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Triangulated Categories

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Abstract

In this paper we discuss triangulated categories, their properties and some examples. We begin by providing a brief introduction to the theory of additive categories, on which we define the structure of triangulated categories, and some elementary properties are proven. Moving on, we discuss the localization of a category and we prove that under the right conditions, the localization of a triangulated category is triangulated. The final chapter is devoted to constructing the homotopy cate-gory and the derived category of an abelian category, and it is proven that these categories are triangulated.

Contents

1 Introduction

One of the most important concepts in homological algebra is the notion of exact sequence. While exact sequences make sense in any category where kernels and images make sense, the most convenient setting for this is in the theory of abelian categories. These categories serve as a generalization of the category of modules over a ring, and almost all concepts and results from classical homological algebra for modules can be expressed in this setting.

For many computational purposes, one replaces an object in an abelian category $\mathcal{A}$ with a projective or injective resolution. This procedure allows one to define more complicated structures such as derived functors, and most importantly these constructions are invariant under the choice of resolutions. A natural question to ask is to what extent we may identify an object with its resolutions, as well as whether this correspondance can be captured explicitly in a categorical language.

A problem with resolutions of an object is that they are not unique, not even up to isomorphism of chain complexes. However, they are equivalent in a weaker sense if we consider chain morphisms up to homotopy. Indeed, if P and P' are projective resolutions of an object X in $\mathcal{A}$, then there are chain morphisms $f : P \to P'$ and $g : P' \to P$ such that $g \circ f$ and $f \circ g$ are homotopic to the corresponding identity morphisms. For this reason, it makes sense to consider the homotopy category $\mathbf{K}(\mathcal{A})$, that is the category of chain complexes in $\mathcal{A}$ where homotopic chain morphisms are identified.

We may also embed the category $\mathcal{A}$ into the corresponding homotopy category by considering each object of $\mathcal{A}$ as a chain complex concentrated in degree zero. Doing this allows us to express a projective resolution of an object X as a chain morphism $P \to X$. Since we want to identify an object with its projective resolution, it would be natural to ask whether this morphism is an isomorphism in the homotopy category. Unfortunately this is not the case in general, but it turns out that the chain morphism $P \to X$ induces isomorphisms on homology. We call such chain morphisms quasi-isomorphisms. This enables us to construct a category $\mathbf{D}(\mathcal{A})$, called the derived category of $\mathcal{A}$, from the homotopy category by formally adding an inverse to each quasi-isomorphism through a process called localization. In this category, each object of $\mathcal{A}$ is isomorphic to its projective resolution as desired.

While these categories solidify the connection between an object and its resolutions, a crucial drawback is that neither the homotopy category nor the derived category are abelian. In fact, the very notion of exactness cannot be made sense of in these categories. Fortunately, there is a weaker structure present in the form of a triangulated category.

In a triangulated category, short exact sequences are replaced by a class of "distinguished triangles", that is sequences of the form $X \to Y \to Z \to \Sigma X$ for some functor $\Sigma : \mathcal{A} \to \mathcal{A}$, subject to certain axioms. In particular these triangles capture the common phenomenon that short exact sequences give rise to long exact sequences. As such, triangulated categories have found applications far beyond the scope of homological algebra in fields such as representation theory, algebraic topology and algebraic geometry.

2 Triangulated Categories

2.1 Additive Categories

We begin our treatment of triangulated categories by describing the general setting for the theory, namely the theory of additive categories. Additive categories are one of the simplest and most general settings for studying homological algebra. While they are too basic to be a rich field of study on their own, they admit enough structure to allow for the construction of a number of familiar concepts from classical homological algebra. This allows them to serve as the foundation on which more complicated structures can be defined.

Definition 2.1.1. Let $\mathcal{A}$ be a category. An object 0 in $\mathcal{A}$ is said to be a *zero object* if for any other object X, there is precisely one morphism $X \to 0$ and $0 \to X$.

It is easily seen that any two zero objects are isomorphic, so whenever one exists it is referred to as *the* zero object. Moreover if $\mathcal{A}$ has a zero object, then for any pair of objects X, Y in $\mathcal{A}$ there is a morphism $X \to Y$ given by the composite $X \to 0 \to Y$ which we call the *zero morphism* from X to Y.

Definition 2.1.2. Let $\mathcal{A}$ be a category. We say that $\mathcal{A}$ is *enriched in abelian groups* if the following axioms are satisfied:

(i) For every pair X, Y of objects in $\mathcal{A}$, the set $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ has the structure of an abelian group, written additively.

(ii) For every triple X, Y, Z of objects in $\mathcal{A}$, the composition map:

$$\circ : \mathrm{Hom}_{\mathcal{A}}(Y, Z) \times \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$$

is biadditive, that is:

$$g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2$$
$$(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f$$

for all $f, f_1, f_2 : X \to Y$ and all $g, g_1, g_2 : Y \to Z$ in $\mathcal{A}$.

If $\mathcal{A}$ is a category enriched in abelian groups, then a zero object in $\mathcal{A}$ satisfies $1_0 = 0$. It is easily seen that the converse holds, namely if $1_X = 0$ for some object X then $X \cong 0$.

Definition 2.1.3. Let $\mathcal{A}$ be a category enriched in abelian groups. Given objects X and Y in $\mathcal{A}$, their *biproduct* is a diagram:

$$\begin{tikzcd} X \arrow[r, shift left, "i_X"] & X \oplus Y \arrow[l, shift left, "p_X"] \arrow[r, shift right, "p_Y"'] & Y \arrow[l, shift right, "i_Y"'] \end{tikzcd}$$

such that the following identities are satisfied:

$$p_X i_X = 1_X \ , \qquad p_Y i_Y = 1_Y \ , \qquad p_X i_Y = 0 \ , \qquad p_Y i_X = 0$$
$$i_X p_X + i_Y p_Y = 1_{X \oplus Y}$$

The morphisms p and i are called the canonical projections and inclusions respectively.

Lemma 2.1.4. ([2] p.194-195) The biproduct satisfies the following universal property. For every object Z in $\mathcal{A}$ and every pair of morphisms $h_X : X \to Z$, $h_Y : Y \to Z$, there is a unique morphism $h : X \oplus Y \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[dr, bend right, "h_X"'] & X \oplus Y \arrow[d, dashed, "h"] & Y \arrow[l, "i_Y"'] \arrow[dl, bend left, "h_Y"] \\ & Z & \end{tikzcd}$$

Proof. Define $h : X \oplus Y \to Z$ by taking $h = h_X p_X + h_Y p_Y$. Then

$$\begin{aligned} hi_X &= (h_X p_X + h_Y p_Y) i_X \\ &= h_X p_X i_X + h_Y p_Y i_X \\ &= h_X p_X i_X = h_X \end{aligned}$$

Similarly, one obtains $hi_Y = h_Y$ so h makes the diagram above commute. For uniqueness, suppose h, h' are two such morphisms. Then by assumption $hi_X = h_X = h'i_X$ and $hi_Y = h_Y = h'i_Y$ so that $hi_X p_X = h'i_X p_X$ and $hi_Y p_Y = h'i_Y p_Y$. Then:

$$h = h(i_X p_X + i_Y p_Y) = h'(i_X p_X + i_Y p_Y) = h'$$

□

Using this property, it is easily seen that the biproduct is unique up to isomorphism whenever it exists.

Biproducts also satisfy the dual universal property, that is for every object Z and every pair of morphisms $f_X : Z \to X$, $f_Y : Z \to Y$ there is a unique morphism $f : Z \to X \oplus Y$ such that the following diagram commutes:

$$\begin{tikzcd} & Z \arrow[dl, bend right, "f_X"'] \arrow[d, dashed, "f"] \arrow[dr, bend left, "f_Y"] & \\ X & X \oplus Y \arrow[l, "p_X"] \arrow[r, "p_Y"'] & Y \end{tikzcd}$$

The proof is analogous to the proof of 2.1.4, so it is omitted.

Now let $f : X \to X'$ and $g : Y \to Y'$ be morphisms. By the universal property, there is a unique morphism $f \oplus g : X \oplus Y \to X' \oplus Y'$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "f"'] & X \oplus Y \arrow[d, dashed, "f \oplus g"] & Y \arrow[l, "i_Y"'] \arrow[d, "g"] \\ X' \arrow[r, "i_{X'}"'] & X' \oplus Y' & Y' \arrow[l, "i_{Y'}"] \end{tikzcd}$$

is commutative. In particular, one sees that $1_X \oplus 1_Y = 1_{X \oplus Y}$ and $0 \oplus 0 = 0$. Moreover, given additional morphisms $f' : X' \to X''$, $g' : Y' \to Y''$ we have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "i_X"] \arrow[d, "f"'] & X \oplus Y \arrow[d, dashed, "f\oplus g"] & Y \arrow[l, "i_Y"'] \arrow[d, "g"] \\
X' \arrow[r, "i_{X'}"'] \arrow[d, "f'"'] & X' \oplus Y' \arrow[d, dashed, "f'\oplus g'"] & Y' \arrow[l, "i_{Y'}"] \arrow[d, "g'"] \\
X'' \arrow[r, "i_{X''}"'] & X'' \oplus Y'' & Y'' \arrow[l, "i_{Y''}"]
\end{tikzcd}$$

Then by the universal mapping property, we have $f'f \oplus g'g = (f' \oplus g') \circ (f \oplus g)$. This implies that the biproduct is functorial if $\mathcal{A}$ has all biproducts.

For any finite family $X_1, ..., X_n$ of objects in $\mathcal{A}$, we can consider the iterated biproduct $\bigoplus_{k=1}^n X_k$. This object satisfies similar universal properties to the ones described in Lemma 2.1.4. Now, given objects $X_1, ..., X_n$ and $Y_1, ..., Y_m$, consider any family of morphisms $f_{lk} : X_k \to Y_l$. Then for each k, there is an induced morphism $f_k : X_k \to \bigoplus_{l=1}^m Y_l$ which in turn induces a morphism $f : \bigoplus_{k=1}^n X_k \to \bigoplus_{l=1}^m Y_l$. Conversely, given any such f, one can recover the family f_{lk} by taking the composites $f_{lk} = p_l f i_k$. Then morphisms between biproducts are uniquely determined by the morphisms between their factors, and we represent a morphism $f : \bigoplus_{k=1}^n X_k \to \bigoplus_{l=1}^m Y_l$ as a matrix of the f_{kl}.

$$f = \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{pmatrix}$$

Moreover, composition of morphisms corresponds to ordinary matrix multiplication in this notation.

Definition 2.1.5. ([2] p.196) A category $\mathcal{A}$ is said to be *additive* if it satisfies the following axioms:

(i) $\mathcal{A}$ is enriched in abelian groups.

(ii) There exists a zero object in $\mathcal{A}$.

(iii) The biproduct of any pair of objects in $\mathcal{A}$ exists.

Remark. In [1] additive categories are defined as categories satisfying axioms (i) and (ii), but where (iii) is replaced by the existance of *coproducts*. A coproduct is an object $X+Y$ equipped with morphisms $i_X : X \to X+Y$, $i_Y : Y \to X+Y$. satisfying the universal mapping property of Lemma 2.1.4. Initially, this makes our definition of additive category seem stronger than the one given in [1] but it can be shown that these two definitions are equivalent. In fact if a category satisfies (i) and (ii) then every coproduct admits the structure of a biproduct. See the full theorem VIII.2 in [2] for more details.

Example 2.1.6.

(i) For any ring R, let R-**Mod** and **Mod**-R denote the categories of left and right R-modules respectively. These categories are additive, with the zero object being given by the trivial module and the biproduct being given by the direct sum.

(ii) The category **Ab** of abelian groups is additive. In fact this is a special case of the previous example, with $R = \mathbb{Z}$.

(iii) Let K be a field and let $\mathbf{Vect}(K)$ be the category of vector spaces over K. The full subcategory $\mathbf{Vect}_2(K)$ of $\mathbf{Vect}(K)$ spanned by even-dimensional vector spaces is additive.

(iv) For any additive category $\mathcal{A}$ the opposite category $\mathcal{A}^{\mathbf{op}}$ is additive. The biproducts in $\mathcal{A}^{\mathbf{op}}$ are the same as in $\mathcal{A}$, but where the projections and inclusions are switched.

For every additive category $\mathcal{A}$, the notions of chain complex and chain morphism in $\mathcal{A}$ can be defined in complete analogy to the corresponding concepts in module theory. This gives rise to a category $\mathbf{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$, which leads to the following proposition:

Proposition 2.1.7. ([1], p.3-4) $\mathbf{Ch}(\mathcal{A})$ is an additive category.

Proof. Too see that $\mathbf{Ch}(\mathcal{A})$ is enriched in abelian groups, consider chain morphisms $f, g : A \to B$ and define $f + g : A \to B$ degreewise by $(f + g)_i = f_i + g_i$. Clearly, this defines an abelian group structure on $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{A})}(A, B)$ and biadditivity of composition follows by definition. Moving on, we see that the zero complex is a zero object in $\mathbf{Ch}(\mathcal{A})$ so it remains to determine the biproduct.

Given complexes A, B, define $A \oplus B$ degreewise as $(A \oplus B)_i = A_i \oplus B_i$ and where the boundary maps $d_i^{A \oplus B}$ are given by $d_i^A \oplus d_i^B$. By functoriality:

$$(d_i^A \oplus d_i^B) \circ (d_{i+1}^A \oplus d_{i+1}^B) = d_i^A d_{i+1}^A \oplus d_i^B d_{i+1}^B = 0 \oplus 0 = 0$$

so we conclude that $A \oplus B$ is a complex.

Now, define chain morphisms $p_A : A \oplus B \to B$ and $i_A : A \to A \oplus B$ as the canonical projections and inclusions in each degree. Then clearly these morphisms satisfy the identities of Definition 2.1.3 so we conclude that $A \oplus B$ is the biproduct in $\mathbf{Ch}(\mathcal{A})$. Thus $\mathbf{Ch}(\mathcal{A})$ is additive. $\square$

We shall now define the appropriate notion of functor between additive categories.

Definition 2.1.8. ([2] p.197) An *additive functor* is a functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories such that

$$F(f + g) = F(f) + F(g)$$

for every $f, g : X \to Y$ in $\mathcal{A}$. In other words, F acts as a group homomorphism on each Hom-set.

Proposition 2.1.9. ([2] p.197) If F is additive, then

$$F(X \oplus Y) \cong F(X) \oplus F(Y)$$

for all $X, Y \in \mathcal{A}$.

Proof. By definition $F(1_X) = 1_{F(X)}$ and $F(0) = 0$, so by applying F to the biproduct identities of Definition 2.1.3, we obtain:

$$F(p_X)F(i_X) = 1_{F(X)} \qquad F(p_Y)F(i_Y) = 1_{F(Y)} \qquad F(p_X)F(i_Y) = 0 \qquad F(p_Y)F(i_Y) = 0$$

$$F(i_X)F(p_X) + F(i_Y)F(p_X) = 1_{F(X\oplus Y)}$$

so that

$$F(X) \underset{F(p_X)}{\overset{F(i_X)}{\rightleftarrows}} F(X\oplus Y) \underset{F(p_Y)}{\overset{F(i_Y)}{\leftrightarrows}} F(Y)$$

is a biproduct diagram. □

We finish this section with some examples of additive functors. For any additive category $\mathcal{A}$, the functors $\mathrm{Hom}_{\mathcal{A}}(-, X)$ and $\mathrm{Hom}_{\mathcal{A}}(X, -)$ are additive since composition is biadditive.

