ASYMPTOTIC FORMULÆ IN COMBINATORY ANALYSIS

By G. H. HARDY AND S. RAMANUJAN.*

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1.

INTRODUCTION AND SUMMARY OF RESULTS.

1 . 1. The present paper is the outcome of an attempt to apply to the principal problems of the theory of partitions the methods, depending upon the theory of analytic functions, which have proved so fruitful in the theory of the distribution of primes and allied branches of the analytic theory of numbers.

The most interesting functions of the theory of partitions appear as the coefficients in the power-series which represent certain elliptic modular functions. Thus $p(n)$, the number of unrestricted partitions of n, is the coefficient of x^n in the expansion of the function

$$(1 . 11) \qquad f(x) = 1 + \sum_1^\infty p(n) x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots}.\dagger$$

If we write

$$(1 . 12) \qquad x = q^2 = e^{2\pi i \tau},$$

where the imaginary part of τ is positive, we see that $f(x)$ is substantially the reciprocal of the modular function called by Tannery and Molk‡ $h(\tau)$; that, in fact,

$$(1 . 13) \qquad h(\tau) = q^{\frac{1}{12}} q_0 = q^{\frac{1}{12}} \prod_1^\infty (1-q^{2n}) = \frac{x^{\frac{1}{24}}}{f(x)}.$$

* A short abstract of the contents of part of this paper appeared under the title "Une formule asymptotique pour le nombre des partitions de n", in the *Comptes rendus*, January 2nd, 1917.

† P. A. MacMahon, *Combinatory Analysis*, Vol. 2, 1916, p. 1.

‡ J. Tannery and J. Molk, *Fonctions elliptiques*, Vol. 2, 1896, pp. 31 *et seq*. We shall follow the notation of this work whenever we have to quote formulæ from the theory of elliptic functions.

The theory of partitions has, from the time of Euler onwards, been developed from an almost exclusively algebraical point of view. It consists of an assemblage of formal identities—many of them, it need hardly be said, of an exceedingly ingenious and beautiful character. Of *asymptotic* formulæ, one may fairly say, there are none.* So true is this, in fact, that we have been unable to discover in the literature of the subject any allusion whatever to the question of the order of magnitude of $p(n)$.

1.2. The function $p(n)$ may, of course, be expressed in the form of an integral

$$p(n) = \frac{1}{2\pi i}\int_\Gamma \frac{f(x)}{x^{n+1}}\,dx, \tag{1.21}$$

by means of Cauchy's theorem, the path Γ enclosing the origin and lying entirely inside the unit circle. The idea which dominates this paper is that of obtaining asymptotic formulæ for $p(n)$ by a detailed study of the

* We should mention one exception to this statement, to which our attention was called by Major MacMahon. The number of partitions of n *into parts none of which exceed* r is the coefficient $p_r(n)$ in the series

$$1+\sum_1^\infty p_r(n)\,x^n = \frac{1}{(1-x)(1-x^2)\dots(1-x^r)}.$$

This function has been studied in much detail, for various special values of r, by Cayley, Sylvester, and Glaisher: we may refer in particular to J. J. Sylvester, "On a discovery in the theory of partitions", *Quarterly Journal*, Vol. 1, 1857, pp. 81–85, and "On the partition of numbers", *ibid.*, pp. 141–152 (Sylvester's *Works*, Vol. 2, pp. 86–89 and 90–99); J. W. L. Glaisher, "On the number of partitions of a number into a given number of parts", *Quarterly Journal*, Vol. 40, 1909, pp. 57–143; "Formulæ for partitions into given elements, derived from Sylvester's Theorem", *ibid.*, pp. 275–348; "Formulæ for the number of partitions of a number into the elements 1, 2, 3, ..., n up to $n = 9$", *ibid.*, Vol. 41, 1910, pp. 94–112: and further references will be found in MacMahon, *l.c.*, pp. 59–71, and E. Netto, *Lehrbuch der Combinatorik*, 1901, pp. 146–158. Thus, for example, the coefficient of x^n in

$$\frac{1}{(1-x)(1-x^2)(1-x^3)}$$

is

$$p_3(n) = \frac{1}{12}(n+3)^2 - \frac{7}{72} + \frac{1}{8}(-1)^n + \frac{2}{9}\cos\frac{2n\pi}{3};$$

as is easily found by separating the function into partial fractions. This function may also be expressed in the orms

$$\tfrac{1}{12}(n+3)^2 + (\tfrac{1}{2}\cos\tfrac{1}{2}\pi n)^2 - (\tfrac{2}{3}\sin\tfrac{1}{3}\pi n)^2,$$

$$1+[\tfrac{1}{12}n(n+6)],\quad \{\tfrac{1}{12}(n+3)^2\},$$

where $[n]$ and $\{n\}$ denote the greatest integer contained in n and the integer nearest to n. These formulæ do, of course, furnish incidentally asymptotic formulæ for the functions in question. But they are, from this point of view, of a very trivial character: the interest which they possess is algebraical.

integral (1 . 21). This idea is an extremely obvious one; it is the idea which has dominated nine-tenths of modern research in the analytic theory of numbers: and it may seem very strange that it should never have been applied to this particular problem before. Of this there are no doubt two explanations. The first is that the theory of partitions has received its most important developments, since its foundation by Euler, at the hands of a series of mathematicians whose interests have lain primarily in algebra. The second and more fundamental reason is to be found in the extreme complexity of the behaviour of the generating function $f(x)$ near a point of the unit circle.

It is instructive to contrast this problem with the corresponding problems which arise for the arithmetical functions $\pi(n)$, $\vartheta(n)$, $\psi(n)$, $\mu(n)$, $d(n)$, ... which have their genesis in Riemann's Zeta-function and the functions allied to it. In the latter problems we are dealing with functions defined by Dirichlet's series. The study of such functions presents difficulties far more fundamental than any which confront us in the theory of the modular functions. These difficulties, however, relate to the distribution of the zeros of the functions and their general behaviour at infinity: no difficulties whatever are occasioned by the crude singularities of the functions in the finite part of the plane. The single finite singularity of $\zeta(s)$, for example, the pole at $s = 1$, is a singularity of the simplest possible character. It is this pole which gives rise to the *dominant* terms in the asymptotic formulæ for the arithmetical functions associated with $\zeta(s)$. To prove such a formula rigorously is often exceedingly difficult; to determine precisely the order of the error which it involves is in many cases a problem which still defies the utmost resources of analysis. But to write down the dominant terms involves, as a rule, no difficulty more formidable than that of deforming a path of integration over a pole of the subject of integration and calculating the corresponding residue.

In the theory of partitions, on the other hand, we are dealing with functions which do not exist at all outside the unit circle. Every point of the circle is an essential singularity of the function, and no part of the contour of integration can be deformed in such a manner as to make its contribution obviously negligible. Every element of the contour requires special study; and there is no obvious method of writing down a "dominant term."

The difficulties of the problem appear then, at first sight, to be very serious. We possess, however, in the formulæ of the theory of the linear transformation of the elliptic functions, an extremely powerful analytical weapon by means of which we can study the behaviour of $f(x)$ near any assigned point of the unit circle.* It is to an appropriate use of these formulæ that the accuracy of our final results, an accuracy which will, we think, be found to be quite startling, is due.

1.3. It is very important, in dealing with such a problem as this, to distinguish clearly the various stages to which we can progress by arguments of a progressively "deeper" and less elementary character. The earlier results are naturally (so far as the particular problem is concerned) superseded by the later. But the more elementary methods are likely to be applicable to other problems in which the more subtle analysis is impracticable.

We have attacked this particular problem by a considerable number of different methods, and cannot profess to have reached any very precise conclusions as to the possibilities of each. A detailed comparison of the results to which they lead would moreover expand this paper to a quite unreasonable length. But we have thought it worth while to include a short account of two of them. The first is quite elementary; it depends only on Euler's identity

$$(1.31) \quad \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots} = 1 + \frac{x}{(1-x)^2} + \frac{x^4}{(1-x)^2(1-x^2)^2} + \ldots,$$

—an identity capable of wide generalisation—and on elementary algebraical reasoning. By these means we show, in section 2, that

$$(1.32) \quad e^{A\sqrt{n}} < p(n) < e^{B\sqrt{n}},$$

where A and B are positive constants, for all sufficiently large values of n.

It follows that

$$(1.33) \quad A\sqrt{n} < \log p(n) < B\sqrt{n};$$

and the next question which arises is the question whether a constant C exists such that

$$(1.34) \quad \log p(n) \sim C\sqrt{n}.$$

We prove that this is so in section 3. Our proof is still, in a sense, "elementary". It does not appeal to the theory of analytic functions, depending only on a general arithmetic theorem concerning infinite series;

* See G. H. Hardy and J. E. Littlewood, "Some problems of Diophantine approximation (II: The trigonometrical series associated with the elliptic Theta-functions)", *Acta Mathematica*, Vol. 37, 1914, pp. 193–238, for applications of the formulæ to different but not unrelated problems.

but this theorem is of the difficult and delicate type which Messrs. Hardy and Littlewood have called "Tauberian". The actual theorem required was proved by us in a paper recently printed in these *Proceedings**. It shows that

$$C = \frac{2\pi}{\sqrt{6}}; \tag{1.35}$$

in other words that

$$p(n) = \exp\left\{\pi\sqrt{\left(\frac{2n}{3}\right)}(1+\epsilon)\right\}, \tag{1.36}$$

where ϵ is small when n is large. This method is one of very wide application. It may be used, for example, to prove that, if $p^{(s)}(n)$ denotes the number of partitions of n into perfect s-th powers, then

$$\log p^{(s)}(n) \sim (s+1)\left\{\frac{1}{s}\Gamma\left(1+\frac{1}{s}\right)\zeta\left(1+\frac{1}{s}\right)\right\}^{s/(s+1)} n^{1/(s+1)}.$$

It is certainly possible to obtain, by means of arguments of this general character, information about $p(n)$ more precise than that furnished by the formula (1.36). And it is equally possible to prove (1.36) by reasoning of a more elementary, though more special, character: we have a proof, for example, based on the identity

$$np(n) = \sum_{\nu=1}^{n} \sigma(\nu)\, p(n-\nu),$$

where $\sigma(\nu)$ is the sum of the divisors of ν, and a process of induction. But we are at present unable to obtain, by any method which does not depend upon Cauchy's theorem, a result as precise as that which we state in the next paragraph, a result, that is to say, which is "vraiment asymptotique".

1.4. Our next step was to replace (1.36) by the much more precise formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left\{\pi\sqrt{\left(\frac{2n}{3}\right)}\right\}.† \tag{1.41}$$

The proof of this formula appears to necessitate the use of much more

* G. H. Hardy and S. Ramanujan, "Asymptotic formulæ concerning the distribution of integers of various types", *Proc. London Math. Soc.*, Ser. 2, Vol. 16, 1917, pp. 112–132.

† In our note in the *Comptes rendus* $4n\sqrt{3}$ is misprinted as $4\pi\sqrt{3}$.

powerful machinery, Cauchy's integral (1 . 21) and the functional relation

$$(1.42) \qquad f(x) = \frac{x^{1/24}}{\sqrt{(2\pi)}} \sqrt{\left(\log \frac{1}{x}\right)} \exp \left\{ \frac{\pi^2}{6 \log(1/x)} \right\} f(x'),$$

where

$$(1.43) \qquad x' = \exp \left\{ -\frac{4\pi^2}{\log(1/x)} \right\}.$$

This formula is merely a statement in a different notation of the relation between $h(\tau)$ and $h(\mathrm{T})$, where

$$\mathrm{T} = \frac{c+d\tau}{a+b\tau}, \quad a = d = 0, \quad b = 1, \quad c = -1;$$

viz.

$$h(\tau) = \sqrt{\left(\frac{i}{\tau}\right)}\, h(\mathrm{T}).*$$

It is interesting to observe the correspondence between (1 . 41) and the results of numerical computation. Numerical data furnished to us by Major MacMahon gave the following results: we denote the right-hand side of (1 . 41) by $\varpi(n)$.

n	$p(n)$	$\varpi(n)$	ϖ/p
10	42	48·104	1·145
20	627	692·385	1·104
50	204226	217590·499	1·065
80	15796476	16606781·567	1·051

It will be observed that the progress of ϖ/p towards its limit unity is not very rapid, and that $\varpi-p$ is always positive and appears to tend rapidly to infinity.

