High-Radix Dividers

Chapter · January 2014

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Contents

Textbook: Computer Arithmetic: Algorithms and Hardware Designs, Oxford University Press, New York, 2000 , by Behrooz Parhami.
Many of the slides are either from the textbook or from Parhami's slides.

1. Basics of High-Radix Division

Radix-r version of division recurrence of Section 13.1

$s^{(j)} = r\, s^{(j-1)} - q_{k-j}\,(r^k d)$ with $s^{(0)} = z$ and $s^{(k)} = r^k s$

High-radix dividers of practical interest have $r = 2^b$

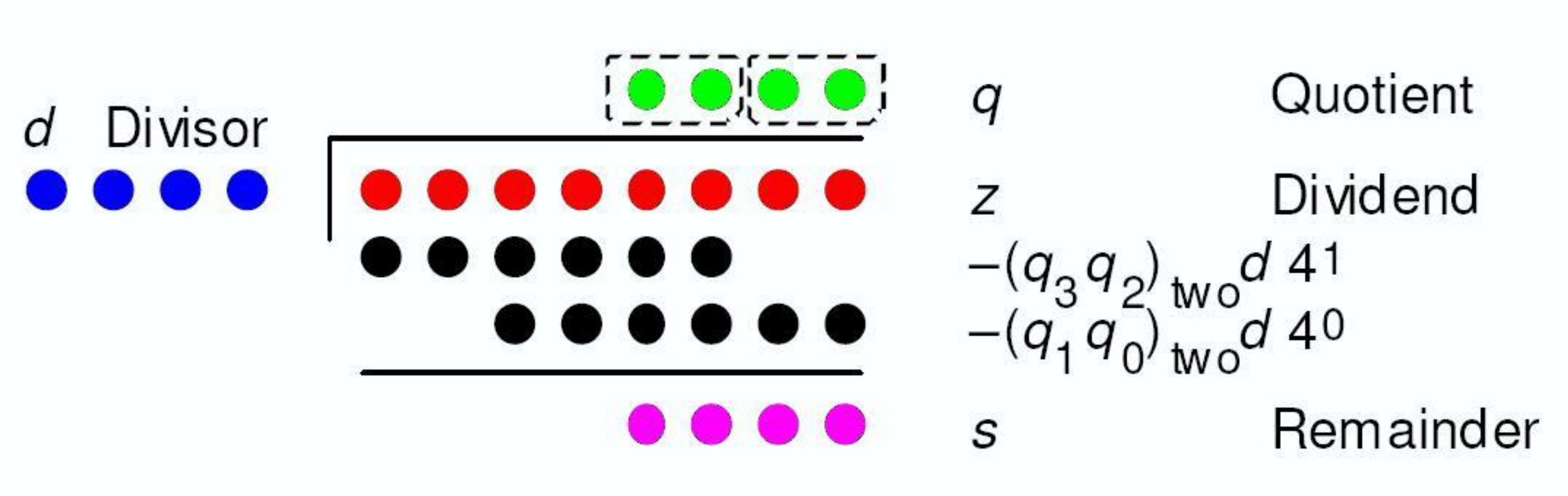

Fig. 14.1 Radix-4 division in dot notation.

Radix-4 integer division

z	0 1 2 3 1 1 2 3	
$4^4 d$	1 2 0 3	
$s^{(0)}$	0 1 2 3 1 1 2 3	
$4s^{(0)}$	0 1 2 3 1 1 2 3	
$-q_3\, 4^4 d$	0 1 2 0 3	$\{q_3 = 1\}$
$s^{(1)}$	0 0 2 2 1 2 3	
$4s^{(1)}$	0 0 2 2 1 2 3	
$-q_2\, 4^4 d$	0 0 0 0 0	$\{q_2 = 0\}$
$s^{(2)}$	0 2 2 1 2 3	
$4s^{(2)}$	0 2 2 1 2 3	
$-q_1\, 4^4 d$	0 1 2 0 3	$\{q_1 = 1\}$
$s^{(3)}$	1 0 0 3 3	
$4s^{(3)}$	1 0 0 3 3	
$-q_0\, 4^4 d$	0 3 0 1 2	$\{q_0 = 2\}$
$s^{(4)}$	1 0 2 1	
s	1 0 2 1	
q	1 0 1 2	

Radix-10 fractional division

z_{frac}	.7 0 0 3	
d_{frac}	.9 9	
$s^{(0)}$	.7 0 0 3	
$10s^{(0)}$	7 .0 0 3	
$-q_{-1} d$	6 .9 3	$\{q_{-1} = 7\}$
$s^{(1)}$	.0 7 3	
$10s^{(1)}$	0 .7 3	
$-q_{-2} d$	0 .0 0	$\{q_{-2} = 0\}$
$s^{(2)}$	.7 3	
s_{frac}	.0 0 7 3	
q_{frac}	.7 0	

Fig. 14.2 Examples of high-radix division with integer and fractional operands.

Radix-4 Restoring

- Select quotient digit q_i
  - 0, 1, 2, or 3
- Subtract $q_i d$ from p
- Shift p left

Divisor	d
X 1	1.301
X 2	3.202
X 3	11.103

- Must consider all digits to select digit q_i
- Must form "awkward multiple" $3d$

Example

p_0	1.2130	q_1	0
$-q_1 d$	- 0.		
	1.2130		
p_1	12.1300	q_2	3
$-q_2 d$	-11.1030		
	1.0210		
p_2	10.2100	q_3	2
$-q_3 d$	- 3.2020		
	1.0020		
	10.0200	q_4	2
$-q_3 d$	- 3.2020		
	1.2120		

$1.213_4/1.301_4 = 0.322_4$, r= 0.0001212_4

$1.609_{10}/1.766_{10} = 0.906_{10}$, r= 0.00585

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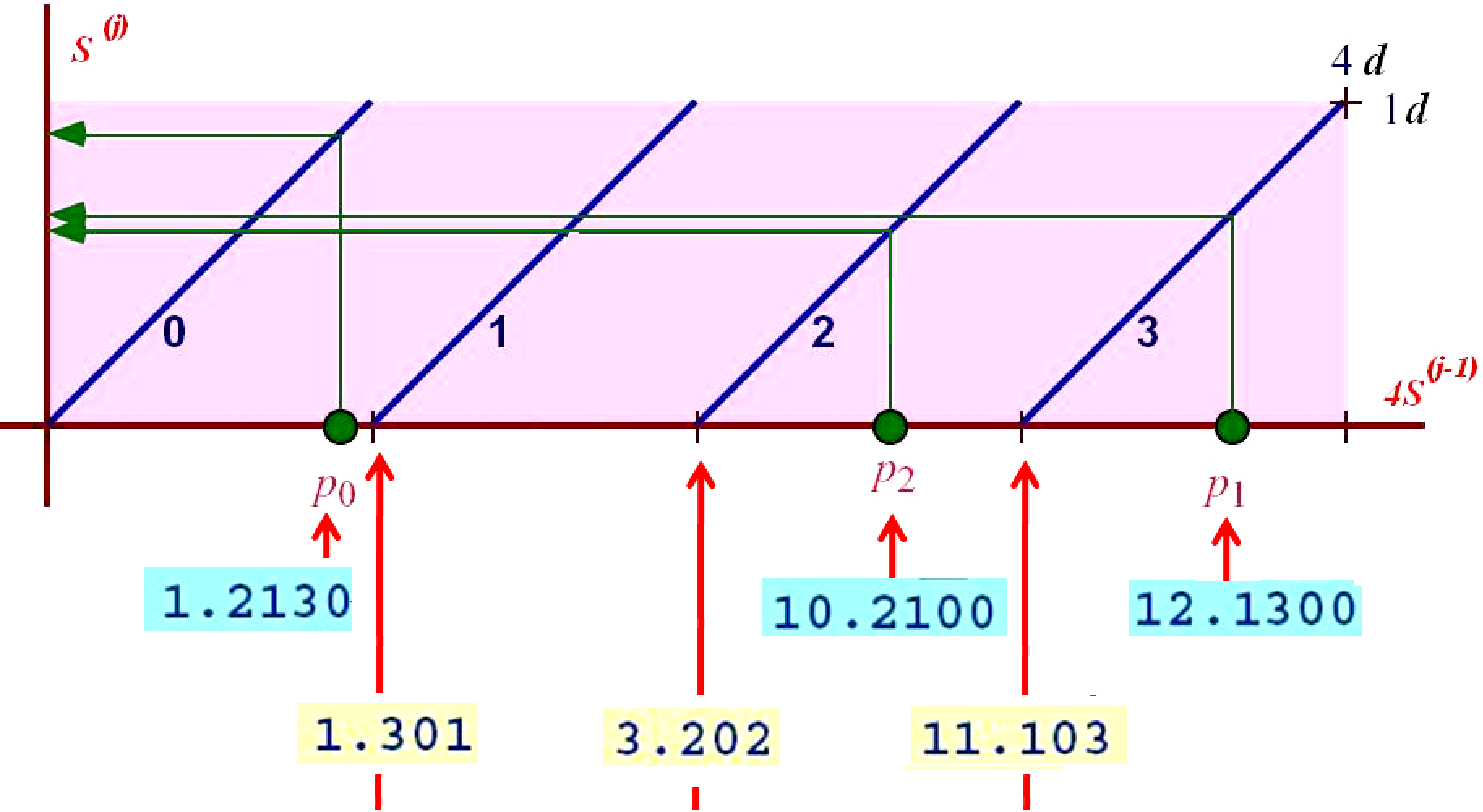
Digit Selection Radix 4 Restoring



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Difficulty of High-Radix Division

- Guessing the correct quotient digit is more difficult.
- Division is naturally a sequential process:

a) guess a quotient digit q_{k-j}

b) compute term $q_{k-j}(r^k d)$

c) compute partial remainder

$s^{(j)} = r s^{(j-1)} - q_{k-j}(r^k d)$

Carry-Save Remainders

• More important for speed than high-radix.

