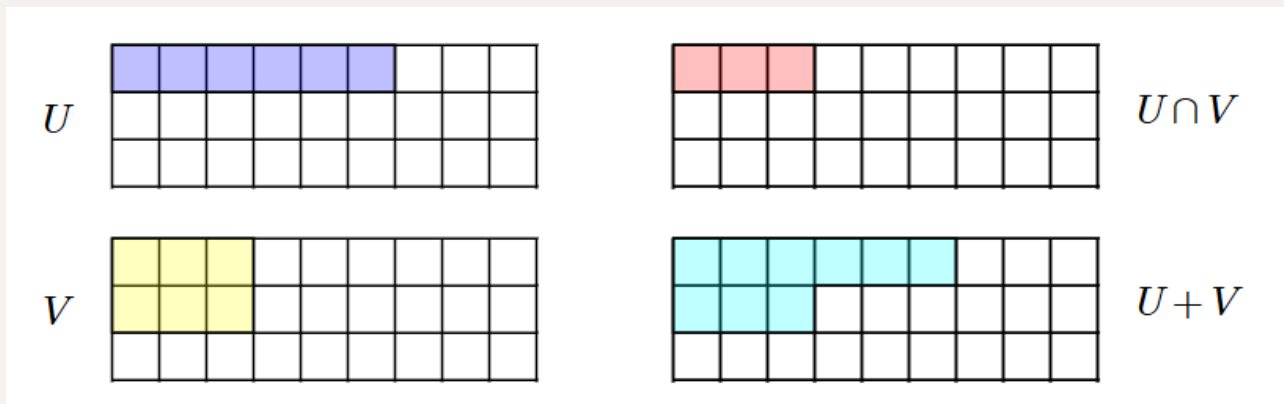


A kindergarten puzzle on Zassenhaus lemma

Throughout, it is *INVALID* to take basis.

Example Let U and V be linear subspaces. Determine (guess) what the following diagram represents.

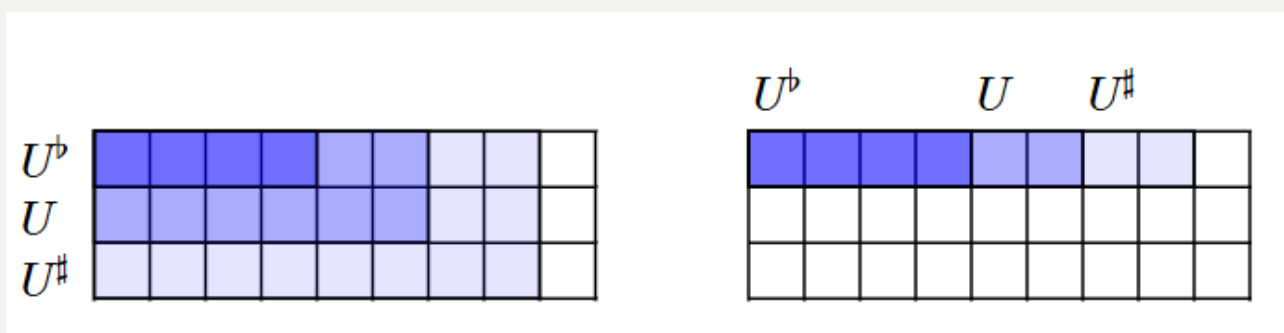


Exercise Prove the following identities of subspaces

1. $(U \cap V) \cap W = U \cap (V \cap W)$;
2. $U + V = V + U$.

Exercise How would you denote the 0-subspace and the entire vector space in the diagram?

