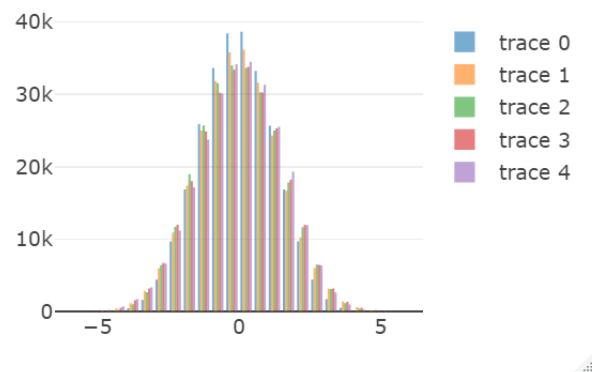
Our goal is to find a pdf for a given number of rare events.

For a given n and J,

$$E_{sig} = \sum_{1 \leq i < j \leq n} J_{i,j} sig_i sig_j.$$

Let D_i be the distribution of E_{sig} over all 2^n sig for the ith realization of J. D_i has mean 0 and variance given by the sum of ${J_{i,j}}^2$. We assume all D_i are normal.

Say E_{sig} is a rare event if it is less than $c\sqrt{n}$ ($\sqrt{n}$ begin the standard deviation of D_i). From our above assumption, the only thing that differs among different D_i is its variance. From 5 realizations of J with $n = 18$, this claim seems reasonable:



We want to find the probability of r rare events:

$$P(r \text{ rare events}) = P(\text{ area } (-\infty, c\sqrt{n}) = r)$$
$$= P(2^n \cdot \tfrac{1}{2}[1 + erf(\tfrac{c\sqrt{n}}{\sqrt{2}\sigma})] = r)$$

$$= P(\sigma = \frac{c\sqrt{n}}{\sqrt{2} erf^{-1}(r/(2^{n-1}) - 1)}) \tag{1}$$

The variance of D_i follows a chi-squared distribution with mean

$$[\text{degrees of freedom}] \cdot (varJ)$$
$$\tfrac{n(n-1)}{2} \cdot \tfrac{2}{n-1}$$

We can directly compute (1) a scaled chi-squared pdf. Let W denote the expression for the desired σ in (1). Then the adjusted quantitiy desired in a chi-squared pdf is $X := W \cdot \frac{n-1}{2}$.

$$P(r \text{ rare events})$$
$$= \tfrac{X^{nC2/2-1} e^{-X/2}}{2^{nC2/2} \Gamma(nC2/2)}$$