偏微分复习(第二部分)

Fourier法

Fourier变换简介

记$\mathbb{R}^n$上的Fourier变换(有处定义不采用$(2\pi)^{-n/2}$)为

$$\mathscr{F}: f(x) \mapsto \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-i\xi\cdot x} \mathrm{d}x.$$

相应地逆变换为

$$\mathscr{F}^{-1}: f(x) \mapsto \check{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{i\xi\cdot x} \mathrm{d}x.$$

对速降空间(Schwarz space)$\mathcal{S}$, $\mathscr{F}: \mathcal{S} \to \mathcal{S}$为双射. 同时, $\mathscr{F}: L^2(\Omega) \to L^2(\Omega)$亦为双射(设函数在相差零测集的意义下相同). 一般地, 对任意$p \in (1, \infty)$, 有双射关系

$$\mathscr{F}: L^p(\Omega) \to L^{p^*}(\Omega).$$

其中共轭指标满足$p^{-1} + (p^*)^{-1} = 1$. 该定理为Riesz-Thorin定理.

当$p = p^* = \dfrac{1}{2}$时$\mathscr{F}$保距, 即对任意$f, g \in L^2(\Omega)$均有

$$\langle f, g \rangle = \langle \mathscr{F}[f], \mathscr{F}[g] \rangle.$$

简单的Fourier变换

考虑$\mathscr{F}: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, 则

- $\mathscr{F}$保持线性, 即保持自变量的加和与数乘.
- $\mathscr{F}[f \circ (-x_0)](\xi) = \hat{f}(\xi) \cdot e^{-ix_0\cdot\xi}$.
- $\mathscr{F}[f \circ (c\cdot)](\xi) = c^{-n} \hat{f}(c^{-1}\xi)$.
- (接上条) 对非奇异常矩阵A, $\mathscr{F}[f \circ (A\cdot)] = (\det A)^{-1} \hat{f}({}^tA^{-1}\xi)$.
- $\mathscr{F}[f * g] = (2\pi)^{n/2} \mathscr{F}[f] \cdot \mathscr{F}[g]$.
- $\mathscr{F}[f \cdot g] = (2\pi)^{-n/2} \mathscr{F}[f](\xi) * \mathscr{F}[g](\xi)$.
- $\mathscr{F}[\partial_{x_j} f] = i\xi_j \cdot \mathscr{F}[f]$. 常以方便故记$\mathcal{D}_{x_j} := \dfrac{\partial_{x_j}}{i}$.
- $\mathscr{F}[(\prod_\alpha i^{-k}\xi_k) \cdot f](\xi) = \partial^\alpha \mathscr{F}[f]$.
- (接上条) 设$\alpha = (\alpha(1), \alpha(2), \ldots, \alpha(n))$为指标, 并定义$\mathcal{D}^\alpha = \prod_k \mathcal{D}_{x_k}^{\alpha(k)}$, $\xi^\alpha = \prod_k \xi_k^{\alpha(k)}$. 则

  $$\mathscr{F}[\mathcal{D}^\alpha f] = \xi^\alpha \mathscr{F}[f].$$

  同理, 对关于若干α的多项式$P(\Lambda) = P(\alpha, \beta, \cdots, \gamma)$, 有

  $$\mathscr{F}[\mathcal{D}^{P(\Lambda)} f] = \xi^{P(\Lambda)} \mathscr{F}[f].$$

- $\mathscr{F}^2: f(x) \mapsto f(-x)$. $\mathscr{F}^4$恒等.

- (广义函数) $\mathscr{F}[\delta](\xi)=\dfrac{1}{\sqrt{2\pi}^n}$.

Fourier变换法应用

对以下方程

$$\begin{cases}u_{tt}+a^2u_{xxxx}=0\\ t=0: u=\varphi(x), u_t=a\psi''(x)\end{cases}$$

关于x做Fourier变换得

$$\begin{cases}\hat{u}_{tt}+a^2\xi^4\hat{u}=0\\ t=0: \hat{u}=\hat{\varphi}(\xi), \hat{u}_t=-a\xi^2\hat{\psi}(\xi)\end{cases}$$

解得$\hat{u}(t,\xi)=\hat{\varphi}(\xi)\cos a\xi^2 t-\hat{\psi}(\xi)\sin a\xi^2 t$. 从而

$$\begin{aligned}
u(t,x)=&\mathscr{F}^{-1}[\hat{u}(t,\cdot)](\xi)\\
=&\mathscr{F}^{-1}[\hat{\varphi}(\xi)\cdot\cos at\xi^2]-\mathscr{F}^{-1}[\hat{\psi}(\xi)\cdot\sin at\xi^2]\\
=&\frac{1}{\sqrt{2\pi}}\left(\varphi*\mathscr{F}^{-1}[\cos at\xi^2]-\psi*\mathscr{F}^{-1}[\sin at\xi^2]\right)\\
=&\frac{1}{2\sqrt{2at}}\int_{\mathbb{R}}\varphi(\xi)\left[\cos\frac{(\xi-x)^2}{4at}+\sin\frac{(\xi-x)^2}{4at}\right]\mathrm{d}\xi\\
&+\frac{1}{2\sqrt{2at}}\int_{\mathbb{R}}\psi(\xi)\left[\cos\frac{(\xi-x)^2}{4at}-\sin\frac{(\xi-x)^2}{4at}\right]\mathrm{d}\xi\\
=&\frac{1}{2\sqrt{at}}\int_{\mathbb{R}}\varphi(\xi)\left[\cos\frac{(\xi-x)^2-at\pi}{4at}\right]\mathrm{d}\xi\\
&+\frac{1}{2\sqrt{at}}\int_{\mathbb{R}}\psi(\xi)\left[\cos\frac{(\xi-x)^2+at\pi}{4at}\right]\mathrm{d}\xi
\end{aligned}$$

