1　ラプラス変換の導入

ラプラス変換の定義 (Definition of the Laplace transform)

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\mathrm{d}t$$

t は時間を表す変数. 時間領域の関数 $f(t)$ をラプラス変換すると周波数領域の関数 $\mathrm{F}(s)$ になる. s は広義積分 (improper integral) を計算する際には定数と考えて積分する.

$\mathcal{L}\{f(t)\}$ は Laplace transform of a function f of t と読む. $\int_0^\infty \mathrm{e}^{-st} f(t)\mathrm{d}t$ は Integral from zero to infinity of e to the minus s t times a function f of t d t と読む.

問 1　関数 $f(t) = 1$ をラプラス変換せよ.

$$\begin{aligned}
\mathcal{L}\{1\} &= \int_0^\infty \mathrm{e}^{-st} \cdot 1\,\mathrm{d}t \\
&= \lim_{A\to\infty} \int_0^A \mathrm{e}^{-st}\mathrm{d}t \\
&= \lim_{A\to\infty} \left[\frac{-1}{s}\mathrm{e}^{-st}\right]_0^A \\
&= \lim_{A\to\infty} \left(\frac{-1}{s}\mathrm{e}^{-sA} - \frac{-1}{s}\right)
\end{aligned}$$

$$\therefore s > 0 \text{ のとき } \mathcal{L}\{1\} = \frac{1}{s}$$

問 2　関数 $f(t) = \mathrm{e}^{at}$ をラプラス変換せよ.

$$\begin{aligned}
\mathcal{L}\{\mathrm{e}^{at}\} &= \int_0^\infty \mathrm{e}^{(-s)t}\mathrm{e}^{at}\,\mathrm{d}t \\
&= \int_0^\infty \mathrm{e}^{(a-s)t}\,\mathrm{d}t \\
&= \left[\frac{1}{a-s}\mathrm{e}^{(a-s)t}\right]_{(t=)0}^{(t=)\infty}
\end{aligned}$$

$$\begin{aligned}
\therefore a - s > 0 \text{ のとき} &\ :\ \text{発散.} \\
a - s = 0 \text{ のとき} &\ :\ \text{未定義.} \\
a - s < 0 \text{ のとき} &\ :\ \lim_{A\to\infty}\left(\mathrm{e}^{(a-s)A}\right) = 0, \mathrm{e}^{(a-s)\cdot 0} = 1, \frac{1}{a-s}(0-1) = \frac{1}{s-a} \text{ より} \\
&\quad\ \mathcal{L}\{\mathrm{e}^{at}\} = \frac{1}{s-a}
\end{aligned}$$

問 3　$\mathcal{L}\{\sin(at)\}$ を求めよ.

$$\mathcal{L}\{\sin(at)\} = \int_0^\infty \mathrm{e}^{(-s)t}\sin(at)\,\mathrm{d}t$$

積の微分公式 (Product rule of differentiation)

$$\frac{\mathrm{d}}{\mathrm{d}t}(u\,v) = u'\,v + u\,v'$$

積の微分公式の両辺を積分することで以下の式を得る.

部分積分 (Integration by parts, IBP)

$$u\,v = \int u'\,v\,\mathrm{d}t + \int u\,v'\,\mathrm{d}t$$
$$\int u'\,v\,\mathrm{d}t = u\,v - \int u\,v'\,\mathrm{d}t$$

$u' = \mathrm{e}^{(-s)t}$, $u = \frac{1}{-s}\mathrm{e}^{-st}$, $v = \sin(at)$, $v' = a \cdot \cos(at)$ として部分積分 $\int u'\,v\,\mathrm{d}t = u\,v - \int u\,v'\,\mathrm{d}t$ を適用する.

$$\begin{aligned}
\mathcal{L}\{\sin(at)\} &= \int_0^\infty \mathrm{e}^{(-s)t}\sin(at)\mathrm{d}t \\
&= \left[\frac{-1}{s}\mathrm{e}^{-st}\sin(at)\right]_0^\infty - \int_0^\infty -\frac{1}{s}\mathrm{e}^{-st}a\cos(at)\,\mathrm{d}t \\
&= \left[\frac{-\mathrm{e}^{-st}}{s}\sin(at)\right]_0^\infty + \frac{a}{s}\int_0^\infty \mathrm{e}^{-st}\cos(at)\,\mathrm{d}t
\end{aligned}$$

$u' = \mathrm{e}^{-st}$, $u = \frac{-1}{s}\mathrm{e}^{-st}$, $v = \cos(at)$, $v' = -a\sin(at)$ として部分積分を適用する.

$$\begin{aligned}
\mathcal{L}\{\sin(at)\} &= \left[\frac{-\mathrm{e}^{-st}}{s}\sin(at) + \frac{a}{s}\left(\frac{-1}{s}\mathrm{e}^{-st}\right)\cos(at)\right]_0^\infty - \frac{a}{s}\int_0^\infty \frac{-1}{s}\mathrm{e}^{-st}\left(-a\sin(at)\right)\mathrm{d}t \\
&= \left[\frac{-\mathrm{e}^{-st}}{s}\sin(at) - \frac{a}{s^2}\mathrm{e}^{-st}\cos(at)\right]_0^\infty - \frac{a^2}{s^2}\int_0^\infty \mathrm{e}^{-st}\sin(at)\mathrm{d}t \\
&= \left[\frac{-\mathrm{e}^{-st}}{s}\sin(at) - \frac{a}{s^2}\mathrm{e}^{-st}\cos(at)\right]_0^\infty - \frac{a^2}{s^2}\mathcal{L}\{\sin(at)\}
\end{aligned}$$

$$\begin{aligned}
\left(1 + \frac{a^2}{s^2}\right)\mathcal{L}\{\sin(at)\} &= \left[-\mathrm{e}^{-st}\left(\frac{\sin(at)}{s} + \frac{a\cos(at)}{s^2}\right)\right]_0^\infty \\
(s > 0\,\text{のとき}):\quad \frac{s^2+a^2}{s^2}\mathcal{L}\{\sin(at)\} &= 0 + 1\cdot\left(0 + \frac{a}{s^2}\right) \\
\frac{s^2+a^2}{s^2}\mathcal{L}\{\sin(at)\} &= \frac{a}{s^2} \\
\mathcal{L}\{\sin(at)\} &= \frac{a}{s^2}\frac{s^2}{s^2+a^2} = \frac{a}{s^2+a^2}
\end{aligned}$$

$$\therefore \mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2} \quad (s > 0)$$

