Orthogonal Structure of Noether Identities

Abstract

Any finite-order gauge-invariant curvature Lagrangian on an $\mathrm{SO}(n+k)$ orthonormal frame bundle $P \to M$ produces an Euler field $\mathcal{E}(\omega^{\oplus}) \in \Omega^1(M, \mathrm{Ad}P)$ satisfying

$$(D^{\oplus})^{\dagger}\mathcal{E}(\omega^{\oplus}) \equiv 0,$$

equivalently $D^{\oplus}(\star\mathcal{E}(\omega^{\oplus})) \equiv 0$. The associated Hodge-dual current is covariantly closed, and the orthogonal splitting $E = TM \oplus N$ yields tangent, mixed, and normal components of this covariant Noether identity. This formulation isolates the canonical orthogonal block decomposition and places the Euler field in the curvature-generated algebra determined by $\omega^{\oplus}$.

1 Geometric Data

1.1 Base and connection

Let M^m be an oriented manifold and $E \to M$ a rank-$(n+k)$ Euclidean bundle. Let

$$P := P_{\mathrm{SO}(E)} \to M$$

be the orthonormal frame bundle. A principal $\mathrm{SO}(n+k)$-connection

$$\omega^{\oplus} \in \Omega^1(P, \mathfrak{so}(n+k)),$$

induces the covariant derivative

$$D^{\oplus} : \Omega^q(M, \mathrm{Ad}P) \to \Omega^{q+1}(M, \mathrm{Ad}P), \qquad \mathrm{Ad}P := P \times_{\mathrm{Ad}} \mathfrak{so}(n+k).$$

Its curvature is

$$\mathcal{F}^{\oplus} = d\omega^{\oplus} + \omega^{\oplus} \wedge \omega^{\oplus} \in \Omega^2(M, \mathrm{Ad}P), \qquad D^{\oplus}\mathcal{F}^{\oplus} = 0,$$

with standard conventions as in [2]. All objects are assumed smooth; variations and gauge parameters are compactly supported unless M is closed.

1.2 Pairing and gauge action

Fix an Ad-invariant nondegenerate pairing

$$\kappa : \mathfrak{so}(n+k) \otimes \mathfrak{so}(n+k) \to \mathbb{R}.$$

Fix background metric and orientation on M, and set

$$\langle U, V \rangle_{\star} := \kappa(U, \star V)$$

for equal-degree $\mathrm{Ad}P$-valued forms. Let

$$\mathcal{G} = \Gamma\big(M, P \times_{\mathrm{Ad}} \mathrm{SO}(n+k)\big), \qquad \mathrm{Lie}(\mathcal{G}) = \Omega^0(M, \mathrm{Ad}P),$$

with infinitesimal action

$$\delta_{\epsilon}\omega^{\oplus} = D^{\oplus}\epsilon, \qquad \delta_{\epsilon}\mathcal{F}^{\oplus} = [\mathcal{F}^{\oplus}, \epsilon].$$

2 Admissible Curvature Densities

Definition 2.1. *A local density $\mathcal{L}$ on $\mathcal{A}(P)$ is admissible if:*

(i) finite order r,

(ii) local and horizontal,

(iii) gauge-invariant up to horizontal exact term: $\delta_\epsilon \mathcal{L} = d\alpha_\epsilon$,

(iv) dependence on $\omega^\oplus$ exclusively through the curvature jets

$$\mathrm{j}_r(\omega^\oplus) = (\mathcal{F}^\oplus, D^\oplus \mathcal{F}^\oplus, \ldots, (D^\oplus)^r \mathcal{F}^\oplus),$$

via invariant contractions built from κ and the Hodge operator.

Write

$$\mathcal{L}(\omega^\oplus) = L(\Phi_0, \ldots, \Phi_r)\,\mathrm{vol}_M, \qquad \Phi_j = (D^\oplus)^j \mathcal{F}^\oplus,$$

and define momenta by

$$\delta L = \sum_{j=0}^{r} \kappa(P_j, \star\delta\Phi_j) + d\vartheta(\delta\omega^\oplus), \qquad \deg P_j = \deg \Phi_j = 2 + j.$$

Here $\vartheta(\delta\omega^\oplus) \in \Omega^{m-1}(M)$ depends linearly on $\delta\omega^\oplus$.

