On the Integrability of Extrinsic Constraint Data

Sina Montazeri[1]
University of North Texas

(*Electronic mail: sinamontazeri@my.unt.edu)

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The Gauss–Codazzi–Ricci relations are equivalent to flatness of a canonical metric connection on $TM \oplus N$. Given a metric, a symmetric normal-bundle–valued second fundamental form, and a metric normal connection, one canonically constructs a metric connection on the direct sum of tangent and normal bundles whose curvature vanishes exactly when the Gauss–Codazzi–Ricci relations hold.

Flatness on a simply connected neighborhood yields a parallel orthonormal frame and a framed local isometric immersion into Euclidean space realizing the given data, unique up to rigid motion and $\nabla^\perp$-parallel $\mathrm{SO}(k)$ gauge. For hypersurfaces, contractions recover the standard constraint identities and, under Einstein's equations, the Hamiltonian and momentum constraints in the 3+1 decomposition.

The Gauss–Codazzi–Ricci equations link intrinsic curvature on a manifold to auxiliary fields that, in an embedded setting, arise from the second fundamental form and the normal connection of a submanifold. Classically they appear as integrability conditions for the existence of an isometric immersion, and they are naturally expressed by Cartan's structure equations for an orthonormal moving frame.

The constraint-data formulation takes a triple $(h, K, \nabla^\perp)$ on (M, N), where h is a Riemannian metric on M, $K \in \Gamma(\mathrm{Sym}^2 T^*M \otimes N)$, and $\nabla^\perp$ is a metric connection on N, and imposes the Gauss–Codazzi–Ricci relations. The natural equivalences are diffeomorphisms of M and orthogonal bundle automorphisms of N preserving $\nabla^\perp$. This provides a gauge-invariant packaging of constraint data compatible with both Euclidean realizations and Lorentzian hypersurface geometry.

The GCR system is exactly the flatness condition for the canonical metric connection on the orthogonal sum bundle $TM \oplus N$. Flatness implies that, on any simply connected neighborhood, one can choose a parallel orthonormal frame of $TM \oplus N$ and integrate a coframing to obtain a map into Euclidean space. This recovers the fundamental theorem of submanifold theory[6,10,11] in a gauge-invariant bundle formulation: GCR data and framed local Euclidean immersions carry the same local content.

Finally, the hypersurface specialization connects directly to Lorentzian constraint geometry. In codimension one, the GCR relations reduce to Gauss and Codazzi constraints on a pair (h, K); when h is induced on a spacelike hypersurface, standard contractions of these relations yield geometric identities whose right-hand sides, upon imposing Einstein's equations, become the Hamiltonian and momentum constraints.

a. Main result. On simply connected $U \subset M$, solutions of the GCR system are in one-to-one correspondence with framed isometric immersions $X : U \to \mathbb{R}^{n+k}$ modulo Euclidean motions, and this correspondence is induced by the flat canonical connection on $TM \oplus N$.

I. CONSTRAINT DATA

Let M^n be a smooth connected manifold (without boundary). Let $N \to M$ be a real vector bundle of rank k equipped with a positive–definite fiber metric $\langle \cdot, \cdot \rangle_N$ and a metric connection $\nabla^\perp$.

a. Conventions. The following conventions are used throughout:

- The Riemann curvature tensor is defined by $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$.
- The curvature 2-form of a connection with connection 1-forms ω_{ij} is $\mathcal{R}_{ij} := d\omega_{ij} + \omega_{ik} \wedge \omega_{kj}$.
- For an embedded hypersurface with unit normal n, the second fundamental form is $K_{ij} = \langle \nabla_{e_i} n, e_j \rangle$ (no minus sign).
- In Lorentzian signature, the metric signature is $(-, +, \dots, +)$.
- Indices i, j, k, ℓ run over tangent frame indices $1, \dots, n$; indices α, β, γ run over normal frame indices $1, \dots, k$.

