偏微分复习(第二部分)

Fourier法

Fourier变换简介

记$\mathbb{R}^n$上的Fourier变换(有处定义不采用$(2\pi)^{-n/2}$)为

$$\mathscr{F}: f(x) \mapsto \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-i\xi\cdot x} \mathrm{d}x.$$

相应地逆变换为

$$\mathscr{F}^{-1}: f(x) \mapsto \check{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{i\xi\cdot x} \mathrm{d}x.$$

对速降空间(Schwarz space)$\mathcal{S}$, $\mathscr{F}: \mathcal{S} \to \mathcal{S}$为双射. 同时, $\mathscr{F}: L^2(\Omega) \to L^2(\Omega)$亦为双射(设函数在相差零测集的意义下相同). 一般地, 对任意$p \in (1, \infty)$, 有双射关系

$$\mathscr{F}: L^p(\Omega) \to L^{p^*}(\Omega).$$

其中共轭指标满足$p^{-1} + (p^*)^{-1} = 1$. 该定理为Riesz-Thorin定理.

当$p = p^* = \dfrac{1}{2}$时$\mathscr{F}$保距, 即对任意$f, g \in L^2(\Omega)$均有

$$\langle f, g\rangle = \langle \mathscr{F}[f], \mathscr{F}[g]\rangle.$$

简单的Fourier变换

考虑$\mathscr{F}: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, 则

- $\mathscr{F}$保持线性, 即保持自变量的加和与数乘.
- $\mathscr{F}[f \circ (-x_0)](\xi) = \hat{f}(\xi) \cdot e^{-ix_0\cdot\xi}$.
- $\mathscr{F}[f \circ (c\cdot)](\xi) = c^{-n}\hat{f}(c^{-1}\xi)$.
- (接上条) 对非奇异常矩阵A, $\mathscr{F}[f \circ (A\cdot)] = (\det A)^{-1}\hat{f}({}^tA^{-1}\xi)$.
- $\mathscr{F}[f * g] = (2\pi)^{n/2}\mathscr{F}[f] \cdot \mathscr{F}[g]$.

- $\mathscr{F}[f\cdot g]=(2\pi)^{-n/2}\mathscr{F}[f](\xi)*\mathscr{F}[g](\xi)$.
- $\mathscr{F}[\partial_{x_j}f]=i\xi_j\cdot\mathscr{F}[f]$. 常以方便故记$\mathcal{D}_{x_j}:=\dfrac{\partial_{x_j}}{i}$.
- $\mathscr{F}[(\prod_\alpha i^{-k}\xi_k)\cdot f](\xi)=\partial^\alpha\mathscr{F}[f]$.
- (接上条) 设$\alpha=(\alpha(1),\alpha(2),\ldots,\alpha(n))$为指标, 并定义$\mathcal{D}^\alpha=\prod_k\mathcal{D}_{x_k}^{\alpha(k)}$, $\xi^\alpha=\prod_k\xi_k^{\alpha(k)}$. 则

  $$\mathscr{F}[\mathcal{D}^\alpha f]=\xi^\alpha\mathscr{F}[f].$$

  同理, 对关于若干α的多项式$P(\Lambda)=P(\alpha,\beta,\cdots,\gamma)$, 有

  $$\mathscr{F}[\mathcal{D}^{P(\Lambda)}f]=\xi^{P(\Lambda)}\mathscr{F}[f].$$

- $\mathscr{F}^2:f(x)\mapsto f(-x)$. $\mathscr{F}^4$恒等.
- (广义函数) $\mathscr{F}[\delta](\xi)=\dfrac{1}{\sqrt{2\pi}^n}$.

Fourier变换法应用

对以下方程

$$\begin{cases}u_{tt}+a^2u_{xxxx}=0\\ t=0:u=\varphi(x),u_t=a\psi''(x)\end{cases}$$

关于x做Fourier变换得

$$\begin{cases}\hat{u}_{tt}+a^2\xi^4\hat{u}=0\\ t=0:\hat{u}=\hat{\varphi}(\xi),\hat{u}_t=-a\xi^2\hat{\psi}(\xi)\end{cases}$$

解得$\hat{u}(t,\xi)=\hat{\varphi}(\xi)\cos a\xi^2t-\hat{\psi}(\xi)\sin a\xi^2t$. 从而

$$\begin{aligned}u(t,x)=&\mathscr{F}^{-1}[\hat{u}(t,\cdot)](\xi)\\ =&\mathscr{F}^{-1}[\hat{\varphi}(\xi)\cdot\cos at\xi^2]-\mathscr{F}^{-1}[\hat{\psi}(\xi)\cdot\sin at\xi^2]\\ =&\frac{1}{\sqrt{2\pi}}\left(\varphi*\mathscr{F}^{-1}[\cos at\xi^2]-\psi*\mathscr{F}^{-1}[\sin at\xi^2]\right)\\ =&\frac{1}{2\sqrt{2at}}\int_{\mathbb{R}}\varphi(\xi)\left[\cos\frac{(\xi-x)^2}{4at}+\sin\frac{(\xi-x)^2}{4at}\right]\mathrm{d}\xi\\ &+\frac{1}{2\sqrt{2at}}\int_{\mathbb{R}}\psi(\xi)\left[\cos\frac{(\xi-x)^2}{4at}-\sin\frac{(\xi-x)^2}{4at}\right]\mathrm{d}\xi\\ =&\frac{1}{2\sqrt{at}}\int_{\mathbb{R}}\varphi(\xi)\left[\cos\frac{(\xi-x)^2-at\pi}{4at}\right]\mathrm{d}\xi\\ &+\frac{1}{2\sqrt{at}}\int_{\mathbb{R}}\psi(\xi)\left[\cos\frac{(\xi-x)^2+at\pi}{4at}\right]\mathrm{d}\xi\end{aligned}$$

