1. 提问：vector spaces "+" "·"

① $\mathbb{F} \subseteq \mathbb{C}$ $\mathbb{F}^n$ — $n=1$ $\mathbb{F}$; $n=2$ $\mathbb{F}^2$; $n=3$ $\mathbb{F}^3$

② $\mathbb{F}^{m \times n}$

} finite dimension

③ $\mathbb{F}[x]$ — ∞ dimension $1, x, x^2 \ldots$

2. 溯源.

$U \xrightarrow{f} V$

$f(ax+by) = af(x) + bf(y)$

eg. $\mathbb{F}[x]_n = \{f(x) \in \mathbb{F}[x] \mid \partial f(x) < n\} = \{\sum_{i=0}^{n} a_i x^i \mid a_i \in F\}$

$\cong \mathbb{F}^n$

$\to f: U \to V$

reduced to $\sigma: \mathbb{F}^n \longrightarrow \mathbb{F}^m$

$x \longmapsto A_{m\times n} x$

3. 情道

·多项式

$\mathbb{F}[x]$ —
- $\mathbb{C}[x]$ - FTOA ($f(x) = a\prod_{i=1}^{s}(x-\alpha_i)$
- $\mathbb{R}[x]$ $f(x) = a\prod_{i=1}^{s}\underset{\text{实根}}{(x-\alpha_i)}\prod_{j=1}^{t}\underset{\text{复根}}{(x^2+p_jx+q_j)}$
- $\{\mathbb{Q}[x], \mathbb{Z}[x]$

<u>Warning!</u> Polynomials are not functions!

eg. $f_1(x) = x \in \mathbb{F}_2[x]$ $f_2(x) \in \mathbb{F}_2[x]$

Polynomial: $f_1(x) \neq f_2(x)$

But $f_1(1) = f_2(1) = 1$, $f_1(0) = f_2(0) = 0$

function: $f_1(x) \equiv f_2(x)$

[$\partial 0 = -\infty$]

Date. / /

eg. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

① $|\sin x| \le 1$, $\forall x \in \mathbb{F}$

② $x = i$, $e^{ix} = \cos x + i\sin x \Rightarrow \sin x = \frac{e^{-1} - e}{2}$

$\Rightarrow |\sin i| = \left|\frac{e^{-2} - e^{2}}{2}\right| > 1$

Liewille's (刘维尔) Thm.

Let $f(z)$ be analytic (解析), given $M > 0$. If $|f(z)| \le M$ $\forall x \in \mathbb{C}$

then $f(z) \equiv f(0)$ $\sum_{i=0} a_i z^i$

FToA: Fundemental Theereom of Algebra

Let $f(x) \in \mathbb{C}[x]$ $\partial f \ge 1$. Then $\exists\, c \in \mathbb{C}$, such that $f(c) = 0$. In other words,

$f(x) = (x - c)\, g(x)$, for some $g(x) \in \mathbb{C}[x]$

pf. (反证) if not, then $\forall c \in \mathbb{C}$ $f(c) \ne 0$

set $g(z) = \frac{1}{f(z)}$ → easy to see $|g(z)| \le M$.

[$\exists N$, $|z| > N$ 时, $\frac{1}{f(z)} < 1$; $|z| \le N$ 时 $g(z) \le \max\{g(z)\} = R$

$\Rightarrow |g(z)| \le R + 1$ (有界)]

by Liewille's Thm: $g(z) \equiv g(0)$ [与 $\partial f \ge 1$ 矛盾]

Division with Remainder (带余除法)

$f(x), g(x) \in \mathbb{F}[x]$, $g(x) \ne 0$. Then $\exists!\, (q(x), r(x)) \in \mathbb{F}[x]^2$, $\partial r < \partial g$. s.t.

$f(x) = g(x)\, q(x) + r(x)$

WLOG, may assume $\partial g(x) \le \partial f(x) = n$ Take $V = \mathbb{F}[x]_{n+1}$, $f(x), g(x) \in V$

a basis of V: $1, x, \cdots, x^{m-1}$, $g(x), xg(x), \cdots$ $x^{n-m} g(x)$ so

~~$f(x) = a_0 + a_1 x + \cdots + a_{m-1} x^{m-1} + a_m g(x) + \cdots + a_n x^{n-m} g(x)$~~

~~$= g(x)(a_m + \cdots + a_n x^{n-m}) + r(x)$~~

$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{m-1} x^{m-1} + a_m g(x) + a_{m+1} x g(x) + \cdots + a_n x^{m-n} g(x)$

$= g(x)\,(a_m + a_{m+1} x + \cdots + a_n x^{m-n}) + r(x)$ ∎

对 $f(x) = (x - c)\, g(x)$ 的证明:

$f(x) = (x - c)\, g(x) + r(x)$

But $f(c) = 0 \Rightarrow r(c) = 0$

$(x - c)$ 为一次, 则 $\partial r(x) < 1 \Rightarrow r(x) \equiv 0$