流形的几何与拓扑

13-Sept-2023

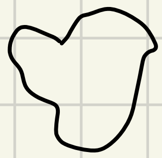
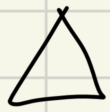
Def Topological space (X, τ) (omit)

Def. Continuous map (omit) f^{-1}: open $\longrightarrow$ open.

X, Y two spaces. If $\exists f: X \longrightarrow Y$ continuous s.t.

$$\begin{cases} (1) & f \text{ is one-to-one}, \\ (2) & f^{-1} \text{ is continuous}. \end{cases} \qquad X \cong Y$$

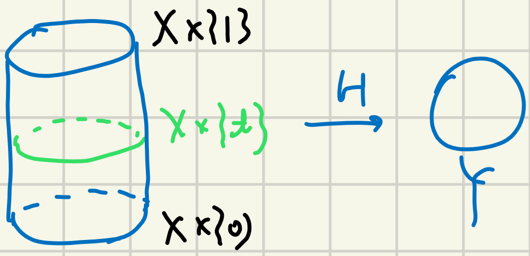
$\Rightarrow f$ is a homeomorphism.

Eg. $(-1,1) \cong (-\infty, +\infty)$,  $\cong$ 

Def. X, Y topo spaces. $f_0, f_1: X \longrightarrow Y$ continuous, If

$\exists H: X \times [0,1] \longrightarrow Y$, s.t. $H(x,0) = f_0(x)$, $H(x,1) = f_1(x)$

$f_0 \simeq f_1$



Eg. X: space $f_0, f_1: X \to E^n$, then $f_0 \simeq f_1$

Consider $H: X \times [0,1] \longrightarrow E^n$,

$(x, t) \longmapsto (1-t) f_0(x) + t f_1(x)$.

Eg. $f_0, f_1: X \to Y$ continuous $A \subset X$. If $\exists$ continuous map

$$H: X \times [0,1] \longrightarrow Y \text{ s.t. } \begin{cases} H(x,0) = f_0(x) \\ H(x,1) = f_1(x) \\ H(a,t) = f_0(a) = f_1(a) \end{cases}$$

$H: f_0 \simeq f_1 \text{ rel} A$

Eg. $f_0, f_1: X \to E^n$ $A = \{x \in X \mid f_0(x) = f_1(x)\} \neq \emptyset$

Then $H: f_0 \simeq f_1$ rel A $\quad (H(x,t) = (1-t) f_0(x) + t f_1(x))$

Def X, Y. $f: X \to Y$, $g: Y \to X$. We call X, Y are homotopy equivalent if $g \circ f \simeq id_X$, $f \circ g \simeq id_Y$ $\quad X \simeq Y$

Eg. $X \times \mathbb{R} \simeq X$ $\quad f: X \times \mathbb{R} \to X$, $(x, s) \mapsto s$ $\quad g: X \to X \times \mathbb{R}$, $x \mapsto (x, 0)$

$g \circ f: (x, s) \mapsto (x, 0)$ $\quad H: (X \times \mathbb{R}) \times I \to (X \times \mathbb{R})$, $((x, s), t) \mapsto (x, ts)$

$f \circ g: x \mapsto x$ is id_X

Def. If $X \simeq \{pt\}$. X is called contractible.

Eg. $E^n \simeq \{pt\}$

Def. If $f: X \to Y \simeq$ constant, f is called null-homotopy

Def X, $A \subset X$. $i: A \to X$, $a \mapsto a$; $r: X \to A$, $x \mapsto r(A)$.

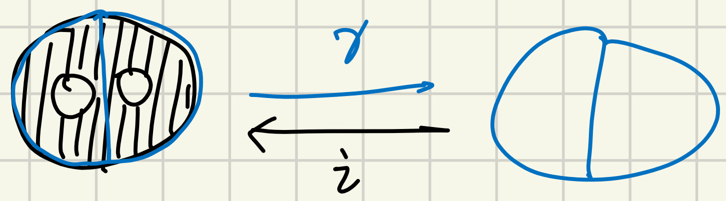
If $\begin{cases} r \circ i = id_A \\ i \circ r \simeq id_X \text{ rel } A \end{cases}$, A is called deformation retract of X

Rmk. Equivalently $H: X \times [0,1] \to X$. s.t. $\begin{cases} H(x, 0) = x \\ H(x, 1) = i \circ r(x) = r(x) \\ H(a, t) = a \quad a \in A, t \in [0,1] \end{cases}$

Eg. $X \times \mathbb{R}$ def retract of $X \times \{0\}$.

