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The Stable Marriage Problem

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The original work of Gale and Shapley on an assignment method using the stable marriage criterion has been extended to find all the stable marriage assignments. The algorithm derived for finding all the stable marriage assignments is proved to satisfy all the conditions of the problem. Algorithm 411 applies to this paper.

Key Words and Phrases: assignment problems, assignment procedures, combinatorics, discrete mathematics, operational research, stable marriage problem, university entrance

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1. Introduction

Several criteria exist for assigning the members of two disjoint sets to one another. The most obvious example is the classical assignment problem where associated with each assignment is a cost, and where the criterion is to maximize (or minimize) the total cost of all the assignments. In some cases this type of assignment criterion is not the most suitable, and in this paper we wish to consider assignment based on the stable marriage criterion.

This concept was first introduced by Gale and Shapley [1], and they used it to look at the problem of the admission of students to college. The term marriage is used because one set can be considered to be men and the other women. In the stable marriage problem each member of a set lists the members of the other set in order of preference. If we assign the men to the women in such a way that there exists a man and a woman who are not assigned to each other but who would both prefer each other to their present partners then the assignment is said to be unstable. If no such unstable assignment exists then the marriage assignment is stable.

Gale and Shapley showed that if the sets are disjoint then at least one set of stable marriages exists. In general there are several such stable sets of marriages, and in order to get a unique stable set of marriages a further constraint, such as the men getting their best possible choices, needs to be introduced. The constructive proof by Gale and Shapley that there is at least one set of stable marriages is in fact an algorithm for finding such a set.

In this paper we confine our attention to the case where the number of members of each set is the same, although the proofs can be easily modified for the more general case. A different algorithm is presented for finding one set of stable marriages and it is compared with the one derived by Gale and Shapley. This paper is concerned with deriving an algorithm for finding all the sets of stable marriages and proving that it gives each stable set of marriages once and only once. ALGOL procedures for the Gale and Shapley algorithm and the two algorithms derived in this paper are given on Algorithm 411. We hope to discuss in a later paper the application of the stable marriage theory to the work of the Universities Central Council on Admissions, which deals with the whole problem of the admission of applicants to British universities.

2. Basic Concepts

2.1 Definition of a Stable Marriage

Consider two distinct sets A and B. An assignment of the members of A to the members of B is said to be a stable marriage if and only if there exist no elements a and b (belonging to A and B respectively) who are not assigned to each other but who would both prefer each other to their present partners.

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In this paper the two sets will have the same number of members and each member will list all the members of the other set in order of preference. Most of the theory can be extended to sets with different numbers of members and partial choice lists as would be required in considering the university entrance problem. However, the theory does require that the sets be distinct. Gale and Shapley [1] have shown by an example that if the sets are not distinct then a stable marriage may not exist.

2.2 Optimality

For any given sets A and B there are in general several stable marriage solutions. Three of these will be defined as follows.

Male optimal stable solution. This is the stable solution when every man is at least as well off under it as under any other stable solution.

Female optimal stable solution. This is similar except that the women get their best possible choices.

Minimum choice stable solution. In this stable solution the sum of the choice numbers of the men and the women is a minimum. This solution may not be unique, but it provides a sort of unselfish optimal solution giving credit to low choice numbers in both sets.

It is possible that all three solutions are the same, or that the minimum choice solution may coincide with either the male optimal or female optimal solution.

Example. Consider the following simple marriage assignment problem with three men α, β, and γ and three women A, B, and C. The men choose the women in the order:

Man α chooses	A	B	C
Man β chooses	B	A	C
Man γ chooses	A	C	B

and the women choose the men in the order:

Woman A chooses	β	α	γ
Woman B chooses	γ	β	α
Woman C chooses	α	γ	β

The pairings αA, βB, and γC give a stable marriage since only one man γ would consider another assignment an improvement, and woman A prefers man α to man γ. This is in fact the male optimal stable solution. The only other stable marriage is given by the pairings αC, βA, and γB and this is the female optimal stable solution. The sum of the choice numbers of the male optimal solution is 10 while that of the female optimal solution is 11. Hence, in this case the minimum choice stable solution is the same as the male optimal stable solution.

3. Simple Marriage Problem

The problem may be stated as follows. Given a community of n men and n women, let each person list those of the opposite sex in accordance with his or her preferences for a marriage partner. A stable marriage assignment in which all the members of the community are married is required.

3.1 Gale and Shapley Algorithm

Gale and Shapley [1] proved that a stable set of marriages which solves the above problem exists. Their proof is a constructive proof and it can be used for finding one set of stable marriages. Their iterative process will be called the Gale and Shapley algorithm in the rest of this paper. A simple example with four men and four women will show how their algorithm works.