• Lead to large performance increases by replacing carry-propagate adder with carry-save adder.

• Key to keeping remainder in carry-save form is:

Redundancy in the representation of q.

– allows less precise guessing of quotient digit based on approximate magnitude of partial remainder

– more redundancy $\rightarrow$ less precision required

2. Radix-2 SRT Division

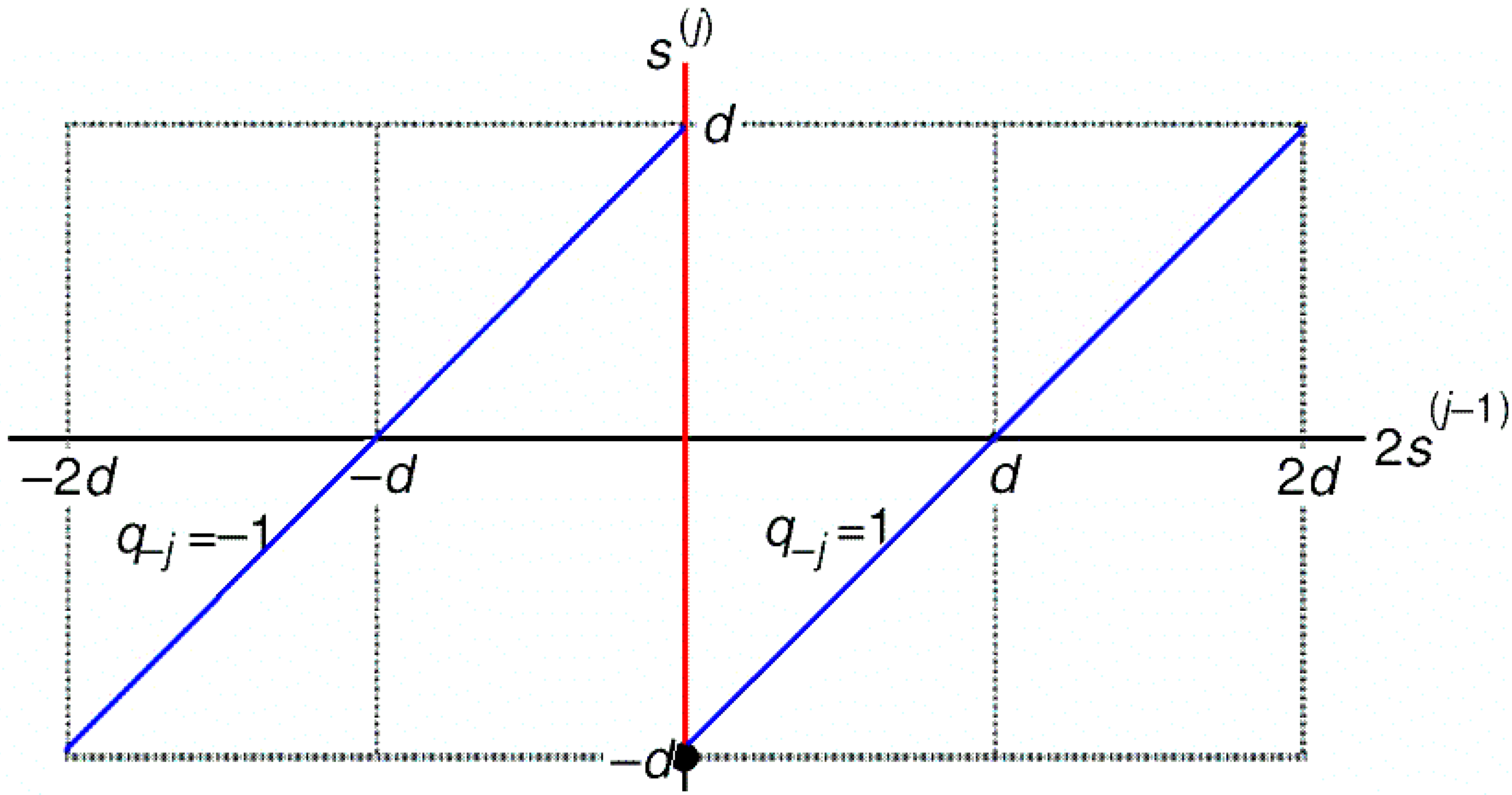


Fig. 14.3 The new partial remainder, $s^{(j)}$, as a function of the shifted old partial remainder, $2s^{(j-1)}$, in radix-2 nonrestoring division.

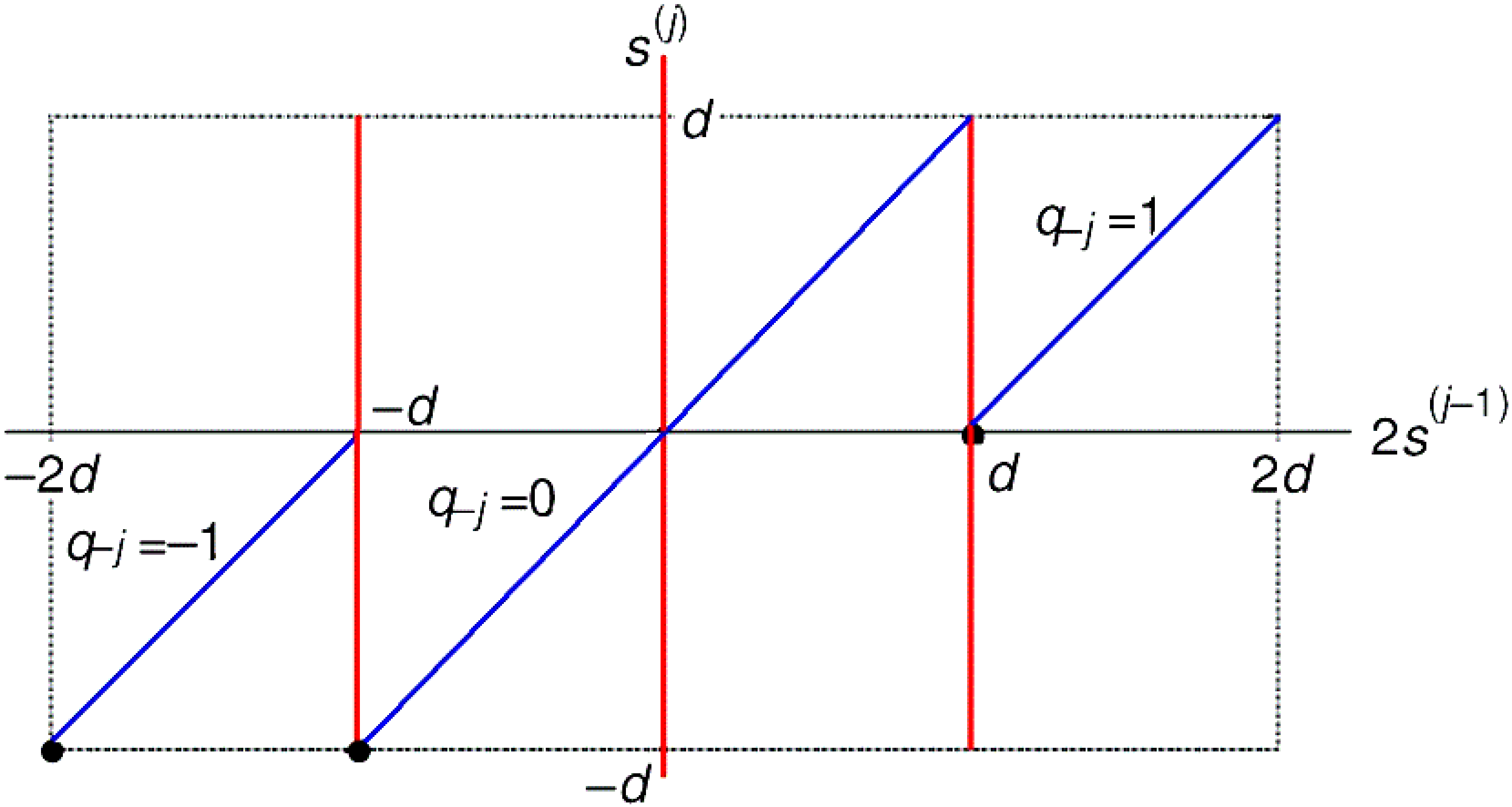


Fig. 14.4 The new partial remainder $s^{(j)}$ as a function of $2s^{(j-1)}$, with q_{-j} in {-1, 0, 1}.

