12-Sept-2023

Notation. $\mathcal{A}$ Abelian category

$(\mathcal{C}, \mathcal{F}, \mathcal{W})$ objects

$(Cof, TCof, Fib, TFib)$ morphism

prop 2.1 $\mathcal{C}$, $\mathcal{C}\cap\mathcal{W}$ are contravariantly finite
$\mathcal{F}$, $\mathcal{F}\cap\mathcal{W}$ are covariantly finite

Thm. 3.6 $\mathcal{A}$ has enough proj and inj, TFAE.

① $(\mathcal{C}\cap\mathcal{W}, \mathcal{F})$ is complete cotorsion pair

② (a) $\forall f \in Fib$, f is epi.
(b) $\forall f \in TFib$, f is monic.
(c) $\forall\ 0 \to A \xrightarrow[f]{} B \to C \to 0$ with $C \in (\mathcal{C}\cap\mathcal{W})$
$\Rightarrow f \in TCof$.

Prop. $(\mathcal{X}, \mathcal{Y})$ is object, closed under direct summand and isomorphism. $w = \mathcal{X}\cap\mathcal{Y}$

$Cof_w = \{f \text{ monic} \mid \operatorname{coker} f \in \mathcal{X}\}$

$TCof_w = \{f \text{ split monic} \mid \operatorname{coker} f \in w\}$

$Fib_w = \{f\ w\text{-epic}\}$

$TFib_w = \{f \text{ epic} \mid \ker f \in \mathcal{Y}\}$

$Weq_w = TFib_w \circ TCof_w$ (proj module structure)

(Cof_w, Fib_w, Weq_w) is closed module structure iff

$(\mathcal{X}, \mathcal{Y})$ is complete cotorsion pair, w is contra. finite.

$[\mathcal{C} = \mathcal{X}, \mathcal{F} = \mathcal{A}, \mathcal{W} = \mathcal{Y}. \mathcal{C}\cap\mathcal{W} = w, \mathcal{F}\cap\mathcal{W} = \mathcal{Y}.]$

$Ho(\mathcal{A}) = \mathcal{X}/w$.

$(\mathcal{C}, \mathcal{F}, \mathcal{W})$ is Frobenius if it is both proj and inj

Thm. 4.10

(i) $\mathcal{A}$ is Frobenius, iff $\mathcal{A}$ admits a Frobenius module structure.

(ii) For closed model structure $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ TFAE.

(a) It is Frobenius

(b) $Cof = \{\text{monomorphism}\}$, $Fib = \{\text{epi}\}$ $Weq = \{\text{iso in } \mathcal{A}/\underline{\mathcal{D}}\}$

2-cluster tilting $(\mathcal{D}, \mathcal{D})$

Eg. (2) AR correspondence: rep finite $\Leftrightarrow$ Auslander algebra

(n-1 cluster tilting) (n-)

Eg. $(\mathcal{X}, \mathcal{Y})_{(\text{pair})} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} (Cof, Fib, Weq)$

$$\begin{cases} \Phi(\mathcal{X}, \mathcal{Y}) = (Cof_w, Fib_w, Weq_w) \\ \Psi(Cof, Fib, Weq) = (\mathcal{C}, \mathcal{F} \cap \mathcal{W}) \end{cases}$$

Satisfying : LHS (above)

RHS (a) $\forall f \in Cof$, f monic

(b) $\forall f \in TFib$, f epic

(c) $\mathcal{F} = \mathcal{A}$

(d) $0 \to A \to B \xrightarrow{f} C \to 0$

$A \in \mathcal{F} \cap \mathcal{W}$, then $f \in TFib$

Def. (Cof_1, Fib_1, Weq_1), (Cof_2, Fib_2, Weq_2) compatible if

(1) $Fib_2 = TFib_1$, which are epic

$Cof_1 = TCof_2$, which are monic

(2) $\mathcal{C}_1 \cap \mathcal{W}_1 = \mathcal{F}_2 \cap \mathcal{W}_2$, $\mathcal{F}_1 = \mathcal{A} = \mathcal{C}_2$.

{ Complete cotorsion pair with ω functorially finite

$\updownarrow$

{ compatible closed model structure

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Df. Compact object X. $\mathrm{Hom}(X,-)$ preserves direct limit

? Thm. compact $\Leftrightarrow$ finite rep.

Goal ① $M \in {}^{\perp}\mathscr{C}$, i.e. $\mathrm{Ext}^1_R(M,\mathscr{C}) = 0$. λ: regular uncountable cardinal.

