



Lecture Notes for K -seminar Recollections from Projective Modules

1 About the Lecture

We strongly recommend the readers to review the basic definitions and facts about

1. groups, rings, modules, field, algebra, etc;
2. the definition of Abelian category \mathcal{A} ;
3. basic knowledge on categories, such as commutative diagrams.

The Lecture Note mainly discusses

1. When free objects are defined and how free objects related to projective objects.
2. Equivalent definitions of projective modules (for small Abelian categories).
3. The chain: Free Mods \rightarrow Stably Free Mods \rightarrow Projective Mods \rightarrow Flat Mods \rightarrow Torison Free Mods.

2 Equivalent definitions of projective modules

Definition 1 (The [adjoint pair](#) of Free and Forgetful functors). For category \mathcal{C} and Set the category of sets, if there exists an adjoint pair $(F \dashv U)$ such that

- $F : \text{Set} \rightarrow \mathcal{C}$ is the [free functor](#) sending the category of sets to \mathcal{C} ;
- U is the [forgetful functor](#) sending the collection of objects (resp., morphisms) of \mathcal{C} to the underlying sets (resp., set mappings);
- the collection $\{\varepsilon_X : FU(X) \rightarrow X\}_{X \in \text{Ob}(\mathcal{C})}$ is called [co-unit](#);
- $\{\eta_X : S \rightarrow UF(X)\}_{S \in \text{Ob}(\text{Set})}$ is called [unit](#).

Remark 1. [Freyd-Mitchell embedding theorem](#) says that every small Abelian category \mathcal{A} is a full subcategory of a category of modules over some ring R , such that the embedding functor $\mathcal{A} \hookrightarrow R\text{-Mod}$ is an exact functor.

SLOGAN 1. Small Abelian categories $\xrightarrow{\text{Longleftarrow}} R\text{-Mod}$ categories.

Definition 2 (projective modules). Suppose that \mathcal{A} is an Abelian category. We say $P \in \mathcal{A}$ is projective, whenever

- for arbitrary epimorphism $X \xrightarrow{\pi} Y$ and $P \xrightarrow{f} Y$, there exists a lift \tilde{f} such that the following diagram commutes

$$\begin{array}{ccc} & & P \\ & \nwarrow \tilde{f} & \downarrow f \\ X & \xrightarrow{\pi} & Y \end{array} .$$

Proposition 1. Try to update