Problem Set for 20-Feb-2025

Problem 1 Find two linear maps

$$\alpha, \beta : \mathbb{F}[x] \to \mathbb{F}[x],$$

such that

$$\alpha(\beta(f)) - \beta(\alpha(f)) = f$$

for any $f \in \mathbb{F}[x]$.

Is it possible to find such $\alpha, \beta : V \to V$ when V is of finite dimension?

Problem 2 Here is a **clarification of irreducibility** over general polynomial rings. Let $\mathbb{A} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots\}$. A polynomial $f \in \mathbb{A}[x]$ is **reducible** if and only if there exists some factorisation $f = g \cdot h$ such that $g^{-1} \notin \mathbb{A}[x]$ and $h^{-1} \notin \mathbb{A}[x]$. For instance:

- $2 \cdot x$ is irreducible in $\mathbb{Q}[x]$, yet reducible in $\mathbb{Z}[x]$;
- $x^2 + 1$ is irreducible in $\mathbb{Q}[x]$, yet reducible in $\mathbb{C}[x]$.

Now consider $f \in \mathbb{Z}[x]$. **Prove** the following:

1. If f is irreducible in $\mathbb{Z}[x]$, then it is irreducible in $\mathbb{Q}[x]$;
2. If f is irreducible in $\mathbb{R}[x]$, then it is irreducible in $\mathbb{Q}[x]$.

> ⚠️ *规范的表述是"多项式 $f(x)$ 在 $\mathbb{A}[x]$ 中可约"，而非"多项式 $f(x)$ 可约"；类似地，规范地表述是" f 是 $\mathbb{F}$-线性映射"，而非" f 是线性映射"。若无歧义，可适当地选用后者以精简表述。*

In fact, one has

$$\text{(domain)} \quad \underbrace{\mathbb{Z}[x] \to (\mathbb{Z}[1/2])[x] \to \cdots \to \mathbb{Q}[x]}_{\text{more irreducible polynomials}} \quad \text{(fractional field)},$$

and

$$\text{(field)} \quad \underbrace{\mathbb{Q}[x] \to (\mathbb{Q}[\sqrt{2}])[x] \to \cdots \to \mathbb{C}[x]}_{\text{less irreducible polynomials}} \quad \text{(algebraic closure)}.$$

(Optional) Find **Gauß's lemma** in any of the textbooks and understand both the statement and the proof. The lemma states that:

> *For any $f(x) \in \mathbb{Z}[x]$, f is irreducible in $\mathbb{Z}[x]$ if and only if f is both irreducible over $\mathbb{Q}[x]$*

> *and f is primitive (i.e., the greatest common divisor of its coefficients is 1).*

Let f be **monic**, i.e., non-zero with leading coefficient 1. From Gauß's lemma, we learn that for any monic $f \in \mathbb{Z}[x]$, f is irreducible in $\mathbb{Z}[x]$ **if and only if** it is irreducible in $\mathbb{Q}[x]$.

Problem 3 Here are some criteria for the irreducibility of polynomials in $\mathbb{C}[x]$:

1. Let $f \in \mathbb{Z}[x]$ be a **monic** polynomial of degree n. Denote the zeros of f in $\mathbb{C}$ by $(z_i)_{i=1}^n$. Show that, if there is exactly one z_i such that $|z_i| \geq 1$ and $f(0) \neq 0$, then f is irreducible in $\mathbb{Q}[x]$.
2. Let $f \in \mathbb{Z}[x]$ be a polynomial such that $f(0)$ is prime. Denote the zeros of f in $\mathbb{C}$ by $(z_i)_{i=1}^n$. Show that, if $|z_i| > 1$ for all i, then f is irreducible.
3. Let $f(x) = \sum_{k=0}^n a_k \cdot x^k \in \mathbb{Z}[x]$ be a polynomial with $f(0)$ prime. Suppose that $|a_0| > \sum_{k=1}^n |a_k|$. Show that f is irreducible.

Problem 4 Find all $f(x) \in \mathbb{C}[x]$ such that

$$f(x) \equiv \begin{cases} 2x \mod (x-1)^2, \\ 3x \mod (x-2)^2. \end{cases}$$

Exercises (optional) The following problems are **optional** but some of the problems are very important.

1. Is there any irreducible $f(x) \in \mathbb{Z}[x]$ such that $f(f(x))$ is reducible?
2. Prove that $1 + \prod_{k=1}^{2025}(x-k)^2$ is irreducible in $\mathbb{Z}[x]$;
3. Prove that $\prod_{i=1}^n (x - x_i) + 1$ is either irreducible in $\mathbb{Z}[x]$, or a perfect square;
    - where $x_1 < x_2 < \cdots < x_n$ are integers.
4. ($f \in \mathbb{Z}[x]$) Prove that if $f(x) = 1$ has ≥ 4 solutions in $\mathbb{Z}$, then $f(x) = -1$ has no solutions in $\mathbb{Z}$.
5. Prove that the partial sum $(e^x)_{\deg \leq n}$ is always irreducible in $\mathbb{Q}[x]$.