Question: $M = \varinjlim M_i$ (M_i λ-presented, $M_i \in {}^{\perp}\mathscr{C}$)?,

how large

② $\{M_i, f_{ji}\}_I$ direct system, with $M_i \in {}^{\perp}\mathscr{C}$.

Question: whether $\varinjlim M_i \in {}^{\perp}\mathscr{C}$?

Df. $M = \{M_i, f_{ji}\}_{i,j\in I}$, directed system is called λ-continuous if

① $\forall$ Chain J, $|J| < \lambda$, $\sup J \in I$ (has upper bound).

② $M_{\sup J} = \varinjlim_J M_i$

Df. M is called λ-directed, if $S \subseteq I$ ($|S| < \lambda$) has an upper bound in I. (e.g. $\mathbb{Q}$ is $\aleph_0$-directed)

Df. M is $<\lambda$-presented if $\mathrm{Hom}_R(M,-)$ preserves λ-directed limits

Df. M, N λ-cont, consisting of $<\lambda$-presented modules $M \xrightarrow{f} N$

$M = \varinjlim_I M_i$, $N = \varinjlim_J N_j$. Define directed system with (need verification)

$\boxed{M_i \xrightarrow{u} N_j}$ s.t. $\begin{array}{ccc} M_i & \xrightarrow{u} & N_j \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$, the set of objects.

$\boxed{M_i \xrightarrow{u} N_j} < \boxed{M_s \xrightarrow{v} N_t}$ iff $\begin{array}{ccc} M_i & \xrightarrow{u} & N_j \\ f_{si}\downarrow & \circlearrowleft & \downarrow g_{tj} \\ M_s & \xrightarrow{v} & N_t \end{array}$

$\boxed{(f_{si}, g_{tj})}$, the set of morphism,

lemma. $M \xrightarrow{f} N$ is direct system.

① $\forall$ $M_i \xrightarrow{u} N_j$, $M_s \xrightarrow{v} N_t$. $\exists$ a, b, w s.t. $\begin{array}{ccc} M_i & \xrightarrow{u} & N_j \\ \downarrow & \circlearrowleft & \downarrow \\ M_a & \xrightarrow{w} & M_b \\ \uparrow & \circlearrowleft & \uparrow \\ M_s & \xrightarrow{v} & M_t \end{array}$ ②

Thm. $M = \varinjlim_I M_i$, $\mathrm{Ext}^1_R(M_i, C) = 0$ $(\forall i)$. Then TFAE

① $\mathrm{Ext}^1_R(M, C) = 0$ $\quad [\ \varinjlim_I M_i \in {}^{\perp}\mathcal{C}\ ]$

② $\forall \{g_{ji} : M_i \to C \mid i<j\}$ s.t. $g_{ki} = g_{ij} + g_{kj} f_{ji}$ $\quad (i<j<k)$

$\exists \{g_i : M_i \to C\}$ s.t. $g_j = g_{ji} + g_i f_{ij}$

pf. Consider

$$\begin{tikzcd}
\bigoplus_{i_0<i_1<i_2} M_{i_0,i_1,i_2} \arrow[r, "\delta_1"] & \overset{D}{\bigoplus_{i_0<i_1} M_{i_0,i_1}} \arrow[rr, "\delta_2"] \arrow[d] & & \overset{B}{\bigoplus_I M_i} \arrow[r] & \overset{A}{\varinjlim_I M} \arrow[r] & 0 \\
& \overset{C}{(0,\dots,x-f(x),\dots 0)} \arrow[urr] & & & &
\end{tikzcd}$$

$$\boxed{\begin{aligned} &\delta_0|_{M_{ij}}(x) = (x, -f_{ji}(x)) \\ &\delta_1|_{M_{ijk}}(x) = (\underset{M_{ik}}{x}, -\underset{M_{ij}}{x}, -\underset{M_{jk}}{f_{ji}(x)}) \end{aligned}}$$

Consider

$$\begin{tikzcd}
D \arrow[r, "g"] \arrow[dr, two heads, "t"'] & B \arrow[r, "f"] & M \arrow[r] & 0 \\
& E \arrow[u, hook, "h"'] & & \\
0 \arrow[ur] & & &
\end{tikzcd}$$

$\xRightarrow{h_C}$ $\big[\ 0 \to (M,C) \to (B,C) \to (E,C) \to \mathrm{Ext}^1(M,C) \to \mathrm{Ext}^1(B,C)$