Example. Let the men be α, β, γ, δ and the women A, B, C, D. Let the men choose the women in the order:

Man α chooses	D	A	C	B
Man β chooses	A	C	D	B
Man γ chooses	C	D	B	A
Man δ chooses	D	A	C	B

Let the women choose the men in the order:

Woman A chooses	α	γ	δ	β
Woman B chooses	δ	α	β	γ
Woman C chooses	γ	β	δ	α
Woman D chooses	β	δ	γ	α

In the first stage all the men propose to their first choice. The women A and C, who have had only one proposal, hold their propoer in suspense. Woman D has the choice of man α or man δ; she therefore chooses her favorite and thus rejects man α. In the second stage all the rejected men propose to the next highest girl in their choice list. In this case man α proposes to woman A. Woman A now has two proposals and so she rejects man β in favor of man α. In the next stage man β proposes to woman C, who rejects him in favor of man γ. Man β proposes again this time to woman D, who rejects man δ in his favor. Man δ then proposes to woman A, who rejects him, to woman C, who rejects him, and finally to woman B, who has not yet had a proposal. She therefore accepts him, and at this stage all the women have had at least one proposal and so the algorithm terminates with each girl accepting the man she is at present holding in suspense. So the stable marriage assignment is Aα, Bδ, Cγ, Dβ.

An ALGOL procedure for finding a stable set of marriages using the Gale and Shapley algorithm is given in Algorithm 411, Pt. 1 [2]. The procedure is explained by ALGOL comments embodied in it.

3.2 A Recursive Algorithm

In this algorithm we define two operations. The first operation *proposal* either makes the next proposal of some man i and calls the second operation *refusal* for the woman to whom man i has just proposed, or *proposal* does nothing if the man who is proposing is the dummy man zero. The second operation *refusal* decides for woman j whether or not to refuse a new proposal from man i, or to refuse the man whe has in suspense and replace him by man i. For whichever of

Table I. Comparison of Whetstone ALGOL Times for Gale and Shapley, and Recursive Algorithms

n	Number of proposals	Time for Gale and Shapley algorithm (secs)	Time for recursive algorithm (secs)	% Improvement
5	11	4	4	0
10	32	14	11	21.4
20	100	53	40	24.5
30	92	90	80	11.1
40	125	155	138	11.0
50	273	310	216	30.3

the two men that woman j refuses, *refusal* calls *proposal* so that this man can make his next proposal.

The simplest way of using these two operations is to perform the operation *proposal* for each man in turn. This method of use gives the same result as the Gale and Shapley algorithm. In fact, each algorithm performs the same proposals and rejections although in general in a different order. We can prove this formally by the following theorem.

THEOREM 1. *The set of stable marriages given by the recursive algorithm is the male optimal stable solution.*

PROOF. Assume that up to a given point in the algorithm the men have had their best possible choices. Suppose at this stage woman A has proposals from man α and man β to consider, and she rejects man α in favor of man β. We must now show that there cannot exist a set of stable marriages with man α married to woman A. If there was such a set, then man β would be married to some woman B. By assumption woman B would be lower in man β's choice list than woman A. Therefore, the set of marriages is unstable since A prefers β to her present partner and β prefers A to his present partner. Obviously the recursive algorithm starts with the men getting their best possible choices, and we have proved that it must continue to do so.

This proof is in essence the same as that given by Gale and Shapley [1].

An ALGOL procedure for finding a stable set of marriages using the recursive algorithm is given in Algorithm 411, Pt. 2 [2]. The procedure is explained by ALGOL comments embodies in it.

3.3 Comparative Efficiency of the Two Algorithms

It has already been shown that the two methods give the same result, and the same proposals are made. However, it can be seen that providing we can use the random access store for all the storage, then the recursive algorithm should be faster than the Gale and Shapley algorithm. This is because at each stage of the latter we have to examine all the men to see if they have been refused, and all the women to see if they have had a new proposal or proposals. In the recursive algorithm each proposal points straight to the girl being proposed to, and each refusal points to the man who has been refused, and so he can make his next choice. However, the extra comparisons being made in the Gale and Shapley algorithm are unlikely to be a large factor in the overall time. This is borne out by the figures given in Table I. The choice matrices were picked by a random selection method and the results show that although the recursive algorithm is always as good as or better than the Gale and Shapley algorithm the improvement is not large.

4. All the Solutions to the Simple Marriage Problem

In Section 3 methods of finding the male optimal solution were derived. It is evident that, by reversing the roles of the men and women and letting the women propose to the men, the female optimal solution can be obtained using these algorithms. If these two solutions are identical then there will be only one stable marriage. However, in general there will be several different solutions to the stable marriage problem and in this section a method for finding all the stable solutions will be described.