基本解理论

基本解

记微分算子$L(\partial_t,\partial_x)=\partial_t^m+\sum_{j=0}^{m-1}\sum_{|x|\le N_j}a_{j,\alpha}(t,x)\partial_t^j\partial_x^\alpha$

考虑方程$Lu=f,t>0,x\in\mathbb{R}^n$, 初值$\partial_t^j u|_{t=0}=\varphi_j$,

$$\begin{aligned}&Lu=f\quad t>0,x\in\mathbb{R}^n\\&\partial_t^j u|_{t=0}=\varphi_j,\quad 1\le j\le m-1\end{aligned}$$

基本解$E=E(t,x)$满足

$$\begin{aligned}&LE=0\quad t>0,x\in\mathbb{R}^n\\&\partial_t^j E|_{t=0}=0,\quad 1\le j\le m-2\\&\partial_t^{m-1}E|_{t=0}=\delta(x)\end{aligned}$$

从而

$$u=\sum_{j=0}^{m-1}\partial_t^{m-1-j}[E(t,\cdot)*\varphi_j](x)+\int_0^t[E(t-\tau,\cdot)*f(t,\cdot)](x)\mathrm{d}\tau$$

热传导方程的基本解

以热方程为例, 记$L(\partial_t,\partial_x):u\mapsto=\partial_t u-a^2\partial_{xx}u$. 则PDE问题为转化为

$$\begin{aligned}&LE=0\quad t>0,x\in\mathbb{R}^n\\&E|_{t=0}=\delta(x)\end{aligned}$$

Fourier变化得

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\hat{E}(t,\cdot)(\xi)+a^2\xi^2\hat{E}(t,\cdot)(\xi)=0\quad t>0,x\in\mathbb{R}^n\\&\hat{E}(0,\cdot)(\xi)=\frac{1}{(\sqrt{2\pi})^n}\end{aligned}$$

解得$\hat{E}(t,\cdot)(\xi)=e^{-ta^2\xi^2}$. Fourier逆变换得

$$
\begin{aligned}
E(t,x) =& \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-ta^2\xi^2} e^{ix\cdot\xi}\mathrm{d}\xi \\
=& \frac{1}{(2\pi)^n}\prod_{k=1}^n\int_{\mathbb{R}} e^{-ta^2(\xi_k - ix_k/2a^2t)}\cdot e^{-x_k^2/4a^2t}\mathrm{d}\xi \\
=& \frac{1}{(2\pi)^n}\cdot\sqrt{\frac{\pi}{a^2t}}\cdot e^{-x^2/4a^2t} \\
=& \frac{e^{-x^2/4a^2t}}{(2a\sqrt{\pi t})^n}
\end{aligned}
$$

波动方程的基本解

考虑全空间内波动方程基本解

$$
\begin{cases}
(\partial_{tt} - a^2\Delta_n)E(t,x) = 0 \\
E(0,x) = 0 \\
E_t(0,x) = \delta(x)
\end{cases}
$$

Fourier变换得

$$
\begin{cases}
(\partial_{tt} + a^2\xi^2)\hat{E} = 0 \\
\hat{E}(0,\xi) = 0 \\
\partial_t\hat{E}(0,\xi) = \dfrac{1}{\sqrt{2\pi}^n}
\end{cases}
$$

解得$\hat{E}(t,\xi) = \dfrac{\sin(a|\xi|t)}{a|\xi|\sqrt{2\pi}^n} = \dfrac{t}{\sqrt{2\pi}^n}\sum_{k\geq 0}\dfrac{(-a^2\xi^2t^2)^k}{(2k+1)!}$. Fourier逆变换得

$$
E(t,x) = \frac{t}{2\pi^n}\sum_{k\geq 0}\frac{(-a^2t^2)^k}{(2k+1)!}\prod_{d=1}^n\int_{\mathbb{R}} e^{(\xi_d^2)^k} e^{i\xi_d x_d}\mathrm{d}\xi_d.
$$

对低维简单情形, 可直接求解.

热传导方程

全空间上的热传导方程

对方程

$$\begin{cases} u_t - a^2\Delta u = f(t,x), t > 0, x \in \mathbb{R}^n \\ t = 0 : u = \varphi(x) \end{cases}$$

考虑对x做Fourier变化所得的PDE问题

$$\begin{cases} \partial_t \hat{u}(t,\xi) + a^2|\xi|^2\hat{u}(t,\xi) = \hat{f}(t,\xi) \\ t = 0 : \hat{u}(t,\xi) = \hat{\varphi}(\xi) \end{cases}$$

解ODE问题得

$$\begin{aligned} u(t,x) =& (2a\sqrt{\pi})^{-n}\int_{\mathbb{R}^n} \frac{e^{-|x-y|^2/4at}}{\sqrt{t}}\varphi(y)\mathrm{d}y \\ &+ (2a\sqrt{\pi})^{-n}\int_0^t\int_{\mathbb{R}^n} \frac{e^{-|x-y|^2/4a(t-\tau)}}{\sqrt{t-\tau}^n} f(\tau,y)\mathrm{d}y\mathrm{d}\tau \end{aligned}$$

设基本解$E(t,x) = \dfrac{\exp\dfrac{-|x|^2}{4at}}{(2a\sqrt{\pi t})^n}$, 从而

$$u(t,x) = [E(t,\cdot) * \varphi](x) + \int_0^t [E(t-\tau,\cdot) * f(\tau,\cdot)](x)\mathrm{d}\tau.$$

基本解关于$t \to 0$为光滑的good kernel, 即满足如下性质:

- $E(t,x) \in C^\infty(\{t > 0\})$.
- $t > 0$时, $\partial_t E(t,x) = a^2\Delta_x E(t,x)$.
- $\int_{\mathbb{R}^n} E(t,x)\mathrm{d}x = 1$. 注意到$E(t,x)$恒正, 故绝对积分一致有界.
- 对任意$\delta > 0$, $\lim_{t\to 0^+}\int_{\mathbb{R}^n - B_n(0,\delta)} |E(t,x)|\mathrm{d}x = 0$.

