A Functional View To Completeness of Algebra

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1 Introduction

Completeness is a important property in the study of algebras. Formally, a class of models are complete for a algebra $\mathcal{A}$ when a equality is always valid in this class of models M implies the equality is derivable using the algebra:

$$M \vDash t_1 = t_2 \Leftrightarrow \mathcal{A} \vdash t_1 = t_2.$$

The importance of completeness basically allows us to move between models and algebra whenever it is convenient:

- We can reason about equality in a simple complete model, and know that such a equality holds in general. For example, when dealing with Kleene Algebra, or Kleene algebra with tests, we typically reason in language models, which is complete (Kozen 1994, 1997).
- When we care about equality in a particular model, like relational model to model programs (Kozen 1997; Smolka et al. 2020). Then we know that all the equalities in the model we care about can be derived just using the equational theory.
- Further more, we can prove or disprove the equality in a complicated model in a simpler model.

Despite the importance of completeness of algebraic theory, completeness is generally very hard to prove from scratch (Kozen 1994; Smolka et al. 2020). Hence this give birth to the idea of reduction, seen in Kozen 1997 and later used in many different works. Pous, Rot, and Wagemaker 2022 generalized the idea of reduction and used to it prove several important completeness theorem for extensions of Kleene Algebras.

2 The Functional Account

It is unsurprising that the free model will be important to this development.

Definition 2.1. A free model $F_{\mathcal{A}}$ of a algebra $\mathcal{A}$ over a set of primitive P (or many sets if the algebra is multi-sorted) is all the terms from over P with operations in $\mathcal{A}$ mod out by provable equalities.

In other words,

$$F_{\mathcal{A}}(P) \vDash t_1 = t_2 \iff \mathcal{A} \vdash t_1 = t_2,$$

by the definition of free model, where t_1 and t_2 are two terms formed by primitives in P. This means that free model is a complete model, by definition.

Free model are typically also used to define a interpretation.

Definition 2.2. A interpretation in model M of algebra $\mathcal{A}$ is a homomorphism (a function that preserves all the operation and sort of $\mathcal{A}$) from the free model to M. i.e. $I : F_{\mathcal{A}}(P) \to \mathcal{A}$.

A interpretation is complete when $M \vDash I(t_1) = I(t_2) \iff \mathcal{A} \vdash t_1 = t_2$ for all term t_1 and t_2.

It is straightforward to see the $\Longleftarrow$ direction for completeness condition will always hold because I is a homomorphism, and the $\Longrightarrow$ direction is true when

$$M \vDash I(t_1) = I(t_2) \implies \mathcal{A} \vdash t_1 = t_2 \implies F_{\mathcal{A}}(P) \vDash t_1 = t_2,$$

which exactly describe that I is injective. Hence:

Corollary 2.3. *a interpretation I is complete iff it is injective.*

We define a extension of a algebraic theory

Definition 2.4. $\mathcal{A}_{ext}$ is a extension of an algebra $\mathcal{A}$, when there exists a homomorphism $[-] : F_{\mathcal{A}}(P) \to F_{\mathcal{A}_{ext}}(P)$ in $\mathcal{A}$.

The map $[-]$ will simply embed a term in $\mathcal{A}$ into $\mathcal{A}_{ext}$. Notice that there is no grantee that such a homomorphism is

- injective, because $\mathcal{A}_{ext}$ can have more equations than $\mathcal{A}$;
- surjective, because $\mathcal{A}_{ext}$ can have more operations than $\mathcal{A}$.

The only thing we know is such a map need to be a homomorphism in $\mathcal{A}$, since $\mathcal{A}_{ext}$ are not allowed to remove equations and operations from $\mathcal{A}$.

Finally, a functional definition of reduction

Definition 2.5. a reduction from an n-sorted extension $\mathcal{A}_{ext}$ to a m-sorted algebra $\mathcal{A}$ is

- a surjective map on primitives $r : Set^m \to Set^n$
- injective $\mathcal{A}_{ext}$-homomorphism $r : F_{\mathcal{A}}(P) \hookrightarrow F_{\mathcal{A}_{ext}}(r(P))$

We are abusing notation here using r for two different map, since which r is used can be inferred from context.

The definition of reduction might seem obscure here, but it is essentially taking advantage of the fact that a injective interpretation is complete.

Theorem 2.6. *Given a complete interpretation I of $\mathcal{A}$ and a reduction to*

3 Acknowledgement

This work is inspired by an unpublished proof by Arthur Azevedo de Amorim of the completeness on TopKAT with respect to language model and general relational models.

References

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