この結果から,

$$
\begin{aligned}
\mathcal{L}\{\sin(t)\} &= \frac{1}{s^2+1} \quad (s>0) \\
\mathcal{L}\{\sin(2t)\} &= \frac{2}{s^2+4} \quad (s>0)
\end{aligned}
$$

2　ラプラス変換の諸性質

2.1　ラプラス変換の線形法則 (Laplace transform as linear operator)

$$
\begin{aligned}
\mathcal{L}\{c_1 f(t) + c_2 g(t)\} &= \int_0^\infty \mathrm{e}^{(-s)t}\left(c_1 f(t) + c_2 g(t)\right)\,\mathrm{d}t \\
&= \int_0^\infty \left(c_1 \mathrm{e}^{(-s)t} f(t) + c_2 \mathrm{e}^{(-s)t} g(t)\right)\,\mathrm{d}t \\
&= c_1 \int_0^\infty \mathrm{e}^{(-s)t} f(t)\,\mathrm{d}t + c_2 \int_0^\infty \mathrm{e}^{(-s)t} g(t)\,\mathrm{d}t \\
&= c_1 \mathcal{L}\{f(t)\} + c_2 \mathcal{L}\{g(t)\}
\end{aligned}
$$

$$
\therefore \mathcal{L}\{c_1 f(t) + c_2 g(t)\} = c_1 \mathcal{L}\{f(t)\} + c_2 \mathcal{L}\{g(t)\}
$$

2.2　ラプラス変換の微分法則 (Laplace transform of derivatives)

$$
\mathcal{L}\{f'(t)\} = \int_0^\infty \mathrm{e}^{-st} f'(t)\,\mathrm{d}t
$$

（ $f'(t)$ は f prime of t と読む．）

$u = \mathrm{e}^{-st}$, $u' = -s\mathrm{e}^{-st}$, $v' = f'(t)$, $v = f(t)$ として部分積分 $\int u\,v'\,\mathrm{d}t = u\,v - \int u'\,v\,\mathrm{d}t$ を適用する.

$$
\begin{aligned}
\mathcal{L}\{f'(t)\} &= \left[\mathrm{e}^{-st}\,f(t)\right]_0^\infty - \int_0^\infty -s\mathrm{e}^{-st} f(t)\,\mathrm{d}t \\
&= \left[\mathrm{e}^{-st}\,f(t)\right]_0^\infty + s\int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t \\
&= \left[\mathrm{e}^{-st}\,f(t)\right]_0^\infty + s\mathcal{L}\{f(t)\} \\
&= \lim_{A\to\infty}\left(\mathrm{e}^{-sA} f(A)\right) - 1\cdot f(0) + s\mathcal{L}\{f(t)\}
\end{aligned}
$$

$f(t)$ が $\lim_{A\to\infty}\left(\mathrm{e}^{-sA} f(A)\right)$ が発散するような指数関数ではなく, $s>0$ のとき $\lim_{A\to\infty}\left(\mathrm{e}^{-sA} f(A)\right) = 0$

$$
\therefore \mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)
$$

問 4 $\mathcal{L}\{\cos(at)\}$ を求めよ.

$$\mathcal{L}\{\sin(at) = \frac{a}{s^2+a^2}$$

$f'(t)=\cos(at), f(t)=\frac{1}{a}\sin(at)$ として, $\mathcal{L}\{f'(t)\}=s\mathcal{L}\{f(t)\}-f(0)$ を適用する.

$$\begin{aligned}
\mathcal{L}\{\cos(at)\} &= s\mathcal{L}\{\frac{1}{a}\sin(at)\}-\frac{1}{a}\sin(0)\\
&= \frac{s}{a}\mathcal{L}\{\sin(at)\}=\frac{s}{s^2+a^2}
\end{aligned}$$

2.3 二階微分のラプラス変換 (Laplace transform of second derivatives)

$$\begin{aligned}
\mathcal{L}\{f''(t)\} &= s\mathcal{L}\{f'(t)\}-f'(0)\\
&= s\left(s\mathcal{L}\{f(t)\}-f(0)\right)-f'(0)\\
&= s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)
\end{aligned}$$

2.4 多項式のラプラス変換 (Laplace transform of polynomials)

問 5 $\mathcal{L}\{t\}$ を求めよ.

$\mathcal{L}\{1\}=\frac{1}{s}$, $\mathcal{L}\{f'(t)\}=s\mathcal{L}\{f(t)\}-f(0)$ を用いる.

$$\begin{aligned}
\mathcal{L}\{f'(t)\} &= s\mathcal{L}\{f(t)\}-f(0)\\
\mathcal{L}\{f'(t)\}+f(0) &= s\mathcal{L}\{f(t)\}\\
\therefore \mathcal{L}\{f(t)\} &= \frac{1}{s}\left(\mathcal{L}\{f'(t)\}+f(0)\right)
\end{aligned}$$

$\mathcal{L}\{1\}=\frac{1}{s},\ f'=1,\ f=t,\ f(0)=0$ を $\mathcal{L}\{f(t)\}=\frac{1}{s}\left(\mathcal{L}\{f'(t)\}+f(0)\right)$ に適用する.

$$\begin{aligned}
\mathcal{L}\{t\} &= \frac{1}{s}\left(\mathcal{L}\{f'\}+f(0)\right)\\
&= \frac{1}{s}\left(\frac{1}{s}+0\right)=\frac{1}{s^2}
\end{aligned}$$

$$\therefore \mathcal{L}\{t\} = \frac{1}{s^2} \quad (s>0)$$

IBP を使用した他の解法.