从物理角度而言, 热方程之解应当具有以下性质(不难验证):

- 齐次热传导方程之解满足$u(t,x) \in [\inf\varphi(x), \sup\varphi(x)]$.

再论迭代法

就以下方程为例

$$\begin{cases} u_t - a^2(u_{xx} + 4u_{yy}) = y^2t^2 \\ t = 0 : u = x^2y \end{cases}$$

记算子$P : u \mapsto \partial_t u - a^2(\partial_{xx} + 4\partial_{yy})u$. 注意到

$$\begin{aligned} \frac{t^3}{3}y^2 &\mapsto y^2t^2 - \frac{8a^2t^3}{3} \\ \frac{2a^2t^4}{3} &\mapsto \frac{8a^2t^3}{3} \\ x^2y &\mapsto -2a^2y \\ 2a^2ty &\mapsto 2a^2y \end{aligned}$$

从而$u = x^2y + 2a^2ty + \dfrac{t^3}{3}y^2 + \dfrac{2a^2t^4}{3}$.

对较复杂的方程(设α与β相关加和在定义域内一致收敛)

$$\begin{cases} u_t - a^2\Delta_n u = \displaystyle\sum_{k\geq 0}\prod_{l=1}^{n}\alpha_{k,l}(t, x^l) \quad x \in \Omega, t > 0 \\ t = 0 : u = \displaystyle\sum_{k\geq 0}\prod_{l=1}^{n}\beta_{k,l}(x^l) \end{cases}$$

则$u = \sum_{k\geq 0}\sum_{l=1}^{n}(v_{k,l} + w_{k.l})$, 其中$v_{k,l}$为方程

$$\begin{aligned} &\partial_t v_{k,l}(t, x^l) - a^2\partial_{x^lx^l}v_{k,l}(t, x^l) = 0 \\ &t = 0 : v_{k,l}(t, x^l) = \beta_{k,l}(x^l) \end{aligned}$$

之解, 即$v_{k,l} = [\beta_{k,l} * E(t, \cdot)](x)$. $v_{k,l}$为方程

$$\begin{aligned} &\partial_t v_{k,l}(t, x^l) - a^2\partial_{x^lx^l}v_{k,l}(t, x^l) = \alpha_{k,l}(t, x^l) \\ &t = 0 : v_{k,l}(t, x^l) = 0 \end{aligned}$$

之解. 综上

$$u = \sum_{k\geq 0}\sum_{l=1}^{n}[\alpha_{k,l} * E(t, \cdot)(x^l) + \int_0^t \beta_{k,l}(\tau, \cdot) * E(t - \tau, \cdot)(x^l)\mathrm{d}\tau].$$

无量纲量法

考虑热方程

$$\begin{gathered}u_t - a^2\Delta_n u = 0 \quad x \in \Omega, t > 0\\ \text{some given boundary conditions}\end{gathered}$$

记无量纲量$\xi = \dfrac{r}{a\sqrt{t}}$, 则

$$\begin{aligned}u_t &= u_\xi \cdot \xi_t = u_\xi \cdot \frac{-r}{2at\sqrt{t}}\\ \Delta_n u &= r^{1-n}\partial_r(r^{n-1}\partial_r u) = \frac{n-1}{ar\sqrt{t}}u_\xi + \frac{r}{a^2t}u_{\xi\xi}\end{aligned}$$

从而PDE化为

$$u_\xi((n-1)/\xi + \xi/2) + u_{\xi\xi} = 0.$$

当$n = 1$时, 解得

$$u = u_0 + (u_\infty - u_0) \cdot \frac{1}{\sqrt{\pi}} \int_0^{x/2a\sqrt{t}} e^{-s^2}\mathrm{d}s.$$

当$n \geq 2$时, 解得

$$u = u_0 + (u_\infty - u_0) \cdot \frac{2}{\Gamma(-n/2)} \cdot \int_0^{x/2a\sqrt{t}} s^{1-n}e^{-s^2}\mathrm{d}s.$$

分离变量法

一维闭区域上情形

对热传导方程

$$\begin{cases}u_t - a^2u_{xx} = 0, \quad 0 < x < l, t > 0\\ t = 0 : u = \varphi(x)\\ \text{some given boundary conditions}\end{cases}$$

StepI: 寻找一个仅满足边值条件的函数v, 下考虑$w = u - v$. 分离变量得特征方程 $\dfrac{X''}{X} = \dfrac{T'}{a^2T} = -\lambda_k$, 考虑正交基$\{e_k\}_{k\geq 0}$使得$e_k(x)$满足边值条件, 且 $e_k''(x) + \lambda_k e_k(x) = 0$. 注意: 当满足Newman条件时应补上0特征值.