基本解理论

基本解

记微分算子$L(\partial_t,\partial_x)=\partial_t^m+\sum_{j=0}^{m-1}\sum_{|x|\le N_j}a_{j,\alpha}(t,x)\partial_t^j\partial_x^\alpha$

考虑方程$Lu=f, t>0, x\in\mathbb{R}^n$, 初值$\partial_t^j u|_{t=0}=\varphi_j$,

$$\begin{aligned}&Lu=f\quad t>0,x\in\mathbb{R}^n\\&\partial_t^j u|_{t=0}=\varphi_j,\quad 1\le j\le m-1\end{aligned}$$

基本解$E=E(t,x)$满足

$$\begin{aligned}&LE=0\quad t>0,x\in\mathbb{R}^n\\&\partial_t^j E|_{t=0}=0,\quad 1\le j\le m-2\\&\partial_t^{m-1}E|_{t=0}=\delta(x)\end{aligned}$$

从而

$$u=\sum_{j=0}^{m-1}\partial_t^{m-1-j}[E(t,\cdot)*\varphi_j](x)+\int_0^t[E(t-\tau,\cdot)*f(t,\cdot)](x)\mathrm{d}\tau$$

热传导方程的基本解

以热方程为例, 记$L(\partial_t,\partial_x):u\mapsto=\partial_t u-a^2\partial_{xx}u$. 则PDE问题为转化为

$$\begin{aligned}&LE=0\quad t>0,x\in\mathbb{R}^n\\&E|_{t=0}=\delta(x)\end{aligned}$$

Fourier变化得

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\hat{E}(t,\cdot)(\xi)+a^2\xi^2\hat{E}(t,\cdot)(\xi)=0\quad t>0,x\in\mathbb{R}^n\\&\hat{E}(0,\cdot)(\xi)=\frac{1}{(\sqrt{2\pi})^n}\end{aligned}$$

解得$\hat{E}(t,\cdot)(\xi)=e^{-ta^2\xi^2}$. Fourier逆变换得

$$\begin{aligned}E(t,x)=&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{-ta^2\xi^2}e^{ix\cdot\xi}\mathrm{d}\xi\\=&\frac{1}{(2\pi)^n}\prod_{k=1}^n\int_{\mathbb{R}}e^{-ta^2(\xi_k-ix_k/2a^2t)}\cdot e^{-x_k^2/4a^2t}\mathrm{d}\xi\\=&\frac{1}{(2\pi)^n}\cdot\sqrt{\frac{\pi}{a^2t}}\cdot e^{-x^2/4a^2t}\\=&\frac{e^{-x^2/4a^2t}}{(2a\sqrt{\pi t})^n}\end{aligned}$$

波动方程的基本解

考虑全空间内波动方程基本解

$$\begin{cases}(\partial_{tt}-a^2\Delta_n)E(t,x)=0\\E(0,x)=0\\E_t(0,x)=\delta(x)\end{cases}$$

Fourier变换得

$$\begin{cases}(\partial_{tt}+a^2\xi^2)\hat{E}=0\\\hat{E}(0,\xi)=0\\\partial_t\hat{E}(0,\xi)=\dfrac{1}{\sqrt{2\pi}^n}\end{cases}$$

解得$\hat{E}(t,\xi)=\dfrac{\sin(a|\xi|t)}{a|\xi|\sqrt{2\pi}^n}=\dfrac{t}{\sqrt{2\cdot\pi}^n}\sum_{k\geq0}\dfrac{(-a^2\xi^2t^2)^k}{(2k+1)!}$. Fourier逆变换得

$$E(t,x)=\frac{t}{2\cdot\pi^n}\sum_{k\geq0}\frac{(-a^2t^2)^k}{(2k+1)!}\prod_{d=1}^n\int_{\mathbb{R}}e^{(\xi_d^2)^k}e^{i\xi_dx_d}\mathrm{d}\xi_d.$$

对低维简单情形, 可直接求解.

热传导方程

全空间上的热传导方程

对方程

$$\begin{cases}u_t-a^2\Delta u=f(t,x),t>0,x\in\mathbb{R}^n\\ t=0:u=\varphi(x)\end{cases}$$

考虑对x做Fourier变化所得的PDE问题

$$\begin{cases}\partial_t\hat{u}(t,\xi)+a^2|\xi|^2\hat{u}(t,\xi)=\hat{f}(t,\xi)\\ t=0:\hat{u}(t,\xi)=\hat{\varphi}(\xi)\end{cases}$$

解ODE问题得

$$\begin{aligned}u(t,x)=&(2a\sqrt{\pi})^{-n}\int_{\mathbb{R}^n}\frac{e^{-|x-y|^2/4at}}{\sqrt{t}}\varphi(y)\mathrm{d}y\\ &+(2a\sqrt{\pi})^{-n}\int_0^t\int_{\mathbb{R}^n}\frac{e^{-|x-y|^2/4a(t-\tau)}}{\sqrt{t-\tau}^n}f(\tau,y)\mathrm{d}y\mathrm{d}\tau\end{aligned}$$

设基本解$E(t,x)=\dfrac{\exp\dfrac{-|x|^2}{4at}}{(2a\sqrt{\pi t})^n}$, 从而

$$u(t,x)=[E(t,\cdot)*\varphi](x)+\int_0^t[E(t-\tau,\cdot)*f(\tau,\cdot)](x)\mathrm{d}\tau.$$

基本解关于$t\to 0$为光滑的good kernel, 即满足如下性质:

- $E(t,x)\in C^\infty(\{t>0\})$.
- $t>0$时, $\partial_tE(t,x)=a^2\Delta_xE(t,x)$.
- $\int_{\mathbb{R}^n}E(t,x)\mathrm{d}x=1$. 注意到$E(t,x)$恒正, 故绝对积分一致有界.
- 对任意$\delta>0$, $\lim_{t\to0^+}\int_{\mathbb{R}^n-B_n(0,\delta)}|E(t,x)|\mathrm{d}x=0$.