1 . 5. In order to obtain more satisfactory results it is necessary to construct some auxiliary function $F(x)$ which is regular at all points of the unit circle save $x = 1$, and has there a singularity of a type as near as possible to that of the singularity of $f(x)$. We may then hope to ob-

* Tannery and Molk, *l.c.*, p. 265 (Table XLV, 5).

tain a much more precise approximation by applying Cauchy's theorem to $f-F$ instead of to F. For although every point of the circle is a singular point of f, $x = 1$ is, to put it roughly, much the *heaviest* singularity. When $x \to 1$ by real values, $f(x)$ tends to infinity like an exponential

$$\exp\left\{\frac{\pi^2}{6(1-x)}\right\};$$

when

$$x = re^{2p\pi i/q},$$

p and q being co-prime integers, and $r \to 1$, $|f(x)|$ tends to infinity like an exponential

$$\exp\left\{\frac{\pi^2}{6q^2(1-r)}\right\};$$

while, if

$$x = re^{2\theta\pi i},$$

where θ is irrational, $|f(x)|$ can become infinite at most like an exponential of the type

$$\exp\left\{o\left(\frac{1}{1-r}\right)\right\}.*$$

The function required is

$$F(x) = \frac{1}{\pi\sqrt{2}} \sum_1^\infty \psi(n)\, x^n, \tag{1.51}$$

where

$$\psi(n) = \frac{d}{dn}\left\{\frac{\cosh C\lambda_n - 1}{\lambda_n}\right\}, \tag{1.52}$$

$$C = 2\pi/\sqrt{6} = \pi\sqrt{(\tfrac{2}{3})}, \qquad \lambda_n = \sqrt{(n-\tfrac{1}{24})}. \tag{1.53}$$

This function may be transformed into an integral by means of a general formula given by Lindelöf† ; and it is then easy to prove that the "principal branch" of $F(x)$ is regular all over the plane except at $x = 1$‡ ;

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* The statements concerning the "rational" points are corollaries of the formulæ of the transformation theory, and proofs of them are contained in the body of the paper. The proposition concerning "irrational" points may be proved by arguments similar to those used by Hardy and Littlewood in their memoir already quoted. It is not needed for our present purpose. As a matter of fact it is *generally* true that $f(x) \to 0$ when θ is irrational, and very nearly as rapidly as $\sqrt{(1-r)}$. It is in reality owing to this that our final method is so successful.

† E. Lindelöf, *Le calcul des résidus et ses applications à la théorie des fonctions* (Gauthier-Villars, Collection Borel, 1905), p. 111.

‡ We speak, of course, of the principal branch of the function, viz. that represented by the series (1.51) when x is small. The other branches are singular at the origin.

and that

$$F(x)-\chi(x),$$

where

$$\chi(x)=\frac{x^{\frac{1}{24}}}{\sqrt{(2\pi)}}\sqrt{\left(\log\frac{1}{x}\right)}\left[\exp\left\{\frac{\pi^2}{6\log(1/x)}\right\}-1\right] \tag{1.54}$$

is regular for $x=1$. If we compare (1 . 42) and (1 . 54), and observe that $f(x')$ tends to unity with extreme rapidity when x tends to 1 along any regular path which does not touch the circle of convergence, we can see at once the very close similarity between the behaviour of f and F inside the unit circle and in the neighbourhood of $x=1$.

It should be observed that the term -1 in (1 . 52) and (1 . 54) is—so far as our present assertions are concerned—otiose: all that we have said remains true if it is omitted; the resemblance between the singularities of f and F becomes indeed even closer. The term is inserted merely in order to facilitate some of our preliminary analysis, and will prove to be without influence on the final result.

Applying Cauchy's theorem to $f-F$, we obtain

$$p(n)=\frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{C\lambda_n}}{\lambda_n}\right)+O(e^{D\sqrt{n}}), \tag{1.55}$$

where D is any number greater than

$$\tfrac{1}{2}C=\tfrac{1}{2}\pi\sqrt{(\tfrac{2}{3})}.$$

1 . 6. The formula (1 . 55) is an asymptotic formula of a type far more precise than that of (1 . 41). The error term is, however, of an exponential type, and may be expected ultimately to increase with very great rapidity. It was therefore with considerable surprise that we found what exceedingly good results the formula gives for fairly large values of n. For $n=61, 62, 63$ it gives*

1121538·972, 1300121·359, 1505535·606,

while the correct values are

1121505, 1300156, 1505499.

The errors 33·972, -34·641, 36·606

are relatively very small, and alternate in sign.

The next step is naturally to direct our attention to the singular

* In the *Comptes rendus* we misstated the second number as 1300111.

point of $f(x)$ next in importance after that at $x=1$, viz., that at $x=-1$; and to subtract from $f(x)$ a second auxiliary function, related to this point as $F(x)$ is to $x=1$. No new difficulty of principle is involved, and we find that

$$(1.61) \qquad p(n)=\frac{1}{2\pi\sqrt{2}}\frac{d}{dn}\left(\frac{e^{C\lambda_n}}{\lambda_n}\right)+\frac{(-1)^n}{2\pi}\frac{d}{dn}\left(\frac{e^{\frac{1}{2}C\lambda_n}}{\lambda_n}\right)+O(e^{D\sqrt{n}}),$$

where D is now any number greater than $\frac{1}{3}C$. It now becomes obvious why our earlier approximation gave errors alternately of excess and of defect.

It is obvious that this process may be repeated indefinitely. The singularities next in importance are those at $x=e^{\frac{2}{3}\pi i}$ and $x=e^{\frac{4}{3}\pi i}$; the next those at $x=i$ and $x=-i$; and so on. The next two terms in the approximate formula are found to be

$$\frac{\sqrt{3}}{\pi\sqrt{2}}\cos\left(\tfrac{2}{3}n\pi-\tfrac{1}{18}\pi\right)\frac{d}{dn}\left(\frac{e^{\frac{1}{3}C\lambda_n}}{\lambda_n}\right)$$

and

$$\frac{\sqrt{2}}{\pi}\cos\left(\tfrac{1}{2}n\pi-\tfrac{1}{8}\pi\right)\frac{d}{dn}\left(\frac{e^{\frac{1}{4}C\lambda_n}}{\lambda_n}\right).$$

As we proceed further, the complexity of the calculations increases. The auxiliary function associated with the point $x=e^{2p\pi i/q}$ involves a certain $24q$-th root of unity, connected with the linear transformation which must be used in order to elucidate the behaviour of $f(x)$ near the point; and the explicit expression of this root in terms of p and q, though known, is somewhat complex. But it is plain that, by taking a sufficient number of terms, we can find a formula in which the error is

$$O(e^{C\lambda_n/\nu}),$$

where ν is a fixed but arbitrarily large integer.

1.7. A final question remains. We have still the recourse of making ν a function of n, that is to say of making the number of terms in our approximate formula itself a function of n. In this way we may reasonably hope, at any rate, to find a formula in which the error is of order less than that of any exponential of the type e^{an}; of the order of a power of n, for example, or even bounded.

When, however, we proceeded to test this hypothesis by means of the numerical data most kindly provided for us by Major MacMahon, we found a correspondence between the real and the approximate values of such astonishing accuracy as to lead us to hope for even more. Taking

$n = 100$, we found that the first six terms of our formula gave

$$\begin{array}{r} 190568944{\cdot}783 \\ +348{\cdot}872 \\ -2{\cdot}598 \\ +{\cdot}685 \\ +{\cdot}318 \\ -{\cdot}064 \\ \hline 190569291{\cdot}996\,, \end{array}$$

while $$p(100) = 190569292\,;$$

so that the error after six terms is only ·004. We then proceeded to calculate p (200), and found

$$\begin{array}{r} 3,\ 972,\ 998,\ 993,\ 185{\cdot}896 \\ +36,\ 282{\cdot}978 \\ -87{\cdot}555 \\ +\ \ 5{\cdot}147 \\ +\ \ 1{\cdot}424 \\ +\ \ 0{\cdot}071 \\ +\ \ 0{\cdot}000^* \\ +\ \ 0{\cdot}043 \\ \hline 3,\ 972,\ 999,\ 029,\ 388{\cdot}004\,, \end{array}$$

and Major MacMahon's subsequent calculations showed that p (200) is, in fact,

$$3,\ 972,\ 999,\ 029,\ 388.$$

These results suggest very forcibly that it is possible to obtain a formula for $p(n)$, which not only exhibits its order of magnitude and structure, but may be used to calculate its *exact* value for any value of n. That this is in fact so is shown by the following theorem.

Statement of the main theorem.

THEOREM.—*Suppose that*

$$(1\,.\,71) \qquad \phi_q(n) = \frac{\sqrt{q}}{2\pi\sqrt{2}}\,\frac{d}{dn}\left(\frac{e^{C\lambda_n/q}}{\lambda_n}\right),$$

* This term vanishes identically.

where C and λ_n are defined by the equations (1 . 53), for all positive integral values of q; that p is a positive integer less than and prime to q; that $\omega_{p,q}$ is a $24q$-th root of unity, defined when p is odd by the formula

(1 . 721)
$$\omega_{p,q} = \left(\frac{-q}{p}\right) \exp\left[-\left\{\tfrac{1}{4}(2-pq-p)+\tfrac{1}{12}\left(q-\frac{1}{q}\right)(2p-p'+p^2p')\right\}\pi i\right],$$

and when q is odd by the formula

(1 . 722)
$$\omega_{p,q} = \left(\frac{-p}{q}\right) \exp\left[-\left\{\tfrac{1}{4}(q-1)+\tfrac{1}{12}\left(q-\frac{1}{q}\right)(2p-p'+p^2p')\right\}\pi i\right],$$

where (a/b) is the symbol of Legendre and Jacobi, and p' is any positive integer such that $1+pp'$ is divisible by q; that*

(1 . 73)
$$A_q(n) = \sum_{(p)} \omega_{p,q}\, e^{-2np\pi i/q};$$

and that a is any positive constant, and ν the integral part of $a\sqrt{n}$.

Then

(1 . 74)
$$p(n) = \sum_1^{\nu} A_q \phi_q + O(n^{-\frac{1}{4}}),$$

so that $p(n)$ is, for all sufficiently large values of n, the integer nearest to

(1 . 75)
$$\sum_1^{\nu} A_q \phi_q.$$

It should be observed that all the numbers A_q are real. A table of A_q from $q = 1$ to $q = 18$ is given at the end of the paper (Table II).

The proof of this theorem is given in section 5: section 4 being devoted to a number of preliminary lemmas. The proof is naturally somewhat intricate; and we trust that we have arranged it in such a form as to be readily intelligible. In section 6 we draw attention to one or two questions which our theorem, in spite of its apparent completeness, still leaves open. In section 7 we indicate some other problems in combinatory analysis and the analytic theory of numbers to which our method may be applied; and we conclude by giving some functional and numerical tables: for the latter we are indebted to Major MacMahon and Mr. H. B. C. Darling. To Major MacMahon in particular we owe many thanks for the

* See Tannery and Molk, *l.c.*, pp. 104–106, for a complete set of rules for the calculation of the value of (a/b), which is, of course, always 1 or -1. When *both* p and q are odd it is indifferent which formula is adopted.

amount of trouble he has taken over very tedious calculations. It is certain that, without the encouragement given by the results of these calculations, we should never have attempted to prove theoretical results at all comparable in precision with those which we have enunciated.

2.

ELEMENTARY PROOF THAT $e^{A\sqrt{n}} < p(n) < e^{B\sqrt{n}}$ FOR SUFFICIENTLY LARGE VALUES OF n.

2 . 1. In this section we give the elementary proof of the inequalities (1 . 32). We prove, in fact, rather more, viz., that positive constants H and K exist such that

$$\frac{H}{n} e^{2\sqrt{n}} < p(n) < \frac{K}{n} e^{2\sqrt{(2n)}} \tag{2 . 11}$$

for $n \geqslant 1$.* We shall use in our proof only Euler's formula (1 . 31) and a debased form of Stirling's theorem, easily demonstrable by quite elementary methods: the proposition that

$$n!\, e^{n} / n^{n+\frac{1}{2}}$$

lies between two positive constants for all positive integral values of n.

2 . 2. The proof of the first of the two inequalities is slightly the simpler. It is obvious that if

$$\Sigma p_r(n)\, x^n = \frac{1}{(1-x)(1-x^2)\ldots(1-x^r)}$$

so that $p_r(n)$ is the number of partitions of n into parts not exceeding r, then

$$p_r(n) = p_{r-1}(n) + p_{r-1}(n-r) + p_{r-1}(n-2r) + \ldots . \tag{2 . 21}$$

* Somewhat inferior inequalities, of the type

$$2^{A[\sqrt{n}]} < p(n) < n^{B[\sqrt{n}]},$$

may be proved by *entirely* elementary reasoning; by reasoning, that is to say, which depends only on the arithmetical definition of $p(n)$ and on elementary finite algebra, and does not presuppose the notion of a limit or the definitions of the logarithmic or exponential functions.

