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Lecture Notes for K-seminar
Recollections from Projective Modules

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1 About the Lecture

We strongly recommend the readers to review the basic definitions and facts about

1. groups, rings, modules, field, algebra, etc;
2. the definition of Abelian category $\mathcal{A}$;
3. basic knowledge on categories, such as commutative diagrams.

The Lecture Note mainly discusses

1. When free objects are defined and how free objects related to projective objects.
2. Equivalent definitions of projective modules (for small Abelian categories).
3. The chain: Free Mods $\to$ Stably Free Mods $\to$ Projective Mods $\to$ Flat Mods $\to$ Torison Free Mods.

2 Equivalent definitions of projective modules

Definition 1 (The adjoint pair of Free and Forgetful functors)**.** For category $\mathcal{C}$ and Set the category of sets, if there exists an adjoint pair $(F \dashv U)$ such that

- $F : \text{Set} \to \mathcal{C}$ is the free functor sending the category of sets to $\mathcal{C}$;
- U is the forgetul functor sending the collection of objects (resp., morphisms) of $\mathcal{C}$ to the underlying sets (resp., set mappings);
- the collection $\{\varepsilon_X : FU(X) \to X\}_{X\in\mathsf{Ob}(\mathcal{C})}$ is called co-unit;
- $\{\eta_X : S \to UF(X)\}_{S\in\mathsf{Ob}(\text{Set})}$ is called unit.

Remark 1. Freyd-Mitchell embedding theorem says that every small Abelian category $\mathcal{A}$ is a full subcategory of a category of modules over some ring R, such that the embedding functor $\mathcal{A} \hookrightarrow R-$Mod is an exact functor.

SLOGAN 1. Small Abelian categories $Longleftarrow R-$Mod categories.

Definition 2 (projective modules)**.** Suppose that $\mathcal{A}$ is an Abelian category. We say $P \in \mathcal{A}$ is projective, whenever

- for arbitrary epimorphism $X \overset{\pi}{\twoheadrightarrow} Y$ and $P \xrightarrow{f} Y$, there exists a lift $\tilde{f}$ such that the following diagram commutes

$$\begin{tikzcd} & P \arrow[dl, dashed, "\tilde{f}"'] \arrow[d, "f"] \\ X \arrow[r, two heads, "\pi"'] & Y \end{tikzcd}.$$