On the Structure of the Orthogonal Compatibility Connection

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Abstract

On $E = TM \oplus N$, the orthogonal compatibility connection $\nabla^\oplus$ encodes the Gauss–Weingarten system as an $\mathrm{SO}(n+k)$-connection. Curvature projection yields the Gauss, Codazzi, and Ricci defect blocks; $\mathcal{F}^\oplus = 0$ is the constraint system. The coupled Bianchi identities and $D^\oplus$-stability of the curvature ideal hold in invariant form. The constraint map $\mathcal{D}(h, K, \nabla^\perp) = (\mathcal{G}, \mathcal{C}, \mathcal{S})$ defines a natural gauge-equivariant PDE operator with explicit principal and gauge symbols. For $n \geq 2$, at $K = 0$ the Codazzi symbol has kernel dimension k for every $\xi \neq 0$, and this kernel survives principal-symbol gauge reduction. Reduced ellipticity fails at $K = 0$.

1 Introduction

1.1 Standing Assumptions

Let M^n be a smooth manifold with Riemannian metric h. Let (N, g_N) be a rank-k Riemannian vector bundle with a metric connection $\nabla^\perp$. Fix extrinsic curvature data

$$K \in \Gamma(\mathrm{Hom}(TM, \mathrm{Hom}(TM, N))), \qquad K(X, Y) = K(Y, X),$$

equivalently $K \in \Gamma(\mathrm{Sym}^2 T^*M \otimes N)$. Define the shape operator $A : N \to \mathrm{End}(TM)$ by

$$\langle A_\nu X, Y \rangle_h = \langle K(X, Y), \nu \rangle_N.$$

No ambient realization (immersion/embedding) or analyticity is assumed; $E = TM \oplus N$ is treated abstractly, and flatness of $\nabla^\oplus$ is only a compatibility statement. Fix orientations so that the orthonormal structure group is $\mathrm{SO}(n+k)$; otherwise replace SO by O. The Levi-Civita connection ∇^h is torsion-free. For any connection ∇, set

$$R(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Call the data *integrable* when the induced $\mathrm{SO}(n+k)$-connection is locally gauge-trivial on simply connected neighborhoods, equivalently when $\nabla^\oplus$ is flat.

Write the Gauss–Weingarten system in block form:

$$\nabla^\oplus = \nabla^h \oplus \nabla^\perp + \Psi, \qquad \Psi = \begin{pmatrix} 0 & -K^T \\ K & 0 \end{pmatrix}.$$

Curvature projection yields the Gauss, Codazzi, and Ricci defect blocks. The metric $h \oplus g_N$ lifts $\nabla^\oplus$ to a principal connection $\omega^\oplus$ on $P_{\mathrm{SO}(E)}$ (cf. [2]).

1.2 Conventions and Notation

Use the composition wedge for bundle-valued forms:

$$(\alpha \wedge \beta)(X, Y) = \alpha_X \circ \beta_Y - \alpha_Y \circ \beta_X.$$

Use graded commutators

$$[\alpha, \beta]_{\mathrm{gr}} := \alpha \wedge \beta - (-1)^{pq} \beta \wedge \alpha, \qquad \alpha \in \Omega^p,\ \beta \in \Omega^q.$$

Interpret transpose as the metric adjoint with respect to $h \oplus g_N$; in particular K^T denotes the $N \to TM$ adjoint block and coincides with the shape-operator block A.

1.3 Main Results and Novelty

Two statements organize the paper. First: a single invariant algebra on $\Omega^\bullet(M, \mathrm{End}(E))$ controls the curvature block decomposition, the coupled Bianchi identities, and closure of the curvature-generated graded ideal. Second: the constraint map $\mathcal{D}(h, K, \nabla^\perp)$ defines a natural gauge-equivariant operator; its block principal symbol and gauge symbol admit explicit formulas.

Freeze $(h, \nabla^\perp)$ and set $K = 0$. The symbol then collapses to the Codazzi block and gives an exact kernel count for all $n \geq 2$ (in particular for surfaces). The full symbol displays the source of non-ellipticity: mixed order and gauge directions, not coordinates. Completeness statements stay within the gauge-natural operator class (KMS sense); they do not assert PDE solvability.

1.4 Relation to Classical Realization Results

The focus stays on operator structure, compatibility identities, and symbol-level consequences. Classical realization (local immersion) belongs to external geometry and serves only as context. The classical fundamental theorem of submanifolds (existence and uniqueness from compatible first and second fundamental data) is not reproved here.

Organization. Section 2 fixes $\nabla^\oplus$, its uniqueness, and the gauge actions. Section 3 gives the curvature block split and type conventions. Section 4 records the coupled Bianchi identities and the defect-complex form. Sections 5 and 6 give gauge-natural completeness and closure of the curvature-generated differential ideal, with a jet-level consequence. Section 7 gives a structural interpretation. Section 8 introduces $\mathcal{D}$ and records its linearization, principal symbol, and gauge symbol. Section 9 records the symbol consequences, including the frozen-background case. Appendices A–C collect component translation, variation formulas, and symbol tables.

2 Foundational Precision

2.1 The Assembled Connection

Fix $E = TM \oplus N$. Define $\nabla^\oplus$ from $(\nabla^h, \nabla^\perp, K)$.

Definition 2.1 (Orthogonal Compatibility Connection)**.** Let ∇^h be the Levi-Civita connection of (M, h). Define $\nabla^\oplus$ by

$$\nabla^\oplus_X (Y \oplus \nu) := \Big(\nabla^h_X Y - A_\nu(X)\Big) \oplus \Big(\nabla^\perp_X \nu + K(X, Y)\Big). \tag{1}$$

Proposition 2.2. *$\nabla^\oplus$ preserves the direct sum metric $g_E = h \oplus g_N$.*

Proof. For sections $s_1 = Y_1 \oplus \nu_1, s_2 = Y_2 \oplus \nu_2$:

$$\begin{aligned} X\langle s_1, s_2\rangle_E &= X(\langle Y_1, Y_2\rangle_h + \langle \nu_1, \nu_2\rangle_N) \\ &= \langle \nabla^h_X Y_1, Y_2\rangle_h + \langle Y_1, \nabla^h_X Y_2\rangle_h + \langle \nabla^\perp_X \nu_1, \nu_2\rangle_N + \langle \nu_1, \nabla^\perp_X \nu_2\rangle_N. \end{aligned}$$