We shall use this equation to prove, by induction, that

(2.22) $$p_r(n) \geqslant \frac{rn^{r-1}}{(r!)^2}.$$

It is obvious that (2.22) is true for $r=1$. Assuming it to be true for $r=s$, and using (2.21), we obtain

$$p_{s+1}(n) \geqslant \frac{s}{(s!)^2}\{n^{s-1}+(n-s-1)^{s-1}+(n-2s-2)^{s-1}+\ldots\}$$

$$\geqslant \frac{s}{(s!)^2}\left\{\frac{n^s-(n-s-1)^s}{s(s+1)}+\frac{(n-s-1)^s-(n-2s-2)^s}{s(s+1)}+\ldots\right\}$$

$$=\frac{n^s}{(s+1)(s!)^2}=\frac{(s+1)n^s}{\{(s+1)!\}^2}$$

This proves (2.22). Now $p(n)$ is obviously not less than $p_r(n)$, whatever the value of r. Take $r=[\sqrt{n}]$: then

$$p(n) \geqslant p_{[\sqrt{n}]}(n) \geqslant \frac{[\sqrt{n}]}{n}\frac{n^{[\sqrt{n}]}}{\{[\sqrt{n}]!\}^2} > \frac{H}{n}e^{2\sqrt{n}},$$

by a simple application of the degenerate form of Stirling's theorem mentioned above.

2.3. The proof of the second inequality depends upon Euler's identity. If we write

$$\Sigma q_r(n)\,x^n=\frac{1}{(1-x)^2(1-x^2)^2\ldots(1-x^r)^2},$$

we have

(2.31) $$q_r(n)=q_{r-1}(n)+2q_{r-1}(n-r)+3q_{r-1}(n-2r)+\ldots,$$

and

(2.32) $$p(n)=q_1(n-1)+q_2(n-4)+q_3(n-9)+\ldots.$$

We shall first prove by induction that

(2.33) $$q_r(n) \leqslant \frac{(n+r^2)^{2r-1}}{(2r-1)!\,(r!)^2}.$$

This is obviously true for $r=1$. Assuming it to be true for $r=s$, and using (2.31), we obtain

$$q_{s+1}(n) \leqslant \frac{1}{(2s-1)!\,(s!)^2}\{(n+s^2)^{2s-1}+2(n+s^2-s-1)^{2s-1}$$
$$+3(n+s^2-2s-2)^{2s-1}+\ldots\}.$$

Now
$$m(m-1)a^{m-2}b^2 \leqslant (a+b)^m - 2a^m + (a-b)^m,$$
if m is a positive integer, and a, b, and $a-b$ are positive, while if $a-b \leqslant 0$, and m is odd, the term $(a-b)^m$ may be omitted. In this inequality write
$$m = 2s+1, \quad a = n+s^2-ks-k \ (k = 0, 1, 2, \ldots), \quad b = s+1,$$
and sum with respect to k. We find that
$$(2s+1)\,2s(s+1)^2\{(n+s^2)^{2s-1}+2(n+s^2-s-1)^{2s-1}+\ldots\} \leqslant (n+s^2+s+1)^{2s+1};$$
and so
$$q_{s+1}(n) \leqslant \frac{(n+s^2+s+1)^{2s+1}}{(2s+1)\,2s(s+1)^2\,(2s-1)!\,(s!)^2} \leqslant \frac{\{n+(s+1)^2\}^{2s+1}}{(2s+1)!\,\{(s+1)!\}^2}.$$
Hence (2 . 33) is true generally.

It follows from (2 . 32) that
$$p(n) = q_1(n-1)+q_2(n-4)+\ldots \leqslant \sum_1^\infty \frac{n^{2r-1}}{(2r-1)!\,(r!)^2}.$$

But, using the degenerate form of Stirling's theorem once more, we find without difficulty that
$$\frac{1}{(2r-1)!\,(r!)^2} < \frac{2^{6r}K}{4r!},$$
where K is a constant. Hence
$$p(n) < 8K \sum_1^\infty \frac{(8n)^{2r-1}}{4r!} < 8K \sum_1^\infty \frac{(8n)^{\frac{1}{2}r-1}}{r!} < \frac{K}{n}e^{2\sqrt{(2n)}}.$$
This is the second of the inequalities (2 . 11).

3.

APPLICATION OF A TAUBERIAN THEOREM TO THE DETERMINATION OF THE CONSTANT C.

3 . 1. The value of the constant
$$C = \lim \frac{\log p(n)}{\sqrt{n}},$$
is most naturally determined by the use of the following theorem.

If $g(x) = \Sigma a_n x^n$ is a power-series with positive coefficients, and
$$\log g(x) \sim \frac{A}{1-x}$$

when $x \to 1$, *then*

$$\log s_n = \log(a_0 + a_1 + \ldots + a_n) \sim 2\sqrt{(An)}$$

when $n \to \infty$.

This theorem is a special case* of Theorem C in our paper already referred to.

Now suppose that

$$g(x) = (1-x) f(x) = \Sigma \{p(n) - p(n-1)\} x^n = \frac{1}{(1-x^2)(1-x^3)(1-x^4)\ldots}.$$

Then

$$a_n = p(n) - p(n-1)$$

is plainly positive. And

$$(3.11) \quad \log g(x) = \sum_2^\infty \log \frac{1}{1-x^\mu} = \sum_1^\infty \frac{1}{\nu} \frac{x^{2\nu}}{1-x^\nu} \sim \frac{1}{1-x} \sum_1^\infty \frac{1}{\nu^2} = \frac{\pi^2}{6(1-x)},$$

when $x \to 1$.† Hence

$$(3.12) \quad \log p(n) = a_0 + a_1 + \ldots + a_n \sim C\sqrt{n},$$

where $C = 2\pi/\sqrt{6} = \pi\sqrt{(\frac{2}{3})}$, as in (1.53).

3.2. There is no doubt that it is possible, by "Tauberian" arguments, to prove a good deal more about $p(n)$ than is asserted by (3.12).

* *L.c.* p. 129 (with $a = 1$).

† This is a special case of much more general theorems: see

K. Knopp, "Grenzwerte von Reihen bei der Annäherung an die Konvergenzgrenze", *Inaugural-Dissertation*, Berlin, 1907, pp. 25 *et seq.*;

K. Knopp, "Über Lambertsche Reihen", *Journal für Math.*, Vol. 142, 1913, pp. 283-315;

G. H. Hardy, "Theorems connected with Abel's Theorem on the continuity of power series", *Proc. London Math. Soc.*, Ser. 2, Vol. 4, 1906, pp. 247-265 (pp. 252, 253);

G. H. Hardy, "Some theorems concerning infinite series", *Math. Ann.*, Vol. 64, 1907, pp. 77-94;

G. H. Hardy, "Note on Lambert's series", *Proc. London Math. Soc.*, Ser. 2, Vol. 13, 1913, pp. 192-198.

A direct proof is very easy: for

$$\nu x^{\nu-1}(1-x) < 1 - x^\nu < \nu(1-x),$$

$$\frac{1}{1-x} \Sigma \frac{x^{2\nu}}{\nu^2} < \log g(x) < \frac{1}{1-x} \Sigma \frac{x^{\nu+1}}{\nu^2}.$$

The functional equation satisfied by $f(x)$ shows, for example, that

$$g(x) \sim \sqrt{\left(\frac{1-x}{2\pi}\right)} \exp\left\{\frac{\pi^2}{6(1-x)}\right\},$$

a relation far more precise than (3 . 11). From this relation, and the fact that the coefficients in $g(x)$ are positive, it is certainly possible to deduce more than (3 . 12). But it hardly seems likely that arguments of this character will lead us to a proof of (1 . 41). It would be exceedingly interesting to know exactly how far they will carry us, since the method is comparatively elementary, and has a much wider range of application than the more powerful methods employed later in this paper. We must, however, reserve the discussion of this question for some future occasion.

4.

LEMMAS PRELIMINARY TO THE PROOF OF THE MAIN THEOREM.

4 . 1. We proceed now to the proof of our main theorem. The proof is somewhat intricate, and depends on a number of subsidiary theorems which we shall state as lemmas.

Lemmas concerning Farey's series.

4 . 21. The *Farey's series of order* m is the aggregate of irreducible rational fractions

$$p/q \quad (0 \leqslant p \leqslant q \leqslant m),$$

arranged in ascending order of magnitude. Thus

$$\tfrac{0}{1},\ \tfrac{1}{7},\ \tfrac{1}{6},\ \tfrac{1}{5},\ \tfrac{1}{4},\ \tfrac{2}{7},\ \tfrac{1}{3},\ \tfrac{2}{5},\ \tfrac{3}{7},\ \tfrac{1}{2},\ \tfrac{4}{7},\ \tfrac{3}{5},\ \tfrac{2}{3},\ \tfrac{5}{7},\ \tfrac{3}{4},\ \tfrac{4}{5},\ \tfrac{5}{6},\ \tfrac{6}{7},\ \tfrac{1}{1}$$

is the Farey's series of order 7.

LEMMA 4 . 21.—*If p/q, p'/q' are two successive terms of a Farey's series, then*

$$p'q-pq' = 1. \tag{4 . 211}$$

This is, of course, a well known theorem, first observed by Farey and first proved by Cauchy.* The following exceedingly simple proof is due to Hurwitz.†

The result is plainly true when $m = 1$. Let us suppose it true for $m = k$; and let p/q, p'/q' be two consecutive terms in the series of order k.

Suppose now that p''/q'' is a term of the series of order $k+1$ which falls between p/q and p'/q'. Let

$$p''q-pq'' = \lambda > 0, \quad p'q''-p''q' = \mu > 0.$$

Solving these equations for p'', q'', and observing that $p'q-pq' = 1$, we obtain

$$p'' = \mu p+\lambda p', \quad q'' = \mu q+\lambda q'.$$

Consider now the aggregate of fractions

$$(\mu p+\lambda p')/(\mu q+\lambda q'),$$

where λ and μ are positive integers without common factor. All of these fractions lie between p/q and p'/q'; and all are in their lowest terms, since a factor common to numerator and denominator would divide

$$\lambda = q(\mu p+\lambda p')-p(\mu q+\lambda q'),$$

and

$$\mu = p'(\mu q+\lambda q')-q'(\mu p+\lambda p').$$

Each of them first makes its appearance in the Farey's series of order $\mu q+\lambda q'$, and the *first* of them to make its appearance must be that for which $\lambda = 1$, $\mu = 1$. Hence

$$p'' = p+p', \quad q'' = q+q',$$

$$p''q-pq'' = p'q''-p''q' = 1.$$

The lemma is consequently proved by induction.

LEMMA 4 . 22.—*Suppose that p/q is a term of the Farey's series of order m, and p''/q'', p'/q' the adjacent terms on the left and the right*:

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* J. Farey, "On a curious property of vulgar fractions", *Phil. Mag.*, Ser. 1, Vol. 47, 1816, pp. 385–386; A. L. Cauchy, "Démonstration d'un théorème curieux sur les nombres", *Exerciçes de mathématiques*, Vol. 1, 1826, pp. 114–116. Cauchy's proof was first published in the *Bulletin de la Société Philomatique* in 1816.

† A. Hurwitz, "Ueber die angenäherte Darstellung der Zahlen durch rationale Brüche", *Math. Ann.*, Vol. 44, 1894, pp. 417–436.

and let $j_{p,q}$ denote the interval

$$\frac{p}{q}-\frac{1}{q(q+q'')},\quad \frac{p}{q}+\frac{1}{q(q+q')}.^{*}$$

Then (i) *the intervals $j_{p,q}$ exactly fill up the continuum* (0, 1), *and* (ii) *the length of each of the parts into which $j_{p,q}$ is divided by p/q*† *is greater than $1/2mq$ and less than $1/mq$.*

(i) Since

$$\frac{1}{q(q+q')}+\frac{1}{q'(q'+q)}=\frac{1}{qq'}=\frac{p'q-pq'}{qq'}=\frac{p'}{q'}-\frac{p}{q},$$

the intervals just fill up the continuum.

(ii) Since neither q nor q' exceeds m, and one at least must be less than m, we have

$$\frac{1}{q(q+q')}>\frac{1}{2mq}.$$

Also $q+q'>m$, since otherwise $(p+p')/(q+q')$ would be a term in the series between p/q and p'/q'. Hence

$$\frac{1}{q(q+q')}<\frac{1}{mq}.$$

Standard dissection of a circle.

4 . 23. The following mode of dissection of a circle, based upon Lemma 4 . 22, is of fundamental importance for our analysis.

Suppose that the circle is defined by

$$x=Re^{2\pi i\theta}\quad(0\leqslant\theta\leqslant 1).$$

Construct the Farey's series of order m, and the corresponding intervals $j_{p,q}$. When these intervals are considered as intervals of variation of θ, and the two extreme intervals, which correspond to abutting arcs on the circle, are regarded as constituting a single interval $\xi_{1,1}$, the circle is divided into a number of arcs

$$\xi_{p,q},$$

---

* When p/q is 0/1 or 1/1, only the part of this interval inside (0, 1) is to be taken; thus $j_{0,1}$ is 0, $1/(m+1)$ and $j_{1,1}$ is $1-1/(m+1)$, 1.

† See the preceding footnote.

where q ranges from 1 to m and p through the numbers not exceeding and prime to q.* We call this dissection of the circle *the dissection* Ξ_m.

Lemmas from the theory of the linear transformation of the elliptic modular functions.