$$\mathcal{L}\{t\} = \int_0^\infty \mathrm{e}^{-st}\,t\,\mathrm{d}t$$

$\int u'\,v\,\mathrm{d}t=u\,v-\int u\,v'\,\mathrm{d}t$, $u=\mathrm{e}^{-st}, u'=\frac{1}{-s}\mathrm{e}^{-st}, v=t, v'=1, \mathcal{L}\{1\}=\frac{1}{s}$ より

$$
\begin{aligned}
\mathcal{L}\{t\} &= \left[(-\frac{1}{s}\mathrm{e}^{-st})\cdot t\right]_0^\infty - \int_0^\infty (\frac{-1}{s}\mathrm{e}^{-st})\cdot 1\,\mathrm{d}t \\
&= \left[-\frac{t}{s}\mathrm{e}^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}t \\
&= \lim_{A\to\infty}(\frac{A}{s}\mathrm{e}^{-sA}) + (\frac{0}{s}\mathrm{e}^{-s0}) + \frac{1}{s}\mathcal{L}\{1\} \\
&= 0 + 0 + \frac{1}{s}\mathcal{L}\{1\} \quad (s>0) \\
&= \frac{1}{s}\mathcal{L}\{1\} = \frac{1}{s^2}
\end{aligned}
$$

$$
\therefore \mathcal{L}\{t\} = \frac{1}{s^2} \quad (s>0)
$$

問 6 $\mathcal{L}\{t^2\}$ を求めよ.

$f = t^2,\ f' = 2t,\ f(0) = 0$ を $\mathcal{L}\{f(t)\} = \frac{1}{s}\left(\mathcal{L}\{f'(t)\} + f(0)\right)$ に適用する.

$$
\mathcal{L}\{t^2\} = \frac{1}{s}\{\mathcal{L}\{2t\} + 0\} = \frac{2}{s}\mathcal{L}\{t\} = \frac{2}{s^3}
$$

$$
\therefore \mathcal{L}\{t^2\} = \frac{2}{s^3} \quad (s>0)
$$

問 7 $\mathcal{L}\{t^3\}$ を求めよ.

$f = t^3,\ f' = 3t^2$, 問 6 と同様にして

$$
\mathcal{L}\{t^3\} = \frac{1}{s}\{\mathcal{L}\{3t^2\} + 0\} = \frac{6}{s^4}
$$

$$
\therefore \mathcal{L}\{t^3\} = \frac{6}{s^4} \quad (s>0)
$$

問 8 $\mathcal{L}\{t^n\}$ （ $n>0$ の整数 ）を求めよ.

$$
\mathcal{L}\{t^n\} = \int_0^\infty t^n \mathrm{e}^{-st}\,\mathrm{d}t
$$

$\int u\,v'\,\mathrm{d}t = u\,v - \int u'\,v\,\mathrm{d}t,\ u = t^n, u' = n\cdot t^{n-1},\ v' = \mathrm{e}^{-st}, v = \frac{1}{-s}\mathrm{e}^{-st}$ より,

$$
\begin{aligned}
\mathcal{L}\{t^n\} &= \left[-t^n\cdot \mathrm{e}^{-st}\right]_0^\infty - \int_0^\infty n\cdot t^{n-1}\cdot\frac{\mathrm{e}^{-st}}{s}\,\mathrm{d}t \\
&= 0 - \frac{-0^n \mathrm{e}^{-s0}}{0} + \frac{n}{s}\int_0^\infty t^{n-1}\mathrm{e}^{-st}\,\mathrm{d}t
\end{aligned}
$$

ここで $\int_0^\infty t^{n-1}\mathrm{e}^{-st}\,\mathrm{d}t = \mathcal{L}\{t^{n-1}\}$ であるから,

$$
\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\}
$$

$$
\begin{aligned}
\mathcal{L}\{t^1\} &= \frac{1}{s^2} \quad (s>0) \\
\mathcal{L}\{t^2\} &= \frac{2}{s}\mathcal{L}\{t^1\} = \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2}{s^3} \\
\mathcal{L}\{t^3\} &= \frac{3}{s}\mathcal{L}\{t^2\} = \frac{3}{s}\cdot\frac{2}{s}\cdot\frac{1}{s^2} = \frac{3!}{s^4} \\
\mathcal{L}\{t^4\} &= \frac{4}{s}\mathcal{L}\{t^3\} = \frac{4}{s}\cdot\frac{3!}{s^4} = \frac{4!}{s^5} \\
&\cdots
\end{aligned}
$$

$$
\therefore \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad (s>0 \text{ かつ } n>0 \text{ の整数})
$$

$n!$ は n factorial と読む.

3 ヘヴィサイドの階段関数 (Unit step function)

ヘヴィサイドの階段関数の定義 (Definition of the unit step function)

$$
u_c(t) = \begin{cases} 0 & t<c \text{ のとき} \\ 1 & t \geq c \text{ のとき} \end{cases}
$$

$u_c(t)$ は u subscript c of t とか unit step function starts at c of t と読む.

$\begin{cases} 0 & t<c \\ 1 & t\geq c \end{cases}$ は defined as zero when t is less than c and defined as one when t is greater than or equal to c と読む.