All signs in subsequent equations are consistent with these conventions.

Definition I.1 (GCR data)**.** A *GCR triple* on (M, N) is a triple $(h, K, \nabla^\perp)$ where:

- h is a Riemannian metric on M with Levi–Civita connection ∇;
- $K \in \Gamma(\mathrm{Sym}^2 T^*M \otimes N)$ is a symmetric N–valued $(0, 2)$–tensor;
- $\nabla^\perp$ is a metric connection on N.

Choose local h–orthonormal tangent frames $\{e_i\}_{i=1}^n$ with dual coframe $\{\theta^i\}$, and local orthonormal frames $\{n_\alpha\}_{\alpha=1}^k$ of N. Write

$$K = \sum_{\alpha=1}^{k} K^{(\alpha)} \otimes n_\alpha, \qquad K^{(\alpha)} \in \Gamma(\mathrm{Sym}^2 T^*M).$$

Let ω_{ij} denote the Levi–Civita connection 1-forms, and $\omega_{\alpha\beta}$ the normal connection 1-forms. The Levi–Civita connection satisfies the torsion-free structure equation

$$d\theta^i + \omega_{ij} \wedge \theta^j = 0. \tag{1}$$

Define the mixed 1-forms

$$\omega_{i\alpha} := \sum_{j=1}^{n} K_{ij}^{(\alpha)}\, \theta^j, \qquad \omega_{\alpha i} := -\omega_{i\alpha}, \tag{2}$$

where $K_{ij}^{(\alpha)} := K^{(\alpha)}(e_i, e_j)$.

b. Covariant derivative of K. The notation ∇K denotes the induced covariant derivative on $\mathrm{Sym}^2 T^*M \otimes N$, defined by

$$(\nabla_X K)(Y,Z) := \nabla_X^\perp\big(K(Y,Z)\big) - K(\nabla_X Y, Z) - K(Y, \nabla_X Z).$$

In an adapted orthonormal frame, denote $(\nabla_{e_i} K)_{jk}^{(\alpha)} := \langle (\nabla_{e_i} K)(e_j, e_k),\, n_\alpha \rangle_N$.

Definition I.2 (Gauss–Codazzi–Ricci equations)**.** The triple $(h, K, \nabla^\perp)$ satisfies the *Gauss–Codazzi–Ricci (GCR) equations* if, in any local adapted orthonormal frames, the following hold:

$$\text{(Gauss)} \quad R_{ijk\ell} = \sum_{\alpha=1}^{k} \left(K_{ik}^{(\alpha)} K_{j\ell}^{(\alpha)} - K_{i\ell}^{(\alpha)} K_{jk}^{(\alpha)} \right), \tag{3}$$

$$\text{(Codazzi)} \quad (\nabla_{e_i} K)_{jk}^{(\alpha)} = (\nabla_{e_j} K)_{ik}^{(\alpha)}, \tag{4}$$

$$\text{(Ricci)} \quad R_{\alpha\beta ij}^\perp = \sum_{m=1}^{n} \left(K_{im}^{(\alpha)} K_{jm}^{(\beta)} - K_{im}^{(\beta)} K_{jm}^{(\alpha)} \right). \tag{5}$$

Here $R_{ijk\ell}$ are the components of the Riemann curvature tensor of (M,h), and $R_{\alpha\beta ij}^\perp$ are the components of the curvature of $\nabla^\perp$.

