A Survey of Gossiping and Broadcasting in Communication Networks

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Gossiping and broadcasting are two problems of information dissemination described for a group of individuals connected by a communication network. In gossiping every person in the network knows a unique item of information and needs to communicate it to everyone else. In broadcasting one individual has an item of information which needs to be communicated to everyone else. We review the results that have been obtained on these and related problems.

1. INTRODUCTION

In 1950, Bavelas [6] studied the effectiveness of different communication patterns in helping small groups of people solve common tasks. A typical task studied was the following: each of five subjects is dealt five playing cards; the cards may not be passed around, but the subjects may communicate with one another according to a given communication pattern by writing messages; the task is considered finished when each subject selects one of their five cards so that the five cards selected comprise the highest-ranking poker hand that can be made by selecting one card from each person. (We assume that each subject has a perfect knowledge of poker hand ratings.) Bavelas considered such measures as the number of messages and the time required to complete the task. He showed that for any communication pattern of a certain type among n people, $2(n - 1)$ messages are required. He also showed that, if any communication pattern is allowed and each message takes unit time, then the time required to complete

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the task is no more than $\lceil \log_2 n \rceil$ time units. Some related experiments were reported by Leavitt in 1949 [82].

In 1951, Shimbel [116] proposed the following problem in group communication as an application of matrix algebra. There are n members in a group. Each member knows initially to whom they can send messages but does not know to whom other members can send messages. The goal of the group communication problem is for each member to learn "the complete structure of their communication system." That is, the problem is for all members to learn who each other member can send messages to. Shimbel raised several interesting questions: Which structures allow solution of the problem? What is the minimum time required to determine the structure? What is the tradeoff between the number of channels and the time required?

In 1954, Landau [80] studied task-oriented groups in which each individual, initially, has one piece of information which must be transmitted to all the others in order to complete a task. At every sending time, each individual sends all the information they have acquired to one other individual. A typical example of a task in this case is the following: "each individual is given a set of colored marbles, only one color being common to all the group. The members of the group must exchange messages about their own colors and what they have learned about the colors of the others, until finally everybody knows the common color. Messages are sent only after everyone has indicated readiness to transmit; the transmissions then take place simultaneously, each individual sending to just one of their possible recipients. Each member of the group knows initially to whom they can send messages, but does not know to whom the others can send" [80]. A major assumption that Landau made was that each individual picks a recipient at random from among those to whom they are permitted to send messages, with equal probability of picking any one of them. For a variety of communication patterns among three and four people, Landau determined the expected value of the time it would take to complete such a task.

A natural step in the progression of studies of communication processes occurred some eighteen years later when Hajnal, Milner, and Szemeredi [64] attributed to Boyd the following problem: "There are n ladies, and each one of them knows an item of scandal which is not known to any of the others. They communicate by telephone, and whenever two ladies make a call, they pass on to each other, as much scandal as they know at the time. How many calls are needed before all ladies know all the scandal?" [64] This problem, which has become known as the Gossip Problem, or the Telephone Problem, has in turn been the source of dozens of research papers that have studied problems concerning the spread of information among a set of people, whether it be by telephone calls, conference calls, letters or even computer networks.

In this paper we survey this steadily growing body of literature, review the results that have been obtained on the Gossip Problem and related problems, and discuss a number of new areas of study that have emerged.

The literature on this subject can be divided primarily into two major areas: one on gossiping, the other on broadcasting. In gossiping every person knows a unique item of information and needs to communicate it to everyone else, while in broadcasting only one person has an item of information which needs to be communicated to everyone else.

In Section 2 of this paper we survey the work on gossiping. In Section 3 we define many of the terms used in the study of broadcasting and then review the literature on this topic. Section 4 contains the definitions of some more recently defined problems and a brief survey of the work in these areas. Section 5 deals with closely related problems which have been studied using somewhat different models. Finally, in Section 6, we suggest some directions for future work in this area.

2. GOSSIPING

Gossiping refers to the information dissemination problem that exists when each member of a set A of n individuals knows a unique piece of information and must transmit it to every other person. The problem is solved by producing a sequence of unordered pairs (i,j), $i,j \in A$, each of which represents a phone call made between a pair of individuals, such that during each call the two people involved exchange all of the information they know at that time; and such that at the end of the sequence of calls, everybody knows everything. Such a calling sequence, which completes gossiping among the n people, is called *complete*.

In this section we review the early history of the Gossip Problem, which consists of three papers, in each of which the minimum number of calls in a complete calling sequence among n people is determined. We then review a number of papers which study variants of the basic Gossip Problem. These variants include:

i) restricting the freedom in placing calls, i.e., a given person can only call a subset of the other people;

ii) restricting the calling process to one-way communication;

iii) allowing k-party or conference calls;

iv) limiting the amount of redundant information which is sent; and

v) considering the amount of time, instead of the number of calls, necessary to complete gossiping.

2.1. GOSSIPING IN A COMPLETE GRAPH

Let $f(n)$ equal the minimum number of calls necessary to complete gossiping among n people where any pair of people may call each other.

$$f(1) = 0; \qquad f(2) = 1; \qquad f(3) = 3 \tag{1}$$

$$f(n) = 2n - 4, \qquad \text{for } n \geqslant 4.$$

This result was proved by Bumby [13] and Spencer (unpublished) (cf. [64]), Hajnal, Milner, and Szemeredi [64], Tijdeman [121], Baker and Shostak [4], and Seress [114]. Each of the published proofs, varying in approach and length, provide interesting perceptions on the nature of the Gossip Problem.

The proof by Hajnal, Milner, and Szemeredi is a particularly ingenious one that makes use of an interchange rule in complete calling sequences. This rule considers conditions under which calls in a particular sequence can be interchanged in time yet still complete gossiping. Their proof also focuses on the positions in a calling sequence by which time an individual knows everything. Let us call such an individual an *expert*. The following two results can be used to prove the main result (1 above).

In any sequence of calls among n people there are no experts after $n - 2$ calls. (2)

In any calling sequence among n people, after $n + k - 4$ calls, there are at most k experts and if there are exactly k experts, then there is an equivalent calling sequence in which the last k calls are made exclusively between the experts. (3)

In Tijdeman's paper the following two results are used to determine the values of $f(n)$.

If two people in a complete calling sequence L are interchanged at a point in time when they have precisely the same information, then the new calling sequence L' is also complete. (4)

If in a sequence of calls L, persons A and B are identified to form a new sequence L', then at any point in L', AB knows at least all of what A and B knew separately at the corresponding point in L, and any other person in L' knows at least as much as they knew at the same time in L. In particular the sequence L' is complete if L is complete. (5)

In addition, Tijdeman attributed to Wirsing the observation that the reverse of a complete sequence of calls is also a complete sequence of calls.

Baker and Shostak presented a third, but very short, proof that $2n - 4$ calls are necessary to complete gossiping among n people. They were the first to introduce the idea that calls may be completed in such a way that no one ever hears their own information from anyone else.

Tijdeman and Hajnal, Milner and Szemeredi also observed that the gossip function $f(n)$ satisfies the following recurrence relation:

$$f(n + 1) \leq f(n) + 2, \qquad \text{for } n \geq 4. \tag{6}$$

This is suggested by the observation:

"Suppose an additional gossip g joins a group of n persons. If g phones any one of the gossips first, the additional information known by g will be transmitted in $f(n)$ calls among the n gossips. After this, one additional call is needed by g to obtain all the gossip." [8]

2.2. Gossiping in Arbitrary Connected Graphs

One of the first variations of the Gossip Problem to be studied restricted the communication patterns among the people, as had Bavelas [6] and Landau [80]. Harary and Schwenk (1974) [65] and Golumbic (1974) [62] considered the situation where an individual can call some but not all other people. The original Gossip Problem, expressed in terms of the underlying communication graph, where vertices represent people and edges represent the allowable communication lines, assumes that the communication graph is complete, i.e., a call can be made between any two people. Harary and Schwenk, in placing restrictions on the underlying communication graphs, obtained the following result.

If the communication graph among a set of n people is a tree, then $f(n) = 2n - 3$, for $n \geq 2$. (7)

The following results were obtained by Harary and Schwenk [65] and also by Golumbic [62].

