Supplementary: Conservation Laws from Geometry

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Overdetermined embeddings constrain curvature and all geometric quantities constructed from it. Physical observables, including energy, momentum, and angular momentum, correspond to geometric quantities on the embedded manifold; hierarchy bounds translate directly into constraints on time evolution. Bounded conservation laws emerge from this structure, reducing to exact conservation in the $K_{\min} \to 0$ limit.

0.1 Bounded Conservation Laws Theorem

Theorem (Bounded Conservation Laws). For any quantity Q constructed from curvature and its derivatives up to order ℓ, all time derivatives satisfy:

$$\left|\frac{d^m Q}{dt^m}\right| \leq \tilde{C}_{m,\ell} K_{\min}^{2+m/2}, \quad m = 1, 2, 3, \ldots$$

where $\tilde{C}_{m,\ell}$ are dimensionless constants depending on m, ℓ, and the geometric configuration.

Proof.

Consider a physical quantity Q constructed from curvature and derivatives:

$$Q = Q(K_{ab}^{(\alpha)}, \nabla K_{ab}^{(\alpha)}, \nabla^2 K_{ab}^{(\alpha)}, \ldots, \nabla^\ell K_{ab}^{(\alpha)}).$$

The time derivative of Q involves spatial and temporal derivatives of the arguments. On a time-dependent embedding where coordinates evolve as $\varphi^a = \varphi^a(t)$:

$$\frac{dQ}{dt} = \sum_{|j| \leq \ell} \frac{\partial Q}{\partial(\nabla^j K)} \cdot \frac{d}{dt}(\nabla^j K).$$

The time derivative of covariant derivatives relates to spatial derivatives through coordinate evolution:

$$\frac{d}{dt}(\nabla^j K) = \sum_{p=0}^{j+1} c_{j,p} \nabla^p K \cdot \dot{\varphi}^a,$$

where $c_{j,p}$ are combinatorial coefficients and $\dot{\varphi}^a = d\varphi^a/dt$ represents coordinate velocity.

From the hierarchy bounds, each term satisfies:

$$|\nabla^p K| \leq C_p K_{\min}^{2+p/2}.$$

Substituting and using dimensional analysis with $|\dot{\varphi}| \sim K_{\min}^{1/2}$ (set by geometric time scale):

$$\left|\frac{dQ}{dt}\right| \lesssim \sum_{p=0}^{\ell+1} K_{\min}^{2+p/2} \cdot K_{\min}^{1/2} \sim K_{\min}^{5/2},$$

matching the $m = 1$ bound with $\tilde{C}_{1,\ell} \sim (\ell + 2)$.

Higher time derivatives follow by induction, each adding a factor $K_{\min}^{1/2}$ from the time evolution operator, yielding the general bound.

0.2 Physical Examples

0.2.1 Energy Conservation

Energy density on the embedded manifold is geometrically defined through the extrinsic curvature:

$$E \sim \int_V K_{ab}^{(\alpha)} K^{ab(\alpha)} \sqrt{h}\, d^n\varphi \sim K_{\min}^2 V,$$

where V is the spatial volume. The time derivative satisfies:

$$\left|\frac{dE}{dt}\right| \leq C_1 K_{\min}^{5/2} V.$$

For cosmological embeddings with $K_{\min} \sim H_0 \sim 10^{-18}$ s^{-1} and $V \sim (10^{26}\,\mathrm{m})^3$:

$$\left|\frac{dE}{dt}\right| \lesssim 10^{-9} J/s.$$

This bound is extraordinarily tight, effectively indistinguishable from exact conservation in laboratory or astrophysical measurements. The difference is fundamental: energy flows between the observable manifold and the embedding space at rate bounded by $K_{\min}^{5/2}$. Exact conservation holds in the full embedding space, not the observable manifold alone.

0.2.2 Momentum Conservation

Spatial momentum components arise from mixed space-time extrinsic curvature. Each component satisfies:

$$\left|\frac{dp^i}{dt}\right| \leq C_1 K_{\min}^{5/2} V, \quad i = 1, 2, 3.$$

The bound applies independently to each spatial direction, preserving rotational symmetry of the geometric constraints.

0.2.3 Angular Momentum Conservation

Angular momentum $L^{ij} = \int (x^i p^j - x^j p^i) dV$ combines position and momentum. The antisymmetric structure ensures:

$$\left|\frac{dL^{ij}}{dt}\right| \leq C_1 K_{\min}^{5/2} V \ell,$$

where ℓ is a characteristic length scale. For isolated systems where $\ell \sim K_{\min}^{-1}$, this gives $|dL/dt| \lesssim K_{\min}^{3/2} V$.

0.3 Noether Limit and Emergence of Exact Conservation

Corollary (Noether Limit). As $K_{\min} \to 0$, bounded conservation laws reduce to exact conservation:

$$\lim_{K_{\min} \to 0} \left|\frac{dQ}{dt}\right| = 0.$$

Physically, the $K_{\min} \to 0$ limit decouples the manifold from the embedding space. Energy exchange between manifold and embedding vanishes, and the manifold becomes a closed system. This limit recovers standard Noether structure where continuous symmetries of the action yield exact conservation laws within the manifold.

Bounded conservation is thus a generalization that:

- Reduces to exact conservation when geometric constraints vanish ($K_{\min} \to 0$)
- Provides quantitative bounds in physically realized embeddings ($K_{\min} \sim H_0$)
- Unifies conservation laws and geometric structure without invoking symmetry postulates

0.4 Universal Geometric Conservation Principle

Corollary (Universality). Every geometric quantity in an overdetermined embedding with k ¿ n$^2 - n - 1satisfiesboundedconservationlaws$.

This universality replaces symmetry postulates as the foundation for conservation. Gauge symmetries arise from redundancy in choosing orthonormal normal vectors: U(1) from single-vector phase freedom, SU(2) and SU(3) from rotations among multiple normals. The embedding constraints, not the action, determine conservation structure.

0.5 Observational Consequences

Bounded conservation differs observationally from exact conservation only at scales where $K_{\min}^{5/2} V$ becomes measurable. For terrestrial experiments with $V \sim (1\,\mathrm{m})^3$ and $K_{\min} \sim H_0$:

$$\left| \frac{dE}{dt} \right| \lesssim 10^{-54}\,\mathrm{W},$$

far below any conceivable measurement precision. Even for galactic scales with $V \sim (10^{21}\,\mathrm{m})^3$, the bound remains $\lesssim 10^{-27}$ W, undetectable with current technology.

At cosmological scales or in extreme environments (early universe, black hole horizons), the bounds become relevant. Testing requires:

- Precision cosmology measuring total energy evolution over Hubble time
- Black hole thermodynamics tracking information and energy transfer
- Early universe nucleosynthesis sensitive to small energy nonconservation

These observational frontiers distinguish bounded conservation laws from exact conservation, providing falsifiable tests.

0.6 Falsification

Bounded conservation predicts violations at the geometric scale $K_{\min}^{5/2}$, while exact conservation predicts zero violations. Detection of systematic conservation violations exceeding geometric bounds would falsify overdetermined embedding. Current observations are consistent with bounded conservation at cosmological scales.