4 . 3. LEMMA 4 . 31.—*Suppose that q is a positive integer; that p is a positive integer not exceeding and prime to q; that p' is a positive integer such that $1+pp'$ is divisible by q; that $\omega_{p,q}$ is defined by the formulæ* (1 . 721) *or* (1 . 722); *that*

$$x = \exp\left(-\frac{2\pi z}{q}+\frac{2p\pi i}{q}\right), \qquad x' = \exp\left(-\frac{2\pi}{qz}+\frac{2p'\pi i}{q}\right),$$

where the real part of z is positive; and that

$$f(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\,\dots}.$$

Then
$$f(x) = \omega_{p,q}\sqrt{z}\exp\left(\frac{\pi}{12qz}-\frac{\pi z}{12q}\right)f(x').$$

This lemma is merely a restatement in a different notation of well known formulæ in the transformation theory.

Suppose, for example, that p is odd. If we take

$$a = p, \qquad b = -q, \qquad c = \frac{1+pp'}{q}, \qquad d = -p',$$

so that $ad-bc = 1$; and write

$$x = q^2 = e^{2\pi i\tau}, \qquad x' = Q^2 = e^{2\pi i T},$$

so that
$$\tau = \frac{p}{q}+\frac{iz}{q}, \qquad T = \frac{p'}{q}+\frac{i}{qz};$$

then we can easily verify that

$$T = \frac{c+d\tau}{a+b\tau}.$$

Also, in the notation of Tannery and Molk, we have

$$f(x) = \frac{q^{\frac{1}{12}}}{h(\tau)}, \qquad f(x') = \frac{Q^{\frac{1}{12}}}{h(T)};$$

* $p = 0$ occurring with $q = 1$ only.

and the formula for the linear transformation of $h(\tau)$ is

$$h(\mathrm{T}) = \left(\frac{b}{a}\right) \exp\left[\left\{\tfrac{1}{4}(a-1) - \tfrac{1}{12}[a(b-c)+bd(a^2-1)]\right\}\pi i\right]\sqrt{(a+b\tau)}\, h(\tau),$$

where $\sqrt{(a+b\tau)}$ has its real part positive.* A little elementary algebra will show the equivalence of this result and ours.

The other formula for $\omega_{p,q}$ may be verified similarly, but in this case we must take

$$a = -p, \quad b = q, \quad c = -\frac{1+pp'}{q}, \quad d = p'.$$

We have included in the Appendix (Table I) a short table of some values of $\omega_{p,q}$, or rather of $(\log \omega_{p,q})/\pi i$.

LEMMA 4 . 32.—*The function $f(x)$ satisfies the equation*

$$(4.321) \quad f(x) = \omega_{p,q}\sqrt{\left\{\frac{q}{2\pi}\log\left(\frac{1}{x_{p,q}}\right)\right\}}\, x_{p,q}^{\frac{1}{24}} \exp\left\{\frac{\pi^2}{6q^2\log(1/x_{p,q})}\right\} f(x'_{p,q}),$$

where

$$(4.322) \quad x_{p,q} = xe^{-2p\pi i/q}, \quad x'_{p,q} = \exp\left\{-\frac{4\pi^2}{q^2\log(1/x_{p,q})} + \frac{2p'\pi i}{q}\right\}.$$

This is an immediate corollary from Lemma 4 . 31, since

$$z = \frac{q}{2\pi}\log\left(\frac{1}{x_{p,q}}\right), \quad e^{-\pi z/12q} = x_{p,q}^{\frac{1}{24}},$$

$$\frac{\pi}{12qz} = \frac{\pi^2}{6q^2\log(1/x_{p,q})}, \quad x' = \exp\left(-\frac{2\pi}{qz} + \frac{2p'\pi i}{q}\right) = x'_{p,q}.$$

If we observe that

$$f(x'_{p,q}) = 1 + p(1)x'_{p,q} + \ldots,$$

we see that, if x tends to $e^{2p\pi i/q}$ along a radius vector, or indeed any regular path which does not touch the circle of convergence, the difference

$$f(x) - \omega_{p,q}\sqrt{\left\{\frac{q}{2\pi}\log\left(\frac{1}{x_{p,q}}\right)\right\}}\, x_{p,q}^{\frac{1}{24}} \exp\left\{\frac{\pi^2}{6q^2\log(1/x_{p,q})}\right\}$$

tends to zero with great rapidity. It is on this fact that our analysis is based.

* Tannery and Molk, *l.c.*, pp. 113, 267.

Lemmas concerning the auxiliary function $F_a(x)$.

4.41. The auxiliary function $F_a(x)$ is defined by the equation

$$F_a(x) = \sum_1^\infty \psi_a(n)\, x^n.$$

where
$$\psi_a(n) = \frac{d}{dn}\,\frac{\cosh a\lambda_n - 1}{\lambda_n},$$

$$\lambda_n = \sqrt{(n-\tfrac{1}{24})}, \qquad a > 0.$$

LEMMA 4.41.—*Suppose that a cut is made along the segment* $(1, \infty)$ *in the plane of* x. *Then* $F_a(x)$ *is regular at all points inside the region thus defined.*

This lemma is an immediate corollary of a general theorem proved by Lindelöf on pp. 109 *et seq.* of his *Calcul des résidus.**

The function
$$\psi_a(z) = \frac{d}{dz}\,\frac{\cosh a\sqrt{(z-\frac{1}{24})}-1}{\sqrt{(z-\frac{1}{24})}}$$

satisfies the conditions imposed upon it by Lindelöf, if the number which he calls a is greater than $\frac{1}{24}$; and

$$F_a(x) = \int_{a-i\infty}^{a+i\infty} \frac{x^z}{1-e^{2\pi i z}}\,\phi(z)\,dz, \tag{4.411}$$

if
$$x = re^{i\theta}, \quad 0 < \theta < 2\pi, \quad x^z = \exp\{z(\log r + i\theta)\}.$$

4.42. LEMMA 4.42.—*Suppose that* D *is the region defined by the inequalities*

$$-\pi < -\theta_0 < \theta < \theta_0 < \pi, \quad r_0 < r, \quad 0 < r_0 < 1,$$

and that $\log(1/x)$ *has its principal value, so that* $\log(1/x)$ *is one-valued, and its square root two-valued, in* D. *Further, let*

$$\chi_a(x) = \sqrt{\{\pi \log(1/x)\}}\, x^{\frac{1}{24}} \left[\exp\left\{\frac{a^2}{4\log(1/x)}\right\} - 1\right],$$

that value of the square root being chosen which is positive when

---

* Lindelöf gives references to Mellin and Le Roy, who had previously established the theorem in less general forms.

$0 < x < 1$. *Then*
$$F_a(x) - \chi_a(x)$$
*is regular inside D.**

We observe first that, when θ has a fixed value between 0 and 2π, the integral on the right-hand side of (4.411) is uniformly convergent for $\frac{1}{24} \leqslant a \leqslant a_0$. Hence we may take $a = \frac{1}{24}$ in (4.411). We thus obtain

$$F_a(x) = ix^{\frac{1}{24}} \int_0^\infty \frac{x^{it}}{1-e^{\frac{1}{2}\pi i - 2\pi t}} \psi_a(\tfrac{1}{24}+it)\,dt + ix^{\frac{1}{24}} \int_0^\infty \frac{x^{-it}}{1-e^{\frac{1}{2}\pi i + 2\pi t}} \psi_a(\tfrac{1}{24}-it)\,dt,$$

where the $\sqrt{(it)}$ and $\sqrt{(-it)}$ which occur in $\psi_a(\frac{1}{24}+it)$ and $\psi_a(\frac{1}{24}-it)$ are to be interpreted as $e^{\frac{1}{4}\pi i}\sqrt{t}$ and $e^{-\frac{1}{4}\pi i}\sqrt{t}$ respectively. We write this in the form

$$(4.421) \qquad F_a(x) = X_a(x) + ix^{\frac{1}{24}} \int_0^\infty \frac{x^{it}}{e^{-\frac{1}{2}\pi i+2\pi t}-1} \psi_a(\tfrac{1}{24}+it)\,dt$$
$$+ ix^{\frac{1}{24}} \int_0^\infty \frac{x^{-it}}{1-e^{\frac{1}{2}\pi i+2\pi t}} \psi_a(\tfrac{1}{24}-it)\,dt$$
$$= X_a(x) + \Theta_1(x) + \Theta_2(x),$$

say, where
$$X_a(x) = ix^{\frac{1}{24}} \int_0^\infty x^{it} \psi_a(\tfrac{1}{24}+it)\,dt.$$

Now, since
$$|x^{it}| = e^{-\theta t}, \quad |x^{-it}| = e^{\theta t},$$

the functions Θ are regular throughout the angle of Lemma 4.42. And

$$X_a(x) = \frac{x^{\frac{1}{24}}}{\sqrt{i}} \int_0^\infty e^{-\lambda t} \frac{d}{dt}\left(\frac{\cosh \mu\sqrt{t} - 1}{\sqrt{t}}\right) dt,$$

where
$$\lambda = i \log \frac{1}{x}, \quad \mu = a\sqrt{i}.$$

The form of this integral may be calculated by supposing λ and μ positive, when we obtain

$$\int_0^\infty e^{-\lambda w^2} \frac{d}{dw}\left(\frac{\cosh \mu w - 1}{w}\right) dw = 2\lambda \int_0^\infty e^{-\lambda w^2}(\cosh \mu w - 1)\,dw$$
$$= \sqrt{(\lambda\pi)}(e^{\mu^2/4\lambda} - 1).$$

Hence

$$(4.422) \qquad X_a(x) = \sqrt{\{\pi \log (1/x)\}}\, x^{\frac{1}{24}} \left[\exp \left\{\frac{a^2}{4 \log(1/x)}\right\} - 1\right] = \chi_a(x),$$

and the proof of the lemma is completed.

---

* Both $F_a(x)$ and $\chi_a(x)$ are two-valued in D. The value of $F_a(x)$ contemplated is naturally that represented by the power series.

Lemmas 4.41 and 4.42 show that $x = 1$ is the sole finite singularity of the principal branch of $F_a(x)$.

4.43. LEMMA 4.43.—*Suppose that* P, θ_1, *and* A *are positive constants,* θ_1 *being less than* π. *Then*

$$|F_a(x)| < K = K(\mathrm{P}, \theta_1, A),$$

for $$0 \leqslant r \leqslant \mathrm{P}, \quad \theta_1 \leqslant \theta \leqslant 2\pi - \theta_1, \quad 0 < a \leqslant A.$$

We use K generally to denote a positive number independent of x and of a. We may employ the formula (4.411). It is plain that

$$\left| \frac{x^z}{1-e^{2\pi i z}} \right| < Ke^{-\theta_1|\eta|},$$

$$|\psi_a(z)| = \left| \frac{d}{dz} \left\{ \frac{\cosh a\sqrt{(z-\frac{1}{24})}-1}{\sqrt{(z-\frac{1}{24})}} \right\} \right| < Ke^{K\sqrt{|\eta|}},$$

where η is the imaginary part of z. Hence

$$|F_a(x)| < K \int_{-\infty}^{\infty} e^{K\sqrt{|\eta|}-\theta_1|\eta|}\, d\eta < K.$$

4.44. LEMMA 4.44.—*Let* c *be a circle whose centre is* $x = 1$, *and whose radius* δ *is less than unity. Then*

$$|F_a(x) - \chi_a(x)| < Ka^2,$$

if x *lies in* c *and* $0 < a \leqslant A$, $K = K(\delta, A)$ *being as before independent of* x *and of* a.

If we refer back to (4.421) and (4.422), we see that it is sufficient to prove that

$$|\Theta_1(x)| < Ka^2, \qquad |\Theta_2(x)| < Ka^2;$$

and we may plainly confine ourselves to the first of these inequalities. We have

$$\Theta_1(x) = \frac{x^{\frac{1}{24}}}{\sqrt{i}} \int_0^{\infty} \frac{x^{it}}{e^{-\frac{1}{12}\pi i+2\pi t}-1} \frac{d}{dt} \left\{ \frac{\cosh a\sqrt{(it)}-1}{\sqrt{t}} \right\} dt.$$

Rejecting the extraneous factor, which is plainly without importance, and integrating by parts, we obtain

$$\Theta(x) = \int_0^{\infty} \Phi(t) \frac{\cosh a\sqrt{(it)}-1}{\sqrt{t}}\, dt,$$

where
$$\Phi(t) = -\frac{ix^{it}\log x}{e^{-\frac{1}{12}\pi i+2\pi t}-1}+\frac{2\pi x^{it}e^{-\frac{1}{12}\pi i+2\pi t}}{(e^{-\frac{1}{12}\pi i+2\pi t}-1)^2}.$$

Now $|\theta| < \frac{1}{2}\pi$ and $|x^{it}| < Ke^{\frac{1}{2}\pi t}$. It follows that

$$|\Phi(t)| < Ke^{-\pi t};$$

and
$$|\Theta(x)| < K\int_0^\infty \frac{e^{-\pi t}}{\sqrt{t}}\,|\sinh^2 \tfrac{1}{2}a\sqrt{(it)}|\,dt$$
$$< K\int_0^\infty \frac{e^{-\pi t}}{\sqrt{t}}\left\{\cosh a\sqrt{(\tfrac{1}{2}t)}-\cos a\sqrt{(\tfrac{1}{2}t)}\right\}dt$$
$$< K\int_0^\infty e^{-\pi w^2}\left(\cosh\frac{aw}{\sqrt{2}}-\cos\frac{aw}{\sqrt{2}}\right)dw$$
$$= K(e^{a^2/8\pi}-e^{-a^2/8\pi}) < Ka^2.$$

5.