$r: X \times \mathbb{R} \to X \times \{0\}$, $(x, s) \mapsto (x, 0)$. i trivial.

Eg.



Def CW complex (cell complex)

$\mathbb{E}^n := \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{R}\}$

$D^n := \{(x_1, \cdots, x_n) \in \mathbb{E}^n \mid x_1^2 + \cdots + x_n^2 \le 1\}$

$S^{n-1} := \partial D^n$

Def. $e^n := \mathring{D}^n$ (interior)

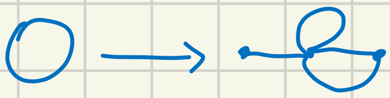
(1) Discrete set X^0, points are 0-cells.

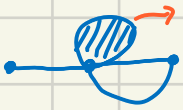
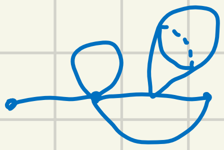
(2) Inductively, for n-skeleton X^n from X^{n-1}

by adding n-cells e^n via maps $\varphi_\alpha^n : S^{n-1} \to X^{n-1}$

$$X^n = X^{n-1} \sqcup (\sqcup D_\alpha^n) / x \sim \varphi_\alpha^n(x),\ x \in S_\alpha^{n-1} \subset D_\alpha^n$$

gluing S_α^{n-1} on X^{n-1}

$\varphi_\alpha^2 :$ 

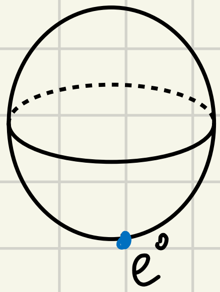
$X^2 :$  $\to \varphi_\alpha^n(x) . x$ (or 

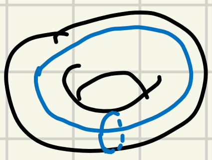
Thus. $X^n = X^{n-1} \sqcup_\alpha e_\alpha^n$

(3) Either stop at X^n, or continue infinitely

$(n = \dim X)$ $\qquad$ $(X = \bigcup_{n=0}^{\infty} X^n)$

A is open . iff $A \cap X^n$ is open in X^n $(\forall n)$

Eg. S^2:  $= e^0 \cup e^2$

T^2:  $= e^0 \cup e^1 \cup e^1 \cup e^2$

Eg.  $S^0 \subset S^1 \subset S^2 \subset S^3 \subset \cdots \subset S^n \subset \cdots$

$$S^n = 2e^0 \cup 2e^1 \cup \cdots \cup 2e^n . = e^0 \cup e^n$$

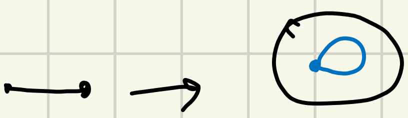
Def. Homotopy group. X: top-space, $x_0 \in X$, I^n $(\cong D^n)$

$f: I^n \to X$, $\partial I^n \mapsto x_0$. Define $[f]$ as follows:

$f_0 \simeq f_1$ rel ∂I^n, iff $H: I^n \times I \to X$,

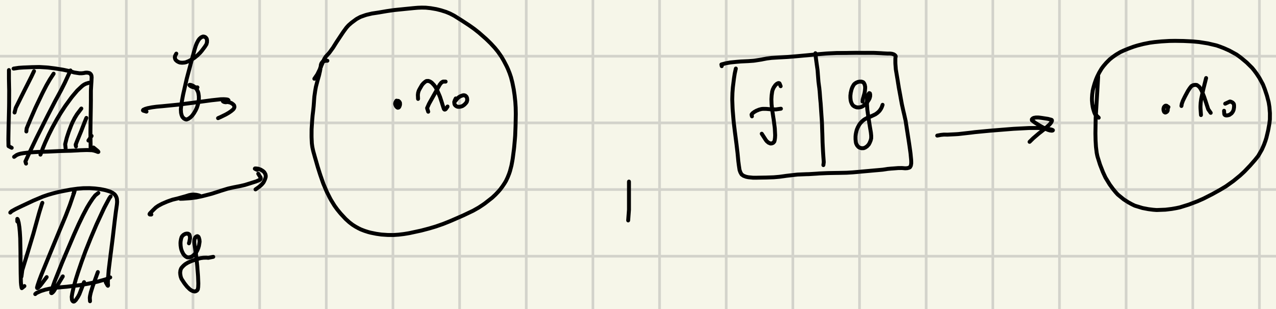
relevative homotopy class $\left\{ \begin{array}{ll} (x,0) \mapsto f_0(x) & (x,1) \mapsto f_1(x). \\ (a,t) \mapsto x_0 & \forall a \in \partial I^n . \end{array} \right.$

Def. $\pi_n(X, x_0) := \{[f]\}$. $(n \geq 0)$

Eg. $\pi_1(X, x_0) = \{ [f] \mid f:$  $\}$

Def. $[f], [g] \in \pi_n(X, x_0)$. $(f+g)(s_1, \dots, s_n) = \begin{cases} f(2s_1, s_2, \dots, s_n) & s_1 \in [0, \frac{1}{2}], \\ g(2s_1 - 1, s_2, \dots, s_n) & s_1 \in [\frac{1}{2}, 1]. \end{cases}$

Eg.



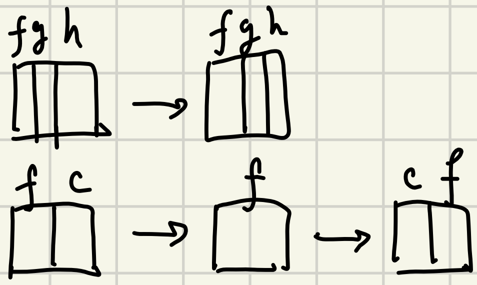
Inverse of f, $\overline{f}$ is defined as

$$\overline{f} : I^n \to X,\ \partial I^n \to x_0,\ (s_1, s_2, \dots, s_n) \mapsto f(1-s_1, s_2, \dots, s_n)$$

lemma. $\pi_n(X, x_0)$ is a group.

① $(f+g)+h = f+(g+h)$

② $f+c = f = c+f$



③ $f+\overline{f} = c = \overline{f}+f$.

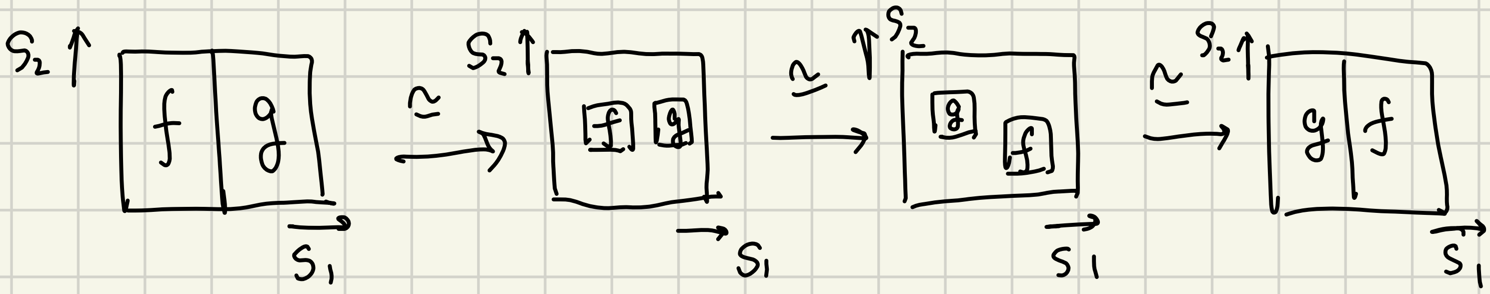
Homotopy group $\pi_n(X, x_0)$

lemma.