SRT division (Sweeney, Robertson, Tocher)

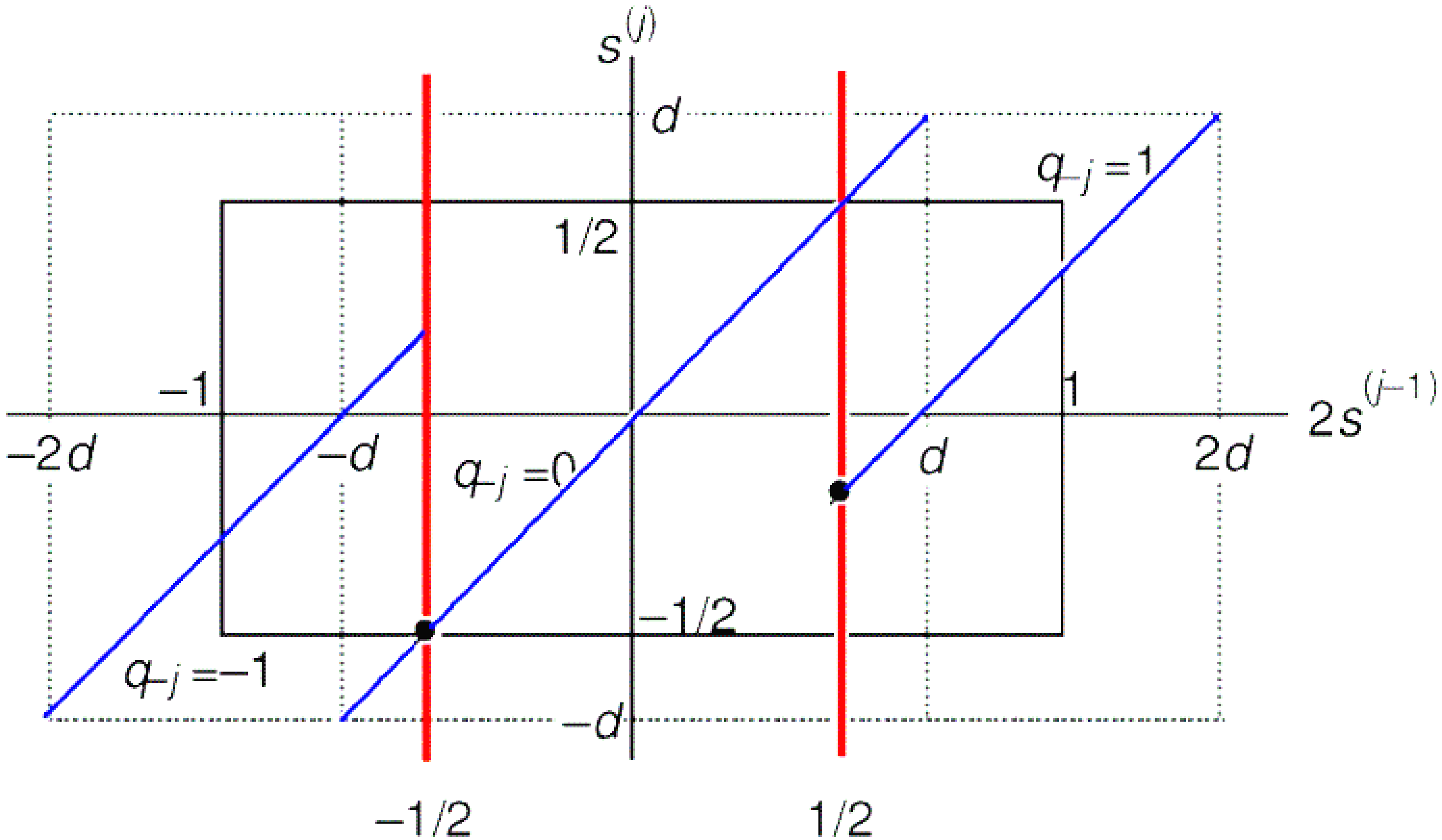


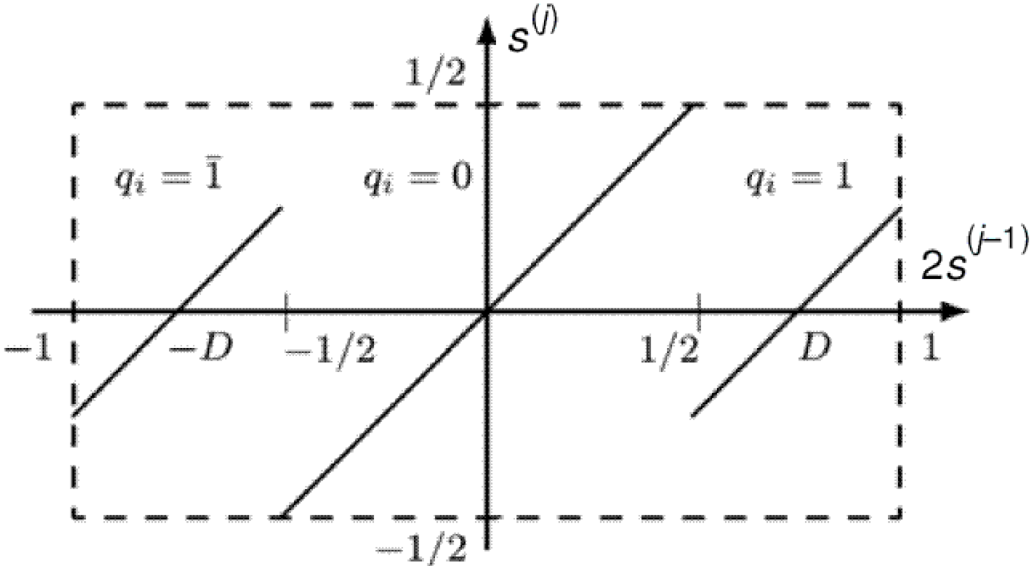
Fig. 14.5 The relationship between new and old partial remainders in radix-2 SRT division. (Sweeney, Robertson, Tocher)

$d \geq 1/2$, $-1/2 \leq s^{(0)} < 1/2$

Quotient Digit Selection

$$q_i = \begin{cases} 1 & if \quad 2S^{j-1} \geq \frac{1}{2} \\ 0 & if \quad -\frac{1}{2} \leq 2S^{j-1} < \frac{1}{2} \\ \bar{1} & if \quad 2S^{j-1} < -\frac{1}{2} \end{cases}$$

SRT Division Algorithm



$$2S^{(j-1)} = 0.1\times\times\times\times \rightarrow 2S^{(j-1)} \geq \frac{1}{2} \rightarrow q_i = 1$$

$$2S^{(j-1)} = 0.0\times\times\times\times \rightarrow 2S^{(j-1)} < \frac{1}{2} \rightarrow q_i = 0$$

$$2S^{(j-1)} = 1.1\times\times\times\times \rightarrow 2S^{(j-1)} \geq -\frac{1}{2} \rightarrow q_i = 0$$

$$2S^{(j-1)} = 1.0\times\times\times\times \rightarrow 2S^{(j-1)} < -\frac{1}{2} \rightarrow q_i = -1$$

z	. 0 1 0 0 0 1 0 1	In [−1/2,1/2), so OK	
d	. 1 0 1 0	In [1/2,1), so OK	
$-d$	1 . 0 1 1 0		
$s^{(0)}$	0 . 0 1 0 0 0 1 0 1		
$2s^{(0)}$	0 . 1 0 0 0 1 0 1	$\geq 1/2$, so set $q_{-1} = 1$	$0.1\times\times\times\times \rightarrow \quad q_i = 1$
$+(-d)$	1 . 0 1 1 0	and subtract	
$s^{(1)}$	1 . 1 1 1 0 1 0 1		
$2s^{(1)}$	1 . 1 1 0 1 0 1	In [−1/2,1/2), so $q_{-2} = 0$	$1.1\times\times\times\times \rightarrow \quad q_i = 0$
$s^{(2)} = 2s^{(1)}$	1 . 1 1 0 1 0 1		
$2s^{(2)}$	1 . 1 0 1 0 1	In [−1/2, 0), so $q_{-3} = 0$	$1.1\times\times\times\times \rightarrow \quad q_i = 0$
$s^{(3)} = 2s^{(2)}$	1 . 1 0 1 0 1		
$2s^{(3)}$	1 . 0 1 0 1	$< -1/2$, so $q_{-4} = -1$	$1.0\times\times\times\times \rightarrow q_i = -1$
$+d$	0 . 1 0 1 0	and add	
$s^{(4)}$	1 . 1 1 1 1	**Negative---**	
$+d$	0 . 1 0 1 0	**correction**	
$s^{(4)}$	0 . 1 0 0 1		
s	0 . 0 0 0 0 1 0 0 1		
q	0 . 1 0 0 -1	**Uncorrected.**	
q	0 . 0 1 1 1	**converted uncorrected**	
q	0 . 0 1 1 0	**corrected**	

Fig. 14.6
Example of unsigned radix-2 SRT division.