PROOF OF THE MAIN THEOREM.

5.1. We write

$$F_{p,q}(x) = \omega_{p,q}\frac{\sqrt{q}}{\pi\sqrt{2}}F_{C/q}(x_{p,q}), \tag{5.11}$$

where $C = \pi\sqrt{\frac{2}{3}}$, $x_{p,q} = xe^{-2p\pi i/q}$; and

$$\Phi(x) = f(x) - \sum_q\sum_p F_{p,q}(x), \tag{5.12}$$

where the summation applies to all values of p not exceeding q and prime to q, and to all values of q such that

$$1 \leqslant q \leqslant \nu = [a\sqrt{n}], \tag{5.13}$$

a being positive and independent of n. If then

$$F_{p,q}(x) = \Sigma c_{p,q,n}x^n, \tag{5.14}$$

we have

$$p(n) - \sum_q\sum_p c_{p,q,n} = \frac{1}{2\pi i}\int_\Gamma \frac{\Phi(x)}{x^{n+1}}\,dx, \tag{5.15}$$

where Γ is a circle whose centre is the origin and whose radius R is less than unity. We take

$$R = 1 - \frac{\beta}{n}, \tag{5.16}$$

where β also is positive and independent of n.

Our object is to show that the integral on the right hand side of (5 . 15) is of the form $O(n^{-\frac{1}{4}})$; the constant implied in the O will of course be a function of a and β. It is to be understood throughout that O's are used in this sense; $O(1)$, for instance, stands for *a function of* x, n, p, q, a, *and* β (or of some only of these variables) *which is less in absolute value than a number* $K = K(a, \beta)$ *independent of* x, n, p, *and* q.

We divide up the circle Γ, by means of the dissection Ξ_ν of 4 . 23, into arcs $\xi_{p,q}$ each associated with a point $Re^{2p\pi i/q}$; and we denote by $\eta_{p,q}$ the arc of Γ complementary to $\xi_{p,q}$. This being so, we have

$$\int_\Gamma \frac{\Phi(x)}{x^{n+1}}\,dx = \Sigma \int_{\xi_{p,q}} \frac{f(x)-F_{p,q}(x)}{x^{n+1}}\,dx - \Sigma \int_{\eta_{p,q}} \frac{F_{p,q}(x)}{x^{n+1}}\,dx \tag{5.17}$$

$$= \Sigma J_{p,q} - \Sigma j_{p,q},$$

say. We shall prove that each of these sums is of the form $O(n^{-\frac{1}{4}})$; and we shall begin with the second sum, which only involves the auxiliary functions F.

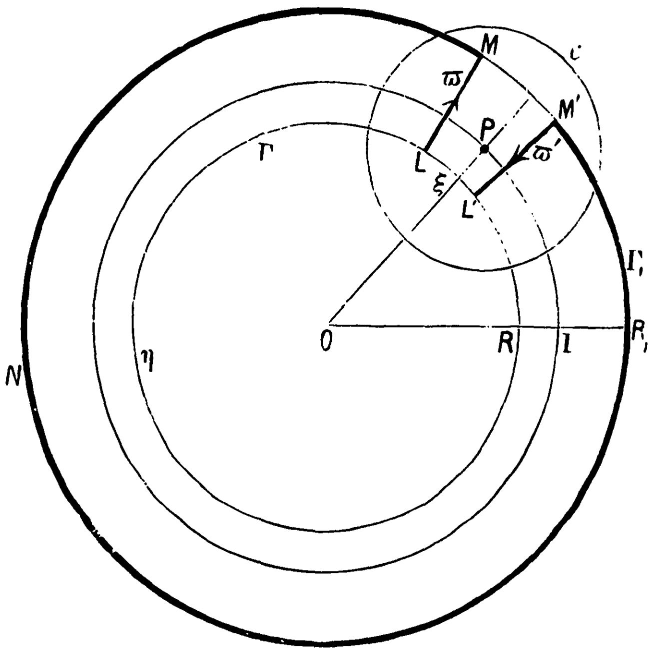
Proof that $\Sigma j_{p,q} = O(n^{-\frac{1}{4}})$.

5 . 21. We have, by Cauchy's theorem,

$$j_{p,q} = \int_{\eta_{p,q}} \frac{F_{p,q}(x)}{x^{n+1}}\,dx = \int_{\zeta_{p,q}} \frac{F_{p,q}(x)}{x^{n+1}}\,dx, \tag{5.211}$$

where $\zeta_{p,q}$ consists of the contour $LMNM'L'$ shown in the figure. Here L and L' are the ends of $\xi_{p,q}$, LM and $M'L'$ are radii vectores, and MNM' is part of a circle Γ_1 whose radius R_1 is greater than 1. P is the point $e^{2p\pi i/q}$; and we suppose that R_1 is small enough to ensure that all points of LM and $M'L'$ are at a distance from P less than $\frac{1}{2}$. The other circle c shown in the figure has P as its centre and radius $\frac{1}{2}$. We denote LM by $\varpi_{p,q}$, $M'L'$ by $\varpi'_{p,q}$, and MNM' by $\gamma_{p,q}$: and we write

$$j_{p,q} = \int_{\zeta_{p,q}} = \int_{\gamma_{p,q}} + \int_{\varpi_{p,q}} + \int_{\varpi'_{p,q}} = j^1_{p,q} + j^2_{p,q} + j^3_{p,q}. \tag{5.212}$$



The contribution of $\Sigma j^1_{p,q}$.

5 . 22. Suppose first that x lies on $\gamma_{p,q}$ and outside c. Then, in virtue of (5 . 11) and Lemma 4 . 43, we have

(5 . 221) $$F_{p,q}(x) = O(\sqrt{q}).$$

If on the other hand x lies on $\gamma_{p,q}$, but inside c, we have, by (5 . 11) and Lemma 4 . 44,

(5 . 222) $$F_{p,q}(x) - \chi_{p,q}(x) = O(q^{-3}),$$

where

(5 . 2221) $$\chi_{p,q}(x) = \omega_{p,q}\frac{\sqrt{q}}{\pi\sqrt{2}}\chi_{C/q}(x_{p,q}).$$

But, if we recur to the definition of $\chi_a(x)$ in Lemma 4 . 42, and observe that

$$\left|\exp\frac{a^2}{4\log(1/x)}\right| = \exp\frac{a^2\log(1/r)}{4[\{\log(1/r)\}^2+\theta^2]} < 1$$

if $x = re^{i\theta}$ and $r > 1$, we see that

(5 . 223) $$\chi_{p,q}(x) = O(\sqrt{q})$$

on the part of $\gamma_{p,q}$ in question. Hence (5.221) holds for all $\gamma_{p,q}$. It follows that

$$j^1_{p,q} = O(R_1^{-n}\sqrt{q}),$$

(5.224) $$\Sigma j_{p,q} = O(R_1^{-n} \sum_q q^{\frac{3}{2}}) = O(n^{\frac{5}{4}} R_1^{-n}).*$$

This sum tends to zero more rapidly than any power of n, and is therefore completely trivial.

The contributions of $\Sigma j^2_{p,q}$ and $\Sigma j^3_{p,q}$.

5.231. We must now consider the sums which arise from the integrals along $\varpi_{p,q}$ and $\varpi'_{p,q}$; and it is evident that we need consider in detail only the first of these two lines. We write

(5.2311) $$j^2_{p,q} = \int_{\varpi_{p,q}} \frac{F_{p,q}(x) - \chi_{p,q}(x)}{x^{n+1}}\,dx + \int_{\varpi_{p,q}} \frac{\chi_{p,q}(x)}{x^{n+1}}\,dx = j'_{p,q} + j''_{p,q},$$

say.

In the first place we have, from (5.222),

$$j'_{p,q} = O\left(q^{-\frac{3}{2}} \int_R^{R_1} \frac{dr}{r^{n+1}}\right) = O(q^{-\frac{3}{2}} n^{-1}),$$

since

(5.2312) $$R^{-n} = \left(1 - \frac{\beta}{n}\right)^{-n} = O(1).$$

Thus

(5.2313) $$\Sigma j'_{p,q} = O\left\{n^{-1} \sum_{q<O(\sqrt{n})} q^{-\frac{1}{2}}\right\} = O(n^{-\frac{3}{4}}).$$

5.232. In the second place we have

$$j''_{p,q} = \omega_{p,q} \frac{\sqrt{q}}{\pi\sqrt{2}} \int_{\varpi_{p,q}} \frac{\chi_{c/q}(x_{p,q})}{x^{n+1}}\,dx.$$

It is plain that, if we substitute y for $xe^{-2p\pi i/q}$, then write x again for y, and finally substitute for $\chi_{c/q}$ its explicit expression as an elementary

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* Here, and in many passages in our subsequent argument, it is to be remembered that the number of values of p, corresponding to a given q, is less than q, and that the number of values of q is of order $\sqrt{n}$. Thus we have generally

$$\Sigma\, O(q^s) = O\left(\sum_{q<O(\sqrt{n})} q^{s+1}\right) = O(n^{\frac{1}{2}s+1}).$$

function, given in Lemma 4 . 42, we obtain

$$(5\,.\,2321)\qquad j''_{p,q} = O(\sqrt{q})\int \{E(x)-1\}\sqrt{\left(\log\frac{1}{x}\right)}\, x^{-n-\frac{23}{24}}\,dx = O(\sqrt{q})\,J,$$

say, where

$$(5\,.\,23211)\qquad E(x) = \exp\left\{\frac{\pi^2}{6q^2\log(1/x)}\right\},$$

and the path of integration is now a line related to $x = 1$ as $\varpi_{p,q}$ is to $x = e^{2p\pi i/q}$: the line defined by $x = re^{i\theta}$, where $R \leqslant r \leqslant R_1$, and θ is fixed and (by Lemma 4 . 22) lies between $1/2q\nu$ and $1/q\nu$.

Integrating J by parts, we find

$$(5\,.\,2322)\qquad (n-\tfrac{1}{24})\,J = -\left[\{E(x)-1\}\sqrt{\left(\log\frac{1}{x}\right)}\, x^{-n+\frac{1}{24}}\right]_{r=R}^{r=R_1}$$

$$-\tfrac{1}{2}\int\{E(x)-1\}\left(\log\frac{1}{x}\right)^{-\frac{1}{2}} x^{-n-\frac{23}{24}}\,dx$$

$$+\frac{\pi^2}{6q^2}\int E(x)\left(\log\frac{1}{x}\right)^{-\frac{3}{2}} x^{-n-\frac{23}{24}}\,dx = J_1+J_2+J_3,$$

say.

5 . 233. In estimating J_1, J_2, and J_3, we must bear the following facts in mind.

(1) Since $|x| \geqslant R$, it follows from (5 . 2312) that $|x|^{-n} = O(1)$ throughout the range of integration.

(2) Since $1-R = \beta/n$ and $1/2q\nu < \theta < 1/q\nu$, where $\nu = [a\sqrt{n}]$, we have

$$\log\left(\frac{1}{x}\right) = O\left(\frac{1}{q\sqrt{n}}\right),$$

when $r = R$, and
$$\frac{1}{\log(1/x)} = O(q\sqrt{n}),$$

throughout the range of integration.

(3) Since
$$|E(x)| = \exp\frac{\pi^2\log(1/r)}{6q^2[\{\log(1/r)\}^2+\theta^2]},$$

$E(x)$ is less than 1 in absolute value when $r > 1$. And, on the part of the path for which $r < 1$, it is of the form

$$\exp O\left(\frac{1}{q^2n\theta^2}\right) = \exp O(1) = O(1).$$

It is accordingly of the form $O(1)$ throughout the range of integration.

5 . 234. Thus we have, first

(5 . 2341) $J_1 = O(1)\, O(1)\, O(R_1^{-n}) + O(1)\, O(q^{-\frac{1}{2}} n^{-\frac{1}{4}})\, O(1) = O(q^{-\frac{1}{2}} n^{-\frac{1}{4}})$,

secondly

(5 . 2342) $$J_2 = O(1)\, O(q^{\frac{1}{2}} n^{\frac{1}{4}}) \int_R^{R_1} \frac{dr}{r^{n+\frac{2}{3}}} = O(q^{\frac{1}{2}} n^{-\frac{3}{4}}),$$

and thirdly

(5 . 2343) $$J_3 = O(q^{-2})\, O(1)\, O(q^{\frac{3}{2}} n^{\frac{3}{4}}) \int_R^{R_1} \frac{dr}{r^{n+\frac{2}{3}}} = O(q^{-\frac{1}{2}} n^{-\frac{1}{4}}).$$

From (5 . 2341), (5 . 2342), (5 . 2343), and (5 . 2322), we obtain

$$J = O(q^{-\frac{1}{2}} n^{-\frac{1}{4}}) + O(q^{\frac{1}{2}} n^{-\frac{3}{4}});$$

and, from (5 . 2321), $j''_{p,q} = O(n^{-\frac{5}{4}}) + O(q n^{-\frac{7}{4}})$.