Compute $\langle \nabla^\oplus_X s_1, s_2\rangle_E + \langle s_1, \nabla^\oplus_X s_2\rangle_E$:

$$-\langle A_{\nu_1} X, Y_2\rangle_h + \langle K(X, Y_1), \nu_2\rangle_N - \langle Y_1, A_{\nu_2} X\rangle_h + \langle \nu_1, K(X, Y_2)\rangle_N,$$

which cancel by the adjoint relation between A and K. Thus $\nabla^\oplus g_E = 0$. □

2.2 Uniqueness

$$\nabla_{product} := \nabla^h \oplus \nabla^\perp.$$

Theorem 2.3 (Uniqueness of the Assembled Connection)**.** *$\nabla^\oplus$ uniquely satisfies: metricity on E, diagonal blocks $\nabla^h \oplus \nabla^\perp$, and mixed block $TM \to N$ equal to K.*

Proof. Impose $\mathrm{pr}_N(\nabla_X(Y \oplus 0)) = K(X, Y)$. Write $\nabla = \nabla_{product} + \Psi$ with $\Psi \in \Omega^1(M, \mathrm{End}(E))$. Metricity forces Ψ_X skew-adjoint with respect to $h \oplus g_N$. Fixing the diagonal blocks gives $\Psi_{TM \to TM} = 0 = \Psi_{N \to N}$ and $\Psi_{TM \to N} = K$. Skew-adjointness forces $\Psi_{N \to TM} = -K^T = -A$. For every X, metricity is

$$\langle \Psi_X u, v\rangle_{h \oplus g_N} + \langle u, \Psi_X v\rangle_{h \oplus g_N} = 0,$$

i.e. Ψ_X is skew-adjoint with respect to $h \oplus g_N$.

$$\Psi_X = \begin{pmatrix} 0 & B_X \\ K_X & 0 \end{pmatrix}, \qquad \Psi_X^* = -\Psi_X$$

with respect to $h \oplus g_N$. Skew-adjointness gives $B_X = -K_X^T$, i.e. $B_X(\nu) = -A_\nu X$. Thus specifying the $TM \to N$ block determines the $N \to TM$ block uniquely. □

2.3 Gauge Actions on Data

Two symmetry classes act on $(h, K, \nabla^\perp)$:

1. Diffeomorphisms $\varphi \in \mathrm{Diff}(M)$:

$$h \mapsto \varphi^* h, \qquad K \mapsto \varphi^* K, \qquad \nabla^\perp \mapsto \varphi^* \nabla^\perp.$$

2. Orthogonal bundle gauge transformations $g \in \Gamma(\mathrm{SO}(N))$:

$$h \mapsto h, \qquad K \mapsto g \cdot K, \qquad \nabla^\perp \mapsto g \nabla^\perp g^{-1}.$$

Proposition 2.4 (Functoriality of the Assembled Connection)**.** *The map $(h, K, \nabla^\perp) \mapsto \nabla^\oplus$ intertwines $\mathrm{Diff}(M) \ltimes \Gamma(\mathrm{SO}(N))$. Curvature projection yields an equivariant defect map $(h, K, \nabla^\perp) \mapsto (\mathcal{G}, \mathcal{C}, \mathcal{S})$.*

Proof. Each block of $\nabla^\oplus = \nabla^h \oplus \nabla^\perp + \Psi$ is functorial: ∇^h under pullback of metrics, $\nabla^\perp$ under pullback and orthogonal conjugation, and Ψ from the tensorial pair (K, K^T). Therefore $\nabla^\oplus$ transforms by pullback/conjugation in the associated principal $\mathrm{SO}(n + k)$ bundle, and curvature transforms covariantly. Hence $(\mathcal{G}, \mathcal{C}, \mathcal{S})$ is equivariant. □

3 Curvature Block Decomposition

Fix the curvature

$$\mathcal{F}^{\oplus} := (\nabla^{\oplus})^2 \in \Omega^2(M, \mathrm{End}(E)).$$

Use the convention $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for all connections. Define the defect blocks by projection:

$$\mathcal{G} := (\mathcal{F}^{\oplus})_{TM}, \quad \mathcal{C} := (\mathcal{F}^{\oplus})_{Mix}, \quad \mathcal{S} := (\mathcal{F}^{\oplus})_N. \tag{2}$$

Regard $(\mathcal{F}^{\oplus})_{Mix}$ as the $TM \to N$ block, so $(\mathcal{F}^{\oplus})_{Mix} \in \Omega^2(M, \mathrm{Hom}(TM, N))$. Then

$$\mathcal{G} \in \Omega^2(M, \mathrm{End}(TM)), \qquad \mathcal{C} \in \Omega^2(M, \mathrm{Hom}(TM, N)), \qquad \mathcal{S} \in \Omega^2(M, \mathrm{End}(N)).$$

Metricity of $\nabla^{\perp}$ forces $R^{\perp} \in \Omega^2(M, \mathfrak{so}(N))$, hence $\mathcal{S} \in \Omega^2(M, \mathfrak{so}(N))$.

Let $D^{product}$ denote the exterior covariant derivative induced by $\nabla_{product} = \nabla^h \oplus \nabla^{\perp}$, and let

$$\mathcal{F}_{product} := (\nabla_{product})^2 = R^h \oplus R^{\perp} \in \Omega^2(M, \mathrm{End}(E)).$$

Thus $R^{\perp} \in \Omega^2(M, \mathfrak{so}(N))$. The perturbation equation is $\mathcal{F}^{\oplus} = \mathcal{F}_{product} + D^{product}\Psi + \Psi \wedge \Psi$. Here $\wedge$ is the composition wedge product:

$$(\alpha \wedge \beta)(X,Y) = \alpha_X \circ \beta_Y - \alpha_Y \circ \beta_X.$$