Step II: 设解具有一般形式($u(t, x) = 0$时$\theta_k \equiv 0$):

$$\sum_{\exists\lambda=0}\varphi(0)+\sum_{k\geq 1}A_k e^{-\lambda_k t}\sin(\sqrt{-\lambda}x+\theta_k).$$

其中

$$A_k=\frac{2}{l}\int_0^l\varphi(x)\sin(\sqrt{-\lambda}x+\theta_k)\mathrm{d}x.$$

二维矩形上情形

考虑方程

$$\begin{aligned}
&u_t-\Delta_2 u=f\\
&t=0:u=\varphi(x)\\
&x=0:u=\mu_1(y)\quad x=a:u=\mu_2(y)\\
&y=0:u=\psi_1(x)\quad y=b:u=\psi_2(x)
\end{aligned}$$

同样, 对含源项采用齐次化原理. 无源时, 考虑$u=T(t)X(x)$, 则

$$\frac{\Delta X(x)}{X}=\frac{T'(t)}{T}=-\lambda.$$

则原问题转化为特征值问题

$$\begin{aligned}
&\Delta_2 u+\lambda u=0\\
&x=0:u=\mu_1(y)\quad x=a:u=\mu_2(y)\\
&y=0:u=\psi_1(x)\quad y=b:u=\psi_2(x)
\end{aligned}$$

拆分$u=v+w$, 其中v在$x\in\{0,a\}$时取值为0, w在$y\in\{0,b\}$时取值为0. 考虑

$$\begin{gathered}
\frac{X''}{X}+\frac{Y''}{Y}+\lambda=0\\
X(0)=X(a)=0
\end{gathered}$$

解得$X(x)\in\operatorname{span}_{k\geq 1}(\sin k\pi x/a)$. 从而

热稳态

(数学物理方法P56-6) 半径为a的半圆形平板, 其表面绝热, 在板的周围边界上保持常温u_0, 而在直径边界上保持常温u_1, 求板的稳恒状态.

解: 稳恒时, 温度分布函数u满足$\partial_t u=0$, 从而$\Delta u=0$. 定解问题为

$$
\begin{cases}
\partial_{rr}u+\dfrac{\partial_r}{r}u+\dfrac{\partial_{\theta\theta}}{r^2}u=0\\
u(a,\theta)=u_0,\quad 0<\theta<\pi\\
u(r,0)=u(r,\pi)=u_1,\quad 0\le r\le a
\end{cases}
$$

令$v=R(r)\Theta(\theta)+u_1$, 从而

$$
r^2\frac{R''}{R}+\frac{rR'}{R}=-\frac{\Theta''}{\Theta}=\lambda.
$$

由$\Theta''+\lambda_k\Theta=0$及$\Theta(0)=\Theta(\pi)=0$知$\lambda_k=k^2$. 解Euler方程

$$
r^2R_k''+rR_k'-\lambda_kR_k=0
$$

得

$$
\begin{cases}
R_k=B_kr^k+C_kr^{-k} & k>0\\
R_0=C_0+D_0\ln r & k-0
\end{cases}
$$

实际上, 由有界性知$C_k=0$. 从而解具有形式

$$
u=u_1+\sum_{k\ge 1}B_kr^k\sin(k\theta).
$$

故

$$
\frac{2}{\pi}\int_0^\pi\sin(k\theta)(u_0-u_1)\mathrm{d}\theta=B_ka^k.
$$

解得$B_k=\dfrac{2(u_0-u_1)}{a^kk\pi}[1-(-1)^k]$. 故

$$
u(r,\theta)=u_1+\frac{4(u_0-u_1)}{\pi}\sum_{n\ge 1}\frac{\sin\left[(2n-1)\theta\right]}{2n-1}\cdot\left(\frac{r}{a}\right)^{2n-1}.
$$

极值原理

无释热源的极值原理

考虑$\Omega \subset \mathbb{R}^n$为有界区域, $\forall T > 0, \Omega_T := (0,T) \times \Omega$. 定义抛物边界

$$\partial' \Omega := \{(t,x) : t = 0 \vee x \in \partial\Omega\}.$$

若$u \in C^0(\overline{\Omega_T}) \cap C^{1,2}(\Omega)$, 则$u$在$\Omega_T$可取到最大值. 若$u$在$\Omega_T$内部满足$\partial_t u - a^2 \Delta u \le 0$, 则根据物理学意义, u最大值在抛物边界取到.

实际上, 若u在非抛物点(t_0, x_0)上取到最大值, 则

1. $\partial_t u(t_0, x_0) \ge 0$, 取大于若且仅若$t_0 = T$.
2. 对固定的t_0, u局部次调和, 即$\Delta u \le 0$.

因此$(\partial_t - a^2 \Delta) u \ge 0$. 取等若且仅若$\Delta u \equiv 0$, 即$u$为常函数.

热方程的极值原理

极值定理的导出

考虑有界区域Ω上的热方程

$$\begin{aligned} &Lu = \partial_t u - \left(\sum_{i,j=1}^n a_{ij} u_{x^i, x^j} + \sum_{i=1}^n b_i u_{x^i} + cu \right) \quad x \in \Omega, t > 0 \\ &\text{Some given initial and boundary conditions} \end{aligned}$$

其中a_{ij}, b_i, c均为关于x, t的连续函数, (a_{ij})恒正定, c恒非负. 则对任意$T > 0$, u在

$$\Gamma_T := \partial(\Omega \times T) - \Omega^\circ \times \{T\}$$

上取非负最大值. 不妨设最大值点$(t_0, x_0) \in (\Omega \times T)^\circ$, 即$\Omega \times T$内部取得最大值$M$. 设$m$为$u$在抛物边界上的最大值, 则$M > m$. 考虑函数

$$v := u + \varepsilon(\|x - x_0\|)^2$$

其中ε可取得充分小使得v在Γ_T上取值不超过$v(t_0, x_0) = M$. 注意到

$$\begin{aligned} Lv =& \partial_t v - \left(\sum_{i,j=1}^n a_{ij} u_{x^i, x^j} + \sum_{i=1}^n b_i u_{x^i} + cu \right) \\ =& - \varepsilon \left(2 \sum_i (a_{ii} + b_i x_i + c(x^i - x_0^i))^2 \right) \end{aligned}$$

而v在内部取最大值, 从而最大值点处Lv为负. 而当v取得非负最大值时, Lv应为非负数, 矛盾.