Summing, we obtain

(5 . 2344) $$\Sigma j''_{p,q} = O\Big(n^{-\frac{5}{4}} \sum_{q < O(\sqrt{n})} q\Big) + O\Big(n^{-\frac{7}{4}} \sum_{q < O(\sqrt{n})} q^2\Big)$$
$$= O(n^{-\frac{1}{4}}) + O(n^{-\frac{1}{4}}) = O(n^{-\frac{1}{4}}).$$

5 . 235. From (5 . 2311), (5 . 2313), and (5 . 2344), we obtain

(5 . 2351) $$\Sigma j^2_{p,q} = O(n^{-\frac{1}{4}});$$

and in exactly the same way we can prove

(5 . 2352) $$\Sigma j^3_{p,q} = O(n^{-\frac{1}{4}}).$$

And from (5 . 212), (5 . 224), (5 . 2351), and (5 . 2352) we obtain, finally,

(5 . 2353) $$\Sigma j_{p,q} = O(n^{-\frac{1}{4}}).$$

Proof that $\Sigma J_{p,q} = O(n^{-\frac{1}{4}})$.

5 . 31. We turn now to the discussion of

(5 . 311) $$J_{p,q} = \int_{\xi_{p,q}} \frac{f(x) - F_{p,q}(x)}{x^{n+1}}\, dx$$
$$= \int_{\xi_{p,q}} \frac{f(x) - X_{p,q}(x)}{x^{n+1}}\, dx - \int_{\xi_{p,q}} \frac{F_{p,q}(x) - \chi_{p,q}(x)}{x^{n+1}}\, dx + \int_{\xi_{p,q}} \frac{\rho_{p,q}(x)}{x^{n+1}}\, dx$$
$$= J^1_{p,q} + J^2_{p,q} + J^3_{p,q},$$

say, where
$$\rho_{p,q}(x) = \omega_{p,q} \sqrt{\left(\frac{q}{2\pi} \log \frac{1}{x_{p,q}}\right)}\, x_{p,q}^{\frac{1}{24}},$$
$$X_{p,q}(x) = \chi_{p,q}(x) + \rho_{p,q}(x) = \rho_{p,q}(x)\, E(x_{p,q}),$$
$E(x)$ being defined as in (5 . 23211).

Discussion of $\Sigma J^2_{p,q}$ and $\Sigma J^3_{p,q}$.

5 . 32. The discussion of these two sums is, after the analysis which precedes, a simple matter. The arc $\xi_{p,q}$ is less than a constant multiple of $1/q\sqrt{n}$; and $x^{-n} = O(1)$ on $\xi_{p,q}$. Also
$$|F_{p,q}(x) - \chi_{p,q}(x)| = O(q^{-\frac{3}{2}}),$$
by (5 . 222); and

(5 . 321)
$$\sqrt{\left(\log \frac{1}{x_{p,q}}\right)} = O(q^{-\frac{1}{2}} n^{-\frac{1}{4}}),$$
since
$$|x_{p,q}| = R = 1 - (\beta/n), \qquad |am\ x_{p,q}| < 1/q\nu.$$
Hence
$$J^2_{p,q} = O(q^{-\frac{5}{2}} n^{-\frac{1}{2}}),$$
(5 . 322)
$$\Sigma J^2_{p,q} = O\left(n^{-\frac{1}{2}} \sum_{q < O(\sqrt{n})} q^{-\frac{5}{2}}\right) = O(n^{-\frac{1}{2}});$$
and
$$J^3_{p,q} = O(q^{-1} n^{-\frac{3}{4}}),$$
(5 . 323)
$$\Sigma J^3_{p,q} = O\left(n^{-\frac{3}{4}} \sum_{q < O(\sqrt{n})} 1\right) = O(n^{-\frac{1}{4}}).$$

Discussion of $\Sigma J^1_{p,q}$.

5 . 33. From (4 . 321) and (5 . 2221), we have

(5 . 331)
$$f(x) - X_{p,q}(x) = \omega_{p,q} \sqrt{\left\{\frac{q}{2\pi} \log\left(\frac{1}{x_{p,q}}\right)\right\}}\, x_{p,q}^{\frac{1}{24}}\, E(x_{p,q})\, \Omega(x'_{p,q}),$$
where
$$\Omega(z) = f(z) - 1 = \prod_{1}^{\infty} \left(\frac{1}{1-z^\nu}\right) - 1 = \sum_{1}^{\infty} p(\nu)\, z^\nu,$$
if $|z| < 1$, and
$$x'_{p,q} = \exp\left\{-\frac{4\pi^2}{q^2 \log(1/x_{p,q})} + \frac{2\pi i p'}{q}\right\}.$$
Now
$$|x'_{p,q}| = \exp\left[-\frac{4\pi^2 \log(1/R)}{q^2 \{[\log(1/R)]^2 + \theta^2\}}\right],$$

where θ is the amplitude of $x_{p,q}$. Also

$$q^2\{[\log(1/R)]^2+\theta^2\} = O\left\{q^2\left(\frac{1}{n^2}+\frac{1}{q^2 n}\right)\right\} = O\left(\frac{1}{n}\right),$$

while $\log(1/R)$ is greater than a constant multiple of $1/n$. There is therefore a positive number δ, less than unity and independent of n and of q, such that

$$|x'_{p,q}| < \delta;$$

and we may write $$\Omega(x'_{p,q}) = O(|x'_{p,q}|).$$

We have therefore

$$E(x_{p,q})\,\Omega(x'_{p,q}) = O(|x'_{p,q}|^{-\frac{1}{24}})\,O(|x'_{p,q}|) = O(|x'_{p,q}|^{\frac{23}{24}}) = O(1);$$

and so, by (5 . 321),

$$f(x)-\chi_{p,q}(x) = O(\sqrt{q})\,O\left(\sqrt{\left|\log\frac{1}{x_{p,q}}\right|}\right)O(1) = O(n^{-\frac{1}{4}}).$$

And hence, as the length of $\xi_{p,q}$ is of the form $O(1/q\sqrt{n})$, we obtain

$$J^1_{p,q} = O(q^{-1}n^{-\frac{3}{4}}),$$

(5 . 332) $$\Sigma J^1_{p,q} = O\left(n^{-\frac{3}{4}}\sum_{q<O(\sqrt{n})} 1\right) = O(n^{-\frac{1}{4}}).$$

5 . 34. From (5 . 311), (5 . 322), (5 . 323), and (5 . 332), we obtain

(5 . 341) $$\Sigma J_{p,q} = O(n^{-\frac{1}{4}}).$$

Completion of the proof.

5 . 4. From (5 . 15), (5 . 17), (5 . 2353), and (5 . 341), we obtain

(5 . 41) $$p(n)-\sum_q\sum_p c_{p,q,n} = O(n^{-\frac{1}{4}}).$$

But $$\sum_p c_{p,q,n} = \frac{\sqrt{q}}{\pi\sqrt{2}}A_q\frac{d}{dn}\frac{\cosh(C\lambda_n/q)-1}{\lambda_n},$$

where $$A_q = \sum_p \omega_{p,q}\,e^{-2np\pi i/q}.$$

All that remains, in order to complete the proof of the theorem, is to show that

$$\cosh(C\lambda_n/q)-1$$

may be replaced by $$\tfrac{1}{2}e^{C\lambda_n/q};$$

and in order to prove this it is only necessary to show that

$$\sum_{q<O(\sqrt{n})} q^{\frac{3}{2}} \frac{d}{dn} \frac{\frac{1}{2}e^{C\lambda_n/q}-\cosh(C\lambda_n/q)+1}{\lambda_n} = O(n^{-\frac{1}{4}}).$$

On differentiating we find that the sum is of the form

$$\sum_{q<O(\sqrt{n})} q^{\frac{3}{2}} \left\{ O\left(\frac{1}{qn}\right)+O\left(\frac{1}{n^{\frac{3}{2}}}\right) \right\} = O\left\{\frac{1}{n} \sum_{q<O(\sqrt{n})} q^{\frac{1}{2}}\right\} = O(n^{-\frac{1}{4}}).$$

Thus the theorem is proved.

6.

ADDITIONAL REMARKS ON THE THEOREM.

6 . 1. The theorem which we have proved gives information about $p(n)$ which is in some ways extraordinarily exact. We are for this reason the more anxious to point out explicitly two respects in which the results of our analysis are incomplete.

6 . 21. We have proved that

$$p(n) = \Sigma A_q \phi_q + O(n^{-\frac{1}{4}}),$$

where the summation extends over the values of q specified in the theorem, for every fixed value of α; that is to say that, when α is given, a number $K = K(\alpha)$ can be found such that

$$|\, p(n) - \Sigma A_q \phi_q \,| < K n^{-\frac{1}{4}}$$

for every value of n. It follows that

$$p(n) = \{\Sigma A_q \phi_q\}, \tag{6 . 211}$$

where $\{x\}$ denotes the integer nearest to x, for $n \geqslant n_0$, where $n_0 = n_0(\alpha)$ is a certain function of α.

The question remains whether we can, by an appropriate choice of α, secure the truth of (6 . 211) for *all* values of n, and not merely for all sufficiently large values. Our opinion is that this is possible, and that it could be proved to be possible without any fundamental change in our analysis. Such a proof would however involve a very careful revision of our argument. It would be necessary to replace all formulæ involving O's by inequalities, containing only numbers expressed explicitly as functions of the various parameters employed. This process would certainly add very considerably to the length and the complexity of our argument. It is, as it stands, sufficient to prove what is, from our point of view, of the greatest interest; and we have not thought it worth while to elaborate it further.

6.22. The second point of incompleteness of our results is of much greater interest and importance. We have not proved either that the series

$$\sum_{1}^{\infty} A_q \phi_q$$

is convergent, or that, if it is convergent, it represents $p(n)$. Nor does it seem likely that our method is one intrinsically capable of proving these results, if they are true—a point on which we are not prepared to express any definite opinion.

It should be observed in this connection that we have not even discovered anything definite concerning the order of magnitude of A_q for large values of q. We can prove nothing better than the absolutely trivial equation $A_q = O(q)$. On the other hand we cannot assert that A_q is, for an infinity of values of q, effectively of an order as great as q, or indeed even that it does not tend to zero (though of course this is most unlikely).

6.3. Our formula directs us, if we wish to obtain the exact value of $p(n)$ for a large value of n, to take a number of terms of order $\sqrt{n}$. The numerical data suggest that a considerably smaller number of terms will be equally effective; and it is easy to see that this conjecture is correct.

Let us write

$$\beta = 4\pi\sqrt{\left(\tfrac{2}{3}\right)} = 4C, \qquad \mu = \left[\frac{\beta\sqrt{n}}{\log n}\right],$$

and let us suppose that $\alpha < 2$. Then

$$\sum_{\mu+1}^{\nu} A_q\phi_q = \sum_{\mu+1}^{\nu} O(q^{\frac{3}{2}})\, O\left(\frac{1}{qn}\right) O(e^{C\sqrt{n}/q}) = O\left(\frac{1}{n}\sum_{\mu+1}^{\nu} \sqrt{q}\, e^{C\sqrt{n}/q}\right)$$

$$= O\left(\frac{1}{n}\int_{\mu}^{\nu} \sqrt{x}\, e^{C\sqrt{n}/x}\, dx\right),$$

since $\sqrt{q}\, e^{C\sqrt{n}/q}$ decreases steadily throughout the range of summation.*

Writing $\sqrt{n}/y$ for x, we obtain

$$O\left(n^{-\frac{1}{4}}\int_{1/\alpha}^{\sqrt{n}/\mu} y^{-\frac{5}{2}} e^{Cy}\, dy\right) = O\left\{n^{-\frac{1}{4}}\left(\frac{\sqrt{n}}{\mu}\right)^{-\frac{5}{2}} e^{C\sqrt{n}/\mu}\right\} = O\left\{n^{-\frac{1}{4}}(\log n)^{-\frac{5}{2}} e^{\frac{1}{4}\log n}\right\}$$

$$= O(\log n)^{-\frac{5}{2}} = o(1).$$

* The minimum occurs when q is about equal to $2C\sqrt{n}$.

It follows that it is enough, when n is sufficiently large, to take

$$\left[\frac{\beta\sqrt{n}}{\log n}\right]$$

terms of the series. It is probably also *necessary* to take a number of terms of order $\sqrt{n}/(\log n)$; but it is not possible to prove this rigorously without a more exact knowledge of the properties of A_q than we possess.