使用例 1: $t<\pi$ の領域は 2 を戻し, 他の領域は 0 を戻す関数.

$$
2 - 2\cdot u_\pi(t)
$$

使用例 2: $t<\pi$ の領域の値は 2 で, $\pi <= t < 2\pi$ の領域の値が 0, $2\pi < t$ の領域の値は 2 の関数.

$$
2 - 2\cdot u_\pi(t) + 2\cdot u_{2\pi}(t)
$$

使用例 3: 関数 $f(t)$ を $t+$ 方向に 3 だけ平行移動し, $t<3$ の領域の値を 0 にした関数 $g(t)$.

$$
g(t) = u_3(t)\cdot f(t-3)
$$

問 9: 関数 $u_c(t)\cdot f(t-c)$ をラプラス変換せよ.

$$
\mathcal{L}\{u_c(t)\cdot f(t-c)\} = \int_{(t=)0}^{(t=)\infty} \mathrm{e}^{-st} u_c(t)\, f(t-c)\,\mathrm{d}t
$$

$t<c$ の領域の値は 0 なので, 積分範囲は $c <= t$ に狭めることができる.

$$
= \int_{(t=)c}^{(t=)\infty} \mathrm{e}^{-st} u_c(t)\, f(t-c)\,\mathrm{d}t
$$

ここで, $u_c(t)$ は全積分範囲で 1 なので,

$$= \int_{(t=)c}^{(t=)\infty} \mathrm{e}^{-st} f(t-c)\,\mathrm{d}t$$

$x=t-c$ と置いて t を x に変数変換する. $t=x+c$, $\frac{\mathrm{d}x}{\mathrm{d}t}=1$, $\mathrm{d}x=\mathrm{d}t$ より,

$$\begin{aligned}
&= \int_{(x=)0}^{(x=)\infty} \mathrm{e}^{-s(x+c)} f(x)\,\mathrm{d}x \\
&= \int_0^{\infty} \mathrm{e}^{-sx-sc} f(x)\,\mathrm{d}x \\
&= \mathrm{e}^{-sc} \int_0^{\infty} \mathrm{e}^{-sx} f(x)\,\mathrm{d}x
\end{aligned}$$

ここで, $\int_{(x=)0}^{(x=)\infty} \mathrm{e}^{-sx} f(x)\,\mathrm{d}x$ は, $\int_{(t=)0}^{(t=)\infty} \mathrm{e}^{-st} f(t)\,\mathrm{d}t$ の積分計算用ループ変数 t が x に変わっただけであり計算の内容は同じである. $\int_{(x=)0}^{(x=)\infty} \mathrm{e}^{-sx} f(x)\,\mathrm{d}x = \int_{(t=)0}^{(t=)\infty} \mathrm{e}^{-st} f(t)\,\mathrm{d}t = \mathcal{L}\{f(t)\}$ より,

$$\begin{aligned}
\mathcal{L}\{u_c(t)\cdot f(t-c)\} &= \mathrm{e}^{-sc}\int_0^{\infty} \mathrm{e}^{-sx} f(x)\,\mathrm{d}x \\
&= \mathrm{e}^{-sc}\mathcal{L}\{f(t)\}
\end{aligned}$$

$$\therefore \mathcal{L}\{u_c(t)\cdot f(t-c)\} = \mathrm{e}^{-sc}\mathcal{L}\{f(t)\}$$

問 10: 関数 $u_\pi(t)\cdot\sin(t-\pi)$ をラプラス変換せよ.

$$\begin{aligned}
\mathcal{L}\{u_\pi(t)\cdot\sin(t-\pi)\} &= \mathrm{e}^{-\pi s}\mathcal{L}\{\sin(t)\} \\
&= \frac{\mathrm{e}^{-\pi s}}{s^2+1}
\end{aligned}$$

4　ディラックの衝撃関数 (Dirac delta function)

4.1　ディラックの衝撃関数の導入

以下の関数 $d_\tau(t)$ の積分を考える.

$$d_\tau(t) = \begin{cases} \frac{1}{2\tau} & -\tau < t < \tau \\ 0 & t \text{ が他の範囲} \end{cases}$$

τ は tau と読む. $d_\tau(t) = \begin{cases} \frac{1}{2\tau} & -\tau < t < \tau \\ 0 & t \text{ が他の範囲} \end{cases}$ は d sub tau equals one over two tau when t is less than tau and greater than minus tau and defined as zero everywhere else などと読む.

関数 $d_\tau(t)$ は $t=0$ 上に幅 2τ, 高さ $\frac{\tau}{2}$ の長方形を作る. その面積は 1 である. よって,

$$\int_{-\infty}^{\infty} d_\tau(t)\,\mathrm{d}t = 1$$

τ の値を 0 に近づけていくと, 面積が 1 で高さが無限に高く幅が無限に狭い長方形ができる. これがディラックの衝撃関数 $\delta(t)$ である. δ は delta と読む.

$$\lim_{\tau\to 0} d_\tau(t) = \delta(t)$$

$\delta(t-3)$ は, 衝撃が $t=3$ の位置に平行移動したものである.

$2\delta(t)$ は, 面積が 2 であり, 衝撃が $\delta(t)$ の 2 倍強い.

4.2 ディラックの衝撃関数のラプラス変換

問 11: 関数 $\delta(t-c)\,f(t)$ をラプラス変換せよ.

$$\mathcal{L}\{\delta(t-c)\,f(t)\} \quad = \quad \int_0^\infty \mathrm{e}^{-st} f(t)\delta(t-c)\,\mathrm{d}t$$

関数 $\mathrm{e}^{-st}f(t)$ は $t=c$ の近傍で $\lim_{\tau\to 0}\mathrm{e}^{-s(c+\tau)}f(c+\tau)=\mathrm{e}^{-cs}f(c)$, $\delta(t-c)$ は $c-\tau<t<c+\tau$ 以外の領域で 0 のため,

$$\begin{aligned} &= \int_0^\infty \mathrm{e}^{-cs} f(c)\delta(t-c)\,\mathrm{d}t \\ &= \mathrm{e}^{-cs} f(c)\int_0^\infty \delta(t-c)\,\mathrm{d}t \end{aligned}$$

ここで $\int_0^\infty \delta(t-c)\,\mathrm{d}t = 1$ より,

$$\therefore \mathcal{L}\{\delta(t-c)\,f(t)\} \quad = \quad \mathrm{e}^{-cs} f(c)$$

問 12: 関数 $\delta(t)$ をラプラス変換せよ.
問 11 の結果に $f(t)=1$, $c=0$ を代入すると,

$$\mathcal{L}\{\delta(t)\} \quad = \quad 1$$

問 13: 関数 $\delta(t-c)$ をラプラス変換せよ.