从物理角度而言, 热方程之解应当具有以下性质(不难验证):

- 齐次热传导方程之解满足$u(t,x)\in[\inf\varphi(x),\sup\varphi(x)]$.

再论迭代法

就以下方程为例

$$\begin{cases}u_t-a^2(u_{xx}+4u_{yy})=y^2t^2\\ t=0:u=x^2y\end{cases}$$

记算子$P:u\mapsto\partial_tu-a^2(\partial_{xx}+4\partial_{yy})u$. 注意到

$$\begin{aligned}\frac{t^3}{3}y^2&\mapsto y^2t^2-\frac{8a^2t^3}{3}\\ \frac{2a^2t^4}{3}&\mapsto\frac{8a^2t^3}{3}\\ x^2y&\mapsto-2a^2y\\ 2a^2ty&\mapsto 2a^2y\end{aligned}$$

从而$u=x^2y+2a^2ty+\dfrac{t^3}{3}y^2+\dfrac{2a^2t^4}{3}$.

对较复杂的方程(设α与β相关加和在定义域内一致收敛)

$$\begin{cases} u_t - a^2\Delta_n u = \sum_{k\ge 0}\prod_{l=1}^{n}\alpha_{k,l}(t,x^l) \quad x\in\Omega, t>0 \\ t=0: u=\sum_{k\ge 0}\prod_{l=1}^{n}\beta_{k,l}(x^l) \end{cases}$$

则$u=\sum_{k\ge 0}\sum_{l=1}^{n}(v_{k,l}+w_{k.l})$, 其中$v_{k,l}$为方程

$$\begin{aligned} &\partial_t v_{k,l}(t,x^l) - a^2\partial_{x^l x^l} v_{k,l}(t,x^l) = 0 \\ &t=0: v_{k,l}(t,x^l) = \beta_{k,l}(x^l) \end{aligned}$$

之解, 即$v_{k,l}=[\beta_{k,l} * E(t,\cdot)](x)$. $v_{k,l}$为方程

$$\begin{aligned} &\partial_t v_{k,l}(t,x^l) - a^2\partial_{x^l x^l} v_{k,l}(t,x^l) = \alpha_{k,l}(t,x^l) \\ &t=0: v_{k,l}(t,x^l) = 0 \end{aligned}$$

之解. 综上

$$u=\sum_{k\ge 0}\sum_{l=1}^{n}[\alpha_{k,l} * E(t,\cdot)(x^l) + \int_0^t \beta_{k,l}(\tau,\cdot) * E(t-\tau,\cdot)(x^l)\mathrm{d}\tau].$$

无量纲量法

考虑热方程

$$\begin{aligned} &u_t - a^2\Delta_n u = 0 \quad x\in\Omega, t>0 \\ &\text{some given boundary conditions} \end{aligned}$$

记无量纲量$\xi=\dfrac{r}{a\sqrt{t}}$, 则

$$\begin{aligned} &u_t = u_\xi\cdot\xi_t = u_\xi\cdot\frac{-r}{2at\sqrt{t}} \\ &\Delta_n u = r^{1-n}\partial_r(r^{n-1}\partial_r u) = \frac{n-1}{ar\sqrt{t}}u_\xi + \frac{r}{a^2 t}u_{\xi\xi} \end{aligned}$$

从而PDE化为

$$u_\xi((n-1)/\xi+\xi/2)+u_{\xi\xi}=0.$$

当$n=1$时, 解得

$$u=u_0+(u_\infty-u_0)\cdot\frac{1}{\sqrt{\pi}}\int_0^{x/2a\sqrt{t}} e^{-s^2}\mathrm{d}s.$$

当$n\ge 2$时, 解得

$$u=u_0+(u_\infty-u_0)\cdot\frac{2}{\Gamma(-n/2)}\cdot\int_0^{x/2a\sqrt{t}} s^{1-n}e^{-s^2}\mathrm{d}s.$$

分离变量法

一维闭区域上情形

对热传导方程

$$\begin{cases} u_t - a^2 u_{xx} = 0, \quad 0 < x < l, t > 0 \\ t = 0 : u = \varphi(x) \\ \text{some given boundary conditions} \end{cases}$$

StepI: 寻找一个仅满足边值条件的函数v, 下考虑$w = u - v$. 分离变量得特征方程 $\dfrac{X''}{X} = \dfrac{T'}{a^2 T} = -\lambda_k$, 考虑正交基$\{e_k\}_{k\geq 0}$使得$e_k(x)$满足边值条件, 且$e_k''(x) + \lambda_k e_k(x) = 0$. 注意: 当满足Newman条件时应补上0特征值.