3 First Variation and Euler Field

Lemma 3.1 (Curvature variation)**.**

$$\delta\mathcal{F}^\oplus = D^\oplus(\delta\omega^\oplus).$$

Lemma 3.2 (Jet recursion)**.** *For $\Phi_j = (D^\oplus)^j \mathcal{F}^\oplus$ and $j \geq 1$,*

$$\delta\Phi_j = D^\oplus(\delta\Phi_{j-1}) + [\delta\omega^\oplus, \Phi_{j-1}].$$

Lemma 3.3 (Formal adjoint)**.** *Let*

$$\mathscr{J}_{\omega^\oplus} : \Omega^1(M, \mathrm{Ad}P) \to \bigoplus_{j=0}^{r} \Omega^{j+2}(M, \mathrm{Ad}P), \quad \mathscr{J}_{\omega^\oplus}(\eta) = (\delta\Phi_0, \ldots, \delta\Phi_r).$$

For compactly supported η, fix the formal adjoint

$$\mathscr{J}^*_{\omega^\oplus} : \bigoplus_{j=0}^{r} \Omega^{j+2}(M, \mathrm{Ad}P) \to \Omega^1(M, \mathrm{Ad}P)$$

such that

$$\int_M \langle \mathscr{J}_{\omega^\oplus}(\eta), Q\rangle_\star \,\mathrm{vol}_M = \int_M \langle \eta, \mathscr{J}^*_{\omega^\oplus}(Q)\rangle_\star \,\mathrm{vol}_M + \int_M d\,\mathcal{B}(\eta, Q).$$

Proposition 3.4 (First Variation)**.**

$$\delta\mathcal{L} = \sum_{j=0}^{r} \langle P_j, \delta\Phi_j\rangle_\star \,\mathrm{vol}_M = \left\langle \mathcal{E}(\omega^\oplus), \delta\omega^\oplus \right\rangle_\star \mathrm{vol}_M + d\Theta, \qquad \mathcal{E}(\omega^\oplus) := \mathscr{J}^*_{\omega^\oplus}(P_0, \ldots, P_r) \in \Omega^1(M, \mathrm{Ad}P).$$

Define

$$\widetilde{\mathcal{E}}(\omega^\oplus) := \star\mathcal{E}(\omega^\oplus) \in \Omega^{m-1}(M, \mathrm{Ad}P), \qquad (D^\oplus)^\dagger := -\star D^\oplus \star.$$

4 Covariant Noether Identity

Theorem 4.1. *For admissible $\mathcal{L}$,*

$$(D^{\oplus})^{\dagger}\mathcal{E}(\omega^{\oplus}) \equiv 0 \qquad \Longleftrightarrow \qquad D^{\oplus}\widetilde{\mathcal{E}}(\omega^{\oplus}) \equiv 0.$$

Proof. With $\delta_\epsilon \omega^{\oplus} = D^{\oplus}\epsilon$,

$$\delta_\epsilon \mathcal{L} = \left\langle \mathcal{E}(\omega^{\oplus}), D^{\oplus}\epsilon \right\rangle_{\star} \mathrm{vol}_M + d\Theta = -\left\langle (D^{\oplus})^{\dagger}\mathcal{E}(\omega^{\oplus}), \epsilon \right\rangle_{\star} \mathrm{vol}_M + d(\Theta + \Xi).$$

Since ϵ is arbitrary, the coefficient of ϵ vanishes identically. □

5 Curvature-Generated Algebra

Define $\mathcal{I}_{\mathrm{curv}} \subset \Omega^{\bullet}(M, \mathrm{End}(E))$ as the graded subalgebra generated by the curvature jets and closed under wedge product, graded commutator, $D^{\oplus}$, and Hodge dual:

$$\mathcal{I}_{\mathrm{curv}} := \left\langle (D^{\oplus})^{j}\mathcal{F}^{\oplus} \ : \ j \geq 0 \right\rangle \subset \Omega^{\bullet}(M, \mathrm{End}(E)),$$

i.e., the smallest graded subalgebra containing the curvature jets and closed under these operations.