This composition wedge is the graded product in $\Omega^{\bullet}(M, \mathrm{End}(E))$; the displayed equation is the degree-1 specialization. Regard

$$A \in \Omega^1(M, \mathrm{Hom}(N, TM)), \qquad K \in \Omega^1(M, \mathrm{Hom}(TM, N)),$$

via $A_X(\nu) := A_\nu X$ and $K_X(Y) := K(X,Y)$. Then

$$(A \wedge K)(X,Y) := A_X \circ K_Y - A_Y \circ K_X \in \mathrm{End}(TM),$$

$$(K \wedge A)(X,Y) := K_X \circ A_Y - K_Y \circ A_X \in \mathrm{End}(N).$$

Transpose Conventions. For each $X \in TM$, define $K_X \in \mathrm{Hom}(TM, N)$ by $K_X(Y) = K(X,Y)$ and its metric adjoint

$$K_X^T \in \mathrm{Hom}(N, TM), \qquad \langle K_X^T \nu, Y \rangle_h = \langle \nu, K_X Y \rangle_N.$$

Then $K_X^T(\nu) = A_\nu X$, so K^T and A are the same $\mathrm{Hom}(N, TM)$-valued 1-form. Also

$$\mathcal{C} \in \Omega^2(M, \mathrm{Hom}(TM, N)), \qquad \mathcal{C}^T \in \Omega^2(M, \mathrm{Hom}(N, TM)),$$

with $(\mathcal{C}^T)_{X,Y} := (\mathcal{C}_{X,Y})^T$.

The induced connection on $\mathrm{Hom}(TM, N)$ is

$$\nabla_X^{\mathrm{Hom}} T = \nabla_X^{\perp} \circ T - T \circ \nabla_X^h, \qquad T \in \Gamma(\mathrm{Hom}(TM, N)),$$

and the induced exterior covariant derivative on $\mathrm{Hom}(TM, N)$-valued 1-forms is

$$(D^{\mathrm{Hom}} K)(X,Y) := \nabla_X^{\mathrm{Hom}} K_Y - \nabla_Y^{\mathrm{Hom}} K_X - K_{[X,Y]}.$$

Theorem 3.1 (Curvature Equations)**.** *Curvature projection gives*

$$\mathcal{G} = R^h - A \wedge K, \tag{3}$$
$$\mathcal{C} = D^{\mathrm{Hom}} K, \tag{4}$$
$$\mathcal{S} = R^{\perp} - K \wedge A. \tag{5}$$

Proof. Write $\nabla^{\oplus} = \nabla_{product} + \Psi$ with $\nabla_{product} = \nabla^h \oplus \nabla^{\perp}$ and

$$\Psi = \begin{pmatrix} 0 & -K^T \\ K & 0 \end{pmatrix}.$$

$\Psi \wedge \Psi$ has only diagonal blocks: since Ψ_X maps $TM \to N$ and $N \to TM$, the composition $\Psi_X \circ \Psi_Y$ preserves TM and N, so mixed blocks of $\Psi \wedge \Psi$ vanish. With curvature convention $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, this yields the Gauss block sign $R^h - A \wedge K$.

$$(\mathcal{F}^{\oplus})_{TM} = R^h + (\Psi \wedge \Psi)_{TM} = R^h - A \wedge K,$$
$$(\mathcal{F}^{\oplus})_{N} = R^{\perp} + (\Psi \wedge \Psi)_{N} = R^{\perp} - K \wedge A,$$
$$(\mathcal{F}^{\oplus})_{Mix} = (D^{product}\Psi)_{Mix}.$$

Since $\nabla_{product} = \nabla^h \oplus \nabla^{\perp}$ preserves TM and N separately, the projections are parallel: $[\nabla^{product}, P_{TM}] = [\nabla^{product}, P_N] = 0$. Together with $P_{TM}\Psi P_{TM} = 0 = P_N \Psi P_N$, this implies $P_{TM}(D^{product}\Psi)P_{TM} = 0$ and $P_N(D^{product}\Psi)P_N = 0$. Here P_{TM}, P_N are the bundle projections associated to $E = TM \oplus N$. For $Z \in TM$,

$$((\Psi \wedge \Psi)_{TM}(X, Y))Z = \Psi_X(\Psi_Y Z) - \Psi_Y(\Psi_X Z) = -A_{K(Y,Z)}X + A_{K(X,Z)}Y = -(A \wedge K)(X, Y)Z.$$
$$((\Psi \wedge \Psi)(X, Y))_{TM \to N} = 0$$

because each composition $\Psi_X \circ \Psi_Y$ sends $TM \to TM$ and $N \to N$, so there is no $TM \to N$ component. For the mixed block, define $K_X \in \mathrm{Hom}(TM, N)$ by $K_X(Y) = K(X, Y)$ and use

$$(D^{product}\Psi)(X, Y) = \nabla_X^{product}\Psi_Y - \nabla_Y^{product}\Psi_X - \Psi_{[X,Y]}.$$

Projecting to $\mathrm{Hom}(TM, N)$ and evaluating on $Z \in \Gamma(TM)$ gives

$$\begin{aligned}((D^{product}\Psi)_{Mix}(X, Y))Z &= (\nabla_X^{\mathrm{Hom}} K_Y)Z - (\nabla_Y^{\mathrm{Hom}} K_X)Z - K_{[X,Y]}Z \\ &= (D^{\mathrm{Hom}} K)(X, Y)Z,\end{aligned}$$

Thus $\mathcal{C} = D^{\mathrm{Hom}} K$. ∎

4 Coupled Bianchi Identities

Let D^T and $D^{\perp}$ denote the exterior covariant derivatives on $\mathrm{End}(TM)$ and $\mathrm{End}(N)$-valued forms, respectively; D^T is induced by ∇^h on TM, and $D^{\perp}$ is induced by $\nabla^{\perp}$ on N. The operator $D^{product}$ acts blockwise as D^T on $\mathrm{End}(TM)$, as $D^{\perp}$ on $\mathrm{End}(N)$, and as D^{Hom} on $\mathrm{Hom}(TM, N)$. All wedges in this section are composition wedges, and all brackets $[\cdot, \cdot]$ are graded commutators in $\Omega^{\bullet}(M, \mathrm{End}(E))$.

$$[\alpha, \beta]_{\mathrm{gr}} := \alpha \wedge \beta - (-1)^{pq}\beta \wedge \alpha, \qquad \alpha \in \Omega^p(M, \mathrm{End}(E)),\ \beta \in \Omega^q(M, \mathrm{End}(E)).$$

The exterior covariant derivative $D^{\oplus}$ is a graded derivation (see [2]):

$$D^{\oplus}(\alpha \wedge \beta) = D^{\oplus}\alpha \wedge \beta + (-1)^p \alpha \wedge D^{\oplus}\beta, \qquad \alpha \in \Omega^p(M, \mathrm{End}(E)).$$