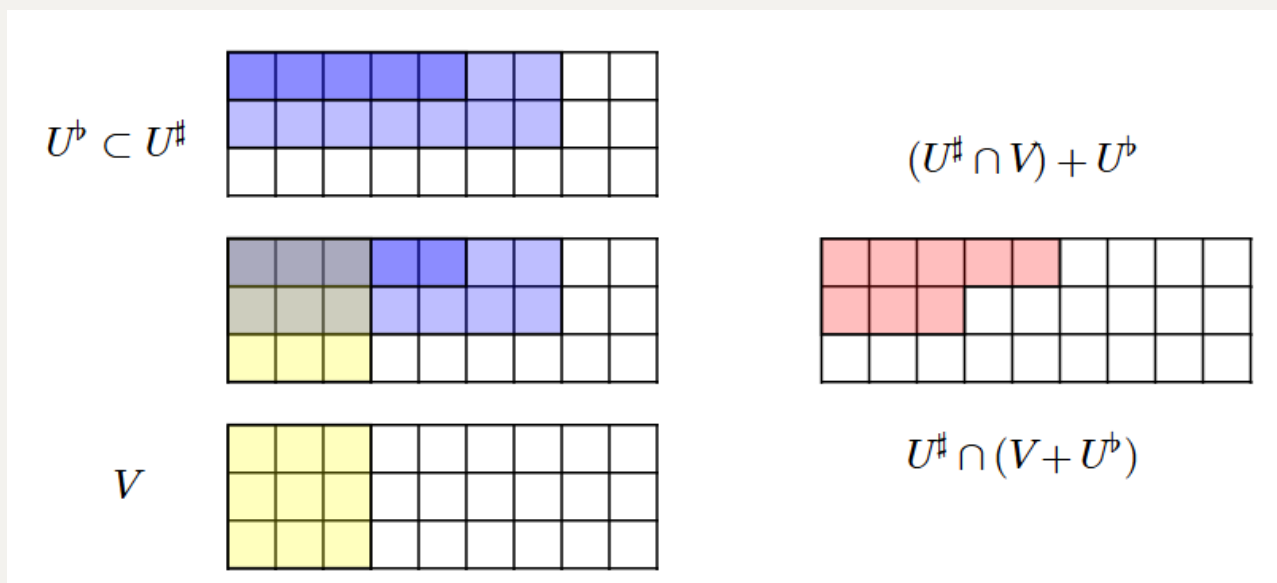
Definition Let $U^b \subset U \subset U^\sharp$ be inclusions of subspaces:



From now onwards, the symbol U^\sharp (respectively U^b) denotes a subspace that contains (respectively is contained by) U in the rest of the adventure.

Theorem (Modular identity, recall the homework months ago)

