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Exact categories

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Abstract

We survey the basics of homological algebra in exact categories in the sense of Quillen. All diagram lemmas are proved directly from the axioms, notably the five lemma, the 3×3-lemma and the snake lemma. We briefly discuss exact functors, idempotent completion and weak idempotent completeness. We then show that it is possible to construct the derived category of an exact category without any embedding into abelian categories and we sketch Deligne's approach to derived functors. The construction of classical derived functors with values in an abelian category painlessly translates to exact categories, i.e., we give proofs of the comparison theorem for projective resolutions and the horseshoe lemma. After discussing some examples we elaborate on Thomason's proof of the Gabriel–Quillen embedding theorem in an appendix.

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1. Introduction

There are several notions of exact categories. On the one hand, there is the notion in the context of additive categories commonly attributed to Quillen [51] with which the present article is concerned; on the other hand, there is the non-additive notion due to Barr [3], to mention but the two most prominent ones. While Barr's definition is intrinsic and an additive category is exact in his sense if and only if it is abelian, Quillen's definition is extrinsic in

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that one has to specify a distinguished class of short exact sequences (an exact structure) in order to obtain an exact category.

From now on we shall only deal with additive categories, so functors are tacitly assumed to be additive. On every additive category $\mathscr{A}$ the class of all split exact sequences provides the smallest exact structure, i.e., every other exact structure must contain it. In general, an exact structure consists of kernel–cokernel pairs subject to some closure requirements, so the class of all kernel–cokernel pairs is a candidate for the largest exact structure. It is quite often the case that the class of all kernel–cokernel pairs is an exact structure, but this fails in general: Rump [53] constructs an example of an additive category with kernels and cokernels whose kernel–cokernel pairs fail to be an exact structure.

It is commonplace that basic homological algebra in categories of modules over a ring extends to abelian categories. By using the Freyd–Mitchell full embedding theorem [17,47], diagram lemmas can be transferred from module categories to general abelian categories, i.e., one may argue by chasing elements around in diagrams. There is a point in proving the fundamental diagram lemmas directly, and be it only to familiarize oneself with the axioms. A careful study of what is actually needed reveals that in most situations the axioms of exact categories are sufficient. An *a posteriori* reason is provided by the Gabriel–Quillen embedding theorem which reduces homological algebra in exact categories to the case of abelian categories, the slogan is "relative homological algebra made absolute" (Freyd [16]). More specifically, the embedding theorem asserts that the Yoneda functor embeds a small exact category $\mathscr{A}$ fully faithfully into the *abelian* category $\mathscr{B}$ of left exact functors $\mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ in such a way that the essential image is closed under extensions and that a short sequence in $\mathscr{A}$ is short exact if and only if it is short exact in $\mathscr{B}$. Conversely, it is not hard to see that an extension-closed subcategory of an abelian category is exact – this is *the* basic recognition principle for exact categories. In Appendix A we present Thomason's proof of the Gabriel–Quillen embedding theorem for the sake of completeness, but we will not apply this result in the present notes. The author is convinced that the embedding theorem should be used to transfer the intuition from abelian categories to exact categories rather than to prove (simple) theorems with it. A direct proof from the axioms provides much more insight than a reduction to abelian categories.

The interest of exact categories is manifold. First of all they are a natural generalization of abelian categories and there is no need to argue that abelian categories are both useful and important. There are several reasons for going beyond abelian categories. The fact that one may *choose* an exact structure gives more flexibility which turns out to be essential in many contexts. Even if one is working with abelian categories one soon finds the need to consider other exact structures than the canonical one, for instance in relative homological algebra [30]. Beyond this, there are quite a few "cohomology theories" which involve functional analytic categories like locally convex modules over a topological group [31,9], locally compact abelian groups [32] or Banach modules over a Banach algebra [33,26] where there is no obvious abelian category around to which one could resort. In more advanced topics of algebra and representation theory (e.g. filtered objects, tilting theory), exact categories arise naturally, while the theory of abelian categories simply does not fit. It is an observation due to Happel [25] that in guise of *Frobenius categories,* exact categories give rise to triangulated categories by passing to the associated stable categories, see Section 13.4. Further fields of application are algebraic geometry (certain categories of vector bundles), algebraic analysis

($\mathscr{D}$-modules) and, of course, algebraic K-theory (Quillen's Q-construction [51], Balmer's Witt groups [2] and Schlichting's Grothendieck–Witt groups [54]). The reader will find a slightly more extensive discussion of some of the topics mentioned above in Section 13.

The author hopes to convince the reader that the axioms of exact categories are quite convenient for giving relatively painless proofs of the basic results in homological algebra and that the gain in generality comes with almost no effort. Indeed, it even seems that the axioms of exact categories are more adequate for proving the fundamental diagram lemmas than Grothendieck's axioms for abelian categories [24]. For instance, it is quite a challenge to find a complete proof (directly from the axioms) of the snake lemma for abelian categories in the literature.

That being said, we turn to a short description of the contents of this paper.

In Section 2 we state and discuss the axioms and draw the basic consequences, in particular we give a characterization of certain push-out squares and Keller's proof of the redundancy of the obscure axiom.

In Section 3 we prove the (short) five lemma, the Noether isomorphism theorem and the 3×3-lemma.

Section 4 briefly discusses quasi-abelian categories, a source of many examples of exact categories. Contrary to the notion of an exact category, the property of being quasi-abelian is intrinsic.

Exact functors are briefly touched upon in Section 5 and after that we treat the idempotent completion and the property of weak idempotent completeness in Sections 6 and 7.

We come closer to the heart of homological algebra when discussing admissible morphisms, long exact sequences, the five lemma and the snake lemma in Section 8. In order for the snake lemma to hold, the assumption of weak idempotent completeness is necessary.

After that we briefly remind the reader of the notions of chain complexes and chain homotopy in Section 9, before we turn to acyclic complexes and quasi-isomorphisms in Section 10. Notably, we give an elementary proof of Neeman's crucial result [48] that the homotopy category of acyclic complexes is triangulated. We do not indulge in the details of the construction of the derived category of an exact category because this is well treated in the literature. We give a brief summary of the derived category of fully exact subcategories and then sketch the main points of Deligne's approach to total derived functors on the level of the derived category as expounded by Keller [39].

On a more leisurely level, projective and injective objects are discussed in Section 11 preparing the grounds for a treatment of classical derived functors (with values in an abelian category) in Section 12, where we state and prove the resolution lemma, the comparison theorem and the horseshoe lemma, i.e., the three basic ingredients for the classical construction.

We end with a short list of examples and applications in Section 13.

In Appendix A we give Thomason's proof of the Gabriel–Quillen embedding theorem of an exact category into an abelian one. Finally, in Appendix B we give a proof of the folklore fact that under the assumption of weak idempotent completeness Heller's axioms for an "abelian" category are equivalent to Quillen's axioms for an exact category.

Historical note. Quillen's notion of an exact category has its predecessors e.g. in Heller [27], Buchsbaum [10], Yoneda [61], Butler–Horrocks [13] and Mac Lane [44, XII.4]. It should be noted that Buchsbaum, Butler–Horrocks and Mac Lane assume the existence of an ambient abelian category and miss the crucial push-out and pull-back axioms, while Heller and Yoneda anticipate Quillen's definition. According to Quillen [51, p. "92/16/100"], assuming idempotent completeness, Heller's notion of an "abelian category" [27, §3], i.e., an additive category equipped with an "abelian class of short exact sequences" coincides with the present definition of an exact category. We give a proof of this assertion in Appendix B. Yoneda's quasi-abelian $\mathscr{S}$-categories are nothing but Quillen's exact categories and it is remarkable that Yoneda proves that Quillen's "obscure axiom" follows from his definition, see [61, p. 525, Corollary], a fact rediscovered thirty years later by Keller in [37, A.1].

Prerequisites. The prerequisites are kept at a minimum. The reader should know what an additive category is and be familiar with fundamental categorical concepts such as kernels, pull-backs, products and duality. Acquaintance with basic category theory as presented in Hilton–Stammbach [29, Chapter II] or Weibel [60, Appendix A] should amply suffice for a complete understanding of the text, up to Section 10 where we assume some familiarity with the theory of triangulated categories.

Disclaimer. This article is written for the reader who *wants* to learn about exact categories and knows *why*. Very few motivating examples are given in this text.

The author makes no claim to originality. All the results are well-known in some form and they are scattered around in the literature. The *raison d'être* of this article is the lack of a systematic *elementary* exposition of the theory. The works of Heller [27], Keller [37,39] and Thomason [58] heavily influenced the present paper and many proofs given here can be found in their papers.

2. Definition and basic properties

In this section we introduce the notion of an exact category and draw the basic consequences of the axioms. We do not use the minimal axiomatics as provided by Keller [37, Appendix A] but prefer to use a convenient self-dual presentation of the axioms due to Yoneda [61, §2] (modulo some of Yoneda's numerous 3×2-lemmas and our Proposition 2.12). The author hopes that the Bourbakists among the readers will pardon this *faux pas*. We will discuss that the present axioms are equivalent to Quillen's [51, §2] in the course of events. The main points of this section are a characterization of certain push-out squares (Proposition 2.12) and the obscure axiom (Proposition 2.16).

Definition 2.1. Let $\mathscr{A}$ be an additive category. A *kernel–cokernel pair* (i, p) in $\mathscr{A}$ is a pair of composable morphisms

$$A' \xrightarrow{i} A \xrightarrow{p} A''$$

such that i is a kernel of p and p is a cokernel of i. If a class $\mathscr{E}$ of kernel–cokernel pairs on $\mathscr{A}$ is fixed, an *admissible monic* is a morphism i for which there exists a morphism p such that $(i, p) \in \mathscr{E}$. *Admissible epics* are defined dually. We depict admissible monics by $\rightarrowtail$ and admissible epics by $\twoheadrightarrow$ in diagrams.