The following simple theorem is required in the development of the algorithm.

THEOREM 2. *In any stable marriage no woman receives a poorer choice than the one she receives in the male optimal solution.*

PROOF. Suppose some woman i receives a poorer choice j than she received in the male optimal solution. In this case she would rather have her husband, k say, given by the male optimal solution than man j. Also man k would rather marry woman i than any other woman he can obtain in any other stable marriage (by definition of the male optimal solution). Therefore, man k and woman i would both rather marry each other than their present partners, and so the proposed marriage is unstable.

4.1 An Algorithm for Finding All the Stable Solutions

It is obvious from the above theorem that if another stable solution exists then at least one woman must have a better choice than in the male optimal solution, and her husband in the male optimal solution must take a poorer choice.

Thus, if we consider using the recursive algorithm of Section 3.2, we can adapt it to finding all the solutions by using a further operation called *breakmarriage*. This operation consists of breaking the marriage of a selected man i, in a stable marriage S, and forcing him, therefore, to take a poorer choice in his list. Similarly, the woman i_s (who was married to man i) has the opportunity of getting a better choice. Once the stable marriage is broken the operations *proposal* and *refusal* come into play as before. The process will terminate either when woman i_s receives a proposal from a man higher in her choice list than man i or when some man runs out of choices (having proposed to all the n women). In the first case we will call the process successful and we will show that the *breakmarriage* operation leads to a new stable marriage solution. In the latter case there is evidently no further stable solution using that particular *breakmarriage* operation. Unrestricted use of the *breakmarriage* operation would lead in general to the same stable solution being obtained many times. We, therefore, impose two further restrictions on the *breakmarriage* operation:

Rule 1. Starting at a stable solution obtained by a succesful *breakmarriage* operation on man i, *breakmarriages* may only be performed on men $\geq i$.

Rule 2. In a *breakmarriage* operation started on man i, only men $\geq i$ may propose (i.e. if after breaking man i's marriage the process attempts to interfere with the marriage of a man $j < i$, then the process is stopped and said to be unsuccessful).

Using the *breakmarriage* operation with these two restrictions we will now prove three theorems which enable us to ensure that our algorithm produces every stable marriage solution once and only once.

THEOREM 3. *When the breakmarriage operation is applied to a set of stable marriages and is terminated successfully the resulting set of marriages is stable.*

PROOF. All pairs will be stable if they are both unaffected by the *breakmarriage* operation since they were stable in the previous set of marriages. If the *breakmarriage* operation affects them and man i is married to woman I at the end of it, and if man i would prefer woman J, then she must have had a proposal from him and rejected him in favor of a better partner. In the operation the women continue to get better choice partners, and so woman J will prefer her present husband to man i, and thus the marriage is stable.

THEOREM 4. *Any stable solution, other than the male optimal stable solution, can be obtained from the male optimal stable solution by successive applications of the breakmarriage operation.*

PROOF. Let the wife of man i in any stable marriage solution M be denoted by i_M. Let S be a stable solution which is not the male optimal solution.

Consider a stable solution T, which is initially the male optimal solution. No man in the solution T has a worse choice than in solution S. Now apply the *breakmarriage* operation to man i in solution T, where man i is the first man whose wife i_T is not the same woman as i_s.

We will now show that no man j can propose to a worse choice than j_s. If this could happen let man k be the first man who, during the *breakmarriage* process, is refused by k_s. Then the woman k_s will have received a proposal from a preferred man l. Man l will not, at this stage, have proposed to woman l_s (since man k

Table II

Problem size = 8

Men choose women in the order:

Man 1 chooses	5	7	1	2	6	8	4	3
Man 2 chooses	2	3	7	5	4	1	8	6
Man 3 chooses	8	5	1	4	6	2	3	7
Man 4 chooses	3	2	7	4	1	6	8	5
Man 5 chooses	7	2	5	1	3	6	8	4
Man 6 chooses	1	6	7	5	8	4	2	3
Man 7 chooses	2	5	7	6	3	4	8	1
Man 8 chooses	3	8	4	5	7	2	6	1

Women choose men in the order:

Woman 1 chooses	5	3	7	6	1	2	8	4
Woman 2 chooses	8	6	3	5	7	2	1	4
Woman 3 chooses	1	5	6	2	4	8	7	3
Woman 4 chooses	8	7	3	2	4	1	5	6
Woman 5 chooses	6	4	7	3	8	1	2	5
Woman 6 chooses	2	8	5	4	6	3	7	1
Woman 7 chooses	7	5	2	1	8	6	4	3
Woman 8 chooses	7	4	1	5	2	3	6	8

