1 Design of Butterworth Lowpass Filter

Butterworth Approximation

1. Define specification of the filter, H_0: DC Gain, H_c: Gain at cutoff frequency, H_s Gain at stopband edge frequency, ω_c: Cutoff frequency (rad/s), ω_s: Stopband frequency (rad/s). Also choose the optimization strategy: stopband edge frequency gain optimized or passband edge gain optimized.
2. Calculate normalized cutoff frequency Ω_s

$$\begin{aligned}\Omega_c &= 1 \\ \Omega_s &= \frac{\omega_s}{\omega_c}\end{aligned}$$

3. Calculate $\beta = \beta_{max}$ on stopband edge frequency gain optimized or $\beta = \beta_{min}$ on passband edge gain optimized.

$$\beta_{max} = \sqrt{\left(\frac{H_0}{H_c}\right)^2 - 1}$$

$$\beta_{min} = \sqrt{\frac{\left(\frac{H_0}{H_s}\right)^2 - 1}{\Omega_s^2}}$$

4. Calculate the order of the filter N by rounding up n_{fmin} to the next integer:

$$n_{fmin} = \frac{\log\left(\frac{\frac{H_0^2}{H_s^2}-1}{\frac{H_0^2}{H_c^2}-1}\right)}{2\log\Omega_s}$$

5. Calculate s_{k+}, the roots of the transfer function denominator polynomial equation:

$$s_{k+} = \sqrt[N]{\frac{1}{\beta}}\mathrm{e}^{j\left(\frac{2k+1}{2N}\pi+\frac{\pi}{2}\right)} \quad k = 0, 1, \cdots N-1$$

Example 1 Determine the transfer function of Butterworth lowpass filter of stopband edge frequency gain optimized, $H_0 = 1$, $H_c = 0.891$ (= -1dB) , $H_s = 0.00398$ (= -48dB) , Cutoff frequency = 125664 (=20kHz), Stopband edge frequency = 1108353 (=176.4kHz)

1. Perform Butterworth Approximation

$$\begin{aligned}
\Omega_c &= 1 \\
\Omega_s &= \frac{\omega_s}{\omega_c} = 8.82 \\
n_{fmin} &= 2.85 \\
N &= Ceiling(2.85) = 3 \\
\beta_{max} &= 0.509 \\
\text{angle of } s_k &= \frac{2k+1}{2N}\pi + \frac{\pi}{2} \\
\text{magnitude of } s_k &= \beta^{\frac{-1}{N}} \\
s_k &= magnitude\left\{\cos(angle) + i\sin(angle)\right\} \\
s_0 &= 1.25\left\{\cos(2.09) + i\sin(2.09)\right\} \\
s_1 &= 1.25\left\{\cos(\pi) + i\sin(\pi)\right\} = -1.25 \\
s_2 &= 1.25\left\{\cos(4.19) + i\sin(4.19)\right\} \\
\text{Transfer function} H(s) &= \frac{1}{(s - 1.25\left\{\cos(2.09) + i\sin(2.09)\right\})(s + 1.25)(s - 1.25\left\{\cos(4.19) + i\sin(4.19)\right\})}
\end{aligned}$$

2. Partial fraction decomposition

$$\begin{aligned}
H(s) &= \frac{1}{(s - 1.25\left\{\cos(2.09) + i\sin(2.09)\right\})(s + 1.25)(s - 1.25\left\{\cos(4.19) + i\sin(4.19)\right\})} \\
&= \frac{c_1}{s - 1.25\left\{\cos(2.09) + i\sin(2.09)\right\}} + \frac{c_2}{s + 1.25} + \frac{c_3}{s - 1.25\left\{\cos(4.19) + i\sin(4.19)\right\}} \\
c_1 &= \frac{1}{(s + 1.25)(s - 1.25\left\{\cos(4.19) + i\sin(4.19)\right\})}, s = 1.25\left\{\cos(2.09) + i\sin(2.09)\right\} \\
&= -0.6263 - 0.3616i \\
c_2 &= \frac{1}{(s - 1.25\left\{\cos(2.09) + i\sin(2.09)\right\})(s - 1.25\left\{\cos(4.19) + i\sin(4.19)\right\})}, s = -1.25 \\
&= 1.253 \\
c_3 &= \frac{1}{(s - 1.25\left\{\cos(2.09) + i\sin(2.09)\right\})(s + 1.25)}, s = 1.25\left\{\cos(4.19) + i\sin(4.19)\right\} \\
&= -0.6263 + 0.3616i \\
\therefore H(s) &= \frac{-0.6263 - 0.3616i}{s + 0.6263 - 1.085i} + \frac{1.253}{s + 1.253} + \frac{-0.6263 + 0.3616i}{s + 0.6263 + 1.085i}
\end{aligned}$$

