

Problem Set for 20-Feb-2025

Problem 1 Find two linear maps

$$\alpha, \beta : \mathbb{F}[x] \rightarrow \mathbb{F}[x],$$

such that

$$\alpha(\beta(f)) - \beta(\alpha(f)) = f$$

for any $f \in \mathbb{F}[x]$.

Is it possible to find such $\alpha, \beta : V \rightarrow V$ when V is of finite dimension?

答: α 是求导, $\beta(f) = x \cdot f(x)$. 那么

$$\alpha(\beta(f)) - \beta(\alpha(f)) = D(xf) - x(Df) = f.$$

有限维无解, 因为不存在 $AB - BA = I$ 的矩阵 (两侧取 tr).

Problem 2 Here is a **clarification of irreducibility** over general polynomial rings. Let $\mathbb{A} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \dots\}$. A polynomial $f \in \mathbb{A}[x]$ is **reducible** if and only if there exists some factorisation $f = g \cdot h$ such that $g^{-1} \notin \mathbb{A}[x]$ and $h^{-1} \notin \mathbb{A}[x]$. For instance:

- $2 \cdot x$ is irreducible in $\mathbb{Q}[x]$, yet reducible in $\mathbb{Z}[x]$;
- $x^2 + 1$ is irreducible in $\mathbb{Q}[x]$, yet reducible in $\mathbb{C}[x]$.

Now consider $f \in \mathbb{Z}[x]$. **Prove** the following:

1. If f is irreducible in $\mathbb{Z}[x]$, then it is irreducible in $\mathbb{Q}[x]$;

答: 对任意 $d \in \mathbb{N}^+$, 多项式 $d \cdot f(x)$ 在 $\mathbb{Z}[x]$ 中无法分解作两个非常值多项式的乘积. 继而使用反证法: 假定 $\mathbb{Q}[x]$ 中存在分解 $f = p \cdot q$, 满足 $\deg p \cdot \deg q \geq 1$. 取 d 使得 $\sqrt{d}p, \sqrt{d}q \in \mathbb{Z}[x]$, 则与 $d \cdot f$ 在 $\mathbb{Z}[x]$ 中的分解方式矛盾.

2. If f is irreducible in $\mathbb{R}[x]$, then it is irreducible in $\mathbb{Q}[x]$.

答: 考虑逆否命题即可.

⚠ 规范的表述是“多项式 $f(x)$ 在 $\mathbb{A}[x]$ 中可约”, 而非“多项式 $f(x)$ 可约”; 类似地, 规范地表

还是“ f 是 \mathbb{F} -线性映射”，而非“ f 是线性映射”。若无歧义，可适当地选用后者以精简表述。

In fact, one has

$$\text{(domain)} \quad \underbrace{\mathbb{Z}[x] \rightarrow (\mathbb{Z}[1/2])[x] \rightarrow \cdots \rightarrow \mathbb{Q}[x]}_{\text{more irreducible polynomials}} \quad \text{(fractional field),}$$

and

$$\text{(field)} \quad \underbrace{\mathbb{Q}[x] \rightarrow (\mathbb{Q}[\sqrt{2}])[x] \rightarrow \cdots \rightarrow \mathbb{C}[x]}_{\text{less irreducible polynomials}} \quad \text{(algebraic closure).}$$

(Optional) Find **Gauß's lemma** in any of the textbooks and understand both the statement and the proof. The lemma states that:

For any $f(x) \in \mathbb{Z}[x]$, f is irreducible in $\mathbb{Z}[x]$ if and only if f is both irreducible over $\mathbb{Q}[x]$ and f is primitive (i.e., the greatest common divisor of its coefficients is 1).

Let f be **monic**, i.e., non-zero with leading coefficient 1. From Gauß's lemma, we learn that for any monic $f \in \mathbb{Z}[x]$, f is irreducible in $\mathbb{Z}[x]$ **if and only if** it is irreducible in $\mathbb{Q}[x]$.

Problem 3 Here are some criteria for the irreducibility of polynomials in $\mathbb{C}[x]$:

1. Let $f \in \mathbb{Z}[x]$ be a **monic** polynomial of degree n . Denote the zeros of f in \mathbb{C} by $(z_i)_{i=1}^n$. Show that, if there is exactly one z_i such that $|z_i| \geq 1$ and $f(0) \neq 0$, then f is irreducible in $\mathbb{Q}[x]$.

答: 依照 Gauß 引理, 首一整系数多项式在 $\mathbb{Z}[x]$ 与 $\mathbb{Q}[x]$ 中的可约性等价. 若 $\mathbb{Z}[x]$ 中存在分解 $f = pq$ ($\deg p \cdot \deg q \geq 1$), 则不妨设 p 的所有根满足 $|z| < 1$. 根据 $p(0) \neq 0$, 以及根乘积的 Vieta 定理, 矛盾.