$$\mathcal{L}\{\delta(t-c)\} \quad = \quad \mathrm{e}^{-cs}$$

5 ラプラス変換の表

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= F(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\mathrm{d}t \\
\mathcal{L}\{1\} &= \frac{1}{s} \\
\mathcal{L}\{\mathrm{e}^{at}\,f(t)\} &= F(s-a) \\
\mathcal{L}\{c_1 f(t) + c_2 g(t)\} &= c_1\mathcal{L}\{f(t)\} + c_2\mathcal{L}\{g(t)\} \\
\mathcal{L}\{f'(t)\} &= s\mathcal{L}\{f(t)\} - f(0) \\
\mathcal{L}\{t^n\} &= \frac{n!}{s^{n+1}} \quad (n>0\text{ の整数}) \\
\mathcal{L}\{\sin(at)\} &= \frac{a}{s^2+a^2} \\
\mathcal{L}\{\cos(at)\} &= \frac{s}{s^2+a^2} \\
\mathcal{L}\{u_c(t)\,f(t-c)\} &= \mathrm{e}^{-cs}F(s) \\
\mathcal{L}\{\delta(t-c)\,f(t)\} &= \mathrm{e}^{-cs}f(c)
\end{aligned}$$

6　逆ラプラス変換 (Inverse Laplace transform)

問 13: 関数 $F(s) = \frac{3!}{(s-2)^4}$ を逆ラプラス変換せよ.

ラプラス変換の表より,

$$\begin{aligned} \mathcal{L}\{t^3\} &= \frac{3!}{s^4} \\ \mathcal{L}\{\mathrm{e}^{2t} f(t)\} &= F(s-2) \end{aligned}$$

よって,

$$\begin{aligned} \mathcal{L}\{\mathrm{e}^{2t} t^3\} &= \frac{3!}{(s-2)^4} \\ \therefore \mathcal{L}^{-1}\left\{\frac{3!}{(s-2)^4}\right\} &= \mathrm{e}^{2t} t^3 \end{aligned}$$

問 14: 関数 $\frac{2(s-1)\mathrm{e}^{-2s}}{s^2-2s+2}$ を逆ラプラス変換せよ.

$s^2 - 2s + 2 = (s^2 - 2s + 1) + 1 = (s-1)^2 + 1$
ラプラス変換の表より,

$$\begin{aligned} \mathcal{L}\{\cos(t)\} &= \frac{s}{s^2+1} \\ \mathcal{L}\{\mathrm{e}^t f(t)\} &= F(s-1) \\ \mathcal{L}\{u_2(t) f(t-2)\} &= \mathrm{e}^{-2s} F(s) \\ \mathcal{L}\{2 \cdot f(t)\} &= 2 \cdot F(s) \end{aligned}$$

よって,

$$\begin{aligned} \mathcal{L}\{\mathrm{e}^t \cos(t)\} &= \frac{s-1}{(s-1)^2+1} \\ \mathcal{L}\{u_2(t)\mathrm{e}^{t-2} \cos(t-2)\} &= \mathrm{e}^{-2s}\frac{s-1}{(s-1)^2+1} \\ \mathcal{L}\{2 \cdot u_2(t)\mathrm{e}^{t-2} \cos(t-2)\} &= \frac{2(s-1)}{(s-1)^2+1}\mathrm{e}^{-2s} \\ \therefore \mathcal{L}^{-1}\left\{\frac{2(s-1)\mathrm{e}^{-2s}}{s^2-2s+2}\right\} &= 2 \cdot u_2(t)\mathrm{e}^{t-2} \cos(t-2) \end{aligned}$$

7　ラプラス変換を用いた微分方程式の解法 (Using the Laplace transform to solve a differential equation)

問 15: $y'' + 5y' + 6y = 0$ を初期条件 $y(0) = 2$, $y'(0) = 3$ のもとで解け.

$$\mathcal{L}\{y''\} + 5\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = 0 (= \mathcal{L}\{0\})$$

$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$ より，

$$
\begin{aligned}
&s\mathcal{L}\{y'\} - y'(0) + 5\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = 0\\
&s(s\mathcal{L}\{y\} - y(0)) - y'(0) + 5(s\mathcal{L}\{y\} - y(0)) + 6\mathcal{L}\{y\} = 0\\
&(s^2 + 5s + 6)\mathcal{L}\{y\} - sy(0) - y'(0) - 5y(0) = 0\\
&(s^2 + 5s + 6)\mathcal{L}\{y\} - 2s - 13 = 0\\
&\mathcal{L}\{y\} = \frac{2s+13}{s^2+5s+6} = \frac{2s+13}{(s+2)(s+3)}\\
&y = \mathcal{L}^{-1}\left\{\frac{2s+13}{(s+2)(s+3)}\right\}
\end{aligned}
$$

部分分数分解 (Partial fraction expansion)

$$
\begin{aligned}
&\frac{2s+13}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3} \text{とおくと，}\\
&\frac{A(s+3)+B(s+2)}{(s+2)(s+3)} = \frac{(A+B)s+3A+2B}{s+2} = \frac{2s+13}{(s+2)(s+3)}\\
&A + B = 2,\ 3A + 2B = 13,\ 3A + 2(2-A) = 13,\\
&A = 13 - 4 = 9,\ B = 2 - 9 = -7\\
&\therefore \frac{2s+13}{(s+2)(s+3)} = \frac{9}{s+2} - \frac{7}{s+3}
\end{aligned}
$$