3. Using Carry-Save Adders

Carry-Save Partial Remainders

$$2s^{(j-1)} = u + v$$

$$u = \left(u_1 u_0 . u_{-1} u_{-2} \cdots\right)_{\text{2's-comp}}$$

$$v = \left(v_1 v_0 . v_{-1} v_{-2} \cdots\right)_{\text{2's-comp}}$$

Let $t = t_1 t_0 . t_{-1} t_{-2} = u_1 u_0 . u_{-1} u_{-2} + v_1 v_0 . v_{-1} v_{-2}$

t is an approximation of $u + v$

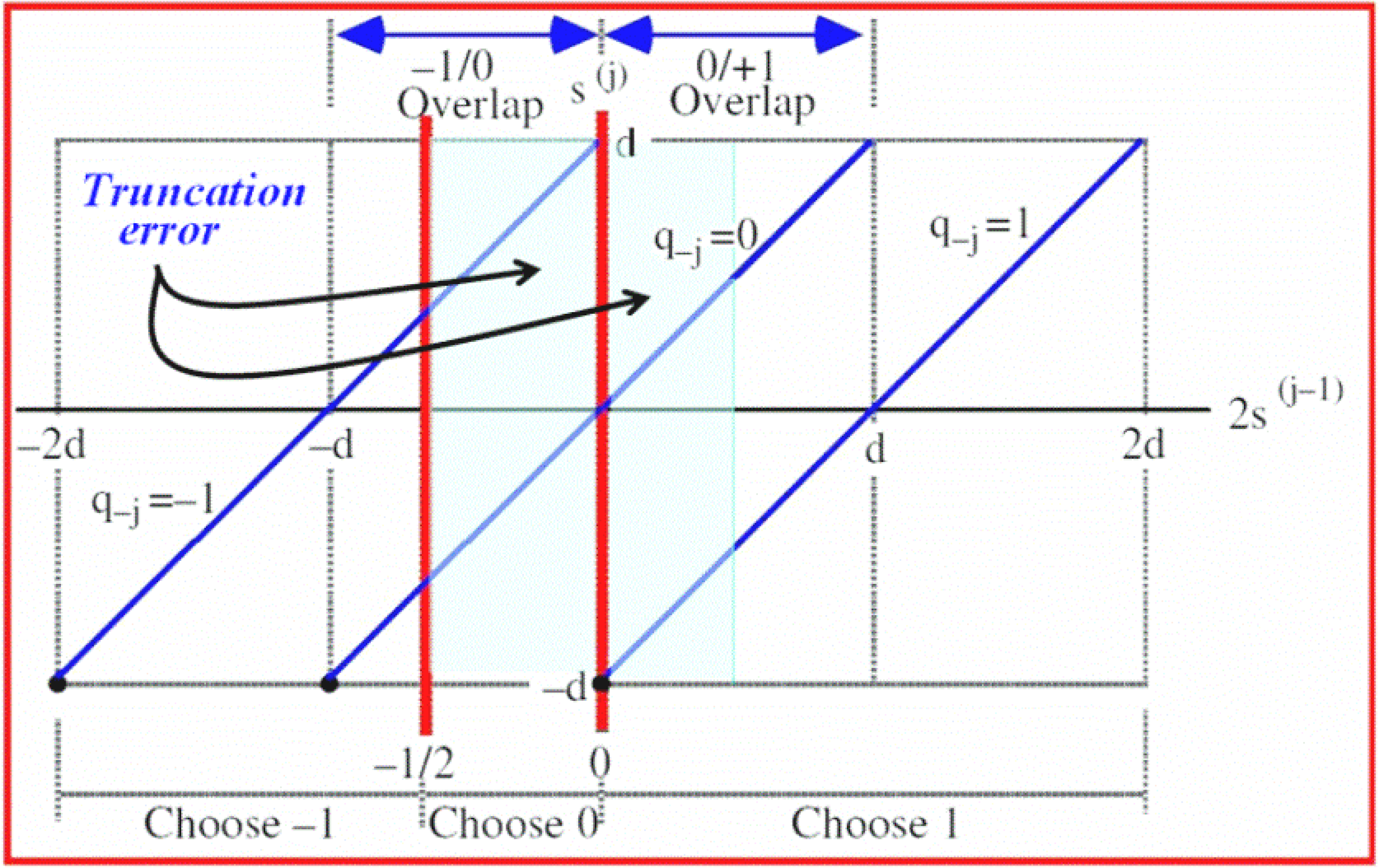
Truncation error is less than $1/4 + 1/4 = 1/2$:

$$0 \le (u + v) - t \le 1/2$$

Digit Selection

$t_1t_0.t_{-1}t_{-2}$	$2s^{(j-1)} = u_0.u_{-1}$	$t_1t_0.t_{-1}t_{-2}$	$2s^{(j-1)} = u_0.u_{-1}$
01.11	$[1.75, 2.0) \rightarrow q_{-j} = 1$	11.11	$[-0.25, 0.00) \rightarrow q_{-j} = 0$
01.10	$[1.5, 1.75) \rightarrow q_{-j} = 1$	11.10	$[-0.5, -0.25) \rightarrow q_{-j} = 0$
01.01	$[1.25, 1.5) \rightarrow q_{-j} = 1$	11.01	$[-0.75, -0.5) \rightarrow q_{-j} = -1$
01.00	$[1.0, 1.25) \rightarrow q_{-j} = 1$	11.00	$[-1.0, -0.75) \rightarrow q_{-j} = -1$
00.11	$[0.75, 1.0) \rightarrow q_{-j} = 1$	10.11	$[-1.25, -1.0) \rightarrow q_{-j} = -1$
00.10	$[0.5, 0.75) \rightarrow q_{-j} = 1$	10.10	$[-1.5, -1.25) \rightarrow q_{-j} = -1$
00.01	$[0.25, 0.5) \rightarrow q_{-j} = 1$	10.01	$[-1.75, -1.5) \rightarrow q_{-j} = -1$
00.00	$[0.0, 0.25) \rightarrow q_{-j} = 1$	10.00	$[-2.0, -1.75) \rightarrow q_{-j} = -1$

Tolerating Truncation Error



Sum part of $2s^{(j-1)}$: $\boxed{u = (u_1 u_0 . u_{-1} u_{-2} \cdots)_{\text{2's-compl}}}$

Carry part of $2s^{(j-1)}$: $\boxed{v = (v_1 v_0 . v_{-1} v_{-2} \cdots)_{\text{2's-compl}}}$

$t = u_{[-2,1]} + v_{[-2,1]}$ {Add the 4 MSBs of u and v}

if $t < -1/2$

then $q_{-j} = -1$

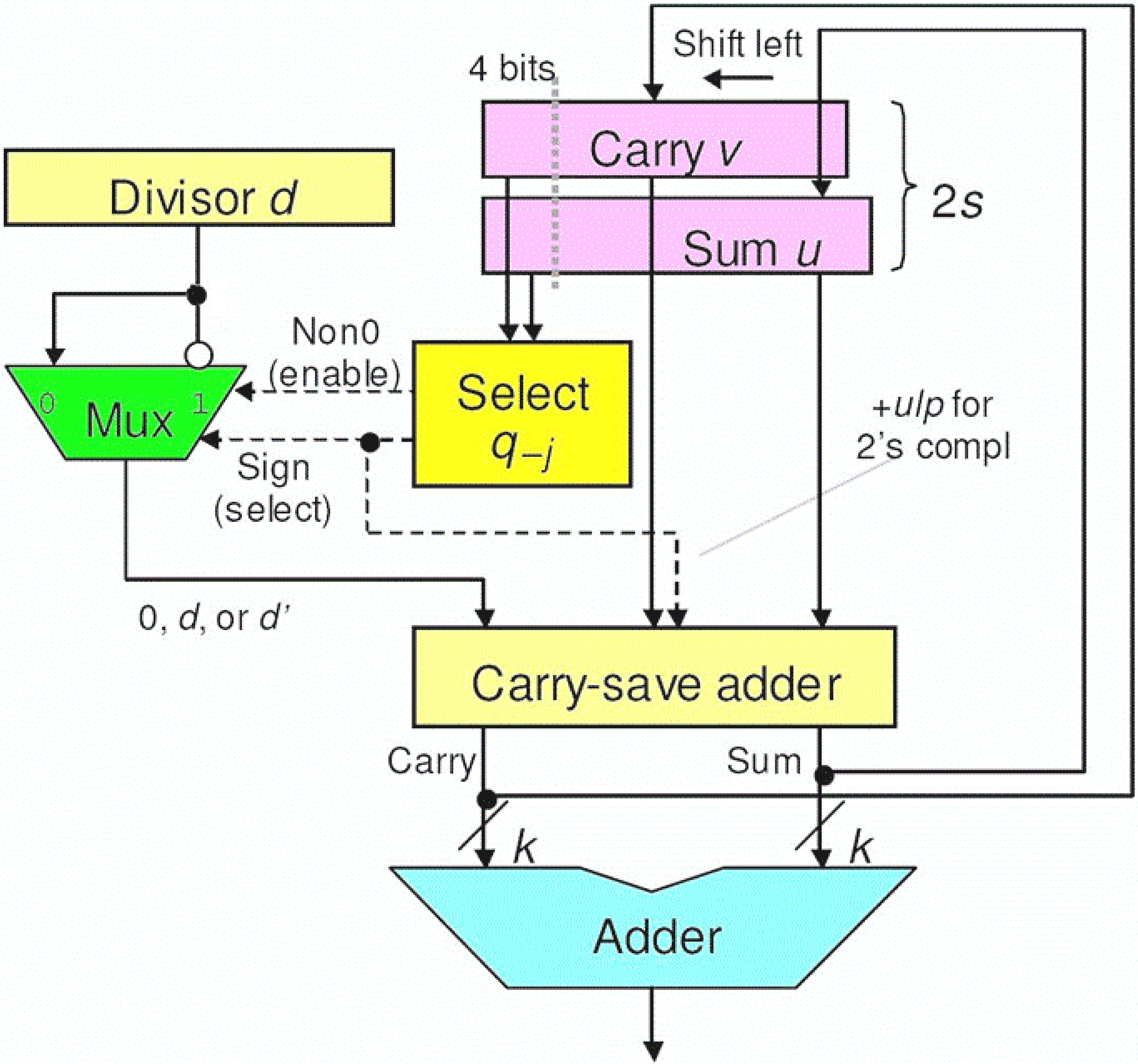
else if $t \geq 0$

then $q_{-j} = 1$

else $q_{-j} = 0$

endif

endif


Shift left
4 bits
Carry v
Sum u
$2s$
Divisor d
Non0
(enable)
Select
q_{-j}
0
Mux
1
Sign
(select)
$+ulp$ for
2's compl
0, d, or d'
Carry-save adder
Carry
Sum
k
k
Adder