For any object X in $\mathcal{A}$, there is a functor $X \oplus - : \mathcal{A} \to \mathcal{A}$ where a morphism f is mapped to $1_X \oplus f$. This functor is additive by the universal property of the biproduct, that is $1_X \oplus (f + g) = 1_X \oplus f + 1_X \oplus g$.

If M is a right R-module, then the tensor product $M \otimes_R -$ is an additive functor from R-**Mod** to the category of abelian groups. Similarly, for any left R-module N, the functor $- \otimes_R N : \mathbf{Mod}\text{-}R \to \mathbf{Ab}$ is additive.

2.2 Triangulated Categories

In this section, we shall introduce the notion of triangulated categories and some elementary properties of these structures will be proven. We shall mainly follow the definitions and proofs as they are given in [1] unless it is stated otherwise.

First, let $\mathcal{A}$ be an additive category and let $\Sigma : \mathcal{A} \to \mathcal{A}$ be an additive endofunctor. A *triangle* in $\mathcal{A}$ with respect to Σ is a sequence of the form:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

A *morphism of triangles* is a triple (u, v, w) of morphisms such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[r, "h"] \arrow[d, "w"'] & \Sigma X \arrow[d, "\Sigma u"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X' \end{tikzcd}$$

With these two concepts, we may state the definition of a triangulated category:

Definition 2.2.1. ([1] p.11-12) A *triangulated category* is an additive category $\mathcal{A}$ equipped with the following data:

I. An additive isomorphism $\Sigma : \mathcal{A} \to \mathcal{A}$ of categories, called the *shift functor*.

II. A class of triangles in $\mathcal{A}$ with respect to Σ, called *distinguished triangles*.

The distinguished triangles are required to satisfy the following axioms:

(A0) Any triangle isomorphic to a distinguished triangle is distinguished.

(A1) For every object $X \in \mathcal{A}$, the following triangle is distinguished:

$$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \Sigma X$$

(A2) Every morphism $f : X \to Y$ in $\mathcal{A}$ can be extended to a distinguished triangle:

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X$$

(A3) (The rotation axiom) The triangle:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is distinguished if and only if the following triangle is distinguished:

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

(A4) For any diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[r, "h"] & \Sigma X \arrow[d, "\Sigma u"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X' \end{tikzcd}$$

where the left square commutes and the rows are distinguished triangles, there is a (not necessarily unique) morphism $w : Z \to Z'$ so that the triple (u, v, w) is a morphism of triangles.

(A5) (The octahedral axiom) Given distinguished triangles:

$$X \xrightarrow{f} Y \longrightarrow Z' \longrightarrow \Sigma X$$

$$Y \xrightarrow{g} Z \longrightarrow X' \longrightarrow \Sigma Y$$

$$X \xrightarrow{gf} Z \longrightarrow Y' \longrightarrow \Sigma X$$

there is a distinguished triangle:

$$Z' \longrightarrow Y' \longrightarrow X' \longrightarrow \Sigma Z'$$

so that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "1_X"'] & Y \arrow[r] \arrow[d, "g"] & Z' \arrow[r] \arrow[d] & \Sigma X \arrow[d, "1_{\Sigma X}"] \\ X \arrow[r, "gf"] \arrow[d, "f"'] & Z \arrow[r] \arrow[d, "1_Z"] & Y' \arrow[r] \arrow[d] & \Sigma X \arrow[d, "\Sigma f"] \\ Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "1_{X'}"] & \Sigma Y \arrow[d] \\ Z' \arrow[r] & Y' \arrow[r] & X' \arrow[r] & \Sigma Z' \end{tikzcd}$$

Remark. Out of all the axioms, the octahedral axiom is by far the most unintuitive. It has many equivalent forms and while we shall not discuss them in this paper, the following argument should provide some intuition on why it is a neccesary, or at the very least useful condition. Begin by taking any commutative triangle:

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z
\end{tikzcd}$$

i.e $h = g \circ f$. Using (A2), we may extend the morphisms f, g, gf into distinguished triangles, forming the following diagram:

$$\begin{tikzcd}
 & & Z' \arrow[rr, bend left, "f''"] & & \Sigma X \arrow[dr, "\Sigma f"] & & \\
 & Y \arrow[ur, "f'"] \arrow[dr, "g"'] & & Y' \arrow[ur, "h''"'] & & \Sigma Y \arrow[dr, "\Sigma f'"] & \\
X \arrow[ur, "f"] \arrow[rr, bend right, "h"'] & & Z \arrow[ur, "h'"'] \arrow[rr, bend right, "g'"'] & & X' \arrow[ur, "g''"'] & & \Sigma Z'
\end{tikzcd}$$

Pictorially, this diagram suggests the existance of a fourth distinguished triangle $Z' \to Y' \to X' \to \Sigma Z'$ making the whole diagram commute. Indeed, by (A4), there is a morphism $j : Z' \to Y'$ making the top left rectangle commute, and by (A2) j can be extended to a distinguished triangle:

$$Z' \xrightarrow{j} Y' \xrightarrow{j'} X'' \xrightarrow{j''} \Sigma Z'$$

However, without the octahedral axiom there is no guarantuee that this triangle can be chosen such that $X' \cong X''$ and such that the following diagram commutes:

$$\begin{tikzcd}
 & & Z' \arrow[rr, bend left, "f''"] \arrow[dr, "j"'] & & \Sigma X \arrow[dr, "\Sigma f"] & & \\
 & Y \arrow[ur, "f'"] \arrow[dr, "g"'] & & Y' \arrow[ur, "h''"'] \arrow[dr, "j'"'] & & \Sigma Y \arrow[dr, "\Sigma f'"] & \\
X \arrow[ur, "f"] \arrow[rr, bend right, "h"'] & & Z \arrow[ur, "h'"'] \arrow[rr, bend right, "g'"'] & & X' \arrow[ur, "g''"'] \arrow[rr, bend right, "j''"'] & & \Sigma Z'
\end{tikzcd}$$

Remark. If $\mathcal{A}$ is a triangulated category, then we may equip the opposite category $\mathcal{A}^{\mathbf{op}}$ with a triangulated structure. To see this, if we apply the rotation axiom in reverse to any distinguished triangle, we obtain a distinguished triangle of the form:

$$\Sigma^{-1} Z \xrightarrow{-\Sigma^{-1} h} X \xrightarrow{f} Y \xrightarrow{g} Z$$

Passing to the opposite category, we obtain a triangle with respect to Σ^{-1}:

$$Z \xrightarrow{g} Y \xrightarrow{f} X \xrightarrow{-\Sigma^{-1}h} \Sigma^{-1}Z$$

and we say that a triangle is distinguished in $\mathcal{A}^{\mathbf{op}}$ if it is obtained from distinguished triangles in $\mathcal{A}$ in this fashion. It is now straighforward to check that these triangles satisfy the axioms (A0)-(A5) so $\mathcal{A}^{\mathbf{op}}$ is triangulated.

We shall now prove some elementary results about triangulated categories, which will highlight the relationship between distinguished triangles and exact sequences in classical homological algebra.

Proposition 2.2.2. ([1] p.14) For any distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

we have $g \circ f = 0$ and $h \circ g = 0$.

Proof. By the rotation axiom, the triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is distinguished so that it suffices to prove that only the first two morphisms in any triangle compose to 0. Now, consider the following diagram:

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "g"'] & Z \arrow[r, "h"] \arrow[d, "1"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y \arrow[d, "\Sigma g"] \\
Z \arrow[r, "1"'] & Z \arrow[r] & 0 \arrow[r] & \Sigma Z
\end{tikzcd}$$

The bottom row is distinguished by (A1) so by (A4) there is a morphism $\Sigma X \to 0$ which completes the diagram. Thus $\Sigma g \circ (-\Sigma f) = -\Sigma(g \circ f) = 0$. Since Σ is an additive isomorphism, this implies that $g \circ f = 0$. □

This result shows that every distinguished triangle is a chain complex, but they need not be exact in the traditional sense since kernels and images need not be present in $\mathcal{A}$. However, distinguished triangles do satisfy a weaker form of exactness which is demonstrated by the following result:

Proposition 2.2.3. ([1] p.14-15) For any object $U \in \mathcal{A}$ and any distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

there is a long exact sequence of abelian groups:

$$\begin{tikzcd}
\ldots \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(U, \Sigma^i X) \arrow[r, "(\Sigma^i f)_*"] & \mathrm{Hom}_{\mathcal{A}}(U, \Sigma^i Y) \arrow[r, "(\Sigma^i g)_*"] & \mathrm{Hom}_{\mathcal{A}}(U, \Sigma^i Z) \arrow[dll, "(\Sigma^i h)_*"] \\
& \mathrm{Hom}_{\mathcal{A}}(U, \Sigma^{i+1} X) \arrow[r, "(\Sigma^{i+1} f)_*"] & \mathrm{Hom}_{\mathcal{A}}(U, \Sigma^{i+1} Y) \arrow[r, "(\Sigma^{i+1} g)_*"] & \mathrm{Hom}_{\mathcal{A}}(U, \Sigma^{i+1} Z) \arrow[r] & \ldots
\end{tikzcd}$$

where f_* denotes the morphism $\mathrm{Hom}_{\mathcal{A}}(U, f)$ defined by $f_*(u) = f \circ u$.

Proof. By the rotation axiom, it suffices to check that the sequence:

$$\mathrm{Hom}_{\mathcal{A}}(U, \Sigma^i X) \xrightarrow{\Sigma^i f_*} \mathrm{Hom}_{\mathcal{A}}(U, \Sigma^i Y) \xrightarrow{\Sigma^i g_*} \mathrm{Hom}_{\mathcal{A}}(U, \Sigma^i Z)$$

is exact. By Proposition 2.2.2, we know that $\Sigma^i g \circ \Sigma^i f = 0$ so in particular $\Sigma^i g_* \circ \Sigma^i f_* = 0$. Now, take $u \in \mathrm{Ker}(\Sigma^i g_*)$. We want to find some morphism $U \to \Sigma^i X$ which maps to u under $\Sigma^i f_*$. To achieve this, consider the following diagram:

$$\begin{tikzcd}
\Sigma^{-i}U \arrow[r] \arrow[d, "\Sigma^{-i}u"'] & 0 \arrow[r] \arrow[d] & \Sigma^{-i+1}U \arrow[r, "-1"] & \Sigma^{-i+1}U \arrow[d, "\Sigma^{-i+1}u"] \\
Y \arrow[r, "g"'] & Z \arrow[r, "h"'] & \Sigma X \arrow[r, "-\Sigma f"'] & \Sigma Y
\end{tikzcd}$$

The upper row is distinguished by (A1) and (A3), and the bottom row is distinguished by (A3). Since $\Sigma^i g \circ u = 0$ by assumption on u, we have $g \circ \Sigma^{-i} u = 0$ so that the left square commutes. Then by (A4) there is a morphism $v : \Sigma^{-i+1}U \to \Sigma X$ making the whole diagram commutative. Thus $-\Sigma f \circ v = -\Sigma^{-i+1}u$ so that $\Sigma^i f \circ \Sigma^{i-1} v = u$. Then $\Sigma^i f_*(\Sigma^{i-1} v) = u$ so in conclusion $\mathrm{Im}(\Sigma^i f_*) = \mathrm{Ker}(\Sigma^i g_*)$. □

If one regards the objects as chain complexes, then this result is reminiscent of the snake lemma in classical homological algebra. In this sense, it is valuable to think of the groups $\mathrm{Hom}_{\mathcal{A}}(U, \Sigma^i X)$ as generalized Ext-groups.

Remark. There is also the dual version of the above result, that is for any U in $\mathcal{A}$ and any distinguished triangle $X \to Y \to Z \to \Sigma X$, there is a long exact sequence of groups:

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(\Sigma^i Z, U) \arrow[r, "(\Sigma^i g)_*"] & \mathrm{Hom}_{\mathcal{A}}(\Sigma^i Y, U) \arrow[r, "(\Sigma^i f)_*"] & \mathrm{Hom}_{\mathcal{A}}(\Sigma^i X, U) \arrow[dll, "(\Sigma^{i-1} h)_*"] & \\
& \mathrm{Hom}_{\mathcal{A}}(\Sigma^{i-1} Z, U) \arrow[r, "(\Sigma^{i-1} g)_*"] & \mathrm{Hom}_{\mathcal{A}}(\Sigma^{i-1} Y, U) \arrow[r, "(\Sigma^{i-1} g)_*"] & \mathrm{Hom}_{\mathcal{A}}(\Sigma^{i-1} X, U) \arrow[r] & \dots
\end{tikzcd}$$

To see this, we note that since the opposite category is triangulated, each distinguished triangle in $\mathcal{A}^{\mathbf{op}}$ gives rise to a long exact sequence of groups as in Proposition 2.2.3. Using the fact that $\mathrm{Hom}_{\mathcal{A}^{\mathbf{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(Y, X)$ then yields the desired sequence.

Proposition 2.2.4. ([1] p.15, The Triangulated Five-Lemma) Given a morphism of distinguished triangles:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[r, "h"] \arrow[d, "w"'] & \Sigma X \arrow[d, "\Sigma u"] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X'
\end{tikzcd}$$

If u and v are isomorphisms, then so is w.

Proof. Applying the functor $\mathrm{Hom}_{\mathcal{A}}(Z', -)$ yields a diagram:

$$\begin{tikzcd}
\mathrm{Hom}(Z',X) \arrow[r] \arrow[d, "u_*"'] & \mathrm{Hom}(Z',Y) \arrow[r] \arrow[d, "v_*"'] & \mathrm{Hom}(Z',Z) \arrow[r] \arrow[d, "w_*"] & \mathrm{Hom}(Z',\Sigma X) \arrow[r] \arrow[d, "\Sigma u_*"] & \mathrm{Hom}(Z',\Sigma Y) \arrow[d, "\Sigma v_*"] \\
\mathrm{Hom}(Z',X') \arrow[r] & \mathrm{Hom}(Z',Y') \arrow[r] & \mathrm{Hom}(Z',Z') \arrow[r] & \mathrm{Hom}(Z',\Sigma X') \arrow[r] & \mathrm{Hom}(Z',\Sigma Y')
\end{tikzcd}$$

where the rows are exact by Proposition 2.2.3. By functoriality of Hom and Σ the morphisms $u_*, v_*, \Sigma u_*$ and Σv_* are isomorphisms so w_* is an isomorphism by the usual five-lemma for abelian groups. Consider the morphism $j = w_*^{-1}(1_{Z'}) : Z' \to Z$. By definition of w_*, we have $w \circ j = 1_{Z'}$ so j is a right inverse to w. To find a left inverse, we apply a similar argument as above with the functor $\mathrm{Hom}_{\mathcal{A}}(-, Z)$ using the dual version of Proposition 2.2.3 discussed in the previous remark. Then w has both a right and left inverse so it is an isomorphism. □

By applying the rotation axiom, one can prove a slightly stronger version of the Triangulated Five-Lemma, namely if any two of the morphisms u, v, w are isomorphisms, then so is the third.

An immediate consequence of the triangulated five-lemma is that the induced triangle of axiom (A2) can be chosen uniquely up to isomorphism. Indeed, let $f : X \to Y$ be a morphism and consider any two distinguished triangles:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

$$X \xrightarrow{f} Y \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X$$

By (A4), we can extend the identity morphisms 1_X and 1_Y to a morphism of triangles:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "1_X"'] & Y \arrow[r, "g"] \arrow[d, "1_Y"] & Z \arrow[r, "h"] \arrow[d, "w"] & \Sigma X \arrow[d, "1_{\Sigma X}"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X
\end{tikzcd}$$

Then w is an isomorphism by Proposition 2.2.4.