ラプラス変換の表より，

$$
\begin{aligned}
\mathcal{L}\{\mathrm{e}^{at} f(t)\} &= F(s-a)\\
\mathcal{L}\{1\} &= \frac{1}{s}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}\{\mathrm{e}^{at}\} &= \frac{1}{s-a}\\
\mathcal{L}\{y\} &= 9\mathcal{L}\left\{\mathrm{e}^{-2t}\right\} - 7\mathcal{L}\left\{\mathrm{e}^{-3t}\right\}\\
&= \mathcal{L}\left\{9\mathrm{e}^{-2t} - 7\mathrm{e}^{-3t}\right\}
\end{aligned}
$$

$$\therefore y = 9\mathrm{e}^{-2t} - 7\mathrm{e}^{-3t}$$

検算

$$
\begin{aligned}
&y' = 9(-2\mathrm{e}^{-2t}) - 7(-3\mathrm{e}^{-3t}) = -18\mathrm{e}^{-2t} + 21\mathrm{e}^{-3t}\\
&y'' = -18(-2\mathrm{e}^{-2t}) + 21(-3\mathrm{e}^{-3t}) = 36\mathrm{e}^{-2t} - 63\mathrm{e}^{-3t}
\end{aligned}
$$

$$
\begin{aligned}
y'' + 5y' + 6y &= 36\mathrm{e}^{-2t} - 63\mathrm{e}^{-3t} \\
&+ 5(-18\mathrm{e}^{-2t} + 21\mathrm{e}^{-3t}) \\
&+ 6(9\mathrm{e}^{-2t} - 7\mathrm{e}^{-3t}) \\
&= 36\mathrm{e}^{-2t} - 63\mathrm{e}^{-3t} \\
&- 90\mathrm{e}^{-2t} + 105\mathrm{e}^{-3t} \\
&+ 54\mathrm{e}^{-2t} - 42\mathrm{e}^{-3t} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
y(0) &= 9\mathrm{e}^0 - 7\mathrm{e}^0 = 2 \\
y'(0) &= -18\mathrm{e}^0 + 21\mathrm{e}^0 = 3
\end{aligned}
$$

問 16: 非斉次方程式 (Nonhomogeneous differential equation) $y'' + y = \sin(2t)$ を初期条件 $y(0) = 2$, $y'(0) = 1$ のもとで解け.

$$
\begin{aligned}
\mathcal{L}\{y''\} &= s\mathcal{L}\{y'\} - y'(0) \\
&= s^2\mathcal{L}\{y\} - s\,y(0) - y'(0) \\
&= s^2 Y(s) - s\,y(0) - y'(0) = s^2 Y(s) - 2s - 1
\end{aligned}
$$

$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$ より,

$$
\begin{aligned}
\mathcal{L}\{y'' + y\} &= \mathcal{L}\{\sin(2t)\} \\
s^2 Y(s) - 2s - 1 + Y(s) &= \frac{2}{s^2+4} \\
(s^2+1)Y(s) &= \frac{2}{s^2+4} + 2s + 1 \\
Y(s) &= \frac{2}{(s^2+4)(s^2+1)} + \frac{2s}{s^2+1} + \frac{1}{s^2+1}
\end{aligned}
$$

部分分数分解

$$\begin{aligned}\frac{2}{(s^2+4)(s^2+1)} &= \frac{As+B}{s^2+4}+\frac{Cs+D}{s^2+1}\\ &= \frac{(As+B)(s^2+1)+(Cs+D)(s^2+4)}{(s^2+4)(s^2+1)}\end{aligned}$$

$As^3+Bs^2+As+B+Cs^3+Ds^2+4Cs+4D=2$
$(A+C)s^3+(B+D)s^2+(A+4C)s+B+4D=2$
$A+C=0,\ B+D=0,\ A+4C=0,\ B+4D=2$ より,
$A=-C,\ -C+4C=0,\ C=0,\ A=0$
$B=-D,\ -D+4D=2,\ 3D=2,\ D=\frac{2}{3},\ B=-\frac{2}{3}$

$$\therefore \frac{2}{(s^2+4)(s^2+1)} = \frac{\frac{-2}{3}}{s^2+4}+\frac{\frac{2}{3}}{s^2+1}$$

$$\begin{aligned}Y(s) &= \frac{\frac{-2}{3}}{s^2+4}+\frac{\frac{2}{3}}{s^2+1}+\frac{2s}{s^2+1}+\frac{1}{s^2+1}\\ &= -\frac{1}{3}\frac{2}{s^2+4}+\frac{2}{3}\frac{1}{s^2+1}+2\frac{s}{s^2+1}+\frac{1}{s^2+1}\end{aligned}$$

$\mathcal{L}\{\sin(a\,t)\}=\frac{a}{s^2+a^2}, \mathcal{L}\{\cos(a\,t)\}=\frac{s}{s^2+a^2}$ より,

$$\begin{aligned}y(t) &= \frac{-1}{3}\sin(2t)+\frac{2}{3}\sin(t)+2\cos(t)+\sin(t)\\ \therefore y(t) &= \frac{-1}{3}\sin(2t)+\frac{5}{3}\sin(t)+2\cos(t)\end{aligned}$$

検算

$$\begin{aligned}y' &= \frac{-2}{3}\cos(2t)+\frac{5}{3}\cos(t)-2\sin(t)\\ y'' &= \frac{4}{3}\sin(2t)-\frac{5}{3}\sin(t)-2\cos(t)\\ y''+y &= \frac{4}{3}\sin(2t)-\frac{5}{3}\sin(t)-2\cos(t)\\ &+ \frac{-1}{3}\sin(2t)+\frac{5}{3}\sin(t)+2\cos(t)=\sin(2t)\end{aligned}$$

$$\begin{aligned}y(0) &= \frac{-1}{3}\sin(0)+\frac{5}{3}\sin(0)+2\cos(0)=2\\ y'(0) &= \frac{-2}{3}\cos(0)+\frac{5}{3}\cos(0)-2\sin(0)=1\end{aligned}$$

問 17: $y''+4y=\sin(t)-u_{2\pi}(t)\sin(t-2\pi)$ を初期条件 $y(0)=0,\ y'(0)=0$ のもとで解け.