For any connected communication graph with n vertices, $2n - 4 \leq f(n) \leq 2n - 3$, for $n \geq 4$. (8)

For any connected communication graph with n vertices which contains a 4-cycle, $f(n) = 2n - 4$, for $n \geq 4$. (9)

Harary and Schwenk and Golumbic both conjectured that $f(n) = 2n - 3$ for any connected graph not containing a 4-cycle. Soon thereafter Cot (1976) [27] found a special subclass of graphs for which the above conjecture is true. This conjecture was later shown to be true by Bumby [13] and Kleitman and Shearer (1980) [72], i.e.,

For any connected communication graph G with n vertices, $f(n) = 2n - 4$ if and only if G contains a 4-cycle. (10)

More recently, Labahn (1986) [78] has studied the structure of optimal gossip schemes in trees.

2.3. Gossiping in Grid Graphs

Grid graphs are a special class of graphs which have received a lot of study from many different perspectives. They have been used, for example, to model games on a checkerboard, networks of city streets, telephone switching networks, geographical data bases, matrix manipulations, the cellular spaces of John von Neumann, and parallel computer architectures. Frequently in these studies one must consider questions of routing or transmitting information from one point to another in a grid. Since the body of literature on this subject is much too large to be surveyed here, we will only mention papers that deal more specifically with issue of gossiping in grid graphs.

In her Ph.D. thesis, Van Scoy (1976) [124] studied the design of several parallel combinatorial and matrix algorithms in cellular spaces (i.e., grids). In particular, in order to construct a parallel matrix multiplication algorithm she, in effect, had to solve the Gossip Problem for grids.

Farley and Proskurowski (1980) [47] studied more specifically the Gossip Problem for grids. They defined the following three functions:

T(G): the minimum number of time units necessary to complete (2-party) gossiping in a graph G;

M(G): the minimum number of calls necessary to complete gossiping in a graph G; and

N(G): the minimum number of calls necessary to complete gossiping in the minimum amount of time in a graph G.

They obtained the following series of results.

For any path P_n of length n, (11)

$$T(P_n) = \begin{cases} n - 1 & \text{for } n \text{ even} \\ n & \text{for } n \text{ odd} \end{cases}$$

$$N(P_n) = M(P_n) = 2n - 3.$$

For any grid $G_{m,n}$, (12)
$T(G_{m,n})$ = the diameter of $G_{m,n}$ (except for $G_{3,3}$)

$N(G_{m,n}) = 2mn - 4$ if m and n are even, and

$M(G_{m,n}) = 2mn - 4$ (this follows from previous results, cf. [65]).

For any simple wrap around grid G, (13)
$T(G)$ = the diameter of G.

For any ILLIAC-grid G, (14)
$T(G)$ = the diameter of G plus 1.

2.4. Gossiping in Directed Graphs

Harary and Schwenk [65] also considered the case of one-way communication, e.g., where communication is by letters or telegrams. In such a case the communication graph is a directed graph.

For any strongly connected communication graph with n vertices, $f(n) = 2n - 2$. (15)

Golumbic [62] provided the following result.

For any weakly connected graph with k strong components, communicating all of the information that can be communicated requires at least $f(n,k) \geq 2n - k - 1$ calls. (16)

2.5 Gossiping in Hypergraphs

Another generalization of the Gossip Problem is obtained by assuming that information is transmitted by 'conference' or "k-party" calls. The underlying communication graph in this case is a hypergraph on n vertices with exactly k vertices in each edge. Such a hypergraph will be referred to as k-uniform. Three questions naturally arise:

What is the minimum number, denoted ($f(n,k)$, of calls necessary to complete gossiping in an arbitrary connected n vertex k-uniform hypergraph?

How many calls, denoted $f(H)$, are necessary to gossip in an arbitrary connected n vertex k-uniform hypergraph H?

Which n vertex k-uniform hypergraphs allow gossiping in $f(n,k)$ calls?

The first question was first answered by Lebensold [83] in 1973, then by Bermond (1976) [10] and still later by Kleitman and Shearer (1980) [72], who reduced Lebensold's 13-page proof to an elegant 3-page proof.

$$f(n,k) = \begin{cases} \left\lceil \frac{n-k}{k-1} \right\rceil + \left\lceil \frac{n}{k} \right\rceil, & \text{for } 1 \leq n \leq k^2 \\ 2\left\lceil \frac{n-k}{k-1} \right\rceil, & \text{for } n \geq k^2 \end{cases} \tag{17}$$

The other questions were addressed by Liestman and Richards (1984) [89]. They showed the following:

$$f(n,k) \leq f(H) \leq 2n - 2k + 1 \tag{18}$$

The upper bound is tight. They also gave a partial answer to the third question. In particular, they characterized the hypergraphs which allow gossiping in $f(n,k)$ calls for those n and k, where $f(n,k)$ is odd. For those n and k where $f(n,k)$ is even, they described a large class of hypergraphs which allow gossiping in $f(n,k)$ calls and conjectured that there are no other such hypergraphs.

2.6. Reducing Redundancy in Gossip Schemes

Cot (1976) [27] studied several interesting variants of the basic Gossip Problem. He defined a call between two people A and B to be *final* if after the call is completed, both A and B know all of the information (i.e., both A and B are experts). He then defined three kinds of final calls. A final call is:

i) *a-redundant* if neither A nor B know everything before the call;

ii) *1-redundant* if either A or B, but not both, knows everything before the call; and

iii) *2-redundant* if A and B know everything before the call.

Cot was interested in the existence of a-redundant calling sequences which complete gossiping in the minimum number ($2n - 4$) of calls. He found two cases for which this was possible, i.e., when $n = 4$ and $n = 8$.

Other variants of the Gossip Problem can be obtained by restricting the allowable calling sequences. One such restriction is to require that no one hears their own information (cf. Baker and Shostak [4]). That is, no one can call a person if the caller already knows the unique piece of information originally known only by the person called. This restriction has been termed "NOHO" (for No One Hears Own) and has been studied extensively by West (1982). In [132], West showed the following.

> Gossip schemes satisfying the NOHO restriction exist for all even $n \geq 4$ and the smallest number of edges in a graph which admits such a scheme is $2n - 4$. (Clearly
> (19) such schemes can not exist for odd $n > 1$.)

West characterized these optimal ($2n - 4$ edge) graphs and examined other graph theoretic properties of these graphs. Some particular examples of such graphs are the 4-cycle and the two 3-regular graphs on n vertices having no triangles. In [133, 134], West investigated quadruples which correspond to these optimal graphs and was able to enumerate the nonisomorphic optimal graphs.

A more severe restriction on the allowable calling sequences is to insist that no transmission of information can ever be duplicated. That is, each message conveys only new information to its receiver. This restriction has been termed "NODUP" (for NO DUPlication). Let $f(n)$ denote the number of calls required for NODUP gossiping among n people in a complete graph. Lenstra et al. (1976) [84] showed that such schemes exist for n divisible by 4 and provided an inductive construction to show that

$$f(n) \leq \tfrac{1}{2} n \log n + 0(1) \qquad \text{for } 4|n \tag{20}$$

West (1982) [134] showed the following lower bound.

$$f(n) \geq 2n - 3 \qquad \text{for } n > 8 \tag{21}$$

West (1982) [135] also provided a construction which established the following upper bound.

$$f(n) \leq \frac{9}{4}n - 6 \qquad \text{for } 4|n \tag{22}$$

Seress (1983) [112] determined the following.

NODUP schemes exist for all even n except for $n = 6,10,14,18$ (and trivially for $n = 1$). (23)

Seress (1986) [114] gave lower bound and provided constructions yielding the following results.

$$\frac{9}{4}n - 4.5 \leq f(n) \leq \frac{9}{4}n - 3.5 \qquad \text{for } n \equiv 2(\text{mod } 4),\ n \geq 22 \tag{24}$$

$$f(n) = \frac{9}{4}n - 6 \qquad \text{for } 4|n,\ n \geq 8 \tag{25}$$

Seress (1986) [113] investigated NODUP gossiping by k person conference calls. He determined for each value of k and n (with finitely many exceptions) for which NODUP conference gossiping is feasible. In [115], he produced calling schemes for these n and k which use a number of calls which is linear in n.

Seress (1986) [113] observed that in a directed graph (one-way calls), if in the initial call A calls B, A must hear its own information when it learns B's. Thus, NOHO and NODUP schemes are not possible with one way calls. He proved the following result.