Step II: 设解具有一般形式($u(t,x) = 0$时$\theta_k \equiv 0$):

$$\sum_{\exists\lambda=0} \varphi(0) + \sum_{k\geq 1} A_k e^{-\lambda_k t} \sin(\sqrt{-\lambda} x + \theta_k).$$

其中

$$A_k = \frac{2}{l} \int_0^l \varphi(x) \sin(\sqrt{-\lambda} x + \theta_k) \mathrm{d}x.$$

二维矩形上情形

考虑方程

$$\begin{aligned} & u_t - \Delta_2 u = f \\ & t = 0 : u = \varphi(x) \\ & x = 0 : u = \mu_1(y) \quad x = a : u = \mu_2(y) \\ & y = 0 : u = \psi_1(x) \quad y = b : u = \psi_2(x) \end{aligned}$$

同样, 对含源项采用齐次化原理. 无源时, 考虑$u = T(t)X(x)$, 则

$$\frac{\Delta X(x)}{X} = \frac{T'(t)}{T} = -\lambda.$$

则原问题转化为特征值问题

$$\begin{aligned} & \Delta_2 u + \lambda u = 0 \\ & x = 0 : u = \mu_1(y) \quad x = a : u = \mu_2(y) \\ & y = 0 : u = \psi_1(x) \quad y = b : u = \psi_2(x) \end{aligned}$$

拆分$u = v + w$, 其中v在$x \in \{0, a\}$时取值为0, w在$y \in \{0, b\}$时取值为0. 考虑

$$\begin{gathered} \frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0 \\ X(0) = X(a) = 0 \end{gathered}$$

解得$X(x) \in \mathrm{span}_{k\geq 1}(\sin k\pi x/a)$. 从而

热稳态

(数学物理方法P56-6) 半径为a的半圆形平板, 其表面绝热, 在板的周围边界上保持常温u_0, 而在直径边界上保持常温u_1, 求板的稳恒状态.

解: 稳恒时, 温度分布函数u满足$\partial_t u = 0$, 从而$\Delta u = 0$. 定解问题为

$$\begin{cases} \partial_{rr} u + \dfrac{\partial_r}{r} u + \dfrac{\partial_{\theta\theta}}{r^2} u = 0 \\ u(a, \theta) = u_0, \quad 0 < \theta < \pi \\ u(r, 0) = u(r, \pi) = u_1, \quad 0 \le r \le a \end{cases}$$

令$v = R(r)\Theta(\theta) + u_1$, 从而

$$r^2 \frac{R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

由$\Theta'' + \lambda_k \Theta = 0$及$\Theta(0) = \Theta(\pi) = 0$知$\lambda_k = k^2$. 解Euler方程

$$r^2 R_k'' + rR_k' - \lambda_k R_k = 0$$

得

$$\begin{cases} R_k = B_k r^k + C_k r^{-k} & k > 0 \\ R_0 = C_0 + D_0 \ln r & k - 0 \end{cases}$$

实际上, 由有界性知$C_k = 0$. 从而解具有形式

$$u = u_1 + \sum_{k \ge 1} B_k r^k \sin(k\theta).$$

故

$$\frac{2}{\pi} \int_0^\pi \sin(k\theta)(u_0 - u_1) \mathrm{d}\theta = B_k a^k.$$

解得$B_k = \dfrac{2(u_0 - u_1)}{a^k k\pi} [1 - (-1)^k]$. 故

$$u(r, \theta) = u_1 + \frac{4(u_0 - u_1)}{\pi} \sum_{n \ge 1} \frac{\sin[(2n-1)\theta]}{2n-1} \cdot \left(\frac{r}{a}\right)^{2n-1}.$$

极值原理

无释热源的极值原理

考虑$\Omega \subset \mathbb{R}^n$为有界区域, $\forall T > 0$, $\Omega_T := (0, T) \times \Omega$. 定义抛物边界

$$\partial' \Omega := \{(t, x) : t = 0 \vee x \in \partial\Omega\}.$$

若$u \in C^0(\overline{\Omega_T}) \cap C^{1,2}(\Omega)$, 则$u$在$\Omega_T$可取到最大值. 若$u$在$\Omega_T$内部满足$\partial_t u - a^2 \Delta u \le 0$, 则根据物理学意义, u最大值在抛物边界取到.

实际上, 若u在非抛物点(t_0, x_0)上取到最大值, 则

1. $\partial_t u(t_0, x_0) \ge 0$, 取大于若且仅若$t_0 = T$.

2. 对固定的t_0, u局部次调和, 即$\Delta u \le 0$.

因此$(\partial_t - a^2\Delta)u \ge 0$. 取等若且仅若$\Delta u \equiv 0$, 即$u$为常函数.

热方程的极值原理

极值定理的导出

考虑有界区域Ω上的热方程

$$\begin{aligned}&Lu = \partial_t u - \left(\sum_{i,j=1}^n a_{ij} u_{x^i,x^j} + \sum_{i=1}^n b_i u_{x^i} + cu\right) \quad x \in \Omega, t > 0\\&\text{Some given initial and boundary conditions}\end{aligned}$$

其中a_{ij}, b_i, c均为关于x, t的连续函数, (a_{ij})恒正定, c恒非负. 则对任意$T > 0$, u在

$$\Gamma_T := \partial(\Omega \times T) - \Omega^\circ \times \{T\}$$

上取非负最大值. 不妨设最大值点$(t_0, x_0) \in (\Omega \times T)^\circ$, 即$\Omega \times T$内部取得最大值$M$. 设$m$为$u$在抛物边界上的最大值, 则$M > m$. 考虑函数

$$v := u + \varepsilon(\|x - x_0\|)^2$$

其中ε可取得充分小使得v在Γ_T上取值不超过$v(t_0, x_0) = M$. 注意到

$$\begin{aligned}Lv =& \partial_t v - \left(\sum_{i,j=1}^n a_{ij} u_{x^i,x^j} + \sum_{i=1}^n b_i u_{x^i} + cu\right)\\=& -\varepsilon\left(2\sum_i (a_{ii} + b_i x_i + c(x^i - x_0^i))^2\right)\end{aligned}$$

而v在内部取最大值, 从而最大值点处Lv为负. 而当v取得非负最大值时, Lv应为非负数, 矛盾.