Definition I.3 (Gauge equivalence)**.** Two triples $(h, K, \nabla^\perp)$ and $(\tilde h, \tilde K, \tilde\nabla^\perp)$ are *equivalent* if there exists a diffeomorphism $\varphi \in \mathrm{Diff}(M)$ and an orthogonal bundle isomorphism $g : N \to N$ covering φ such that

$$\tilde h = \varphi^* h, \qquad \tilde K = g \cdot (\varphi^* K), \qquad \tilde\nabla^\perp = g \circ (\varphi^* \nabla^\perp) \circ g^{-1}.$$

Remark I.4 (Ambient signatures)**.** Everything below is presented for Euclidean ambient group $\mathrm{SO}(n+k)$. The same formalism extends to pseudo-Riemannian ambient signatures by replacing $\mathrm{SO}(n+k)$ with $\mathrm{SO}(p,q)$ and $\mathfrak{so}(n+k)$ with $\mathfrak{so}(p,q)$, and adjusting sign conventions accordingly. All proofs are formal and carry over verbatim with $\mathrm{SO}(p,q)$.

II. FLATNESS

A. Metric connection on the sum bundle

Consider the orthogonal sum bundle $E = TM \oplus N$ equipped with the direct sum metric. An adapted orthonormal frame for E is a tuple $(e_1, \ldots, e_n, n_1, \ldots, n_k)$ where $\{e_i\}$ frames TM and $\{n_\alpha\}$ frames N.

Given a GCR triple $(h, K, \nabla^\perp)$, define a metric connection $\nabla^\oplus$ on $E = TM \oplus N$ by the Gauss–Weingarten formulas: for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(N)$,

$$\nabla_X^\oplus(Y, 0) = (\nabla_X Y,\ K(X,Y)), \tag{6}$$

$$\nabla_X^\oplus(0, \xi) = (-A_\xi X,\ \nabla_X^\perp \xi), \tag{7}$$

where the shape operator $A_\xi : TM \to TM$ is defined by

$$h(A_\xi X, Y) = \langle K(X,Y), \xi \rangle_N.$$

A direct check using the defining relation for A_ξ shows that $\nabla^\oplus$ preserves the direct sum metric on $TM \oplus N$.

In an adapted orthonormal frame, the connection 1-form matrix of $\nabla^\oplus$ is given block-wise by:

$$\Omega = \begin{pmatrix} \omega_{ij} & \omega_{i\alpha} \\ \omega_{\alpha i} & \omega_{\alpha\beta} \end{pmatrix}, \qquad \omega_{\alpha i} = -\omega_{i\alpha}. \tag{8}$$

Here

- ω_{ij} are the Levi–Civita connection 1-forms of (M,h),
- $\omega_{\alpha\beta}$ are the connection 1-forms of $\nabla^\perp$,
- $\omega_{i\alpha} = \sum_j K_{ij}^{(\alpha)} \theta^j$ are the mixed 1-forms defined by the second fundamental form.

Since Ω takes values in $\mathfrak{so}(n+k)$, $\nabla^\oplus$ is a metric compatible connection on $TM \oplus N$. Its curvature 2-form is $\mathcal{F} = d\Omega + \Omega \wedge \Omega$.

B. Flatness and the GCR equations

Proposition II.1 (GCR $\iff$ flatness)**.** *The assembled connection $\nabla^\oplus$ is flat (i.e., $\mathcal{F} = 0$) if and only if the triple $(h, K, \nabla^\perp)$ satisfies the Gauss–Codazzi–Ricci equations.*

Proof. In adapted orthonormal frames, the curvature 2-form has block components

$$\mathcal{F}_{ij} = d\omega_{ij} + \omega_{ik} \wedge \omega_{kj} + \omega_{i\alpha} \wedge \omega_{\alpha j}, \tag{9}$$

$$\mathcal{F}_{i\alpha} = d\omega_{i\alpha} + \omega_{ik} \wedge \omega_{k\alpha} + \omega_{i\beta} \wedge \omega_{\beta\alpha}, \tag{10}$$

$$\mathcal{F}_{\alpha\beta} = d\omega_{\alpha\beta} + \omega_{\alpha i} \wedge \omega_{i\beta} + \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta}. \tag{11}$$

It suffices to verify that $\mathcal{F} = 0$ if and only if (3)–(5) hold.