2.3 Localization of a Triangulated Category

In this section we describe the localization of a triangulated category. Informally, localization is the process of adding inverses to a class of morphisms in a category. If this class of morphisms satisfies certain compatibility criteria, then the localization can be equipped with a canonical triangulated structure inherited from its parent category.

We shall begin by describing localization over arbitrary categories. First, we say that a functor $F : \mathcal{C} \to \mathcal{D}$ makes a morphism f in $\mathcal{C}$ invertible if $F(f)$ is an isomorphism in $\mathcal{D}$.

Definition 2.3.1. Let $\mathcal{C}$ be a category and let S be a class of morphisms in $\mathcal{C}$. The *localization* of $\mathcal{C}$ with respect to S is defined to be a category $S^{-1}\mathcal{C}$ equipped with a functor $L : \mathcal{C} \to S^{-1}\mathcal{C}$, called the *localization functor*, which makes the morphisms in S invertible, such that the following universal property is satisfied. Suppose $F : \mathcal{C} \to \mathcal{D}$ is another functor which makes the morphisms in S invertible. Then there is a unique functor $\tilde{F} : S^{-1}\mathcal{C} \to \mathcal{D}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "L"] \arrow[dr, "F"'] & S^{-1}\mathcal{C} \arrow[d, dashed, "\tilde{F}"] \\ & \mathcal{D} \end{tikzcd}$$

Remark. The localization of a category need not exist in general, but when it does exist the universal property guarantuees that it is unique up to a unique isomorphism. Indeed, if $\mathcal{S}$ and $\mathcal{S}'$ are both localizations with respect to S, then there are functors $F : \mathcal{S} \to \mathcal{S}'$ and $G : \mathcal{S}' \to \mathcal{S}$ with $G \circ L' = L$ and $F \circ L = L'$ so in particular $G \circ F \circ L = L$ and $F \circ G \circ L' = L'$, hence $G \circ F = \mathrm{Id}_{\mathcal{S}}$ and $F \circ G = \mathrm{Id}_{\mathcal{S}'}$ by uniqueness.

Under the right assumptions on the class S, we may describe the localization in terms of a calculus of fractions. More precisely, we can take $S^{-1}\mathcal{C}$ to be the category comprised of the same objects as $\mathcal{C}$ but where morphisms are "formal fractions" $f \circ s^{-1}$, which we represent by diagrams of the form:

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

where $s \in S$. The construction of this category is a very technical procedure, and the majority of this section will be devoted to giving a full description of this construction. Throughout this section, we shall follow the constructions given in chapters 4 and 13 of [3], albeit with modifed proofs.

Definition 2.3.2. Let $\mathcal{C}$ be a category. A *right multiplicative system* in $\mathcal{C}$ is a class S of morphisms satisfying the following axioms:

(M1) $1_X \in S$ for every $X \in \mathcal{C}$ and if $f, g \in S$ is a composable pair of morphisms, then $g \circ f \in S$.

(M2) Any diagram $X \xrightarrow{f} Y \xleftarrow{t} Y'$ where $t \in S$ can be completed to a commutative square:

$$\begin{tikzcd} X' \arrow[r, dashed, "f'"] \arrow[d, dashed, "t'"'] & Y' \arrow[d, "t"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where $t' \in S$.

(M3) Let $f, g : X \to Y$ be morphisms in $\mathcal{C}$. If $t \circ f = t \circ g$ for some $t \in S$, then there is a $s \in S$ such that $f \circ s = g \circ s$.

Similarly, a *left multiplicative system* in $\mathcal{C}$ is a class S of morphisms in $\mathcal{C}$ satisfying (M1) as well as the following two axioms:

(M2') Any diagram $X' \xleftarrow{s} X \xrightarrow{f} Y$ where $s \in S$ can be completed to a commutative square:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[d, dashed, "s'"] \\
X' \arrow[r, dashed, "f'"'] & Y'
\end{tikzcd}$$

where $s' \in S$.

(M3') Let $f, g : X \to Y$ be morphisms in $\mathcal{C}$. If $f \circ s = g \circ s$ for some $s \in S$, then there is a $t \in S$ such that $t \circ f = t \circ g$.

Definition 2.3.3. Let S be a right multiplicative system in $\mathcal{C}$. A *right fraction* is a diagram:

$$\begin{tikzcd}
 & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & & Y
\end{tikzcd}$$

where $s \in S$. We shall sometimes use the shorthand $(f, s) : X \to Y$ for such a diagram.

Definition 2.3.4. Two right fractions $(f_1, s_1), (f_2, s_2) : X \to Y$ are said to be equivalent if there is a commutative diagram:

$$\begin{tikzcd}
 & X_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\
X & X_3 \arrow[l, "s_3"'] \arrow[r, "f_3"] \arrow[u] \arrow[d] & Y \\
 & X_2 \arrow[ul, "s_2"] \arrow[ur, "f_2"'] &
\end{tikzcd}$$

such that $s_3 \in S$. We write $(f_1, s_1) \sim (f_2, s_2)$ for this relation.

Lemma 2.3.5. Equivalence of right fractions as defined above is an equivalence relation.

Before we can prove that this is an equivalence relation, we introduce a new relation E on right fractions by taking $(f_1, s_1)E(f_2, s_2)$ if and only if there is a morphism $g : X_1 \to X_2$ making the following diagram commute:

$$\begin{tikzcd}
 & X_1 \arrow[dl, "s_1"'] \arrow[dd, "g"] \arrow[dr, "f_1"] & \\
X & & Y \\
 & X_2 \arrow[ul, "s_2"] \arrow[ur, "f_2"'] &
\end{tikzcd}$$

Clearly this relation is transitive and reflexive, though it is not symmetric, so E is not an equivalence relation. However, we note that $(f_1, s_1) \sim (f_2, s_2)$ if and only if there exists an (f_3, s_3) such that $(f_3, s_3)E(f_1, s_1)$ and $(f_3, s_3)E(f_2, s_2)$. Moreover, we have the following claim:

Claim: If $(f_1, s_1)E(f_3, s_3)$ and $(f_2, s_2)E(f_3, s_3)$, then $(f_1, s_1) \sim (f_2, s_2)$.

Proof. The conditions $(f_1, s_1)E(f_3, s_3)$ and $(f_2, s_2)E(f_3, s_3)$ yields a commutative diagram:

$$\begin{tikzcd}
 & X_1 \arrow[dl, "s_1"'] \arrow[d, "g_{13}"] \arrow[dr, "f_1"] & \\
X & X_3 \arrow[l, "s_3"'] \arrow[r, "f_3"] & Y \\
 & X_2 \arrow[ul, "s_2"] \arrow[u, "g_{23}"'] \arrow[ur, "f_2"'] &
\end{tikzcd}$$

By (M2), there is a commutative square:

$$\begin{tikzcd}
X_4 \arrow[r, dashed, "g_{41}"] \arrow[d, dashed, "g_{42}"'] & X_1 \arrow[d, "s_1"] \\
X_2 \arrow[r, "s_2"'] & X
\end{tikzcd}$$

where $g_{42} \in S$. Using the commutativity of the diagrams above, we obtain:

$$s_3 \circ g_{23} \circ g_{42} = s_2 \circ g_{42} = s_1 \circ g_{41} = s_3 \circ g_{13} \circ g_{41}$$

Then by (M3), there is a $s_{44} : X_4' \to X_4$ in S such that:

$$g_{23} \circ g_{42} \circ s_{44} = g_{13} \circ g_{41} \circ s_{44}$$

Setting $g_{41}' := g_{41} \circ s_{44}$ and $g_{42}' := g_{42} \circ s_{44}$, this becomes:

$$g_{23} \circ g_{42}' = g_{13} \circ g_{41}'$$

and we note that g_{42}' is in S. Now, define $f_4 : X_4' \to Y$ and $s_4 : X_4' \to X$ in S by:

$$f_4 := f_1 \circ g_{41}' = f_3 \circ g_{13} \circ g_{41}' = f_3 \circ g_{23} \circ g_{42}' = f_2 \circ g_{42}'$$

$$s_4 := s_1 \circ g_{41}' = s_2 \circ g_{42}'$$

Then by definition, the following diagram commutes:

$$\begin{tikzcd}
 & X_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\
X & X_4' \arrow[l, "s_4"'] \arrow[r, "f_4"] \arrow[u, "g_{41}'"'] \arrow[d, "g_{42}'"] & Y \\
 & X_2 \arrow[ul, "s_2"] \arrow[ur, "f_2"'] &
\end{tikzcd}$$

Hence $(f_1, s_1) \sim (f_2, s_2)$. $\square$

Proof of Lemma 2.3.5. It is clear that $\sim$ is reflexive and symmetric, so it remains to prove transitivity. Suppose then that $(f_1, s_1) \sim (f_2, s_2)$ and $(f_2, s_2) \sim (f_3, s_3)$. Then there are $(f_4, s_4), (f_5, s_5)$ such that:

$$(f_4, s_4)E(f_1, s_1), \quad (f_4, s_4)E(f_2, s_2), \quad (f_5, s_5)E(f_2, s_2), \quad (f_5, s_5)E(f_3, s_3)$$

By the previous claim, this implies that $(f_4, s_4) \sim (f_5, s_5)$, so there is an (f_6, s_6) such that $(f_6, s_6)E(f_4, s_4)$ and $(f_6, s_6)E(f_5, s_5)$. Since E is transitive, this implies that $(f_6, s_6)E(f_1, s_1)$ and $(f_6, s_6)E(f_3, s_3)$ so $(f_1, s_1) \sim (f_3, s_3)$. Thus $\sim$ is an equivalence relation. $\square$

We may now define the *category of right fractions* $S^{-1}\mathcal{C}_R$ to be the category whose objects are those of $\mathcal{C}$ and whose morphisms are the equivalence classes of right fractions. For any pair $(f, s) : X \to Y$ and $(g, t) : Y \to Z$ of right fractions, there exists a commutative diagram:

$$\begin{tikzcd}
 & & X'' \arrow[dl, dashed, "t'"'] \arrow[dr, dashed, "f'"] & & \\
 & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

where $t' \in S$. The composite $(g, t) \circ (f, s)$ is then defined to be the equivalence class of the right fraction $(g \circ f', s \circ t') : X \to Z$ obtained by composing the outer roof. Moreover, we define the identity morphism to be the equivalence class of the right fraction $(1_X, 1_X)$. It remains to verify that this composition law is well-defined and satisfies the associativity and unit laws of a category.

Theorem 2.3.6. The structure $S^{-1}\mathcal{C}_R$ as defined above is a category.

Proof. To see that composition is well-defined on the equivalence classes, we need to prove that it is independent of the choice of the dotted arrows in the diagram above and that compositions of equivalent fractions remain equivalent. Suppose then that for right fractions $(f, s) : X \to Y$ and $(g, t) : Y \to Z$, we are given two commutative diagrams:

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "t_1"'] & Y' \arrow[d, "t"] \\ X' \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X_2 \arrow[r, "f_2"] \arrow[d, "t_2"'] & Y' \arrow[d, "t"] \\ X' \arrow[r, "f"'] & Y \end{tikzcd}$$

with $t_1, t_2 \in S$. We aim to prove that the corresponding compositions are equivalent, that is $(g \circ f_1, s \circ t_1) \sim (g \circ f_2, s \circ t_2)$. By (M2), there is a commutative square:

$$\begin{tikzcd} X_3 \arrow[r, dashed, "t_{32}"] \arrow[d, dashed, "t_{31}"'] & X_2 \arrow[d, "t_2"] \\ X_1 \arrow[r, "t_1"'] & X' \end{tikzcd}$$

where $t_{31} \in S$. By commutativity of these three diagrams, we have:

$$t \circ f_1 \circ t_{31} = f \circ t_1 \circ t_{31} = f \circ t_2 \circ t_{32} = t \circ f_2 \circ t_{32}$$

Then by (M3), there is a $q : X_3' \to X_3$ in S such that:

$$f_1 \circ t_{31} \circ q = f_2 \circ t_{32} \circ q$$

Now, setting $t_{31}' = t_{31} \circ q$ and $t_{32}' = t_{32} \circ q$, we define $f_3 : X_3' \to Y'$ and $t_3 : X_3' \to X'$ by:

$$f_3 := f_1 \circ t_{31}' = f_2 \circ t_{32}'$$

$$t_3 := t_1 \circ t_{31}' = t_2 \circ t_{32}'$$

Note that $t_3 \in S$ since t_{31}' and t_1 are in S. By construction, this gives a commutative diagram:

$$\begin{tikzcd} & X_1 \arrow[dl, "s \circ t_1"'] \arrow[dr, "g \circ f_1"] & \\ X & X_3' \arrow[l, "s \circ t_3"'] \arrow[r, "g \circ f_3"] \arrow[u, "t_{31}'"'] \arrow[d, "t_{32}'"] & Y \\ & X_2 \arrow[ul, "s \circ t_3"] \arrow[ur, "g \circ f_2"'] & \end{tikzcd}$$

where $s \circ t_3 \in S$. Thus $(g \circ f_1, s \circ t_1) \sim (g \circ f_2, s \circ t_2)$.

Now, let $(f, s) : X \to Y$ and $(g_1, t_1), (g_2, t_2) : Y \to Z$ be right fractions such that $(g_1, t_1) \sim (g_2, t_2)$. The aim is to prove that $(g_1, t_1) \circ (f, s) \sim (g_2, t_2) \circ (f, s)$. We know from before that $(g_1, t_1) \sim (g_2, t_2)$ if and only if there is a (g_3, t_3) such that $(g_3, t_3) E (g_1, t_1)$ and $(g_3, t_3) E (g_2, t_2)$. Suppose then that $(g_3, t_3) E (g_1, t_1)$. By definition, there is a commutative diagram:

$$\begin{tikzcd} & Y_3 \arrow[dl, "t_3"'] \arrow[dr, "g_3"] \arrow[d, "h"] & \\ Y & & Z \\ & Y_1 \arrow[ul, "t_1"] \arrow[ur, "g_1"'] & \end{tikzcd}$$

Composing (f, s) with (g_3, t_3) means choosing a commutative square:

$$\begin{tikzcd} Y_4 \arrow[r, "f'"] \arrow[d, "t_3'"'] & Y_3 \arrow[d, "t_3"] \\ X' \arrow[r, "f"'] & Y \end{tikzcd}$$

where $t_3' \in S$. Since $t_1 \circ h = t_3$, we obtain another commutative square:

$$\begin{tikzcd} Y_4 \arrow[r, "h \circ f'"] \arrow[d, "t_3'"'] & Y_1 \arrow[d, "t_1"] \\ X' \arrow[r, "f"'] & Y \end{tikzcd}$$

Hence:

$$(g_3, t_3) \circ (f, s) = (g_3 \circ f', s \circ t_3') = (g_1 \circ h \circ f', s \circ t_3') = (g_1, t_1) \circ (f, s)$$

So in particular, $(g_3, t_3) \circ (f, s) \sim (g_1, t_1) \circ (f, s)$. Similarly, we obtain $(g_3, t_3) \circ (f, s) \sim (g_2, t_2) \circ (f, s)$ so that $(g_1, t_1) \circ (f, s) \sim (g_2, t_2) \circ (f, s)$. By an analogous proof, one finds that if $(f_1, s_1) \sim (f_2, s_2)$, then $(g, t) \circ (f_1, s_1) \sim (g, t) \circ (f_2, s_2)$ so we conclude that composition is well-defined on equivalence classes of right fractions.

