P such that the six angles such as APB are equal; and then gave a geometric proof, different from the above, that P gives the minimum Σ.

3918 [1939, 363]. *Proposed by B. M. Stewart, University of Wisconsin.*

Given a block in which are fixed k pegs and a set of n washers, no two alike in size, and arranged on one peg so that no washer is above a smaller washer. What is the minimum number of moves in which the n washers can be placed on another peg, if the washers must be moved one at a time, subject always to the condition that no washer be placed above a smaller washer?

For $k=3$ this problem is called "The tower of Hanoï" in Ball's *Mathematical Recreations*, and the solution is given as 2^n-1.

I. *Solution by J. S. Frame, Brown University, Providence, R. I.*

Halfway through the process of moving the n washers, the largest washer lies alone on its original peg, and the chosen final peg is free to receive it. The other $n-1$ washers are distributed among the $h=k-2$ auxiliary pegs, and we may assume that the n_1 largest of these washers are on the first peg, the next n_2 on the next, *etc.* $\cdots$ and the n_h smallest ones on the last. In some cases the solution requiring the least number of moves is unique; in others it is not. We shall describe one of these "most economical" methods, understanding that others may be equally short but not shorter. In the trivial case $h>n-1$, only $n-1$ auxiliary pegs need be used, so the number of moves required is the same as for $h=n-1$. Otherwise, if the smallest washer is to cover n_h-1 others at this stage, it is a most economical method to have these be the smallest washers, so that these in turn do not block other pegs. Similarly in each of the h auxiliary piles the washers may be arranged in consecutive order according to size. It is also a most economical method to have the larger piles contain the smaller washers, since the latter have access to more pegs at the time of their transfer. Hence

$$(1) \qquad n = 1 + n_1 + n_2 + \cdots + n_h, \qquad 1 \leqq n_1 \leqq n_2 \leqq \cdots \leqq n_h.$$

To complete the transfer we move the largest washer to its destination, then move the n_1 next largest washers onto it using one auxiliary peg, then the next n_2 using two auxiliary pegs, *etc.* $\cdots$, and finally move the n_h smallest washers using h auxiliary pegs. The minimum number of moves, $m(h, n)$, required to move n washers using $h=k-2$ auxiliary pegs is thus given by

$$(2) \qquad m(h, n) = 1 + 2[m(1, n_1) + m(2, n_2) + \cdots + m(h, n_h)],$$

if the best partition of n is chosen. A possible partition of $n-1$ is given by $1+n_1+ \cdots +(n_r-1)+ \cdots +n_h$, but since this is not necessarily the most economical partition, we have, instead of the equality (2), the inequality

$$(3) \qquad m(h, n-1) \leqq 1 + 2[m(1, n_1) + \cdots + m(r, n_r - 1) + \cdots + m(h,n_h)].$$

We define the cost of moving the nth washer to be

$$(4) \qquad c(h, n) = m(h, n) - m(h, n-1),$$

and from (2), (3), and (4), we obtain the inequalities

$$c(h, n) \geqq 2c(r, n_r), \qquad (r = 1, 2, \cdots, h). \tag{5}$$

To minimize $c(h, n)$, we choose the partition of n so as to minimize the largest of the quantities $c(r, n_r)$. Then $c(h, n)$ can be taken to be twice this value. By induction we see at once that $c(h, n)$ is a power of 2, say 2^s. For fixed h, as n increases, s is non-decreasing, but may have constant stretches. We denote by $n_{h,s}$ the largest n for given h and s. Then

$$c(h, n) = 2^s, \qquad n_{h,s-1} < n \leqq n_{h,s}. \tag{6}$$

The maximum value of $c(r, n_r)$ must be 2^{s-1}. Without increasing this, we may choose our partition in a unique manner so that

$$n_r = n_{r,s-1}, \quad r < h; \qquad n_h = n - (1 + n_1 + n_2 + \cdots + n_{h-1}). \tag{7}$$