$$(U^\sharp \cap V) + U^b = U^\sharp \cap (V + U^b).$$

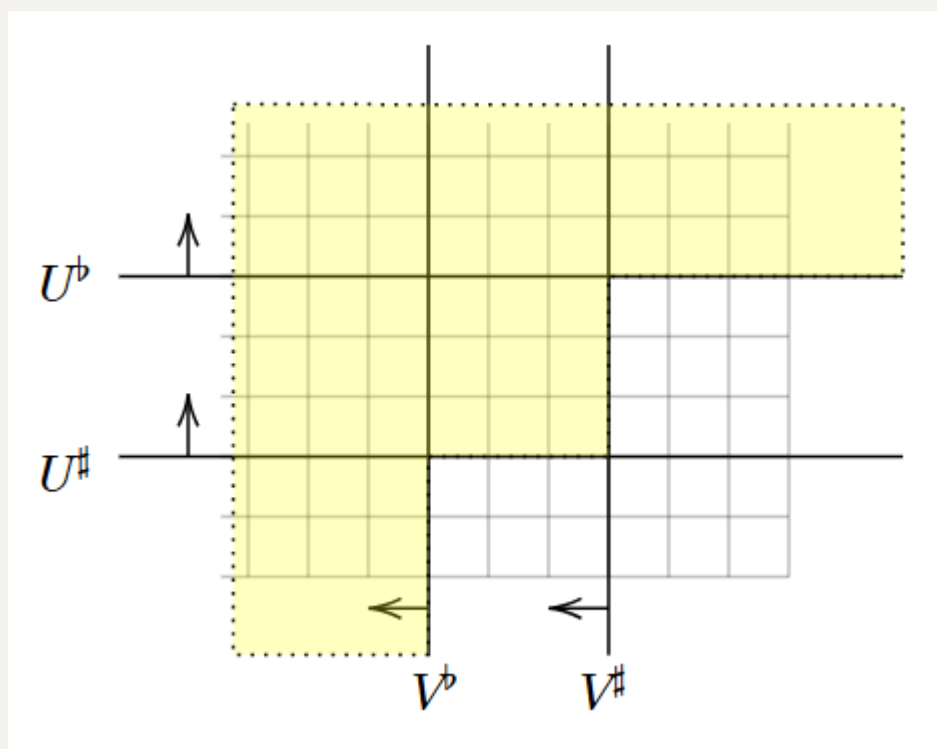


Exercise Prove (in diagrams) that the subspace

$$U^b + V^\# \cap U^\# + V^b$$

is well-defined. In other words, there is no need to add any round brackets (parentheses in Ameliclish).

Hint



Exercise Hunt for a prove **Isomorphism theorem A** in *done right*, which states that

- Let $f : U \rightarrow V$ be linear map. Then there exists an isomorphism

$$\frac{U}{\ker(f)} \rightarrow \text{im}(f), \quad [u] \mapsto f(u),$$

where:

- a. $U/\ker(f)$ is quotient set of U , the equivalence classes of U characterised by the relation $[u_1 \sim u_2] \iff [u_1 - u_2 \in \ker(f)]$, and

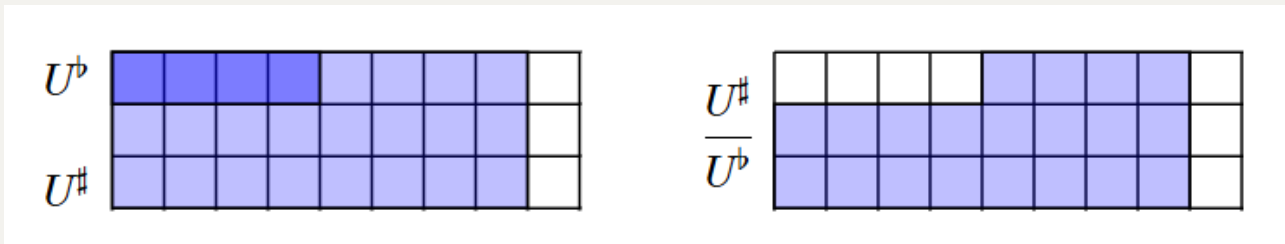
$$[u] := \{u' \in U \mid u \sim u'\}.$$

- b. $\text{im}(f)$ is the subset of V , where

$$[v \in \text{im}(f)] \iff [\exists u \in U (v = f(u))].$$

- Notice that both the quotient set $U/\ker(f)$ and the subset $\text{im}(f)$ are automatically linear spaces.

Definition (Quotient spaces in diagram) The rectangles are assumed to be unbounded in the left \leftarrow and upward \uparrow directions:



Exercise (Isomorphism theorem B) For any subspaces U and V , there exists an isomorphism:

$$\frac{U + V}{U} \simeq \frac{V}{U \cap V}.$$

Restate the theorem in diagrams.