从而u在边界上取得非负最大值及非正最小值.

解的稳定性分析

为方便起见, 以下讨论标准形式的热方程.

Dirichlet条件

考虑一般热方程

$$
\begin{aligned}
&(\partial_t - a^2\Delta)u = f(t,x) \quad x\in\Omega\subset\mathbb{R}^n, t>0\\
&t=0: u=\varphi(x)\\
&u|_{(0,\infty)\times\partial\Omega} = \mu(t,x)
\end{aligned}
$$

方程解至多唯一.

记$u=\mathscr{S}(f,\varphi,\mu)$为解, 则当发生微扰$\delta f, \delta\varphi$与$\delta\mu$时, 新方程满足

$$
\begin{aligned}
&(\partial_t - a^2\Delta)u = f(t,x)+\delta f(t,x)\\
&t=0: u=\varphi(x)+\delta\varphi(x)\\
&u|_{(0,\infty)\times\partial\Omega} = \mu(t,x)+\delta\mu(t,x)
\end{aligned}
$$

记$\|g\| := \sup_{(t,x)\in\Omega_T}|g|$. 因此对任意$T>0$均有(对源分析采用齐次化原理)

$$
\begin{aligned}
&\|\mathscr{S}(f,\varphi,u)-\mathscr{S}(f+\delta f,\varphi+\delta\varphi,\mu+\delta\mu)\|\\
\le&\|\delta\varphi\|+\|\delta\mu\|+\int_0^T\|\delta f(\tau,x)\|\mathrm{d}\tau\\
\le&\|\delta\varphi\|+\|\delta\mu\|+T\|\delta f(\tau,x)\|
\end{aligned}
$$

即解关于初边值与源稳定.

Robin条件

$$
\begin{aligned}
&(\partial_t - a^2\Delta)u = f(t,x) \quad x\in\Omega\subset\mathbb{R}^n, t>0\\
&t=0: u=\varphi(x)\\
&(\sigma u+\partial_n u)|_{\partial\Omega\times(0,\infty)} = \mu(t,x)
\end{aligned}
$$

对任意$T>0$, u在Γ_T上取得最大值. 若在$\partial\Omega\times[0,T]$上取得最大值, 则最大值点$x_0$处函数的法向导数应不小于0, 因此$u(x)\le\dfrac{\|\mu\|}{\sigma}$. 从而

$$
\begin{aligned}
&\|\mathscr{S}(f,\varphi,u)-\mathscr{S}(f+\delta f,\varphi+\delta\varphi,\mu+\delta\mu)\|\\
\le&\|\delta\varphi\|+\sigma^{-1}\|\delta\mu\|+\int_0^T\|\delta f(\tau,x)\|\mathrm{d}\tau\\
\le&\|\delta\varphi\|+\sigma^{-1}\|\delta\mu\|+T\|\delta f(\tau,x)\|
\end{aligned}
$$

实际上, 若$\sigma(x)$作为x的函数在$\partial\Omega$上有正下界或负上界, 则稳定性仍得证.

Newmann条件

$$
\begin{aligned}
&(\partial_t-a^2\Delta)u=f(t,x)\quad x\in\Omega\subset\mathbb{R}^n,t>0\\
&t=0:u=\varphi(x)\\
&(\partial_n u)|_{\partial\Omega\times(0,\infty)}=\mu(t,x)
\end{aligned}
$$

不妨设O在Ω内部, 令$v=ue^{-r^2}$. 原方程化为

$$
\begin{aligned}
&v_t-\left(\Delta v+\sum_i 4x_iv_{x_i}+(2+4|\nabla v|^2)v\right)=f(t,x)e^{-r^2}\quad x\in\Omega\subset\mathbb{R}^n,t>0\\
&t=0:v=\varphi(x)e^{-r^2}\\
&(\partial_n u-2rv)|_{\partial\Omega\times(0,\infty)}=\mu(t,x)e^{-r^2}
\end{aligned}
$$

由于$(2+4|\nabla v|)$恒正, $2r$有严格大于零的下界, 故

$$
\begin{aligned}
&\|\mathscr{S}(f,\varphi,u)-\mathscr{S}(f+\delta f,\varphi+\delta\varphi,\mu+\delta\mu)\|\\
\le&\|\delta\varphi\|+(\sigma^{-1})_{\min}\|\delta\mu\|+\int_0^T\|\delta f(\tau,x)\|\mathrm{d}\tau\\
\le&\|\delta\varphi\|+r_{\min}^{-1}\|\delta\mu\|+T\|\delta f(\tau,x)\|
\end{aligned}
$$

基本解理论

记微分算子$L(\partial_t,\partial_x)=\partial_t^m+\sum_{j=0}^{m-1}\sum_{|x|\le N_j}a_{j,\alpha}(t,x)\partial_t^j\partial_x^\alpha$

考虑方程$Lu=f,t>0,x\in\mathbb{R}^n$, 初值$\partial_t^ju|_{t=0}\varphi_j$,

$$
\begin{aligned}
&Lu=f\quad t>0,x\in\mathbb{R}^n\\
&\partial_t^ju|_{t=0}=\varphi_j,\quad 1\le j\le m-1
\end{aligned}
$$