Proposition 5.1.

$$\mathcal{E}(\omega^{\oplus}) \in \mathcal{I}_{\mathrm{curv}}, \qquad D^{\oplus}\widetilde{\mathcal{E}}(\omega^{\oplus}) \in \mathcal{I}_{\mathrm{curv}}, \qquad D^{\oplus}\widetilde{\mathcal{E}}(\omega^{\oplus}) \equiv 0.$$

Proof. Admissibility gives P_j as invariant contractions of curvature jets. Construction of $\mathcal{E}(\omega^{\oplus}) = \mathscr{J}^{*}_{\omega^{\oplus}}(P_0, \ldots, P_r)$ uses only $D^{\oplus}$, wedge, graded commutator, and $\star$, so $\mathcal{E}(\omega^{\oplus}) \in \mathcal{I}_{\mathrm{curv}}$. Therefore $\widetilde{\mathcal{E}}(\omega^{\oplus}) = \star\mathcal{E}(\omega^{\oplus}) \in \mathcal{I}_{\mathrm{curv}}$ and $D^{\oplus}\widetilde{\mathcal{E}}(\omega^{\oplus}) \in \mathcal{I}_{\mathrm{curv}}$. The universal identity gives $D^{\oplus}\widetilde{\mathcal{E}}(\omega^{\oplus}) \equiv 0$. □

6 Orthogonal Block Projection

Under the orthogonal splitting $E = TM \oplus N$, the adjoint bundle $\mathrm{Ad}P \subset \mathrm{End}(E)$ inherits the block decomposition

$$\mathrm{End}(E) = \mathrm{End}(TM) \oplus \mathrm{Hom}(N, TM) \oplus \mathrm{End}(N).$$

Fix $E = TM \oplus N$ and write

$$\widetilde{\mathcal{E}}(\omega^{\oplus}) = \begin{pmatrix} E_T & E_{\mathrm{mix}} \\ E_{\mathrm{mix}}^{\top} & E_N \end{pmatrix},$$

with $E_{\mathrm{mix}} \in \mathrm{Hom}(N, TM)$ and $E_{\mathrm{mix}}^{\top} \in \mathrm{Hom}(TM, N)$.

Proposition 6.1.

$$D^{\oplus}\widetilde{\mathcal{E}}(\omega^{\oplus}) = 0 \quad \Longleftrightarrow \quad \Pi_T(D^{\oplus}\widetilde{\mathcal{E}}(\omega^{\oplus})) = \Pi_{\mathrm{mix}}(D^{\oplus}\widetilde{\mathcal{E}}(\omega^{\oplus})) = \Pi_N(D^{\oplus}\widetilde{\mathcal{E}}(\omega^{\oplus})) = 0.$$

Here D^{prod} denotes the block-diagonal covariant derivative induced by $\omega^{\oplus}$. With

$$\Psi = \begin{pmatrix} 0 & -A \\ K & 0 \end{pmatrix}, \qquad D^{\oplus}\widetilde{\mathcal{E}}(\omega^{\oplus}) = D^{\mathrm{prod}}\widetilde{\mathcal{E}}(\omega^{\oplus}) + [\Psi, \widetilde{\mathcal{E}}(\omega^{\oplus})],$$

one projected equation is

$$D^{h}E_T - (A\, E_{\mathrm{mix}}^{\top} + E_{\mathrm{mix}}\, K) = 0.$$

Commutator channels.

$$[\Psi,\widetilde{\mathcal{E}}(\omega^{\oplus})]_{TT}=-(A\,E_{\rm mix}^{\top}+E_{\rm mix}\,K),\qquad [\Psi,\widetilde{\mathcal{E}}(\omega^{\oplus})]_{TN}=-A\,E_N+E_T\,A,\qquad [\Psi,\widetilde{\mathcal{E}}(\omega^{\oplus})]_{NN}=E_{\rm mix}^{\top}A+K\,E_{\rm mix}.$$

7 Yang–Mills Channel

$$S_{\rm YM}[\omega^{\oplus}]:=\frac{1}{2}\int_M\big\langle\mathcal{F}^{\oplus},\mathcal{F}^{\oplus}\big\rangle_{\star}\,{\rm vol}_M.$$