*If you are very familiar with quotient sets, then it is valid to use $=$ instead of \simeq . However, one must write \simeq in **isomorphism theorem A**, since there is an external bijection induced by f .*

Exercise (Isomorphism theorem C) There exists an isomorphism

$$\frac{U^\# / U^b}{U / U^b} \simeq \frac{U^\#}{U}.$$

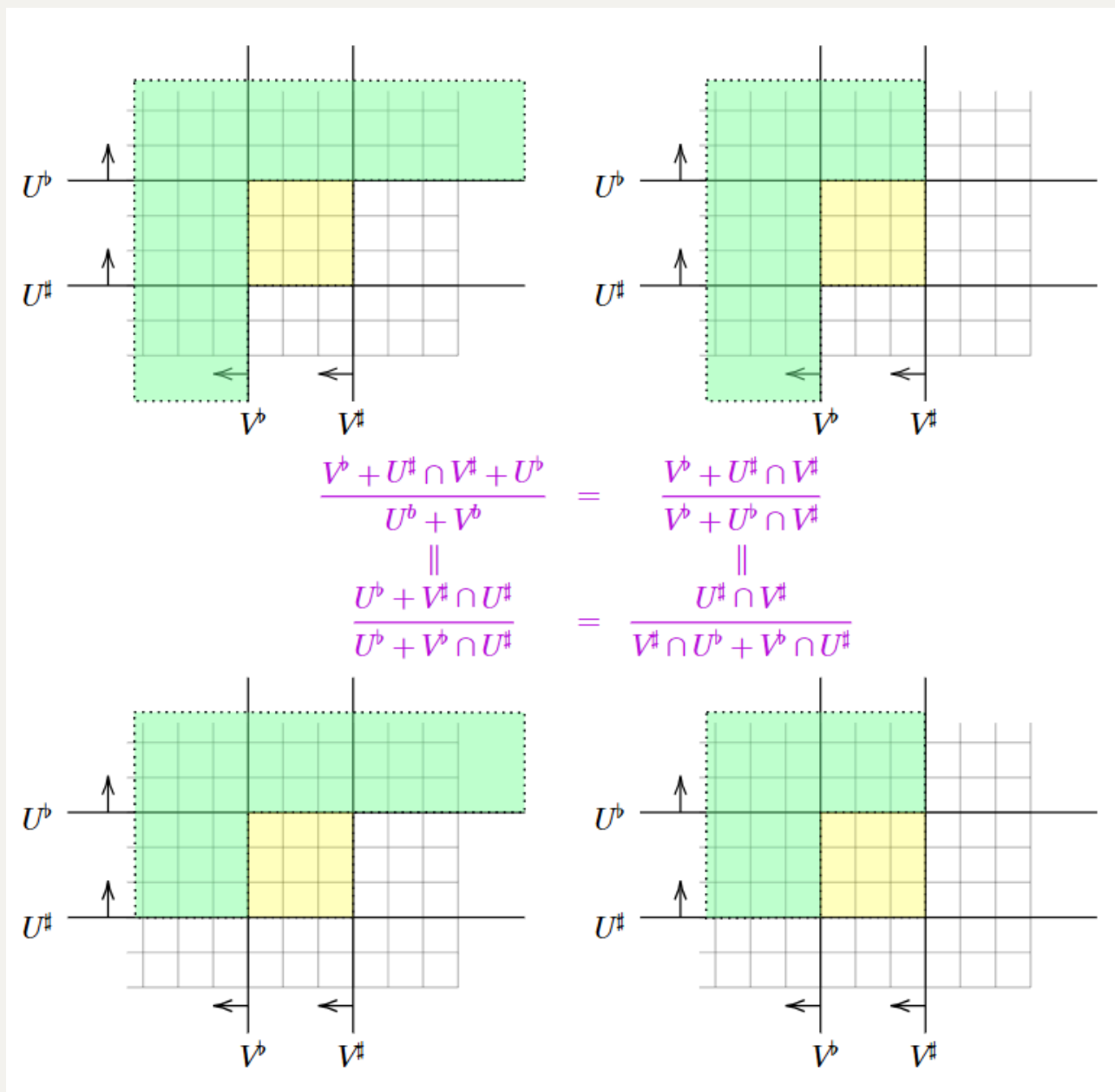
Restate the theorem in diagrams.

(Repeat) If you are familiar with quotient sets, then it is valid to use $=$ instead of \simeq .