Theorem 4.1 (Universal Bianchi Identities)**.** *The defect tensors satisfy the following coupled identities:*

$$D^T\mathcal{G} + [K^T, \mathcal{C}] = 0 \tag{6}$$

$$D^{\mathrm{Hom}}\mathcal{C} + K \wedge \mathcal{S} - \mathcal{G} \wedge K = 0 \tag{7}$$

$$D^{\perp}\mathcal{S} + [K, \mathcal{C}^T] = 0 \tag{8}$$

where $[K^T, \mathcal{C}] = K^T \wedge \mathcal{C} - \mathcal{C}^T \wedge K$. *The first identity lies in* $\Omega^3(M, \mathrm{End}(TM))$*, the second in* $\Omega^3(M, \mathrm{Hom}(TM, N))$*, and the third in* $\Omega^3(M, \mathfrak{so}(N)) \subset \Omega^3(M, \mathrm{End}(N))$*.*

Proof. Write the connection perturbation and curvature in block form:

$$\Psi = \begin{pmatrix} 0 & -K^T \\ K & 0 \end{pmatrix}, \qquad \mathcal{F}^{\oplus} = \begin{pmatrix} \mathcal{G} & -\mathcal{C}^T \\ \mathcal{C} & \mathcal{S} \end{pmatrix}.$$

Using $D^{\oplus} = D^{product} + [\Psi, \cdot]_{\mathrm{gr}}$, the Bianchi identity is

$$0 = D^{\oplus}\mathcal{F}^{\oplus} = D^{product}\mathcal{F}^{\oplus} + [\Psi, \mathcal{F}^{\oplus}]_{\mathrm{gr}}.$$

Since $\deg \Psi = 1$ and $\deg \mathcal{F}^{\oplus} = 2$ (so $1 \cdot 2$ is even), $[\Psi, \mathcal{F}^{\oplus}]_{\mathrm{gr}} = \Psi \wedge \mathcal{F}^{\oplus} - \mathcal{F}^{\oplus} \wedge \Psi$. A direct block computation gives

$$[\Psi, \mathcal{F}^{\oplus}]_{\mathrm{gr}} = \begin{pmatrix} K^T \wedge \mathcal{C} - \mathcal{C}^T \wedge K & K^T \wedge \mathcal{S} - \mathcal{G} \wedge K^T \\ \mathcal{S} \wedge K - K \wedge \mathcal{G} & K \wedge \mathcal{C}^T - \mathcal{C} \wedge K^T \end{pmatrix}.$$

For example, the (TM, TM) block is

$$([\Psi, \mathcal{F}^{\oplus}]_{\mathrm{gr}})_{TM} = (\Psi \wedge \mathcal{F}^{\oplus})_{TM} - (\mathcal{F}^{\oplus} \wedge \Psi)_{TM} = [(-K^T) \wedge \mathcal{C}] - [(-\mathcal{C}^T) \wedge K] = K^T \wedge \mathcal{C} - \mathcal{C}^T \wedge K.$$

Equating (TM, TM), (Mix), and (N, N) blocks in $D^{product}\mathcal{F}^{\oplus} + [\Psi, \mathcal{F}^{\oplus}]_{\mathrm{gr}} = 0$ yields the three displayed identities. $\square$

4.1 Defect-Complex Viewpoint

The three coupled identities are exactly the block components of one graded equation:

$$D^{\oplus}\mathcal{F}^{\oplus} = 0 \quad \text{in} \quad \Omega^3(M, \mathrm{End}(E)).$$

This packages the Gauss, Codazzi, and Ricci differential compatibilities into a single off-shell identity in the non-commutative graded algebra of $\mathrm{End}(E)$-valued forms.

5 Gauge-Natural Completeness

Completeness of $\mathcal{F}^{\oplus}$ as an obstruction is formulated below.

Theorem 5.1 (Completeness of the Obstruction)**.** *Let* $\omega^{\oplus}$ *be the principal* $\mathrm{SO}(n+k)$*-connection induced by* $\nabla^{\oplus}$ *on* $P_{\mathrm{SO}(E)}$*, and let* P *be a local finite-order* (r) *gauge-natural differential operator in the KMS sense: namely, a principal-automorphism-equivariant operator from finite jets of* $\omega^{\oplus}$ *to an associated tensor bundle. If* P *vanishes whenever* $\mathcal{F}^{\oplus} = 0$*, then there exists finite* $s = s(r)$ *and a bundle map* Φ *such that*

$$P = \Phi\big(\mathcal{F}^{\oplus},\ D^{\oplus}\mathcal{F}^{\oplus},\ \ldots,\ (D^{\oplus})^s\mathcal{F}^{\oplus}\big).$$

Equivalently, after identifying curvature jets with curvature and its iterated covariant derivatives, P *factors through* $j^s\mathcal{F}^{\oplus}$*.*

Proof. The connection $\nabla^{\oplus}$ determines uniquely a principal connection $\omega^{\oplus}$ on $P_{\mathrm{SO}(E)}$. By the KMS regularity theorem (cf. Theorem 19.7 in [1]), a local order-r gauge-natural operator depends on finite jets of $\omega^{\oplus}$, equivalently on curvature and finitely many covariant derivatives up to order $s = s(r)$. Here "flat data" means exactly $\mathcal{F}^{\oplus} = 0$. Therefore any such operator vanishing on $\mathcal{F}^{\oplus} = 0$ factors through $(\mathcal{F}^{\oplus}, D^{\oplus}\mathcal{F}^{\oplus}, \ldots, (D^{\oplus})^s\mathcal{F}^{\oplus})$, equivalently through $j^{s(r)}\mathcal{F}^{\oplus}$ under the standard jet/covariant-derivative identification. This is a classification of local invariants; it does not imply formal integrability, involutivity, or PDE solvability for immersion equations. □

5.1 Corollary Class

Any local gauge-natural scalar or tensor invariant built from $\omega^{\oplus}$ and vanishing on flat data factors through finite jets of $\mathcal{F}^{\oplus}$. This isolates curvature jets as the complete local generating source for such invariants in the KMS sense.

