Analysis (李[illegible])

12-Sept-2023

Ref. Rudin (Real Complex analysis), Lax (functional analysis), Folland (Real analysis)

§ Measure

Motivation. Lebesgue measure : volume $\longleftrightarrow$ measure ($\mathcal{L}^n$)

$\Rightarrow$ we expect $\mathcal{L}^n$ satisfy

(1) $\mathcal{L}^n(\sqcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mathcal{L}^n(E_i)$ preserve countable disjoint union,

(2) $\mathcal{L}^n$ fixed under Euclidean rigid motion,

(3) [normalization] $\mathcal{L}^n([0,1]^n) = 1$

Question : (1)-(3) holds if $\mathcal{L}^n$ is defined on $P(\mathbb{R}^n)$? No

Counterexample depends on AC. See ___?___ .

$\Rightarrow$ $\mathcal{L}^n$ is defined on some $\mathcal{F} \subsetneq P(\mathbb{R}^n)$

Definition (σ-algebra) [$\cap$, $\cup$ are addition, multiplication (resp.)]

X non-empty set. (1) $\mathcal{A}(X) \subset P(X)$ is an algebra if it is closed under finite $\cup$ and complement.

(2) ——— σ-algebra ——— countable $\cup$ and completement.

Remark. $\cup(A_i^c) = (\cap A_i)^c$ $\left(\begin{matrix}\text{finite} \\ \text{countable}\end{matrix}\ \cup\right) + (\ )^c \Rightarrow \left(\begin{matrix}\text{finite} \\ \text{countable}\end{matrix}\ \cap\right)$

Example (0) $\mathcal{A} = \{\phi, X\}$ is an σ-algebra.

$\mathcal{A} = P(X)$ is an σ-algebra.

(1) X uncountable, $\mathcal{F} := \{E \subset X \mid \text{either } |E| \le |\mathbb{N}| \text{ or } |E^c| \le |\mathbb{N}|\}$.

(2) $\{\mathcal{F}_i\}_{i \in I}$ is the set of σ-algebra. So is $\cap_{i\in I} \mathcal{F}_i$.

Def. (Smallest σ-alg) $\mathcal{E} \subset P(X)$ be any subset. $\sigma(\mathcal{E}) = \cap \{\mathcal{F} \subset P(X) \mid \mathcal{F} \text{ is } \sigma\text{-alg}, \mathcal{E} \subset \mathcal{F}\}$.

Rmk. $\sigma(\mathcal{E})$ is the σ-alg generated by $\mathcal{E}$.

Prop. $\mathcal{E}_1 \subset \sigma(\mathcal{E}_2) \Rightarrow \sigma(\mathcal{E}_1) \subset \sigma(\mathcal{E}_2)$

Def. (X, τ) is topological space. The Borel set is defined to be $\mathcal{B}_X := \sigma(\tau)$.

Eg. Euclidean Topology $(\mathbb{R}, \tau)$. $\sigma(\tau)$ is generated intervals.

Def. ($\mathcal{F}_1 \otimes \mathcal{F}_2$) $\{X_i\}_{i\in I}$ non-empty. $X := \prod_{i\in I} X_i$, $\mathcal{F}_i$ σ-alg on X_i.

$\pi_i : X \to X_i$, $(x_i)_{i\in I} \mapsto x_i$ is the projection.

$\mathcal{F}_X := \sigma\left(\{\pi_i^{-1}(E_i) \mid E_i \in \mathcal{F}_i,\ i\in I\}\right)$.

Rmk. $|I| \le |\mathbb{N}|$. Then $\mathcal{F}_X = \sigma\left(\{\prod_{i\in I} E_i \mid E_i \in \mathcal{F}_i\}\right)$, $\otimes_{i\in I}\mathcal{F}_i$

Prop $\{\mathcal{F}_i\}_{i\in I}$, $\mathcal{F}_i = \sigma(\mathcal{E}_i)$. Then $\otimes_{i\in I}\mathcal{F}_i = \sigma\left(\underbrace{\{\pi_i^{-1}(E_i) \mid E_i \in \mathcal{E}_i,\ i\in I\}}_{\hat{\mathcal{E}}}\right)$.

