Problem Set for 3rd March 2025

Exercise: Determine linear maps $\alpha, \beta : V \to V$ such that

$$\alpha \circ \beta = \mathrm{id}_V, \quad \text{but} \quad \beta \circ \alpha \neq \mathrm{id}_V.$$

Problem: Let $\mathbb{F}$ denote the ground field, and let S be any finite set.

1. Demonstrate that $\mathbf{Hom}_{\mathrm{Sets}}(S, \mathbb{F})$ forms a vector space.
2. Construct a linear bijection (hereinafter referred to as a *linear isomorphism*)

$$\mathbf{Hom}_{\mathrm{Sets}}(S, \mathbb{F}) \to \mathbb{F}^{|S|}.$$

3. Demonstrate that following function constitutes an injection of sets:

$$\varphi : S \to \mathbf{Hom}_{\mathbb{F}}(\mathbf{Hom}_{\mathrm{Sets}}(S, \mathbb{F}), \mathbb{F})$$

$$s \mapsto \begin{bmatrix} \mathbf{Hom}_{\mathrm{Sets}}(S, \mathbb{F}) & \to & \mathbb{F} \\ f & \mapsto & f(s) \end{bmatrix}$$

4. Demonstrate that the image $\varphi(S)$ forms a basis for $\mathbf{Hom}_{\mathbb{F}}(\mathbf{Hom}_{\mathrm{Sets}}(S, \mathbb{F}), \mathbb{F})$.
5. This is how we define

$$\mathbb{F}s_1 \oplus \mathbb{F}s_2 \oplus \cdots \oplus \mathbb{F}s_n \quad S = \{s_1, \ldots, s_n\}.$$