Differentiable Manifold.

11-Sept-2023

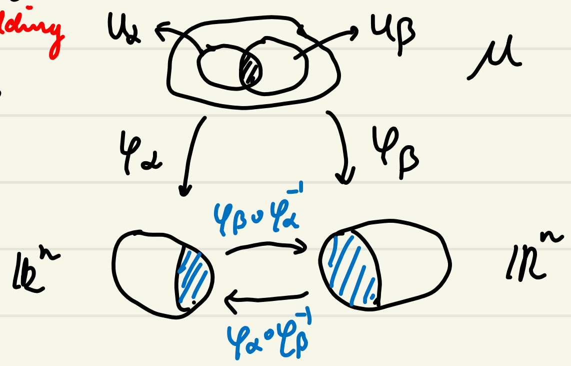
(has countable base)

Def. M Hausdorff, C_2. We say

① M is differentiable manifold if $\{U_\alpha\}_{\alpha\in\Lambda}$. $\{\varphi_\alpha\}$. $\varphi_\alpha: U_\alpha \to \mathbb{R}^n$ s.t.

(a) φ_α is homomorphism, eg. embedding

(b) if $U_\alpha \cap U_\beta \neq \emptyset$. e.g. $\longrightarrow$



$\varphi_\alpha\circ\varphi_\beta^{-1}$ and $\varphi_\beta\circ\varphi_\alpha^{-1}$ are differentiable

② C^0, C^r, C^∞

Rmk. $(U_\alpha, \varphi_\alpha)$ is a local coordinate system.

Eg $\mathbb{R}^n$ is d.m. $\left(id_{\mathbb{R}^n}: \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n\right)$

Eg S^2 $(\subset \mathbb{R}^3)$ is d.m. $U_1^{\pm} = \{(x^1, x^2, x^3) \mid \pm(x^1) > 0\}$,

$\varphi_1^{\pm}: U_1^{\pm} \to \mathbb{R}^2$, $(x^1, x^2, x^3) \longmapsto (x^2, x^3)$

Thus $\varphi_2^{+}\circ\varphi_1^{+-1}: (x^2, x^3) \mapsto (\sqrt{1-x^2x^2-x^3x^3}, x^3)$ is C^∞.

Eg. S^n is similar

Eg. $P^n(\mathbb{R}) = S^n/\sim$ $\quad (x \sim -x)$

Def. Differenciable structure: $\{(U_\alpha, \varphi_\alpha)\}_{-} =: \mathcal{S}$

Def. M: n-diff manifold. $f: M \to \mathbb{R}$ is continuous. f is diff if
$\forall p \in M$. $\exists (U, \varphi)$ local chart at p, s.t. $f\circ\varphi^{-1}: \varphi(U) \to \mathbb{R}$ is diff.

Rmk. For another coordinate φ' at U. $f\circ\varphi^{-1} = (f\circ\varphi'^{-1})\circ(\varphi'\circ\varphi^{-1})$
on some $p \in U_p \subset U$. $(\varphi'^{-1}\circ\varphi)$ is an embedding.

Def. $F: N^n \to M^m$ is diff at $q \in N$, if

$(V_q, \psi) \xrightarrow{F} (U_{F(q)}, \varphi)$

$$\begin{tikzcd} N \arrow[r, "F"] \arrow[d, "\psi"'] & M \arrow[d, "\varphi"] \\ \mathbb{R}^n \arrow[r, "\text{diff}"] & \mathbb{R}^m \end{tikzcd}$$

Rmk. Still well-defined w.r.t. coordinate.

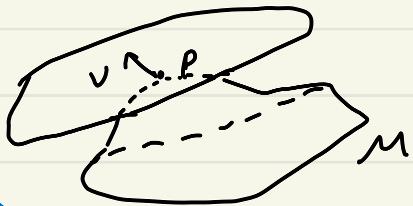
Rmk. $\gamma: (a,b) \to M$ is differentiable curve if ---- .

Def. $F: N \to M$ diffeomorphism if F is homeo, diff. so is F^{-1}.

14-Sept-2023.