A one-way gossip scheme exists for all n such that $n - 1$ members hear their own information and everyone hears every other piece of information exactly once. Furthermore, there is no such scheme in which at most $n - 2$ members hear their own information. (26)

2.7. Gossiping in Minimum Time

Another variant of the Gossip Problem is obtained by considering the function $t_0(n,k)$ which equals the minimum amount of time required to complete gossiping among n people in a complete communication graph, where we assume that each call requires one unit of time, and each call involves $k \geq 2$ people. The values for $t_0(n,2)$ appear to have been obtained independently by many people, e.g., Bavelas (1950) [6], Landau (1954) [80], and Knodel (1975) [74], while the values for $t_0(n,k)$, $k \geq 2$ were obtained by Schmitt (1976) [111] as follows:

$$t_0(n,2) = \begin{cases} \lceil \log_2 n \rceil, & \text{for } n \text{ even} \\ \lceil \log_2 n \rceil + 1, & \text{for } n \text{ odd} \end{cases} \tag{27}$$

$$t_0(n,k) = \begin{cases} \lceil \log_k n \rceil, & \text{if } k \text{ divides } n \\ \lceil \log_k (n/(k - 1)) \rceil + 1, & \text{otherwise} \end{cases} \tag{28}$$

Labahn (1986) [78] investigated the time required for n people to gossip if the underlying communication network is a tree. Let $t_L(T)$ be the time required to gossip in tree T on n vertices. Labahn showed the following:

$$2\lceil \log n \rceil - 1 \leq t_L(T) \leq 2n - 3 \tag{29}$$

In addition, he characterized the trees which allow gossiping in $2\lceil \log n \rceil - 1$ time and described how to construct these trees and their minimum time calling schemes.

Landau (1954) [80] obtained the values for a slightly different time function $t_r(n)$ which equals the minimum time necessary to complete gossiping among n people under the assumptions that communication is one-way only and that every person sends messages to at most r others during each time unit. Landau proved the following result.

$$t_r(n) = \lceil \log_{r+1} n \rceil. \tag{30}$$

Shimbel's early paper (1951) [116] can be viewed as a variant of gossiping in a directed graph in which each member sends messages to each of its neighbors at each time unit. Let $t_S(G)$ be the minimum number of time units necessary to complete gossiping in an n vertex digraph G. Shimbel showed how to calculate $t_S(G)$ from the adjacency matrix of G.

Entringer and Slater (1976) [38] have considered the minimum amount of time necessary to complete gossiping in complete digraphs, under two different conditions. First, let $t_1(n,k)$ equal the minimum number of time units necessary to complete gossiping subject to the constraint that during each period of time each person can send all the information they know to each of at most k other people *and* each of at most k other people can send information to them. Second, let $t_2(n,k)$ equal the minimum number of time units necessary to complete gossiping subject to the constraint that during each time unit a person can either send all the information they know to at most k other people *or* they can receive information from at most k other people. No one can both send and receive information during a time unit. They obtained the following two results.

$$t_1(n,k) = \lceil \log_{k+1} n \rceil \tag{31}$$

$$\lceil \log_{k+1} n \rceil \leq t_2(n,k) \leq 2 \lceil \log_{k+1} n \rceil \tag{32}$$

Schmitt (1976) [111] considered the time, denoted $t_3(n,k)$, necessary to gossip in an n vertex hypergraph with each edge containing no more than k vertices. (Note that these hypergraphs are not k-uniform.) He proved the following:

$$t_3(n,k) = \begin{cases} \lceil \log_k n \rceil & \text{if } k \mid n \\ \left\lceil \log_k\left(\dfrac{n}{k-1}\right) \right\rceil + 1 & \text{otherwise} \end{cases} \tag{33}$$

2.8. Gossiping with Randomly Placed Calls

Another variant concerns the process of gossiping by placing random phone calls. As previously mentioned, Landau [80] has studied the case in which, at each sending time each person randomly selects one of their possible recipients to receive their message. Landau determined the expected time $E(t)$ it would take to complete gossiping in each of several small one way communication graphs.

Moon (1972) [96] has studied random 2-way gossiping, but with different assumptions. "Whenever all [communication] lines are free, a switchboard operator chooses two people at random from the n people and connects them. We assume, as before, that each caller tells the other caller all the information he knows at each stage." [96] Let C_n denote the number of calls made before each of the n persons knows everything, and let $E(C_n)$ denote the expected value of C_n. Moon proved the following result.

$(1 - \varepsilon)n \ln n < E(C_n) < (2 + \varepsilon)n \cdot \ln^2 n$ for all sufficiently large values of n, where ε denotes any positive constant. (34)

This result has been sharpened by Boyd and Steele (1979) [12] who showed that:

$$E(C_n) = \frac{3}{2} n \ln n + 0(n(\ln n)^{1/2}). \tag{35}$$

2.9. Other Variants

Cot (1976) [27] considered briefly the case of gossiping in which each call has associated with it a given cost, depending on such factors as the distance between the two vertices or the number of messages transmitted. In particular, one is interested in finding a calling sequence which minimizes the total cost of gossiping.

Harary and Schwenk (1974) [66] considered gossip schemes which utilize every edge of the graph at least once. They gave sufficient conditions for a graph to admit such a scheme using $2n - 4$ calls, for a graph to admit a scheme using $2n - 3$ calls, and for a strongly connected digraph to admit such a scheme using $2n - 2$ calls. They conjecture that these conditions are also necessary.

Cederbaum (1980) [16] proposed the following variation of the gossip problem. In a connected graph G on n vertices (labeled $1,2, \ldots, n$), each vertex has an item of information known only to itself. At each time unit $t = 1,2, \ldots,$ vertex $i \equiv t (\text{mod } n)$ sends all of the information that it knows to all of its neighbors. A round consists of vertices 1 through n each sending their information once. The problem is to find a relabeling of the vertices so that the time required to gossip in G by this method is minimized. $c(G)$ is used to denote the minimum number of rounds (over all possible relabelings) needed to gossip in G. Cederbaum gave partial results for this problem. Assman and Kleitman (1983) [2] gave the following complete solution.

$c(G) = 1$ if G contains a vertex of degree $n - 1$. (36)

$c(G) = 2$ if G is 2-connected and has no vertex of degree $n - 1$. (37)

$c(G) = \text{rad}(G')$ otherwise, where G' is an associated graph based on the 2-connected components of G. (38)

Richards and Liestman (1988) [107] proposed the following generalization of gossiping. Determine the number of calls which are necessary for each member to learn at least k items of information in a complete communication graph on n vertices. If "at least" is replaced by "exactly," this is not always possible. Richards and Liestman determined the feasible values of n and k in this case. For both versions of the problem, they determined the exact number of calls necessary for $k \leq 3$ and presented calling schemes which established upper bounds for $k \geq 4$.

Burosch, Gorlow, Labahn, and Szegedy (1984) [14] defined and investigated a number of variations of the gossip problem. A gossip scheme corresponds to a numeration ϕ of the edges of a graph G. The numeration consists of the set of times at which each edge is used in the scheme. Determining the time required to gossip on a particular graph G corresponds to finding the minimum value of $\Sigma_e\ |\phi(e)|$ over all numerations ϕ on G. The orig inal gossip problem corresponds to finding the minimum of these values over all graphs G on n vertices. Burosch, Gorlow, Labahn, and Szegedy consider the related problems which are obtained by changing either (or both) of the minimums to maximums. The goal of the original gossip problem is to make "the most profitable use of a most suitable network." They investigate finding "the least profitable use of a most suitable network," "the most profitable use of a least suitable network," and "the least profitable use of a least suitable network." They give results concerning the number of calls, the number of edges and the time of transmission for graphs, digraphs and hypergraphs in this context.

Miklos, Newman, Seress, and West (1988) [94] defined the addition game, a problem motivated by and related to NODUP gossiping. Each of n players has value 1. During the game, whenever two players communicate, their values both become the sum of their previous values. The object of the game is for every player to achieve the value n. The authors investigated both the minimum and maximum number of calls required to reach this final state. They also investigated a more general version of the addition game in d dimensions.

Berman and Hawrylycz (1986) [9] investigated a problem concerning the reliability of gossip schemes. Consider a calling scheme on multigraph G. We say that a vertex v communicates k-failure-safe with vertex w if there is a sequence of calls from v to w even when any k edges of G are deleted. Thus, a gossip scheme is k-failure-safe if gossiping is completed by the scheme even when any k edges are deleted. Let $e(n,k)$ $(a(n,k))$ be the minimum number of edges in a multigraph (multidigraph) on n vertices which allows a k-failure-safe gossip scheme. Berman and Hawrylycz proved the following.

$$\left\lceil \left(\frac{k+4}{2}\right)\left(n-1\right) \right\rceil - 2\left\lceil \sqrt{n} \right\rceil + 1, \tag{39}$$

$$\leq e(n,k) \leq \left\lfloor \left(k+\frac{3}{2}\right)(n-1) \right\rfloor, \quad k \leq n-2$$

$$\left\lceil \left(\frac{k+3}{2}\right)n \right\rceil - 2\left\lceil \sqrt{n} \right\rceil \leq e(n,k) \leq \left\lfloor \left(k+\frac{3}{2}\right)(n-1) \right\rfloor, \quad k \geq n-2 \tag{40}$$

$$a(n,k) = (k+2)(n-2) \tag{41}$$

These results have recently been improved by Haddad, Roy, and Schaffer [63] (1987) who showed that

$$e(n,k) \leq \left(\frac{k}{2} + 2m\right)\left((n - 1) + \frac{n - 1}{2^m - 1} + 2^m\right). \tag{42}$$

Haddad, Roy, and Schaffer also showed that the time used by their scheme is within a small multiplicative factor of optimal.