2. Let $f \in \mathbb{Z}[x]$ be a polynomial such that $f(0)$ is prime. Denote the zeros of f in \mathbb{C} by $(z_i)_{i=1}^n$. Show that, if $|z_i| > 1$ for all i , then f is irreducible.

答: 若有分解 $f = pq$ ($\deg p \cdot \deg q \geq 1$), 则 $|p(0)| \cdot |q(0)|$ 是素数. 不妨设 $|p(0)| = 1$, 与根乘积的 Vieta 定理矛盾.

3. Let $f(x) = \sum_{k=0}^n a_k \cdot x^k \in \mathbb{Z}[x]$ be a polynomial with $f(0)$ prime. Suppose that $|a_0| > \sum_{k=1}^n |a_k|$. Show that f is irreducible.

答: 三角不等式 $|a_0| > \sum_{k=1}^n |a_k|$ 说明所有根模长大于 1, 往后同上一问.

Problem 4 Find all $f(x) \in \mathbb{C}[x]$ such that

$$f(x) \equiv \begin{cases} 2x & \text{mod } (x-1)^2, \\ 3x & \text{mod } (x-2)^2. \end{cases}$$

答: 见张贤科 (1.17)

此题题干和张贤科书中有出入, 正确答案是

$$-3x^3 + 14x^2 - 17x + 8 + P(x)(x-1)^2(x-2)^2.$$

Exercises (optional) The following problems are **optional** but some of the problems are very important.

1. Is there any irreducible $f(x) \in \mathbb{Z}[x]$ such that $f(f(x))$ is reducible?

答: 经耐心计算, 可以发现 $f(x) = x^2 + 10x + 17$, 且

$$f(f(x)) = (x^2 + 8x + 14)(x^2 + 12x + 34).$$

2. Prove that $1 + \prod_{k=1}^{2025} (x-k)^2$ is irreducible in $\mathbb{Z}[x]$;

答: 若可约, 则存在次数 ≤ 2025 的子式 $f \in \mathbb{Z}[x]$. 对 $1 \leq k \leq 2025$, $f(k)$ 是 1 的约数. 由于原多项式无零点, 其因子 f 亦无零点. 此时, 不妨设假设 $f-1$ 以 $\{k\}_{k=1}^{2025}$ 为零点. 比较次数, 只能有 $f = \prod_{k=1}^{2025} (x-k) + 1$, 矛盾.

3. Prove that $\prod_{i=1}^n (x-x_i) + 1$ is either irreducible in $\mathbb{Z}[x]$, or a perfect square;
 ◦ where $x_1 < x_2 < \cdots < x_n$ are integers.

答: 若可约, 则 $f^2 = g^2 h^2$. 不妨设 $\deg g^2 \leq n$. 由零点数量知 $(g^2 - 1) = (f - 1)$. 从而 f 是完全平方式.

由初等数学, f 是平方式当且仅当 x_i 是长度为 2 或 4 的等差数列, 公差为 1.

4. ($f \in \mathbb{Z}[x]$) Prove that if $f(x) = 1$ has ≥ 4 solutions in \mathbb{Z} , then $f(x) = -1$ has no solutions in \mathbb{Z} .

答: 也就是说明 $(x-a)(x-b)(x-c)(x-d)g(x) = 2$ 无解. 显然.

5. Prove that the partial sum $(e^x)_{\deg \leq n}$ is always irreducible in $\mathbb{Q}[x]$.

答: 考虑多项式 $f_n(x) = x^n + nx^{n-1} + \cdots + n!$, 记素因子分解 $n = \prod p_i^{n_i}$.

依照补充材料中的小技巧, 对任意素数 p_i , 相应的 $\bmod p$ -凸包的底部折线段的斜率总是形如 $\frac{1-p_i^k}{p_i^k \cdot (p_i-1)}$, 其中 $k \geq n_i$. 用格点将凸包底线切分成最细单元, 由于 $\frac{(p^k-1)/(p-1)}{p^k}$ 是既约分数, 故这些最细单元的最短横向长度必然是 p^{n_i} . 上述最细单元的横向长度在多项式乘法下不会变得 longer.

作为推论, $f_n(x)$ 的每个不可约子式的次数一定是 p^{n_i} 的整数倍 (使用反证即可). 今遍历所有 $p_i^{n_i}$, 则 $f_n(x)$ 不可约子式的次数一定是各 $p_i^{n_i}$ 的倍数. 由是观之, f_n 不可约.