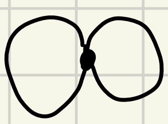
$$\begin{tikzcd} f : (I^n, \partial I^n) \arrow[rr] \arrow[dr] & & (X, x_0) \\ & (S^n, s_0) \arrow[ur] & \end{tikzcd} \qquad (S^n \cong I^n / \partial I^n)$$

Prop. $n \geq 2$. $\pi_n(X, x_0)$ is Abelian.

Pf. $f, g : (I^n, \partial I^n) \to (X, x_0)$ To prove that $f+g \simeq g+f$ rel ∂I^n.



Eg. $\pi_1(X, x_0)$ need not to be commutative

 $\pi_1(X, x_0) = \mathbb{Z} * \mathbb{Z}$ non-Abelian

Eg. $\pi_n(S^n, x_0) \simeq \mathbb{Z}$

20-Sept-2023

Def ($\pi_n(X, x_0)$) $f : (I^n, \partial I^n) \to (X, x_0)$, $(n \geq 1)$ $f_0 \simeq f_1$ rel $\partial I^n \iff [f_0] = [f_1]$.

$\Rightarrow \mathcal{F}/\sim$ is the group $\pi_n(X, x_0)$

Prop. By definition,

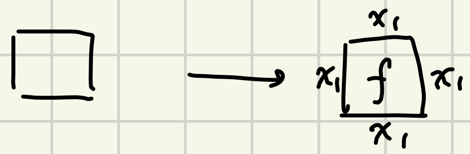
$$\begin{tikzcd}
(I^n, \partial I^n) \arrow[rr, "\varphi"] \arrow[dr, "i_n"'] & \arrow[d, phantom, "\circlearrowright"] & (X, x_0) \\
& (S^n, s_0) \arrow[ur, "\tilde{\varphi}"'] &
\end{tikzcd}$$

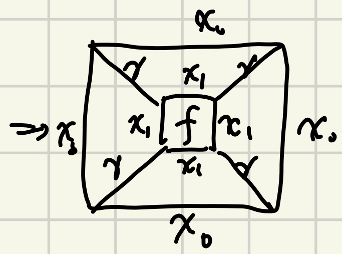
Prop. If X is path connected, then $\pi_n(X, x_0) \simeq \pi_n(X, x_1)$

proof. $n = 1$,  then $\gamma f \bar{\gamma} : I^1 \to X$, $\partial I^1 \to x_0$

$\longrightarrow \gamma(-)\bar{\gamma} : \pi(X, x_1) \xrightarrow{\sim} \pi(X, x_0)$. $[f] \mapsto [\gamma f \bar{\gamma}]$.

$n \geq 2$, $f : (I^n, \partial I^n) \longrightarrow (X, x_1)$





γf

$[f] \longmapsto [\gamma f]$ is an isomorphism

Def. (loop) $\gamma : [0,1] \to X$, $\gamma(0) = \gamma(1)$ (continuous)

(inner)

$n = 1$ $\gamma(-)\bar{\gamma} : [f] \mapsto [\gamma][f][\gamma]^{-1}$ is an isomorphism

$n \geq 2$ $[f] \mapsto [\gamma f]$ need not to be inner iso.

Rmk. π_n as a functor $\left[(X, x_0) \xrightarrow{f} (Y, y_0)\right] \Rightarrow \left[\begin{matrix} f_* \; [a] \longmapsto [f \circ a] \\ \pi(X, x_0) \longmapsto \pi(Y, y_0) \end{matrix}\right]$

$$\begin{tikzcd}
& (X, x_0) \arrow[d, "f"] \arrow[dl, phantom, "\circlearrowright"] \\
(I^n, \partial I^n) \arrow[ur, "a"] \arrow[r, "f \circ a"'] & (Y, y_0)
\end{tikzcd}$$

Prop $\pi_n(X \times Y, (x_0, y_0)) \simeq \pi_n(X, x_0) \times \pi_n(Y, y_0)$ $n \geq 1$

pf. $p_1 : X \times Y \to X$, $p_2 : X \times Y \to Y$. Consider $(p_1)_*, (p_2)_*$.