6.4. We add a word on certain simple approximate formulæ for $\log p(n)$ found empirically by Major MacMahon and by ourselves. Major MacMahon found that if

$$\log_{10} p(n) = \sqrt{(n+4)} - a_n, \tag{6.41}$$

then a_n is approximately equal to 2 within the limits of his table of values of $p(n)$ (Table IV). This suggested to us that we should endeavour to find more accurate formulæ of the same type. The most striking that we have found is

$$\log_{10} p(n) = \tfrac{10}{9}\{\sqrt{(n+10)} - a_n\}; \tag{6.42}$$

the mode of variation of a_n is shown in Table III.

In this connection it is interesting to observe that the function

$$13^{-\sqrt{n}}\, p(n)$$

(which ultimately tends to infinity with exponential rapidity) is equal to $\cdot 973$ for $n = 30000000000$.

7.

FURTHER APPLICATIONS OF THE METHOD.

7.1. We shall conclude with a few remarks concerning actual or possible applications of our method to other problems in Combinatory Analysis or the Analytic Theory of Numbers.

The class of problems in which the method gives the most striking results may be defined as follows. Suppose that $q(n)$ is the coefficient of x^n in the expansion of $F(x)$, where $F(x)$ is a function of the form

$$\frac{\{f(\pm x^a)\}^{\alpha}\{f(\pm x^{a'})\}^{\alpha'}\dots}{\{f(\pm x^b)\}^{\beta}\{f(\pm x^{b'})\}^{\beta'}\dots};* \tag{7.11}$$

* Since $$f(-x) = \frac{\{f(x^2)\}^3}{f(x)\, f(x^4)},$$ the arguments with a negative sign may be eliminated if this is desired.

$f(x)$ being the function considered in this paper, the a's, b's, α's, and β's being positive integers, and the number of factors in numerator and denominator being finite; and suppose that $|F(x)|$ tends exponentially to infinity when x tends in an appropriate manner to some or all of the points $e^{2p\pi i/q}$. Then our method may be applied in its full power to the asymptotic study of $q(n)$, and yields results very similar to those which we have found concerning $p(n)$.

Thus, if

$$F(x) = \frac{f(x)}{f(x^2)} = (1+x)(1+x^2)(1+x^3)\ldots = \frac{1}{(1-x)(1-x^3)(1-x^5)\ldots},$$

so that $q(n)$ is the number of partitions of n into odd parts, or into unequal parts*, we find that

$$q(n) = \frac{1}{\sqrt{2}}\frac{d}{dn} J_0[i\pi\sqrt{\{\tfrac{1}{3}(n+\tfrac{1}{24})\}}]$$
$$+\sqrt{2}\cos(\tfrac{2}{3}n\pi - \tfrac{1}{9}\pi)\frac{d}{dn} J_0[\tfrac{1}{3}i\pi\sqrt{\{\tfrac{1}{3}(n+\tfrac{1}{24})\}}]+\ldots.$$

The error after $[\alpha\sqrt{n}]$ terms is of the form $O(1)$. We are not in a position to assert that the *exact* value of $q(n)$ can always be obtained from the formula (though this is probable); but the error is certainly bounded.

If $$F(x) = \frac{f(x^2)}{f(-x)} = \frac{f(x)f(x^4)}{\{f(x^2)\}^2} = (1+x)(1+x^3)(1+x^5)\ldots,$$

so that $q(n)$ is the number of partitions of n into parts which are both odd and unequal, then

$$q(n) = \frac{d}{dn} J_0[i\pi\sqrt{\{\tfrac{1}{6}(n-\tfrac{1}{24})\}}]$$
$$+2\cos(\tfrac{2}{3}n\pi - \tfrac{2}{9}\pi)\frac{d}{dn} J_0[\tfrac{1}{3}i\pi\sqrt{\{\tfrac{1}{6}(n-\tfrac{1}{24})\}}]+\ldots.$$

The error is again bounded (and probably tends to zero).

If $$F(x) = \frac{\{f(x)\}^2}{f(x^2)} = \frac{1}{1-2x+2x^4-2x^9+\ldots},$$

$q(n)$ has no very simple arithmetical interpretation; but the series is none the less, as the direct reciprocal of a simple ϑ-function, of particular

* Cf. MacMahon, *loc. cit.*, p. 11. We give at the end of the paper a table (Table V) of the values of $q(n)$ up to $n = 100$. This table was calculated by Mr. Darling.

interest. In this case we find

$$q(n) = \frac{1}{4\pi}\frac{d}{dn}\frac{e^{\pi\sqrt{n}}}{\sqrt{n}} + \frac{\sqrt{3}}{2\pi}\cos(\tfrac{2}{3}n\pi - \tfrac{1}{6}\pi)\frac{d}{dn}\frac{e^{\frac{1}{3}\pi\sqrt{n}}}{\sqrt{n}} + \dots .$$

The error here is (as in the partition problem) of order $O(n^{-\frac{1}{4}})$, and the exact value can always be found from the formula.

7 . 2. The method may also be applied to products of the form (7 . 11) which have (to put the matter roughly) no exponential infinities. In such cases the approximation is of a much less exact character. On the other hand the problems of this character are of even greater arithmetical interest.

The standard problem of this category is that of the representation of a number as the sum of s squares, s being any positive integer odd or even.* We must reserve the application of our method to this problem for another occasion; but we can indicate the character of our main result as follows.

If $r_s(n)$ is the number of representations of n as the sum of s squares, we have

$$F(x) = \Sigma r_s(n)\, x^n = (1+2x+2x^4+\dots)^s = \frac{\{f(x^2)\}^s}{\{f(-x)\}^{2s}} = \frac{\{f(x)\}^{2s}\{f(x^4)\}^{2s}}{\{f(x^2)\}^{5s}}.$$

We find that

(7 . 21) $$r_s(n) = \frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)} n^{\frac{1}{2}s-1}\, \Sigma \frac{c_q}{q^{\frac{1}{2}s}} + O(n^{\frac{1}{4}s}),$$

where c_q is a function of q and of n of the same general type as the function A_q of this paper. The series

(7 . 22) $$\Sigma \frac{c_q}{q^{\frac{1}{2}s}}$$

is absolutely convergent for sufficiently large values of s, and the summation in (7 . 21) may be regarded indifferently as extended over all values of q or only over a range $1 \leqslant q \leqslant a\sqrt{n}$. It should be observed that the series (7 . 22) is quite different in form from any of the infinite series which are already known to occur in connection with this problem.

7 . 3. There is also a wide range of problems to which our methods

* As s well known, the arithmetical difficulties of the problem are much greater when s is odd.

are *partly* applicable. Suppose, for example, that

$$F(x) = \Sigma\, p^2(n)\, x^n = \frac{1}{(1-x)(1-x^4)(1-x^9)\ldots},$$

so that $p^2(n)$ is the number of partitions of n into *squares*. Then $F(x)$ is not an elliptic modular function; it possesses no general transformation theory: and the full force of our method cannot be applied. We can still, however, apply some of our preliminary methods. Thus the "Tauberian" argument shows that

$$\log p^2(n) \sim 2^{-\frac{4}{3}}\, 3\pi^{\frac{1}{3}} \left\{\zeta\left(\tfrac{3}{2}\right)\right\}^{\frac{2}{3}} n^{\frac{1}{3}}.$$

And although there is no general transformation theory, there is a formula which enables us to specify the nature of the singularity at $x = 1$. This formula is

$$\frac{1}{f(e^{-\pi z})} = 2\sqrt{\left(\frac{\pi}{z}\right)} \exp\left\{\frac{2\pi}{\sqrt{z}}\,\zeta\left(-\tfrac{1}{2}\right)\right\}$$

$$\times \prod_1^\infty \left\{1 - 2e^{-2\pi\sqrt{(n/z)}} \cos 2\pi\sqrt{(n/z)} + e^{-4\pi\sqrt{(n/z)}}\right\}.$$

By the use of this formula, in conjunction with Cauchy's theorem, it is certainly possible to obtain much more precise information about $p^2(n)$, and in particular the formula

$$p^2(n) \sim 3^{-\frac{1}{2}} (4\pi n)^{-\frac{7}{6}} \left\{\zeta\left(\tfrac{3}{2}\right)\right\}^{\frac{2}{3}} e^{2^{-\frac{4}{3}} 3\pi^{\frac{1}{3}} \{\zeta(\frac{3}{2})\}^{\frac{2}{3}} n^{\frac{1}{3}}}.$$

The corresponding formula for $p^s(n)$, the number of partitions of n into perfect s-th powers, is

$$p^s(n) \sim (2\pi)^{-\frac{1}{2}(s+1)} \sqrt{\left(\frac{s}{s+1}\right)}\, k n^{\frac{1}{s+1} - \frac{3}{2}}\, e^{(s+1)\, k n^{1/(s+1)}},$$

where

$$k = \left\{\frac{1}{s}\,\Gamma\left(1 + \frac{1}{s}\right) \zeta\left(1 + \frac{1}{s}\right)\right\}^{\frac{s}{s+1}}.$$

[The series in (7.21) may be written in the form

$$\frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)}\, n^{\frac{1}{2}s-1} \sum_{p,q} \frac{\omega_{p,q}^s}{q^{\frac{1}{2}s}}\, e^{-np\pi i/q},$$

where $\omega_{p,q}$ is always one of the five numbers 0, $e^{\frac{1}{4}\pi i}$, $e^{-\frac{1}{4}\pi i}$, $-e^{\frac{1}{4}\pi i}$, $-e^{-\frac{1}{4}\pi i}$. When s is even it begins

$$\frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)}\, n^{\frac{1}{2}s-1} \left\{1^{-\frac{1}{2}s} + 2\cos\left(\tfrac{1}{2}n\pi - \tfrac{1}{4}s\pi\right) 2^{-\frac{1}{2}s} + 2\cos\left(\tfrac{2}{3}n\pi - \tfrac{1}{2}s\pi\right) 3^{-\frac{1}{2}s} + \ldots\right\}.$$

It has been proved by Ramanujan that the series gives an *exact* representation of $r_s(n)$ when $s = 4, 6, 8$; and by Hardy that this is also true when $s = 3, 5, 7$. See Ramanujan, "On certain trigonometrical sums and their applications in the Theory of Numbers"; Hardy, "On the expression of a number as the sum of any number of squares, and in particular of five or seven".*—*Added April* 19*th*, 1918.]

TABLE I: $\omega_{p,q}$.

p	q	$\log \omega_{p,q}/\pi i$	p	q	$\log \omega_{p,q}/\pi i$	p	q	$\log \omega_{p,q}/\pi i$
1	1	0	3	11	3/22	8	15	7/18
1	2	0	4	,,	3/22	11	,,	−19/90
1	3	1/13	5	,,	−5/22	13	,,	−7/18
2	,,	−1/13	6	,,	5/22	14	,,	−1/90
1	4	1/8	7	,,	−3/22	1	16	−29/32
3	,,	−1/8	8	,,	−3/22	3	,,	−27/32
1	5	1/5	9	,,	−5/22	5	,,	−5/32
2	,,	0	10	,,	−15/22	7	,,	−3/32
3	,,	0	1	12	55/72	9	,,	3/32
4	,,	−1/5	5	,,	−1/72	11	,,	5/32
1	6	5/18	7	,,	1/72	13	,,	27/32
5	,,	−5/18	11	,,	−55/72	15	,,	29/32
1	7	5/14	1	13	11/13	1	17	−14/17
2	,,	1/14	2	,,	4/13	2	,,	8/17
3	,,	−1/14	3	,,	1/13	3	,,	5/17
4	,,	1/14	4	,,	−1/13	4	,,	0
5	,,	−1/14	5	,,	0	5	,,	1/17
6	,,	−5/14	6	,,	−4/13	6	,,	5/17
1	8	7/16	7	,,	4/13	7	,,	1/17
3	,,	1/16	8	,,	0	8	,,	−8/17
5	,,	−1/16	9	,,	1/13	9	,,	8/17
7	,,	−7/16	10	,,	−1/13	10	,,	−1/17
1	9	14/27	11	,,	−4/13	11	,,	−5/17
2	,,	4/27	12	,,	−11/13	12	,,	−1/17
4	,,	−4/27	1	14	13/14	13	,,	0
5	,,	4/27	3	,,	3/14	14	,,	−5/17
7	,,	−4/27	5	,,	3/14	15	,,	−8/17
8	,,	−14/27	9	,,	−3/14	16	,,	14/17
1	10	3/5	11	,,	−3/14	1	18	−20/27
3	,,	0	13	,,	−13/14	5	,,	2/27
7	,,	0	1	15	1/90	7	,,	−2/27
9	,,	−3/5	2	,,	7/18	11	,,	2/27
1	11	15/22	4	,,	19/90	13	,,	−2/27
2	,,	5/22	7	,,	−7/18	17	,,	20/27

* The first paper is in course of publication in the *Transactions of the Cambridge Philosophical Society*. An account of the second is to appear shortly in the *Proceedings of the National Academy of Sciences* (Washington, D.C.); see also *Records of Proceedings at Meetings*, March 1918.