6 Hierarchy Closure

Theorem 6.1 ($D^{\oplus}$-Stable Graded Two-Sided Ideal)**.** *Let $\mathcal{I}$ be the graded two-sided ideal in $\Omega^{\bullet}(M, \mathrm{End}(E))$ generated by $\mathcal{F}^{\oplus}$:*

$$\mathcal{I} = \Big\{ \sum_i \alpha_i \wedge \mathcal{F}^{\oplus} \wedge \beta_i : \alpha_i, \beta_i \in \Omega^{\bullet}(M, \mathrm{End}(E)) \Big\}.$$

Then $\mathcal{I}$ is stable under $D^{\oplus}$.

Proof. By Bianchi, $D^{\oplus}\mathcal{F}^{\oplus} = 0$. Since $D^{\oplus}$ is a graded derivation for composition wedge, for homogeneous α, β one has

$$D^{\oplus}(\alpha \wedge \mathcal{F}^{\oplus} \wedge \beta) = (D^{\oplus}\alpha) \wedge \mathcal{F}^{\oplus} \wedge \beta + (-1)^{|\alpha|}\alpha \wedge (D^{\oplus}\mathcal{F}^{\oplus}) \wedge \beta + (-1)^{|\alpha|+2}\alpha \wedge \mathcal{F}^{\oplus} \wedge D^{\oplus}\beta.$$

The middle term vanishes, and the remaining terms are again in the two-sided ideal generated by $\mathcal{F}^{\oplus}$. Hence $D^{\oplus}\mathcal{I} \subseteq \mathcal{I}$. No commutativity assumption is used: the argument only uses graded derivation and two-sided ideal structure in the non-commutative algebra $\Omega^{\bullet}(M, \mathrm{End}(E))$. □

Remark 6.2. For associated bundles (here $\mathrm{End}(E)$-valued forms), the curvature representation identity $(D^{\oplus})^2\eta = [\mathcal{F}^{\oplus}, \eta]$ also holds.

6.1 Jet-Level Closure Statement

The $D^{\oplus}$-stability of the curvature-generated ideal implies that all iterated covariant derivatives $(D^{\oplus})^r\mathcal{F}^{\oplus}$ remain in the same graded two-sided ideal. In prolongation language, universal differential consequences generated from curvature remain algebraically controlled by the curvature ideal itself. This is an algebraic closure statement and does not assert involutivity or formal integrability of the PDE system.

7 Structural Interpretation

Proposition 7.1 (Structural Interpretation of Compatibility)**.** *Let $E = TM \oplus N$ with metric $h \oplus g_N$, let ∇^h be the torsion-free Levi-Civita connection on TM, let $\nabla^{\perp}$ be a metric connection on N, and let $K \in \Gamma(\mathrm{Sym}^2 T^*M \otimes N)$ define the mixed block via the canonical inclusion $TM \hookrightarrow E$. Then $\nabla^{\oplus}$ induces a principal $\mathrm{SO}(n+k)$-connection $\omega^{\oplus}$ on $P_{\mathrm{SO}(E)}$ such that:*

1. *local flatness/integrability of the Gauss–Weingarten system is equivalent to $\mathcal{F}^{\oplus} = 0$;*

2. *$\mathcal{F}^{\oplus}$ is complete as a local structure function within the gauge-natural class, i.e. every local gauge-natural obstruction factors through finite jets of $\mathcal{F}^{\oplus}$.*

Proof. Item (1) is exactly the Gauss–Codazzi–Ricci flatness criterion encoded by the block equations above. Flatness yields local principal-bundle trivializations with local parallel adapted orthonormal frames on simply connected neighborhoods. This is an integrability statement for the abstract $\mathrm{SO}(n+k)$-connection on E, not yet an immersion statement. Under additional classical hypotheses of the fundamental theorem of submanifolds (compatible first/second fundamental data on a simply connected local domain), this specializes to local immersion realization into Euclidean space [4, 3]. No global embedding claim is made. Item (2) is the gauge-natural completeness theorem in the previous section, applied to $\omega^{\oplus}$. □

8 The Constraint Operator as a Natural PDE

8.1 Definition of the Constraint Operator

Definition 8.1 (Nonlinear Constraint Operator)**.**

$$\mathcal{D}:\ (h, K, \nabla^{\perp}) \longmapsto (\mathcal{G}, \mathcal{C}, \mathcal{S}),$$

with

$$\mathcal{G} = R^h - A \wedge K, \qquad \mathcal{C} = D^{\mathrm{Hom}} K, \qquad \mathcal{S} = R^{\perp} - K \wedge A.$$

Thus $\mathcal{D} = 0$ is exactly the Gauss–Codazzi–Ricci system in invariant block form.

Definition 8.2 (Linearization and Frozen Operator)**.** At background $(h, K, \nabla^{\perp})$, write

$$\mathbf{L}_{(h,K,\nabla^{\perp})} := D\mathcal{D}\big|_{(h,K,\nabla^{\perp})}.$$

At frozen background $(h, 0, \nabla^{\perp})$, the restricted Codazzi operator is

$$L : \Gamma(\mathrm{Sym}^2 T^*M \otimes N) \to \Omega^2(M, \mathrm{Hom}(TM, N)), \qquad L(K) := D^{\mathrm{Hom}} K.$$

Definition 8.3 (Block Principal Symbol and Reduced Symbol)**.** At $x \in M$, define

$$U_x := \mathrm{Sym}^2 T_x^*M \oplus (\mathrm{Sym}^2 T_x^*M \otimes N_x) \oplus (T_x^*M \otimes \mathfrak{so}(N_x)),$$

$$V_x := \Lambda^2 T_x^*M \otimes \mathrm{End}(T_xM) \oplus \Lambda^2 T_x^*M \otimes \mathrm{Hom}(T_xM, N_x) \oplus \Lambda^2 T_x^*M \otimes \mathfrak{so}(N_x).$$

For $\xi \neq 0$, take $\sigma_\xi(\mathbf{L}_{(h,K,\nabla^{\perp})}) : U_x \to V_x$ blockwise from highest-order terms in each variable block. For

$$\sigma_\xi(\delta_{\mathrm{gauge}}) : \mathfrak{g}_x \to U_x, \qquad \mathfrak{g}_x := T_xM \oplus \mathfrak{so}(N_x),$$

the reduced symbol is

$$\overline{\sigma}_\xi(\mathbf{L}_{(h,K,\nabla^{\perp})}) : U_x / \operatorname{im} \sigma_\xi(\delta_{\mathrm{gauge}}) \to V_x.$$

Equivalently, this equals the principal symbol of the linearization viewed on the direct sum of jet bundles with variable weights $(2, 1, 1)$ for $(h, K, \nabla^{\perp})$.