3. BROADCASTING

A major variant of the Gossip Problem was introduced by Slater in 1977 when he, along with Cockayne and Hedetniemi, studied the problem of determining the minimum amount of time required for one person to transmit one piece of information to everyone else in a communication graph. By comparison, gossiping is an all-to-all information dissemination process; we call this one-to-all process broadcasting. This seemingly simple variation touched off a substantial amount of research into the theory and technology of broadcasting in communication, information and computer networks. In the remaining sections of this paper, we review this work.

Broadcasting refers to the process of message dissemination in a communication network whereby a message, originated by one member, is transmitted to all members of the network. Broadcasting is accomplished by placing a series of calls over the communication lines of the network. This is to be completed as quickly as possible subject to the constraints that:

i) each call involves only two vertices;
ii) each call requires one unit of time;
iii) a vertex can participate in only one call per unit of time; and
iv) a vertex can only call a vertex to which it is adjacent.

Restriction (iv) above leads to the term "local" broadcasting.

Given a connected graph G and a message originator, vertex u, it is natural to ask, "what is the minimum number of time units required to complete broadcasting from vertex u?" We define the *broadcast time of vertex u, b(u),* to equal this minimum time. It is easy to see that for any vertex u in a connected graph G with n vertices, $b(u) \geq \lceil \log_2 n \rceil$, since during each time unit the number of informed vertices can at most double.

3.1. Broadcasting in Trees

Slater, Cockayne, and Hedetniemi [117] designed a linear algorithm for finding the broadcast center $BC(T)$ of any tree T, i.e., the set of vertices u in T for which $b(u)$ is minimum. They also obtained the following two results.

For any tree T with $n \geq 2$ vertices, the broadcast center $BC(T)$ consists of a star with at least two vertices. (43)

For any vertex v in a tree T, $b(v) = k + b(BC(T))$, where k is the minimum distance from b to a vertex in $BC(T)$ and $b(BC(T))$ is the broadcast time from a vertex in $BC(T)$. (44)

TABLE III.1 The Broadcast Function, $B(n)$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$B(n)$	0	1	2	4	5	6	8	12	10	12	13	15	18	21	24	32	22	23

We define a *minimum broadcast tree* to be a rooted tree with n vertices and root u such that $b(u) = \lceil \log_2 n \rceil$. Proskurowski (1981) [101] studied minimum broadcast trees (mbts). In particular, he designed an $O(n)$ algorithm for deciding if an arbitrary rooted tree with n vertices is an mbt, designed an algorithm for generating all mbts with n vertices, and developed a recurrence relation to count a number of subsets of mbts. By coincidence, all minimum broadcast trees with 2^k vertices are binomial trees B_k (cf. Vuillemin (1978) [126]), a class of trees which occur naturally in the study of various data manipulation problems, in particular those involving priority queues.

3.2. Broadcasting in Connected Graphs

In any connected graph G, a broadcast from a vertex u determines a spanning tree rooted at u. We define the *broadcast time of a graph* G, $b(G)$ to equal the maximum broadcast time of any vertex u in G, i.e., $b(G) = \max \{b(u) \mid u \in V(G)\}$. For the complete graph K_n with $n \geq 2$ vertices, $b(K_n) = \lceil \log_2 n \rceil$, yet K_n may not be minimal with respect to this property. That is, we may be able to remove edges from K_n and still have a graph G with n vertices such that $b(G) = \lceil \log_2 n \rceil$. We define a *minimal broadcast graph* to be a graph G with n vertices such that $b(G) = \lceil \log_2 n \rceil$, but for every proper spanning subgraph $G' \subset G$, $b(G') > \lceil \log_2 n \rceil$. Note that every vertex of a minimal broadcast graph is the root of a minimum broadcast tree which contains all of the vertices of G.

We define the *broadcast function* $B(n)$ to equal the minimum number of edges in any minimal broadcast graph on n vertices. A *minimum broadcast graph* (mbg) is a minimal broadcast graph on n vertices having $B(n)$ edges. From an applications perspective, minimum broadcast graphs represent the cheapest possible communication networks (having the fewest communication lines) in which broadcasting can be accomplished, from any vertex, as fast as theoretically possible.

Shortly after the results of Slater, Cockayne and Hedetniemi were obtained, Johnson and Garey (1978) showed that the problem of determining whether $b(v) \geq k$, for a vertex v in an arbitrary graph G for fixed $k \geq 4$ is NP-complete, (cf. [57] and [117]). Scheuerman and Wu (1984) [110] presented a dynamic programming formulation for determining $b(v)$ and a corresponding broadcasting scheme for a vertex v in an arbitrary graph. Scheuerman and Edelberg (1981) [109] implemented a backtracking algorithm based on this formulation. Since this exact algorithm is not efficient for large graphs, Scheuerman and Wu also presented several heuristics for achieving efficient near-optimal schemes.

In [46] Farley, Hedetniemi, Mitchell, and Proskurowski (1979) studied the broadcast

TABLE III-2. Number of mbg's

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Number	1	1	1	1	4	1	1	4	6	21	2	1*	1*	2*	3*	8*	5*	1*

*Indicates number known.

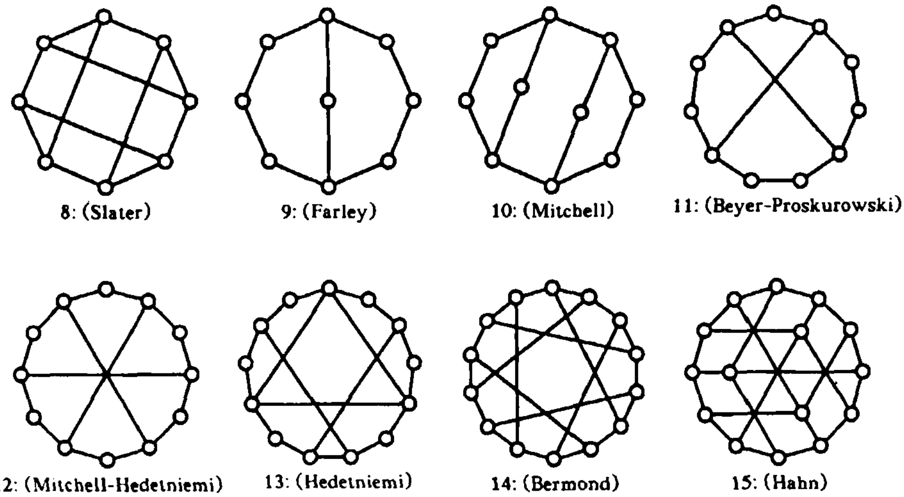

FIG. 3-1. Examples of minimum broadcast graphs.

function $B(n)$, i.e., the number of edges in a minimum broadcast graph with n vertices. In particular, they determined the values of $B(n)$ for $n \leqslant 15$ and showed that $B(2^k) = k \cdot 2^{k-1}$. Mitchell and Hedetniemi [95] (1980) determined the value for $B(17)$ and Wang [131] (1986) found the value of $B(18)$. For $1 \leqslant n \leqslant 18$, the values of the broadcast function $B(n)$ are shown in Table III-1.

Farley, Hedetniemi, Mitchell, and Proskurowski [46] also found examples of minimum broadcast graphs for each value of n, $1 \leqslant n \leqslant 15$. These examples were extended by Mitchell and Hedetniemi (1980) [95] who found all mbgs with $n \leqslant 11$ vertices and presented the only known examples of mbgs with $n \leqslant 17$ vertices. Their census was later augmented by Bermond (1981) [11] who found a second mbg with 14 vertices and by Wang (1986) [131] who found three additional mbg's on 17 vertices and one mbg on 18 vertices. For $1 \leqslant n \leqslant 18$, the number of mbgs with n vertices is shown in Table III-2 where * indicates that the value given is the number of mbgs known to exist.[1]

The results of these studies suggest that mbgs are extremely difficult to find; in fact, no mbg with n vertices is known for any value of $n \geqslant 19$, except for the values $n = 2^k$, where mbg's are easy to construct. Figure 3-1 illustrates several examples of minimum broadcast graphs (with the author's name indicated in parentheses).