从而u在边界上取得非负最大值及非正最小值.

解的稳定性分析

为方便起见, 以下讨论标准形式的热方程.

Dirichlet条件

考虑一般热方程

$$\begin{aligned}&(\partial_t - a^2\Delta)u = f(t,x) \quad x \in \Omega \subset \mathbb{R}^n, t > 0\\&t = 0 : u = \varphi(x)\\&u|_{(0,\infty)\times\partial\Omega} = \mu(t,x)\end{aligned}$$

方程解至多唯一.

记$u = \mathscr{S}(f, \varphi, \mu)$为解, 则当发生微扰$\delta f$, $\delta\varphi$与$\delta\mu$时, 新方程满足

$$\begin{aligned}&(\partial_t - a^2\Delta)u = f(t,x) + \delta f(t,x)\\&t = 0 : u = \varphi(x) + \delta\varphi(x)\\&u|_{(0,\infty)\times\partial\Omega} = \mu(t,x) + \delta\mu(t,x)\end{aligned}$$

记$\|g\| := \sup_{(t,x)\in\Omega_T} |g|$. 因此对任意$T > 0$均有(对源分析采用齐次化原理)

$$
\begin{aligned}
&\|\mathscr{S}(f,\varphi,u)-\mathscr{S}(f+\delta f,\varphi+\delta\varphi,\mu+\delta\mu)\|\\
\le&\|\delta\varphi\|+\|\delta\mu\|+\int_0^T\|\delta f(\tau,x)\|\mathrm{d}\tau\\
\le&\|\delta\varphi\|+\|\delta\mu\|+T\|\delta f(\tau,x)\|
\end{aligned}
$$

即解关于初边值与源稳定.

Robin条件

$$
\begin{aligned}
&(\partial_t-a^2\Delta)u=f(t,x)\quad x\in\Omega\subset\mathbb{R}^n,t>0\\
&t=0:u=\varphi(x)\\
&(\sigma u+\partial_n u)|_{\partial\Omega\times(0,\infty)}=\mu(t,x)
\end{aligned}
$$

对任意$T>0$, u在Γ_T上取得最大值. 若在$\partial\Omega\times[0,T]$上取得最大值, 则最大值点$x_0$处函数的法向导数应不小于0, 因此$u(x)\le\dfrac{\|\mu\|}{\sigma}$. 从而

$$
\begin{aligned}
&\|\mathscr{S}(f,\varphi,u)-\mathscr{S}(f+\delta f,\varphi+\delta\varphi,\mu+\delta\mu)\|\\
\le&\|\delta\varphi\|+\sigma^{-1}\|\delta\mu\|+\int_0^T\|\delta f(\tau,x)\|\mathrm{d}\tau\\
\le&\|\delta\varphi\|+\sigma^{-1}\|\delta\mu\|+T\|\delta f(\tau,x)\|
\end{aligned}
$$

实际上, 若$\sigma(x)$作为x的函数在$\partial\Omega$上有正下界或负上界, 则稳定性仍得证.

Newmann条件

$$
\begin{aligned}
&(\partial_t-a^2\Delta)u=f(t,x)\quad x\in\Omega\subset\mathbb{R}^n,t>0\\
&t=0:u=\varphi(x)\\
&(\partial_n u)|_{\partial\Omega\times(0,\infty)}=\mu(t,x)
\end{aligned}
$$

不妨设O在Ω内部, 令$v=ue^{-r^2}$. 原方程化为

$$
\begin{aligned}
&v_t-\left(\Delta v+\sum_i 4x_iv_{x_i}+(2+4|\nabla v|^2)v\right)=f(t,x)e^{-r^2}\quad x\in\Omega\subset\mathbb{R}^n,t>0\\
&t=0:v=\varphi(x)e^{-r^2}\\
&(\partial_n u-2rv)|_{\partial\Omega\times(0,\infty)}=\mu(t,x)e^{-r^2}
\end{aligned}
$$

由于$(2+4|\nabla v|)$恒正, $2r$有严格大于零的下界, 故

$$
\begin{aligned}
&\|\mathscr{S}(f,\varphi,u)-\mathscr{S}(f+\delta f,\varphi+\delta\varphi,\mu+\delta\mu)\|\\
\le&\|\delta\varphi\|+(\sigma^{-1})_{\min}\|\delta\mu\|+\int_0^T\|\delta f(\tau,x)\|\mathrm{d}\tau\\
\le&\|\delta\varphi\|+r_{\min}^{-1}\|\delta\mu\|+T\|\delta f(\tau,x)\|
\end{aligned}
$$

基本解理论

记微分算子$L(\partial_t,\partial_x)=\partial_t^m+\sum_{j=0}^{m-1}\sum_{|x|\le N_j}a_{j,\alpha}(t,x)\partial_t^j\partial_x^\alpha$

考虑方程$Lu=f,t>0,x\in\mathbb{R}^n$, 初值$\partial_t^ju|_{t=0}\varphi_j$,

$$
\begin{aligned}
&Lu=f\quad t>0,x\in\mathbb{R}^n\\
&\partial_t^ju|_{t=0}=\varphi_j,\quad 1\le j\le m-1
\end{aligned}
$$

基本解$E=E(t,x)$满足

$$
\begin{aligned}
&LE = 0 \quad t > 0, x \in \mathbb{R}^n \\
&\partial_t^j E|_{t=0} = 0, \quad 1 \le j \le m-2 \\
&\partial_t^{m-1} E|_{t=0} = \delta(x)
\end{aligned}
$$

从而

$$
u = \sum_{j=0}^{m-1} \partial_t^{m-1-j} [E(t,\cdot) * \varphi_j](x) + \int_0^t [E(t-\tau,\cdot) * f(t,\cdot)](x) \mathrm{d}\tau
$$

调和方程

Lapalce方程

设$\Omega \subset \mathbb{R}^n$为有界区域, 且区域上$\Delta u(x) \equiv 0$, 且$u$满足相应边界条件. 若满足上述条件的解$u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$, 则$u$为调和方程的解. 该类在$\Omega$内部求解的方程为内问题.