Definition 8.4 (Reduced Ellipticity Convention)**.** Fix the convention: ellipticity means injectivity of $\overline{\sigma}_\xi(\mathbf{L}_{(h,K,\nabla^{\perp})})$ for every $\xi \neq 0$.

Remark 8.5. This is the standard overdetermined (injective-symbol) notion (cf. [5]); no surjectivity or Fredholm claim is made here.

Lemma 8.6 (Invariance of the Block Symbol Construction)**.** *The block principal symbol and reduced symbol above are coordinate-independent and depend only on (x, ξ).*

Proof. For each block, the principal symbol is obtained by retaining only the highest-derivative terms in local coordinates; these coefficients transform tensorially under coordinate change, so each block symbol is well-defined as a linear map depending only on (x, ξ). The same coordinate-change rule applies to the infinitesimal gauge map, so both $\sigma_\xi(\mathbf{L}_{(h,K,\nabla^\perp)})$ and $\sigma_\xi(\delta_{\text{gauge}})$ are equivariant under $\mathrm{GL}(T_xM) \times \mathrm{SO}(N_x)$. Hence $\operatorname{im}\sigma_\xi(\delta_{\text{gauge}}) \subset U_x$ is intrinsic, and the induced quotient map $\overline{\sigma}_\xi(\mathbf{L}_{(h,K,\nabla^\perp)})$ is intrinsic as well. □

Proposition 8.7 (Gauge Equivariance)**.** *$\mathcal{D}$ intertwines* $\mathrm{Diff}(M) \ltimes \Gamma(\mathrm{SO}(N))$.

Proof. Equivariance follows from functoriality of $\nabla^\oplus$ and covariance of curvature under pullback/conjugation, then projection to the three bundle-valued blocks. □

8.2 Linearization of $\mathcal{D}$

Let $(\delta h, \delta K, a)$ be a variation of $(h, K, \nabla^\perp)$, where $a \in \Omega^1(M, \mathfrak{so}(N))$ is the connection variation. The linearization decomposes blockwise:

$$\delta\mathcal{D} = (\delta\mathcal{G}, \delta\mathcal{C}, \delta\mathcal{S}).$$

Principal-order terms are:

$$\delta\mathcal{G} = \delta R^h + (\text{terms of order} \leq 1 \text{ in } \delta h, \text{ order } 0 \text{ in } \delta K, a),$$

$$\delta\mathcal{C} = D^{\text{Hom}}\delta K + (\text{terms of order} \leq 1 \text{ in } \delta h, \text{ order } 0 \text{ in } a, \delta K),$$

$$\delta\mathcal{S} = \delta R^\perp + (\text{terms of order } 0 \text{ in } \delta h, \delta K, a).$$

Hence the mixed-order structure is intrinsic: second order in h, first order in K and $\nabla^\perp$. Since $\mathcal{D}$ is not pure-order, principal symbols are taken blockwise by highest-order contributions in each variable block.

8.3 Principal Symbol of the Full Operator

For $\xi \neq 0$, the symbol is computed from highest-order terms:

$$\sigma_\xi(\mathbf{L}_{(h,K,\nabla^\perp)})(\delta h, \delta K, a) = \Big(\mathcal{R}_\xi(\delta h),\ \xi \wedge \delta K + \mathcal{B}_\xi(\delta h; K),\ \xi \wedge a\Big),$$

where:

1. $\mathcal{R}_\xi$ is the Riemann-curvature symbol in the metric variation, e.g.

   $$(\mathcal{R}_\xi(\delta h))_{ijkl} = \frac{1}{2}(\xi_k\xi_i\delta h_{jl} + \xi_l\xi_j\delta h_{ik} - \xi_k\xi_j\delta h_{il} - \xi_l\xi_i\delta h_{jk}).$$

   Using h_x to raise an index, identify

   $$\Lambda^2 T_x^*M \otimes \mathrm{End}(T_xM) \cong \Lambda^2 T_x^*M \otimes \Lambda^2 T_x^*M.$$

   Under this identification, $\mathcal{R}_\xi(\delta h)$ has the standard Riemann symmetries.

2. $\xi \wedge \delta K$ is the Codazzi principal symbol, defined by

$$((\xi \wedge \delta K)(X,Y))Z := \xi(X)\,\delta K(Y,Z) - \xi(Y)\,\delta K(X,Z).$$

3. $\xi \wedge a$ is the curvature symbol of the normal connection variation, defined by

$$((\xi \wedge a)(X,Y)) := \xi(X)\,a(Y) - \xi(Y)\,a(X) \in \mathfrak{so}(N).$$

4. $\mathcal{B}_\xi(\delta h; K)$ is first order in δh and vanishes at $K = 0$.

8.4 Gauge Symbol and Reduced Symbol

The infinitesimal gauge action generated by $(V,\phi) \in \Gamma(TM) \oplus \Gamma(\mathfrak{so}(N))$ is

$$\delta_{(V,\phi)} h = \mathcal{L}_V h, \qquad \delta_{(V,\phi)} K = \mathcal{L}_V K + \phi \cdot K, \qquad \delta_{(V,\phi)} \nabla^\perp = \mathcal{L}_V \nabla^\perp + d^{\nabla^\perp}\phi.$$