Since mbg's seem to be difficult to find for arbitrary n, several authors have shown how to construct sparse graphs which allow minimum time broadcasting from each vertex. These graphs also provide upper bounds on $B(n)$.

In [40] Farley (1979) designed several techniques for constructing minimal broadcast graphs with n vertices and approximately $(n/2)\log_2 n$ edges, for arbitrary values of n.

[1]The original value given for the number of mbg's on 8 vertices was incorrect. The correct value was determined by Kusalik [77]. Two of the mbg's on 17 vertices which appear in [95] were shown to be isomorphic by Wang [131].

He also showed that many star polygons are minimal broadcast graphs. Chau and Liestman (1987) [17] presented constructions based on Farley's techniques which yield somewhat sparser graphs for most values of n. Some additional improvements were noted by Wang (1986) [131]. MacGillivray (1987) [92] gave several bounds on $B(n)$.

In [99] Peleg (1987) gave new upper and lower bounds for $B(n)$. In particular, he showed that

$B(n) \in \Theta(L(n-1)n)$ for $n \geq 1$, where $L(k)$ denotes the exact number of consecutive leading 1's in the binary representation of k. (45)

Asymtotically, Peleg's construction (which establishes his upper bound) produces the best known graphs for most values of n.

Peleg also showed that *relaxed broadcast graphs*, in which broadcast can be completed in at most one time unit more than optimal from any vertex, can be constructed for every $n \geq 1$ using fewer than $2n$ edges.

It should be noted that Peleg's construction produces graphs with some vertices of degree $O(n)$ while the graphs constructed by Farley and by Chau and Liestman have maximum degree $O(\log_2 n)$. Liestman and Peters [88] have recently (1988) investigated the problem of broadcasting in time $O(\log_2 n)$ in graphs with fixed maximum degree.

3.3. Broadcasting in Hypergraphs

Richards and Liestman (1988) [107] began the study of broadcasting by conferences. Let $t_k(n)$ be the time required to broadcast in the complete k-uniform hypergraph on n vertices. Note that since each conference requires k participants, the minimum time for such a broadcast is not always $\lceil \log_k n \rceil$. Richards and Liestman showed the following.

$$\text{Let } n = k^i + p(k-1) + q \quad \text{with } 1 \leq p < k^i \text{ and } r = k^i - \text{p} \tag{46}$$

$$t_k = \begin{cases} \lceil \log_k n \rceil & \text{for } \max(0, k - \text{r}) \leq q < k \\ \lceil \log_k n \rceil + 1 & \text{for } 1 \leq q < k - r \end{cases}$$

They also studied the hypergraph broadcast function $B_k(n)$, i.e., the minimum number of edges in a k-uniform hypergraph that allows each vertex to be the originator of a $t_k(n)$ time broadcast. In particular, they determined values of $B_k(n)$ for several small values of n with $k = 3$ and $k = 4$, gave bounds on $B_k(n)$ for other small values of n with $k = 3$ and $k = 4$, and showed that $B_k(k^i) = ik^{i-1}$.

Another version of conference call broadcasting was investigated by Peleg [99] (1987). In his model, the network is represented as a graph and a member may inform up to c neighbors in a single call. Since this version, called *c-broadcasting*, allows calls involving fewer than $c + 1$ members, this process can be completed in time $\lceil \log_{c+1} n \rceil$ in an n vertex complete graph. Let $B_c(n)$ denote the minimum number of edges in a graph on n vertices that allows each vertex to be the originator of a $\lceil \log_{c+1} n \rceil$ time c-broadcast. Peleg showed that

$$B_c((c+1)^k) = \tfrac{1}{2} ckn \qquad \text{for } n = (c+1)^k,\ k \geq 1 \tag{47}$$

and that

$B_c(n) \in \Theta\,(cL_c(n-1)n)$ for $n \geq 1$, where $L_c(k)$ is the exact number of consecutive leading c's in the $(c+1)$-ary representation of k. (48)

Peleg also showed that *relaxed* c*-broadcast graphs*, in which c-broadcast can be completed in at most one time unit more than optimal from any vertex, can be constructed for every $n \geq 1$ and $c \geq 1$ using fewer than $2n$ edges.

3.4. Broadcasting in Grids

Farley and Hedetniemi (1978) were the first to study broadcasting in grids, where in [45] they showed the following.

Let v be a vertex in a grid graph $G_{m,n}$. Then

if v is a corner vertex then (49)
$b(v) = m + n - 2$;

if v is a side vertex then
$b(v) =$ the maximum distance from v to a corner vertex;

if v is an interior vertex at position (i,j), then
$b(v) =$ the maximum distance from v to a corner vertex

plus 1 if $j = \dfrac{n+1}{2}$, or

plus 2 if $j = \dfrac{n+1}{2} = i = \dfrac{m+1}{2}$.

They also determined the broadcasting times required in simple wrap-around grids and ILLIAC-grids and considered broadcasting in an infinite 2-dimensional grid. Let $f(n,t)$ be the maximum number of vertices which can be informed of a single message originated by a given "cell" in the infinite n-dimensional grid, after t time units, by any local broadcasting scheme. They gave a scheme which yielded the following upper bound on $f(2,t)$ and conjectured equality.

$$f(2,t) \leq 2t^2 - 6t + 8, \qquad \text{for } t \geq 2. \tag{50}$$

The conjecture gave rise to several subsequent papers. Cockayne and Hedetniemi (1978) [24] made the following conjecture.

Conjecture 1: In the n-dimensional grid, $f(n,t) = 2^t$, when $t \leq 2n$, and $f(n,2n+k+1) = 2^n \sum_{j=0}^{n} 2^j \binom{n-j+k}{k}$, for other values of t.

Ko (1979) [75, 76] and Peck (1980) [98] independently verified the 2-dimensional conjecture of Farley and Hedetniemi.

$$f(2,t) = 2t^2 - 6t + 8, \qquad \text{for } t \geq 2. \tag{51}$$

Ko also evaluated $f(3,t)$, disproving Conjecture 1.

$$f(3,t) = \frac{4}{3}t^3 - 11t^2 + \frac{101}{3}t - 28, \qquad \text{for } t \geq 9. \tag{52}$$

In addition to the above, Ko obtained the following upper bound on $f(n,t)$.

$$f(n,t) \leq \frac{2^n}{n!} t^n + \frac{2 - n2^n}{(n-1)!} t^{n-1} + \frac{1}{(n-2)!} (4 - n - \frac{1}{3} (2^{n+1} - n2^{n-2}) + n^2 2^{n-1}) t^{n-2} + O(t^{n-3}). \tag{53}$$

A broadcast scheme for an n-dimensional grid which achieves this bound to within the two highest degree terms was obtained by Peck, who also gave an upper bound.

$$f(n,t) = \frac{2^n}{n!} t^n - \frac{n2^n - 2}{(t-1)!} t^{n-1} + O(t^{n-2}) \qquad \text{for } n \geq 4 \text{ and large } t. \tag{54}$$

Note that although this result also implies that Conjecture 1 is incorrect, it does not imply (52).

Ko also obtained optimal schemes for broadcasting in infinite 2-dimensional hexagonal arrays and infinite 2-dimensional triangular arrays.

Several authors have studied the rate of spread of information (broadcasting) in n-dimensional infinite grids obtained by whispering (in which a cell can only call one neighbor per unit of time) and shouting (in which a cell can simultaneously call all of its neighbors). (These terms are due to Stout (1981) [120].)

In an early paper, Gentleman (1978) [58] showed that $2t^2 + 2t + 1$ cells could be informed in t time units by shouting in a 2-dimensional infinite grid.

Other authors have considered whispering and shouting in a more general situation. Given an originator in Z^n and a set of vectors F, a member $a \in Z^n$ is connected to all members $a + f$, where $f \in F$. Let $\sigma(t)$ and $\omega(t)$ denote the maximum number of cells of such a grid which can be informed in t time units by shouting and whispering, respectively. Stout (1981) [120] made four conjectures concerning $\sigma(t)$ and $\omega(t)$.

Conjecture 2. $\sigma(t)$ is ultimately an nth degree polynomial.

Conjecture 3. There is a single whispering scheme which is optimal for all t.

Conjecture 4. $\omega(t)$ is ultimately an nth degree polynomial.