Table III

Stable solutions

	Women								
Man	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
1 marries	5	8	3	3	3	3	8	8	8
2 marries	3	3	6	6	6	6	3	3	3
3 marries	8	5	5	1	2	1	1	2	1
4 marries	6	6	8	8	8	8	6	6	6
5 marries	7	7	7	7	1	2	7	1	2
6 marries	1	1	1	5	5	5	5	5	5
7 marries	2	2	2	2	7	7	2	7	7
8 marries	4	4	4	4	4	4	4	4	4
Number of proposals	16	22	31	35	43	38	26	34	29
Choice count	48	49	51	50	54	51	48	52	49

was the first to have been refused by his woman in marriage S, and if l_s had accepted man l he could not have been retained by woman k_s). Therefore, man l and woman k_s prefer each other to their partners in the solution S, which would make S unstable. Thus man k cannot exist.

Hence the *breakmarriage* operation on man i of solution T produces a stable marriage T' in which no man has a worse choice than in S. If T' is not identical with S, then repeat the *breakmarriage* operations with T' replacing T and redefine man i as before. Since after each application of the *breakmarriage* operation at least one man gets a worse choice, the solution S will be obtained after a finite number of applications of the *breakmarriage* operation.

THEOREM 5. *The breakmarriage operation with the two restrictions given above allows each stable solution to be obtained only once.*

PROOF. Suppose a sequence of *breakmarriage* operations different from that defined in Theorem 4 could be used to obtain one of the stable sets of marriages.

Consider the first occasion when the new sequence requires a *breakmarriage* operation to be performed on a man j, who is different from man i in the old sequence. Now $j > i$ since all the men less than i have already got their correct wives for the solution required. Man i will not yet have proposed to his wife in the required solution under the new sequence of *breakmarriage* operations. The rules (since $j > i$) will now prevent him from proposing again. Thus man i cannot get his wife in the required solution, and so such a new sequence cannot lead to the same set of stable marriages.

The *breakmarriage* operation and the two rules that restrict its application give us by repeated application a unique ordering of the stable marriage solutions. This is used in the ALGOL procedure for finding all the stable solutions which is given in Algorithm 411, Pt. 3 [2]. It will be seen that the three subprocedures carry out the basic operations *breakmarriage*, *proposal*, and *refusal*. The procedure is explained by ALGOL comments embodied in it.

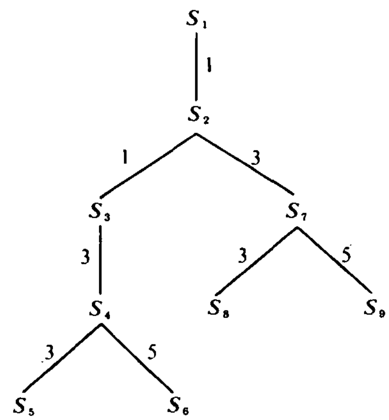
4.2 Results and Discussion

The procedure for finding all the stable marriages has been used for many sizes of problem. In Table II the male and female choice arrays are given for an 8×8 problem. The results for this problem are given in Table III. There are nine stable sets of marriages and these are labeled $S_1, S_2, \cdots, S_9$ in the order in which the procedure of Algorithm 411, Pt. 3, will derive them. Also given are the number of proposals made by the men to arrive at each stable marriage. The smallest number of proposals is the S_1 solution, which is the male optimal stable solution. The largest number of proposals is the S_5 solution, which is, therefore, the female optimal stable solution. The final row of Table III gives the choice count, and from it we can find the minimum choice stable solution as defined in Section 2.2. In this case both the solutions S_1 and S_7 give a minimum choice count of 48. Either of these, therefore, can be taken as the minimum choice stable solution.

In order to illustrate the process of *breakmarriage*, the nine solutions $S_1, \cdots, S_9$ have been drawn as a treelike structure in Figure 1. The figures on the branches between two stable marriages indicate which marriage is broken to get from the higher to the lower stable solution. S_1, the male optimal stable solution, is at the top of the tree. The female optimal stable solution (S_5 in this example) will always be at the bottom left-hand corner.

We can make Algorithm 411, Pt. 3, more efficient for large values of n if we restrict the searching in some cases. Since after the female optimal stable solution has been found, we only need to search the male choice array up to the male choice values of the female optimal stable solution. This is because the men can never do

Fig. 1. An example of the tree-structure for the stable marriage solutions. The full size numbers represent the marriage that was broken to get from the higher solution to the lower.



worse than in this solution. Useful savings in time can, therefore, be made in the cases where n is large and there are many stable solutions and the female optimal stable solution is obtained quite early. The original algorithm was modified to incorporate this change, and in an example with $n = 30$, which had 24 stable solutions, the fifth of these being the female optimal solution, the Whetstone ALGOL run-time was reduced from 310 sec to 270 sec.

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