$\mathrm{Ext}^1(B,C) = \varprojlim_I \mathrm{Ext}^1(M_i,C) = 0$

For $\underline{0 \to (M,C) \to (B,C) \to (E,C) \to \mathrm{Ext}^1(B,C) \to 0}$

$\mathrm{Ext}^1(B,C) = 0 \iff (B,C) \twoheadrightarrow (E,C)$

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Compact but not Generated module

Def. Compact: $\oplus (M, N_i) \hookrightarrow (M, \oplus N_i)$ is an iso.

$$(f_i) \longmapsto \Sigma_i e_i f_i$$

(Fact R-module is compact $\Longleftrightarrow$ finitely presented)

Equivalently, $\forall f \in (M, \oplus_i N_i)$ factors through $\overbrace{\oplus_{j \in J_f} N_j}^{\text{some}}$, $J_f \subseteq I$ is finite.

Ex. X uncountable, k field, $\underline{R := \{f \mid f: X \to k\}\ \text{a ring}}$

$M = \{f \in R \mid \text{supp } f \text{ is countable}\}$, $M \triangleleft R$ is $\underline{\text{not f.g.}}$

To see that M is compact, if $\exists \varphi \in (M, \oplus_{i \in I} N_i)$ $(|I| = \infty)$

s.t. $\nexists$ finite J,
$$\begin{tikzcd} & \oplus_J N_i \arrow[dr] & \\ M \arrow[ur] \arrow[rr, "?"', "\varphi"] & & \oplus_I N_i \end{tikzcd}$$
Then $\exists$ countable many

$\pi_i \circ \varphi \neq 0$, $\forall i \in J$, $\pi_i : \oplus_I N_i \to N_i$.

$\Rightarrow \exists$ countable $f_i \in M$, $i \in J$ s.t. $0 \neq (\pi_i \circ \varphi)(f_i) \in N_i$

Now consider $f: X \to k$. $x \longmapsto \begin{cases} 0 & , x \notin \cup_{i \in J} \text{supp } f_i \\ f_i(x) & x \in \text{some supp } f_i. \end{cases}$

$\Rightarrow f \in M$. Take $r_i \in R$ s.t. $r_i f = f_i$

$\Rightarrow 0 \neq (\pi_i \circ \varphi) f_i = r_i (\pi_i \circ \varphi) f \quad \forall i \in J$

$\Rightarrow (\pi_i \circ \varphi)(f) \neq 0$, $\forall i \in J$. contradiction

$\overleftarrow{\Delta}$ category. $(\mathcal{C}, \Omega, \Delta)$ $\mathcal{C}$ add cat, $\Omega : \mathcal{C} \to \mathcal{C}$ add functor.

(LTr 1) (i) Δ is closed under iso.

(ii) $\forall\, u : X \to Y$ in $\mathrm{Mor}(\mathcal{C})$ is embedding into $\Omega Y \to W \to X \xrightarrow{u} Y$.

(iii) $\forall X \in \mathcal{C}$. $0 \to X \xrightarrow{id} X \to 0 \in \Delta$

(Left Shift)

(LTr 2) $\Omega Y \xrightarrow{w} W \xrightarrow{v} X \xrightarrow{u} Y \in \Delta$, $\Rightarrow$ $\Omega X \xrightarrow{-\Omega u} \Omega Y \xrightarrow{w} W \xrightarrow{v} X \in \Delta$

(LTr 3)
$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "u'"] & Y' \end{tikzcd} \Rightarrow \begin{tikzcd} \Omega X \arrow[r, "w"] \arrow[d, "\Omega f"'] & W \arrow[r, "v"] \arrow[d, dashed, "h"'] & X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"'] \\ \Omega X' \arrow[r, "w'"] & W' \arrow[r, "v'"] & X' \arrow[r, "u'"] & Y' \end{tikzcd}$$
$uv = 0$, $vw = 0$

(LTr 4)
$$\begin{tikzcd} & \Omega X' \arrow[r, "\Omega i"] \arrow[d, "k'\Omega i"'] & \Omega Y \arrow[d, "k'"] & & \\ \Omega Y \arrow[r, "k'"] \arrow[d, "\Omega v"'] & Z' \arrow[r, equal] \arrow[d, dashed, "g"] & Z' \arrow[d, "k"] & & \\ \Omega Z \arrow[r, "j'"] \arrow[d, equal] & Y \arrow[r, "j"] \arrow[d, dashed, "\gamma"] & X \arrow[r, "uv"] \arrow[d, "u"] & Z \arrow[d, equal] & \star \\ \Omega Z \arrow[r, "i'"] & X' \arrow[r, "i"] & Y \arrow[r, "v"] & Z & \star \\ & \circledast & \star & & \end{tikzcd}$$