基本解$E=E(t,x)$满足

$$
\begin{aligned}
&LE=0\quad t>0,x\in\mathbb{R}^n\\
&\partial_t^jE|_{t=0}=0,\quad 1\le j\le m-2\\
&\partial_t^{m-1}E|_{t=0}=\delta(x)
\end{aligned}
$$

从而

$$u=\sum_{j=0}^{m-1}\partial_t^{m-1-j}[E(t,\cdot)*\varphi_j](x)+\int_0^t[E(t-\tau,\cdot)*f(t,\cdot)](x)\mathrm{d}\tau$$

调和方程

Lapalce方程

设$\Omega\subset\mathbb{R}^n$为有界区域, 且区域上$\Delta u(x)\equiv 0$, 且$u$满足相应边界条件. 若满足上述条件的解$u\in C^0(\overline{\Omega})\cap C^2(\Omega)$, 则$u$为调和方程的解. 该类在$\Omega$内部求解的方程为内问题.

相应地外问题满足

$$\begin{cases}\Delta u\equiv 0\quad x\in(\overline{\Omega})^c\\ \text{some boundary conditions}\\ \lim\limits_{|x|\to\infty}u(x)=0\end{cases}$$

Green公式与基本解

设$\Omega\subset\mathbb{R}^n$为有界区域, 且$\partial\Omega\in C^1$. 设$\vec{n}$为单位外法向量. 则对任意$u,v\in C^1(\overline{\Omega})\cap C^2(\Omega)$都有Green第一公式成立

$$\begin{aligned}\int_\Omega u\Delta v&=\int_\Omega u\nabla\cdot(\nabla v)\\&=\int_\Omega\nabla\cdot(u\nabla v)-\nabla u\cdot\nabla v\\&=\int_{\partial\Omega}u\nabla v\cdot\vec{n}\cdot\mathrm{d}S-\int_\Omega\nabla u\cdot\nabla v\\&=\int_{\partial\Omega}u\frac{\partial v}{\partial n}\mathrm{d}S-\int_\Omega\nabla u\cdot\nabla v\end{aligned}$$

Poisson公式(Green第二公式)为

$$\int_\Omega u\Delta v-v\Delta u=\int_{\partial\Omega}u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}$$

据基本解理论, 今考察n维Poisson方程基本解$E_n(x)$满足$\Delta E_n(x)=\delta(x)$.

显然E_n径向对称, 不妨设$\Delta E_n(r)$为径向函数, 相应的Laplace算子为$r^{1-n}\partial_r r^{n-1}\partial_r$. 解得(假设$E_n$非常数)

$$E_n(r)=\begin{cases}\dfrac{1}{2\pi}\ln\dfrac{1}{r} & n=2,\\ \dfrac{1}{n(n-2)|B_n(0,1)|}\cdot\dfrac{1}{r^{n-2}} & n\geq 3.\end{cases}$$

从而对任意$y\in\Omega^\circ$, $\int_\Omega u(x)\Delta E_n(y-x)\mathrm{d}x=u(y)$. 代入Poisson方程, 解得

$$
\begin{aligned}
u(x)&=\int_\Omega \Delta u(y)E_n(x-y)\mathrm{d}x+\int_{\partial\Omega}\frac{\partial u(y)}{\partial n}E_n(x-y)-\frac{\partial E_n(y)}{\partial n}u(x-y)\mathrm{d}S\\
&=\int_{\partial\Omega}\frac{\partial u(y)}{\partial n}E_n(x-y)-\frac{\partial E_n(y)}{\partial n}u(x-y)\mathrm{d}S
\end{aligned}
$$

实际上, 定义x_0在Ω内的测度:

$$
\chi(x_0)=\begin{cases}1 & x_0\in\Omega^\circ\\ 1/2 & x_0\in\partial\Omega\\ 0 & x_0\in(\overline{\Omega})^c\end{cases}
$$

则

$$
\chi(x)\cdot u(x)=\int_{\partial\Omega}\frac{\partial u(y)}{\partial n}E_n(x-y)-\frac{\partial E_n(x-y)}{\partial n}u(y)\mathrm{d}S.
$$

平均值公式与极值定理

置公式

$$
\int_\Omega u\Delta v-v\Delta u=\int_{\partial\Omega}u\partial_n v-v\partial_n u
$$

中$\Omega=B(x,R)$, u为调和函数, $v=1$, 则$\int_{\partial B(x,R)}\partial_n u=0$. 据基本解公式

$$
u(x)=\int_{\partial B(x,R)}E_n(x-y)\partial_n u(y)-u(y)\partial_n E_n(x-y)\mathrm{d}S_y.
$$

从而$u(x)$为$\dfrac{\int_{\partial B(x,R)}u(y)\mathrm{d}S_y}{\int_{\partial B(x,R)}\mathrm{d}S_y}$, 即球面上的平均积分.

容易见得, 球面平均与球体平均等价. 设$\varphi(x)$为仅与$r=|x|$相关之函数, 则在积分收敛时有

$$
[\varphi*u](x)=u(x)\int_{\mathbb{R}^n}\varphi.
$$

对任意开集$\Omega\subset\mathbb{R}^n$上具有局部均值性质的函数$u$, 置$\varphi_\varepsilon$为以$B(0,\varepsilon)$为紧支撑的磨光函数, 则可证得$u$在$\{x:d(x,\partial\Omega)<\varepsilon\}$上光滑. 令$\varepsilon\to0$即得$u\in C^\infty(\Omega)$. 由于

$$
\Delta_n u(x)=r^{1-n}\partial_r r^{n-1}\partial_r\int_{B(0,r)}u=0.
$$

从而u调和. 因此开集上的调和函数等价于满足局部均值性质的函数.