An *exact structure* on $\mathscr{A}$ is a class $\mathscr{E}$ of kernel–cokernel pairs which is closed under isomorphisms and satisfies the following axioms:

[E0] For all objects $A \in \mathscr{A}$, the identity morphism 1_A is an admissible monic.

[E0$^{\mathrm{op}}$] For all objects $A \in \mathscr{A}$, the identity morphism 1_A is an admissible epic.

[E1] The class of admissible monics is closed under composition.

[E1$^{\mathrm{op}}$] The class of admissible epics is closed under composition.

[E2] The push-out of an admissible monic along an arbitrary morphism exists and yields an admissible monic.

[E2$^{\mathrm{op}}$] The pull-back of an admissible epic along an arbitrary morphism exists and yields an admissible epic.

Axioms [E2] and [E2$^{\mathrm{op}}$] are subsumed in the diagrams

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\text{PO}"] & B \arrow[d, dotted] \\ A' \arrow[r, tail, dotted] & B' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A' \arrow[r, two heads, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\text{PB}"] & B' \arrow[d] \\ A \arrow[r, two heads] & B \end{tikzcd}$$

respectively.

An *exact category* is a pair $(\mathscr{A}, \mathscr{E})$ consisting of an additive category $\mathscr{A}$ and an exact structure $\mathscr{E}$ on $\mathscr{A}$. Elements of $\mathscr{E}$ are called *short exact sequences*.

Remark 2.2. Note that $\mathscr{E}$ is an exact structure on $\mathscr{A}$ if and only if $\mathscr{E}^{\mathrm{op}}$ is an exact structure on $\mathscr{A}^{\mathrm{op}}$. This allows for reasoning by dualization.

Remark 2.3. Isomorphisms are admissible monics and admissible epics. Indeed, this follows from the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f", "\cong"'] \arrow[d, "1_A"', "\cong"] & B \arrow[r] \arrow[d, "f^{-1}"', "\cong"] & 0 \arrow[d, "\cong"'] \\ A \arrow[r, tail, "1_A"] & A \arrow[r, two heads] & 0, \end{tikzcd}$$

the fact that exact structures are assumed to be closed under isomorphisms and that the axioms are self-dual.

Remark 2.4 (*Keller [37, App. A]*). The axioms are somewhat redundant and can be weakened. For instance, let us assume instead of [E0] and [E0$^{\mathrm{op}}$] that 1_0, the identity of the zero object, is an admissible epic. For any object A there is the pull-back diagram

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d] \arrow[dr, phantom, "\text{PB}"] & A \arrow[d] \\ 0 \arrow[r, "1_0"] & 0 \end{tikzcd}$$

so [E2$^{\mathrm{op}}$] together with our assumption on 1_0 shows that [E0$^{\mathrm{op}}$] holds. Since 1_0 is a kernel of itself, it is also an admissible monic, so we conclude by [E2] that [E0] holds as well.

More importantly, Keller proves in [37] (A.1, proof of the proposition, step 3) that one can also dispose of one of [E1] or [E1$^{\mathrm{op}}$]. Moreover, he mentions (A.2, Remark), that one may also weaken one of [E2] or [E2$^{\mathrm{op}}$] – this is a straightforward consequence of Proposition 3.1.

Remark 2.5. Keller [37,39] uses *conflation*, *inflation* and *deflation* for what we call short exact sequence, admissible monic and admissible epic. This terminology stems from Gabriel–Roĭter [21, Chapter 9] who give a list of axioms for exact categories whose underlying additive category is weakly idempotent complete in the sense of Section 7, see Keller's appendix to [15] for a thorough comparison of the axioms. A variant of the Gabriel–Roĭter-axioms appear in Freyd's book on abelian categories [17, Chapter 7, Exercise G, p. 153] (the Gabriel–Roĭter-axioms are obtained from Freyd's axioms by adding the dual of Freyd's condition (2)).

Exercise 2.6. An admissible epic which is additionally monic is an isomorphism.

Lemma 2.7. *The sequence*

$$A \overset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\rightarrowtail} A \oplus B \overset{[\,0\ 1\,]}{\twoheadrightarrow} B$$

is short exact.

Proof. The following diagram is a push-out square:

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\text{PO}"] & B \arrow[d, tail, "\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]"] \\ A \arrow[r, tail, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"] & A \oplus B. \end{tikzcd}$$

The top arrow and the left-hand arrow are admissible monics by [E0$^{\mathrm{op}}$] while the bottom arrow and the right-hand arrow are admissible monics by [E2]. The lemma now follows from the facts that the sequence in question is a kernel–cokernel pair and that $\mathscr{E}$ is closed under isomorphisms. $\square$

Remark 2.8. Lemma 2.7 shows that Quillen's axiom a) [51, §2] stating that split exact sequences belong to $\mathscr{E}$ follows from our axioms. Conversely, Quillen's axiom a) obviously implies [E0] and [E0$^{\mathrm{op}}$]. Quillen's axiom b) coincides with our axioms [E1], [E1$^{\mathrm{op}}$], [E2] and [E2$^{\mathrm{op}}$]. We will prove that Quillen's axiom c) follows from our axioms in Proposition 2.16.

Proposition 2.9. *The direct sum of two short exact sequences is short exact.*

Proof. Let $A' \rightarrowtail A \twoheadrightarrow A''$ and $B' \rightarrowtail B \twoheadrightarrow B''$ be two short exact sequences. First observe that for every object C the sequence

$$A' \oplus C \rightarrowtail A \oplus C \twoheadrightarrow A''$$

is exact – the second morphism is an admissible epic because it is the composition of the admissible epics $[1 \;\; 0] : A \oplus C \twoheadrightarrow A$ and $A \twoheadrightarrow A''$; the first morphism in the sequence is a kernel of the second one, hence it is an admissible monic. Now it follows from [E1] that

$$A' \oplus B' \rightarrowtail A \oplus B$$

is an admissible monic because it is the composition of $A' \oplus B' \rightarrowtail A \oplus B'$ with $A \oplus B' \rightarrowtail A \oplus B$. It is obvious that

$$A' \oplus B' \rightarrowtail A \oplus B \twoheadrightarrow A'' \oplus B''$$

is a kernel–cokernel pair, hence the proposition is proved. □

Corollary 2.10. *The exact structure $\mathscr{E}$ is an additive subcategory of the additive category $\mathscr{A}^{\to\to}$ of composable morphisms of $\mathscr{A}$.*

Remark 2.11. In Exercise 3.9 the reader is asked to show that $\mathscr{E}$ is exact with respect to a natural exact structure.

Proposition 2.12. *Consider a commutative square*

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"'] & B \arrow[d, "f'"] \\ A' \arrow[r, tail, "i'"] & B' \end{tikzcd}$$

in which the horizontal arrows are admissible monics. The following assertions are equivalent:

(i) *The square is a push-out.*
(ii) *The sequence $A \overset{\left[\begin{smallmatrix} i \\ -f \end{smallmatrix}\right]}{\rightarrowtail} B \oplus A' \overset{[f' \;\; i']}{\twoheadrightarrow} B'$ is short exact.*
(iii) *The square is bicartesian, i.e., both a push-out and a pull-back.*
(iv) *The square is part of a commutative diagram*

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"'] & B \arrow[r, two heads, "p"] \arrow[d, "f'"] & C \arrow[d, equal] \\ A' \arrow[r, tail, "i'"] & B' \arrow[r, two heads, "p'"] & C \end{tikzcd}$$

with exact rows.

Proof. (i) ⇒ (ii): The push-out property is equivalent to the assertion that $\left[f' \;\; i'\right]$ is a cokernel of $\left[\begin{smallmatrix} i \\ -f \end{smallmatrix}\right]$, so it suffices to prove that the latter is an admissible monic. But this follows from [E1] since $\left[\begin{smallmatrix} i \\ -f \end{smallmatrix}\right]$ is equal to the composition of the morphisms

$$A \overset{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]}{\rightarrowtail} A \oplus A' \xrightarrow[\cong]{\left[\begin{smallmatrix} 1 & 0 \\ -f & 1 \end{smallmatrix}\right]} A \oplus A' \overset{\left[\begin{smallmatrix} i & 0 \\ 0 & 1 \end{smallmatrix}\right]}{\rightarrowtail} B \oplus A'$$

which are all admissible monics by Lemma 2.7, Remark 2.3 and Proposition 2.9, respectively.

(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i): obvious.

(i) $\Rightarrow$ (iv): Let $p : B \twoheadrightarrow C$ be a cokernel of i. The push-out property of the square yields that there is a unique morphism $p' : B' \to C$ such that $p'f' = p$ and $p'i' = 0$. Observe that $p'f' = p$ implies that p' is epic. In order to see that p' is a cokernel of i', let $g : B' \to X$ be such that $gi' = 0$. Then $gf'i = gi'f = 0$, so gf' factors uniquely over a morphism $h : C \to X$ such that $gf' = hp$. We claim that $hp' = g$: this follows from the push-out property of the square because $hp'f' = hp = gf'$ and $hp'i' = 0 = gi'$. Since p' is epic, the factorization h of g is unique, so p' is a cokernel of i'.