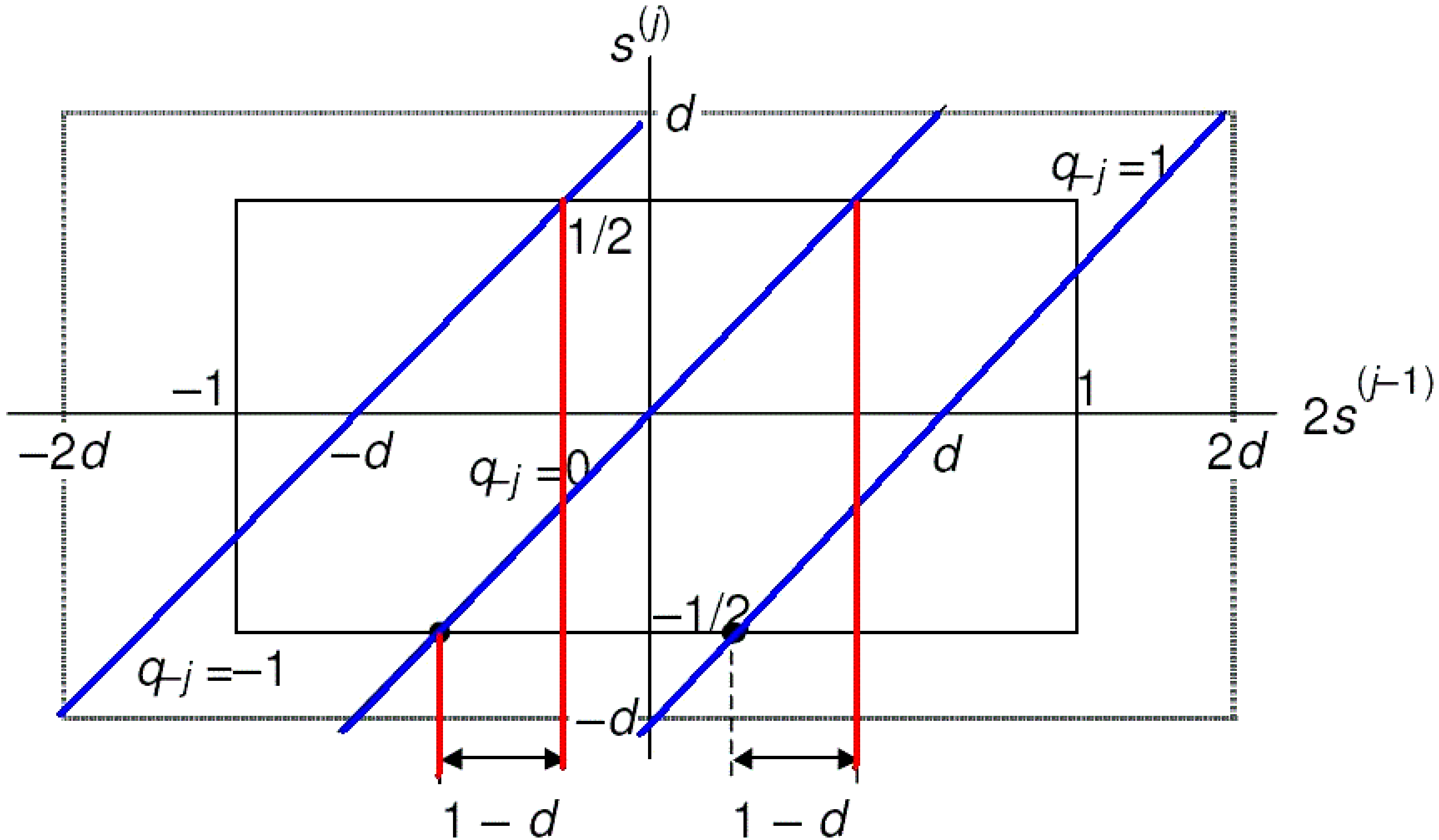


Fig. 14.9 Overlap regions in radix-2 SRT division.

4. Choosing the Quotient Digits

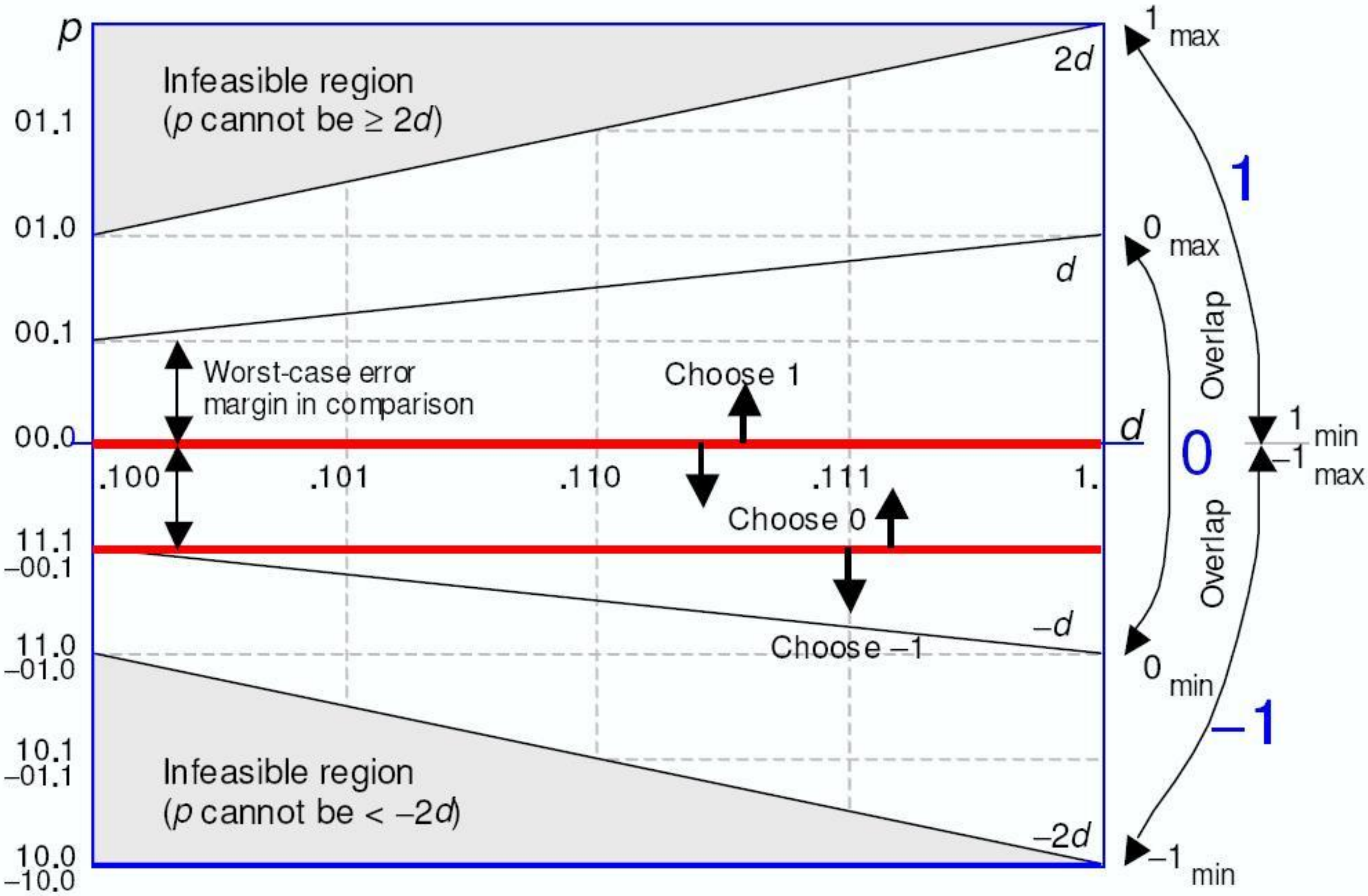
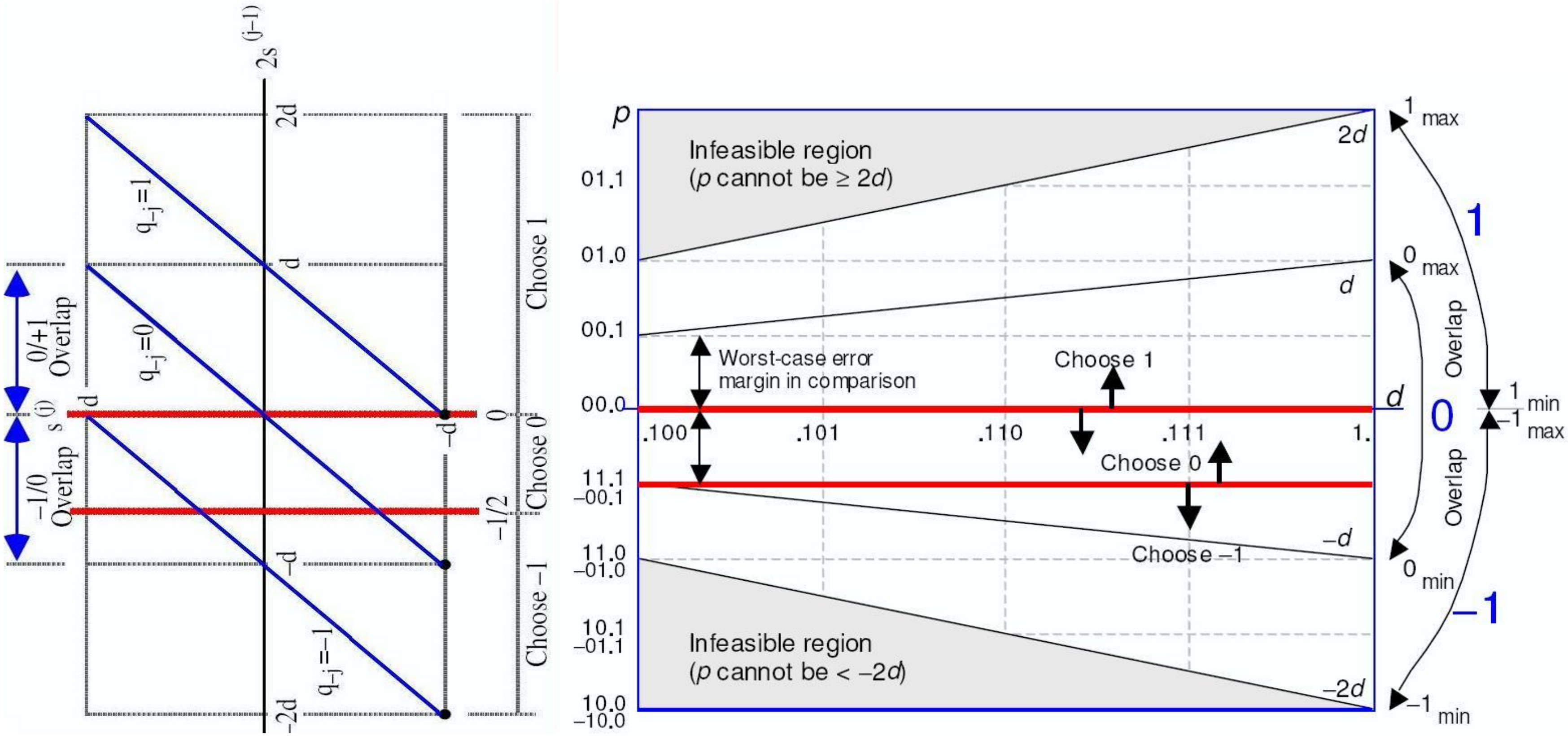