Diffeomorphisms act on $\nabla^\perp$ by pullback of connections on $N \to M$, and $\mathcal{L}_V \nabla^\perp$ denotes this infinitesimal pullback action. Concretely, φ sends $(N, \nabla^\perp)$ to $(\varphi^* N, \varphi^* \nabla^\perp)$; we implicitly identify $\varphi^* N$ with N in local trivializations when writing the infinitesimal action. Its principal symbol at $\xi \neq 0$ is

$$\sigma_\xi(\delta_{\text{gauge}})(V,\phi) = \big(\xi \odot V^\flat,\ \mathcal{K}_\xi(V,\phi;K),\ \xi \otimes \phi\big),$$

where $\xi \odot V^\flat := \frac{1}{2}(\xi \otimes V^\flat + V^\flat \otimes \xi)$ is the symmetric product (principal symbol of $\mathcal{L}_V h$), and $\mathcal{K}_\xi(V,\phi;K)$ is linear in K and vanishes at $K = 0$. The reduced symbol is the quotient

$$\overline{\sigma}_\xi(\mathbf{L}_{(h,K,\nabla^\perp)}) := \sigma_\xi(\mathbf{L}_{(h,K,\nabla^\perp)}) \bmod \operatorname{im} \sigma_\xi(\delta_{\text{gauge}}),$$

and the residual symbol-kernel is

$$\ker \overline{\sigma}_\xi(\mathbf{L}_{(h,K,\nabla^\perp)}) \cong \frac{\ker \sigma_\xi(\mathbf{L}_{(h,K,\nabla^\perp)}) + \operatorname{im} \sigma_\xi(\delta_{\text{gauge}})}{\operatorname{im} \sigma_\xi(\delta_{\text{gauge}})} \cong \frac{\ker \sigma_\xi}{\ker \sigma_\xi \cap \operatorname{im} \sigma_\xi(\delta_{\text{gauge}})}.$$

The reduced characteristic set is defined by failure of injectivity of $\overline{\sigma}_\xi(\mathbf{L}_{(h,K,\nabla^\perp)})$.

Lemma 8.8 (Principal Gauge Symbol Computation)**.** *In a local frame at x, the principal blocks are*

$$\sigma_\xi(\mathcal{L}_V h) = \xi \odot V^\flat, \qquad \sigma_\xi(d^{\nabla^\perp}\phi) = \xi \otimes \phi,$$

and the K-block principal symbol is K-linear, hence vanishes at $K = 0$.

Proof. Using $\mathcal{L}_V h_{ij} = \nabla_i V_j + \nabla_j V_i$, principal order gives $\xi_i V_j + \xi_j V_i = \xi \odot V^\flat$. For the normal gauge block, $(d^{\nabla^\perp}\phi)_i = \nabla^\perp_i \phi + \cdots$, so principal order is $\xi_i \phi$, i.e. $\xi \otimes \phi$. For K, $\delta_{(V,\phi)} K = \mathcal{L}_V K + \phi \cdot K$, whose principal part is linear in coefficients of K; therefore it vanishes at $K = 0$. $\square$

9 Symbol Consequences and Non-Ellipticity

9.1 Frozen-Background Linearization (Warm-Up)

Linearize at $K = 0$ with h and $\nabla^\perp$ fixed. Then $\delta R^h = 0$, $\delta R^\perp = 0$, and

$$\delta(A \wedge K)\big|_{K=0} = 0, \qquad \delta(K \wedge A)\big|_{K=0} = 0.$$

Only the first-order Codazzi block remains:

$$L : \Gamma(\mathrm{Sym}^2 T^* M \otimes N) \to \Omega^2(\mathrm{Hom}(TM,N)), \qquad L(K) := D^{\mathrm{Hom}} K.$$

Using $\mathrm{Hom}(T_x M, N_x) \cong T^*_x M \otimes N_x$, we identify the frozen Codazzi target with $\Lambda^2 T^*_x M \otimes T^*_x M \otimes N_x$.

Theorem 9.1 (All-Dimension Frozen Codazzi Kernel)**.** *Let $n \geq 2$ and N have rank k. For nonzero covector ξ, consider*

$$\sigma_\xi(L) : \ \mathrm{Sym}^2(T_x^*M) \otimes N_x \to \Lambda^2(T_x^*M) \otimes T_x^*M \otimes N_x,$$

given by $(\sigma_\xi \delta K)^\alpha_{ij|\ell} = \xi_i \delta K^\alpha_{j\ell} - \xi_j \delta K^\alpha_{i\ell}$. Then $\dim\ker\sigma_\xi(L) = k$.

Proof. By linear invariance, choose $\xi = e^1$. Then

$$(\sigma_\xi \delta K)^\alpha_{ij|\ell} = \delta_{i1}\delta K^\alpha_{j\ell} - \delta_{j1}\delta K^\alpha_{i\ell}.$$

For $i = 1 < j$, all equations force $\delta K^\alpha_{j\ell} = 0$ for $j > 1$ and all ℓ. By symmetry in (j, ℓ), every component with at least one index > 1 vanishes. The only free component per normal direction is δK^α_{11}. Hence per normal direction nullity is 1, and total nullity is k. Equivalently, invariantly

$$\ker\sigma_\xi(L) \cong \mathrm{span}\{\xi \otimes \xi\} \otimes N_x.$$

□

Corollary 9.2 (Surface Frozen-Background Kernel Dimension)**.** *Let $n = 2$ and* $\mathrm{rank}(N) = k$. *For any non-zero covector $\xi \in T_x^*M$, one has* $\dim\ker\sigma_\xi(L) = k$.

Proposition 9.3 (Frozen Gauge Symbol: Vanishing δK Block)**.** *Fix $(h, \nabla^\perp)$ and linearize at $K = 0$. For $\xi \in T_x^*M \setminus \{0\}$, the principal symbol of the infinitesimal gauge action satisfies*

$$(\sigma_\xi(\delta_{\mathrm{gauge}})(V, \phi))_{\delta K} = 0.$$

Equivalently,

$$\mathrm{im}\,\sigma_\xi(\delta_{\mathrm{gauge}}) \cap (\delta K\textit{-subspace}) = \{0\}.$$

Proof. At $K = 0$, one has

$$\delta_{(V,\phi)}K = \mathcal{L}_V K + \phi \cdot K = 0.$$

Hence the principal gauge symbol has no δK-component. □

Theorem 9.4 (Reduced Frozen Symbol Has k-Dimensional Kernel)**.** *Fix $(h, \nabla^\perp)$ and linearize at $K = 0$. Let $\overline{\sigma}_\xi(L)$ denote the Codazzi principal symbol on the δK-sector induced after quotienting by* $\mathrm{im}\,\sigma_\xi(\delta_{\mathrm{gauge}})$. *Then for every $\xi \neq 0$,*

$$\dim\ker\overline{\sigma}_\xi(L) = k.$$

Consequently, the compatibility operator is not elliptic at $K = 0$ even after principal-symbol gauge reduction.