For associativity, suppose $(f, s) : X \to Y$, $(g, t) : Y \to Z$ and $(h, r) : Z \to W$ are right fractions. By repeated application of (M2), we may construct the following commutative diagram:

$$\begin{tikzcd}[column sep=small] & & & X''' \arrow[dl, "s''"'] \arrow[dr, "f''"] & & & \\ & & X'' \arrow[dl, "s'"'] \arrow[dr, "f'"] & & Y'' \arrow[dl, "t'"'] \arrow[dr, "g'"] & & \\ & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & & Z' \arrow[dl, "r"'] \arrow[dr, "h"] & \\ X & & Y & & Z & & W \end{tikzcd}$$

where $s', s'', t' \in S$. From this diagram, it is clear that the composites $(h, r) \circ ((g, t) \circ (f, s))$ and $((h, r) \circ (g, t)) \circ (f, s)$ belong to the same equivalence class. Thus composition of right fractions is associative.

Finally, take any right fraction (f, s) and consider the diagram:

$$\begin{tikzcd}[column sep=small] & & X' \arrow[dl, "1_{X'}"'] \arrow[dr, "f"] & & \\ & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y \arrow[dl, "1_Y"'] \arrow[dr, "1_Y"] & \\ X & & Y & & Y \end{tikzcd}$$

Since $1_{X'} \in S$, this gives $(1_Y, 1_Y) \circ (f, s) = (f, s)$. By a similar argument, one finds that $(f, s) \circ (1_X, 1_X) = (f, s)$ so the equivalence classes $(1_X, 1_X)$ are indeed the identity morphisms in $S^{-1}\mathcal{C}_R$, which completes the proof. $\square$

Definition 2.3.7. Define the *localization functor* $L_R : \mathcal{C} \to S^{-1}\mathcal{C}_R$ to be the identity on objects, and where each morphism $f : X \to Y$ is mapped to the equivalence class of the right fraction $(f, 1_X)$.

Proposition 2.3.8. The localization functor L_R makes the morphisms in S invertible.

Proof. Let $s : X \to Y$ be a morphism in S, so that $L_R(s) = (s, 1_X) : X \to Y$. Then $(1_X, s) : Y \to X$ is a right fraction, and we consider the composite $(1_X, s) \circ (s, 1_X)$. We have the following commutative diagram:

$$\begin{tikzcd}
& & X \arrow[dl, "1_X"'] \arrow[dr, "1_X"] & & \\
& X \arrow[dl, "1_X"'] \arrow[dr, "s"] & & X \arrow[dl, "s"'] \arrow[dr, "1_X"] & \\
X & & Y & & X
\end{tikzcd}$$

From this, it is clear that $(1_X, s) \circ (s, 1_X) = (1_X, 1_X)$. Now take $(s, 1_X) \circ (1_X, s)$. We have the following commutative diagram:

$$\begin{tikzcd}
& & X \arrow[dl, "1_X"'] \arrow[dr, "1_X"] & & \\
& X \arrow[dl, "s"'] \arrow[dr, "1_X"] & & X \arrow[dl, "1_X"'] \arrow[dr, "s"] & \\
Y & & Y & & X
\end{tikzcd}$$

so that $(s, 1_X) \circ (1_X, s) = (s, s)$. But $(s, s) \sim (1_Y, 1_Y)$ via the diagram:

$$\begin{tikzcd}
& X \arrow[dl, "s"'] \arrow[dr, "s"] & \\
Y & X \arrow[l, "s"'] \arrow[r, "s"] \arrow[u, "1_X"'] \arrow[d, "s"] & Y \\
& Y \arrow[ul, "1_Y"] \arrow[ur, "1_Y"'] &
\end{tikzcd}$$

so $L(s) = (s, 1_X)$ is an isomorphism in $S^{-1}\mathcal{C}_R$. $\square$

Proposition 2.3.9. The pair $(S^{-1}\mathcal{C}_R, L_R)$ satisfies the universal property of Definition 2.3.1.

Proof. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor which makes the morphisms in S invertible. We construct a functor $\tilde{F} : S^{-1}\mathcal{C}_R \to \mathcal{D}$ on objects by taking $\tilde{F}(X) = F(X)$ for every $X \in \mathcal{C}$. For a right fraction (f, s), we define $\tilde{F}(f, s)$ by $F(f) \circ F(s)^{-1}$. To see that this indeed defines a functor, we must prove that $\tilde{F}$ is well-defined on equivalence classes of right fractions and that composition and identities are preserved.

Suppose that $(f_1, s_1) \sim (f_2, s_2)$ for some $(f_1, s_1), (f_2, s_2) : X \to Y$. By definiton, there is a commutative diagram:

$$\begin{tikzcd}
 & X_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\
X & X_3 \arrow[l, "s_3"'] \arrow[r, "f_3"] \arrow[u, "g_{31}"'] \arrow[d, "g_{32}"] & Y \\
 & X_2 \arrow[ul, "s_2"] \arrow[ur, "f_2"'] &
\end{tikzcd}$$

where $s_3 \in S$. Applying the functor F to this diagram, we see that $F(s_1)^{-1} = F(g_{31}) \circ F(s_3)^{-1}$ and $F(s_2)^{-1} = F(g_{32}) \circ F(s_3)^{-1}$. This yields:

$$\begin{aligned}
\tilde{F}(f_1, s_1) &= F(f_1) \circ F(s_1)^{-1} \\
&= F(f_1) \circ F(g_{31}) \circ F(s_3)^{-1} \\
&= F(f_3) \circ F(s_3)^{-1} \\
&= F(f_2) \circ F(g_{32}) \circ F(s_3)^{-1} \\
&= F(f_2) \circ F(s_2)^{-1} = \tilde{F}(f_2, s_2)
\end{aligned}$$

So $\tilde{F}$ is well-defined. Now consider $(f, s) : X \to Y$ and $(g, t) : Y \to Z$. Composing yields a commutative diagram:

$$\begin{tikzcd}
 & & X'' \arrow[dl, dashed, "t'"'] \arrow[dr, dashed, "f'"] & & \\
 & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

where $t' \in S$. By commutativity, we have $F(f) \circ F(t') = F(t) \circ F(f')$ which implies that $F(t)^{-1} \circ F(f) = F(f') \circ F(t')^{-1}$. Then:

$$\begin{aligned}
\tilde{F}((g, t) \circ (f, s)) &= F(g \circ f') \circ F(s \circ t')^{-1} \\
&= F(g) \circ F(f') \circ F(t')^{-1} \circ F(s)^{-1} \\
&= F(g) \circ F(t)^{-1} \circ F(f) \circ F(s)^{-1} \\
&= \tilde{F}(g, t) \circ \tilde{F}(f, s)
\end{aligned}$$

So $\tilde{F}$ preserves the composition law. Finally, we have $\tilde{F}(1_X, 1_X) = F(1_X) \circ F(1_X)^{-1} = 1_{F(X)}$ so $\tilde{F}$ preserves identities. Thus $\tilde{F}$ is a functor $S^{-1}\mathcal{C}_R \to \mathcal{D}$ and it is clear from the definition that $\tilde{F} \circ L = F$.

For uniqueness, suppose there is a functor $G : S^{-1}\mathcal{C}_R \to \mathcal{D}$ such that $G \circ L = F$. Then $G(X) = F(X)$ and $G(f, 1) = F(f)$ for all objects X and all morphisms f in $\mathcal{C}$. Moreover, given any right fraction (f, s) it is easily seen that $(f, s) = (f, 1) \circ (1, s)$ and we saw in Proposition 2.3.8 that $(1, s) = (s, 1)^{-1}$. Using functoriality of G, we see that

$$G(f, s) = G(f, 1) \circ G(1, s) = G(f, 1) \circ G(s, 1)^{-1} = F(f) \circ F(s)^{-1} = \tilde{F}(f, s)$$

Hence $\tilde{F} = G$ which completes the proof. $\square$

Remark. This result shows that the category of right fractions is indeed equivalent to the localization of $\mathcal{C}$ with respect to S. Now, if S is a left multiplicative system, there is the dual notion of *left fraction*, that is a diagram of the form:

$$\begin{tikzcd} & Y' & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \end{tikzcd}$$

where $s \in S$. By a similar procedure to the one we described for right fractions, one may construct the *category of left fractions* $S^{-1}\mathcal{C}_L$ and it can be proven that it too is equivalent to the localization. In particular, if S is a *multiplicative system*, that is S is both a left and right multiplicative system, then there is a canonical isomorphism $S^{-1}\mathcal{C}_L \cong S^{-1}\mathcal{C}_R$ by the universal property of the localization. Throughout this paper however, we shall always assume that we are working with the category of right fractions when we discuss the localization.

From now on, let $\mathcal{A}$ be an additive category and let S be a multiplicative system in $\mathcal{A}$. For now, the main aim will be to prove that the localization of an additive category is additive. To achieve this, we first observe the following:

Lemma 2.3.10. Let $(f_1, s_1), (f_2, s_2) : X \to Y$ be right fractions. Then there exists a morphism $s \in S$ and morphisms f_1', f_2' such that $(f_1, s_1) \sim (f_1', s)$ and $(f_2, s_2) \sim (f_2', s)$.

Proof. By (M2) there exists a commutative square:

$$\begin{tikzcd} X_3 \arrow[r, "s_{32}"] \arrow[d, "s_{31}"'] & X_2 \arrow[d, "s_2"] \\ X_1 \arrow[r, "s_1"'] & X \end{tikzcd}$$

where $s_{31} \in S$. Taking $s = s_1 \circ s_{31} = s_2 \circ s_{32}$, $f_1' = f_1 \circ s_{31}$ and $f_2' = f_2 \circ s_{32}$ we see that the following diagrams are commutative:

$$\begin{tikzcd} & X_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\ X & X_3 \arrow[l, "s"'] \arrow[r, "f_1'"] \arrow[u, "s_{31}"'] \arrow[d, "1"] & Y \\ & X_3 \arrow[ul, "s"] \arrow[ur, "f_1'"'] & \end{tikzcd} \qquad \begin{tikzcd} & X_2 \arrow[dl, "s_2"'] \arrow[dr, "f_2"] & \\ X & X_3 \arrow[l, "s"'] \arrow[r, "f_2'"] \arrow[u, "s_{32}"'] \arrow[d, "1"] & Y \\ & X_3 \arrow[ul, "s"] \arrow[ur, "f_2'"'] & \end{tikzcd}$$

hence $(f_1, s_1) \sim (f_1', s)$ and $(f_2, s_2) \sim (f_2', s)$ as desired. $\square$

This result means that we can assume without loss of generality that $s_1 = s_2$ whenever we are given two parallell right fractions. In fact, by induction using similar argument one finds that any finite set of parallell right fractions can be assumed to be of the form (f_i, s) for some $s \in S$. Using this, we state the following lemma:

Lemma 2.3.11. Two fractions $(f_1, s), (f_2, s) : X \to Y$ are equivalent if and only if there exists a morphism $t \in S$ such that $f_1 \circ t = f_2 \circ t$.

Proof. Suppose $(f_1, s) \sim (f_2, s)$. Then there is a commutative diagram:

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f_1"] & \\ X & X'' \arrow[l, "s_3"'] \arrow[u, "s_1"'] \arrow[r, "f_3"] \arrow[d, "s_2"] & Y \\ & X' \arrow[ul, "s"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

where $s_3 \in S$. Since $s \circ s_1 = s \circ s_2$, there exists a morphism $s_4 \in S$ such that $s_1 \circ s_4 = s_2 \circ s_4$. Setting $t = s_1 \circ s_4$ we obtain $f_1 \circ t = f_2 \circ t$ as desired.

Conversely suppose such a morphism t exists. Then the following diagram commutes:

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f_1"] & \\ X & X'' \arrow[l, "s \circ t"'] \arrow[u, "t"'] \arrow[r, "f_1 \circ t"] \arrow[d, "t"] & Y \\ & X' \arrow[ul, "s"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

Hence $(f_1, s) \sim (f_2, s)$. □

Now, using Lemma 2.3.10 we define the addition of two parallell morphisms in $S^{-1}\mathcal{A}$ by taking $(f_1, s) + (f_2, s)$ to be the equivalence class of the right fraction $(f_1 + f_2, s)$. It is clear from the definition that this operation satisfies the axioms of an abelian group, provided it is well-defined on equivalence classes. This leads to the following proposition:

Proposition 2.3.12. The localization $S^{-1}\mathcal{A}$ is enriched in abelian groups and the localization functor $L : \mathcal{A} \to S^{-1}\mathcal{A}$ is additive.

Proof. First we check that the addition is well-defined. Suppose that $(f_1, s) \sim (f_2, s)$ and $(g_1, s) \sim (g_2, s)$. Then by Lemma 2.3.11 there are morphisms $t_1, t_2 \in S$ such that $f_1 \circ t_1 = f_2 \circ t_1$ and $g_1 \circ t_2 = g_2 \circ t_2$. By (M2), there is a commutative square:

$$\begin{tikzcd} X'' \arrow[r, "t_2'"] \arrow[d, "t_1"'] & X_2 \arrow[d, "t_2"] \\ X_1 \arrow[r, "t_1"'] & X' \end{tikzcd}$$

Setting $t = t_1 \circ t'_1 = t_2 \circ t'_2$, we find that:

$$\begin{aligned}(f_1 + g_1) \circ t &= f_1 \circ t_1 \circ t'_1 + g_1 \circ t_2 \circ t'_2 \\ &= f_2 \circ t_1 \circ t'_1 + g_2 \circ t_2 \circ t'_2 = (f_2 + g_2) \circ t\end{aligned}$$

Hence $(f_1+g_1, s) \sim (f_2+g_2, s)$ by Lemma 2.3.11 so addition is well defined on equivalence classes.

Next, we prove that composition is biadditive. Take $(f_1, s), (f_2, s) : X \to Y$ and $(g, t) : Y \to Z$. Computing the composites $(g, t) \circ (f_1, s)$ and $(g, t) \circ (f_2, s)$ means choosing commutative squares:

$$\begin{tikzcd} X_1 \arrow[r, "f'_1"] \arrow[d, "t_1"'] & Y' \arrow[d, "t"] \\ X' \arrow[r, "f_1"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X_2 \arrow[r, "f'_2"] \arrow[d, "t_2"'] & Y' \arrow[d, "t"] \\ X' \arrow[r, "f_2"'] & Y \end{tikzcd}$$

where $t_1, t_2 \in S$. This gives composites $(g \circ f'_1, s \circ t_1)$ and $(g \circ f'_2, s \circ t_2)$. By picking a commutative square:

$$\begin{tikzcd} X'' \arrow[r, "t'_2"] \arrow[d, "t'_1"'] & X_2 \arrow[d, "t_2"] \\ X_1 \arrow[r, "t_1"'] & X' \end{tikzcd}$$

where $t'_1 \in S$, we obtain right fractions $(g \circ f'_1 \circ t'_1, s \circ t_1 \circ t'_1)$ and $(g \circ f'_2 \circ t'_2, s \circ t_2 \circ t'_2)$ such that:

$$(g \circ f'_1, s \circ t_1) \sim (g \circ f'_1 \circ t'_1, s \circ t_1 \circ t'_1)$$
$$(g \circ f'_2, s \circ t_2) \sim (g \circ f'_2 \circ t'_2, s \circ t_2 \circ t'_2)$$

Since $s \circ t_1 \circ t'_1 = s \circ t_2 \circ t'_2$, we obtain the sum:

$$(g, t) \circ (f_1, s) + (g, t) \circ (f_2, s) = (g \circ f'_1 \circ t'_1 + g \circ f'_2 \circ t'_2, s \circ t_1 \circ t'_1)$$

Next, we examine the composite $(g, t) \circ (f_1 + f_2, s)$. Let $\tilde{t}$ denote the morphism $t_1 \circ t'_1 = t_2 \circ t'_2$. Then $\tilde{t} \in S$ and by using commutativity of the three diagrams above, we obtain a commutative square:

$$\begin{tikzcd} X'' \arrow[r, "f'_1 \circ t'_1 + f'_2 \circ t'_2"] \arrow[d, "\tilde{t}'"'] & Y' \arrow[d, "t"] \\ X' \arrow[r, "f_1 + f_2"'] & Y \end{tikzcd}$$

Thus $(g, t) \circ (f_1 + f_2, s) = (g \circ (f'_1 \circ t'_1 + f'_2 \circ t'_2), s \circ \tilde{t})$ so we conclude that:

$$(g, t) \circ (f_1, s) + (g, t) \circ (f_2, s) = (g, t) \circ (f_1 + f_2, s) = (g, t) \circ ((f_1, s) + (f_2, s))$$

By a similar argument, one shows that if $(g_1, t), (g_2, t) : Y \to Z$ and $(f, s) : X \to Y$ are morphisms in $S^{-1}\mathcal{A}$, then $((g_1, t) + (g_2, t)) \circ (f, s) = (g_1, t) \circ (f, s) + (g_2, t) \circ (f, s)$ so composition is biadditive. Thus $S^{-1}\mathcal{A}$ is enriched in abelian groups.