(i,j)-block (Gauss): By definition of the curvature 2-form, $\mathcal{R}_{ij} := d\omega_{ij} + \omega_{ik} \wedge \omega_{kj}$ is the Riemann curvature 2-form of the Levi–Civita connection. In components, $\mathcal{R}_{ij} = \frac{1}{2} R_{ijk\ell}\, \theta^k \wedge \theta^\ell$.

For the quadratic term, substitute $\omega_{i\alpha} = K_{im}^{(\alpha)} \theta^m$ from (2):

$$\begin{aligned} \omega_{i\alpha} \wedge \omega_{\alpha j} &= \sum_{\alpha,m,p} K_{im}^{(\alpha)} K_{jp}^{(\alpha)}\, \theta^m \wedge \theta^p \\ &= \sum_{\alpha, m<p} \left(K_{im}^{(\alpha)} K_{jp}^{(\alpha)} - K_{ip}^{(\alpha)} K_{jm}^{(\alpha)} \right) \theta^m \wedge \theta^p. \end{aligned}$$

Hence $\mathcal{F}_{ij} = 0$ if and only if

$$R_{ijk\ell} = \sum_{\alpha} \left(K^{(\alpha)}_{ik} K^{(\alpha)}_{j\ell} - K^{(\alpha)}_{i\ell} K^{(\alpha)}_{jk} \right),$$

which is (3).

(i, α)-block (Codazzi): The component $\mathcal{F}_{i\alpha}$ is the covariant exterior derivative $D\omega_{i\alpha}$ with respect to the Levi–Civita and normal connections. In components,

$$\mathcal{F}_{i\alpha}(e_j, e_k) = (\nabla_{e_j} K)^{(\alpha)}_{ik} - (\nabla_{e_k} K)^{(\alpha)}_{ij},$$

so $\mathcal{F}_{i\alpha} = 0$ is equivalent to the Codazzi symmetry (4).

(α, β)-block (Ricci): By definition, $d\omega_{\alpha\beta} + \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta}$ is the curvature 2-form $R^{\perp}_{\alpha\beta}$ of $\nabla^{\perp}$. The quadratic term is

$$\omega_{\alpha i} \wedge \omega_{i\beta} = -\omega_{i\alpha} \wedge \omega_{i\beta} = -\sum_{i,m,p} K^{(\alpha)}_{im} K^{(\beta)}_{ip}\, \theta^m \wedge \theta^p.$$

Expanding and using antisymmetry of the wedge product, $\mathcal{F}_{\alpha\beta} = 0$ if and only if

$$R^{\perp}_{\alpha\beta ij} = \sum_{m} \left(K^{(\alpha)}_{im} K^{(\beta)}_{jm} - K^{(\beta)}_{im} K^{(\alpha)}_{jm} \right),$$

which is (5). □

Remark II.2 (Cartan's method of moving frames)**.** The matrix Ω has the block structure of a connection 1-form on $\mathrm{SO}(n+k)$: the diagonal blocks lie in $\mathfrak{so}(n) \oplus \mathfrak{so}(k)$, while the off-diagonal blocks lie in the complement $\mathfrak{m} \cong \mathbb{R}^n \otimes \mathbb{R}^k$. In a local trivialization, the flatness condition $d\Omega + \Omega \wedge \Omega = 0$ is the Maurer–Cartan equation, hence Ω is locally of the form $F^{-1} dF$ for some $F : U \to \mathrm{SO}(n+k)$.

III. INTEGRATION

Flatness integrates to a local realization on simply connected domains.