Set $\varphi : \pi_n(X \times Y, (x_0, y_0)) \longrightarrow \pi_n(X, x_0) \times \pi_n(Y, y_0)$

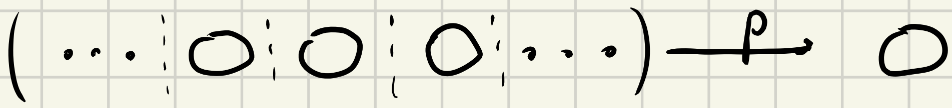
Ext $[h] \longmapsto (p_{1*}([h]), p_{2*}([h]))$

φ is inj and surj.

Covering Space

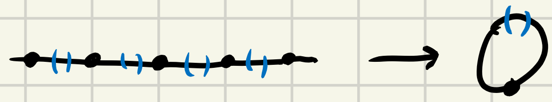
Def. E, B path connected and locally path connected. $E \xrightarrow{p} B$ is a covering map if

$\forall b \in B$, $\exists$ path connected nbh U_b s.t. $p^{-1}(U_b)$ is disjoint union of open sets.



$$\begin{tikzcd} & E \arrow[r, "p"] & B \\ & X \arrow[u, dashed, "\tilde f"] \arrow[ur, "f"'] & \end{tikzcd}$$

Eg. $p: E^1 \to S^1$, $x \mapsto e^{2\pi i x}$.



Def. $p: E \to B$, $f: X \to B$. $\tilde f: X \to E$ is a lift of f s.t. $p \circ \tilde f = f$

Thm. $\hat f_1(x_0) = \hat f_2(x_0) \Rightarrow \hat f_1(x) = \hat f_2(x)$ unique lifting thm

Thm.

$$0 \longmapsto b_0$$

$$\begin{tikzcd} {[0,1]} \arrow[r, "a"] \arrow[dr, "\tilde a"'] & B & b_0 \\ & E \arrow[u, "p"] & e_0 \arrow[u] \end{tikzcd}$$

Path lifting thm.

Thm $H: I^n \times [0,1] \to B$ $\quad a \simeq b$ rel ∂I^n $\quad \hat a(\partial I^n) = \hat b(\partial I^n)$

$\Rightarrow \exists\, \tilde H: I^n \times [0,1] \to E$ s.t. $\tilde a \simeq \tilde b$ rel ∂I^n. Homotopy lifting thm

Thm. X p.c. l.p.c. $\quad f: (X, x_0) \to (B, b_0)$ $\quad e_0 \in p^{-1}(b_0)$

$\exists\, \tilde f \iff f_* \pi_1(X, x_0) \subset p_* \pi_1(E, e_0)$

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde f"] \arrow[r, "f"'] & B \end{tikzcd}$$

pf. "$\Rightarrow$" $f_*(\pi_1(X, x_0)) = (p \circ \tilde f)_*(\pi_1(X, x_0))$
$\subset p_*(\pi_1(E, e_0))$

"$\Leftarrow$" $\forall x \in X$, $x_0 \xrightarrow{w} x$ a path.

$f \circ w: [0,1] \to B$ is a path from b_0 to $f(x)$

$\Rightarrow f \circ w$ has a lift $\tilde w: [0,1] \to E$ $\quad \tilde w(0) = e_0$.

Define $\tilde f: X \to E$, $x \mapsto \tilde w(1)$ Then $p \circ \tilde f(x) = f(x)$

Rmk. $\tilde f(x)$ is independent of $\tilde w$.

$\tilde f$ is continuous.

null homotopic

$$\begin{tikzcd} & E^1 \arrow[d, "p"] \\ S^n \arrow[ur, "\tilde f"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

Eg. $n \geq 2$. $f: S^n \to S^1$ is null homotopic

$\Rightarrow \pi_n(S^1) = \begin{cases} \mathbb{Z} & n = 1 \\ 0 & n \geq 2 \end{cases}$

universal covering $\quad \pi_1(T^3) = \mathbb{Z}^3$. $\quad \pi_n(T^n) = 0 \quad (n \geq 2)$.