(iv) $\Rightarrow$ (ii): Form the pull-back over p and p' in order to obtain the commutative diagram

$$\begin{tikzcd}
& A \arrow[r, equal] \arrow[d, tail, "j"] & A \arrow[d, tail, "i"] \\
A' \arrow[r, tail, "j'"] \arrow[d, equal] & D \arrow[r, two heads, "q'"] \arrow[d, two heads, "q"] \arrow[dr, phantom, "\mathrm{PB}"] & B \arrow[d, two heads, "p"] \\
A' \arrow[r, tail, "i'"] & B' \arrow[r, two heads, "p'"] & C
\end{tikzcd}$$

with exact rows and columns using the dual of the implication (i) $\Rightarrow$ (iv). Since the square

$$\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "f'"] & B \arrow[d, two heads, "p"] \\
B' \arrow[r, two heads, "p'"] & C
\end{tikzcd}$$

is commutative, there is a unique morphism $k : B \to D$ such that $q'k = 1_B$ and $qk = f'$. Since $q'(1_D - kq') = 0$, there is a unique morphism $l : D \to A'$ such that $j'l = 1_D - kq'$. Note that $lk = 0$ because $j'lk = (1_D - kq')k = 0$ and j' is monic, while the calculation $j'lj' = (1_D - kq')j' = j'$ implies $lj' = 1_{A'}$, again because j' is monic. Furthermore

$$i'lj = (qj')lj = q(1_D - kq')j = -(qk)(q'j) = -f'i = -i'f$$

implies $lj = -f$ since i' is monic.

The morphisms

$$[k \quad j'] : B \oplus A' \to D \quad \text{and} \quad \begin{bmatrix} q' \\ l \end{bmatrix} : D \to B \oplus A'$$

are mutually inverse since

$$[k \quad j'] \begin{bmatrix} q' \\ l \end{bmatrix} = kq' + j'l = 1_D \quad \text{and} \quad \begin{bmatrix} q' \\ l \end{bmatrix} [k \quad j'] = \begin{bmatrix} q'k & q'j' \\ lk & lj' \end{bmatrix} = \begin{bmatrix} 1_B & 0 \\ 0 & 1_{A'} \end{bmatrix}.$$

Now

$$[f' \quad i'] = q[k \quad j'] \quad \text{and} \quad \begin{bmatrix} i \\ -f \end{bmatrix} = \begin{bmatrix} q' \\ l \end{bmatrix} j$$

show that $A \xrightarrow{\left[\begin{smallmatrix} i \\ -f \end{smallmatrix}\right]} B \oplus A' \xrightarrow{[f' \quad i']} B'$ is isomorphic to $A \xrightarrow{j} D \xrightarrow{q} B'$. $\square$

Remark 2.13. Consider the push-out diagram

$$\begin{tikzcd} A' \arrow[r, tail, "i'"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{PO}"] & B' \arrow[d, "b"] \\ A \arrow[r, tail, "i"] & B. \end{tikzcd}$$

If $j' : B' \twoheadrightarrow C'$ is a cokernel of i' then the unique morphism $j : B \to C'$ such that $ji = 0$ and $jb = j'$ is a cokernel of i. If $j : B \twoheadrightarrow C$ is a cokernel of i then $j' = jb$ is a cokernel of i'.

The first statement was established in the proof of the implication (i) $\Rightarrow$ (iv) of Proposition 2.12 and we leave the easy verification of the second statement as an exercise for the reader.

The following simple observation will only be used in the proof of Lemma 10.3. We state it here for ease of reference.

Corollary 2.14. *The surrounding rectangle in a diagram of the form*

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{PB}"] & B \arrow[r, tail, "g"] \arrow[d, "b"] \arrow[dr, phantom, "\mathrm{PO}"] & C \arrow[d, "c"] \\ A' \arrow[r, two heads, "f'"] & B' \arrow[r, tail, "g'"] & C' \end{tikzcd}$$

is bicartesian and $A \overset{\left[\begin{smallmatrix} -a \\ gf \end{smallmatrix}\right]}{\rightarrowtail} A' \oplus C \overset{[g'f' \;\; c]}{\twoheadrightarrow} C'$ *is short exact.*

Proof. It follows from Proposition 2.12 and its dual that both squares are bicartesian. Gluing two bicartesian squares along a common arrow yields another bicartesian square, which entails the first part and the fact that the sequence of the second part is a kernel–cokernel pair. The equation $\begin{bmatrix} g'f' & c \end{bmatrix} = \begin{bmatrix} g' & c \end{bmatrix} \begin{bmatrix} f' & 0 \\ 0 & 1_C \end{bmatrix}$ exhibits $\begin{bmatrix} g'f' & c \end{bmatrix}$ as a composition of admissible epics by Propositions 2.9 and 2.12. $\square$

Proposition 2.15. *The pull-back of an admissible monic along an admissible epic yields an admissible monic.*

Proof. Consider the diagram

$$\begin{tikzcd} A' \arrow[r, "i'"] \arrow[d, two heads, "e'"'] \arrow[dr, phantom, "\mathrm{PB}"] & B' \arrow[r, two heads, "pe"] \arrow[d, two heads, "e"] & C \arrow[d, equal] \\ A \arrow[r, tail, "i"] & B \arrow[r, two heads, "p"] & C. \end{tikzcd}$$

The pull-back square exists by axiom [E2$^{\mathrm{op}}$]. Let p be a cokernel of i, so it is an admissible epic and pe is an admissible epic by axiom [E1$^{\mathrm{op}}$]. In any category, the pull-back of a monic is a monic (if it exists). In order to see that i' is an admissible monic, it suffices to prove that i' is a kernel of pe. Suppose that $g' : X \to B'$ is such that $peg' = 0$. Since i is a kernel of p, there exists a unique $f : X \to A$ such that $eg' = if$. Applying the universal property of the pull-back square, we find a unique $f' : X \to A'$ such that $e'f' = f$ and $i'f' = g'$. Since i' is monic, f' is the unique morphism such that $i'f' = g'$ and we are done. □

Proposition 2.16 (*Obscure axiom*). *Suppose that $i : A \to B$ is a morphism in $\mathscr{A}$ admitting a cokernel. If there exists a morphism $j : B \to C$ in $\mathscr{A}$ such that the composite $ji : A \rightarrowtail C$ is an admissible monic then i is an admissible monic.*

Remark 2.17. The statement of the previous proposition (and its dual) is given as axiom c) in Quillen's definition of an exact category [51, §2]. At that time, it was already proved to be a consequence of the other axioms by Yoneda [61, Corollary, p. 525]. The redundancy of the obscure axiom was rediscovered by Keller [37, A.1]. Thomason baptized axiom c) the "obscure axiom" in [58, A.1.1].

A convenient and quite powerful strengthening of the obscure axiom holds under the rather mild additional hypothesis of weak idempotent completeness, see Proposition 7.6.

Proof of Proposition 2.16. (*Keller*). Let $k : B \to D$ be a cokernel of i. From the push-out diagram

$$\begin{tikzcd} A \arrow[r, tail, "ji"] \arrow[d, "i"'] \arrow[dr, phantom, "\mathrm{PO}"] & C \arrow[d] \\ B \arrow[r, tail] & E \end{tikzcd}$$

and Proposition 2.12 we conclude that $\begin{bmatrix} i \\ ji \end{bmatrix} : A \rightarrowtail B \oplus C$ is an admissible monic. Because $\begin{bmatrix} 1_B & 0 \\ -j & 1_C \end{bmatrix} : B \oplus C \to B \oplus C$ is an isomorphism, it is in particular an admissible monic, hence $\begin{bmatrix} i \\ 0 \end{bmatrix} = \begin{bmatrix} 1_B & 0 \\ -j & 1_C \end{bmatrix} \begin{bmatrix} i \\ ji \end{bmatrix}$ is an admissible monic as well. Because $\begin{bmatrix} k & 0 \\ 0 & 1_C \end{bmatrix}$ is a cokernel of $\begin{bmatrix} i \\ 0 \end{bmatrix}$, it is an admissible epic. Consider the following diagram:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, equal] & B \arrow[r, "k"] \arrow[d, "\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]"] \arrow[dr, phantom, "\mathrm{PB}"] & D \arrow[d, "\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]"] \\ A \arrow[r, tail, "\left[\begin{smallmatrix} i \\ 0 \end{smallmatrix}\right]"'] & B \oplus C \arrow[r, two heads, "\left[\begin{smallmatrix} k & 0 \\ 0 & 1_C \end{smallmatrix}\right]"'] & D \oplus C. \end{tikzcd}$$

Since the right-hand square is a pull-back, it follows that k is an admissible epic and that i is a kernel of k, so i is an admissible monic. □

Corollary 2.18. *Let (i, p) and (i', p') be two pairs of composable morphisms. If the direct sum $(i \oplus i', p \oplus p')$ is short exact then (i, p) and (i', p') are both short exact.*

Proof. It is clear that (i, p) and (i', p') are kernel–cokernel pairs. Since i has p as a cokernel and since

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} i = \begin{bmatrix} i & 0 \\ 0 & i' \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is an admissible monic, the obscure axiom implies that i is an admissible monic. □

Exercise 2.19. Suppose that the commutative square

$$\begin{tikzcd} A' \arrow[r, tail, "f'"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{PO}"] & B' \arrow[d, tail, "b"] \\ A \arrow[r, tail, "f"] & B \end{tikzcd}$$

is a push-out. Prove that a is an admissible monic.

Hint: Let $b' : B \twoheadrightarrow B''$ be a cokernel of $b : B' \rightarrowtail B$. Prove that $a' = b'f : A \to B''$ is a cokernel of a, then apply the obscure axiom.