Proof. The all-dimension kernel theorem gives $\dim\ker\sigma_\xi(L) = k$. By the preceding proposition, $\mathrm{im}\,\sigma_\xi(\delta_{\mathrm{gauge}})$ has zero δK-component at $K = 0$, so the quotient projection is injective on the δK-subspace, in particular on $\ker\sigma_\xi(L)$. Therefore, as a map on the δK-sector, $\ker\overline{\sigma}_\xi(L)$ identifies with $\ker\sigma_\xi(L)$ and has dimension k. □

9.2 Non-Ellipticity Regimes and Rank Structure

The full operator has mixed order and persistent symbol kernels:

1. metric directions enter at order 2 through $\mathcal{R}_\xi$;
2. K- and $\nabla^\perp$-directions enter at order 1;
3. gauge directions survive at symbol level through $\xi \odot V^\flat$ and $\xi \otimes \phi$.

Consequently, ellipticity fails generically before gauge reduction; after gauge reduction, injectivity still fails in the frozen background for the codimension-k family above.

Theorem 9.5 (Reduced Non-Ellipticity at Frozen Background)**.** *Fix $(h, \nabla^\perp)$ and background $K = 0$. For every nonzero $\xi \in T_x^*M$,*

$$\dim\ker \overline{\sigma}_\xi(\mathbf{L}_{(h,0,\nabla^\perp)}) \geq k.$$

Hence $\mathbf{L}_{(h,0,\nabla^\perp)}$ is not reduced-elliptic.

Proof. Restrict to the δK-subspace and use the frozen Codazzi block L. By the all-dimension kernel theorem, $\dim\ker \sigma_\xi(L) = k$. By the frozen gauge-symbol proposition, the gauge image has zero δK-component at $K = 0$, so these kernel directions survive in the quotient. Therefore the reduced kernel of the full linearization has dimension at least k. □

9.3 Minimal Prolongation Identities from Bianchi

Theorem 9.6 (First Universal Compatibility Conditions)**.** *For the system $\mathcal{D} = 0$, the only universal off-shell identities obtained functorially from $\nabla^\oplus$ are the coupled Bianchi identities from Section 4.*

Proof. The system is equivalent to $\mathcal{F}^\oplus = 0$. Applying $D^\oplus$ gives $D^\oplus \mathcal{F}^\oplus = 0$, whose block projections are precisely the coupled identities for $(\mathcal{G}, \mathcal{C}, \mathcal{S})$. Section 6 shows closure under repeated covariant differentiation in the curvature-generated ideal, so these identities generate the first prolongation-level universal consequences. □

Remark 9.7. Here "universal" means gauge-natural jet-level identities holding for all $(h, K, \nabla^\perp)$, not identities tied to specific solution subclasses.

A Component Translation

Restricting to an orthonormal frame e_i and normal frame n_α, the invariant blocks correspond to the classical Gauss-Codazzi-Ricci equations: here $\nabla^{h,\perp}$ denotes the induced connection on $\mathrm{Sym}^2 T^*M \otimes N$ (Levi-Civita on tangent indices and $\nabla^\perp$ on the normal index). The second fundamental form components satisfy $K^\alpha_{ij} = K^\alpha_{ji}$. Define

$$\mathcal{G}_{ijkl} := \langle \mathcal{G}(e_i, e_j)e_k, e_l\rangle_h, \qquad \mathcal{C}^\alpha_{ijk} := \langle \mathcal{C}(e_i, e_j)e_k, n_\alpha\rangle_N, \qquad \mathcal{S}_{\alpha\beta ij} := \langle \mathcal{S}(e_i, e_j)n_\alpha, n_\beta\rangle_N.$$

1. **Gauss:** $\mathcal{G}_{ijkl} = R^h_{ijkl} - \sum_\alpha (K^\alpha_{ik} K^\alpha_{jl} - K^\alpha_{il} K^\alpha_{jk})$.
2. **Codazzi:** $\mathcal{C}^\alpha_{ijk} = \nabla^{h,\perp}_k K^\alpha_{ij} - \nabla^{h,\perp}_j K^\alpha_{ik}$.
3. **Ricci:** $\mathcal{S}_{\alpha\beta ij} = R^\perp_{\alpha\beta ij} - \sum_k (K^\alpha_{ik} K^\beta_{jk} - K^\beta_{ik} K^\alpha_{jk})$.

Remark A.1. With the curvature convention fixed above, these Gauss/Ricci sign conventions agree with standard conventions.

B Variation Formulas

For metric variation δh on (M, h), the Levi-Civita variation has principal part

$$\delta\Gamma^i_{jk} = \frac{1}{2}h^{i\ell}\big(\nabla_j \delta h_{k\ell} + \nabla_k \delta h_{j\ell} - \nabla_\ell \delta h_{jk}\big),$$

and curvature variation has principal part

$$(\delta R^h)^i{}_{jkl} = \nabla_k(\delta\Gamma^i_{jl}) - \nabla_l(\delta\Gamma^i_{jk}) + \text{l.o.t.}$$

For normal connection variation $a = \delta\nabla^\perp \in \Omega^1(M, \mathfrak{so}(N))$:

$$\delta R^\perp = D^\perp a + \text{l.o.t.}$$

For mixed block variation:

$$\delta(D^{\mathrm{Hom}} K) = D^{\mathrm{Hom}} \delta K + (\delta\nabla^\perp)\cdot K - K \cdot (\delta\nabla^h) + \text{l.o.t.}$$

These formulas justify the principal symbols stated in Sections 8 and 9.

C Symbol Tables

Block	δh	δK	$a = \delta\nabla^\perp$
$\sigma_\xi(\delta\mathcal{G})$	$\mathcal{R}_\xi(\delta h)$	0	0
$\sigma_\xi(\delta\mathcal{C})$	$\mathcal{B}_\xi(\delta h; K)$	$\xi \wedge \delta K$	0
$\sigma_\xi(\delta\mathcal{S})$	0	0	$\xi \wedge a$

Gauge Generator	Field Variation	Principal Symbol
$V \in \Gamma(TM)$	$\delta h = \mathcal{L}_V h$	$\xi \odot V^\flat$
$\phi \in \Gamma(\mathfrak{so}(N))$	$\delta\nabla^\perp = d^{\nabla^\perp}\phi + \cdots$	$\xi \otimes \phi$

At $K = 0$, $\mathcal{B}_\xi(\delta h; K) = 0$ and the K-gauge symbol contribution vanishes, recovering the frozen-background symbol used in Section 9.1.

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