Finally, consider any pair $f, g : X \to Y$ of morphisms in $\mathcal{A}$. Then:

$$L(f+g) = (f+g, 1_X) = (f, 1_X) + (g, 1_X) = L(f) + L(g)$$

so the localization functor is additive. $\square$

Corollary 2.3.13. Every pair of objects in $S^{-1}\mathcal{A}$ has a biproduct.

Proof. Let X, Y be objects in $\mathcal{A}$. Since $\mathcal{A}$ is additive, there is a biproduct diagram:

$$\begin{tikzcd} X \arrow[r, shift left, "i_X"] & X \oplus Y \arrow[l, shift left, "p_X"] \arrow[r, shift right, "p_Y"'] & Y \arrow[l, shift right, "i_Y"'] \end{tikzcd}$$

Since L is an additive functor, the following diagram is a biproduct diagram in $S^{-1}\mathcal{A}$:

$$\begin{tikzcd} L(X) \arrow[r, shift left, "L(i_X)"] & L(X \oplus Y) \arrow[l, shift left, "L(p_X)"] \arrow[r, shift right, "L(p_Y)"'] & L(Y) \arrow[l, shift right, "L(i_Y)"'] \end{tikzcd}$$

Now, since L is the identity on objects, this shows that every pair of objects has a biproduct in $S^{-1}\mathcal{A}$. $\square$

Proposition 2.3.14. The the image of the zero object in $\mathcal{A}$ under the localization functor is the zero object in $S^{-1}\mathcal{A}$.

Proof. We need to show for every object $X \in S^{-1}\mathcal{A}$, there exists a unique morphism $0 \to X$ and $X \to 0$. For existance, we have the right fractions $0 \longleftarrow 0 \longrightarrow X$ and $X \xleftarrow{1_X} X \longrightarrow 0$ which gives a morphism $0 \to X$ and $X \to 0$ respectively. For uniqueness, take any right fraction $0 \longleftarrow X' \xrightarrow{f} X$. Then the following diagram commutes:

$$\begin{tikzcd} & X' \arrow[dl] \arrow[dr, "f"] & \\ 0 & 0 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & X \\ & 0 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

so $(f, 0) \sim (0, 0)$. Thus there is only one equivalence class of right fractions $0 \to X$. By a similar argument, one proves that there is only one equivalence class of right fractions $X \to 0$. Thus $S^{-1}\mathcal{A}$ has a zero object. $\square$

Combining these results, we conclude that $S^{-1}\mathcal{A}$ is an additive category. For the remainder of this section, assume that $\mathcal{A}$ is a triangulated category.

Definition 2.3.15. A multiplicative system S is said to be *compatible* with the triangulated structure of $\mathcal{A}$ if the following holds:

(i) $\Sigma^i s \in S$ for every $s \in S$ and every $i \in \mathbb{Z}$.

(ii) Given a commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[r, "h"] & \Sigma X \arrow[d, "\Sigma u"] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X'
\end{tikzcd}$$

where the rows are distinguished triangles and $u, v \in S$, there is a morphism $w : Z \to Z'$ in S such that (u, v, w) is a morphism of triangles.

Proposition 2.3.16. If S is a multiplicative system compatible with the triangulated structure of $\mathcal{A}$, then the shift functor $\Sigma : \mathcal{A} \to \mathcal{A}$ induces a unique additive isomorphism $\widetilde{\Sigma} : S^{-1}\mathcal{A} \to S^{-1}\mathcal{A}$ such that $\widetilde{\Sigma} \circ L = L \circ \Sigma$.

Proof. Consider the functor $L \circ \Sigma : \mathcal{A} \to S^{-1}\mathcal{A}$. Since S is compatible with the triangulated structure of $\mathcal{A}$, this functor makes the morphisms in S invertible. Then by the universal property of the localization, there is a unique functor $\widetilde{\Sigma} : S^{-1}\mathcal{A} \to S^{-1}\mathcal{A}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "L"] \arrow[d, "\Sigma"'] & S^{-1}\mathcal{A} \arrow[d, "\widetilde{\Sigma}"] \\
\mathcal{A} \arrow[r, "L"'] & S^{-1}\mathcal{A}
\end{tikzcd}$$

By a similar argument, there exists a unique functor $\widetilde{\Sigma^{-1}} : S^{-1}\mathcal{A} \to S^{-1}\mathcal{A}$ such that $\widetilde{\Sigma^{-1}} \circ L = L \circ \Sigma^{-1}$ and by uniqueness, one finds that $\widetilde{\Sigma} \circ \widetilde{\Sigma^{-1}} = \mathrm{Id}_{S^{-1}\mathcal{A}} = \widetilde{\Sigma^{-1}} \circ \widetilde{\Sigma}$ so it is an ismorphism of categories. To see that this functor is additive, recall from Proposition 2.3.9 that the induced functor $\widetilde{\Sigma}$ was defined on morphisms by $\widetilde{\Sigma}(f, s) = L(\Sigma f) \circ L(\Sigma s)^{-1}$. This gives:

$$L(\Sigma f) \circ L(\Sigma s)^{-1} = (\Sigma f, 1) \circ (\Sigma s, 1)^{-1} = (\Sigma f, 1) \circ (1, \Sigma s) = (\Sigma f, \Sigma s)$$

Now take right fractions $(f_1, s), (f_2, s)$. Then by additivity of Σ, we obtain:

$$\begin{aligned}
\widetilde{\Sigma}((f_1, s) + (f_2, s)) &= \widetilde{\Sigma}(f_1 + f_2, s) \\
&= (\Sigma(f_1 + f_2), \Sigma s) \\
&= (\Sigma f_1, \Sigma s) + (\Sigma f_2, \Sigma s) = \widetilde{\Sigma}(f_1, s) + \widetilde{\Sigma}(f_2, s)
\end{aligned}$$

so $\widetilde{\Sigma}$ is additive. □

Definition 2.3.17. Let $\mathcal{A}$ and $\mathcal{B}$ be triangulated categories. A functor $F : \mathcal{A} \to \mathcal{B}$ is said to be *exact* if there exists a natural isomorphism $\tau : F \circ \Sigma_{\mathcal{A}} = \Sigma_{\mathcal{B}} \circ F$ and if for every distinguished triangle:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma_{\mathcal{A}} X$$

in $\mathcal{A}$, the following triangle is distinguished in $\mathcal{B}$:

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{\tau_X \circ F(h)} \Sigma_{\mathcal{B}} F(X)$$

We are now ready to prove the main theorem of this chapter:

Theorem 2.3.18. If S is a multiplicative system compatible with the triangulated structure of $\mathcal{A}$, then there is a triangulated structure on $S^{-1}\mathcal{A}$ such that the localization functor $L : \mathcal{A} \to S^{-1}\mathcal{A}$ is exact.

Proof. We say that a triangle in $S^{-1}\mathcal{A}$ is distinguished if it is isomorphic to the image of a distinguished triangle in $\mathcal{A}$ under the localization functor, that is triangles of the form:

$$X \xrightarrow{(f,1)} Y \xrightarrow{(g,1)} Z \xrightarrow{(h,1)} \Sigma X$$

We call such triangles *standard* triangles in $S^{-1}\mathcal{A}$. It remains to verify the axioms (A0)-(A5). Note that (A0) follows by definition and it is clear that the triangle

$$X \xrightarrow{(1,1)} X \xrightarrow{(0,1)} 0 \xrightarrow{(0,0)} \Sigma X$$

is distinguished, so (A1) holds.

For (A2), take any right fraction $X \xleftarrow{s} X' \xrightarrow{f} Y$. Then $f : X' \to Y$ can be extended to a distinguished triangle in $\mathcal{A}$:

$$X' \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X'$$

Now consider the triangle:

$$X \xrightarrow{(f,s)} Y \xrightarrow{(g,1)} Z \xrightarrow{(\Sigma s,1)\circ(h,1)} \Sigma X$$

in $S^{-1}\mathcal{A}$. Then the following diagram commutes:

$$\begin{tikzcd}
X' \arrow[r, "{(f,1)}"] \arrow[d, "{(s,1)}"'] & Y \arrow[r, "{(g,1)}"] \arrow[d, "1"'] & Z \arrow[r, "{(h,1)}"] \arrow[d, "1"] & \Sigma X' \arrow[d, "{(\Sigma s,1)}"] \\
X \arrow[r, "{(f,s)}"'] & Y \arrow[r, "{(g,1)}"'] & Z \arrow[r, "{(\Sigma s,1)\circ(h,1)}"'] & \Sigma X
\end{tikzcd}$$

where the columns are isomorphisms and the upper row is distinguished. Thus the bottom row is distinguished which proves (A2).

For (A3), it suffices to prove the axiom for standard triangles. Given a standard triangle:

$$X \xrightarrow{(f,1)} Y \xrightarrow{(g,1)} Z \xrightarrow{(h,1)} \Sigma X$$

the rotated triangle:

$$Y \xrightarrow{(g,1)} Z \xrightarrow{(h,1)} \Sigma X \xrightarrow{(-\Sigma f,1)} \Sigma Y$$

is distinguished since it is the image of the distinguished triangle:

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

in $\mathcal{A}$. Thus (A3) holds.

Again, for (A4) it suffices to prove the axiom for standard triangles. Consider a diagram:

$$\begin{tikzcd} X \arrow[r, "{(f,1)}"] \arrow[d, "{(u,s)}"'] & Y \arrow[r, "{(g,1)}"] \arrow[d, "{(v,t)}"] & Z \arrow[r, "{(h,1)}"] & \Sigma X \arrow[d, "{(\Sigma u,\Sigma s)}"] \\ X' \arrow[r, "{(f',1)}"'] & Y' \arrow[r, "{(g',1)}"'] & Z' \arrow[r, "{(h',1)}"'] & \Sigma X' \end{tikzcd}$$

such that the left square commutes. We want to construct a right fraction $(w, r) : Z \to Z'$ making the whole diagram commute. To achieve this, we require the following technical lemma:

Lemma 2.3.19. Suppose the following square commutes in $S^{-1}\mathcal{A}$:

$$\begin{tikzcd} X \arrow[r, "{(f,1)}"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ X' \arrow[r, "{(f',1)}"'] & Y' \end{tikzcd}$$

where g and h are arbitrary morphisms in $S^{-1}\mathcal{A}$, i.e equivalence classes of right fractions. Then there are right fractions $(u, s) : X \to X'$ and $(v, t) : Y \to Y'$ representing the morphisms g and h respectively, and a morphism $f'' : X'' \to Y''$ such that the following diagram commutes in $\mathcal{A}$:

$$\begin{tikzcd} X \arrow[d, "f"'] & X'' \arrow[l, "s"'] \arrow[r, "u"] \arrow[d, "f''"] & X' \arrow[d, "f'"] \\ Y & Y'' \arrow[l, "t"] \arrow[r, "v"'] & Y' \end{tikzcd}$$

Proof. First, let (u, s) and (v, t) be any representatives of g and h respectively. By (M2), there is a commutative diagram:

$$\begin{tikzcd} X \arrow[d, "f"'] & \widetilde{X} \arrow[l, "\tilde{t}"'] \arrow[d, "\tilde{f}"] & X' \arrow[d, "f'"] \\ Y & Y'' \arrow[l, "t"] \arrow[r, "v"'] & Y' \end{tikzcd}$$

where $\tilde{t} \in S$. This gives the composite $(v, t) \circ (f, 1) = (v \circ \tilde{f}, \tilde{t})$. By assumption, we have $(v, t) \circ (f, 1) = (f', 1) \circ (u, s)$ which implies that $(v \circ \tilde{f}, \tilde{t}) \sim (f' \circ u, s)$. This amounts to a commutative diagram:

$$\begin{tikzcd} & X'' \arrow[dl, "s"'] \arrow[dr, "f'\circ u"] & \\ X & X''' \arrow[u, "t'"] \arrow[d, "s'"] & Y' \\ & \widetilde{X} \arrow[ul, "\tilde{t}"] \arrow[ur, "v\circ\tilde{f}"'] & \end{tikzcd}$$

where $s \circ t' = \tilde{t} \circ s' \in S$. In particular, we see that $(u, s) \sim (u \circ t', s \circ t')$ which yields the following commutative diagram:

$$\begin{tikzcd}
X \arrow[d, "f"'] & X''' \arrow[l, "s\circ t'"'] \arrow[r, "u\circ t'"] \arrow[d, "\tilde{f}\circ s'"] & X' \arrow[d, "f'"] \\
Y & Y'' \arrow[l, "t"] \arrow[r, "v"'] & Y'
\end{tikzcd}$$

as desired. $\square$

To prove (A4), we may now assume that the morphism f'' exists as described in the above lemma. Since (A2) holds in $\mathcal{A}$, we may extend this morphism to a distinguished triangle:

$$X'' \xrightarrow{f''} Y'' \xrightarrow{g''} Z'' \xrightarrow{h''} \Sigma X''$$

and by applying (A4), we obtain a right fraction $(w, r) : Z \to Z'$ so that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \\
X'' \arrow[r, "f''"] \arrow[u, "s"] \arrow[d, "u"'] & Y'' \arrow[r, "g''"] \arrow[u, "t"'] \arrow[d, "v"] & Z'' \arrow[r, "h''"] \arrow[u, "r"'] \arrow[d, "w"] & \Sigma X'' \arrow[u, "\Sigma s"'] \arrow[d, "\Sigma u"] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X'
\end{tikzcd}$$

Note that $r \in S$ by compatibility of S with the triangulation on $\mathcal{A}$. It remains to verify that $(w, r) \circ (g, 1) = (g', 1) \circ (v, t)$ and $(h', 1) \circ (w, r) = (\Sigma u, \Sigma s) \circ (h, 1)$. From this diagram, we see that:

$$(w, r) \circ (g, 1) = (w \circ g'', t) = (g' \circ v, t) = (g', 1) \circ (v, t)$$

Similarly, we find that:

$$(\Sigma u, \Sigma s) \circ (h, 1) = (\Sigma u \circ h'', r) = (h' \circ w, r) = (h', 1) \circ (w, r)$$

Thus the following diagram commutes in $S^{-1}\mathcal{A}$:

$$\begin{tikzcd}
X \arrow[r, "(f,1)"] \arrow[d, "(u,s)"'] & Y \arrow[r, "(g,1)"] \arrow[d, "(v,t)"] & Z \arrow[r, "(h,1)"] \arrow[d, "(w,r)"] & \Sigma X \arrow[d, "(\Sigma u,\Sigma s)"] \\
X' \arrow[r, "(f',1)"'] & Y' \arrow[r, "(g',1)"'] & Z' \arrow[r, "(h',1)"'] & \Sigma X'
\end{tikzcd}$$

which completes the proof of (A4).