Theorem III.1 (Local realization from GCR data)**.** *Let $(h, K, \nabla^{\perp})$ satisfy the Gauss–Codazzi–Ricci equations on an open set $U \subset M$. Assume U is simply connected. Then there exists a smooth map*

$$X : U \to \mathbb{R}^{n+k}$$

and smooth fields $(E_1, \ldots, E_n, N_1, \ldots, N_k)$ along X forming an orthonormal frame of $\mathbb{R}^{n+k}$ at each point, such that:

1. *X is a local isometric immersion: $dX(e_i) = E_i$ and $\langle dX(\cdot), dX(\cdot) \rangle = h$;*
2. *the second fundamental form and normal connection induced by the frame (N_α) coincide with $(K, \nabla^{\perp})$ on U.*

Proof. Fix an adapted orthonormal frame $\mathbf{u}$ of $E|_U$. In this trivialization, $\nabla^{\oplus}$ is represented by the $\mathfrak{so}(n+k)$-valued 1-form Ω defined in (8). By Proposition II.1, $d\Omega + \Omega \wedge \Omega = 0$. Since U is simply connected, there exists a smooth map

$$F : U \to \mathrm{SO}(n+k)$$

satisfying $F^{-1} dF = \Omega$.

Choose a parallel frame of the trivial bundle $U \times \mathbb{R}^{n+k}$ induced by the gauge F. Writing the columns of F as $\mathbb{R}^{n+k}$-valued functions gives an orthonormal frame

$$(E_1, \ldots, E_n, N_1, \ldots, N_k)$$

satisfying

$$dE_i = \omega_{ij} E_j + \omega_{i\alpha} N_\alpha, \qquad (12)$$

$$dN_\alpha = -\omega_{i\alpha} E_i + \omega_{\alpha\beta} N_\beta. \qquad (13)$$

Define an $\mathbb{R}^{n+k}$-valued 1-form on U by

$$\eta := \sum_{i=1}^{n} E_i\, \theta^i.$$

Using (12) and the torsion-free structure equation (1), one computes

$$d\eta = \sum_i dE_i \wedge \theta^i + \sum_i E_i\, d\theta^i = \sum_{i,\alpha} \omega_{i\alpha} N_\alpha \wedge \theta^i + \sum_{i,j} (\omega_{ij} E_j \wedge \theta^i - E_i\, \omega_{ij} \wedge$$

The key cancellation is that

$$\sum_{i,\alpha} \omega_{i\alpha} N_\alpha \wedge \theta^i = \sum_{i,j,\alpha} K^{(\alpha)}_{ij} N_\alpha\, \theta^j \wedge \theta^i = 0$$

because $K^{(\alpha)}_{ij} = K^{(\alpha)}_{ji}$ is symmetric while $\theta^j \wedge \theta^i$ is antisymmetric. Combined with the skew-symmetry $\omega_{ij} = -\omega_{ji}$ and relabeling of indices in the second sum, this yields $d\eta = 0$. Hence η is closed. Since U is simply connected, there exists $X : U \to \mathbb{R}^{n+k}$ with $dX = \eta$. Then $dX(e_i) = E_i$ and $\langle E_i, E_j \rangle = \delta_{ij}$ implies X is an isometric immersion.

Finally, differentiating $dX(e_i) = E_i$ and projecting the derivative of E_i onto the normal span of $\{N_\alpha\}$ using (12) yields the second fundamental form coefficients $K^{(\alpha)}_{ij}$ (via (2)), while (13) shows that the normal connection in the frame N_α is precisely $\omega_{\alpha\beta}$, i.e. $\nabla^{\perp}$. □

Remark III.2 (Euclidean connection)**.** The integration of the position vector X can be packaged alongside the frame F by considering the Euclidean group $E(n+k) = \mathbb{R}^{n+k} \rtimes \mathrm{SO}(n+k)$. We define the extended connection form $\widehat{\Omega}$ taking values in $\mathfrak{e}(n+k)$ by

$$\widehat{\Omega} = \begin{pmatrix} \Omega & \eta \\ 0 & 0 \end{pmatrix},$$

(in the standard matrix representation of $\mathfrak{e}(n+k)$), so that the bracket includes the semidirect product action of $\mathfrak{so}(n+k)$ on $\mathbb{R}^{n+k}$), where $\eta = \sum E_i\theta^i$ (pulled back to U) represents the soldering form. The condition that $\widehat{\Omega}$ is flat, $d\widehat{\Omega} + \widehat{\Omega}\wedge\widehat{\Omega} = 0$, decomposes into $d\Omega + \Omega\wedge\Omega = 0$ (GCR) and $d\eta + \Omega\wedge\eta = 0$ (torsion-free condition), offering a unified flat-connection description of the immersion geometry.