3. Some diagram lemmas

In this section we will prove variants of diagram lemmas which are well-known in the context of abelian categories, in particular we will prove the five lemma and the 3×3-lemma. Further familiar diagram lemmas will be proved in Section 8. The proofs will be based on the following simple observation:

Proposition 3.1. *Let $(\mathscr{A}, \mathscr{E})$ be an exact category. A morphism from a short exact sequence $A' \rightarrowtail B' \twoheadrightarrow C'$ to another short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ factors over a short exact sequence $A \rightarrowtail D \twoheadrightarrow C'$*

$$\begin{tikzcd} A' \arrow[r, tail, "f'"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{BC}"] & B' \arrow[r, two heads, "g'"] \arrow[d, "b'"] & C' \arrow[d, equal] \\ A \arrow[r, tail, "m"] \arrow[d, equal] & D \arrow[r, two heads, "e"] \arrow[d, "b''"'] \arrow[dr, phantom, "\mathrm{BC}"] & C' \arrow[d, "c"] \\ A \arrow[r, tail, "f"] & B \arrow[r, two heads, "g"] & C \end{tikzcd}$$

in such a way that the two squares marked BC *are bicartesian. In particular there is a canonical isomorphism of the push-out $A \cup_{A'} B'$ with the pull-back $B \times_C C'$.*

Proof. Form the push-out under f' and a in order to obtain the object D and the morphisms m and b'. Let $e : D \to C'$ be the unique morphism such that $eb' = g'$ and $em = 0$ and let $b'' : D \to B$ be the unique morphism $D \to B$ such that $b''b' = b : B' \to B$ and $b''m = f$. It is easy to see that e is a cokernel of m (Remark 2.13) and hence the result follows from Proposition 2.12 since the square $DC'BC$ is commutative (this is a consequence of the push-out property of the upper left hand square). $\square$

Corollary 3.2 (*Five lemma, I*). *Consider a morphism of short exact sequences*

$$\begin{tikzcd} A' \arrow[r, tail] \arrow[d, "a"] & B' \arrow[r, two heads] \arrow[d, "b"] & C' \arrow[d, "c"] \\ A \arrow[r, tail] & B \arrow[r, two heads] & C. \end{tikzcd}$$

If a and c are isomorphisms (*or admissible monics*, *or admissible epics*) *then so is b.*

Proof. Assume first that a and c are isomorphisms. Because isomorphisms are preserved by push-outs and pull-backs, it follows from the diagram of Proposition 3.1 that b is the composition of two isomorphisms $B' \to D \to B$. If a and c are both admissible monics, it follows from the diagram of Proposition 3.1 together with [E2] and Proposition 2.15 that b is the composition of two admissible monics. The case of admissible epics is dual. □

Exercise 3.3. If in a morphism of short exact sequences as in the five lemma 3.2 two out of a, b, c are isomorphisms then so is the third.

Hint: Use e.g. that c is uniquely determined by a and b.

Remark 3.4. The reader insisting that Corollary 3.2 should be called "three lemma" rather than "five lemma" is cordially invited to give the details of the proof of Lemma 8.9 and to solve Exercise 8.10. We will, however, use the more customary name five lemma.

Lemma 3.5 (*Noether isomorphism $C/B \cong (C/A)/(B/A)$*). *Consider the diagram*

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d, tail] & X \arrow[d, tail, dotted] \\ A \arrow[r, tail] & C \arrow[r, two heads] \arrow[d, two heads] & Y \arrow[d, two heads, dotted] \\ & Z \arrow[r, equal] & Z \end{tikzcd}$$

in which the first two horizontal rows and the middle column are short exact. Then the third column exists, is short exact, and is uniquely determined by the requirement that it makes the diagram commutative. Moreover, the upper right-hand square is bicartesian.

Proof. The morphism $X \to Y$ exists since the first row is exact and the composition $A \to C \to Y$ is zero while the morphism $Y \to Z$ exists since the second row is exact and the composition $B \to C \to Z$ vanishes. By Proposition 2.12 the square containing $X \to Y$ is bicartesian. It follows that $X \to Y$ is an admissible monic and that $Y \to Z$ is its cokernel. The uniqueness assertion is obvious. □

Corollary 3.6 (3×3-*Lemma*). *Consider a commutative diagram*

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, tail, "a"] & B' \arrow[r, "g'"] \arrow[d, tail, "b"] & C' \arrow[d, tail, "c"] \\ A \arrow[r, "f"] \arrow[d, two heads, "a'"] & B \arrow[r, "g"] \arrow[d, two heads, "b'"] & C \arrow[d, two heads, "c'"] \\ A'' \arrow[r, "f''"] & B'' \arrow[r, "g''"] & C'' \end{tikzcd}$$

in which the columns are exact and assume in addition that one of the following conditions holds:

(i) *the middle row and either one of the outer rows is short exact*;
(ii) *the two outer rows are short exact and $gf = 0$.*

Then the remaining row is short exact as well.

Proof. Let us prove (i). The two possibilities are dual to each other, so we need only consider the case that the first two rows are exact. Apply Proposition 3.1 to the first two rows so as to obtain the commutative diagram

$$\begin{tikzcd}
A' \arrow[r, tail, "f'"] \arrow[d, tail, "a"'] \arrow[dr, phantom, "\mathrm{BC}"] & B' \arrow[r, two heads, "g'"] \arrow[d, tail, "i"] & C' \arrow[d, equal] \\
A \arrow[r, tail, "\bar{f}"] \arrow[d, equal] & D \arrow[r, two heads, "\bar{g}"] \arrow[d, tail, "j"] \arrow[dr, phantom, "\mathrm{BC}"] & C' \arrow[d, tail, "c"] \\
A \arrow[r, tail, "f"] & B \arrow[r, two heads, "g"] & C
\end{tikzcd}$$

where $ji = b$ – notice that i and j are admissible monics by axiom [E2] and Proposition 2.15, respectively. By Remark 2.13 the morphism $i' : D \to A''$ determined by $i'i = 0$ and $i'\bar{f} = a'$ is a cokernel of i, while the morphism $j' : B \twoheadrightarrow C''$ given by $j' = c'g = g''b'$ is a cokernel of j.

If we knew that the diagram

$$\begin{tikzcd}
B' \arrow[r, tail, "i"] \arrow[d, equal] & D \arrow[r, two heads, "i'"] \arrow[d, tail, "j"] & A'' \arrow[d, "f''"] \\
B' \arrow[r, tail, "b"] & B \arrow[r, two heads, "b'"] \arrow[d, two heads, "j'"] & B'' \arrow[d, "g''"] \\
& C'' \arrow[r, equal] & C''
\end{tikzcd}$$

is commutative then we would conclude from the Noether isomorphism 3.5 that (f'', g'') is a short exact sequence. It therefore remains to prove that $f''i' = b'j$ since the other commutativity relations $b = ji$ and $g''b' = j'$ hold by construction. We are going to apply the push-out property of the square $A'B'AD$. We have

$$(f''i')i = 0 = b'b = (b'j)i \quad \text{and} \quad (b'j)\bar{f} = b'f = f''a' = (f''i')\bar{f}$$

which together with

$$(f''i'\bar{f})a = (f''i'i)f' = 0 \quad \text{and} \quad (b'j\bar{f})a = f''a'a = 0 = b'bf' = (b'ji)f'$$

show that both $f''i'$ and $b'j$ are solutions to the same push-out problem, hence they are equal. This settles case (i).

In order to prove (ii) we start by forming the push-out under g' and b so that we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
A' \arrow[r, tail, "f'"] \arrow[d, equal] & B' \arrow[r, two heads, "g'"] \arrow[d, tail, "b"] \arrow[dr, phantom, "\mathrm{PO}"] & C' \arrow[d, tail, "k"] \\
A \arrow[r, tail, "i"] & B \arrow[r, two heads, "j"] \arrow[d, two heads, "b'"] & D \arrow[d, two heads, "k'"] \\
& B'' \arrow[r, equal] & B''
\end{tikzcd}$$

in which the cokernel k' of k is determined by $k'j = b'$ and $k'k = 0$, while $i = bf'$ is a kernel of the admissible epic j, see Remark 2.13 and Proposition 2.15. The push-out property of the square $B'C'BD$ applied to the square $B'C'BC$ yields a unique morphism $d' : D \to C$ such that $d'k = c$ and $d'j = g$. The diagram

$$\begin{tikzcd}
C' \arrow[r, equal] \arrow[d, tail, "k"] & C' \arrow[d, tail, "c"] \\
D \arrow[r, "d'"] \arrow[d, two heads, "k'"] & C \arrow[d, two heads, "c'"] \\
B'' \arrow[r, two heads, "g''"] & C''
\end{tikzcd}$$

has exact rows and it is commutative: Indeed, $c = d'k$ holds by construction of d', while $c'd' = g''k'$ follows from $c'd'j = c'g = g''b' = g''k'j$ and the fact that j is epic. We conclude from Proposition 2.12 that $DCB''C''$ is a pull-back, so d' is an admissible epic and so is $g = d'j$. The unique morphism $d : A'' \to D$ such that $k'd = f''$ and $d'd = 0$ is a kernel of d'. By the pull-back property of $DCB''C''$ the diagram

$$\begin{tikzcd}
A' \arrow[r, equal] \arrow[d, tail, "a"] & A' \arrow[d, tail, "i"] & \\
A \arrow[r, "f"] \arrow[d, two heads, "a'"] & B \arrow[r, two heads, "g"] \arrow[d, two heads, "j"] & C \arrow[d, equal] \\
A'' \arrow[r, tail, "d"] & D \arrow[r, two heads, "d'"] & C
\end{tikzcd}$$

is commutative as $k'(da') = f''a' = b'f = k'(jf)$ and $d'(da') = 0 = gf = d'(jf)$. Notice that the hypothesis that $gf = 0$ enters at this point of the argument. It follows from the dual of Proposition 2.12 that $ABA''D$ is bicartesian, so f is a kernel of g by Proposition 2.15. $\square$

Exercise 3.7. Consider the solid arrow diagram

$$\begin{tikzcd}
A' \arrow[r, tail] \arrow[d, tail] & B' \arrow[r, two heads] \arrow[d, tail] & C' \arrow[d, tail, dotted] \\
A \arrow[r, tail] \arrow[d, two heads] & B \arrow[r, two heads] \arrow[d, two heads] & C \arrow[d, two heads, dotted] \\
A'' \arrow[r, tail] & B'' \arrow[r, two heads] & C''
\end{tikzcd}$$

with exact rows and columns. Strengthen the Noether isomorphism 3.5 to the statement that there exist unique maps $C' \to C$ and $C \to C''$ making the diagram commutative and the sequence $C' \rightarrowtail C \twoheadrightarrow C''$ is short exact.