Fig. 14.10 ***A p-d* plot for radix-2 division with *d* ∈ [1/2,1), partial remainder in [–*d*, *d*), and quotient digits in [–1, 1].**

Choosing the Quotient Digits



5. Radix-4 SRT division

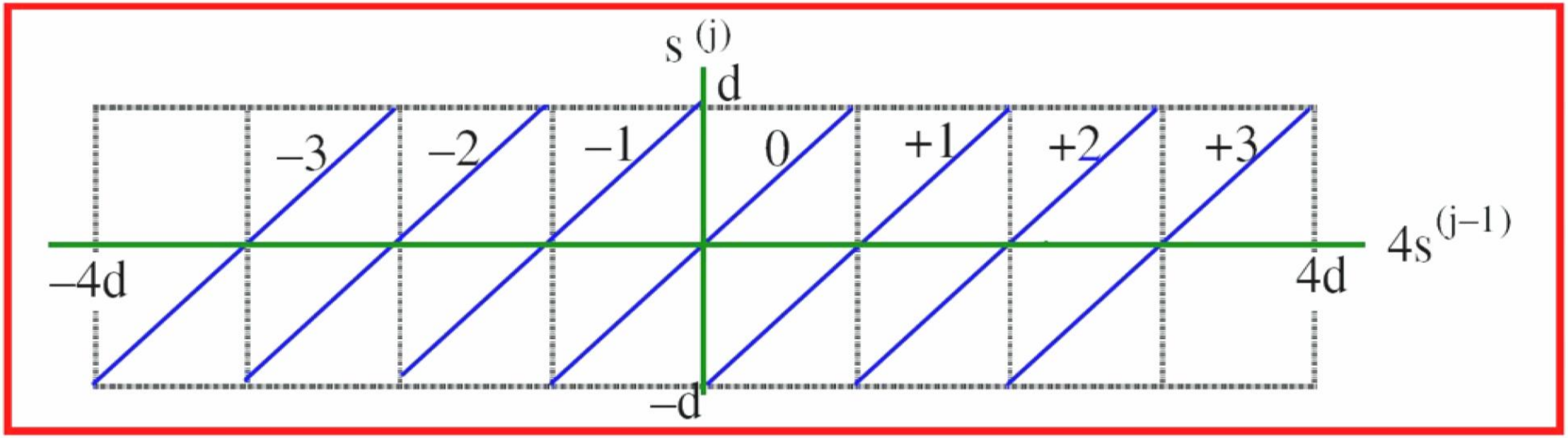


Fig. 14.11 New versus shifted old partial remainder in radix-4 division with q_{-j} in [–3, 3].

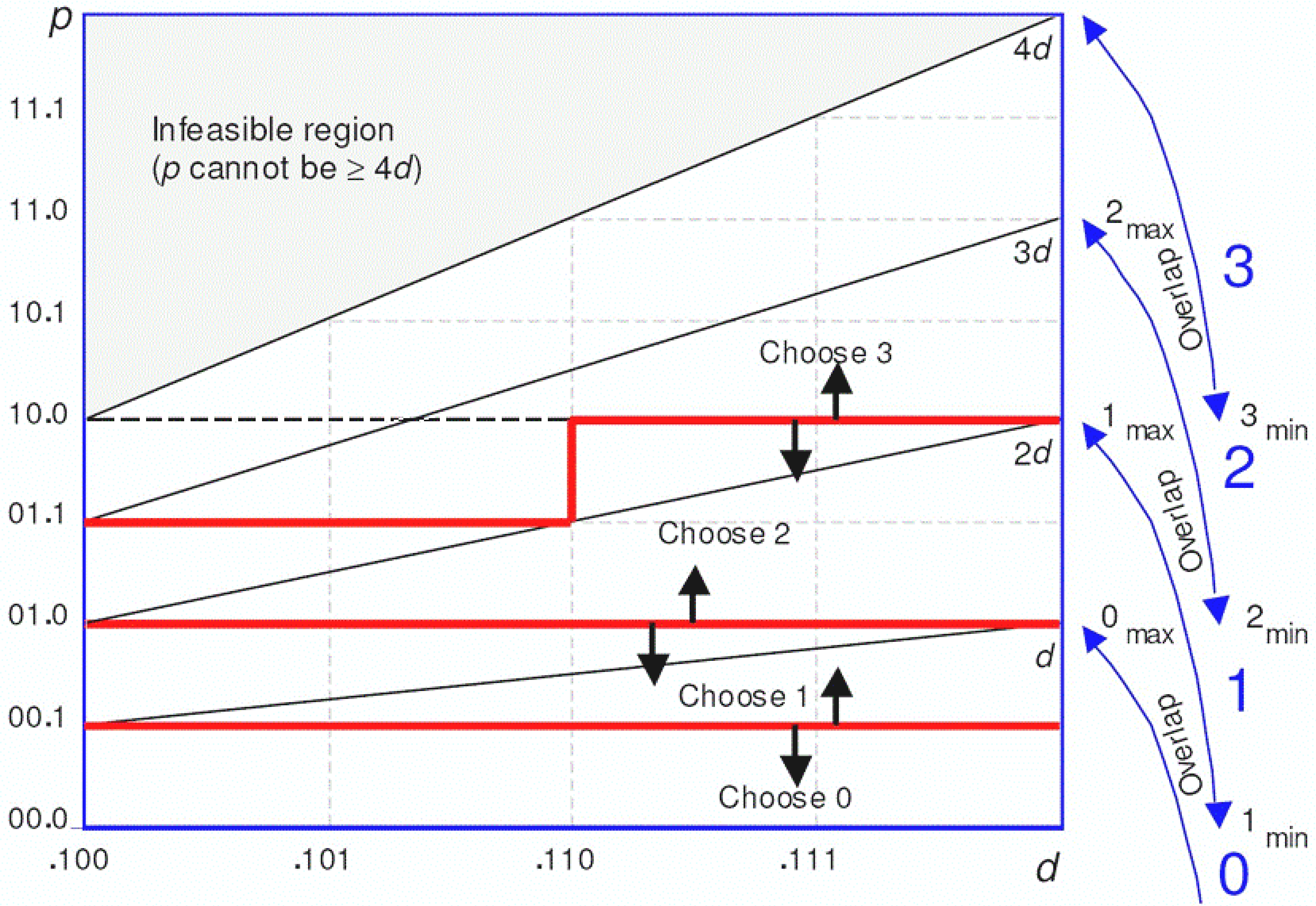


Fig. 14.12 ***p-d* plot for radix-4 SRT division with quotient digit set [–3, 3].**

