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Notation $\mathcal{A}$, $(\mathcal{X},\mathcal{Y})$ cotorsion pair, $\omega = \mathcal{X}\cap\mathcal{Y}$

$CoFib_\omega = \{f \text{ monic} \mid f \in \mathcal{X}\}$ $\quad TCoFib_\omega \{f \text{ split monic} \mid f\in\omega\}$

$Fib_\omega = \{f \mid Hom(A,f) \text{ epic}, \forall A\in\omega\}$

$TFib_\omega = \{f \text{ epic} \mid \ker f \in \mathcal{Y}\}$

$Weq_\omega = TFib_\omega \circ TCoFib_\omega$

Thm 4.2 TFAE

① $(Cofib_\omega, Fib_\omega, Weq_\omega)$ is a closed model structure

② $CoF_\omega = \mathcal{X}$, $CoF_\omega\cap Weq_\omega = \omega$. $F_\omega = \mathcal{A}$. $F_\omega\cap Weq_\omega = \mathcal{Y}$, $W_\omega = \mathcal{Y}$

$(\mathcal{X},\mathcal{Y})$ is a complete cotorsion pair, ω is contra finite.

$CoF_\omega = \{0\to X \in CoFib_\omega\} \Leftrightarrow X\in\mathcal{X}$

$F_\omega = \{X\to 0 \in Fib_\omega\} \Leftrightarrow X\in\mathcal{A}$

$W_\omega = \{W\to 0 \in Weq_\omega\}$ and consider

$$\begin{tikzcd} W \arrow[rr] \arrow[dr, "f"'] & & 0 \\ & Y \arrow[ur, "g"'] & \end{tikzcd}$$

$g\in TFib_\omega$, $f\in TCoFib_\omega$

$\boxed{W_\omega = \mathcal{Y}}$ "$\Rightarrow$" $Y\in\mathcal{Y}$. f split monic $\Rightarrow$ W is summand of Y $\Rightarrow$ $W_\omega\subseteq\mathcal{Y}$

"$\Leftarrow$" Consider

$$\begin{tikzcd} Y \arrow[r] \arrow[d, equal] & 0 \\ Y \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$(\forall Y\in\mathcal{Y})$.

Eg. For $(\mathcal{P},\mathcal{A},\mathcal{A})$. $\Rightarrow$ (split monic and coker in $\mathcal{P}$, surjective, Mor $\mathcal{A}$)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\binom{1}{0}"'] & Y \\ X\oplus P \arrow[ur, "(f{,}\pi)"'] & \end{tikzcd}$$

proof $1\Rightarrow 2$ depends on Prop 2.1. Prop 3.4 (1)

Prop 3.4 (1) If [1] every cofibration is monic, [2] trivial fibration is epic, [3] any epic with kernel in TFib is trivial fibration.

Then $(CoF, TFib)$ is a complete cotorsion pair.