Def. M is n-dimension d.f. If $\{U_\alpha, \varphi_\alpha\}$. $\varphi_\alpha: U_\alpha \xrightarrow{\sim} \mathbb{R}^n$

Def. Tangent space at $p \in M$



if $\forall f: \mathbb{R}^n \to \mathbb{R}$, $\frac{\partial f}{\partial v}$ is determined $\Rightarrow v \in T_pM$

$$\left\{\begin{array}{l} \left.\frac{\partial f}{\partial v}\right|_p := \frac{d}{dt} f(p+vt)\Big|_{t=0} \\ \\ \text{or } \gamma: (-\varepsilon, \varepsilon) \to M,\ 0 \mapsto p \text{ diff curve s.t. } \gamma'(0) = v. \\ \Rightarrow \left.\frac{\partial f}{\partial v}\right|_p := \frac{d}{dt} f\circ\gamma(t)\Big|_{t=0} \end{array}\right.$$

Rmk. $\gamma_1, \gamma_2: (-\varepsilon, \varepsilon) \to M$, $0 \mapsto p$.

$\gamma_1'(0) = \gamma_2'(0) \Leftrightarrow (f\circ\gamma_1)'(0) = (f\circ\gamma_2)'(0)$, $\forall f: \mathbb{R}^n \to \mathbb{R}$.

Def $\gamma_i: (-\varepsilon_i, \varepsilon_i) \to M$ are 2 diff curve. If

① $\gamma_i'(0) = v$, $i = 1, 2$

② $\forall f: M \to \mathbb{R}$ diff, $(f\circ\gamma_1)'(0) = (f\circ\gamma_2)'(0)$. defined via dual space.

then γ_1, γ_2 are tangent at $p \in M$.

Def $\boxed{T_pM}$ $D_p: \{\gamma: (-\varepsilon, \varepsilon) \to M.\ \text{diff}.\ \gamma(0) = p\}$.

$D_p / \sim_{\text{tangent}} =: T_pM$, tangent space.

Question $D_p/\sim$ a linear space? canonical space?

prop. $(U, \varphi, (x^1, \cdots, x^n))$, $\varphi(p) = (0, \cdots, 0)$

① $[\gamma_1] + [\gamma_2] \triangleq [\varphi^{-1}(\varphi\circ\gamma_1 + \varphi\circ\gamma_2)]$

To see $\varphi^{-1}(\varphi\circ\gamma_1 + \varphi\circ\gamma_2)$ and $\bar\varphi^{-1}(\bar\varphi\circ\bar\gamma_1 + \bar\varphi\circ\bar\gamma_2)$ are tangent, i.e.

$$[f(\varphi^{-1}(\varphi\circ\gamma_1 + \varphi\circ\gamma_2))]'(0) = [\underbrace{f(\varphi^{-1}}_{\text{trivial}}(\varphi\circ\bar\gamma_1 + \varphi\circ\bar\gamma_2))]'(0) = [f(\underbrace{\bar\varphi^{-1}}_{\text{use } f\circ\bar\varphi^{-1} = f\circ\varphi^{-1}\circ\varphi\circ\bar\varphi^{-1}.}(\bar\varphi\circ\bar\gamma_1 + \bar\varphi\circ\bar\gamma_2))]'(0)$$

② $\lambda[\gamma_1] \triangleq [\varphi^{-1}(\lambda\varphi\circ\gamma)]$ Similar to ①.

Prop. Now take $\gamma_i: (-\varepsilon, \varepsilon) \to M$. by $\gamma_i(t) = \varphi^{-1}((0, \cdots, 0, \overset{i\text{th}}{t}, 0, \cdots, 0)) \in M$

Then $\{[\gamma_i]\}_{1 \le i \le n}$ is a basis of T_pM.

① $[\gamma] \in D_p/\sim$, $(\varphi\circ\gamma)(t) = (x^1(t), \cdots, x^n(t))$, $a_i := \frac{dx^i}{dt}\Big|_{t=0}$

Then $[\gamma] = [\varphi^{-1}(x^1(t), \cdots, x^n(t))] = [\varphi^{-1}(a_1 t, \cdots, a_n t)] = \sum_{i=1}^n a_i [\gamma_i]$

② $\{[\gamma_i]\}_{1\le i\le n}$ is linear independent.