[Rmk. $|I| \le |\mathbb{N}|$. Then $\mathcal{F}_X = \sigma\left(\{\prod_{i\in I} E_i \mid E_i \in \mathcal{E}_i\}\right)$]

Pf. To prove $\mathcal{F}_X = \sigma(\hat{\mathcal{E}})$. ① $\hat{\mathcal{E}} \subseteq \mathcal{F}_X$, thus $\sigma(\hat{\mathcal{E}}) \subset \mathcal{F}_X$.

② fix an $i \in I$. consider $\mathcal{G}_i := \{A \subset X_i \mid \pi_i^{-1}(A) \in \sigma(\hat{\mathcal{E}})\}$, σ-alg on X_i

since $\mathcal{E}_i \subset \mathcal{G}_i$, $\sigma(\mathcal{E}_i) \subset \mathcal{G}_i$. Thus $\otimes_{i\in I}\mathcal{F}_i \subset \sigma(\hat{\mathcal{E}})$. #

Prop $X_1, \dots, X_n$ metric spaces. $X = X_1 \times \dots \times X_n$. Then $\otimes_{i\in I} \mathcal{B}_{X_i} \subseteq \mathcal{B}_X$

Equality holds if X_i's are seperable has countable dense subset.

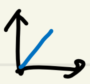
Pf (Ex) ① $\otimes_{i\in I} \mathcal{B}_i = \sigma\left(\{\pi_i^{-1}(E_i) \mid E_i \in \tau_i,\ i\in I\}\right) \subseteq \mathcal{B}_X$ (The key)

② If X_i's are seperable, then $\tau_i = \{$Balls, countable union$\}$

The topological basis of X consists of countable union of $\prod$ Balls.

✲ If S is discrete, uncountable, then

$\begin{cases} \mathcal{F}_S \otimes \mathcal{F}_S = \{\text{countable union of boxes}\} \\ \mathcal{F}_{S\times S} = \mathcal{P}(S\times S) \end{cases}$

For $S = [0,1]$,  $\in \mathcal{F}_{S\times S} \setminus (\mathcal{F}_S \otimes \mathcal{F}_S)$.

Cor. $\otimes_{i=1}^n \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^n}$.

§ Measure

Def $(X, \mathcal{F}, \mu) = (\text{Set}, \sigma\text{-alg}, \text{measure})$ is called a measure space if

$\mu : \mathcal{F} \to [0, \infty]$ s.t.

① $\mu(\emptyset) = 0$, (See ②)

② $\mu(\sqcup_{i=1}^\infty E_i) = \sum_{i=1}^\infty \mu(E_i)$ (σ-additivity)

Rmk. If $\mu(X) < \infty$, then μ is called finite. ($[0,1]^n$)

If X is countable union of μ-finite sets $\Rightarrow$ μ is σ-finite ($\mathbb{R}^n$)

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Eg. Measure space $(X, \mathcal{F}, \mathbb{P})$. $\mathbb{P} : \mathcal{F} \to [0,1]$ is probability measure.

Eg. $\mu\llcorner A : \mathcal{F} \to [0, \mu(A)]$. $X \mapsto \mu(X \cap A)$.

Prop. ① $\mu(E) \le \mu(F)$, if $E \subseteq F$.

② $\{E_i\}_{i=1}^{\infty} \subset \mathcal{F}$, then $\mu(\bigcup_{i=1}^{\infty} E_i) \le \sum_{i=1}^{\infty} \mu(E_i)$.

③ $E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots$, then $\mu(\bigcup_{i=1}^{\infty} E_i) = \lim_{i\to\infty} \mu(E_i)$,

④ $E_1 \supset E_2 \supset \cdots \supset E_n \supset \cdots$, then $\mu(\bigcap_{i=1}^{\infty} E_i) \le \lim_{i\to\infty} \mu(E_i)$,

※ e.g. $E_n = [n, +\infty[$. If $\mu(E_i) < \infty$, then "=" holds.