Conjecture 5. $\sigma(t) - \omega(t) = O(t^{n-1})$ if F has at least two vectors.

These conjectures are true for the special case of 2-dimensional rectangular grids which can be seen from the results quoted above. Conjecture 2 was proved by Klarner (1981) [70, 71] who determined $\sigma(t)$. In [120], Stout gave the following result which proves Conjecture 5.

Whispering is asymptotically as efficient as shouting in the sense that $\lim (\sigma(t)/\omega(t)) = 1$. (55)

Conjectures 3 and 4 were shown to be true for the 1-dimensional case by Hell and Liestman (1984) [68, 69].

In another paper on grid graphs, Van Scoy (1979) [125] considered broadcasting multiple messages in complete grids $G_{m,n}$. In particular, she designed an algorithm for broadcasting k messages in $G_{m,n}$ in $2n + 2k - 4$ units, where $k \leq n - 2$ if n is odd and $k \leq n - 1$ if n is even.

3.5. Reliable Broadcasting

Recently a few papers have been written concerning problems of broadcasting in the presence of faults, i.e., the problems which arise when one or more communication lines fail or one or more communication sites (vertices) are inoperative. Liestman (1981) [86, 87], for example, has studied several parameters related to fault-tolerant broadcasting. In his model, it is assumed that faults are not detected during the broadcast and, in order to tolerate k faults, the broadcast scheme must consist of $k + 1$ edge-disjoint calling paths from the originator to every other member. Let $T_{0,k}(n)$ equal the minimum time required to broadcast in the presence of k faults in any graph on n vertices. Liestman showed the following.

$$T_{0,1}(n) = \lceil \log_2 n \rceil + 1. \tag{56}$$

$$T_{0,2}(n) = \lceil \log_2 n \rceil + 2, \text{ for } n \neq 4i + 3. \tag{57}$$

$$T_{0,k}(n) \geqslant \lceil \log_2 n \rceil + k, \text{ for appropriate values of } k \text{ and } n. \tag{58}$$

Let $G_{i,k}(n)$ equal the set of graphs with n vertices in which k fault-tolerant broadcasting can be completed in i time units beyond the minimum possible time, and which have the minimum number of edges. Although no general method is known for constructing these graphs. Liestman has determined $G_{i,k}(n)$ for various small values of i, k, and n. He also observed that the sets $G_{i,k}(n)$ for large i are known to contain only the minimum $k + 1$ edge-connected graphs. Liestman also showed how to construct 1 fault-tolerant and 2 fault-tolerant broadcast graphs with a small number of edges. More recently, Chau and Liestman (1987) [18] presented constructions for sparser 1 fault-tolerant and 2 fault-tolerant broadcast graphs. In another paper Liestman (1980) [85] initiated the study of fault-tolerant broadcasting in grid, or grid-like graphs.

Farley (1979) and Farley and Proskurowski (1980) have studied communication problems in networks in the presence of certain line and/or site failures. A set of failures is called *independent* or *isolated* if no two of them are adjacent or incident. In [43] Farley defines independently reliable networks to be those networks in which all message transfers between operative sites (vertices) can be completed as long as network failures occur independently. The class of graphs known as 2-trees are shown to be minimum independently reliable networks. In [50] Farley and Proskurowski presented an algorithm for converting an arbitrary tree into a minimum network which is immune to certain isolated line failures. Wald and Colbourn (1983) [127] showed that all minimum independently reliable networks are 2-trees and presented a linear time algorithm to convert partial 2-trees into 2-trees. In a related paper [48], Farley and Proskurowski describe classes of graphs with a minimum number of edges, which have no disconnecting independent set of vertices or edges.

More recently, Farley (1987) [44] has investigated broadcasting in networks with independent site failures. In Farley's model, failures can be detected by neighboring sites and the broadcast scheme can be modified dynamically in order to avoid redundant calls. Farley has shown that in certain star polygons on n vertices, broadcast can be completed in the presence of a single site failure in time $\lceil \log_2 n \rceil$. He also showed that in the same networks, broadcast can be completed in the presence of k isolated site failures in at most $\lceil \log_2 n \rceil + k$ time.

Berman and Hawrylycz (1986) [9] considered a version of reliable broadcasting on multigraphs. We say that a vertex v communicates k-failure-safe with vertex w if there is a sequence of calls from v to w even when any k edges of G are deleted. Let $f(n,k)$ ($g(n,k)$) be the minimum number of edges in a multigraph (multidigraph) on n vertices in which one vertex can communicate k-failure-safe with every other vertex. This is similar to our usual notion of broadcasting except that there is a single specified originator. Berman and Hawrylycz proved the following.

$$f(n,k) = \begin{cases} \left\lceil \left(\frac{k+2}{2}\right)(n-1) \right\rceil, & k \leq n-2 \\ \left\lceil \left(\frac{k+1}{2}\right)n \right\rceil, & k \geq n-2 \end{cases} \tag{59}$$

$$g(n,k) = (k+1)(n-1) \tag{60}$$

3.6. Multiple Originator Broadcasting

The next variant of local broadcasting to be studied considered the problem of determining the minimum number of message originators necessary to complete broadcasting in a specified amount of time. Hedetniemi and Hedetniemi (1979) [67] observed that if broadcasting must be completed in 1 time unit in an arbitrary communication graph, then the graph must be decomposed into a minimum number of subgraphs, each of which is either a K_1 or a K_2 (this is equivalent to the Maximum Matching Problem). If broadcasting must be completed in 2 time units, then the graph must be decomposed into a minimum number of paths, each of which is of length at most 3. In [67] they present a linear algorithm for decomposing an arbitrary tree into a minimum number of paths, each of which has length $\leq k$, for arbitrary values of k. Shortly thereafter, Farley and Proskurowski (1980) [49] completely settled the question of determining the minimum number of originators necessary to complete broadcasting in an arbitrary tree in at most t time units, for arbitrary values of t. They present a linear algorithm for decomposing a tree into a minimum number of subtrees such that broadcasting can be completed in at most t time units in each subtree.

The results of Scheuerman and Wu (1984) [110] which were mentioned in Section 3.2 also allow for the case of multiple originators.

3.7. Multiple Message Broadcasting

The next work on broadcasting reconsidered the assumption that each call requires one unit of time. In particular, if one is broadcasting large files of information over the lines of a computer network, then this assumption is not very realistic. It was quickly realized that if one needed to broadcast more than one message, then simple modifications of one-message broadcast schemes would be too inefficient. For example, neither repeating a one-message broadcast scheme k times, nor having each call take k time units produces a very efficient k-message broadcast scheme. It is easy, for example, to construct "mixed" calling schemes which complete multiple-message broadcasting in less time than either of these two schemes.

Consider the following functions:

P(m,t): the maximum number of people who can be informed of m messages in t time units;
M(p,t): the maximum number of messages which can be broadcast to p people in t time units; and
T(m,p): the minimum number of time units necessary to broadcast m messages to p people.

Chinn, Hedetniemi, and Mitchell (1979) [23] and Farley [42] were the first to study multiple-message broadcasting in complete graphs. In [23] the values of $P(m,t)$ were determined for $1 \leq m \leq 8$ and $1 \leq t \leq 9$. The following general results were also obtained.

$$P(1,t) = 2^t \tag{61}$$

$$P(2,t) = 2^{t-2}, \quad \text{for } t > 2 \tag{62}$$

$$P(3,t) = 2^{t-4} + 2, \quad \text{for } t > 4 \tag{63}$$

$$P(4,t) = 2^{t-6} + 2, \quad \text{for } t > 6 \quad \text{(attributed to Odlyzko)} \tag{64}$$

In [42] the value for $T(m,p)$ was determined for odd values of p.

$$T(m,p) = 2m - 1 + \lfloor \log_2 p \rfloor, \quad \text{for } p \text{ odd.} \tag{65}$$

Soon thereafter Cockayne and Thomason (1979) [25] completely settled the problems of multiple-message broadcasting in complete graphs, as follows.

$$T(m,p) = 2m + \lfloor \log_2 p \rfloor - \left\lfloor \frac{m - 1 + 2^{(\log_2 p)}}{p/2} \right\rfloor, \quad \text{for } p \text{ even.} \tag{66}$$

$$\text{For } p \text{ odd, } M(p,t) = \left\lfloor \frac{t - q + 1}{2} \right\rfloor, \quad \text{where } q = \lfloor \log_2 p \rfloor. \tag{67}$$

$$\text{For } p \text{ even, } M(p,t) = \left\lfloor \frac{p^{t-q} + 2^{q+1} - 2}{2(p - 1)} \right\rfloor. \tag{68}$$

For p odd, m messages can be broadcast to p people if and only if (69)

$p \leq 2^{t-2(m-1)} - 1$

For p even, m messages can be broadcast to p people if and only if $p \leq P(m,t)$, where $P(m,t)$ is the largest even integer u such that

$$t \geq 2m + q - \left\lceil \frac{m - 1 + 2^q}{u/2} \right\rceil, \text{ where } q = \lfloor \log_2 u \rfloor. \tag{70}$$

3.8. Broadcasting with Randomly Placed Calls

The study of the dissemination of information has a long history in probability theory. The various models which have been investigated assume (in our terminology) that calls are made randomly from some set of possible calls.