Exercise 3.8. In the situation of the 3×3-lemma prove that there are two exact sequences $A' \rightarrowtail A \oplus B' \to B \twoheadrightarrow C''$ and $A' \rightarrowtail B \to B'' \oplus C \twoheadrightarrow C''$ in the sense that the morphisms $\to$

factor as $\twoheadrightarrow\rightarrowtail$ in such a way that consecutive $\rightarrowtail\twoheadrightarrow$ are short exact (compare also with Definition 8.8).

Hint: Apply Proposition 3.1 to the first two rows in order to obtain a short exact sequence $A' \rightarrowtail A \oplus B' \twoheadrightarrow D$ using Proposition 2.12. Conclude from the push-out property of $DC'BC$ that $D \rightarrowtail B$ has C'' as cokernel.

Exercise 3.9 (*Heller [27, 6.2]*). Let $(\mathscr{A}, \mathscr{E})$ be an exact category and consider $\mathscr{E}$ as a full subcategory of $\mathscr{A}^{\to\to}$. We have shown that $\mathscr{E}$ is additive in Corollary 2.10. Let $\mathscr{F}$ be the class of short sequences

$$\begin{tikzcd}
(A' \arrow[r, tail] \arrow[d, tail] & B' \arrow[r, two heads] \arrow[d, tail] & C') \arrow[d, tail] \\
(A \arrow[r, tail] \arrow[d, two heads] & B \arrow[r, two heads] \arrow[d, two heads] & C) \arrow[d, two heads] \\
(A'' \arrow[r, tail] & B'' \arrow[r, two heads] & C'')
\end{tikzcd}$$

over $\mathscr{E}$ with short exact columns [we write $(A \rightarrowtail B \twoheadrightarrow C)$ to indicate that we think of the sequence as an object of $\mathscr{E}$]. Prove that $(\mathscr{E}, \mathscr{F})$ is an exact category.

Remark 3.10. The category of short exact sequences in a nonzero abelian category is *not* abelian, see [44, XII.6, p. 375].

Exercise 3.11 (*Künzer's axiom, cf. e.g. [42]*).

(i) Let $f : A \to B$ be an arbitrary morphism and let $g : B \twoheadrightarrow C$ be an admissible epic. If $h = gf : A \rightarrowtail C$ is an admissible monic, then f is an admissible monic and the induced morphism $\operatorname{Ker} g \to \operatorname{Coker} f$ is an admissible monic as well.
Hint: Form the pull-back P over h and g, use Proposition 2.15 and factor f over P to see the first part (see also Remark 7.4). For the second part use the Noether isomorphism 3.5.

(ii) Let $\mathscr{E}$ be a class of kernel–cokernel pairs in the additive category $\mathscr{A}$. Assume that $\mathscr{E}$ is closed under isomorphisms and contains the split exact sequences. If $\mathscr{E}$ enjoys the property of point (i) and its dual then it is an exact structure.
Hint: Let $f : A \rightarrowtail B$ be an admissible monic and let $a : A \to A'$ be arbitrary. The push-out axiom follows from the commutative diagram

$$\begin{tikzcd}
A \arrow[rr, tail, "f"] \arrow[dr, tail, "{\left[\begin{smallmatrix} a \\ f \end{smallmatrix}\right]}"'] & & B \\
& A' \oplus B \arrow[ur, two heads, "{[\,0\ 1\,]}"'] \arrow[dr, two heads, "{[\,f'\ -b\,]}"] & \\
A' \arrow[ur, tail, "{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]}"] \arrow[rr, tail, "f'"] & & B'
\end{tikzcd}$$

in which $\left[\begin{smallmatrix} a \\ f \end{smallmatrix}\right]$ and f' are admissible monics by (i) and $[f' \quad -b]$ is a cokernel of $\left[\begin{smallmatrix} a \\ f \end{smallmatrix}\right]$. Next observe that the dual of (i) implies that if in addition a is an admissible epic then so is b. In order to prove the composition axiom, let f and g be admissible monics and choose a cokernel f' of f. Form the push-out under g and f' and verify that gf is a kernel of the push-out of f'.

4. Quasi-abelian categories

Definition 4.1. An additive category $\mathscr{A}$ is called *quasi-abelian* if:

(i) Every morphism has a kernel and a cokernel.
(ii) The class of kernels is stable under push-out along arbitrary morphisms and the class of cokernels is stable under pull-back along arbitrary morphisms.

Remark 4.2. The concept of quasi-abelian categories is self-dual, that is to say $\mathscr{A}$ is quasi-abelian if and only if $\mathscr{A}^{\mathrm{op}}$ is quasi-abelian.

Exercise 4.3. Let $\mathscr{A}$ be an additive category with kernels. Prove that every pull-back of a kernel is a kernel.

Proposition 4.4 (*Schneiders [55, 1.1.7]*). *The class $\mathscr{E}_{\max}$ of all kernel–cokernel pairs in a quasi-abelian category is an exact structure.*

Proof. It is clear that $\mathscr{E}_{\max}$ is closed under isomorphisms and that the classes of kernels and cokernels contain the identity morphisms. The pull-back and push-out axioms are part of the definition of quasi-abelian categories. By duality it only remains to show that the class of cokernels is closed under composition. So let $f : A \twoheadrightarrow B$ and $g : B \twoheadrightarrow C$ be cokernels and put $h = gf$. In the diagram

$$\begin{tikzcd}
\operatorname{Ker} f \arrow[r, dotted, "u"] \arrow[d, equal] & \operatorname{Ker} h \arrow[r, dotted, "v"] \arrow[d, tail, "\ker h"] & \operatorname{Ker} g \arrow[d, tail, "\ker g"] \\
\operatorname{Ker} f \arrow[r, tail, "\ker f"] & A \arrow[r, two heads, "f"] \arrow[d, "h"] & B \arrow[d, two heads, "g"] \\
& C \arrow[r, equal] & C
\end{tikzcd}$$

there exist unique morphisms u and v making it commutative. The upper right-hand square is a pull-back, so v is a cokernel and u is its kernel. But then it follows by duality that the upper right-hand square is also a push-out and this together with the fact that h is epic implies that h is a cokernel of $\ker h$. $\square$

Remark 4.5. Note that we have just re-proved the Noether isomorphism 3.5 in the special case of quasi-abelian categories.

Definition 4.6. The *coimage* of a morphism f in a category with kernels and cokernels is $\mathrm{Coker}(\ker f)$, while the *image* is defined to be $\mathrm{Ker}(\mathrm{coker}\, f)$. The *analysis* (cf. [44, IX.2]) of f is the commutative diagram

$$\begin{tikzcd}
& A \arrow[rrr, "f"] \arrow[dr, two heads, "\mathrm{coim}\, f"'] & & & B \arrow[dr, two heads, "\mathrm{coker}\, f"] & \\
\mathrm{Ker}\, f \arrow[ur, tail, "\ker f"] & & \mathrm{Coim}\, f \arrow[r, "\hat{f}"] & \mathrm{Im}\, f \arrow[ur, tail, "\mathrm{im}\, f"'] & & \mathrm{Coker}\, f
\end{tikzcd}$$

in which $\hat{f}$ is uniquely determined by requiring the diagram to be commutative.

Remark 4.7. The difference between quasi-abelian categories and abelian categories is that in the quasi-abelian case the canonical morphism $\hat{f}$ in the analysis f is not in general an isomorphism. Indeed, it is easy to see that a quasi-abelian category is abelian *provided* that $\hat{f}$ is always an isomorphism. In other words, in a general quasi-abelian category not every monic is a kernel and not every epic is a cokernel.

Proposition 4.8 (*[55, 1.1.5]*). *Let f be a morphism in the quasi-abelian category $\mathscr{A}$. The canonical morphism $\hat{f} : \mathrm{Coim}\, f \to \mathrm{Im}\, f$ is monic and epic.*

Proof. By duality it suffices to check that the morphism $\bar{f}$ in the diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, two heads, "j"'] & & B \\
& \mathrm{Coim}\, f \arrow[ur, "\bar{f}"'] &
\end{tikzcd}$$

is monic. Let $x : X \to \mathrm{Coim}\, f$ be a morphism such that $\bar{f}x = 0$. The pull-back $y : Y \to A$ of x along j satisfies $fy = 0$, so y factors over $\mathrm{Ker}\, f$ and hence $jy = 0$. But then the map $Y \twoheadrightarrow X \to \mathrm{Coim}\, f$ is zero as well, so $x = 0$. $\square$

Remark 4.9. Every morphism f in a quasi-abelian category $\mathscr{A}$ has two epic-monic factorizations, one over $\mathrm{Coim}\, f$ and one over $\mathrm{Im}\, f$. The quasi-abelian category $\mathscr{A}$ is abelian if and only if the two factorizations coincide for all morphisms f.

Remark 4.10. An additive category with kernels and cokernels is called *semi-abelian* if the canonical morphism $\mathrm{Coim}\, f \to \mathrm{Im}\, f$ is always monic and epic. We have just proved that quasi-abelian categories are semi-abelian. It may seem obvious that the concept of semi-abelian categories is strictly weaker than the concept of a quasi-abelian category. However, it is surprisingly delicate to come up with an explicit example. This led Raĭkov to conjecture that every semi-abelian category is quasi-abelian. A counterexample to this conjecture was recently found by Rump [53].

Remark 4.11. We do not develop the theory of quasi-abelian categories any further. The interested reader may consult Schneiders [55], Rump [52] and the references therein.

5. Exact functors

Definition 5.1. Let $(\mathscr{A}, \mathscr{E})$ and $(\mathscr{A}', \mathscr{E}')$ be exact categories. An (additive) functor $F : \mathscr{A} \to \mathscr{A}'$ is called *exact* if $F(\mathscr{E}) \subset \mathscr{E}'$. The functor F *reflects exactness* if $F(\sigma) \in \mathscr{E}'$ implies $\sigma \in \mathscr{E}$ for all $\sigma \in \mathscr{A}^{\to\to}$.