Thm $p_* \pi_n(E,e_0) \xrightarrow{\sim} \pi_n(B,b_0)$ $\quad n\geq 2$ $\quad$ if $E \to B$ is covering space

$p_* \pi_1(E,e_0) \hookrightarrow \pi_1(B,b_0)$ $\quad n=1$

pf $n \geq 2$ p_* is surjective $[f] \in \pi_n(B,b_0)$

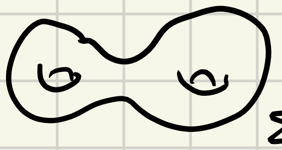
$$\begin{tikzcd} (I^n,\partial I^n) \arrow[r, "q"] \arrow[rr, bend left, "f"] \arrow[rr, bend right, "\tilde f"'] & (S^n,s_0) \arrow[r, "g"] \arrow[dr, "\tilde g"'] & (B,b_0) \\ & & (E,e_0) \arrow[u, "p"'] \end{tikzcd}$$

$p_*([\tilde g \circ q]) = [f]$

p_* is inj. $[h] \in \pi_n(E,e_0)$, $p_*[h] = 0 \in \pi_n(B,b_0)$

$\Rightarrow p \circ h \simeq c_{b_0}$ rel ∂I^n in B. $\xRightarrow{\text{h.l.t}}$ $h \simeq c_{e_0}$ rel ∂I^n

Def $p: E \to B$ is universal cover, if $\pi_1(E) = \{1\}$.

Eg.  Σ_g

$\pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g \mid [a_i, b_i] \rangle$

$\pi_n(\Sigma_g) = 0 \quad (n \geq 2)$

$\mathbb{H}^2$ is a universal covering space. tiling multi-gon on Poincaré disk.

Eg. $p: S^n \to \mathbb{RP}^n$, $\pi_k(S^n) \simeq \pi_k(\mathbb{RP}^n)$ $(k \geq 2)$

Eg.

i	1	2	3	4	5	6	7	8	9	10	11	12
$\pi_i(S^2)$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$

Def. X, Y cell complex. X^n, Y^n n-skeleton consists of cells of dim $\leq n$

$f: X \to Y$ is cellular if $f(X^n) \subset Y^n$ $(n \geq 0)$

Thm Cellular approximating thm $f: X \to Y$ is homotopic to some cellular map

Rmk $\pi_i(S^n) = 0$ $(i < n)$

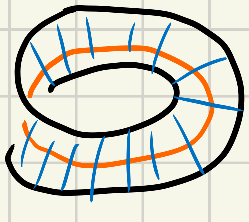
Def (Fibre Bundle) $F \to E$ Total space $\exists U_b$ s.t. $p^{-1}(U_b) \cong F \times U_b$

Fibre $\quad \downarrow p$

B Base space $\ni b$

$$\begin{tikzcd} \mathbb{N} \arrow[r, "\iota"] & \mathbb{R} \arrow[d, "p"] \\ & S^1 \end{tikzcd}$$

Eg. $p: E \to B$ is a fibre bundle. F discrete point set



F	S^1		S^1	$\{e^{i\theta}\}$
E	S^{2n+1}	$\xrightarrow{n=1}$	S^3	$\{(z_1,z_2) \mid \lvert z_1\rvert^2 + \lvert z_2\rvert^2 = 1\}$
B	$\mathbb{CP}^n$		S^2	$\{z_1/z_2\} = \hat{\mathbb{C}}$

Thm $F\to E\to B$ is a fibre bundle. E, B path connected, then

$$\pi_n(F,x_0)\to\pi_n(E,e_0)\to\pi_n(B,b_0)\xrightarrow{\partial}\pi_{n-1}(F,e_0)\to\cdots\to 0$$

Eg. $S^1\to S^3\to S^2$, then $\underset{\underset{0}{\|}}{\pi_n(S^1)}\to \underline{\underline{\pi_n(S^3)\xrightarrow{\sim}\pi_n(S^2)}}\to\underset{\underset{0}{\|}}{\pi_{n-1}(S^1)}$ $\quad n\geq 3$

Eg. $SO(n-1)\to SO(n)\to S^{n-1}$ $\quad 0\neq v\in\mathbb{E}^n$

Eg. Use $\pi_1(SO(n))=\mathbb{Z}_2$ $n\geq 3$ to determine S^2 bundles over S^n