In 1952, Rapoport and Rebhun [106] suggested that the theory of random nets (see [102], [119] and [118]) could be applied to the spread of rumors. Their work was apparently motivated by experimental work in message spreading by the Washington Public Opinion Laboratory [35]. In their model, one or more "starters" know a message initially. Rapoport and Rebhun assume that the population is "thoroughly mixed," that is that the probability of transmitting a message from one individual to another is the same for all pairs of individuals in the population. Each "knower" tells the message to a other individuals on average. They are interested in the fraction of the population which is eventually informed and in the number of individuals at each "distance" from the starters. They modify the equations used for random nets and compare their predictions with the experimental evidence.

In 1953, Landau and Rapoport [81] observed that the spread of rumors is similar to the spread of a communicable disease. They generalized the work of Kermack and McKendrick [73] on epidemics, introducing a probability of transmission which depends on the age of the rumor and on the time since the potential transmitter heard the rumor. In the same year, Landahl [79] introduced a slightly different model in which whenever an individual hears a piece of information, he transmits it an average of f times, where f may be a function of time. Thus, an individual may become a transmitter repeatedly, not only on first hearing the message.

Rapoport (1953) [103] questioned the assumption of a thoroughly mixed population. In particular, he pointed out that the variance between the predicted size of generations of knowers (those at a fixed distance from a starter) and the experimental results may be due to the "structure of the population." In other words, a knower is more likely to tell the rumor to an acquaintance than to a randomly chosen individual. In [103] and [104], he introduced modifying parameters to account for this phenomenon. In [105], he suggested further experiments to be conducted to test this theory.

In his text book on probability theory [54], Feller included the following problem:

> "*Spread of Rumors*. In a town of $n + 1$ inhabitants, a person tells a rumor to a second person, who in turn repeats it to a third person, etc. At each step the recipient of the rumor is chosen at random from the n people available. Find the probability that the rumor will be told r times without: (a) returning to the originator, (b) being repeated to any person. Do the same problem when at each step the rumor is told by one person to a gathering of N randomly chosen people. (The first question is the special case $N = 1$.)"
>
> The answers are (a) $(1 - N/n)^{r-1}$ and (b) $P(n,Nr)/(P(n,N))^r$, where $P(n,k)$ denotes $n(n - 1) \ldots (n - k + 1)$.

Apparently unaware of the work of Landau and Rapoport, Goffman and Newill (1964) [60] suggested applying epidemic theory to the transmission of ideas. (See Bailey [3] for a detailed account of the mathematical theory of epidemics.) Their work was continued in [61] and by Goffman alone in [59].

Daley and Kendall (1964) [31, 32] countered by suggesting that the spread of information may behave differently than the spread of infectious disease. They proposed a model similar to Feller's except that as the rumor spreading proceeds, a person who knows the rumor will eventually decide that it is no longer "news" and will stop actively spreading it. (This is suggestive of Landau and Rapoport's model but differs in details.) During the rumor spreading process, the $n + 1$ people can be partitioned into three classes: ignorants, active spreaders and stiflers (those who have heard the rumor but are no longer spreading it). Daley and Kendall were interested in determining the fraction f of the total population that is informed of the rumor when the number of active spreaders becomes zero. Under the assumption that a spreader becomes a stifler when he encounters either another spreader or a stifler, they found that for large values of n, $f \approx .80$. They also studied generalizations of this problem in which a) a spreader tells the rumor to any individual he meets with probability p, $0 < p \leq 1$ ($p = 1$ above), b) when a spreader tells the rumor to someone who already knows it (spreader or stifler) he becomes a stifler with probability a, $0 < a \leq 1$ ($a = 1$ above), or c) a spreader becomes a stifler after encountering $k \geq 1$ individuals who already know the message.

Cane (1966) [15] also discussed the similarities between rumor spreading and epidemics. Daley (1967) [30], Dunstan (1982) [37] and Pittel (1987) [100] have investigated the distribution of the sizes of the generations of knowers under various assumptions. Osei and Thompson (1977) [97] considered the problem of two opposing rumors being spread in a closed population using a model similar to that of Daley and Kendall.

More recently, Berg (1983) [7] studied a variant of this problem which begins with two individuals knowing the rumor. At each stage, each newly informed person contacts one other person chosen at random. Only those persons which were newly informed in the last stage are active in spreading the rumor. The process stops when all contacts are with previously informed individuals. Berg studied the duration of the process, the ultimate number of people informed and the rate at which the rumor spreads.

Frieze and Grimmett (1985) [56] appear to be the first investigators of rumor spreading with random calls to be concerned with informing all of the members of the population. In their model, one of the n people in a town initially knows a rumor. At each stage, every knower tells the rumor to someone else chosen randomly and independently of all other choices. S_n is defined to be the number of stages before the whole town knows the rumor. Frieze and Grimmett showed that in probability

$$S_n = \log_2 n + \ln n + o(\log_2 n) \qquad \text{as } n \to \infty. \tag{71}$$

More recently, Pittel (1987) [100] showed that in probability

$$S_n = \log_2 n + \ln n + O(\log_2 n) \qquad \text{as } n \to \infty \tag{72}$$

Demers et al. [33] (1987) have reported experimental and analytical results regarding the use of several variations of rumor spreading in updating replicated databases.

The work of Daley and Kendall is included in the survey article of Dietz (1967)

[34] which briefly mentions the papers of Rapoport, Landahl, Landau, and Goffman and Newill cited above. Three textbooks on mathematical modeling have included rumor spreading. Maki and Thompson (1973) [93] contains two short sections dealing with rumor spreading problems. Frauenthal (1980) [55] includes a retelling of one of Maki and Thompson's problems. Bartholomew (1982) [5] contains some of the work of Rapoport and also of Daley and Kendall. In addition, he gives a good historical account of the mathematical theory of epidemics, rumor spreading, and other types of diffusion.

3.9. Other Variants

In [41] Farley (1980) modified the local broadcasting restriction by permitting "long distance" calls, i.e., a vertex u may call any vertex w if there exists a path from u to w, no edge of which is being used in any other call. He obtained a constructive proof of the following, rather nice result.

In any connected graph G with n vertices, every vertex can complete long distance broadcasting in $\lceil \log_2 n \rceil$ time units. (73)

Consider the optimal broadcast schemes for several originators in a given network. In these schemes, a particular vertex may forward the message to its neighbors in different orders depending on which vertex was the originator. That is, a member of the network may need to determine the originator of the message before being able to relay it. Rosenthal (1981) [108] has considered broadcasting schemes for trees in which each vertex calls its neighbors in a fixed order regardless of originator. He was able to show that such orderings exist which yield optimal broadcast schemes under two cost measures.

Farley and Schachum (1983) [52, 53] introduced a variant on broadcasting. In their model, each member may 'shout' to all of its neighbors but a member may only receive one message at a time. If two or more messages arrive simultaneously, there is a collision and no message is received. One question which arises in this context is: How many members can send messages concurrently so that each message is received by at least one other member? The problem of determining the maximum size of such sets in a general graph was shown to be NP-complete by Even, Goldreich, and Tong (1981) [39]. Linear time algorithms for finding such a maximum sized set in trees and in partial 2-trees have been given by Farley and Proskurowski (1984) [51] and by Colbourn and Proskurowski (1983) [26], respectively.

Dresner and Barak (1984) [36] investigated the time required to broadcast in a complete digraph under the assumption that at each time unit every informed processor randomly chooses another processor to call. They showed that for large values of n, it is possible with probability tending to 1 to broadcast to a set of n nodes in time $(1 + \ln 2)\log n$.

Alon, Barak, and Manber (1987) [1] considered broadcasting in rounds in a complete digraph. Each member sends a message to another member in each round. If the members are numbered $1, \ldots, n$, the calls made in each round can be written as vectors containing permutations of $1, \ldots, n$. A calling scheme corresponds to a type of truncated Latin square. The main result of Alon, Barak, and Manber is that for $n = 2^k$ (and for $n =$ certain primes) a broadcasting scheme exists which uses only

$\lceil \log n \rceil$ rounds. For other n, they describe a scheme which takes $2\lceil \log n \rceil$ rounds. In addition, these schemes can be augmented to allow detection of failed processors.