Restricting the Range of s

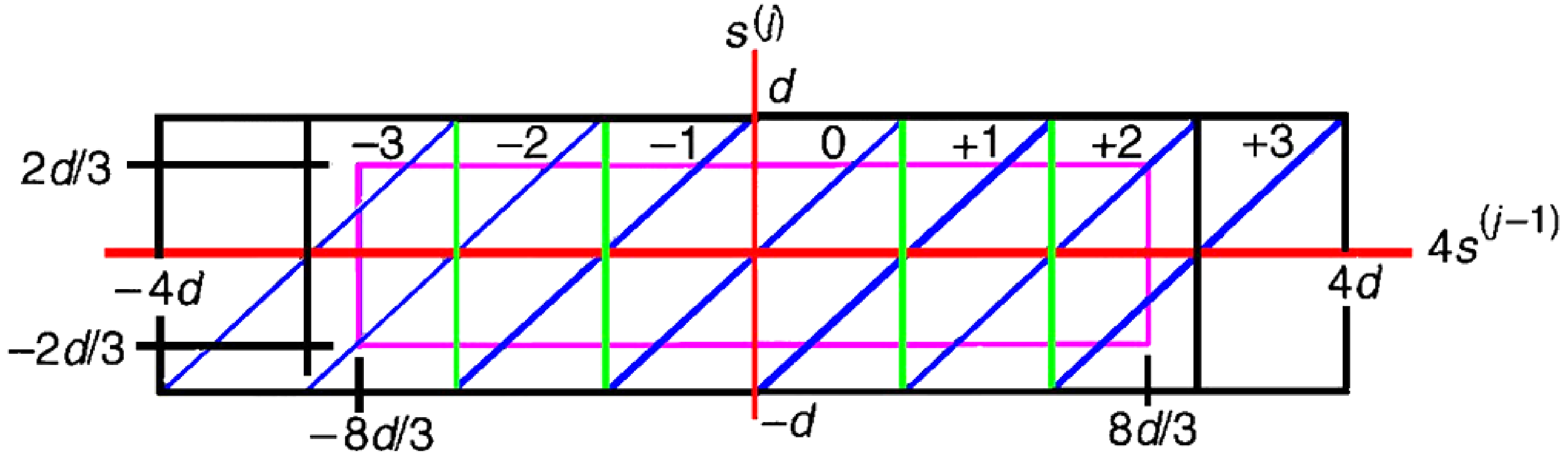


Fig. 14.13 New versus shifted old partial remainder in radix-4 division with q_{-j} in [–2, 2].

$$-hd \le s^{(j-1)} < hd, \quad \text{for some } h < 1$$

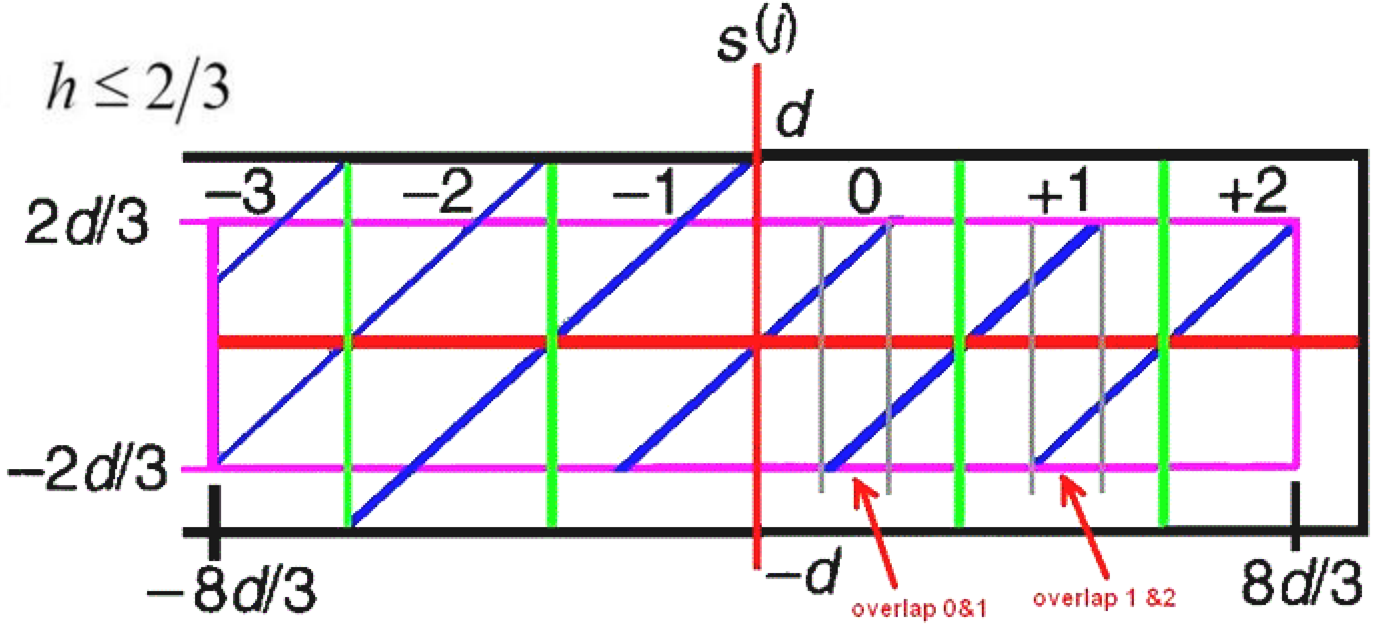
$$-4hd \le 4s^{(j-1)} < 4hd$$

$$-4hd - q_{-j}d \le 4s^{(j-1)} - q_{-j}d < 4hd - q_{-j}d$$

$$\underbrace{-4hd + 2d}_{q_{-j}=-2} \le \underbrace{4s^{(j-1)} - q_{-j}d}_{s^{(j)}} < \underbrace{4hd - 2d}_{q_{-j}=2}$$