(A5) Once again, it suffices to prove the octahedral axiom for standard triangles. Suppose we are given standard triangles:

$$X \xrightarrow{(f,1)} Y \longrightarrow Z' \longrightarrow \Sigma X$$

$$Y \xrightarrow{(g,1)} Z \longrightarrow X' \longrightarrow \Sigma Y$$

$$X \xrightarrow{(gf,1)} Z \longrightarrow Y' \longrightarrow \Sigma X$$

Then since the octahedral axiom holds in $\mathcal{A}$, there is a distinguished triangle in $\mathcal{A}$ of the form:

$$Z' \longrightarrow X' \longrightarrow Y' \longrightarrow \Sigma Z'$$

making the relevant diagram commute. Applying the localization functor yields the desired triangle in $S^{-1}\mathcal{A}$. Thus $S^{-1}\mathcal{A}$ is a triangulated category.

Finally, exactness of $L : \mathcal{A} \to S^{-1}\mathcal{A}$ follows from the fact that $L \circ \Sigma = \widetilde{\Sigma} \circ L$ and that the image of a distinguished triangle in $\mathcal{A}$ is a distinguished triangle in $S^{-1}\mathcal{A}$. $\square$

3 Derived Categories

3.1 The Homotopy Category and its Triangulation

Definition 3.1.1. Given chain complexes A, B in $\mathcal{A}$ and chain morphisms $f, g : A \to B$, a *chain homotopy* $s : f \to g$ is a family of morphisms $s_i : A_i \to B_{i+1}$ such that:

$$d^B_{i+1}s_i + s_{i-1}d^A_i = f_i - g_i$$

for all i. We say that f is homotopic to g if there is a chain homotopy from f to g and we write $f \simeq g$.

Lemma 3.1.2. The relation $\simeq$ is an equivalence relation on the set $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{A})}(A, B)$.

Proof. For reflexivity, take any $f : A \to B$. Then one obtains a homotopy from f to itself by taking $s_i = 0$ for all i. For symmetry, suppose f is homotopic to g via some chain homotopy s. Then the family $-s_i$ is a chain homotopy from g to f. Finally for transitivity, suppose $s : f \to g$ and $t : g \to h$ are chain homotopies. Then the family $s_i + t_i$ satisfies:

$$d^B_{i+1}(s_i + t_i) + (s_{i-1} + t_{i-1})d^A_i = (f_i - g_i) + (g_i - h_i) = f_i - h_i$$

so f is homotopic to h. □

Lemma 3.1.3. Suppose $f, f' : A \to B$ and $g, g' : B \to C$ are chain morphisms. Then we have the following:

$$f \simeq f' \Rightarrow g \circ f \simeq g \circ f'$$
$$g \simeq g' \Rightarrow g \circ f \simeq g' \circ f$$

Proof. Let $s : f \to f'$ be a homotopy. We construct a homotopy $gs : g \circ f_1 \to g \circ f_2$ by taking the family $(gs)_i = g_{i+1}s_i$. Then:

$$\begin{aligned} d^C_{i+1}g_{i+1}s_i + g_i s_{i-1}d^A_i &= g_i d^B_{i+1}s_i + g_i s_{i-1}d^A_i \\ &= g_i(d^B_{i+1}s_i + s_{i-1}d^A_i) \\ &= g_i(f_i - f'_i) = g_i \circ f_i - g_i \circ f'_i \end{aligned}$$

so $g \circ f \simeq g \circ f'$. By an analogous proof, one obtains $g \circ f \simeq g' \circ f$. □

These lemmas allow us to state the following definition:

Definition 3.1.4. Let $\mathcal{A}$ be an additive category. The *homotopy category* of $\mathcal{A}$ is the category $\mathbf{K}(\mathcal{A})$ whose objects are chain complexes in $\mathcal{A}$ and whose morphisms are the equivalence classes of chain morphisms with respect to homotopy.

Proposition 3.1.5. $\mathbf{K}(\mathcal{A})$ is an additive category.

Proof. First we show that $\mathbf{K}(\mathcal{A})$ is enriched in abelian groups. For chain morphisms $f, g : A \to B$ define $[f] + [g] := [f + g]$ where $[f]$ denotes the homotopy class of f. This is well-defined since if $s : f \to f'$ and $t : g \to g'$ are homotopies, then the family $s_i + t_i$ satisfies

$$d^B_{i+1}(s_i + t_i) + (s_{i-1} + t_{i-1})d^A_i = f_i - f'_i + g_i - g'_i = (f_i + g_i) - (f'_i + g'_i)$$

hence $f + g \simeq f' + g'$. This clearly defines a group structure on $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(A, B)$ and biadditivity of composition follows from the additive structure on $\mathbf{Ch}(\mathcal{A})$.

It is clear that the zero complex is the zero object in $\mathbf{K}(\mathcal{A})$ so it remains to determine the biproduct.

Consider the biproducts in $\mathbf{Ch}(\mathcal{A})$. By definition, the canonical projections and inclusions satisfy the biproduct identities of 2.1.3, so the corresponding homotopy classes in $\mathbf{K}(\mathcal{A})$ also satisfy these identities. Hence $\mathbf{K}(\mathcal{A})$ has all biproducts. □

The remainder of this section will be devoted to describing the triangulated structure on $\mathbf{K}(\mathcal{A})$.

Definition 3.1.6. Let $\mathcal{A}$ be an additive category. The *shift functor* is the functor $[1] : \mathbf{Ch}(\mathcal{A}) \to \mathbf{Ch}(\mathcal{A})$ which maps each complex A to the shifted complex $A[1]_i = A_{i-1}$ with boundary $d^{A[1]}_i = -d^A_{i-1}$ and where each chain morphism $f : A \to B$ is mapped to the chain morphism $f[1] : A[1] \to B[1]$ given by $f[1]_i = f_{i-1}$.

The shift functor preserves homotopies between chain morphisms, so the it induces an additive functor $[1] : \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$. In particular, this functor is an isomorphism of categories.

Definition 3.1.7. Given a chain morphism $f : A \to B$, the *mapping cone* of f is the complex $M(f)_i = A_{i-1} \oplus B_i$ with differential:

$$d^{M(f)}_i = \begin{pmatrix} -d^A_{i-1} & 0 \\ f_{i-1} & d^B_i \end{pmatrix}$$

Given f, we can define chain morphisms $\alpha(f) : B \to M(f)$ and $\beta(f) : M(f) \to A[1]$ by taking the inclusions and projections in each degree respectively. Using these morphisms, we can form a triangle:

$$A \xrightarrow{f} B \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} A[1]$$

A triangle of this form is called a *standard triangle*. We say that a triangle is *distinguished* if it isomorphic in $\mathbf{K}(\mathcal{A})$ to a standard triangle. We are now ready to state and prove the main result of this section:

Theorem 3.1.8. ([1] p.19-23) $\mathbf{K}(\mathcal{A})$ is a triangulated category.

Proof. For the sake of brevity, we shall omit the verifications that certain families of morphisms form chain morphisms and chain homotopies as their proofs are all done by straighforward computation using their respective definitions.

(A0) and (A2) are satisfied by construction. For (A1), we have a distinguished triangle:

$$X \xrightarrow{1_X} X \xrightarrow{\alpha(1_X)} M(1_X) \xrightarrow{\beta(1_X)} X[1]$$

for every complex X. We want to show that $M(1_X)$ is isomorphic to 0 in $\mathbf{K}(\mathcal{A})$. To see this, consider the identity chain morphism on $M(1_X)$ and define a homotopy $s : 1_{M(1_X)} \to 0$ by taking the family:

$$s_i = \begin{pmatrix} 0 & 1_{X_i} \\ 0 & 0 \end{pmatrix} : X_{i-1} \oplus X_i \to X_i \oplus X_{i+1}$$

Thus $1_{M(f)} \simeq 0$ so $M(f) \cong 0$ in $\mathbf{K}(\mathcal{A})$.

For (A3), we observe that if we are given an isomorphism of triangles, rotation yields an isomorphism of the rotated triangles. Hence it suffices to prove the rotation axiom for standard triangles. Take a standard triangle:

$$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1]$$

We want to show that the rotated triangle

$$Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1] \xrightarrow{-f[1]} Y[1]$$

is isomorphic in $\mathbf{K}(\mathcal{A})$ to the standard triangle:

$$Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\alpha(\alpha(f))} M(\alpha(f)) \xrightarrow{\beta(\alpha(f))} Y[1]$$

We construct an isomorphism between these triangles by first taking the diagram:

$$\begin{tikzcd}
Y \arrow[r, "\alpha(f)"] \arrow[d, "1_Y"'] & M(f) \arrow[r, "\beta(f)"] \arrow[d, "1_{M(f)}"] & X[1] \arrow[r, "-f[1]"] & Y[1] \arrow[d, "1_{Y[1]}"] \\
Y \arrow[r, "\alpha(f)"'] & M(f) \arrow[r, "\alpha(\alpha(f))"'] & M(\alpha(f)) \arrow[r, "\beta(\alpha(f))"'] & Y[1]
\end{tikzcd}$$

Define morphisms $u : X[1] \to M(\alpha(f))$ and $v : M(\alpha(f)) \to X[1]$ by:

$$u_i = \begin{pmatrix} -f_{i-1} \\ 1_{X_{i-1}} \\ 0 \end{pmatrix} : X_{i-1} \to M(\alpha(f))_i = Y_{i-1} \oplus X_{i-1} \oplus Y_i$$

$$v_i = \begin{pmatrix} 0, & 1_{X_{i-1}}, & 0 \end{pmatrix} : M(\alpha(f))_i \to X_{i-1}$$

We see that $\beta(\alpha(f)) \circ u = -f[1]$ by construction. Consider the family:

$$s_i = \begin{pmatrix} 0 & -1_{X_i} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} : M(f)_i \to M(\alpha(f))_{i+1}$$

It is easily verified that this is a homotopy $u \circ \beta(f) \simeq \alpha(\alpha(f))$ so that the triple $(1_Y, 1_{M(f)}, u)$ is a morphism of triangles in $\mathbf{K}(\mathcal{A})$.

For the morphism v, we have $\beta(f) = v \circ \alpha(\alpha(f))$ by construction, and we see that $-f[1] \circ v \simeq \beta(\alpha(f))$ via the homotopy:

$$t_i = \begin{pmatrix} 0 & 0 & -1_{Y_i} \end{pmatrix} : M(\alpha(f))_i \to Y[1]_{i+1} = Y_i$$

Hence $(1_X, 1_{M(f)}, v)$ is a morphism of triangles in $\mathbf{K}(\mathcal{A})$. It remains to show that u and v are isomorphisms in $\mathbf{K}(\mathcal{A})$. We see from the definition that $v \circ u = 1_{X[1]}$, so we consider the composite $u \circ v$. The family:

$$\begin{pmatrix} 0 & 0 & -1_{Y_i} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : M(\alpha(f))_i \to M(\alpha(f))_{i+1}$$

is a homotopy $u \circ v \to 1_{M(\alpha(f))}$, hence u and v are isomorphisms so the triples $(1_X, 1_{M(f)}, u)$ and $(1_X, 1_{M(f)}, v)$ are isomorphisms of triangles. Thus the rotated triangle is isomorphic to a standard triangle, so it is distinguished.

For (A4), it once again suffices to prove the axiom for standard triangles. Consider a diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "\alpha(f)"] \arrow[d, "v"] & M(f) \arrow[r, "\beta(f)"] & X[1] \arrow[d, "u[1]"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "\alpha(f')"'] & M(f') \arrow[r, "\beta(f')"'] & X'[1] \end{tikzcd}$$

where the left square is commutative in $\mathbf{K}(\mathcal{A})$. This means that there exists a homotopy $s : v \circ f \to f' \circ u$, i.e a family of morphisms $s_i : X_i \to Y'_{i+1}$ such that:

$$f'_i u_i - v_i f_i = d^{Y'}_{i+1} s_i + s_{i-1} d^X_i$$

Now define $w : M(f) \to M(f')$ by:

$$w_i = \begin{pmatrix} u_{i-1} & 0 \\ s_{i-1} & v_i \end{pmatrix}$$

This defines a chain morphism since:

$$\begin{aligned} w_i d^{M(f)}_{i+1} &= \begin{pmatrix} u_{i-1} & 0 \\ s_{i-1} & v_i \end{pmatrix} \begin{pmatrix} -d^X_i & 0 \\ f_i & d^Y_{i+1} \end{pmatrix} \\ &= \begin{pmatrix} -u_{i-1} d^X_i & 0 \\ s_{i-1} d^X_i + v_i f_i & v_i d^Y_{i+1} \end{pmatrix} \\ &= \begin{pmatrix} -d^{X'}_i u_i & 0 \\ f'_i u_i - d^{Y'}_{i+1} s_i & d^{Y'}_i v_{i+1} \end{pmatrix} \\ &= \begin{pmatrix} -d^{X'}_i & 0 \\ f'_i & d^{Y'}_{i+1} \end{pmatrix} \begin{pmatrix} u_i & 0 \\ s_i & v_{i+1} \end{pmatrix} = d^{M(f')}_{i+1} w_{i+1} \end{aligned}$$

Now, by construction we have $\beta(f') \circ w = u[1] \circ \beta(f)$ and $w \circ \alpha(f) = \alpha(f') \circ v$ so the triple (u, v, w) is a morphism of triangles.