Green函数

端详公式Ω内Dirichlet问题解所满足的方程

$$u(x)=\int_{\partial\Omega}\frac{\partial u(y)}{\partial n}E_n(x-y)-\frac{\partial E_n(x-y)}{\partial n}u(y)\mathrm{d}S.$$

对给定的初边值问题而言, u与$\partial_n u$不可兼得. 就Dirichlet问题而言, 应当消去公式中的$\partial_n u$项. 下推导Dirichlet问题的Green函数.

考虑函数$g(x,y)$使得对任意给定的$x\in\Omega$, $g(x,y)$在$\partial\Omega$上取值与$E(x-y)$相同. 且$g(x,y)$在Ω内部调和. 从而

$$\int_{\Omega}u(y)\Delta g(x,y)-g(x,y)\Delta u(y)=\int_{\partial\Omega}u(y)\partial_n g(x,y)-g(x,y)\partial_n(y)=0.$$

令$G(x,y)=E_n(x-y)-g(x,y)$, 则有

$$u(x)=\int_{\partial\Omega}G(x,y)\partial_n u(y)-u(y)\partial_n G(x,y)\mathrm{d}S_y.$$

其中$G(x,y)|_{\partial\Omega}\equiv 0$. 从而当$u(x)|_{\partial\Omega}=\mu(x)$时, 解得

$$u(x)=-\int_{\partial\Omega}\mu(y)\partial_n G(x,y)\mathrm{d}S_y.$$

对Newman条件, 只需构造满足$\partial_n[g(x,y)-E_n(y-x)]_{\partial\Omega}\equiv 0$的调和函数$g(x,y)$即可. 记$G(x,y)=E(x-y)-g(x,y)$, $\partial_n u\equiv\mu(x)$, 则

$$u(x)=\int_{\partial\Omega}\mu(y)G(x,y)\mathrm{d}S_y.$$

对Robin条件, 构造g满足$(\partial_n+\sigma\cdot\mathrm{id})[g(x,y)-E_n(y-x)]\equiv 0$即可. 下不赘述.

检验得Green函数满足以下性质(以Dirichlet问题对应的Green函数为例)

1. 对给定的x, $G(x,y)$在$\Omega-\{x\}$调和, 在$x'\to x$时以$E(x-x')\sim[\|x-x'\|^{n-1}]'$速度趋向无穷. 同时$\Omega$内有

   $$0<G(x,y)<E(x-y).$$

   边界上有$G(x,y)\equiv 0$.
2. 由于Green函数与u无关. 置$u|_{\partial\Omega}\equiv 1$, 得

   $$\int_{\partial\Omega}\partial_n G(x,y)\mathrm{d}S_y=-1.$$

3. Green函数$G(x,y)$指标可交换. 任取$x,y\in\Omega$, $0<\varepsilon\ll\mathrm{diam}(\Omega)$, 则

$$
\begin{aligned}
0 =& \int_{\Omega - B(x,\varepsilon) - B(y,\varepsilon)} G(x,t)\Delta G(y,t) - G(y,t)\Delta G(z,t)\mathrm{d}t \\
=& \int_{\partial[B(x,\varepsilon)\cup B(y,\varepsilon)]} G(x,t)\partial_n G(y,t) - G(y,t)\partial_n(x,t)\mathrm{d}S_t \\
&+ \int_{\partial\Omega} G(x,t)\partial_n G(y,t) - G(y,t)\partial_n(x,t)\mathrm{d}S_t \\
\overset{\varepsilon\to 0}{\to}& \int_{\partial B(x,\varepsilon)} G(x,t)\partial_n G(y,x)\mathrm{d}S_t - \int_{\partial B(y,\varepsilon)} G(y,t)\partial_n G(x,y)\mathrm{d}S_t \\
& \int_{\partial B(y,\varepsilon)} G(x,y)\partial_n G(y,t)\mathrm{d}S_t - \int_{\partial B(x,\varepsilon)} G(y,x)\partial_n G(x,t)\mathrm{d}S_t \\
=& 0 + 0 + (-1)G(x,y) - (-1)G(y,x)
\end{aligned}
$$

其中$\int_{\partial B(x,\varepsilon)} G(x,t)\mathrm{d}S_t \sim E_n(\varepsilon)\cdot|\partial B(0,\varepsilon)| \sim \varepsilon^1 \to 0.$

特别地, 对球面$B(O,R)$中点x, $G(x,y)$为点电荷x在空球导体内的生成电场. 设 $x' = \dfrac{R^2}{x^2}\cdot x$, 则

$$
G(x,y) = E_n(y-x) - E_n(\frac{R}{|x|}(y-x')).
$$

解得

$$
u(x) = \frac{R - \|x\|^2/R}{|\partial B_n(0,1)|}\cdot\int_{\partial B_n(O,R)} \frac{f(y)}{\|y-x\|^n}\mathrm{d}S_y.
$$

特别地, 上半平面的Green函数即$G(x,y) = E_n(y-x) - E_n(y-x')$. 其中 $x = (x_1,\ldots,x_n)$, $x' = (x_1,\ldots,x_{n-1},-x_n)$. 解得

$$
u(x) = \frac{x_n\Gamma(n/2)}{\pi^{n/2}}\int_{\mathbb{R}^{n-1}} \frac{f(y)}{\sqrt{(\tilde{x}_{n-1}-y)^2 + (x_n)^2}^n}\mathrm{d}y.
$$

二维单连通区域的Dirichlet问题

考虑Dirichlet问题

$$
\begin{aligned}
&\Delta u = 0 \quad x\in\Omega, t>0 \\
&u|_{\partial\Omega} = \varphi(x)
\end{aligned}
$$