$$-hd \le s^{(j)} < hd$$

$$hd \le 4hd - 2d \;\Rightarrow\; h \le 2/3$$



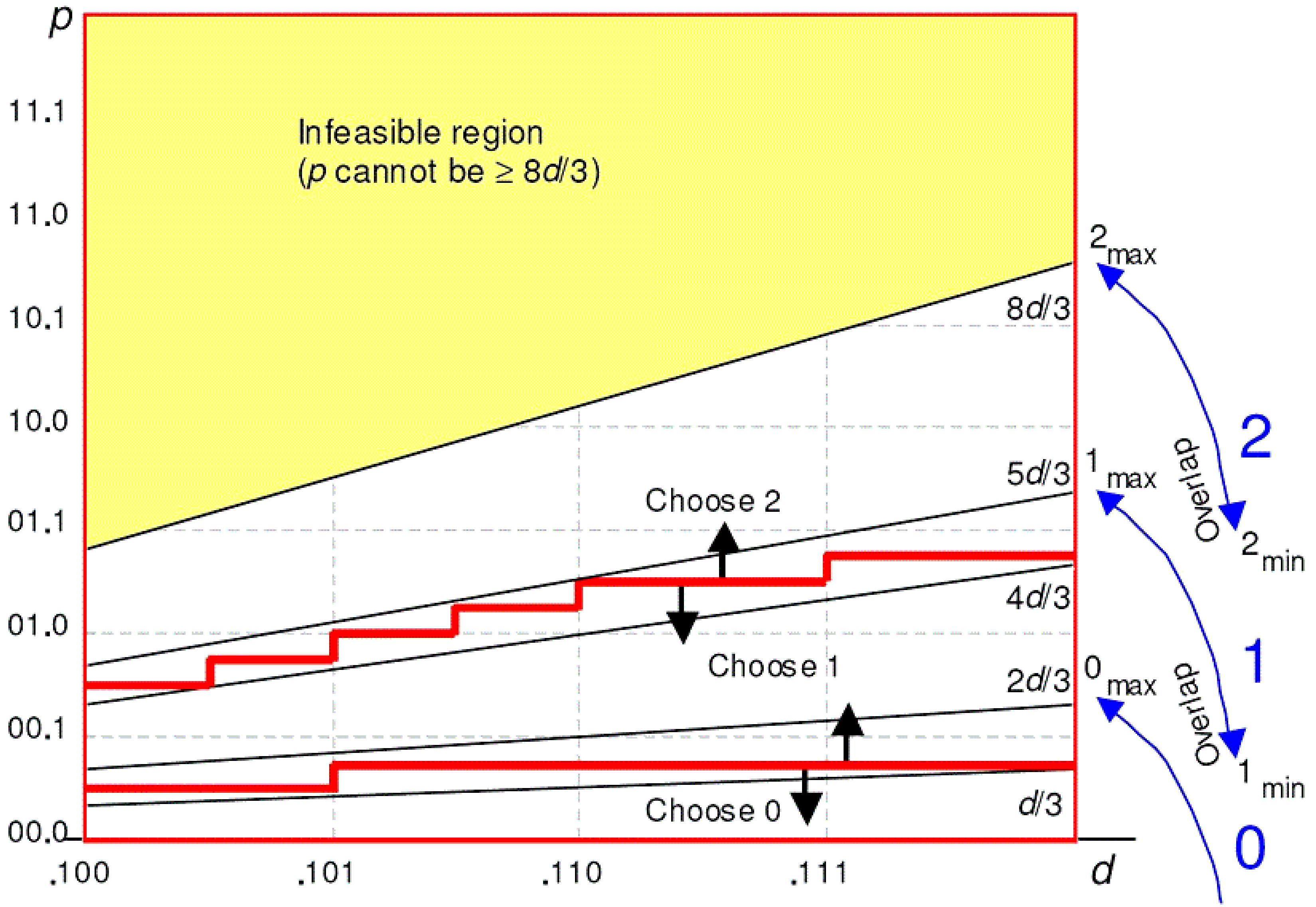


Fig. 14.14 *A p-d plot for radix-4 SRT division with quotient digit set [−2, 2].*

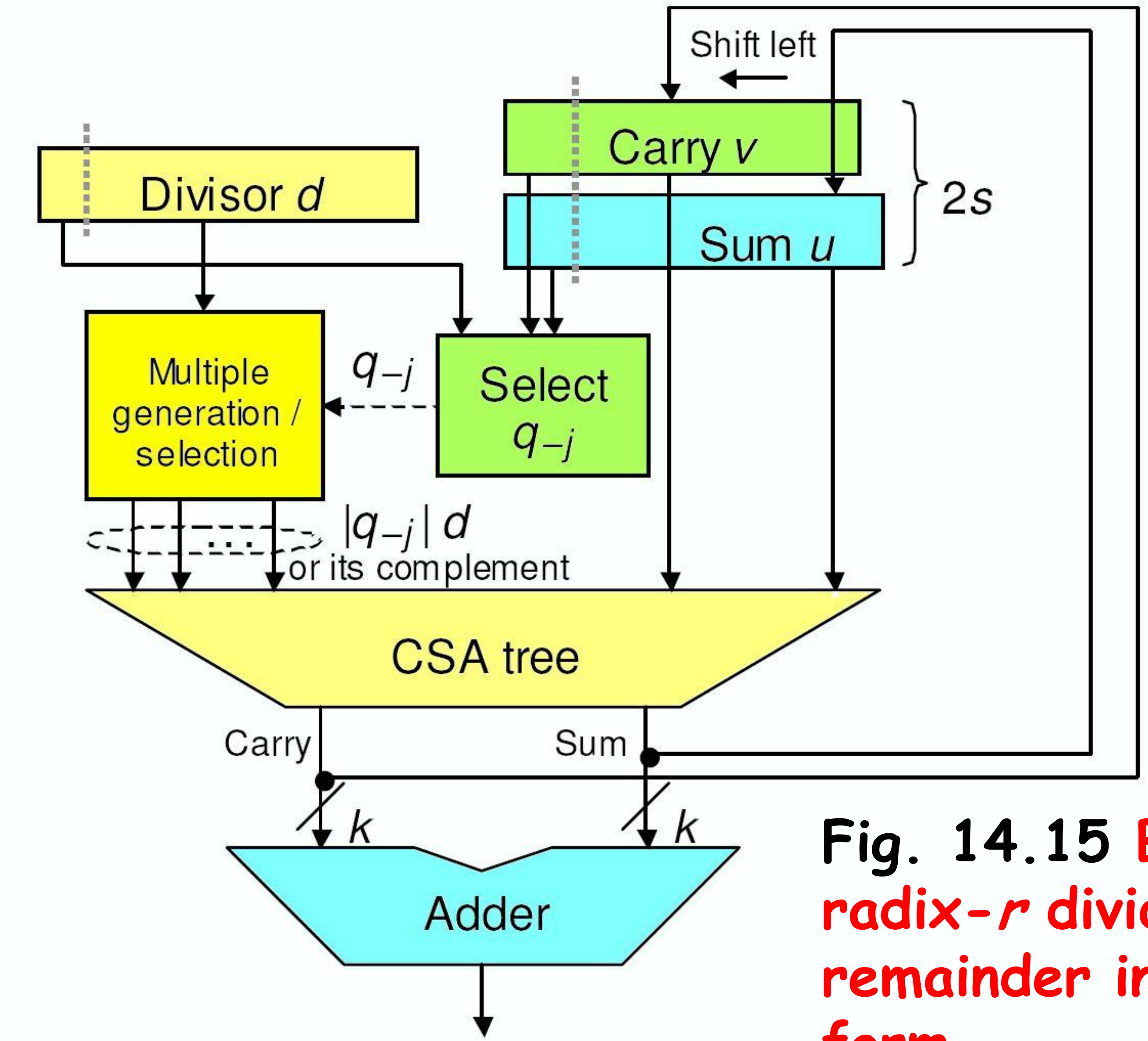
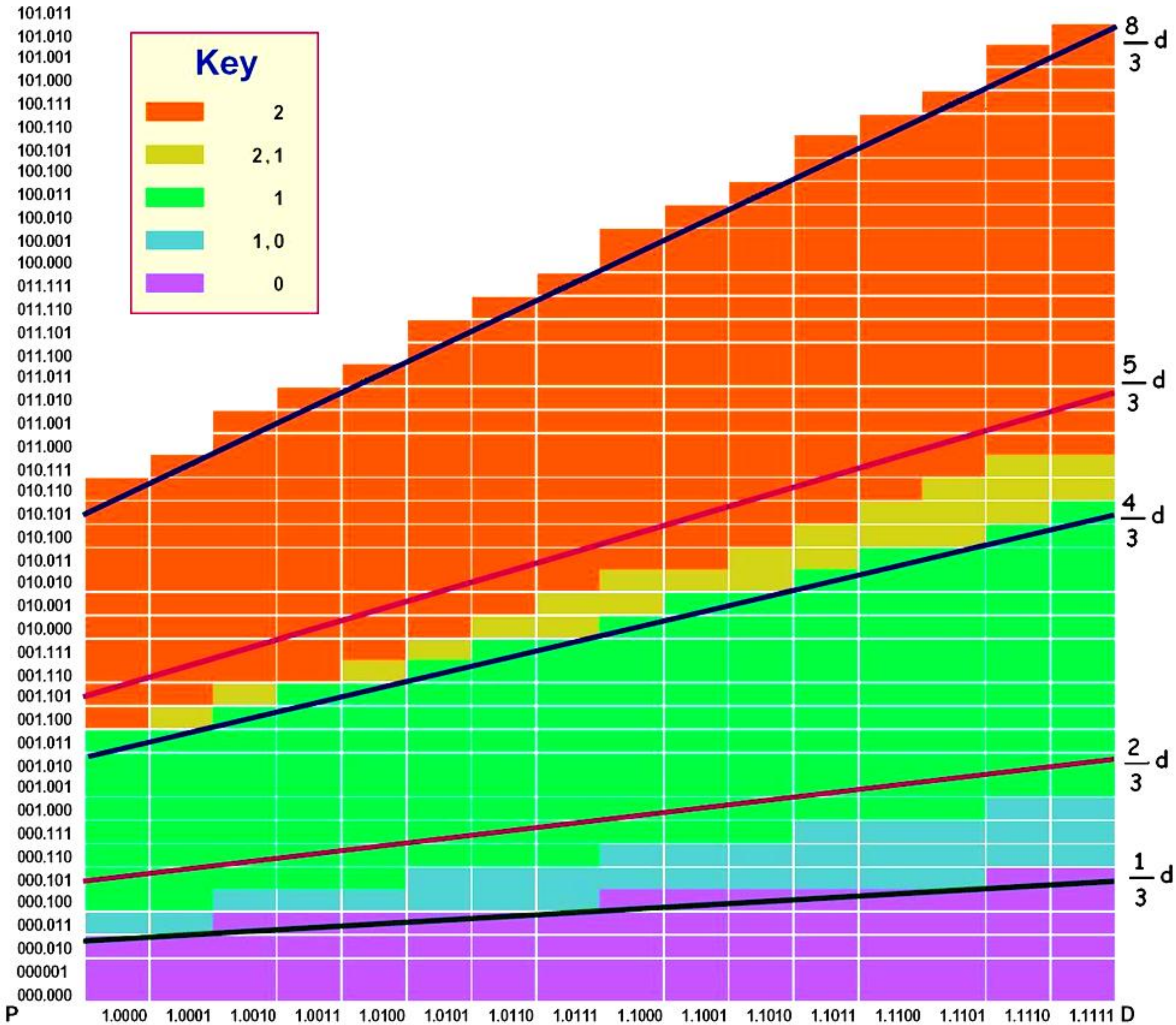


Fig. 14.15 **Block diagram of radix-*r* divider with partial remainder in stored-carry form.**



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