相应地外问题满足

$$
\begin{cases}
\Delta u \equiv 0 \quad x \in (\overline{\Omega})^c \\
\text{some boundary conditions} \\
\lim\limits_{|x| \to \infty} u(x) = 0
\end{cases}
$$

Green公式与基本解

设$\Omega \subset \mathbb{R}^n$为有界区域, 且$\partial\Omega \in C^1$. 设$\vec{n}$为单位外法向量. 则对任意$u, v \in C^1(\overline{\Omega}) \cap C^2(\Omega)$都有Green第一公式成立

$$
\begin{aligned}
\int_\Omega u \Delta v &= \int_\Omega u \nabla \cdot (\nabla v) \\
&= \int_\Omega \nabla \cdot (u \nabla v) - \nabla u \cdot \nabla v \\
&= \int_{\partial\Omega} u \nabla v \cdot \vec{n} \cdot \mathrm{d}S - \int_\Omega \nabla u \cdot \nabla v \\
&= \int_{\partial\Omega} u \frac{\partial v}{\partial n} \mathrm{d}S - \int_\Omega \nabla u \cdot \nabla v
\end{aligned}
$$

Poisson公式(Green第二公式)为

$$
\int_\Omega u \Delta v - v \Delta u = \int_{\partial\Omega} u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}
$$

据基本解理论, 今考察n维Poisson方程基本解$E_n(x)$满足$\Delta E_n(x) = \delta(x)$.

显然E_n径向对称, 不妨设$\Delta E_n(r)$为径向函数, 相应的Laplace算子为$r^{1-n} \partial_r r^{n-1} \partial_r$. 解得(假设$E_n$非常数)

$$
E_n(r) = \begin{cases}
\dfrac{1}{2\pi} \ln \dfrac{1}{r} & n = 2, \\
\dfrac{1}{n(n-2)|B_n(0,1)|} \cdot \dfrac{1}{r^{n-2}} & n \ge 3.
\end{cases}
$$

从而对任意$y \in \Omega^\circ$, $\int_\Omega u(x) \Delta E_n(y-x) \mathrm{d}x = u(y)$. 代入Poisson方程, 解得

$$
\begin{aligned}
u(x) &= \int_\Omega \Delta u(y) E_n(x-y) \mathrm{d}x + \int_{\partial\Omega} \frac{\partial u(y)}{\partial n} E_n(x-y) - \frac{\partial E_n(y)}{\partial n} u(x-y) \mathrm{d}S \\
&= \int_{\partial\Omega} \frac{\partial u(y)}{\partial n} E_n(x-y) - \frac{\partial E_n(y)}{\partial n} u(x-y) \mathrm{d}S
\end{aligned}
$$

实际上, 定义x_0在Ω内的测度:

$$\chi(x_0)=\begin{cases}1 & x_0\in\Omega^\circ\\ 1/2 & x_0\in\partial\Omega\\ 0 & x_0\in(\overline{\Omega})^c\end{cases}$$

则

$$\chi(x)\cdot u(x)=\int_{\partial\Omega}\frac{\partial u(y)}{\partial n}E_n(x-y)-\frac{\partial E_n(x-y)}{\partial n}u(y)\mathrm{d}S.$$

平均值公式与极值定理

置公式

$$\int_\Omega u\Delta v-v\Delta u=\int_{\partial\Omega}u\partial_n v-v\partial_n u$$

中$\Omega=B(x,R)$, u为调和函数, $v=1$, 则$\int_{\partial B(x,R)}\partial_n u=0$. 据基本解公式

$$u(x)=\int_{\partial B(x,R)}E_n(x-y)\partial_n u(y)-u(y)\partial_n E_n(x-y)\mathrm{d}S_y.$$

从而$u(x)$为$\dfrac{\int_{\partial B(x,R)}u(y)\mathrm{d}S_y}{\int_{\partial B(x,R)}\mathrm{d}S_y}$, 即球面上的平均积分.

容易见得, 球面平均与球体平均等价. 设$\varphi(x)$为仅与$r=|x|$相关之函数, 则在积分收敛时有

$$[\varphi*u](x)=u(x)\int_{\mathbb{R}^n}\varphi.$$

对任意开集$\Omega\subset\mathbb{R}^n$上具有局部均值性质的函数$u$, 置$\varphi_\varepsilon$为以$B(0,\varepsilon)$为紧支撑的磨光函数, 则可证得$u$在$\{x:d(x,\partial\Omega)<\varepsilon\}$上光滑. 令$\varepsilon\to 0$即得$u\in C^\infty(\Omega)$. 由于

$$\Delta_n u(x)=r^{1-n}\partial_r r^{n-1}\partial_r\int_{B(0,r)}u=0.$$

从而u调和. 因此开集上的调和函数等价于满足局部均值性质的函数.