ラプラス変換の表より,

$$\begin{aligned}
\mathcal{L}\{f'(t)\} &= s\mathcal{L}\{f(t)\} - f(0) \\
\mathcal{L}\{f''(t)\} &= s^2\mathcal{L}\{f(t)\} - s\cdot f(0) - f'(0) \\
\mathcal{L}\{\sin(t)\} &= \frac{1}{s^2+1} \\
\mathcal{L}\{\sin(at)\} &= \frac{2}{s^2+4} \\
\mathcal{L}\{u_{2\pi}(t)\, f(t-2\pi)\} &= \mathrm{e}^{-2\pi s}F(s)
\end{aligned}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 4\mathcal{L}\{y\} = \frac{1}{s^2+1} - \mathcal{L}\{u_{2\pi}(t)\sin(t-2\pi)\}$$

$\mathcal{L}\{u_{2\pi}(t)\sin(t-2\pi)\} = \mathrm{e}^{-2\pi s}\frac{1}{s^2+1}$ より,

$$\begin{aligned}
s^2\mathcal{L}\{y\} + 4\mathcal{L}\{y\} &= \frac{1}{s^2+1} - \mathrm{e}^{-2\pi s}\frac{1}{s^2+1} \\
\mathcal{L}\{y\}\frac{1}{s^2+4} &= (1-\mathrm{e}^{-2\pi s})\frac{1}{s^2+1} \\
\mathcal{L}\{y\} &= (1-\mathrm{e}^{-2\pi s})\frac{1}{(s^2+1)(s^2+4)}
\end{aligned}$$

部分分数分解

$$\begin{aligned}
\frac{1}{(s^2+1)(s^2+4)} &= \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \\
&= \frac{(As+B)(s^2+4) + (Cs+D)(s^2+1)}{(s^2+1)(s^2+4)}
\end{aligned}$$

$As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2 + Cs + D = 1$
$(A+C)s^3 + (B+D)s^2 + (4A+C)s + 4B + D = 1$
$A+C=0,\ B+D=0,\ 4A+C=0,\ 4B+D=1$ より,
$A=-C,\ -4C+C=0,\ C=0,\ A=0$
$B=-D,\ -4D+D=1,\ -3D=1,\ D=\frac{-1}{3},\ B=\frac{1}{3}$

$$\therefore \frac{1}{(s^2+1)(s^2+4)} = \frac{\frac{1}{3}}{s^2+1} - \frac{\frac{1}{3}}{s^2+4}$$

$$\begin{aligned}
\mathcal{L}\{y\} &= (1-\mathrm{e}^{-2\pi s})\left(\frac{1}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{1}{s^2+4}\right) \\
&= (1-\mathrm{e}^{-2\pi s})\left(\frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4}\right) \\
&= \frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4} - \mathrm{e}^{-2\pi s}\frac{1}{3}\frac{1}{s^2+1} + \mathrm{e}^{-2\pi s}\frac{1}{6}\frac{2}{s^2+4}
\end{aligned}$$

ラプラス変換の表より,

$$\begin{aligned}
\mathcal{L}\{\sin(at)\} &= \frac{a}{s^2+a^2} \\
\mathcal{L}\{u_{2\pi}(t)\, f(t-2\pi)\} &= \mathrm{e}^{-2\pi s}F(s)
\end{aligned}$$

$$y = \frac{1}{3}\sin(t) - \frac{1}{6}\sin(2t) - \frac{1}{3}u_{2\pi}(t)\sin(t-2\pi) + \frac{1}{6}u_{2\pi}(t)\sin(2(t-2\pi))$$

8 畳み込み積分 (Convolution integral)

畳み込み積分の定義

$$(f*g)(t) = \int_{\tau=0}^{\tau=t} f(t-\tau)g(\tau)\mathrm{d}\tau$$

$f*g$ は convolution of f with g とか f star g などと読む。τ は tau と読む.

問 18: $f(t) = \sin(t)$, $g(t) = \cos(t)$ として $f*g(t)$ を求めよ.

$$(f*g)(t) = \int_0^t \sin(t-\tau)\cos(\tau)\mathrm{d}\tau$$

$\sin(t-\tau) = \sin(t)\cos(\tau) - \sin(\tau)\cos(t)$ より,

$$\begin{aligned}
(f*g)(t) &= \int_0^t (\sin(t)\cos(\tau) - \sin(\tau)\cos(t) - \tau)\cos(\tau)\mathrm{d}\tau \\
&= \int_0^t \sin(t)\cos^2(\tau) - \cos(t)\sin(\tau)\cos(\tau)\mathrm{d}\tau \\
&= \int_0^t \sin(t)\cos^2(\tau)\mathrm{d}\tau - \int_0^t \cos(t)\sin(\tau)\cos(\tau)\mathrm{d}\tau
\end{aligned}$$

τ で積分しているので $\sin(t)$ や $\cos(t)$ は外に出せる.

$$(f*g)(t) = \sin(t)\int_0^t \cos^2(\tau)\mathrm{d}\tau - \cos(t)\int_0^t \sin(\tau)\cos(\tau)\mathrm{d}\tau$$

$\cos^2(\tau) = \frac{1}{2}(1+\cos(2\tau))$, $u = \sin(\tau)$, $\frac{\mathrm{d}u}{\mathrm{d}\tau} = \cos(\tau)$, $\mathrm{d}u = \cos(\tau)\mathrm{d}\tau$, $\sin(2\tau) = 2\sin(\tau)\cos(\tau)$ を適用すると,

$$\begin{aligned}
(f*g)(t) &= \frac{1}{2}\sin(t)\int_0^t (1+\cos(2\tau))\mathrm{d}\tau - \cos(t)\int_{\tau=0}^{\tau=t} u\mathrm{d}u \\
&= \frac{1}{2}\sin(t)\int_0^t (1+\cos(2\tau))\mathrm{d}\tau - \cos(t)\left[\frac{1}{2}u^2\right]_{\tau=0}^{\tau=t} \\
&= \frac{1}{2}\sin(t)\left[\tau + \frac{1}{2}\sin(2\tau)\right]_0^t - \cos(t)\left[\frac{1}{2}\sin^2(\tau)\right]_0^t \\
&= \frac{1}{2}\sin(t)(t + \frac{1}{2}\sin(2t) - 0 - \frac{1}{2}\sin(0)) - \cos(t)(\frac{1}{2}\sin^2(t) - 0) \\
&= \frac{1}{2}t\cdot\sin(t) + \frac{1}{4}\sin(t)\sin(2t) - \frac{1}{2}\sin^2(t)\cos(t)
\end{aligned}$$