Theorem III.3 (Local uniqueness)**.** *Under the hypotheses of Theorem III.1, let (X, F) and $(\tilde{X}, \tilde{F})$ be two realizations constructed from the same connection form Ω (i.e., using the same adapted frame). Then there exists a constant $A \in \mathrm{SO}(n+k)$ and a constant vector $b \in \mathbb{R}^{n+k}$ such that $\tilde{X} = AX + b$ and $\tilde{F} = AF$.*

Equivalently, uniqueness holds modulo rigid motions once a gauge (trivialization) of $E|_U$ is fixed.

Proof. If $F^{-1}dF = \Omega = \tilde{F}^{-1}d\tilde{F}$ on a simply connected domain, then $d(\tilde{F}F^{-1}) = 0$, hence $\tilde{F} = AF$ for constant $A \in \mathrm{SO}(n+k)$. Since $d\tilde{X} = \sum_i \tilde{E}_i\theta^i = \sum_i (AE_i)\theta^i = A\,dX$, it follows that $d(\tilde{X} - AX) = 0$, so $\tilde{X} = AX + b$ for constant b. □

Remark III.4 (Gauge freedom)**.** Changing the choice of adapted normal frame corresponds to a gauge transformation $g : U \to \mathrm{SO}(k)$. This transforms the connection form $\Omega \mapsto \tilde{\Omega}$ and consequently the realizing frame $F \mapsto \tilde{F}$. If the triple $(h, K, \nabla^\perp)$ is held fixed (as a section K of $\mathrm{Sym}^2T^*M \otimes N$ with connection $\nabla^\perp$), then the residual normal gauge freedom consists precisely of $\nabla^\perp$-parallel $\mathrm{SO}(k)$-automorphisms (i.e. $\nabla^\perp g = 0$).

Corollary III.5 (Equivalence of local realization problems)**.** *On simply connected $U \subset M$, specifying a GCR triple $(h, K, \nabla^\perp)$ is equivalent to specifying a framed isometric immersion $X : U \to \mathbb{R}^{n+k}$ up to rigid motion, and the equivalence is compatible with diffeomorphisms of U and $\mathrm{SO}(k)$-changes of normal frame.*

Remark III.6 (On non-simply-connected domains)**.** On a general domain U, flatness yields a locally defined parallel frame and hence local realizations; global realizability is controlled by holonomy (monodromy) of the flat connection.

IV. LORENTZIAN HYPERSURFACES

The codimension one reduction isolates the standard contractions in Lorentzian hypersurface geometry.

A. Codimension one reduction

Assume $\mathrm{rank}(N) = 1$ and fix a unit normal field n (locally). Then $\nabla^\perp$ is trivial and K becomes an ordinary symmetric $(0, 2)$-tensor on M, $K_{ij} = K(e_i, e_j)$. The Ricci equation is identically satisfied, leaving only Gauss and Codazzi.

In an *ambient space form* of constant curvature κ, the Gauss equation reads

$$R_{ijk\ell} = \kappa\,(h_{ik}h_{j\ell} - h_{i\ell}h_{jk}) + K_{ik}K_{j\ell} - K_{i\ell}K_{jk},$$

and Codazzi reads $(\nabla_{e_i}K)_{jk} = (\nabla_{e_j}K)_{ik}$. In the Euclidean case $\kappa = 0$.