TABLE II: A_q.

$A_1 = 1.$

$A_2 = \cos n\pi.$

$A_3 = 2\cos(\frac{2}{3}n\pi - \frac{1}{18}\pi).$

$A_4 = 2\cos(\frac{1}{2}n\pi - \frac{1}{8}\pi).$

$A_5 = 2\cos(\frac{2}{5}n\pi - \frac{1}{5}\pi) + 2\cos\frac{4}{5}n\pi.$

$A_6 = 2\cos(\frac{1}{3}n\pi - \frac{5}{18}\pi).$

$A_7 = 2\cos(\frac{2}{7}n\pi - \frac{5}{14}\pi) + 2\cos(\frac{4}{7}n\pi - \frac{1}{14}\pi) + 2\cos(\frac{6}{7}n\pi + \frac{1}{14}\pi).$

$A_8 = 2\cos(\frac{1}{4}n\pi - \frac{7}{16}\pi) + 2\cos(\frac{3}{4}n\pi - \frac{1}{16}\pi).$

$A_9 = 2\cos(\frac{2}{9}n\pi - \frac{14}{27}\pi) + 2\cos(\frac{4}{9}n\pi - \frac{4}{27}\pi) + 2\cos(\frac{8}{9}n\pi + \frac{4}{27}\pi).$

$A_{10} = 2\cos(\frac{1}{5}n\pi - \frac{3}{5}\pi) + 2\cos\frac{3}{5}n\pi.$

$A_{11} = 2\cos(\frac{2}{11}n\pi - \frac{15}{22}\pi) + 2\cos(\frac{4}{11}n\pi - \frac{5}{22}\pi) + 2\cos(\frac{6}{11}n\pi - \frac{3}{22}\pi) + 2\cos(\frac{8}{11}n\pi - \frac{3}{22}\pi)$
$\qquad + 2\cos(\frac{10}{11}n\pi + \frac{5}{22}\pi).$

$A_{12} = 2\cos(\frac{1}{6}n\pi - \frac{55}{72}\pi) + 2\cos(\frac{5}{6}n\pi + \frac{1}{72}\pi).$

$A_{13} = 2\cos(\frac{2}{13}n\pi - \frac{11}{13}\pi) + 2\cos(\frac{4}{13}n\pi - \frac{4}{13}\pi) + 2\cos(\frac{6}{13}n\pi - \frac{1}{13}\pi) + 2\cos(\frac{8}{13}n\pi + \frac{1}{13}\pi) + 2\cos\frac{10}{13}n$
$\qquad + 2\cos(\frac{12}{13}n\pi + \frac{4}{13}\pi).$

$A_{14} = 2\cos(\frac{1}{7}n\pi - \frac{13}{14}\pi) + 2\cos(\frac{3}{7}n\pi - \frac{3}{14}\pi) + 2\cos(\frac{5}{7}n\pi - \frac{3}{14}\pi).$

$A_{15} = 2\cos(\frac{2}{15}n\pi - \frac{1}{90}\pi) + 2\cos(\frac{4}{15}n\pi - \frac{7}{18}\pi) + 2\cos(\frac{8}{15}n\pi - \frac{19}{90}\pi) + 2\cos(\frac{14}{15}n\pi + \frac{7}{18}\pi).$

$A_{16} = 2\cos(\frac{1}{8}n\pi + \frac{26}{32}\pi) + 2\cos(\frac{3}{8}n\pi + \frac{27}{32}\pi) + 2\cos(\frac{5}{8}n\pi + \frac{5}{32}\pi) + 2\cos(\frac{7}{8}n\pi + \frac{3}{32}\pi).$

$A_{17} = 2\cos(\frac{2}{17}n\pi + \frac{14}{17}\pi) + 2\cos(\frac{4}{17}n\pi - \frac{8}{17}\pi) + 2\cos(\frac{6}{17}n\pi - \frac{5}{17}\pi) + 2\cos\frac{8}{17}n\pi + 2\cos(\frac{10}{17}n\pi - \frac{1}{17}\pi)$
$\qquad + 2\cos(\frac{12}{17}n\pi - \frac{5}{17}n\pi) + 2\cos(\frac{14}{17}n\pi - \frac{1}{17}\pi) + 2\cos(\frac{16}{17}n\pi + \frac{8}{17}\pi).$

$A_{18} = 2\cos(\frac{1}{9}n\pi + \frac{20}{27}\pi) + 2\cos(\frac{5}{9}n\pi - \frac{2}{27}\pi) + 2\cos(\frac{7}{9}n\pi + \frac{2}{27}\pi).$

It may be observed that

$A_5 = 0\ \big(n \equiv 1, 2 \pmod 5\big)$, $\qquad A_7 = 0\ \big(n \equiv 1, 3, 4 \pmod 7\big)$,

$A_{10} = 0\ \big(n \equiv 1, 2 \pmod 5\big)$, $\qquad A_{11} = 0\ \big(n \equiv 1, 2, 3, 5, 7 \pmod{11}\big)$,

$A_{13} = 0\ \big(n \equiv 2, 3, 5, 7, 9, 10 \pmod{13}\big)$, $\qquad A_{14} = 0\ \big(n \equiv 1, 3, 4 \pmod 7\big)$,

$A_{16} = 0\ \big(n \equiv 0 \pmod 2\big)$, $\qquad A_{17} = 0\ \big(n \equiv 1, 3, 4, 6, 7, 9, 13, 14 \pmod{17}\big)$;

while A_1, A_2, A_3, A_4, A_6, A_8, A_9, A_{12}, A_{15}, and A_{18} never vanish.

TABLE III: $\log_{10} p(n) = \frac{10}{9}\{\sqrt{(n+10)} - a_n\}$.

n	a_n	n	a_n
1	3·317	10000	4·148
3	3·176	30000	4·364
10	3·011	100000	4·448
30	2·951	300000	4·267
100	3·036	1000000	3·554
300	3·237	3000000	2·072
1000	3·537	10000000	−1·188
3000	3·838	30000000	−6·796
		∞	$-\infty$

TABLE IV*: $p(n)$.

1 ...	1	39 ...	31185	77 ...	10619863	115 ...	1064144451
2 ...	2	40 ...	37338	78 ...	12132164	116 ...	1188908248
3 ...	3	41 ...	44583	79 ...	13848650	117 ...	1327710076
4 ...	5	42 ...	53174	80 ...	15796476	118 ...	1482074143
5 ...	7	43 ...	63261	81 ...	18004327	119 ...	1653668665
6 ...	11	44 ...	75175	82 ...	20506255	120 ...	1844349560
7 ...	15	45 ...	89134	83 ...	23338469	121 ...	2056148051
8 ...	22	46 ...	105558	84 ...	26543660	122 ...	2291320912
9 ...	30	47 ...	124754	85 ...	30167357	123 ...	2552338241
10 ...	42	48 ...	147273	86 ...	34262962	124 ...	2841940500
11 ...	56	49 ...	173525	87 ...	38887673	125 ...	3163127352
12 ...	77	50 ...	204226	88 ...	44108109	126 ...	3519222692
13 ...	101	51 ...	239943	89 ...	49995925	127 ...	3913864295
14 ...	135	52 ...	281589	90 ...	56634173	128 ...	4351078600
15 ...	176	53 ...	329931	91 ...	64112359	129 ...	4835271870
16 ...	231	54 ...	386155	92 ...	72533807	130 ...	5371315400
17 ...	297	55 ...	451276	93 ...	82010177	131 ...	5964539504
18 ...	385	56 ...	526823	94 ...	92669720	132 ...	6620830889
19 ...	490	57 ...	614154	95 ...	104651419	133 ...	7346629512
20 ...	627	58 ...	715220	96 ...	118114304	134 ...	8149040695
21 ...	792	59 ...	831820	97 ...	133230930	135 ...	9035836076
22 ...	1002	60 ...	966467	98 ...	150198136	136 ...	10015581680
23 ...	1255	61 ...	1121505	99 ...	169229875	137 ...	11097645016
24 ...	1575	62 ...	1300156	100 ...	190569292	138 ...	12292341831
25 ...	1958	63 ...	1505499	101 ...	214481126	139 ...	13610949895
26 ...	2436	64 ...	1741630	102 ...	241265379	140 ...	15065878135
27 ...	3010	65 ...	2012558	103 ...	271248950	141 ...	16670689208
28 ...	3718	66 ...	2323520	104 ...	304801365	142 ...	18440293320
29 ...	4565	67 ...	2679689	105 ...	342325709	143 ...	20390982757
30 ...	5604	68 ...	3087735	106 ...	384276336	144 ...	22540654445
31 ...	6842	69 ...	3554345	107 ...	431149389	145 ...	24908858009
32 ...	8349	70 ...	4087968	108 ...	483502844	146 ...	27517052599
33 ..	10143	71 ...	4697205	109 ...	541946240	147 ...	30388671978
34 ...	12310	72 ...	5392783	110 ...	607163746	148 ...	33549419497
35 ...	14883	73 ...	6185689	111 ...	679903203	149 ...	37027355200
36 ...	17977	74 ...	7089500	112 ...	761002156	150 ...	40853235313
37 ...	21637	75 ...	8118264	113 ...	851376628	151 ...	45060624582
38 ...	26015	76 ...	9289091	114 ...	952050665	152 ...	49686288421

* The numbers in this table were calculated by Major MacMahon, by means of the recurrence formulæ obtained by equating coefficients in the identity

$$(1-x-x^2+x^5+x^7-x^{12}-x^{15}+\ldots)\sum_0^\infty p(n)\,x^n = 1.$$

We have verified the table by direct calculation up to $n = 158$. Our calculation of $p(200)$ from the asymptotic formula then seemed to render further verification unnecessary.

TABLE IV.—*Continued.*

153 ...	54770336324	165 ...	172389800255	177 ...	522115831195	189 ...	1527273599625
154 ...	60356673280	166 ...	189334822579	178 ...	571701605655	190 ...	1667727404093
155 ...	66493182097	167 ...	207890420102	179 ...	625846753120	191 ...	1820701100652
156 ...	73232243759	168 ...	228204732751	180 ...	684957390936	192 ...	1987276856363
157 ...	80630964769	169 ...	250438925115	181 ...	749474411781	193 ...	2168627105469
158 ...	88751778802	170 ...	274768617130	182 ...	819876908323	194 ...	2366022741845
159 ...	97662728555	171 ...	301384802048	183 ...	896684817527	195 ...	2580840212973
160 ...	107438159466	172 ...	330495499613	184 ...	980462880430	196 ...	2814570987591
161 ...	118159068427	173 ...	362326859895	185 ...	1071823774337	197 ...	3068829878530
162 ...	129913904637	174 ...	397125074750	186 ...	1171432692373	198 ...	3345365983698
163 ...	142798995930	175 ...	435157697830	187 ...	1280011042268	199 ...	3646072432125
164 ...	156919475295	176 ...	476715857290	188 ...	1398341745571	200 ..	3972999029388

TABLE V: $q(n)$.*

n	c_n	n	c_n	n	c_n	n	c_n
1 ...	1	26 ...	165	51 ...	4097	76 ...	53250
2 ...	1	27 ...	192	52 ...	4582	77 ...	58499
3 ...	2	28 ...	222	53 ...	5120	78 ...	64234
4 ...	2	29 ...	256	54 ...	5718	79 ...	70488
5 ...	3	30 ...	296	55 ...	6378	80 ...	77312
6 ...	4	31 ...	340	56 ...	7108	81 ...	84756
7 ...	5	32 ...	390	57 ...	7917	82 ...	92864
8 ...	6	33 ...	448	58 ...	8808	83 ...	101698
9 ...	8	34 ...	512	59 ...	9792	84 ...	111322
10 ...	10	35 ...	585	60 ...	10880	85 ...	121792
11 ...	12	36 ...	668	61 ...	12076	86 ...	133184
12 ...	15	37 ...	760	62 ...	13394	87 ...	145578
13 ...	18	38 ...	864	63 ...	14848	88 ...	159046
14 ...	22	39 ...	982	64 ...	16444	89 ...	173682
15 ...	27	40 ...	1113	65 ...	18200	90 ...	189586
16 ...	32	41 ...	1260	66 ...	20132	91 ...	206848
17 ..	38	42 ...	1426	67 ...	22250	92 ...	225585
18 ...	46	43 ...	1610	68 ...	24576	93 ...	245920
19 ...	54	44 ...	1816	69 ...	27130	94 ...	267968
20 ...	64	45 ...	2048	70 ...	29927	95 ...	291874
21 ...	76	46 ...	2304	71 ...	32992	96 ...	317788
22 ...	89	47 ...	2590	72 ...	36352	97 ...	345856
23 ...	104	48 ...	2910	73 ...	40026	98 ...	376256
24 ...	122	49 ...	3264	74 ...	44046	99 ...	409174
25 ...	142	50 ...	3658	75 ...	48446	100 ...	444793

* We are indebted to Mr. Darling for this table.