Def $\mathcal{A}$: Abel cat. $(\mathcal{C}, \Omega, \Delta)$ $\overleftarrow{\Delta}$ cat. $H : \mathcal{C} \to \mathcal{A}$ is a cohomological functor if $T \in \Delta$, $H(T)$ is long exact sequence.

ends.

$$H(\Omega^i W) \to H(\Omega^i X) \to H(\Omega^i Y) \to H(\Omega^{i-1} W) \to \cdots \to H(W) \to H(X) \to H(Y) \,\|$$

Thm. $\mathrm{Hom}(M, -)$ $\mathrm{Hom}(-, M)$ are cohomological functors.

Q. $\mathrm{Hom}(M, -)$ gives the iso. exact sequence $\Rightarrow T \simeq T'$?

Pure exact sequence ?

Λ Artin alg. $\mathcal{C} \subseteq \text{mod}\,\Lambda$ is of finite type (has finite simple iso class)

closed under $\oplus$. Then $\mathcal{C}$ is functorially finite.

Pf. $\text{ind}(\mathcal{C}) = \{C_1, \cdots, C_n\}$ $C = C_1 \oplus \cdots \oplus C_n$. $\forall M \in \text{mod}\,\Lambda$,
$\text{Hom}(C, M)$ is f.g. as $\text{End}(C^{op})$ module (via $\{f_1, \cdots, f_n\}$)

21-Sept-2023

Notation $\mathcal{A}$, $(\mathcal{X},\mathcal{Y})$ cotorsion pair, $\omega = \mathcal{X}\cap\mathcal{Y}$

$CoFib_\omega = \{f \text{ monic} \mid f\in \mathcal{X}\}$ $TCoFib_\omega \{f \text{ split monic} \mid f\in\omega\}$

$Fib_\omega = \{f \mid Hom(A,f) \text{ epic}, \forall A\in\omega\}$

$TFib_\omega = \{f \text{ epic} \mid \ker f\in\mathcal{Y}\}$

$Weq_\omega = TFib_\omega \circ TCoFib_\omega$

Thm 4.2 TFAE

① $(Cofib_\omega, Fib_\omega, Weq_\omega)$ is a closed model structure

② $CoF_\omega = \mathcal{X}$, $CoF_\omega\cap Weq_\omega = \omega$, $F_\omega = \mathcal{A}$, $F_\omega\cap Weq_\omega = \mathcal{Y}$, $W_\omega = \mathcal{Y}$

$(\mathcal{X},\mathcal{Y})$ is a complete cotorsion pair, ω is contra finite.

$CoF_\omega = \{0\to X \in CoFib_\omega\} \Leftrightarrow X\in\mathcal{X}$

$F_\omega = \{X\to 0 \in Fib_\omega\} \Leftrightarrow X\in\mathcal{A}$

$W_\omega = \{W\to 0 \in Weq_\omega\}$ and consider

$$\begin{tikzcd} W \arrow[rr] \arrow[dr, "f"'] & & 0 \\ & Y \arrow[ur, "g"'] & \end{tikzcd}$$

$g\in TFib_\omega$, $f\in TCoFib_\omega$

$\boxed{W_\omega = \mathcal{Y}}$ "⇒" $Y\in\mathcal{Y}$, f split monic ⇒ W is summand of Y ⇒ $W_\omega\subseteq\mathcal{Y}$

"⇐" Consider

$$\begin{tikzcd} Y \arrow[r] \arrow[d, equal] & 0 \\ Y \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$(\forall\, Y\in\mathcal{Y})$.

Eg. For $(\mathcal{P},\mathcal{A},\mathcal{A})$. ⇒ (split monic and coker in $\mathcal{P}$, surjective, also $\mathcal{A}$)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\binom{1}{0}"'] & Y \\ X\oplus P \arrow[ur, "{(f,x)}"'] & \end{tikzcd}$$

proof $1\Rightarrow 2$ depends on Prop 2.1, Prop 3.4(1)

Prop 3.4(1) If [1] every cofibration is monic, [2] trivial fibration is epic, [3] any epic with kernel in TFib is trivial fibration.

Then $(CoF, TFib)$ is a complete cotorsion pair.