MacGillivray (1986) [91] has studied minimum panic graphs. These are minimum sized graphs in which it is possible to broadcast in minimum time from any originator regardless of which neighbor the originator calls first.

4. RELATED PROBLEMS

4.1. Receiving

Cheston and Hedetniemi (1984) [20] defined receiving to be the process in which every member sends a distinct message to a specified member called the receiver. Although this appears to be the opposite of broadcasting, it is important to note that the messages sent are all distinct. Furthermore, unlike the gossip problem, it is not allowed to send more than one distinct message during a call. For vertex u, let $r(u)$ denote the minimum time necessary for u to receive messages from every other vertex. Define the receiving center of graph G, denoted $RC(G)$, to be the set of vertices for which $r(u)$ is a minimum.

Cheston and Hedetniemi presented algorithms to compute $r(u)$ and a minimum time receiving scheme for any vertex u of a tree and to determine the receiving center of a tree. They also showed that the receiving center of a biconnected graph is the entire graph. Cheston (1984) [19] gave a linear time algorithm to construct a spanning tree of a biconnected graph (with a specified receiver) which can be used for receiving in minimum time. Ling (1985) [90] produced an optimal receiving scheme for an arbitrary receiver in unicyclic graphs. He also presented an algorithm to produce a receiving scheme for an arbitrary receiver in a general connected graph that requires no more than $\frac{5}{4}$ times the minimum time.

4.2. Polling

Polling, defined by Cheston and Hedetniemi (1984) [21, 22], is an information accumulation process. A specific originating member sends a request to each other member of the network as in broadcasting. Each of these members, upon receiving the request, returns a response to the originator as in receiving. The goal is to accomplish this in the minimum amount of time which is called the polling time. The polling center of the network is the set of vertices with minimum polling time.

Cheston and Hedetniemi gave an algorithm for determining an optimal polling scheme for an arbitrary originator in a tree. They also gave a linear time algorithm to determine the polling center of a tree. Ling (1985) [90] determined optimal polling schemes for arbitrary vertices in Hamiltonian graphs and in complete k-partite graphs. He also gave an algorithm to produce a good (but not necessarily optimal) polling scheme for an arbitrary originator in a biconnected graph.

4.3. Set to Set Broadcasting

Richards and Liestman (1988) [107] proposed a generalization of both broadcasting and gossiping which they called set to set broadcasting. Two sets of vertices A and B

are specified. The goal is to send a message from each member of A to each member of B using the minimum number of calls. Let $|A| = a$, $|B| = b$, and $i = |A \cap B|$. Since the reverse of a calling scheme from A to B is a calling scheme from B to A, we can assume that $a \leq b$. Let $F(A,B)$ be the number of calls necessary to complete set to set broadcasting from A to B in a complete graph. If the set of allowed calls is constrained to the edges of a given graph G, $F(A,B,G)$ is defined analogously, Richards and Liestman showed:

$$F(A,B) = \begin{cases} a + b - i - 1 & \text{for } 0 \leq i \leq 2 \\ a + b - 3 & \text{for } i = 3 \end{cases} \tag{74}$$

$$F(A,B) \leq a + b - 4 \qquad \text{for } i \geq 4 \tag{75}$$

They also characterized the graphs G for which $F(A,B) = F(A,B,G)$ for all disjoint A and B.

5. BROADCASTING IN COMPUTER NETWORKS

Several papers have been published which are closely related to the study of gossiping and broadcasting, but which consider somewhat different models of information dissemination. Perhaps the most notable of these is the work of Dalal (1977) [28], Dalal and Metcalfe (1978) [29], Wall and Owicki (1980) [129], and Wall (1980) [128], who have considered special problems of broadcasting in packet-switched computer networks, like ARPANET. A fundamental problem with networks like ARPANET is that they are not built with a mechanism to handle broadcasting. Consequently, in order to carry out broadcasting from a given vertex an individual message must be routed to each of the other vertices in the network.

Dalal [28] and Dalal and Metcalfe [29] proposed several different broadcasting algorithms for overcoming this deficiency. A basic assumption in these algorithms is that once a vertex receives a message along a given communication line, it can, in effect, "shout" that message (using Stout's terminology [120]), simultaneously to any of several other vertices to which it is connected by communication lines. Their algorithms for broadcasting can be described briefly as follows:

i) transmission of separately addressed packets—this is the simplest and perhaps most often used broadcasting scheme, whereby one copy of the message is routed to each of the $n - 1$ other vertices;

ii) multidestination addressing—in this case fewer total messages are sent, but each of them is routed to a subset of the other vertices;

iii) hot potato forwarding—in this case, once a message is received by a vertex, it is automatically transmitted to *all* of its other neighbors; this can produce what is called *message flooding;*

iv) spanning tree forwarding—once a spanning tree T for the network is determined, broadcasting is accomplished by sending messages only along the lines of T;

v) source based forwarding—once a shortest path spanning tree T is determined from the originating vertex, messages are only sent along the lines of T;

vi) reverse path forwarding—once a message is received by a vertex, it is sent only to those neighbors it considers to lie farther away from the perceived source of the message; and

vii) extended reverse path forwarding—a variant of (vi) above.

For each of these broadcasting algorithms, Dalal and Metcalfe studied factors such as reliability, cost, speed, and total number of messages transmitted.

This work has since been extended by Wall and Owicki [130] and Wall [128], who focused primarily on minimizing the total delay in carrying out broadcasting, whereas Dalal and Metcalfe concentrated more on minimizing the total cost of broadcasting. Much like Dalal and Metcalfe, Wall and Owicki decided to broadcast using only the lines of a single spanning tree, a tree which was a shortest-path spanning tree whose root was located somewhere near the "center" of the network (loosely defined). This gave rise to their term "center based broadcasting." Topkis (1985) [122] has investigated broadcasting by flooding and has proposed a variant of flooding which eliminates redundant messages. He has also studied all-to-all broadcasting (gossiping) in trees [122] and general graphs [123]. The reader is referred to the thesis of Wall [128] for an interesting discussion of several instances where the need to broadcast arises in the use of computer networks.

6. FUTURE STUDIES

Although a moderate amount of work has been done to date on gossiping and broadcasting in communication networks, a substantial amount of work remains to be done, not only in extending the results in each of the existing areas, but in exploring a number of new areas. For example, the great majority of work to date has adopted the constraint that a vertex can only place one call per unit of time. Although a few papers have made the assumption that a vertex can simultaneously call all of its neighbors (shout) this variant is essentially unstudied.

Virtually all of the work to date has been done on local broadcasting and local gossiping, i.e., a vertex can only call a neighbor. Although Farley [41] showed that long distance broadcasting can always be completed in $\lceil \log_2 n \rceil$ time, he did not, for example, consider the *costs* of long distance broadcasting. Except for Farley's paper, long distance information dissemination appears to be totally unexplored.

Another aspect of gossiping that has yet to be considered is the time required to complete a phone call. For the most part, all calls are assumed to require one time unit of time. If, however, it requires n units of time to communicate n messages, then it is natural to ask: how long does it take to complete gossiping? Preliminary investigations of this question have been carried out by Landau [80] and by Cot [27].

The vast majority of work on broadcasting has assumed that the communication network in question is an undirected graph, i.e., communication is 2-way. Yet in many applications only one-way communication is available. Except for the work in [6], [36], [38], [62], [65], and [80], the restriction to directed graphs is mostly unexplored.

Still another area of study is *multi-destination broadcasting* (as suggested by Dalal and Metcalfe [29]), in which it is necessary to transmit information to only a subset of other vertices (one-to-many instead of one-to-all).

In conclusion, several things seem fairly clear relative to this survey. First, the subjects of broadcasting, gossiping and related information dissemination processes are terribly rich in real-world applications. Second, as we enter the age of the Information Society in which more than 70% of the work force will be occupied in one way or another with information processing, increasing degrees of sophistication will be required in our methods of information dissemination. Third, the mathematics of information dissemination has not yet been well developed, but it appears that it will involve a blend of combinatorics, graph theory, probability and computing. And finally, as the results in this survey suggest, our understanding of this subject at present is at best primitive.

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Note:
The authors are aware of several related papers which have appeared since this survey was completed. We intend to continue to collect results in this area. Please inform the third author of any related results not included in this survey.

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