Again, we only need to consider standard triangles to prove (A5). Suppose then we are given standard triangles:

$$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1]$$

$$Y \xrightarrow{g} Z \xrightarrow{\alpha(g)} M(g) \xrightarrow{\beta(g)} Y[1]$$

$$X \xrightarrow{gf} Z \xrightarrow{\alpha(gf)} M(gf) \xrightarrow{\beta(gf)} X[1]$$

Collecting the morphisms yields the following diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "1_X"'] & Y \arrow[r, "\alpha(f)"] \arrow[d, "g"] & M(f) \arrow[r, "\beta(f)"] & X[1] \arrow[d, "1_{X[1]}"] \\
X \arrow[r, "gf"] \arrow[d, "f"'] & Z \arrow[r, "\alpha(gf)"] \arrow[d, "1_Z"] & M(gf) \arrow[r, "\beta(gf)"] & X[1] \arrow[d, "f[1]"] \\
Y \arrow[r, "g"] \arrow[d, "g"'] & Z \arrow[r, "\beta(g)"] \arrow[d, "\alpha(gf)"] & M(g) \arrow[r, "\beta(g)"] \arrow[d, "1_{M(g)}"] & Y[1] \arrow[d, "\alpha(f)[1]"] \\
M(f) & M(gf) & M(g) & M(f)[1]
\end{tikzcd}$$

where the top left squares commute. The aim is to find a distinguished triangle of the form:

$$M(f) \xrightarrow{u} M(gf) \xrightarrow{v} M(g) \xrightarrow{w} M(f)[1]$$

making the entire diagram commute. To this end, we define the morphisms u, v, w by:

$$u_i = \begin{pmatrix} 1_{X_{i-1}} & 0 \\ 0 & g_i \end{pmatrix} : X_{i-1} \oplus Y_i \to X_{i-1} \oplus Z_i$$

$$v_i = \begin{pmatrix} u_{i-1} & 0 \\ 0 & 1_{Z_i} \end{pmatrix} : X_{i-1} \oplus Z_i \to Y_{i-1} \oplus Z_i$$

$$w_i = \begin{pmatrix} 0 & 0 \\ 1_{Y_{i-1}} & 0 \end{pmatrix} : Y_{i-1} \oplus Z_i \to X_{i-2} \oplus Y_{i-1}$$

It is easily seen that these morphisms make the above diagram commutative, so it remains to show that this forms a distinguished triangle. Consider the following standard triangle:

$$M(f) \xrightarrow{u} M(gf) \xrightarrow{\alpha(u)} M(u) \xrightarrow{\beta(u)} M(f)[1]$$

We define morphisms $\phi : M(g) \to M(u)$ and $\psi : M(u) \to M(g)$ by:

$$\phi_i = \begin{pmatrix} 0 & 0 \\ 1_{Y_{i-1}} & 0 \\ 0 & 0 \\ 0 & 1_{Z_i} \end{pmatrix} : Y_{i-1} \oplus Z_i \to X_{i-2} \oplus Y_{i-1} \oplus X_{i-1} \oplus Z_i$$

$$\psi_i = \begin{pmatrix} 0 & 1_{Y_{i-1}} & f_{i-1} & 0 \\ 0 & 0 & 0 & 1_{Z_i} \end{pmatrix} : X_{i-2} \oplus Y_{i-1} \oplus X_{i-1} \oplus Z_i \to Y_{i-1} \oplus Z_i$$

We want to prove that the triples $(1_{M(f)}, 1_{M(gf)}, \phi)$ and $(1_{M(f)}, 1_{M(gf)}, \psi)$ are morphisms of triangles. By straightforward calculation, we see that $\psi \circ \alpha(u) = v$ and $\beta(u) \circ \phi = w$

so it remains to show that $\alpha(u) \simeq \phi \circ v$ and $\beta(u) \simeq w \circ \psi$. The family:

$$\begin{pmatrix} 1_{X_{i-1}} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} : X_{i-1} \oplus Z_i \to X_{i-1} \oplus Y_i \oplus X_i \oplus Z_{i+1}$$

defines a homotopy from $\alpha(u)$ to $\phi \circ v$, hence the triple $(1_{M(f)}, 1_{M(gf)}, \phi)$ is a morphism of triangles. Similarly, we define a homotopy $\beta(u) \to w \circ \psi$ by:

$$\begin{pmatrix} 0 & 0 & 1_{X_{i-1}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : X_{i-2} \oplus Y_{i-1} \oplus X_{i-1} \oplus Z_i \to X_{i-1} \oplus Y_i$$

so that $(1_{M(f)}, 1_{M(gf)}, \psi)$ is a morphism of triangles. Finally, we show that these morphisms are isomorphisms of triangles in $\mathbf{K}(\mathcal{A})$. By definition of ϕ and ψ we have $\psi \circ \phi = 1_{M(g)}$. The composite $\phi \circ \psi : M(u) \to M(u)$ is given in each degree by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1_{Y_{i-1}} & f_{i-1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{Z_i} \end{pmatrix}$$

and it can be easily verified that the morphisms:

$$\begin{pmatrix} 0 & 0 & -1_{X_{i-1}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is a homotopy from $\phi \circ \psi \to 1_{M(u)}$. Thus $\phi \circ \psi = 1_{M(u)}$ in $\mathbf{K}(\mathcal{A})$ so that $(1_{M(f)}, 1_{M(gf)}, \phi)$ is an isomorphism of triangles which completes the proof of the octahedral axiom. $\square$

3.2 The Derived Category and its Triangulation

In order to define the derived category, we first provide a brief overview of the theory of abelian categories. Abelian categories provide one of the most convenient settings for homological algebra, as they admit enough structure to treat familiar concepts such as exact sequences, homology as well as exact and derived functors in full generality.

Using these concepts, we shall define the derived category of an abelian category $\mathcal{A}$ as the localization of the homotopy category $\mathbf{K}(\mathcal{A})$ with respect to a certain class Q of chain morphisms. To this end, we shall devote the majority of this section to setting up the relevant theory required to define the class Q as well as showing that it is multiplicative system. We will then see that Q is compatible with the triangulated structure on $\mathbf{K}(\mathcal{A})$, which equips the derived category with the canonical triangulated structure inherited from $\mathbf{K}(\mathcal{A})$.

Definition 3.2.1. ([1] p.6) Let $\mathcal{A}$ be an additive category and let $f : X \to Y$ be a morphism in $\mathcal{A}$.

(i) The *kernel* of f is an object $\mathrm{Ker}(f)$ and a morphism $k : \mathrm{Ker}(f) \to X$ with $f \circ k = 0$, satisfying the following universal property. For any other morphism $m : Z \to X$ with $f \circ m = 0$, there is a unique morphism $\tilde{m} : Z \to \mathrm{Ker}(f)$ such that the following diagram commutes:

$$
\begin{tikzcd}
Z \arrow[drr, bend left, "0"] \arrow[dr, dashed, "\tilde{m}"] \arrow[ddr, bend right, "m"'] & & \\
& \mathrm{Ker}(f) \arrow[r, "0"] \arrow[d, "k"'] & Y \\
& X \arrow[ur, bend right, "f"'] &
\end{tikzcd}
$$

(ii) The *cokernel* of f is an object $\mathrm{Coker}(f)$ and a morphism $c : Y \to \mathrm{Coker}(f)$, with $c \circ f = 0$, satisfying the following universal property. For any other morphism $q : Y \to Z$, there is a unique morphism $\tilde{q} : \mathrm{Coker}(f) \to Z$ such that the following diagram commutes:

$$
\begin{tikzcd}
& Y \arrow[d, "c"] \arrow[ddr, bend left, "q"] & \\
X \arrow[ur, bend left, "f"] \arrow[r, "0"'] \arrow[ddrr, bend right, "0"'] & \mathrm{Coker}(f) \arrow[dr, dashed, "\tilde{q}"'] & \\
& & Z
\end{tikzcd}
$$

Clearly, whenever the kernel or cokernel of a morphism exists, they are unique up to canonical isomorphism. Moreover, it is easily proven from the universal properties that the morphisms $k : \mathrm{Ker}(f) \to X$ and $c : Y \to \mathrm{Coker}(f)$ is a monomorphism and an epimorphism respectively.

Example 3.2.2. From these definitions, it is easily proven that:

(i) If f is a monomorphism, then $\mathrm{Ker}(f) = 0$.

(ii) If f is an epimorphism, then $\mathrm{Coker}(f) = 0$.

(iii) For $0 : X \to Y$, we have $\mathrm{Ker}(0) = X$ and $\mathrm{Coker}(0) = Y$.

Definition 3.2.3. ([1], p.7) Let $\mathcal{A}$ be an additive category and let $f : X \to Y$ be a morphism in $\mathcal{A}$.

(i) If the cokernel of f exists and the morphism $c : Y \to \mathrm{Coker}(f)$ has a kernel, then the *image* of f is defined by:

$$\mathrm{Im}(f) := \mathrm{Ker}(c)$$

provided the kernel of c exists.

(ii) If the kernel of f exists and the morphism $k : \mathrm{Ker}(f) \to X$ has a cokernel, then the *coimage* of f is defined by:

$$\mathrm{Coim}(f) := \mathrm{Coker}(k : \mathrm{Ker}(f) \to X)$$

Proposition 3.2.4. ([1] p.7) Let $\mathcal{A}$ be an additive category and let $f : X \to Y$ be a morphism in $\mathcal{A}$. Suppose that the kernel, cokernel, image, and coimage of f exists. Then there exists a unique morphism $h : \mathrm{Coim}(f) \to \mathrm{Im}(f)$ such that f factors as the composite:

$$X \xrightarrow{\tilde{c}} \mathrm{Coim}(f) \xrightarrow{h} \mathrm{Im}(f) \xrightarrow{\tilde{k}} Y$$

Proof. Since the composite $c \circ f : X \to \mathrm{Coker}(f)$ is zero by definition, there is an induced morphism $\tilde{f} : X \to \mathrm{Im}(f)$ making the following diagram commute:

$$\begin{tikzcd}
X \arrow[dr, dashed, "\tilde{f}"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "f"'] & & \\
& \mathrm{Im}(f) \arrow[r, "0"] \arrow[d, "\tilde{k}"'] & \mathrm{Coker}(f) \\
& Y \arrow[ur, bend right, "c"'] &
\end{tikzcd}$$

Moreover, since $f \circ k = 0$ we have that $\tilde{f} \circ k : \mathrm{Ker}(f) \to \mathrm{Im}(f)$ is zero since $\tilde{k}$ is a monomorphism. This induces a unique morphism $h : \mathrm{Coim}(f) \to \mathrm{Im}(f)$ making the following diagram commute:

$$\begin{tikzcd}
& X \arrow[d, "\tilde{c}"] \arrow[ddr, bend left, "\tilde{f}"] & \\
\mathrm{Ker}(f) \arrow[ur, bend left, "k"] \arrow[r, "0"'] \arrow[drr, bend right, "0"'] & \mathrm{Coim}(f) \arrow[dr, dashed, "h"] & \\
& & \mathrm{Im}(f)
\end{tikzcd}$$

Finally, collecting the morphisms we obtain $f = \tilde{k} \circ h \circ \tilde{c}$ as desired. Uniqueness of h follows from the fact that $\tilde{k}$ is a monomorphism and $\tilde{c}$ is an epimorphism. □

Definition 3.2.5. ([1] p.8) An additive category $\mathcal{A}$ is said to be *abelian* if the following holds:

(i) Every morphism in $\mathcal{A}$ has a kernel and a cokernel.

(ii) The canonical morphism $h : \mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism for every f in $\mathcal{A}$.

Example 3.2.6.

(i) The categories R-**Mod** and **Mod**-R are abelian for any ring R.

(ii) If $\mathcal{A}$ is abelian, then so is the opposite category $\mathcal{A}^{\mathbf{op}}$.

(iii) The category $\mathbf{Vect}_2(K)$ is not abelian.

Proposition 3.2.7. ([1] p.9) If $\mathcal{A}$ is abelian, then $\mathbf{Ch}(\mathcal{A})$ is abelian.

Proof. We know from before that $\mathbf{Ch}(\mathcal{A})$ is additive. Let $f : A \to B$ be a chain morphism and consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Ker}(f_i) \arrow[r, "k_i"] & A_i \arrow[r, "f_i"] \arrow[d, "d_i^A"'] & B_i \arrow[d, "d_i^B"] \\
\mathrm{Ker}(f_{i-1}) \arrow[r, "k_{i-1}"] & A_{i-1} \arrow[r, "f_{i-1}"'] & B_{i-1}
\end{tikzcd}$$

Since $f_{i-1} \circ d_i^A \circ k_i = d_i^B \circ f_i \circ k_i = 0$, there is a unique morphism d_i^K such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Ker}(f_i) \arrow[r, "k_i"] \arrow[d, "d_i^K"'] & A_i \arrow[d, "d_i^A"] \\
\mathrm{Ker}(f_{i-1}) \arrow[r, "k_{i-1}"'] & A_{i-1}
\end{tikzcd}$$

Using commutativity of this diagram, we have

$$k_{i-1} \circ d_i^K \circ d_{i+1}^K = d_i^A \circ k_i \circ d_{i+1}^K = d_i^A \circ d_{i+1}^A \circ k_{i+1} = 0$$

so it follows that $d_i^K \circ d_{i+1}^K = 0$ since k_{i-1} is a monomorphism. Thus the objects $\mathrm{Ker}(f_i)$ equipped with the morphisms d_i^K defines a chain complex $\mathrm{Ker}(f)$ in $\mathcal{A}$.

It remains to check that $\mathrm{Ker}(f)$ satisfies the universal property of the kernel. By construction of $\mathrm{Ker}(f)$, the family $k_i : \mathrm{Ker}(f_i) \to A_i$ is a chain morphism $k : \mathrm{Ker}(f) \to A$ with the property that $f \circ k = 0$. Suppose then that $m : C \to A$ is a chain morphism such that $f \circ m = 0$. Then by the universal property of the kernel in $\mathcal{A}$, there is a unique morphism $\tilde{m}_i : C_i \to \mathrm{Ker}(f_i)$ for each i such that $k_i \circ \tilde{m}_i = m_i$. This is a chain morphism since:

$$\begin{aligned}
k_{i-1} \circ \tilde{m}_{i-1} \circ d_i^C &= m_{i-1} \circ d_i^C \\
&= d_i^A \circ m_i \\
&= d^A \circ k_i \circ \tilde{m}_i = k_{i-1} \circ d_i^K \circ \tilde{m}_i
\end{aligned}$$

so it follows that $d_i^{\mathrm{Ker}(f)} \circ \tilde{m}_i = \tilde{m}_{i-1} \circ d_i^C$ as k_{i-1} is mono.

Dually, one constructs the cokernel complex $\mathrm{Coker}(f)$ for each chain morphism f, which is given degreewise by $\mathrm{Coker}(f_i)$.

Finally, the canonical morphism $h : \mathrm{Coim}(f) \to \mathrm{Im}(f)$ in $\mathbf{Ch}(\mathcal{A})$ is given in each degree by the canonical morphism $h_i : \mathrm{Coim}(f_i) \to \mathrm{Im}(f_i)$ in $\mathcal{A}$. Since $\mathcal{A}$ is abelian, each h_i is an isomorphism so h is an isomorphism in $\mathbf{Ch}(\mathcal{A})$. □

We shall now briefly discuss homology theory in an arbitrary abelian category. It can easily be proven that for every chain complex A in $\mathbf{Ch}(\mathcal{A})$, the chain complex property $d_i \circ d_{i+1} = 0$ induces a monomorphism $\rho_i : \mathrm{Im}(d_{i+1}) \to \mathrm{Ker}(d_i)$ for each i. Using this, we define:

Definition 3.2.8.

(i) The i'th *homology* of A is given by $H_i(A) := \mathrm{Coker}(\rho_i : \mathrm{Im}(d_{i+1}) \to \mathrm{Ker}(d_i))$.

(ii) We say that A is *exact* at i if $H_i(A) = 0$, that is if ρ_i is an isomorphism. If all $H_i(A)$ are zero, then A is said to be an *exact sequence*.

We shall state the next few results without proofs as they are beyond the scope of this paper, for a more comprehensive treatment of abelian categories, see chapter 12 in [3] or chapter VIII in [2].

Proposition 3.2.9. For every chain morphism $f : A \to B$, there is an induced morphism $H_i(f) : H_i(A) \to H_i(B)$. In particular, this defines an additive functor $H_i : \mathbf{Ch}(\mathcal{A}) \to \mathcal{A}$.

Many important results about exactness and homology from classical homological algebra hold in any abelian category, most notably the five-lemma and the snake-lemma. Because of this, we also have the following result:

Theorem 3.2.10. Given a short exact sequence in $\mathbf{Ch}(\mathcal{A})$:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

There is a long exact sequence in homology:

$$\begin{tikzcd} \cdots \arrow[r] & H_i(A) \arrow[r, "H_i(f)"] & H_i(B) \arrow[r, "H_i(g)"] & H_i(C) \arrow[dll, rounded corners, "\delta_i"] & \\ & H_{i-1}(A) \arrow[r, "H_{i-1}(f)"] & H_{i-1}(B) \arrow[r, "H_{i-1}(g)"] & H_{i-1}(C) \arrow[r] & \cdots \end{tikzcd}$$

Proposition 3.2.11. Let $f, g : A \to B$ be chain morphisms and suppose that $f \simeq g$. Then $H_i(f) = H_i(g)$.