Green函数

端详公式Ω内Dirichlet问题解所满足的方程

$$u(x)=\int_{\partial\Omega}\frac{\partial u(y)}{\partial n}E_n(x-y)-\frac{\partial E_n(x-y)}{\partial n}u(y)\mathrm{d}S.$$

对给定的初边值问题而言, u与$\partial_n u$不可兼得. 就Dirichlet问题而言, 应当消去公式中的$\partial_n u$项. 下推导Dirichlet问题的Green函数.

考虑函数$g(x,y)$使得对任意给定的$x\in\Omega$, $g(x,y)$在$\partial\Omega$上取值与$E(x-y)$相同. 且$g(x,y)$在Ω内部调和. 从而

$$\int_\Omega u(y)\Delta g(x,y)-g(x,y)\Delta u(y)=\int_{\partial\Omega}u(y)\partial_n g(x,y)-g(x,y)\partial_n(y)=0.$$

令$G(x,y)=E_n(x-y)-g(x,y)$, 则有

$$u(x)=\int_{\partial\Omega}G(x,y)\partial_n u(y)-u(y)\partial_n G(x,y)\mathrm{d}S_y.$$

其中$G(x,y)|_{\partial\Omega}\equiv 0$. 从而当$u(x)|_{\partial\Omega}=\mu(x)$时, 解得

$$u(x)=-\int_{\partial\Omega}\mu(y)\partial_n G(x,y)\mathrm{d}S_y.$$

对Newman条件, 只需构造满足$\partial_n[g(x,y)-E_n(y-x)]_{\partial\Omega}\equiv 0$的调和函数$g(x,y)$即可. 记$G(x,y)=E(x-y)-g(x,y)$, $\partial_n u\equiv\mu(x)$, 则

$$u(x)=\int_{\partial\Omega}\mu(y)G(x,y)\mathrm{d}S_y.$$

对Robin条件, 构造g满足$(\partial_n+\sigma\cdot\mathrm{id})[g(x,y)-E_n(y-x)]\equiv 0$即可. 下不赘述.

检验得Green函数满足以下性质(以Dirichlet问题对应的Green函数为例)

1. 对给定的x, $G(x,y)$在$\Omega-\{x\}$调和, 在$x'\to x$时以$E(x-x')\sim[\|x-x'\|^{n-1}]'$速度趋向无穷. 同时$\Omega$内有

$$0<G(x,y)<E(x-y).$$

   边界上有$G(x,y)\equiv 0$.
2. 由于Green函数与u无关. 置$u|_{\partial\Omega}\equiv 1$, 得

$$\int_{\partial\Omega}\partial_n G(x,y)\mathrm{d}S_y=-1.$$

3. Green函数$G(x,y)$指标可交换. 任取$x,y\in\Omega$, $0<\varepsilon\ll\mathrm{diam}(\Omega)$, 则

$$\begin{aligned}0=&\int_{\Omega-B(x,\varepsilon)-B(y,\varepsilon)}G(x,t)\Delta G(y,t)-G(y,t)\Delta G(z,t)\mathrm{d}t\\
=&\int_{\partial[B(x,\varepsilon)\cup B(y,\varepsilon)]}G(x,t)\partial_n G(y,t)-G(y,t)\partial_n(x,t)\mathrm{d}S_t\\
&+\int_{\partial\Omega}G(x,t)\partial_n G(y,t)-G(y,t)\partial_n(x,t)\mathrm{d}S_t\\
\overset{\varepsilon\to 0}{\to}&\int_{\partial B(x,\varepsilon)}G(x,t)\partial_n G(y,x)\mathrm{d}S_t-\int_{\partial B(y,\varepsilon)}G(y,t)\partial_n G(x,y)\mathrm{d}S_t\\
&\int_{\partial B(y,\varepsilon)}G(x,y)\partial_n G(y,t)\mathrm{d}S_t-\int_{\partial B(x,\varepsilon)}G(y,x)\partial_n G(x,t)\mathrm{d}S_t\\
=&0+0+(-1)G(x,y)-(-1)G(y,x)\end{aligned}$$

   其中$\int_{\partial B(x,\varepsilon)}G(x,t)\mathrm{d}S_t\sim E_n(\varepsilon)\cdot|\partial B(0,\varepsilon)|\sim\varepsilon^1\to 0$.

特别地, 对球面$B(O,R)$中点x, $G(x,y)$为点电荷x在空球导体内的生成电场. 设$x'=\dfrac{R^2}{x^2}\cdot x$, 则

$$G(x,y)=E_n(y-x)-E_n(\frac{R}{|x|}(y-x')).$$

解得

$$u(x)=\frac{R-\|x\|^2/R}{|\partial B_n(0,1)|}\cdot\int_{\partial B_n(O,R)}\frac{f(y)}{\|y-x\|^n}\mathrm{d}S_y.$$

特别地, 上半平面的Green函数即$G(x,y)=E_n(y-x)-E_n(y-x')$. 其中$x=(x_1,\dots,x_n)$, $x'=(x_1,\dots,x_{n-1},-x_n)$. 解得

$$u(x)=\frac{x_n\Gamma(n/2)}{\pi^{n/2}}\int_{\mathbb{R}^{n-1}}\frac{f(y)}{\sqrt{(\tilde{x}_{n-1}-y)^2+(x_n)^2}^n}\mathrm{d}y.$$

二维单连通区域的Dirichlet问题

考虑Dirichlet问题

$$\begin{aligned}&\Delta u = 0 \quad x \in \Omega, t > 0\\ &u|_{\partial\Omega} = \varphi(x)\end{aligned}$$