$\sin(2t) = 2\sin(t)\cos(t)$ より,

$$
\begin{aligned}
&= \frac{1}{2}t\cdot\sin(t)+\frac{1}{4}\sin(t)(2\sin(t)\cos(t))-\frac{1}{2}\sin^2(t)\cos(t)\\
&= \frac{1}{2}t\cdot\sin(t)+\frac{1}{2}\sin^2(t)\cos(t)-\frac{1}{2}\sin^2(t)\cos(t)\\
&= \frac{1}{2}t\cdot\sin(t)
\end{aligned}
$$

$$\therefore (\sin(t)*\cos(t))(t)=\frac{1}{2}t\cdot\sin(t)$$

8.1 畳み込み積分とラプラス変換

$\mathcal{L}\{f(t)\}=F(s)$, $\mathcal{L}\{g(t)\}=G(s)$ のとき,

$$
\begin{aligned}
\mathcal{L}\{f*g\} &= F(s)G(s)\\
f*g &= \mathcal{L}^{-1}\{F(s)G(s)\}
\end{aligned}
$$

問 19: $H(s)=\frac{2s}{(s^2+1)^2}$ のとき $\mathcal{L}^{-1}\{H(s)\}$ を求めよ.

$$\mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\}$$

$$\frac{2s}{(s^2+1)^2}=2\cdot\frac{1}{s^2+1}\cdot\frac{s}{s^2+1}$$

$$\mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{2\cdot\frac{1}{s^2+1}\cdot\frac{s}{s^2+1}\right\}$$

$F(s)=\frac{2}{s^2+1}$ とおくと $f(t)=2\sin(t)$, $G(s)=\frac{s}{s^2+1}$ とおくと $g(t)=\cos(t)$ より,

$$
\begin{aligned}
\mathcal{L}^{-1}\{H(s)\} &= \mathcal{L}^{-1}\{F(s)G(s)\}\\
&= \mathcal{L}^{-1}\{F(s)\}*\mathcal{L}^{-1}\{G(s)\}\\
&= \mathcal{L}^{-1}\left\{\frac{2}{s^2+1}\right\}*\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}\\
&= 2\sin(t)*\cos(t)
\end{aligned}
$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+1}\cdot\frac{s}{s^2+1}\right\}=2\sin(t)*\cos(t)$$

$f*g=\int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau$ より,

$$
\begin{aligned}
\mathcal{L}^{-1}\{H(s)\} &= 2\int_0^t \sin(t-\tau)\cos(\tau)\mathrm{d}\tau\\
&= 2\cdot\frac{1}{2}t\cdot\sin(t)\\
&= t\sin(t)
\end{aligned}
$$

問 20: $y'' + 2y' + 2y = \sin(\alpha t)$ を初期条件 , $y(0)$, $y'(0) = 0$ のもとで解け.

α は alpha と読む.

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 2Y(s) &= \frac{\alpha}{s^2+\alpha^2} \\ s^2Y(s) + 2sY(s) + 2Y(s) &= \frac{\alpha}{s^2+\alpha^2} \\ (s^2+2s+2)Y(s) &= \frac{\alpha}{s^2+\alpha^2} \end{aligned}$$

$$\begin{aligned} Y(s) &= \frac{\alpha}{s^2+\alpha^2} \cdot \frac{1}{s^2+2s+2} \\ &= \frac{\alpha}{s^2+\alpha^2} \cdot \frac{1}{s^2+2s+1+1} \\ &= \frac{\alpha}{s^2+\alpha^2} \cdot \frac{1}{(s+1)^2+1} \end{aligned}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{\alpha}{s^2+\alpha^2} \cdot \frac{1}{(s+1)^2+1}\right\}$$

ラプラス変換の表より,

$$\begin{aligned} \mathcal{L}\{\sin(\alpha t)\} &= \frac{\alpha}{s^2+\alpha^2} \\ \mathcal{L}\{\sin(t)\} &= \frac{1}{s^2+1} \\ \mathcal{L}\{\mathrm{e}^{at} f(t)\} &= F(s-a) \\ \mathcal{L}\{\mathrm{e}^{-t}\sin(t)\} &= \frac{1}{(s+1)^2+1} \end{aligned}$$

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{\alpha}{s^2+\alpha^2} \cdot \frac{1}{(s+1)^2+1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{\alpha}{s^2+\alpha^2}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} \\ &= \sin(\alpha t) * \mathrm{e}^{-t}\sin(t) \end{aligned}$$

$$(f*g)(t) = \int_{\tau=0}^{\tau=t} f(t-\tau)g(\tau)\mathrm{d}\tau$$

$$\begin{aligned} y(t) &= \sin(\alpha t) * \mathrm{e}^{-t}\sin(t) = \int_0^t \sin((t-\tau)\alpha)\mathrm{e}^{-\tau}\sin(\tau)\mathrm{d}\tau \\ &= \mathrm{e}^{-t}\sin(t) * \sin(\alpha t) = \int_0^t \mathrm{e}^{-(t-\tau)}\sin(t-\tau)\sin(\alpha\tau)\mathrm{d}\tau \end{aligned}$$

$$\begin{aligned} \therefore y(t) &= \int_0^t \sin((t-\tau)\alpha)\mathrm{e}^{-\tau}\sin(\tau)\mathrm{d}\tau \quad \text{または} \\ y(t) &= \int_0^t \mathrm{e}^{-(t-\tau)}\sin(t-\tau)\sin(\alpha\tau)\mathrm{d}\tau \end{aligned}$$

(答はどちらでも良い.)