Algebraic topology

11-Sept-2023

Ref ① Munkres ② Miller ③ Hatch.

Plan ① Fundamental group ② Homology Theory

③ Cohomology Theory ④ Poincaré duality

Ch1 Fundamental group

Q. X, Y Top spaces. Prove $X \underset{top}{\simeq} Y$ (homeomorphic)

Attempt 1 conti with conti inverse?

Q. ———— $\not\simeq$ ——— ?

Attempt 1 ～～～

Attempt 2 Topological invariants

Ex. $S^1 \not\simeq \mathbb{R}$? pf. Using compactness.

§1.1 Homotopy of paths.

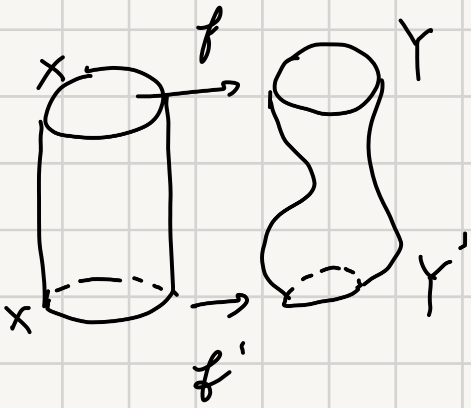
Df 1.1.1 $f, f' : X \longrightarrow Y$ continuous

$f \overset{F}{\simeq} f'$ (homotopic) iff

$\exists\ F : X \times [0,1] \longrightarrow Y$ continuous (I)

$(x, 0) \longmapsto f(x)$

$(x, 1) \longmapsto f'(x)$

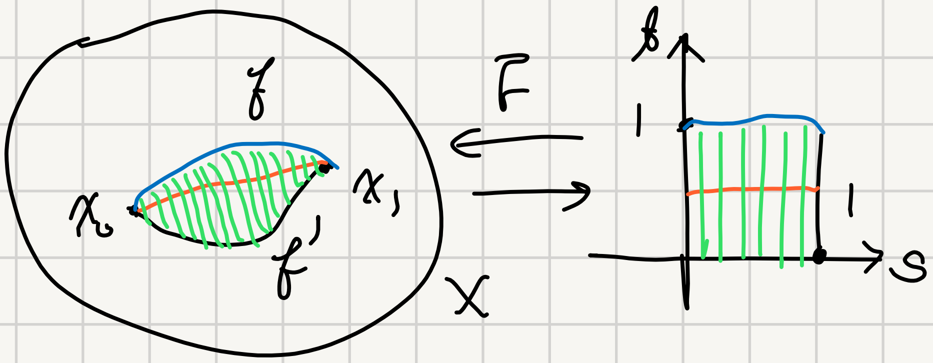


* In part, f is constant, we call it null-homotopic

Df. 1.1.2 f, f' are paths with $f(0) = f'(0)$ (x_0), $f(1) = f'(1)$ (x_1)

i.e. $f, f' : [0,1] \longrightarrow X$. Whenever $f \overset{F}{\simeq} f'$,

f, f' are path homotopic.



where $F(t,s)=f_t(s)$. $(f_0=f,\ f_1=f')$

Lemma 1.1.3 path homotopic = homotopic for paths,

$\Longrightarrow [f]=[f'] \Leftrightarrow f \simeq f'$

Ex.

(well-defined equivalence relation)

Ex $f, g: X \to \mathbb{R}^n$ are continuous, then $X \simeq g$.

in fact, can be any convex subspace

Hint: $F(t,x) := t\cdot f(x) + (1-t)g(x)$

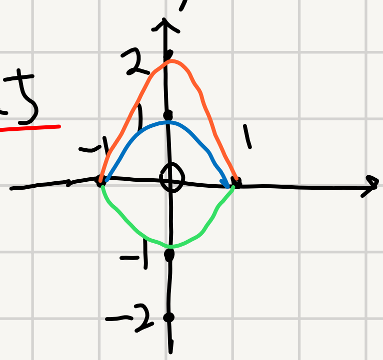
Rmk. A convex subspace of $\mathbb{R}^n$, $x_0 \in A$.

Set $e_{x_0}: I \longrightarrow A,\ x \longmapsto x_0$,

$\forall f: I \longrightarrow A$ is homotopic to e_{x_0}.

The proof is similar.

Ex 1.1.5



$X := \mathbb{R}^2 \setminus \{0\}$

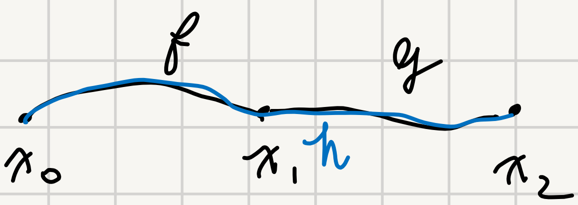
$[f]=[g] \neq [h]$.

Def 1.1.6 f. path in X. from x_0 to x_1

g. path ——— x_1 — x_2

We define $f*g =: h:$ $\begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$

Thm 1.1.7 h is continuous.



Def 1.1.8 $[f*g] = [f]*[g]$ is well-defined.