$(0,\cdots,0,1,0,\cdots,0)^T$

Rmk. $[\gamma_i] = \frac{\partial}{\partial x^i}$ $\quad (f\circ\gamma_i)'(0) = \left((f\circ\varphi^{-1})\circ(\varphi\circ\gamma_i)\right)'(0)$

$$= \left(\frac{\partial(f\circ\varphi^{-1})}{\partial x^1}, \cdots, \frac{\partial(f\circ\varphi^{-1})}{\partial x^n}\right)\cdot\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix} = \frac{\partial(f\circ\varphi^{-1})}{\partial x^i}(\varphi(p)) = \frac{\partial f}{\partial x^i}(0)$$

Prop Change basis $\left\{\frac{\partial}{\partial x^i}\right\} \to \left\{\frac{\partial}{\partial y^j}\right\}$. | Notation $(f\circ\gamma)'(0) = \frac{\partial f}{\partial v} =: vf$.

$(U,\varphi,(x^1,\cdots,x^n))$ and $(V,\psi,(y^1,\cdots,y^n))$

$$\Rightarrow \frac{\partial}{\partial y^i} = \sum_{k=1}^{n} \frac{\partial(\varphi\circ\psi^{-1})^k}{\partial y^i}\Big|_0 \cdot \frac{\partial}{\partial x^k}$$

pf. $\forall f\in C^\infty(M)$ $\quad \frac{\partial}{\partial x^i}f\Big|_p = \frac{\partial(f\circ\varphi^{-1})}{\partial x^i}\Big|_{\varphi(p)} = \frac{\partial(f\circ\psi^{-1}\circ\psi\circ\varphi^{-1})}{\partial x^i}\Big|_{\varphi(p)}$

$$= \sum_{k=1}^{n} \frac{\partial(f\circ\psi^{-1})}{\partial y^k}\Big|_{\underset{0}{\psi(p)}} \cdot \frac{\partial(\psi\circ\varphi^{-1})^k}{\partial x^i}\Big|_{\underset{0}{\varphi(p)}}$$

$$= \sum_{k=1}^{n} \frac{\partial}{\partial y^k}f\Big|_p \cdot \frac{\partial(\psi\circ\varphi^{-1})^k}{\partial x^i}.$$

18-Sept-2023

Def. We equal $\frac{\partial f}{\partial v}\Big|_p$, $(f\circ\gamma)'(0)$, $vf|_p$, $[\gamma]f|_p$ $\quad (f\circ\gamma_i)'(0) = \frac{\partial f}{\partial x^i}\Big|_p$

$\hookrightarrow$ $\{[\gamma_i]\}_{1\le i\le n}$ is the standard basis w.r.t. $(U,\varphi,(x^1,\cdots,x^n))$

Def (Co tangent space T_p^*M) $\quad T_p^*M := (T_pM)^*$ Given $(U,\varphi,(x^1,\cdots,x^n))$

with basis $\{x^i\}_{1\le i\le n}$, $\quad x^i(q) := (\varphi(q))^i$, $q\in U$ (coordinate function)

Prop $\left\langle \frac{\partial}{\partial x^i}, x^j\right\rangle = \frac{\partial}{\partial x^i}x^j\Big|_p = \frac{\partial(x^j\circ\varphi^{-1})}{\partial x^i}\Big|_p = \frac{\partial(\varphi(\varphi^{-1}))^j}{\partial x^i}\Big|_p = \delta_i^j$

Def $F: M^n \to N^m$ (diff)

$dF_p: T_pM \to T_{F(p)}N$

$v \longmapsto \frac{d}{dt}F(p+tv)$

$$\begin{array}{ccccc}
\gamma & \longrightarrow & [\gamma] & \gamma'(0) & = v\\
F\downarrow & & F\downarrow & \downarrow dF_p & \downarrow dF_p\\
F\circ\gamma & \longrightarrow & [F\circ\gamma] & (F\circ\gamma)'(0) & = (dF)(v)
\end{array}$$

Prop $\left(dF_p\left(\frac{\partial}{\partial x^i}\right)\right)(y^\alpha) = [F\circ\gamma_i](y^\alpha) = \frac{d}{dt}\left(y^\alpha(F\circ\gamma_i)\right)\Big|_{t=0} = \frac{d}{dt}\left(\psi(F\circ\gamma_i)\right)^\alpha$

$$= \frac{d}{dt}\left((\psi\circ F\circ\varphi^{-1})\circ(\varphi\circ\gamma_i)\right)^\alpha = \sum_{k=1}^{n}\frac{\partial(\psi\circ F\circ\varphi^{-1})^\alpha}{\partial x^k}\Big|_{\varphi(p)} \cdot \boxed{\frac{d(\varphi\circ\gamma_i)^k}{dt}\Big|_0} = \delta_i^k$$