Proposition 5.2. *An exact functor preserves push-outs along admissible monics and pull-backs along admissible epics.*

Proof. An exact functor preserves admissible monics and admissible epics, in particular it preserves diagrams of type

$$\begin{tikzcd} {} \arrow[r, tail] \arrow[d] & {} \arrow[d] \\ {} \arrow[r, tail] & {} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} {} \arrow[r, two heads] \arrow[d] & {} \arrow[d] \\ {} \arrow[r, two heads] & {} \end{tikzcd}$$

so the result follows immediately from Proposition 2.12 and its dual. □

The following exercises show how one can induce new exact structures using functors satisfying certain exactness properties.

Exercise 5.3 (*Heller [27, 7.3]*). Let $F : (\mathscr{A}, \mathscr{E}) \to (\mathscr{A}', \mathscr{E}')$ be an exact functor and let $\mathscr{F}'$ be another exact structure on $\mathscr{A}'$. Then $\mathscr{F} = \{\sigma \in \mathscr{E} : F(\sigma) \in \mathscr{F}'\}$ is an exact structure on $\mathscr{A}$.

Remark 5.4 (*Heller*). The prototypical application of the previous exercise is the following: A (unital) ring homomorphism $\varphi : R' \to R$ yields an exact functor $\varphi^* : {}_R\mathbf{Mod} \to {}_{R'}\mathbf{Mod}$ of the associated module categories. Let $\mathscr{F}'$ be the class of split exact sequences on ${}_{R'}\mathbf{Mod}$. The induced structure $\mathscr{F}$ on ${}_R\mathbf{Mod}$ consisting of sequences σ such that $\varphi^*(\sigma)$ is split exact is the *relative exact structure* with respect to φ. This structure is used in particular to define the relative derived functors such as the relative Tor and Ext functors.

Exercise 5.5 (*Künzer*). Suppose $F : (\mathscr{A}, \mathscr{E}) \to (\mathscr{A}', \mathscr{E}')$ preserves admissible kernels, i.e., for every morphism $f : B \to C$ with an admissible monic $k : A \rightarrowtail B$ as kernel, the morphism $F(k)$ is an admissible monic and a kernel of $F(f)$. Let $\mathscr{F} = \{\sigma \in \mathscr{E} : F(\sigma) \in \mathscr{E}'\}$ be the largest subclass of $\mathscr{E}$ on which F is exact. Prove that $\mathscr{F}$ is an exact structure.

Hint: Axioms [E0], [E0$^{\mathrm{op}}$] and [E1$^{\mathrm{op}}$] are easy. To check axiom [E1] use the obscure axiom 2.16 and the 3×3-lemma 3.6. Axiom [E2] follows from the obscure axiom 2.16 and Proposition 2.12(iv), while axiom [E2$^{\mathrm{op}}$] follows from the fact that F preserves certain pull-back squares.

Exercise 5.6. Let $\mathscr{P}$ be a set of objects in the exact category $(\mathscr{A}, \mathscr{E})$. Consider the class $\mathscr{E}_{\mathscr{P}}$ consisting of the sequences $A' \rightarrowtail A \twoheadrightarrow A''$ in $\mathscr{E}$ such that

$$\mathrm{Hom}_{\mathscr{A}}(P, A') \rightarrowtail \mathrm{Hom}_{\mathscr{A}}(P, A) \twoheadrightarrow \mathrm{Hom}_{\mathscr{A}}(P, A'')$$

is an exact sequence of abelian groups for all $P \in \mathscr{P}$. Prove that $\mathscr{E}_{\mathscr{P}}$ is an exact structure on $\mathscr{A}$.

Hint: Use Exercise 5.5.

6. Idempotent completion

In this section we discuss Karoubi's construction of 'the' idempotent completion of an additive category, see [34, 1.2], and show how to extend it to exact categories. Admittedly, the constructions and arguments presented here are quite obvious (once the definitions are given) and thus rather boring, but as the author is unaware of a reasonably complete exposition it seems worthwhile to outline the details. A different account for small categories (not necessarily additive) is given in Borceux [5, Proposition 6.5.9, p. 274]. It appears that the latter approach is due to Bunge [12].

Definition 6.1. An additive category $\mathscr{A}$ is *idempotent complete* [34, 1.2.1, 1.2.2] if for every idempotent $p : A \to A$, *i.e.*, $p^2 = p$, there is a decomposition $A \cong K \oplus I$ of A such that $p \cong \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Remark 6.2. The additive category $\mathscr{A}$ is idempotent complete if and only if every idempotent has a kernel.

Indeed, suppose that every idempotent has a kernel. Let $k : K \to A$ be a kernel of the idempotent $p : A \to A$ and let $i : I \to A$ be a kernel of the idempotent $1 - p$. Because $p(1-p) = 0$, we have $(1-p) = kl$ for a unique $l : A \to K$ and because $(1-p)p = 0$ we have $p = ij$ for a unique $j : A \to I$. Since k is monic and $kli = (1-p)i = 0$ we have $li = 0$ and because $klk = (1-p)k = pk + (1-p)k = k$ we have $lk = 1_K$. Similarly, $jk = 0$ and $ji = 1_I$. Therefore $[k \;\; i] : K \oplus I \to A$ and $\begin{bmatrix} l \\ j \end{bmatrix} : A \to K \oplus I$ are inverse to each other and $\begin{bmatrix} l \\ j \end{bmatrix} p [k \;\; i] = \begin{bmatrix} l \\ j \end{bmatrix} ij[k \;\; i] = \begin{bmatrix} 0 & 0 \\ 0 & 1_I \end{bmatrix}$ as desired. Notice that we have constructed an analysis of p:

$$\begin{tikzcd}
 & A \arrow[rr, "p"] \arrow[dr, two heads, "j"'] & & A \arrow[dr, two heads, "l"] & \\
K \arrow[ur, tail, "k"] & & I \arrow[ur, tail, "i"'] & & K,
\end{tikzcd}$$

in particular $k : K \rightarrowtail A$ is a kernel of p and $i : I \rightarrowtail A$ is an image of p. The converse direction is even more obvious.

Remark 6.3. Every additive category $\mathscr{A}$ can be fully faithfully embedded into an idempotent complete additive category $\mathscr{A}^\wedge$.

The objects of $\mathscr{A}^\wedge$ are the pairs (A, p) consisting of an object A of $\mathscr{A}$ and an idempotent $p : A \to A$ while the sets of morphisms are

$$\mathrm{Hom}_{\mathscr{A}^\wedge}((A, p), (B, q)) = q \circ \mathrm{Hom}_{\mathscr{A}}(A, B) \circ p$$

with the composition induced by the composition in $\mathscr{A}$. It is easy to see that $\mathscr{A}^\wedge$ is additive with biproduct $(A, p) \oplus (A', p') = (A \oplus A', p \oplus p')$ and obviously the functor

$i_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}^\wedge$ given by $i_{\mathscr{A}}(A) = (A, 1_A)$ and $i_{\mathscr{A}}(f) = f$ is fully faithful. In order to see that $\mathscr{A}^\wedge$ is idempotent complete, suppose pfp is an idempotent of (A, p) in $\mathscr{A}^\wedge$. *A fortiori* pfp is an idempotent of $\mathscr{A}$ and the object (A, p) is isomorphic to the direct sum $(A, p - pfp) \oplus (A, pfp)$ via the morphisms $\left[\begin{smallmatrix} p-pfp \\ pfp \end{smallmatrix}\right]$ and $[p - pfp \quad pfp]$. The equation $\left[\begin{smallmatrix} p-pfp \\ pfp \end{smallmatrix}\right] pfp\, [p - pfp \quad pfp] = \left[\begin{smallmatrix} 0 & 0 \\ 0 & pfp \end{smallmatrix}\right]$ proves $\mathscr{A}^\wedge$ to be idempotent complete.

Definition 6.4. The pair $(\mathscr{A}^\wedge, i_{\mathscr{A}})$ constructed in Remark 6.3 is called the *idempotent completion* of $\mathscr{A}$.

The next goal is to characterize the pair $(\mathscr{A}^\wedge, i_{\mathscr{A}})$ by a universal property (Proposition 6.10). We first work out some nice properties of the explicit construction.

Remark 6.5. If $\mathscr{A}$ is idempotent complete then $i_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}^\wedge$ is an equivalence of categories. In order to construct a quasi-inverse functor of $i_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}^\wedge$, choose for each idempotent $p : A \to A$ a kernel K_p, an image I_p and morphisms i_p, j_p, k_p, l_p as in Remark 6.2 and map the object (A, p) of $\mathscr{A}^\wedge$ to I_p. A morphism $(A, p) \to (B, q)$ of $\mathscr{A}^\wedge$ can be written as qfp and map it to $j_q qfp i_p = j_q f i_p$. Obviously, this yields a quasi-inverse functor of $i_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}^\wedge$.

Remark 6.6. A functor $F : \mathscr{A} \to \mathscr{B}$ yields a functor $F^\wedge : \mathscr{A}^\wedge \to \mathscr{B}^\wedge$, simply by setting $F^\wedge(A, p) = (F(A), F(p))$ and $F^\wedge(qfp) = F(q)F(f)F(p)$. Obviously, $F^\wedge i_{\mathscr{A}} = i_{\mathscr{B}} F$ and $(GF)^\wedge = G^\wedge F^\wedge$.

Remark 6.7. A natural transformation $\alpha : F \Rightarrow G$ of functors $\mathscr{A} \to \mathscr{B}$ yields a unique natural transformation $\alpha^\wedge : F^\wedge \Rightarrow G^\wedge$.