A direct consequence of this proposition is that the homology functor $H_i : \mathbf{Ch}(\mathcal{A}) \to \mathcal{A}$ lifts to a functor $H_i : \mathbf{K}(\mathcal{A}) \to \mathcal{A}$.

Definition 3.2.12. Let $f : A \to B$ be a chain morphism. f is called a *quasi-isomorphism* if the induced morphism $H_i(f) : H_i(A) \to H_i(B)$ is an isomorphism for all i.

Note that this definition makes sense in both $\mathbf{Ch}(\mathcal{A})$ and $\mathbf{K}(\mathcal{A})$ by Proposition 3.2.11.

Example 3.2.13. An object P in $\mathcal{A}$ is said to be *projective* if there is a commutative diagram:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\tilde{p}"'] \arrow[d, "p"] & \\ X \arrow[r, "f"'] & Y \arrow[r] & 0 \end{tikzcd}$$

for every p and every f such that the bottom row is exact. Now, given an object X in $\mathcal{A}$, a *projective resolution* of X is an exact sequence:

$$\dots \longrightarrow P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} X \longrightarrow 0$$

with all P_i projective. If we regard X as a chain complex concentrated in degree zero, then a projective resolution of X can be expressed as a chain morphism $p : P \to X$ as in the diagram:

$$\begin{tikzcd}
P : \arrow[d, "p"] & \dots \arrow[r] & P_2 \arrow[r, "p_2"] \arrow[d] & P_1 \arrow[r, "p_1"] \arrow[d] & P_0 \arrow[r] \arrow[d, "p_0"] & 0 \\
X : & \dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

We claim that this is a quasi-isomorphism. In both rows in the diagram above, the homology is zero everywhere except in degree zero, and we see that $H_0(X) = X$. To compute $H_0(P)$, we see that $\mathrm{Im}(p_1) = \mathrm{Ker}(p_0)$ by exactness of the resolution. Then:

$$\begin{aligned}
H_0(P) &= \mathrm{Coker}(\mathrm{Im}(p_1) \hookrightarrow P_0) \\
&= \mathrm{Coker}(\mathrm{Ker}(p_0) \hookrightarrow P_0) \\
&= \mathrm{Coim}(p_0) \\
&\cong \mathrm{Im}(p_0)
\end{aligned}$$

By exactness, p_0 is an epimorphism, hence $H_0(P) \cong X$. In this case, $H_0(p) : H_0(P) \to H_0(X)$ is the canonical isomorphism $\mathrm{Coim}(p_0) \to \mathrm{Im}(p_0)$, so we conclude that $p : P \to X$ is a quasi-isomorphism.

We may also define injective objects and injective resolutions in a dual way to their projective counterparts. In this case, an injective resolution is a quasi-isomorphism $q : X \to I$ where I is a complex of injective modules concentrated in negative degrees.

Lemma 3.2.14. Given a chain morphism $f : A \to B$ in $\mathbf{K}(\mathcal{A})$, the mapping cone gives rise to an exact sequence in $\mathbf{Ch}(\mathcal{A})$:

$$0 \longrightarrow B \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} A[1] \longrightarrow 0$$

Then by Theorem 3.2.10 there is a long exact sequence of homology objects:

$$\cdots \longrightarrow H_{i+1}(A[1]) \xrightarrow{\delta_{i+1}} H_i(B) \xrightarrow{H_i(\alpha(f))} H_i(M(f)) \xrightarrow{H_i(\beta(f))} H_i(A[1]) \xrightarrow{\delta_i} H_{i-1}(B) \longrightarrow \cdots$$

and we claim that $\delta_i = H_{i-1}(f)$ for all i.

While we omit the proof of this lemma, it should be noted that it makes sense to assert that $\delta_i = H_{i-1}(f)$ since $H_i(A[1]) = H_{i-1}(A)$. Using this result, we prove the following useful proposition:

Proposition 3.2.15. ([1], p.27) A chain morphism $f : A \to B$ in $\mathbf{Ch}(\mathcal{A})$ is a quasi-isomorphism if and only if the mapping cone $M(f)$ is exact.

Proof. The mapping cone gives rise to a short exact sequence of complexes:

$$0 \longrightarrow B \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} A[1] \longrightarrow 0$$

Then by Lemma 3.2.14 there is a long exact sequence of homology:

$$\cdots \longrightarrow H_{i+1}(A[1]) \xrightarrow{H_i(f)} H_i(B) \xrightarrow{H_i(\alpha(f))} H_i(M(f)) \xrightarrow{H_i(\beta(f))} H_i(A[1]) \xrightarrow{H_{i-1}(f)} H_{i-1}(B) \longrightarrow \cdots$$

If f is a quasi-isomorphism, then $H_i(f)$ is an isomorphism so exactness of the sequence above tells us that $H_i(\alpha(f)) = 0$ and $H_i(\beta(f)) = 0$. Then $H_i(M(f)) = 0$ so $M(f)$ is exact as desired.

Conversely, suppose $M(f)$ is exact. Then for each i, we obtain the following exact sequence:

$$0 \longrightarrow H_i(A) \xrightarrow{H_i(f)} H_i(B) \longrightarrow 0$$

so $H_i(f)$ is an isomorphism. Thus f is a quasi-isomorphism. $\square$

Remark. If we now consider a distinguished triangle in $\mathbf{K}(\mathcal{A})$:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

Then an immediate consequence of the preceding result is that f is a quasi-isomorphism if and only if the complex Z is exact.

Proposition 3.2.16. ([1], p.30-31) The class Q of quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$ is a multiplicative system.

Proof. (M1) follows from functoriality of the homology functors.

(M2) Consider any morphism $f : X \to Y$ and any quasi-isomorphism $t : Y' \to Y$. We may extend t to a distinguished triangle:

$$Y' \xrightarrow{t} Y \xrightarrow{\alpha(t)} M(t) \xrightarrow{\beta(t)} Y'[1]$$

We also extend the morphism $\alpha(t) \circ f$ to a distinguished triangle of the form:

$$X' \xrightarrow{s} X \xrightarrow{\alpha(t)\circ f} M(t) \xrightarrow{u} X'[1]$$

If we apply the rotation axiom to these triangles, we obtain commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "\alpha(t)\circ f"] \arrow[d, "f"'] & M(t) \arrow[r, "u"] \arrow[d, "1"] & X'[1] \arrow[r, "-s[1]"] & X[1] \arrow[d, "f[1]"] \\
Y \arrow[r, "\alpha(t)"'] & M(t) \arrow[r, "\beta(t)"'] & Y'[1] \arrow[r, "-t[1]"'] & Y[1]
\end{tikzcd}$$

so by (A4), there is a morphism $f'[1] : X'[1] \to Y'[1]$ making the whole diagram commute. Applying the rotation axiom in reverse, we obtain a commutative diagram:

$$\begin{tikzcd} X' \arrow[r, "s"] \arrow[d, "f'"'] & X \arrow[r, "\alpha(t)\circ f"] \arrow[d, "f"'] & M(t) \arrow[r, "u"] \arrow[d, "1"] & X'[1] \arrow[d, "f'[1]"] \\ Y' \arrow[r, "t"'] & Y \arrow[r, "\alpha(t)"'] & M(t) \arrow[r, "\beta(t)"'] & Y'[1] \end{tikzcd}$$

Now, $M(t)$ is exact since t is a quasi-isomorphism. Then s is a quasi-isomorphism, so we obtain the commutative square:

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "s"'] & Y' \arrow[d, "t"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

as desired.

(M3) Suppose $t \circ f = t \circ g$ where $t : Y \to Y'$ is a quasi-isomorphism and $f, g : X \to Y$ are abitrary morphisms. By (A2) and the rotation axiom, we can find a distinguished triangle of the form:

$$Z \xrightarrow{u} Y \xrightarrow{t} Y' \xrightarrow{v} Z[1]$$

Since t is a quasi-isomorphism, both Z and $Z[1]$ are exact. By applying the rotation axiom and (A4), we can find a morphism $w : X \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "w"] & X \arrow[r] \arrow[d, "f-g"] & 0 \arrow[r] \arrow[d] & X[1] \arrow[d, "w[1]"] \\ Z \arrow[r, "u"'] & Y \arrow[r, "t"'] & Y' \arrow[r, "v"'] & Z[1] \end{tikzcd}$$

Now, if we extend w to a distinguished triangle:

$$X' \xrightarrow{s} X \xrightarrow{w} Z \xrightarrow{h} X'[1]$$

Then s is a quasi-isomorphism since Z is exact. In particular, we see that $(f - g) \circ s = u \circ w \circ s$ and $w \circ s = 0$ by Proposition 2.2.2. Thus $f \circ s = g \circ s$ which proves (M3). This shows that Q is a right multiplicative system. It follows from a similar proof that Q is a left multiplicative system, so we conclude that Q is multiplicative. $\square$

Since Q is a multiplicative system, we may consider the localization $Q^{-1}\mathbf{K}(\mathcal{A})$. We shall refer to this category as the *derived category* of $\mathcal{A}$ and we denote it by $\mathbf{D}(\mathcal{A})$. By Proposition 2.3.12, this category is additive and comes equipped with an additive localization functor $L : \mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$.

Proposition 3.2.17. ([3], Section 13.5) Quasi-isomorphisms are compatible with the triangulated structure on $\mathbf{K}(\mathcal{A})$.

Proof. If $f : A \to B$ is a quasi-isomorphism then clearly $f[1]$ is a quasi-isomorphism as well. Next, consider the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"'] & Z \arrow[r, "h"] \arrow[d, "w"] & X[1] \arrow[d, "u[1]"] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & X'[1]
\end{tikzcd}$$

By Lemma 3.2.14, passing to homology yields a commutative diagram in $\mathcal{A}$:

$$\begin{tikzcd}
H_i(X) \arrow[r, "H_i(f)"] \arrow[d, "H_i(u)"'] & H_i(Y) \arrow[r, "H_i(g)"] \arrow[d, "H_i(v)"'] & H_i(Z) \arrow[r, "H_i(h)"] \arrow[d, "H_i(w)"] & H_i(X[1]) \arrow[r, "H_i(f[1])"] \arrow[d, "H_i(u[1])"] & H_i(Y[1]) \arrow[d, "H_i(v[1])"] \\
H_i(X') \arrow[r, "H_i(f')"'] & H_i(Y') \arrow[r, "H_i(g')"'] & H_i(Z') \arrow[r, "H_i(h')"'] & H_i(X'[1]) \arrow[r, "H_i(f'[1])"'] & H_i(Y'[1])
\end{tikzcd}$$

where the rows are exact. It now follows from the five-lemma that $H_i(w)$ is an isomorphism, hence w is a quasi-isomorphism. □

By Theorem 2.3.18, this result tells us that $\mathbf{D}(\mathcal{A})$ can be equipped with the structure of a triangulated category such that the localization functor $L : \mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ is exact.

We have now produced two examples of triangulated categories; the homotopy category and the derived category. Other important examples of triangulated categories arise as subcategories of these two. Quite often, one only considers *bounded* complexes in $\mathcal{A}$, that is complexes that vanish for sufficiently large degrees. More precisely, we can consider the following subcategories of $\mathbf{K}(\mathcal{A})$ and $\mathbf{Ch}(\mathcal{A})$:

- The subcategories $\mathbf{K}^+(\mathcal{A})$ and $\mathbf{D}^+(\mathcal{A})$ spanned by complexes that are bounded below, that is the complexes that vanish for all degrees $i \ll 0$.
- The subcategories $\mathbf{K}^-(\mathcal{A})$ and $\mathbf{D}^-(\mathcal{A})$ spanned by complexes that are bounded above, that is the complexes that vanish for all degrees $i \gg 0$.
- The subcategories $\mathbf{K}^b(\mathcal{A})$ and $\mathbf{D}^b(\mathcal{A})$ spanned by bounded complexes, that is the complexes that vanish for all degrees $|i| \gg 0$.

It an easy exercise to check that these subcategories are triangulated, indeed it follows from the fact that if f is a chain morphism between complexes satisfying one of the above boundedness criteria, then the mapping cone satisfies the same boundedness criterion.

We finish this paper with a result demonstrating of the usefulness of the derived category. First, we say that an abelian category as *enough projectives* if every object admits a projective resolution. Similarly, an abelian category has *enough injectives* if every object admits an injective resolution.

Theorem 3.2.18. Let $\mathcal{A}$ be an abelian category with enough projectives and let M, N be objects in $\mathcal{A}$. Then for every integer i, there is an isomorphism

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(M, N[i]) \cong \mathrm{Ext}^i_{\mathcal{A}}(M, N)$$

where M and N are regarded as complexes concentrated in degree zero.

Proof. First, take any projective resolution $p : P \to M$ of M. As discussed in Example 3.2.13, this is a quasi-isomorphism in $\mathbf{K}(\mathcal{A})$ and therefore an isomorphism in $\mathbf{D}(\mathcal{A})$. Thus

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(M, N[i]) \cong \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(P, N[i])$$

Next, we require the following technical lemma:

Lemma 3.2.19. ([4], Corollary 10.4.7) Let P be a complex of projective objects that is bounded below. Then

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(P, X) \cong \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(P, X)$$

for every complex X.

By applying this lemma, we obtain:

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(P, N[i]) \cong \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(P, N[i])$$

Now, note that by definition hom-sets in $\mathbf{K}(\mathcal{A})$ are obtained from $\mathbf{Ch}(\mathcal{A})$ by taking the quotients of the subgroups consisting of the null-homotopic chain maps. To compute $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(P, N[i])$, we first consider an arbitrary chain morphism $f : P \to N[i]$ in $\mathbf{Ch}(\mathcal{A})$. Since $N[i]$ is concentrated in degree i, this amounts to a morphism $f_i : P_i \to N$ in $\mathcal{A}$ characterized by the property $f_i \circ d_{i+1} = 0$. In other words, $f_i \in \mathrm{Ker}(- \circ d_{i+1})$, hence $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{A})}(P, N[i]) \cong \mathrm{Ker}(- \circ d_{i+1})$.

Next, suppose f is null-homotopic. By the above, a homotopy from f to zero amounts to a morphism $s_i : P_{i-1} \to N$ such that $f_i = s_i \circ d_i$. In other words f is null-homotopic if and only if $f_i \in \mathrm{Im}(- \circ d_i)$. Thus

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(P, N[i]) \cong \frac{\mathrm{Ker}(- \circ d_{i+1})}{\mathrm{Im}(- \circ d_i)} \cong \mathrm{Ext}^i_{\mathcal{A}}(M, N)$$

as desired. □

This result is remarkable, as it demonstrates that the derived category is the natural setting for studying derived functors. In particular, we see that $\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(M, N) \cong \mathrm{Ext}^0_{\mathcal{A}}(M, N) = \mathrm{Hom}_{\mathcal{A}}(M, N)$. An immediate consequence of this is that the functor $\mathcal{A} \to \mathbf{D}(\mathcal{A})$ mapping each object in $\mathcal{A}$ to the corresponding complex concentrated in degree zero is fully faithful. In this sense, the derived category is really an extension of $\mathcal{A}$ that naturally encodes resolutions and derived functors on $\mathcal{A}$.

References

[1] Thorsten Holm, Peter Jørgensen, and Raphaël Rouquier. *Triangulated categories*, volume 375. Cambridge University Press, 2010.

[2] Saunders Mac Lane. *Categories for the working mathematician.* Springer Verlag, 2nd edition, 1998.

[3] The Stacks Project Authors. *Stacks Project.* `https://stacks.math.columbia.edu`, 2022.

[4] Charles A Weibel. *An introduction to homological algebra.* Number 38. Cambridge university press, 1995.