Differentiable Manifold.

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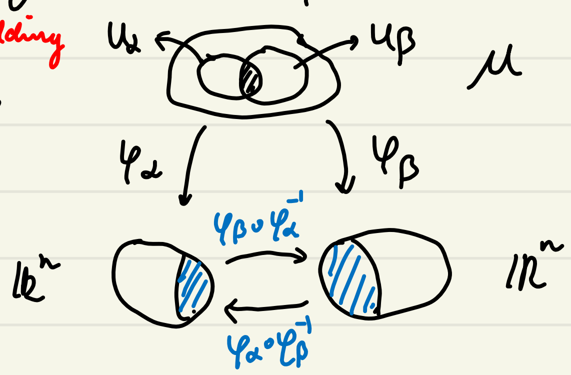
Def. $\mathcal{M}$ Hausdorff, C_2 (has countable base). We say

① $\mathcal{M}$ is differentiable manifold if $\{U_\alpha\}_{\alpha\in\Lambda}$. $\{\varphi_\alpha\}$. $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ s.t.

(a) φ_α is homomorphism, eq. embedding

(b) if $U_\alpha \cap U_\beta \neq \phi$. e.g. $\longrightarrow$

$\varphi_\alpha \circ \varphi_\beta^{-1}$ and $\varphi_\beta \circ \varphi_\alpha^{-1}$ are differentiable



② C^0, C^r, C^∞

Rmk. $(U_\alpha, \varphi_\alpha)$ is a local coordinate system.

Eg $\mathbb{R}^n$ is d.m. $\left(id_{\mathbb{R}^n} : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n\right)$

Eg S^2 $(\subset \mathbb{R}^3)$ is d.m. $U_1^{\pm} = \{(x^1, x^2, x^3) \mid \pm(x^1) > 0\}$,

$\varphi_1^{\pm} : U_1^{\pm} \to \mathbb{R}^2$, $(x^1, x^2, x^3) \longmapsto (x^2, x^3)$

Thus $\varphi_2^+ \circ \varphi_1^{+-1} : (x^2, x^3) \mapsto (\sqrt{1 - x^2x^2 - x^3x^3}, x^3)$ is C^∞.

Eg. S^n is similar

Eg. $P^n(\mathbb{R}) = S^n/\sim$ $(x \sim -x)$

Def. Differenciable structure: $\{(U_\alpha, \varphi_\alpha)\}_- =: \mathcal{S}$

Def. $\mathcal{M}$: n-diff manifold. $f : \mathcal{M} \to \mathbb{R}$ is continuous. f is diff if $\forall p \in \mathcal{M}$. $\exists$ (U, φ) local chart at p, s.t. $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is diff.

Rmk. For another coordinate φ' at U. $f \circ \varphi^{-1} = (f \circ \varphi'^{-1}) \circ (\varphi' \circ \varphi^{-1})$ on some $p \in U_p \subset U$. $(\varphi'^{-1} \circ \varphi)$ is an embedding.

Def. $F : N^n \to \mathcal{M}^m$ is diff at $q \in N$, if $(V_q, \psi) \xrightarrow{F} (U_{F(q)}, \varphi)$

$$\begin{tikzcd} N \arrow[r, "F"] \arrow[d, "\psi"'] & \mathcal{M} \arrow[d, "\varphi"] \\ \mathbb{R}^n \arrow[r, "\text{diff}"] & \mathbb{R}^m \end{tikzcd}$$

Rmk. Still well-defined w.r.t. coordinate.

Rmk. $\gamma : (a, b) \to \mathcal{M}$ is differentiable curve if ——.

Def. $F : N \to \mathcal{M}$ diffeomorphism if F is homeo, diff. so is F^{-1}.