Def (null-set) $\mu(E) = 0$, $E \subset X$. $\emptyset$ is null

Def a.e./a.s. $\{\neg \text{property}\}$ is null,

Def (Complete $(X, \mathcal{F}, \mu)$) $\mu(X) = 0 \Rightarrow \mu(Y) = 0$ $(\forall Y \subseteq X)$.

Eg. $\mathbb{R}^n$ are complete. $\mu(\mathbb{Q}) = 0$

Prop $(X, \mathcal{F}, \mu)$ be any measure space $\mathcal{Z} := \{N \in \mathcal{F} \mid \mu(N) = 0\}$

$$\begin{cases} \overline{\mathcal{F}} = \{B \cup E \mid E \in \mathcal{F},\ B \subseteq N \text{ for some } N \in \mathcal{Z}\}, \\ \overline{\mu}(B \cup E) = \overline{\mu}(B) = \mu(B), \end{cases}$$

is the completion.

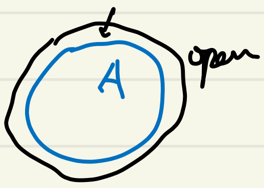
§ Construction of measures

Def. $\mathcal{A} \subset P(X)$, $\mu_0 : \mathcal{A} \to [0, +\infty]$ is called a pre measure iff

$\mu_0(\emptyset) = 0$, $\mu_0(\bigsqcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu_0(A_i)$ if $\bigsqcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

Def $\mu^* : \boxed{P(X)} \to [0, \infty]$ is an outer measure if

$$\begin{cases} \mu^*(\emptyset) = 0. \\ \mu^*(A) \le \mu^*(B) \text{ if } A \subseteq B. \\ \mu^*(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} \mu^*(A_i). \end{cases}$$



Def (premeasure space) $\mu^*(E) := \inf\{\sum_{i=1}^{\infty} \mu^*(O_i) \mid \bigcup_{i=1}^{\infty} O_i \supseteq E,\ O_i \in \mathcal{A}\}$.

Thm (Carathéodory) μ^* an outer measure on X. $A \subset X$ is said to obey Carathéodory separation condition [C], if $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$, $\forall E \subset X$.

Then $\mathcal{F}^* := \{A \subseteq X \mid A \text{ satisfy } [C]\}$ is σ-alg. $\mu^*|_{\mathcal{F}^*}$ is complete on $\mathcal{F}^*$.

A is called μ^*-measurable. [C]

Eg. $(\mathbb{R}^n, \square, \mu_0)$

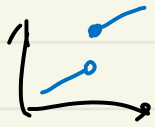
$$\begin{tikzcd}
(\sim, \mathcal{A}, \mu_0) \text{ pre-measure} \arrow[rr, "\text{complete!}"] \arrow[dr, "\text{minimal covering}"'] \arrow[ddr, bend right, "\text{Hahn-extension}"'] & & (\mathcal{F}^*, \mu^*) \\
& (P(X), \mu^*)\text{, outer measure} \arrow[ur, "{[C]}"'] \arrow[d, "\text{restriction}"] & \\
& (\sigma(\mathcal{A}), \mu = \mu^*|_{\sigma(\mathcal{A})}) \arrow[uur, dashed, bend right, "\varphi\ ?"'] &
\end{tikzcd}$$

Lebesgue measurable set

(Borel-σ-alg. Lebesgue measure)

If μ is σ-finite, then φ is canonical.

§ Lebesgue Stieltjes measure on $\mathbb{R}^n$

Eg. $F: \mathbb{R} \to \mathbb{R}$ be increasing, right-continuous 

h-intervals

Consider intervals of the type $]a,b]$, $]a,\infty[$, $\emptyset$. $-\infty \le a < b < \infty$
$\mathcal{A} := \{$ finite disjoint of h-intervals $\}$ is an algebra.