Then

$$\delta S_{\rm YM}=\int_M\big\langle D^{\oplus}(\delta\omega^{\oplus}),\mathcal{F}^{\oplus}\big\rangle_{\star}\,{\rm vol}_M=-\int_M\big\langle\delta\omega^{\oplus},\star D^{\oplus}(\star\mathcal{F}^{\oplus})\big\rangle_{\star}\,{\rm vol}_M+\int_M d\,\Theta_{\rm YM},$$

with

$$\Theta_{\rm YM}=\kappa(\delta\omega^{\oplus}\wedge\star\mathcal{F}^{\oplus}),\qquad \mathcal{E}(\omega^{\oplus})_{\rm YM}:=-\star D^{\oplus}(\star\mathcal{F}^{\oplus}),\qquad \widetilde{\mathcal{E}}(\omega^{\oplus})_{\rm YM}:=\star\mathcal{E}(\omega^{\oplus})_{\rm YM}.$$

$$(D^{\oplus})^{\dagger}\mathcal{E}(\omega^{\oplus})_{\rm YM}\equiv 0\qquad\Longleftrightarrow\qquad D^{\oplus}\widetilde{\mathcal{E}}(\omega^{\oplus})_{\rm YM}\equiv 0.$$

8 First Higher-Jet Channel

$$S_1[\omega^{\oplus}]:=\frac{1}{2}\int_M\big\langle D^{\oplus}(\star\mathcal{F}^{\oplus}),D^{\oplus}(\star\mathcal{F}^{\oplus})\big\rangle_{\star}\,{\rm vol}_M.$$

Set $X:=\star\mathcal{F}^{\oplus}$ and $\Psi:=D^{\oplus}X$. Then

$$\delta X=\star D^{\oplus}\eta,\qquad \delta\Psi=D^{\oplus}(\star D^{\oplus}\eta)+[\eta,\star\mathcal{F}^{\oplus}].$$

So

$$\delta S_1=\int_M\big\langle\eta,\mathcal{E}(\omega^{\oplus})_1\big\rangle_{\star}\,{\rm vol}_M+\int_M d\Theta_1,\qquad \mathcal{E}(\omega^{\oplus})_1:=\mathcal{J}_1^{*}(\Psi),\quad \mathcal{J}_1(\eta):=D^{\oplus}(\star D^{\oplus}\eta)+[\eta,\star\mathcal{F}^{\oplus}].$$

Substituting $\delta_\epsilon\omega^{\oplus}=D^{\oplus}\epsilon$ and integrating by parts yields

$$(D^{\oplus})^{\dagger}\mathcal{E}(\omega^{\oplus})_1\equiv 0.$$

9 Comparison with Classical Noether Theory

The identity $(D^{\oplus})^{\dagger}\mathcal{E}(\omega^{\oplus})\equiv 0$ is the gauge Noether identity in curvature form [3, 4, 5]. The orthogonal splitting expresses this identity as tangent, mixed, and normal block components. While gauge-natural Noether identities are classical, the present formulation isolates their orthogonal block structure within the compatibility-connection framework.

A Sign Conventions

$$D^{\oplus}\alpha=d\alpha+[\omega^{\oplus},\alpha],\qquad \mathcal{F}^{\oplus}=d\omega^{\oplus}+\omega^{\oplus}\wedge\omega^{\oplus},$$

$$[\alpha,\beta]=\alpha\wedge\beta-(-1)^{pq}\beta\wedge\alpha,\qquad \alpha\in\Omega^p,\ \beta\in\Omega^q.$$

B Covariant Integration by Parts

For compactly supported $\mathrm{Ad}P$-valued forms U, V,

$$\kappa(U, \star D^{\oplus} V)\,\mathrm{vol}_M = d\,\mathcal{B}(U,V) - (-1)^{\deg U}\kappa(D^{\oplus}U, \star V)\,\mathrm{vol}_M,$$

with

$$\mathcal{B}(U,V) := \kappa(U \wedge \star V), \qquad \kappa(U, \star V)\,\mathrm{vol}_M = \kappa(U \wedge \star V).$$

C Graded Derivation Identities

$$D^{\oplus}(\alpha \wedge \beta) = (D^{\oplus}\alpha) \wedge \beta + (-1)^{\deg \alpha}\alpha \wedge (D^{\oplus}\beta), \tag{1}$$

$$D^{\oplus}[\alpha, \beta] = [D^{\oplus}\alpha, \beta] + (-1)^{\deg \alpha}[\alpha, D^{\oplus}\beta]. \tag{2}$$

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