Observe first that a natural transformation $\alpha' : F' \Rightarrow G'$ of functors $\mathscr{A}^\wedge \to \mathscr{B}^\wedge$ is completely determined by its values on $i_{\mathscr{A}}(\mathscr{A})$ by the following argument. Every object (A, p) of $\mathscr{A}^\wedge$ is canonically a retract of $(A, 1_A)$ via the morphisms $s : (A, p) \to (A, 1_A)$ and $r : (A, 1_A) \to (A, p)$ given by $p \in \operatorname{Hom}_{\mathscr{A}}(A, A)$. Therefore, by naturality, we must have

$$\alpha'_{(A,p)} = \alpha'_{(A,p)} F'(r) F'(s) = G'(r) \alpha'_{(A,1_A)} F'(s),$$

so α' is completely determined by its values on $i_{\mathscr{A}}(\mathscr{A})$. Now given a natural transformation $\alpha : F \Rightarrow G$ of functors $\mathscr{A} \to \mathscr{B}$ put

$$\alpha^\wedge_{(A,p)} = G^\wedge(r) i_{\mathscr{B}}(\alpha_A) F^\wedge(s)$$

which is simply the element $G(p)\alpha_A F(p)$ in

$$\operatorname{Hom}_{\mathscr{B}^\wedge}(F^\wedge(A, p), G^\wedge(A, p)) = G(p) \circ \operatorname{Hom}_{\mathscr{B}}(F(A), G(A)) \circ F(p).$$

The reader will readily check that this definition of $\alpha^\wedge$ yields a natural transformation $F^\wedge \Rightarrow G^\wedge$ as desired.

Remark 6.8. The assignment $\alpha \mapsto \alpha^\wedge$ is compatible with vertical and horizontal composition (see e.g. [45, II.5, p. 42]): For functors $F, G, H : \mathscr{A} \to \mathscr{B}$ and natural transformations $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ we have $(\beta \circ \alpha)^\wedge = \beta^\wedge \circ \alpha^\wedge$ while for functors $F, G : \mathscr{A} \to \mathscr{B}$ and $H, K : \mathscr{B} \to \mathscr{C}$ with natural transformations $\alpha : F \Rightarrow G$ and $\beta : H \Rightarrow K$ we have $(\beta * \alpha)^\wedge = \beta^\wedge * \alpha^\wedge$.

Remark 6.9. A functor $F : \mathscr{A}^\wedge \to \mathscr{I}$ to an idempotent complete category is determined up to unique isomorphism by its values on $i_{\mathscr{A}}(\mathscr{A})$. A natural transformation $\alpha : F \Rightarrow G$ of functors $\mathscr{A}^\wedge \to \mathscr{I}$ is determined by its values on $i_{\mathscr{A}}(\mathscr{A})$.

Indeed, exhibit each (A, p) as a retract of $(A, 1_A)$ as in Remark 6.7. Consider p as an idempotent of $(A, 1_A)$, so $F(p)$ is an idempotent of $F(A, 1_A)$. Choosing an image $I_{F(p)}$ of $F(p)$ as in Remark 6.2, it is clear that the functor F must map (A, p) to $I_{F(p)}$ and is thus determined up to a unique isomorphism. The claim about natural transformations is analogous to the argument in Remark 6.7.

Proposition 6.10. *The functor $i_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}^\wedge$ is 2-universal among functors from $\mathscr{A}$ to idempotent complete categories*:

(i) *For every functor $F : \mathscr{A} \to \mathscr{I}$ to an idempotent complete category $\mathscr{I}$ there exists a functor $\widetilde{F} : \mathscr{A}^\wedge \to \mathscr{I}$ and a natural isomorphism $\alpha : F \Rightarrow \widetilde{F} i_{\mathscr{A}}$.*
(ii) *Given a functor $G : \mathscr{A}^\wedge \to \mathscr{I}$ and a natural transformation $\gamma : F \Rightarrow G i_{\mathscr{A}}$ there is a unique natural transformation $\beta : \widetilde{F} \Rightarrow G$ such that $\gamma = \beta * \alpha$.*

Sketch of the proof. Given a functor $F : \mathscr{A} \to \mathscr{I}$, the construction outlined in Remark 6.9 yields candidates for $\widetilde{F} : \mathscr{A}^\wedge \to \mathscr{I}$ and $\alpha : F \Rightarrow \widetilde{F} i_{\mathscr{A}}$ and we leave it to the reader to check that this works. Once $\widetilde{F}$ and α are fixed, $\widetilde{\beta} := \gamma * \alpha^{-1}$ yields a natural transformation $\widetilde{F} i_{\mathscr{A}} \Rightarrow G i_{\mathscr{A}}$ and the procedure in Remark 6.9 shows what an extension $\beta : \widetilde{F} \Rightarrow G$ of $\widetilde{\beta}$ must look like and again we leave it to the reader to check that this works. □

Corollary 6.11. *Let $\mathscr{A}$ be a small additive category. The functor $i_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}^\wedge$ induces an equivalence of functor categories*

$$(i_{\mathscr{A}})^* : \mathrm{Hom}(\mathscr{A}^\wedge, \mathscr{I}) \xrightarrow{\simeq} \mathrm{Hom}(\mathscr{A}, \mathscr{I})$$

for every idempotent complete category $\mathscr{I}$.

Proof. Point (i) of Proposition 6.10 states that $(i_{\mathscr{A}})^*$ is essentially surjective and it follows from point (ii) that it is fully faithful, hence it is an equivalence of categories. □

Example 6.12. Let $\mathscr{F}$ be the category of (finitely generated) free modules over a ring R. Its idempotent completion $\mathscr{F}^\wedge$ is equivalent to the category of (finitely generated) projective modules over R.

Let now $(\mathscr{A}, \mathscr{E})$ be an exact category. Call a sequence in $\mathscr{A}^\wedge$ short exact if it is a direct summand in $(\mathscr{A}^\wedge)^{\to\to}$ of a sequence in $\mathscr{E}$ and denote the class of short exact sequences in $\mathscr{A}^\wedge$ by $\mathscr{E}^\wedge$.

Proposition 6.13. *The class $\mathscr{E}^\wedge$ is an exact structure on $\mathscr{A}^\wedge$. The inclusion functor $i_{\mathscr{A}} : (\mathscr{A}, \mathscr{E}) \to (\mathscr{A}^\wedge, \mathscr{E}^\wedge)$ preserves and reflects exactness and is 2-universal among exact functors to idempotent complete exact categories*:

(i) *Let $F : \mathscr{A} \to \mathscr{I}$ be an exact functor to an idempotent complete exact category $\mathscr{I}$. There exists an exact functor $\widetilde{F} : \mathscr{A}^\wedge \to \mathscr{I}$ together with a natural isomorphism $\alpha : F \Rightarrow \widetilde{F} i_{\mathscr{A}}$.*
(ii) *Given another exact functor $G : \mathscr{A}^\wedge \to \mathscr{I}$ together with a natural transformation $\gamma : F \Rightarrow G i_{\mathscr{A}}$, there is a unique natural transformation $\beta : \widetilde{F} \Rightarrow G$ such that $\gamma = \beta * \alpha$.*

Proof. To prove that $\mathscr{E}^\wedge$ is an exact structure is straightforward but rather tedious, so we skip it.[1] Given this, it is clear that the functor $\mathscr{A} \to \mathscr{A}^\wedge$ is exact and reflects exactness. If $F : \mathscr{A} \to \mathscr{I}$ is an exact functor to an idempotent complete exact category then, as every sequence in $\mathscr{E}^\wedge$ is a direct summand of a sequence in $\mathscr{E}$, an extension $\widetilde{F}$ of F as provided by Proposition 6.10 must carry exact sequences in $\mathscr{A}^\wedge$ to exact sequences in $\mathscr{I}$. Thus statements (i) and (ii) follow from the corresponding statements in Proposition 6.10. □

Corollary 6.14. *For a small exact category $(\mathscr{A}, \mathscr{E})$, the exact functor $i_{\mathscr{A}}$ induces an equivalence of the categories of exact functors*

$$(i_{\mathscr{A}})^* : \mathrm{Hom}_{\mathrm{Ex}}((\mathscr{A}^\wedge, \mathscr{E}^\wedge), \mathscr{I}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Ex}}((\mathscr{A}, \mathscr{E}), \mathscr{I})$$

to every idempotent complete exact category $\mathscr{I}$.

7. Weak idempotent completeness

Thomason introduced in [58, A.5.1] the notion of an exact category with "weakly split idempotents". It turns out that this is a property of the underlying additive category rather than the exact structure.

Recall that in an arbitrary category a morphism $r : B \to C$ is called a *retraction* if there exists a *section* $s : C \to B$ of r in the sense that $rs = 1_C$. Dually, a morphism $c : A \to B$ is a *coretraction* if it admits a section $s : B \to A$, i.e., $sc = 1_A$. Observe that retractions are epics and coretractions are monics. Moreover, a section of a retraction is a coretraction and a section of a coretraction is a retraction.

Lemma 7.1. *In an additive category $\mathscr{A}$ the following are equivalent*:

(i) *Every coretraction has a cokernel.*
(ii) *Every retraction has a kernel.*

Definition 7.2. If the conditions of the previous lemma hold then $\mathscr{A}$ is said to be *weakly idempotent complete*.

[1] Thomason [58, A.9.1 (b)] gives a short argument relying on the embedding into an abelian category, but it can be done by completely elementary means as well.

Remark 7.3. Freyd [19] uses the more descriptive terminology *retracts have complements* for weakly idempotent complete categories. He proves in particular that a weakly idempotent complete category with countable coproducts is idempotent complete.

Remark 7.4. Assume that $r : B \to C$ is a retraction with section $s : C \to B$. Then $sr : B \to B$ is an idempotent. Let us prove that this idempotent gives rise to a splitting of B if r admits a kernel $k : A \to B$.

