$d: A^r(U) \to A^{r+1}(U)$ is defined as follows: $\forall \omega \in A^r(U)$, writing $\omega = \frac{1}{r!} \sum_{1 \le i_1, \cdots, i_r \le n} \omega_{i_1 \cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r}$, then

$$d\omega \triangleq \frac{1}{r!} \sum_{1 \le i_1 \cdots i_r \le n} d\omega_{i_1 \cdots i_r} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r}.$$

<u>Prop.</u> d is independent of the choice of coordinate.

<u>Proof.</u> Taking another coordinate system $(y^1, \cdots, y^n)$ and writing

$$\omega = \frac{1}{r!} \sum_{1 \le j_1 \cdots j_r \le n} \omega'_{j_1 \cdots j_r} dy^{j_1} \wedge \cdots \wedge dy^{j_r},$$

One needs to prove

$$\sum_{1 \le i_1 \cdots i_r \le n} d\omega_{i_1 \cdots i_r} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r} = \sum_{1 \le j_1 \cdots j_r \le n} d\omega'_{j_1 \cdots j_r} \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_r}.$$

First, one has the following relation between the coefficients: For $1 \le i_1, \cdots, i_r \le n$,

$$\omega_{i_1 \cdots i_r} = \sum_{1 \le j_1 \cdots j_r \le n} \omega'_{j_1 \cdots j_r} \frac{\partial y^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{j_r}}{\partial x^{i_r}}.$$

Compute

$$\begin{aligned}
&\sum_{1 \le i_1 \cdots i_r \le n} d\omega_{i_1 \cdots i_r} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r} \\
&= \sum_{1 \le i_1 \cdots i_r \le n} \Bigg[\sum_{1 \le j_1 \cdots j_r \le n} \left(d\omega'_{j_1 \cdots j_r} \frac{\partial y^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{j_r}}{\partial x^{i_r}} + \omega'_{j_1 \cdots j_r} \left(\sum_{s=1}^{r} \frac{\partial y^{j_1}}{\partial x^{i_1}} \cdots \widehat{\frac{\partial y^{j_s}}{\partial x^{i_s}}} \cdots \frac{\partial y^{j_r}}{\partial x^{i_r}} \, d \frac{\partial y^{j_s}}{\partial x^{i_s}} \right) \right) \\
&\quad \wedge \sum_{1 \le k_1, \cdots, k_r \le n} \left(\frac{\partial x^{i_1}}{\partial y^{k_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{k_r}} \right) dy^{k_1} \wedge \cdots \wedge dy^{k_r} \Bigg] \\
&= \sum_{1 \le j_1 \cdots j_r \le n} d\omega'_{j_1 \cdots j_r} \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_r} \\
&\quad + \sum_{1 \le i_1 \cdots i_r \le n} \sum_{1 \le j_1 \cdots j_r \le n} \omega'_{j_1 \cdots j_r} \left(\sum_{s=1}^{r} \frac{\partial y^{j_1}}{\partial x^{i_1}} \cdots \widehat{\frac{\partial y^{j_s}}{\partial x^{i_s}}} \cdots \frac{\partial y^{j_r}}{\partial x^{i_r}} \, d \frac{\partial y^{j_s}}{\partial x^{i_s}} \right) \\
&\qquad \wedge \sum_{1 \le k_1 \cdots k_r \le n} \left(\frac{\partial x^{i_1}}{\partial y^{k_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{k_r}} \right) dy^{k_1} \wedge \cdots \wedge dy^{k_r}
\end{aligned}$$

$$= \sum_{1\le j_1\cdots j_r\le n} d\omega'_{j_1\cdots j_r}\wedge dy^{j_1}\wedge\cdots\wedge dy^{j_r}$$

$$+\sum_{1\le j_1\cdots j_r\le n}\ \sum_{1\le k_1\cdots k_r\le n}\omega'_{j_1\cdots j_r}$$

$$\left(\sum_{s=1}^{r}\sum_{1\le i_s\le n}\delta^{j_1}_{k_1}\cdots\widehat{\delta^{j_s}_{k_s}}\cdots\delta^{j_r}_{k_r}\cdot\frac{\partial x^{i_s}}{\partial y^{k_s}}\, d\frac{\partial y^{j_s}}{\partial x^{i_s}}\right)\wedge dy^{k_1}\wedge\cdots\wedge dy^{k_r}$$

$$= \sum_{1\le j_1\cdots j_r\le n} d\omega'_{j_1\cdots j_r}\wedge dy^{j_1}\wedge\cdots\wedge dy^{j_r}$$

$$+\sum_{1\le j_1\cdots j_r\le n}\omega'_{j_1\cdots j_r}\left(\sum_{s=1}^{r}\sum_{1\le i_s,k_s\le n}\frac{\partial x^{i_s}}{\partial y^{k_s}}\, d\frac{\partial y^{j_s}}{\partial x^{i_s}}\right)$$

$$\wedge dy^{j_1}\wedge\cdots\wedge dy^{j_{s-1}}\wedge dy^{k_s}\wedge dy^{j_{s+1}}\wedge\cdots\wedge dy^{j_r}$$

$$= \sum_{1\le j_1\cdots j_r\le n} d\omega'_{j_1\cdots j_r}\wedge dy^{j_1}\wedge\cdots\wedge dy^{j_r}$$

$$+\sum_{1\le j_1\cdots j_r\le n}\omega'_{j_1\cdots j_r}\left(\sum_{s=1}^{r}(-1)^{s-1}\sum_{1\le i_s,k_s\le n}\frac{\partial x^{i_s}}{\partial y^{k_s}}\, d\frac{\partial y^{j_s}}{\partial x^{i_s}}\wedge dy^{k_s}\right)$$

$$\wedge dy^{j_1}\wedge\cdots\wedge dy^{j_{s-1}}\wedge\widehat{dy^{j_s}}\wedge dy^{j_{s+1}}\wedge\cdots\wedge dy^{j_r}.$$

Observe

$$\sum_{1\le i_s,k_s\le n}\frac{\partial x^{i_s}}{\partial y^{k_s}}\, d\frac{\partial y^{j_s}}{\partial x^{i_s}}\wedge dy^{k_s}$$

$$=\sum_{1\le i_s,k_s\le n}\left[d\left(\frac{\partial x^{i_s}}{\partial y^{k_s}}\frac{\partial y^{j_s}}{\partial x^{i_s}}\right)\wedge dy^{k_s}-\frac{\partial y^{j_s}}{\partial x^{i_s}}\left(d\frac{\partial x^{i_s}}{\partial y^{k_s}}\right)\wedge dy^{k_s}\right]$$

$$=0. \quad \left(\text{Note } 0=d(dx^{i_s})=d\left(\sum_{k_s=1}^{n}\frac{\partial x^{i_s}}{\partial y^{k_s}}dy^{k_s}\right)\right.$$

$$\left.=\sum_{k_s=1}^{n}\left(d\frac{\partial x^{i_s}}{\partial y^{k_s}}\right)\wedge dy^{k_s}.\right)$$

The prop. is proved.