX(e_i) = E_i$ and $\langle dX(\cdot), dX(\cdot) \rangle = h$;*
2. *the second fundamental form and normal connection induced by the frame (N_α) coincide with (K, ∇^\perp) on U .*

Proof. Fix an adapted orthonormal frame \mathbf{u} of $E|_U$. In this trivialization, ∇^\oplus is represented by the $\mathfrak{so}(n+k)$ -valued 1-form Ω defined in (8). By Proposition II.1, $d\Omega + \Omega \wedge \Omega = 0$. By standard integration for flat principal connections on simply connected domains (e.g.⁹), there exists a smooth map

$$F : U \rightarrow \text{SO}(n+k)$$

satisfying $F^{-1}dF = \Omega$.

Choose a parallel frame of the trivial bundle $U \times \mathbb{R}^{n+k}$ induced by the gauge F . Writing the columns of F as \mathbb{R}^{n+k} -valued functions gives an orthonormal frame

$$(E_1, \dots, E_n, N_1, \dots, N_k)$$

satisfying

$$dE_i = \omega_{ij} E_j + \omega_{i\alpha} N_\alpha, \quad (12)$$

$$dN_\alpha = -\omega_{i\alpha} E_i + \omega_{\alpha\beta} N_\beta. \quad (13)$$

Define an \mathbb{R}^{n+k} -valued 1-form on U by

$$\eta := \sum_{i=1}^n E_i \theta^i.$$

Using (12) and the torsion-free structure equation (1), one computes

$$\begin{aligned} d\eta &= \sum_i dE_i \wedge \theta^i + \sum_i E_i d\theta^i \\ &= \sum_{i,\alpha} \omega_{i\alpha} N_\alpha \wedge \theta^i + \sum_{i,j} (\omega_{ij} E_j \wedge \theta^i - E_i \omega_{ij} \wedge \theta^j). \end{aligned}$$

Since

$$\sum_{i,\alpha} \omega_{i\alpha} N_\alpha \wedge \theta^i = \sum_{i,j,\alpha} K_{ij}^{(\alpha)} N_\alpha \theta^j \wedge \theta^i = 0,$$

because $K_{ij}^{(\alpha)} = K_{ji}^{(\alpha)}$ while $\theta^j \wedge \theta^i$ is antisymmetric, and since $\omega_{ij} = -\omega_{ji}$, one gets $d\eta = 0$. Hence η is closed. Since U is simply connected, there exists $X : U \rightarrow \mathbb{R}^{n+k}$ with $dX = \eta$. Then $dX(e_i) = E_i$ and $\langle E_i, E_j \rangle = \delta_{ij}$ implies X is an isometric immersion.

Differentiating $dX(e_i) = E_i$ and projecting the derivative of E_i onto the normal span of $\{N_\alpha\}$ using (12) yields the second fundamental form coefficients $K_{ij}^{(\alpha)}$ (via (2)), while (13) gives the normal connection forms $\omega_{\alpha\beta}$, i.e. ∇^\perp . \square

Theorem III.2 (Local uniqueness). *Under the hypotheses of Theorem III.1, let (X, F) and (\tilde{X}, \tilde{F}) be two realizations constructed from the same connection form Ω (i.e., using the same adapted frame). Then there exists a constant $A \in \text{SO}(n+k)$ and a constant vector $b \in \mathbb{R}^{n+k}$ such that $\tilde{X} = AX + b$ and $\tilde{F} = AF$.*

Equivalently, uniqueness holds modulo rigid motions once a gauge (trivialization) of $E|_U$ is fixed.

Proof. If $F^{-1}dF = \Omega = \tilde{F}^{-1}d\tilde{F}$ on a simply connected domain, then $d(\tilde{F}F^{-1}) = 0$, hence $\tilde{F} = AF$ for constant $A \in \text{SO}(n+k)$. Since $d\tilde{X} = \sum_i \tilde{E}_i \theta^i = \sum_i (AE_i) \theta^i = A dX$, it follows that $d(\tilde{X} - AX) = 0$, so $\tilde{X} = AX + b$ for constant b . \square

Remark III.3 (Gauge freedom). Changing adapted normal frame corresponds to $g : U \rightarrow \text{SO}(k)$. If (h, K, ∇^\perp) is fixed, residual normal gauge freedom is exactly ∇^\perp -parallel $\text{SO}(k)$ -automorphisms.

IV. LORENTZIAN HYPERSURFACES

The codimension one reduction isolates the standard contractions in Lorentzian hypersurface geometry.

A. Codimension one reduction

Assume $\text{rank}(N) = 1$ and fix a unit normal field n (locally). Then ∇^\perp is trivial and K is a symmetric $(0, 2)$ -tensor on M , so Ricci is identically satisfied and only Gauss/Codazzi remain.

B. Lorentzian ambient geometry and constraint equations

Let (\mathcal{M}^{n+1}, g) be a Lorentzian manifold with signature $(-, +, \dots, +)$, and let $M^n \hookrightarrow \mathcal{M}$ be a *spacelike* hypersurface with induced Riemannian metric h and second fundamental form K (defined by the conventions of Section I). Let n^μ be the future-pointing unit normal and $h^\mu{}_\nu = \delta^\mu{}_\nu + n^\mu n_\nu$ the tangential projector.

Standard contractions of the Gauss and Codazzi equations yield the geometric identities:

$$^{(n)}R + K^2 - K_{ij}K^{ij} = 2G_{\mu\nu}n^\mu n^\nu, \quad (14)$$

$$\nabla_j (K^j{}_i - K \delta^j{}_i) = -G_{\mu\nu}n^\mu h^\nu{}_i, \quad (15)$$

where $^{(n)}R$ is the scalar curvature of (M, h) , $K = K^i{}_i$ is the trace, and $G_{\mu\nu}$ is the Einstein tensor of (\mathcal{M}, g) .

Remark IV.1 (Sign conventions). The sign of the extrinsic term $K^2 - K_{ij}K^{ij}$ in (14) depends on the chosen signatures for the Riemann tensor, the normal n , and the definition of K . With our conventions fixed in Section I, this contraction matches the standard form found in^{2,7}.

Equations (14)–(15) are geometric Gauss–Codazzi contractions. Under Einstein’s equations ($G_{\mu\nu} = 8\pi T_{\mu\nu}$), the right-hand sides become energy and momentum densities:

$$2G_{\mu\nu}n^\mu n^\nu = 16\pi\rho, \quad -G_{\mu\nu}n^\mu h^\nu{}_i = 8\pi j_i.$$

In vacuum ($G_{\mu\nu} = 0$) they vanish, yielding the familiar Hamiltonian and momentum constraints of the 3+1 decomposition.

Remark IV.2 (Canonical momentum in ADM). In a 3+1 split, the momentum conjugate to h_{ab} is

$$\pi^{ab} = \sqrt{h} (K^{ab} - K h^{ab}),$$

and the constraints are $\mathcal{H}(h, \pi) = 0$ and $\mathcal{H}_a(h, \pi) = 0$.

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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