$$= \frac{\partial(\psi\circ F\circ\varphi^{-1})^\alpha}{\partial x^i}$$

$$\Rightarrow \left(dF_p\left(\frac{\partial}{\partial x^i}\right)\right) = \sum_{\beta=1}^{n}\frac{\partial(\psi\circ F\circ\varphi^{-1})^\beta}{\partial x^i}\frac{\partial}{\partial y^\beta} =: \sum \frac{\partial F^\beta}{\partial x^i}\cdot\frac{\partial}{\partial y^\beta}.$$

Rmk. $dF_p = \left(\frac{\partial F^j}{\partial x^i} \right)_{i,j}$

Prop. $\mathrm{Hom}(T_pM, T_{f(p)}N) \simeq T_p^*M \otimes T_pN$

$$\begin{array}{ccc} \Downarrow & & \Downarrow \\ dF_p & \longmapsto & \sum \frac{\partial F^\alpha}{\partial x^i} dx^i \otimes \frac{\partial}{\partial y^\alpha} \end{array}$$

21 - Sept - 2023

Ex. $F: M^n \longrightarrow N^m$

$$\begin{array}{ccc} dF_p: T_pM & \longrightarrow & T_{F(p)}N \\ [\gamma] & \longmapsto & [F\circ\gamma] \end{array} \qquad dF_p = \sum_{\substack{1\le i\le n \\ 1\le \alpha\le m}} \frac{\partial(\psi\circ F\circ\varphi^{-1})^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha} \otimes dx^i$$

Def The rank of F is $\mathrm{rank}(dF_p)$.

(resp. global)

Def $F: M^n \to N^m$ ① $\mathrm{rank}(dF_p) = n$ $(\le m)$, F is an immersion at p

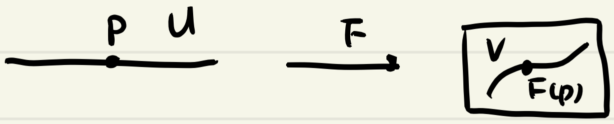
②

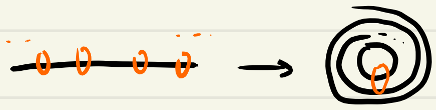
Rmk. locally $F: U_p \to V_{F(p)}$, dF is immersion at p

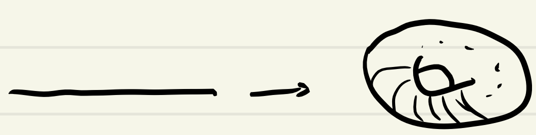
Consider $F(x^1, \dots, x^n) = (y^1(x^1, \dots, x^n), \dots, y^m(x^1, \dots, x^n))$.

|| inverse function thm.

$V \Leftarrow (\bar{y}^1(x^1, \dots, x^n), \dots, \bar{y}^n(x^1, \dots, x^n), \text{constant})$



Eg. $(1, +\infty) \to \mathbb{R}^2$, 

$\mathbb{R} \to T^2$, 

Def The pair (M, F) is called an immersed submanifold if F is an immersion

embedding embedding

Thm Injective immersion + compact domain $\Rightarrow$ embedding.

Def (tangent bundle) $TM = \bigcup_{p\in M} T_pM$

Def (Vector field) $V: M \longrightarrow TM$. $p \longmapsto \bullet \in T_pM$.
we say V is differentiable if v^i's are differentiable, $V(q) = \sum \frac{\partial}{\partial x^i} \cdot V^i(q)$.

Def (Lie Bracket) $\mathfrak{X}(M) = (\{\text{diff vector bundle on } M\}, [\,,\,])$
$[X, Y]f = X(Yf) - Y(Xf)$. (Ex)

Eg $[fX, gY](h) = (f(Xg)Y - g(Yf)X)h$

Eg $X = \sum X^i \frac{\partial}{\partial x^i}$, $Y = \sum Y^j \frac{\partial}{\partial x^j}$ $[X, Y] = \sum X^i \frac{\partial Y^j}{\partial x^i} \cdot \frac{\partial}{\partial x^j} - Y^i \frac{\partial X^j}{\partial x^i} \cdot \frac{\partial}{\partial x^j}$.