Problem set for 27-Feb-2025

Exercise 1.1 Let $P_n := \{f(x \in \mathbb{F}[x]) \mid \deg f(x) < n\}$. Pick $\{a_i\}_{i=1}^n \in \mathbb{F}$ such that $a_i \neq a_j$ for any $i \neq j$. Show that

$$f_j(x) := \prod_{i \neq j}(x - a_i) \quad (1 \leq j \leq n)$$

form a basis of P_n.

> *答: 只需证明 $\{f_j\}_{j=1}^n$ 是大小为 n 的线性无关组.对线性无关性证明如下, 定义线性映射*
>
> $$\varphi_j : P_n \mapsto \mathbb{F}, \quad f \mapsto f(a_j).$$
>
> *依照定义, $f_j(x_i) = \delta_{i,j}$. 对任意线性组合式 $\sum_{i=1}^n c_i f_i$, 总有*
>
> $$\varphi_j : \sum_{i=1}^n c_i f_i \mapsto c_j f_j(x_j).$$
>
> *今取遍所有 φ_j, 上述线性组合式为 0 当且仅当一切 c_i 为 0. 这说明了线性无关性.*

Exercise 4.1 Assume $f(x) = x^3 + px + q \in \mathbb{Z}[x]$ is irreducible and $\alpha \in \mathbb{C}$ is a root of f.

1. Prove that $\mathbb{Q}[\alpha] := \{g(\alpha) \mid g(x) \in \mathbb{Q}[x]\}$ is a linear space over $\mathbb{Q}$ and $1, \alpha, \alpha^2$ form a basis.
2. Prove that $\varphi : \beta \mapsto f'(\alpha)\beta$ gives a linear map on $\mathbb{Q}[\alpha]$ and find its matrix under $1, \alpha, \alpha^2$.

> *答: 为证明 $1, \alpha, \alpha^2$ 构成基, 下依次证明 $\mathrm{Span}(1, \alpha, \alpha^2) = \mathbb{Q}[\alpha]$, 以及线性无关性.*
>
> 1. *依照 $f(\alpha) = 0$, $\alpha^{\geq 3}$ 可以由更低次项的线性组合表示. 这说明 $\mathrm{Span}(1, \alpha, \alpha^2) = \mathbb{Q}[\alpha]$.*
> 2. *我们断言 $1, \alpha, \alpha^2$ 线性无关. 若存在非零线性组合使得 $a + b\alpha + c\alpha^2 = 0$, 则 α 具有次数小于 3 的零化多项式, 这和 f 不可约矛盾.*
>
> *为找到 φ 的矩阵表示, 直接计算得*

$$
\begin{aligned}
\varphi(1 \mid \alpha \mid \alpha^2) &= (\varphi(1) \mid \varphi(\alpha) \mid \varphi(\alpha^2)) \\
&= (3\alpha^2 + p \mid 3\alpha^3 + p\alpha \mid 3\alpha^4 + p\alpha^2) \\
&= (3\alpha^2 + p \mid -2p\alpha - 3q \mid -2p\alpha^2 - 3q\alpha) \\
&= (1 \mid \alpha \mid \alpha^2) \cdot \begin{pmatrix} p & -3q & 0 \\ 0 & -2p & -3q \\ 3 & 0 & -2p \end{pmatrix}.
\end{aligned}
$$