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Def (In case of Lebesgue Stieltjes measure on $\mathbb{R}$)
$\mu_0(\emptyset)=0$, $\mu_0(\amalg_{j=1}^{n}]a_j,b_j]) = \sum_{j=1}^{n}(F(b_j)-F(a_j))$

Rmk. $F = id \Rightarrow$ Lebesgue measure   Ex. μ_0 is premeasure on $\mathcal{A}$

Def. μ_F or its completion $\overline{\mu_F} = \mu_F$, is called Lebesgue-S measure

Note $\sigma(\mathcal{A}) = \mathcal{B}_{\mathbb{R}}$. (trivial)

Lemma. $(\mathcal{A}, \mu_0)$ extends to a unique Borel measure

$$\begin{tikzcd}
\text{outermeasure } (\mathcal{P}(\mathbb{R}), \mu^*) \arrow[r] & (\mathcal{F}^*, \bar{\mu}) \; (\mathcal{M}_{\mathcal{L}}, \mathcal{L}') \\
\text{premeasure } (\mathcal{A}, \mu_0) \arrow[u] \arrow[r] & (\sigma(\mathcal{A}), \mu) \; (\mathcal{B}_{\mathbb{R}}, \mathcal{L}') \arrow[u, "\sigma\text{-finite } \varphi"', "\text{completion}"]
\end{tikzcd}$$

Prop $\mu_F = \mu_G \iff F - G = $ constant.

proof. Recover F from μ_F via $F(x) = \begin{cases} \mu_F(]0,x]) & x>0 \\ 0 & x=0 \\ -\mu_F(]0,-x]) & x<0 \end{cases}$

Write $\mathcal{M}_\mu = dom(\bar{\mu}) = \{E \subset X \mid E \text{ satisfy [C]}\}$

Then $\forall E \in \mathcal{M}_\mu$, $\bar{\mu}(E) = \inf\left(\sum_{j=1}^{\infty} \mu(]a_j,b_j]) \mid \bigcup_{j=1}^{\infty}]a_j,b_j] \supset E\right)$ ⊗

Lemma In ⊗ $]a_j,b_j]$ can be replaced by $]a_j,b_j[$.

pf. ① $]a,b[=]a,\frac{a+b}{2}] \cup]\frac{a+b}{2}, \frac{a+2b}{3}] \cup \cdots \cup]\frac{a+kb}{k+1}, \frac{a+kb+b}{k+2}] \cup \cdots$
$\Rightarrow \mu_{],]} \le \mu_{],[}$

② $\left|\mu_{],]}(E) - \sum_{j=1}^{\infty}(F(b_j)-F(a_j))\right| < \varepsilon/2$ Select δ_j s.t.

$|F(b_j+\delta_j) - F(b_j)| < \frac{\varepsilon}{2^{j+1}}$ by right continuity of F.

$\xrightarrow{(a_i,\, b_j+\delta_j)} \mu_{J,I}(E) \le \mu_{J,J}(E) + \underbrace{\frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \cdots}_{\varepsilon}$

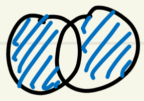
Thm, μ L-S measure. $\mathcal{M}_\mu : \{[C]\}$. $\forall E \in \mathcal{M}_\mu$. we have that

$\mu(E) = \inf \{ \mu(O) \mid O \supset E,\ O \text{ open} \}$ (by lemma ↑)

outer regular

inner regular $= \sup \{ \mu(K) \mid K \subset E,\ K \text{ compact} \} \leftarrow$ Ex

Littlewood's 1st principle: Borel set Δ good set = null



Good sets includes

$\begin{cases} G_\delta\text{-set} & \text{countable } \cap \text{ of open sets} \\ F_\sigma\text{-set} & \sim\ \cup\ \sim \text{closed} \sim \end{cases}$

Thm $E \subset \mathbb{R}^n$, TFAE

① $E \in \mathcal{M}_\mu$;

② $E = V \setminus N_1$, V is G_δ, $\mu(N_1) = 0$;

③ $E = H \cup N_2$, H is F_σ, $\mu(N_2) = 0$.

pf. ② or ③ $\Rightarrow$ ① μ is complete in $\mathcal{M}_\mu$. $G_\delta, F_\sigma \in \mathcal{B}$.

①, ② $\Rightarrow$ ③ By ↑ $K_j \subseteq E \subseteq O_j$. $\mu(K_j) \le \mu(E) \le \mu(O_j)$ For $\mu(E) < \infty$.