其中Ω可通过全纯函数f共性映照至$\mathbb{D}$, i.e., $f : \Omega \to \mathbb{D}$全纯且同胚. 从而原PDE转化为(等同$g(x, y)$与$g(x + iy)$):

$$\begin{aligned}&\Delta u \circ f^{-1} = 0 \quad x \in \mathbb{D}, t > 0\\ &u|_{\partial\mathbb{D}} = \varphi \circ f^{-1}(x)\end{aligned}$$

对任意$f(x_0) \in \mathbb{D}$, 由Poisson积分公式得

$$u(x_0) = \frac{1}{2\pi}\int_{S^1}\frac{1 - |f(x_0)|^2}{|z - f(x_0)|^2}\varphi \circ f^{-1}(z)\mathrm{d}z$$

记$z = f(x)$, 则

$$u(x_0) = \frac{1}{2\pi}\int_{\partial\Omega}\frac{1 - |f(x_0)|^2}{|f(x) - f(x_0)|^2}\varphi(x)\frac{\partial f}{\partial t}(x)\mathrm{d}x.$$

实际上, $G_\Omega(\xi, \xi_0) = G_{\mathbb{D}}(f(\xi), f(\xi_0))$. 特别地, 若共性映照

$$\Phi(\cdot, z_0) : \Omega \to \mathbb{D}, z_0 \mapsto 0, \partial\Omega \to U_1.$$

则$G(z, z_0) = \dfrac{1}{2\pi}\ln\dfrac{1}{|f(z, z_0)|}$.

可去奇点定理

可取奇点定理本质上反应了调和函数阶数的某种间断性, 其本质仍是解析性. 若n维调和函数u在$B(x, r) - \{x\}$内调和, 且$x' \to x$时满足$\dfrac{u(x')}{E_n(x - x')} \to 0$, 则$u(x)$可定义(即奇点可去).

不妨设$r = 2$, $x = O$. 依Poisson积分公式延拓$u|_{\partial B(O,1)}$为函数$v|_{\overline{B(O,1)}}$. 作$w_\varepsilon(x) := u(x) - v(x) + \varepsilon E_n(x)$, 其中$\varepsilon \in (-1, 1) - \{0\}$. 对足够小的$\delta(\varepsilon) \ll 1$, $\varepsilon E_n(x)$项主导w_ε. 记$\delta(|\varepsilon|) := \inf_{t \geq |\varepsilon|}\min\{\delta(t), \delta(-t)\}$(单调递减至0), 则$w_{|\varepsilon|}(x)$在$\partial B(O, 1) \cup \partial B(O, \delta(|\varepsilon|))$上非负, 即$w_\varepsilon$在$B(O, 1) - B(O, \delta(|\varepsilon|))$上非负. 同理$w_{-|\varepsilon|}$在$B(O, 1) - B(O, \delta(|\varepsilon|))$上非正. 令$\varepsilon \to 0$即得

$$(u - v)|_{B(O,1) - B(O,\delta(|\varepsilon|))} \equiv 0.$$

而$\delta \overset{\varepsilon\to 0}{\to} 0$, 故$u \equiv v$在$B(O, 1)$上恒成立.

可采用可去奇点定理可估计定义域为$B(O, R)$外的调和函数的衰减速度. 对任意$u \in \mathrm{Har}(\overline{B(O, R)^c})$, $u|_{\partial B(O,R)} = \varphi(x)$, 考虑Kevin变换

$$v(x) = \frac{\|x\|^{n-2}}{R^{n-2}}u(\frac{R^2}{\|x\|^2}x) \quad \|x\| \geq R.$$

从而$v(x)$在$0 < \|x\| < R$时调和, 且边界上取值$v(x) = u(x)|_{x \in \partial B(O,R)}$. 再若

$$\lim_{x\to\infty} u(x) = 0.$$

则

$$\lim_{x\to 0} v(x)\cdot (E_n(x)) = 0.$$

据可去奇点定理, $v(0)$为可取间断点. 据Poisson积分公式得

$$u(x) = \frac{\|x\|^{n-2}}{R^{n-2}}\cdot \frac{1}{|S^{n-1}|}\int_{S^{n-1}} \frac{R^2 - R^4/\|x\|^2}{(y - xR^2/\|x\|^2)^n} u(y)\mathrm{d}S_y.$$

Harnack不等式

由于$u(x)$在$\overline{B_n(O,R)}$内一致有界, 故不妨设$u|_{\overline{B_n(O,R)}} \geq 0$. 考察Poisson积分公式

$$u(x) = \frac{R - \|x\|^2/R}{|\partial B_n(0,1)|}\cdot \int_{\partial B_n(O,R)} \frac{f(y)}{\|y-x\|^n}\mathrm{d}S_y.$$

注意到$R - \|x\| \leq \|y - x\| \leq R + \|x\|$, 从而

$$\frac{R^2 - \|x\|^2}{(R + \|x\|)^n} u(O) \leq u(x) \leq \frac{R^2 + \|x\|^2}{(R - \|x\|)^n} u(O).$$

同理, 对外问题有

$$u(x) = \frac{\|x\|^{n-2}}{R^{n-2}}\cdot \frac{1}{|S^{n-1}|}\int_{S^{n-1}} \frac{R^2 - R^4/\|x\|^2}{(y - xR^2/\|x\|^2)^n} u(y)\mathrm{d}S_y.$$

$$\frac{\|x\|^{2n-4}}{R^{2n-4}}\cdot \frac{\|x\|^2 - R^2}{(\|x\| + R)^n} u(O) \leq u(x) \leq \frac{\|x\|^{2n-4}}{R^{2n-4}}\cdot \frac{\|x\|^2 - R^2}{(\|x\| - R)^n} u(O)$$