The largest n satisfying (6), namely $n_{h,s}$, is obtained by choosing $n_h = n_{h,s-1}$, so that all the costs $c(r, n_r)$ are equal. We thus obtain a recursion formula and two initial conditions for the function $n_{h,s}$, which define it for positive integral values of h and s,

$$n_{h,s} = \sum_{r=0}^{h} n_{r,s-1}, \qquad n_{0,s} = n_{h,0} = 1. \tag{8}$$

These same formulas define the binomial coefficients $(h+s)!/h!s!$. Hence,

$$n_{h,s} = (h + s)!/h!s!. \tag{9}$$

For given values of n and h, the number of washers costing 2^t moves may be written

$$n_{h,t} - n_{h,t-1} = n_{h-1,t}, \qquad t < s. \tag{10}$$

Each of the last $n - n_{h,s-1}$ washers will cost 2^s moves. Hence, by (4), (10), and (9), the minimum number of moves required to move the n washers is given by the formula

$$m(h, n) = \sum_{t=0}^{s-1} 2^t \frac{(h - 1 + t)!}{(h - 1)!t!} + 2^s\left[n - \frac{(h + s - 1)!}{h!(s - 1)!}\right], \tag{11}$$

where s is the largest integer for which the last term on the right of equation (11) is positive. In the classical case $k=3$, we have $h=1$, $s=n-1$, and $m(1, n)=2^n-1$.

II. *Solution by the Proposer.*

It will be shown that the minimum number of moves for $k \geqq 3$ is given by

$$(1) \qquad \begin{aligned} {}_kX_n &= 2^{s+1}(n - {}_kQ_s) + \sum_{j=0}^{s} 2^j\, {}_{k-1}Q_j, \\ {}_kQ_s &= \binom{k-2+s}{s}, \qquad n \subset {}_kI_s, \quad \text{that is,} \quad {}_kQ_s \leqq n < {}_kQ_{s+1}. \end{aligned}$$

Consider a rectangular array of squares with coördinates (k, n), $k \geqq 3$, $n \geqq 1$.

With each square (k, n) there is a corresponding rectangle made up of squares (k', n') such that both $k' \leqq k$, $n' \leqq n$. The proof is by an induction which assumes the formula (1) true for all the squares of the rectangle except (k, n) and then establishes the formula for this corner square.

As a basis for the induction we use the facts that the theorem is true for $k=3$ and any n; and for any k with $n < {}_kQ_1 = k-1$, which are easily seen directly without the formula.

Essentially any possible best way of moving the washers can be described in three steps: move the n_1 uppermost washers to another peg, using all k pegs; move the n_2 remaining washers to a second peg, using the available $k-1$ pegs; and finally move the n_1 washers to this second peg, once again using k pegs. We define the function

$$(2) \qquad {}_kY_n(n_1) = 2{}_kX_{n_1} + {}_{k-1}X_{n_2}, \qquad n = n_1 + n_2,$$

where n_1 and n_2 are positive integers. At first thought, when $k>4$, an extension of this reasoning ought to be considered, dividing the n washers into three (or more) sets and by the symmetry of the problem examining a function

$$ {}_kY_n(n_1, n_2) = 2{}_kX_{n1} + 2{}_{k-1}X_{n_2} + {}_{k-2}X_{n_3}, \qquad n = n_1 + n_2 + n_3.$$

But the last two terms, representing a way of moving n_2+n_3 washers using $k-1$ pegs, can best be replaced by ${}_{k-1}X_{n_2+n_3}$.