($\mu(E) = \infty$, use σ-finite)

Convention/Def. $\mu_F =: \mathcal{L}^1$ in (1D case) $\mathcal{M}_{\mu_F} =: \mathcal{M}_{\mathcal{L}^1}$

Eg. (Pathological Ex) Top big $\not\Rightarrow$ Measure big

(i) open dense $E \subset I$, with arbitrary small $\mathcal{L}^1(-)$

(ii) uncountable null set (Cantor)

(iii) $\exists$ non Borel measurable set.

HW1

Q1: $\{w_n\}$ be sequence of non-negative real numbers. For $E \subset \mathbb{N}$, set

$$\mu_w(E) = \sum_{n\in E} w_n .$$

(1) Show that μ_w is a measure on $(\mathbb{N}, P(\mathbb{N}))$.

(2) $\forall \nu$ a measure on $(\mathbb{N}, P(\mathbb{N}))$, Then $\nu = \mu_w$ with $w_n = \nu(\{n\})$.

pf. (1) $\mu_w(\emptyset) = \emptyset$, $\mu_w(\sqcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu_w(E_i)$ since $w_n \geq 0$

(2) ν is determined by $(\nu(\{k\}))_{k\in\mathbb{N}}$. Here $\nu(\{k\}) \geq 0$.

Q2: E open in $\mathbb{R}$, $x, y \in E$, $I_{xy} :=$ closed interval between x and y.

(1) $x \sim y$ iff $I_{x,y} \subset E$, $\sim$ is equi relation.

(2) E is the disjoint union of most countably many open intervals.

pf (1) $\forall x \in E$, $I_{x,x} \in E$, ①

$I_{x,y} = I_{y,x}$, ②

$I_{x,y}, I_{y,z} \in E$. Then $I_{x,z} \in E$. ③ $\Rightarrow \sim$.

$E/\sim \; = \pi_0(E)$. $[x]$ is the connected component (open interval)

(2) If not, then there are $> |\mathbb{N}|$ many rational numbers.

Q3: $\mathcal{F}$ is a σ-alg on $\mathbb{R}$ containing all intervals $]a, +\infty[$ for $a \in \mathbb{R}$.
Then $\mathcal{F}$ contains the Borel σ-alg on $\mathbb{R}$.

pf Open sets in $\mathbb{R}$ is generated by $]a, +\infty[$, i.e.

$$(a,b) = \left((]a,+\infty[)^c \cup [b,+\infty[\right)^c, \qquad [b,+\infty[= \left(\bigcup_{n=1}^{\infty}\left(]b+\tfrac{1}{n}, +\infty[\right)^c\right)^c$$

Q4 $(X, \mathcal{F}, \mu)$. $f: X \to Y$. $f_\#(\mathcal{F}) := \{B \subset Y \mid f^{-1}(B) \in \mathcal{F}\}$,
$\forall B \in f_\#(\mathcal{F})$, $[f_\#\mu](B) := \mu(f^{-1}(B))$

$$\begin{tikzcd} B \arrow[r, "f^{-1}"] \arrow[dr, "f_\#\mu"'] & f^{-1}(B) \arrow[d, "\mu"] \\ & {} \end{tikzcd}$$

(1) $(Y, f_\#(\mathcal{F}), f_\#\mu)$ is m.s.

(2) $Y = \mathbb{R}$, $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathcal{L}')$ be $(X, \mathcal{F}, \mu)$, determine $(\mathbb{R}, f_\#(\mathcal{F}), f_\#\mu)$ when

(a) $f(x) = \tan x$ if $\cos x \neq 0$, otherwise $f(x) = 0$

(b) $f(x) = \arctan x$.

pf. (1) First $f_\#(\mathcal{F})$ is an σ-algebra,

(i) $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(f(X)) = X$.

(ii) $(f^{-1}(E))^c = f^{-1}(f(X) \setminus f(E))$.

(iii) $\sqcup_{n=1}^{\infty} f^{-1}(E_n) = f^{-1}(\sqcup_{n=1}^{\infty} E_n)$

Then $f_\#\mu$ is measure on $f_\#(\mathcal{F})$.