Supplementary: Embedded Geometry Details

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0.1 Dual Sources of Overdetermination

Overdetermination arises from two independent constraint systems, making curvature bounds robust.

0.1.1 Source I: Embedding Compatibility

Gauss-Codazzi-Ricci equations constitute a system of partial differential equations relating intrinsic and extrinsic geometry. For $(n, k) = (2, 3)$, 13 constraints on 12 degrees of freedom yields overdetermination by 1.

Constraint breakdown:

- Gauss equations: 1 constraint
- Codazzi equations: 6 constraints
- Ricci equations: 6 constraints
- Total: 13 constraints

Degrees of freedom:

- Intrinsic metric h_{ab}: 3 components
- Extrinsic curvature $K_{ab}^{(\alpha)}$ for 3 normal directions: $3 \times 3 = 9$ components
- Total: 12 degrees of freedom

0.1.2 Source II: Normal Bundle Structure

Requiring k normal directions be orthonormal imposes algebraic constraints on extrinsic curvature before embedding equations are considered.

Component count:

- Extrinsic curvature has $k \cdot n(n+1)/2$ components
- For $(n,k) = (2,3)$: $3 \times 3 = 9$ components

Constraint count:

- Orthonormality $n^{(\alpha)} \cdot n^{(\beta)} = \delta^{\alpha\beta}$: $k(k+1)/2 = 6$ constraints
- Perpendicularity $n^{(\alpha)} \cdot e_a = 0$: $nk = 6$ constraints
- Total: 12 constraints on 9 components

0.2 Robustness

Both constraint systems must be satisfied simultaneously. Embedding compatibility restricts which metrics can be embedded; orthonormality restricts which extrinsic curvatures can realize those embeddings. Together they force curvature bounds from two independent directions, making $K_G \geq K_{\min}^2$ robust against perturbations of either constraint system alone.

0.3 Geometric Intuition

Consider constraints as partial differential equations for embedding X. When underdetermined ($k \leq n^2 - n - 1$), more unknowns than equations leave freedom to choose curvature arbitrarily. When overdetermined ($k > n^2 - n - 1$), more equations than unknowns require strong relationships between derivatives.

Bianchi identities ensure higher derivatives of curvature satisfy consistency conditions. In an overdetermined system these conditions become bounds; to prevent contradictions, all curvature derivatives must remain within ranges determined by lowest-order curvature.

0.3.1 Nash's Theorem Comparison

Nash's theorem guarantees any Riemannian manifold admits isometric embedding into sufficiently high-dimensional Euclidean space, but places no restrictions on normal bundle structure. Requiring k orthonormal normal directions is a constraint not addressed by Nash's theorem. This additional structure generates physical laws.

Key distinction:

- Nash: existence of embedding guaranteed

- This work: embedding with structured normal bundle forces curvature bounds