First if $n \subset {}_kI_s$ we can find $n_1 \subset {}_kI_{s-1}$ such that

$$(3) \qquad {}_kY_n(n_1) = {}_kX_n.$$

Either $n - {}_kQ_s < {}_{k-1}Q_s$, and $n_1 = n - {}_{k-1}Q_s$ will serve; or $n - {}_kQ_s \geqq {}_{k-1}Q_s$, and $n_1 = {}_kQ_s - 1$ will serve; in both cases $n_1 \subset {}_kI_{s-1}$ with either $n_2 \subset {}_kI_s$ or $n_2 = {}_{k-1}Q_{s+1}$. The proof of (3) is made by use of the above formulas for ${}_kQ_s$, where ${}_kQ_0 = 1$ and ${}_kQ_s = 0$ if s is negative, and by use of the relation ${}_kQ_{s+1} = {}_kQ_s + {}_{k-1}Q_{s+1}$.

We prove next that ${}_kY_n(n_1)$, for values of n_1 other than those above, is never less than ${}_kX_n$ by considering the variation of ${}_kY_n(N_1)$ as N_1 increases by unity from an n_1 chosen under the conditions above, and similarly when N_1 decreases.

We can easily show that

$$(4) \qquad \Delta_k X_n = 2^{s+1}, \qquad n \subset {}_kI_s.$$

Then for an increasing N_1 we have

$$(5) \qquad \delta_k Y_n(N_1) = 2\Delta_k X_{N_1} - \Delta_{k-1}X_{N_2-1}, \qquad N_1 + N_2 = n,$$

where N_1, N_2 are positive integers admitting the considered variations. From (4) we have $\Delta_k X_{N_1} \geqq 2^s$, ${}_{k-1}X_{N_2-1} \leqq 2^{s+1}$; hence $\delta_k Y_n(N_1) \geqq 0$. For a decreasing N_1 we have

$$(6) \qquad \delta_k Y_n(N_1) = \Delta_{k-1}X_{N_2} - 2\Delta_k X_{N_1-1}, \qquad N_1 + N_2 = n.$$

Here $\Delta_{k-1}X_{N_2} \geqq 2^{s+1}$, $\Delta_k X_{N_1-1} \leqq 2^s$; hence $\delta_k Y_n(N_1) \geqq 0$.

Thus if the problem of moving n washers on k pegs is solved in the shortest way, the total number of moves is given by ${}_kY_n(n_1)$ which has been shown to be a minimum for n_1 and n_2 chosen in any one of the ways described in (3). But if the assumption of the induction is applied to (2), then by (3) the theorem is true for ${}_kX_n$, for the squares with coördinates (k, n_1) and $(k-1, n_2)$ are in the rectangle corresponding to (k, n). A basis for the induction has already been noted; hence the theorem is true.

Editorial Note. The analysis in each of the above solutions depends upon a preliminary lemma in the statements above equation (2) in each. It would be desirable to have a brief and rigorous proof of these lemmas. It will suffice to prove the following lemma: If the first n_h washers from the top of the initial peg are placed on a single auxiliary peg, say peg h; the next n_{h-1} on peg $h-1$; and so on until the largest washer is placed with one move on the final peg where it is alone; then, for suitable values of n_i, this plan for the removal of all the washers from the initial peg requires as small a number of moves as any other.

In the second solution the induction may be made in steps from $s-1$ to s. The equation (11) in the first solution may also be written

$$m(h, n) = (-1)^h + 2^s\left[n - \sum_{t=0}^{[h/2]} n_{h-2t,s-2}\right]. \tag{12}$$

This is a slight modification of a similar equation given by Frame in the first draft of his solution. The proposer gave the following results for $n=64$ and values of h in the parentheses: (1) 18,446,744,073,709,551,615; (2) 18,433; (3) 1535; (4) 673; (5) 479; (6) 385; (7) 351. We add two other computations, using equation (12): $m(2, 128)=720{,}897$; $m(2, 192)=10{,}485{,}761$.

The set of integers n_i may be chosen in any way so that $n_{i,s-2} \leqq n_i \leqq n_{i,s-1}$, where on the left we must have at least one inequality so that n is in the given interval. For $h=1$ and any n there is only one choice, and for this reason this case is quite simple. If $n=n_{h,s}$, there is only one choice, but there are cases where there may be a large number of different selections of these integers each giving the same total number of moves.