其中Ω可通过全纯函数f共性映照至$\mathbb{D}$, i.e., $f:\Omega\to\mathbb{D}$全纯且同胚. 从而原PDE转化为(等同$g(x,y)$与$g(x+iy)$):

$$
\begin{aligned}
&\Delta u\circ f^{-1} = 0 \quad x\in\mathbb{D}, t>0 \\
&u|_{\partial\mathbb{D}} = \varphi\circ f^{-1}(x)
\end{aligned}
$$

对任意$f(x_0)\in\mathbb{D}$, 由Poisson积分公式得

$$u(x_0)=\frac{1}{2\pi}\int_{S^1}\frac{1-|f(x_0)|^2}{|z-f(x_0)|^2}\varphi\circ f^{-1}(z)\mathrm{d}z$$

记$z=f(x)$, 则

$$u(x_0)=\frac{1}{2\pi}\int_{\partial\Omega}\frac{1-|f(x_0)|^2}{|f(x)-f(x_0)|^2}\varphi(x)\frac{\partial f}{\partial t}(x)\mathrm{d}x.$$

实际上, $G_\Omega(\xi,\xi_0)=G_{\mathbb{D}}(f(\xi),f(\xi_0))$. 特别地, 若共性映照

$$\Phi(\cdot,z_0):\Omega\to\mathbb{D},z_0\mapsto 0,\partial\Omega\to U_1.$$

则$G(z,z_0)=\dfrac{1}{2\pi}\ln\dfrac{1}{|f(z,z_0)|}$.

可去奇点定理

可取奇点定理本质上反应了调和函数阶数的某种间断性, 其本质仍是解析性. 若n维调和函数u在$B(x,r)-\{x\}$内调和, 且$x'\to x$时满足$\dfrac{u(x')}{E_n(x-x')}\to 0$, 则$u(x)$可定义(即奇点可去).

不妨设$r=2,x=O$. 依Poisson积分公式延拓$u|_{\partial B(O,1)}$为函数$v|_{\overline{B(O,1)}}$. 作
$w_\varepsilon(x):=u(x)-v(x)+\varepsilon E_n(x)$, 其中$\varepsilon\in(-1,1)-\{0\}$. 对足够小的$\delta(\varepsilon)\ll 1,\varepsilon E_n(x)$项主导$w_\varepsilon$. 记$\delta(|\varepsilon|):=\inf_{t\geq|\varepsilon|}\min\{\delta(t),\delta(-t)\}$(单调递减至0), 则$w_{|\varepsilon|}(x)$在$\partial B(O,1)\cup\partial B(O,\delta(|\varepsilon|))$上非负, 即$w_\varepsilon$在$B(O,1)-B(O,\delta(|\varepsilon|))$上非负. 同理$w_{-|\varepsilon|}$在$B(O,1)-B(O,\delta(|\varepsilon|))$上非正. 令$\varepsilon\to 0$即得

$$(u-v)|_{B(O,1)-B(O,\delta(|\varepsilon|))}\equiv 0.$$

而$\delta\overset{\varepsilon\to 0}{\to}0$, 故$u\equiv v$在$B(O,1)$上恒成立.

可采用可去奇点定理可估计定义域为$B(O,R)$外的调和函数的衰减速度. 对任意$u\in\mathrm{Har}(\overline{B(O,R)^c}),u|_{\partial B(O,R)}=\varphi(x)$, 考虑Kevin变换

$$v(x)=\frac{\|x\|^{n-2}}{R^{n-2}}u(\frac{R^2}{\|x\|^2}x)\quad\|x\|\geq R.$$

从而$v(x)$在$0<\|x\|<R$时调和, 且边界上取值$v(x)=u(x)|_{x\in\partial B(O,R)}$. 再若

$$\lim_{x\to\infty}u(x)=0.$$

则

$$\lim_{x\to 0} v(x)\cdot(E_n(x))=0.$$

据可去奇点定理, $v(0)$为可取间断点. 据Poisson积分公式得

$$u(x)=\frac{\|x\|^{n-2}}{R^{n-2}}\cdot\frac{1}{|S^{n-1}|}\int_{S^{n-1}}\frac{R^2-R^4/\|x\|^2}{(y-xR^2/\|x\|^2)^n}u(y)\mathrm{d}S_y.$$

Harnack不等式

由于$u(x)$在$\overline{B_n(O,R)}$内一致有界, 故不妨设$u|_{\overline{B_n(O,R)}}\geq 0$. 考察Poisson积分公式

$$u(x)=\frac{R-\|x\|^2/R}{|\partial B_n(0,1)|}\cdot\int_{\partial B_n(O,R)}\frac{f(y)}{\|y-x\|^n}\mathrm{d}S_y.$$

注意到$R-\|x\|\leq\|y-x\|\leq R+\|x\|$, 从而

$$\frac{R^2-\|x\|^2}{(R+\|x\|)^n}u(O)\leq u(x)\leq\frac{R^2+\|x\|^2}{(R-\|x\|)^n}u(O).$$

同理, 对外问题有

$$u(x)=\frac{\|x\|^{n-2}}{R^{n-2}}\cdot\frac{1}{|S^{n-1}|}\int_{S^{n-1}}\frac{R^2-R^4/\|x\|^2}{(y-xR^2/\|x\|^2)^n}u(y)\mathrm{d}S_y.$$

$$\frac{\|x\|^{2n-4}}{R^{2n-4}}\cdot\frac{\|x\|^2-R^2}{(\|x\|+R)^n}u(O)\leq u(x)\leq\frac{\|x\|^{2n-4}}{R^{2n-4}}\cdot\frac{\|x\|^2-R^2}{(\|x\|-R)^n}u(O)$$