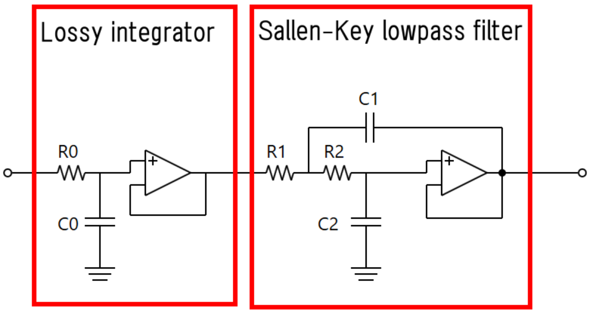

Figure 1

3. Multiply conjugate complex 1st order rational polynomial to get 2nd order rational polynomial with real value coefficients

$$
\begin{aligned}
H(s) &= \frac{-0.6263-0.3616i}{s+0.6263-1.085i} + \frac{1.253}{s+1.253} + \frac{-0.6263+0.3616i}{s+0.6263+1.085i} \\
&= \frac{1.253}{s+1.253} + \frac{-0.6263-0.3616i}{(s+0.6263-1.085i)(s+0.6263-1.085i)} \\
&= \frac{1.253}{s+1.253} + \frac{0.523}{s^2+1.253s+1.569}
\end{aligned}
$$

4. Create lossy integrator from 1st order rational polynomial

$$
\begin{aligned}
H_{s1} &= \frac{A}{s+a} \\
\omega_c &= 20000 * 2 * \pi = 125663 \\
R_{n0} &= 1 \\
C_{n0} &= \frac{a}{\omega_c * R_0} \\
R_0 &= R_{n0} * 10000 = 10(k\Omega) \quad \text{(frequency scaling)} \\
C_0 &= \frac{C_0}{10000} = 996(\text{pF}) \quad \text{(frequency scaling)}
\end{aligned}
$$

Bug: resulted circuit does not reflect A of the rational polynomial!

5. Create Sallen-Key lowpass filter from 2nd order rational polynomial

$$
\begin{aligned}
H_{s2} &= \frac{A}{s^2 + as + b} \\
\omega_0 &= \sqrt{b} \\
\omega_c &= 20000 * 2 * \pi = 125663 \\
Q &= \frac{\omega_0}{a} \\
C_{n2} &= 1 \\
C_{n1} &= \sqrt{3} * Q * C_{n2} \\
R_{n1} &= \frac{1}{\omega_0 * Q * C_{n2}} \\
R_{n2} &= \frac{1}{\sqrt{(3)} * \omega_0 * C_{n2}} \\
C_1 &= \frac{C_{n1}}{\omega_c * 10000} = 1.38(\text{nF}) \quad \text{(frequency scaling)} \\
C_2 &= \frac{C_{n2}}{\omega_c * 10000} = 796(\text{nF}) \quad \text{(frequency scaling)} \\
R_1 &= R_{n1} * 10000 = 7.98(k\Omega) \quad \text{(frequency scaling)} \\
R_2 &= R_{n2} * 10000 = 4.61(k\Omega) \quad \text{(frequency scaling)}
\end{aligned}
$$

Bug: resulted circuit does not reflect A of the rational polynomial!