Exercise (Isomorphism theorem D) Let $U \subset V$. There is a bijective correspondence between:

1. the subspaces of V containing U , i.e. those W such that $U \subset W \subset V$,
2. the subspaces of V/U .

Exercise State and prove Zassenhaus lemma



You are expected to explain how we obtain the diagram(s), quotient spaces, and all equalities (or isomorphisms, you decide).

Once you find Zassenhaus theorem trivial, proceed with the following exercises.

Non-kindergarten part

The concept of linear maps is beyond the scope of students (precisely, babies) from kindergarten, as it concerns homomorphisms of diagrams.

Definition (Linear maps) A linear map is an affine transformation of diagrams, which preserves 0. Let f be a linear map. We restate the definition of image and pre-image (as mentioned in mathematical analysis, I believe) in formal and symmetric terms:

1. (the image) $f_*U = \{f(u) \mid u \in U\}$, where U is the subspace of the domain (domain: where f starts/maps from/originates);
2. (the inverse image) $f^*V = \{u \mid f(u) \in V\}$, where V is the subspace of the codomain (codomain: where f ends/maps to/terminates).

Exercise The most elementary part of linear algebra is to study the operations $\{f_*, f^*, +, \cap\}$. It is an enjoyable time to observe how these simple operations generate identities.

Prove the following (when we write f_*X , it is assumed that the linear space X is a subspace of the domain, and similarly for f^*Y):

1. $U \subset f^*f_*U$, when does equality hold for all U ?
2. $f_*f^*V \subset V$, when does equality hold for all V ?
3. $f_*f^*f_* = f_*$,
4. $f^*f_*f^* = f^*$,
5. $f_*(U + V) = f_*U + f_*V$;
6. $f^*(U \cap V) = f^*U \cap f^*V$;
7. if $f_*(U \cap V) = f_*U \cap f_*V$ always holds, then f is surjective, does the converse hold?
8. if $f^*(U + V) = f^*U + f^*V$ always holds, then f is injective, does the converse hold?
9. $f_*((f^*U) \cap V) = U \cap (f_*V)$;
10. $f^*((f_*U) + V) = U + (f^*V)$.

*The final two identities come from basic logical facts (guess what they are). There is an analogue known as the **projection formula** if you are familiar with sheaf theory.*

Theorem (Zassenhaus lemma with a linear map) The following are identities and isomorphisms

$$\begin{array}{ccccc}
 & & \frac{V^b + f^*U^\sharp \cap V^\sharp + f^*U^b}{f^*U^b + V^b} & \xlongequal{\quad} & \frac{V^b + f^*U^\sharp \cap V^\sharp}{V^b + f^*U^b \cap V^\sharp} \\
 & \nearrow \sim & \parallel & & \nearrow \sim \\
 \frac{f_*V^b + U^\sharp \cap f_*V^\sharp + U^b}{U^b + f_*V^b} & \xlongequal{\quad} & & \xlongequal{\quad} & \frac{f_*V^b + U^\sharp \cap f_*V^\sharp}{f_*V^b + U^b \cap f_*V^\sharp} \\
 \parallel & & \parallel & & \parallel \\
 & \nearrow \sim & \frac{f^*U^b + V^\sharp \cap f^*U^\sharp}{f^*U^b + V^b \cap f^*U^\sharp} & \xlongequal{\quad} & \frac{f^*U^\sharp \cap V^\sharp}{V^\sharp \cap f^*U^b + V^\sharp \cap f^*U^\sharp} \\
 & & \parallel & & \nearrow \sim \\
 \frac{U^b + f_*V^\sharp \cap U^\sharp}{U^b + f_*V^b \cap U^\sharp} & \xlongequal{\quad} & & \xlongequal{\quad} & \frac{U^\sharp \cap f_*V^\sharp}{f_*V^\sharp \cap U^b + f_*V^\sharp \cap U^\sharp}
 \end{array}$$