B. Lorentzian ambient geometry and constraint equations

Let $(\mathcal{M}^{n+1}, g)$ be a Lorentzian manifold with signature $(-, +, \ldots, +)$, and let $M^n \hookrightarrow \mathcal{M}$ be a *spacelike* hypersurface with induced Riemannian metric h and second fundamental form K (defined by the conventions of Section I). Let n^μ be the future-pointing unit normal and $h^\mu{}_\nu = \delta^\mu{}_\nu + n^\mu n_\nu$ the tangential projector.

Standard contractions of the Gauss and Codazzi equations yield the geometric identities:

$$^{(n)}R + K^2 - K_{ij}K^{ij} = 2\,G_{\mu\nu}n^\mu n^\nu, \tag{14}$$

$$\nabla_j\left(K^j{}_i - K\,\delta^j{}_i\right) = -\,G_{\mu\nu}n^\mu h^\nu{}_i, \tag{15}$$

where $^{(n)}R$ is the scalar curvature of (M, h), $K = K^i{}_i$ is the trace, and $G_{\mu\nu}$ is the Einstein tensor of $(\mathcal{M}, g)$.

Remark IV.1 (Sign conventions)**.** The sign of the extrinsic term $K^2 - K_{ij}K^{ij}$ in (14) depends on the chosen signatures for the Riemann tensor, the normal n, and the definition of K. With our conventions fixed in Section I, this contraction matches the standard form found in[2,7].

Equations (14)–(15) are purely geometric Gauss–Codazzi contractions. Imposing Einstein's equations ($G_{\mu\nu} = 8\pi T_{\mu\nu}$) relates the right-hand sides to energy and momentum densities:

$$2G_{\mu\nu}n^\mu n^\nu = 16\pi\rho, \qquad -G_{\mu\nu}n^\mu h^\nu_i = 8\pi j_i.$$

In vacuum ($G_{\mu\nu} = 0$) they vanish, yielding the familiar Hamiltonian and momentum constraints of the 3+1 decomposition.

Remark IV.2 (Canonical momentum in ADM)**.** If one chooses the Einstein–Hilbert action (with appropriate boundary terms) and performs a 3+1 split, the momentum conjugate to h_{ab} is the density

$$\pi^{ab} = \sqrt{h}\,(K^{ab} - Kh^{ab}),$$

and the constraints can be written as a scalar constraint $\mathcal{H}(h, \pi) = 0$ and a vector constraint $\mathcal{H}_a(h, \pi) = 0$. The appearance of π^{ab} is variational; the appearance of (14)–(15) is purely hypersurface–geometric.

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

[1]R. Arnowitt, S. Deser, and C. W. Misner, *Republication of: The Dynamics of General Relativity*, Gen. Relativ. Gravit. **40** (2008), 1997–2027. (Originally in *Gravitation: an introduction to current research*, ed. L. Witten, Wiley, 1962.) See also arXiv:gr-qc/0405109.
[2]E. Gourgoulhon, 3+1 *Formalism and Bases of Numerical Relativity*, arXiv:gr-qc/0703035 (2007).
[3]R. Bartnik and J. Isenberg, *The Constraint Equations*, arXiv:gr-qc/0405092 (2004).
[4]C. Teitelboim, *How commutators of constraints reflect the spacetime structure*, Ann. Phys. **79** (1973), 542–557.
[5]P. A. M. Dirac, *The theory of gravitation in Hamiltonian form*, Proc. Roy. Soc. Lond. A **246** (1958), 333–343.
[6]S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol. II*, Interscience/Wiley, 1969.
[7]R. M. Wald, *General Relativity*, University of Chicago Press, 1984.
[8]T. A. Ivey and J. M. Landsberg, *Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Systems*, Graduate Studies in Mathematics, Vol. 61, AMS, 2003.
[9]R. W. Sharpe, *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program*, Graduate Texts in Mathematics 166, Springer, 1997.
[10]M. Spivak, *A Comprehensive Introduction to Differential Geometry, Vol. IV*, Publish or Perish, 3rd ed.
[11]M. P. do Carmo, *Riemannian Geometry*, Birkhäuser, 1992.