Indeed, since $r(1_B - sr) = 0$, there is a unique morphism $t : B \to A$ such that $kt = 1_B - sr$. It follows that k is a coretraction because $ktk = (1_B - sr)k = k$ implies that $tk = 1_A$. Moreover $kts = 0$, so $ts = 0$, hence $[k \;\; s] : A \oplus C \to B$ is an isomorphism with inverse $\left[\begin{smallmatrix} t \\ r \end{smallmatrix}\right]$. In particular, the sequences $A \to B \to C$ and $A \to A \oplus C \to C$ are isomorphic.

Proof of Lemma 7.1. By duality it suffices to prove that (ii) implies (i).

Let $c : C \to B$ be a coretraction with section s. Then s is a retraction and, assuming (ii), it admits a kernel $k : A \to B$. By the discussion in Remark 7.4, k is a coretraction with section $t : B \to A$ and it is obvious that t is a cokernel of c. $\square$

Corollary 7.5. *Let $(\mathscr{A}, \mathscr{E})$ be an exact category. The following are equivalent*:

(i) *The additive category $\mathscr{A}$ is weakly idempotent complete.*
(ii) *Every coretraction is an admissible monic.*
(iii) *Every retraction is an admissible epic.*

Proof. It follows from Remark 7.4 that every retraction $r : B \to C$ admitting a kernel gives rise to a sequence $A \to B \to C$ which is isomorphic to the split exact sequence $A \rightarrowtail A \oplus C \twoheadrightarrow C$, hence r is an admissible epic by Lemma 2.7, whence (i) implies (iii). By duality (i) implies (ii) as well. Conversely, every admissible monic has a cokernel and every admissible epic has a kernel, hence (ii) and (iii) both imply (i). $\square$

In a weakly idempotent complete exact category the obscure axiom (Proposition 2.16) has an easier statement – this is Heller's cancellation axiom [27, (P2), p. 492]:

Proposition 7.6. *Let $(\mathscr{A}, \mathscr{E})$ be an exact category. The following are equivalent*:

(i) *The additive category $\mathscr{A}$ is weakly idempotent complete.*
(ii) *Consider two morphisms $g : B \to C$ and $f : A \to B$. If $gf : A \twoheadrightarrow C$ is an admissible epic then g is an admissible epic.*

Proof. (i) $\Rightarrow$ (ii): Form the pull-back over g and gf and consider the diagram

$$\begin{tikzcd}
A \arrow[drr, bend left, "f"] \arrow[dr, dotted, "\exists!"] \arrow[ddr, bend right, "1_A"'] & & \\
& B' \arrow[r, two heads] \arrow[d, "g'"'] \arrow[dr, phantom, "\mathrm{PB}"] & B \arrow[d, "g"] \\
& A \arrow[r, two heads, "gf"] & C
\end{tikzcd}$$

which proves g' to be a retraction, so g' has a kernel $K' \to B'$. Because the diagram is a pull-back, the composite $K' \to B' \to B$ is a kernel of g and now the dual of Proposition 2.16 applies to yield that g is an admissible epic.

For the implication (ii) ⇒ (i) simply observe that (ii) implies that retractions are admissible epics. □

Corollary 7.7. *An exact category is weakly idempotent complete if and only if it has the following property: all morphisms $g : B \to C$ for which there is a commutative diagram*

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[dr, two heads, "gf"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

are admissible epics.

Proof. A weakly idempotent complete exact category enjoys the stated property by Proposition 7.6.

Conversely, let $r : B \to C$ and $s : C \to B$ be such that $rs = 1_C$. We want to show that r is an admissible epic. The sequences

$$B \overset{\left[\begin{smallmatrix} 1 \\ -r \end{smallmatrix}\right]}{\rightarrowtail} B \oplus C \overset{[r\ 1]}{\twoheadrightarrow} C \quad \text{and} \quad C \overset{\left[\begin{smallmatrix} -s \\ 1 \end{smallmatrix}\right]}{\rightarrowtail} B \oplus C \overset{[1\ s]}{\twoheadrightarrow} B$$

are split exact, so $[r \quad 1]$ and $[1 \quad s]$ are admissible epics. But the diagram

$$\begin{tikzcd} B \oplus C \arrow[r, two heads, "{[1\ s]}"] \arrow[dr, two heads, "{[r\ 1]}"'] & B \arrow[d, "r"] \\ & C \end{tikzcd}$$

is commutative, hence r is an admissible epic. □

Remark 7.8 (Neeman [48, 1.12]). Every small additive category $\mathscr{A}$ has a *weak idempotent completion* $\mathscr{A}'$. Objects of $\mathscr{A}'$ are the pairs (A, p), where $p : A \to A$ is an idempotent factoring as $p = cr$ for some retraction $r : A \to X$ and coretraction $c : X \to A$ with $rc = 1_X$, while the morphisms are given by

$$\operatorname{Hom}_{\mathscr{A}'}((A, p), (B, q)) = q \circ \operatorname{Hom}_{\mathscr{A}}(A, B) \circ p.$$

It is easy to see that the functor $\mathscr{A} \to \mathscr{A}'$ given on objects by $A \mapsto (A, 1_A)$ is 2-universal among functors from $\mathscr{A}$ to a weakly idempotent complete category. Moreover, if $(\mathscr{A}, \mathscr{E})$ is exact then so is $(\mathscr{A}', \mathscr{E}')$, where the sequences in $\mathscr{E}'$ are the direct summands in $\mathscr{A}'$ of sequences in $\mathscr{E}$, and the functor $\mathscr{A} \to \mathscr{A}'$ preserves and reflects exactness and is 2-universal among exact functors to weakly idempotent complete exact categories.

Remark 7.9. Contrary to the construction of the idempotent completion, there is the set-theoretic subtlety that the weak idempotent completion might not be well-defined if $\mathscr{A}$ is not small: it is *not* clear a priori that the objects (A, p) form a class – essentially for the same reason that the monics in a category need not form a class, see e.g. the discussion in Borceux [6, p. 373].

Exercise 7.10.

(i) Let $R = \mathbb{R}[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$ be the coordinate ring of the 2-sphere S^2. Recall that an algebraic vector bundle over S^2 is the same thing as a finitely generated projective R-module P and the bundle represented by P is trivial if and only if P is free. The tangent bundle

$$TS^2 = \{(f, g, h) \in R^3 : Xf + Yg + Zh = 0\}$$

satisfies $TS^2 \oplus R \cong R^3$ ("the sum of the tangent bundle with the normal bundle is trivial"), in particular TS^2 is projective. Deduce from the hairy ball theorem that TS^2 is not free.

(ii) Let R be a ring for which there exists a finitely generated projective module P which is not free, but stably free, i.e., there exists $n \geqslant 1$ such that $P \oplus R^{\oplus n}$ is free. Let $\mathscr{A}$ be the category of finitely generated free R-modules. Then $\mathscr{A}$ is not weakly idempotent complete because the morphism $\begin{bmatrix} 0 \\ 1 \end{bmatrix} : R^{\oplus n} \to P \oplus R^{\oplus n}$ is a coretraction without cokernel (otherwise P would have to be free).

Exercise 7.11. Consider the ring $R = \mathbb{Q} \times \mathbb{Q}$. Prove that the idempotent

$$1 \times 0 : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q} \times \mathbb{Q}$$

does not split in the category $\mathscr{A}$ of finitely generated free R-modules. Prove that every coretraction in $\mathscr{A}$ has a cokernel, hence $\mathscr{A}$ is weakly idempotent complete but not idempotent complete.

8. Admissible morphisms and the snake lemma

Definition 8.1. A morphism $f : A \to B$ in an exact category

$$\begin{tikzcd} A \arrow[rr, "f", "\circ" marking] \arrow[dr, two heads, "e"'] & & B \\ & I \arrow[ur, tail, "m"'] & \end{tikzcd}$$

is called *admissible* if it factors as a composition of an admissible monic with an admissible epic. Admissible morphisms will sometimes be displayed as $\overset{\circ}{\longrightarrow}$ in diagrams.

Remark 8.2. Let f be an admissible morphism. If e' is an admissible epic and m' is an admissible monic then $m' f e'$ is admissible if the composition is defined. However, admissible morphisms are *not* closed under composition in general, see Excercise 8.6.

Remark 8.3. We choose the terminology *admissible morphism* even though *strict morphism* seems to be more standard (see e.g. [52,55]). By Exercise 2.6 an admissible monic is the same thing as an admissible morphism which happens to be monic.

Lemma 8.4 (*Heller [27, 3.4]*). *The factorization of an admissible morphism is unique up to unique isomorphism. More precisely, in a commutative diagram of the form*

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d, two heads, "e'"'] & I \arrow[d, tail, "m"] \arrow[dl, dotted, bend left=10, "i"] \\ I' \arrow[r, tail, "m'"'] \arrow[ur, dotted, bend left=10, "i'"] & B \end{tikzcd}$$

there exist unique morphisms i, i' making the diagram commutative. In particular, i and i' are mutually inverse isomorphisms.

Proof. Let k be a kernel of e. Since $m'e'k = mek = 0$ and m' is monic we have $e'k = 0$, hence there exists a unique morphism $i : I \to I'$ such that $e' = ie$. Moreover, $m'ie = m'e' = me$ and e epic imply $m'i = m$. Dually for i'. $\square$

Remark 8.5. An admissible morphism has an *analysis* (cf. [44, IX.2])

$$\begin{tikzcd} & A \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & B \arrow[dr, two heads, "c"] & \\ K \arrow[ur, tail, "k"] & & I \arrow[ur, tail, "m"'] & & C \end{tikzcd}$$

where k is a kernel, c is a cokernel, e is a coimage and m is an image of f and the isomorphism classes of K, I and C are well-defined by Lemma 8.4.

Exercise 8.6.

(i) If $\mathscr{A}$ is an exact category in which every morphism is admissible then $\mathscr{A}$ is abelian. [A solution is given by Freyd in [18, Proposition 3.1].]

(ii) The class of admissible morphisms is closed under composition if and only if $\mathscr{A}$ is abelian.
Hint: For every morphism $g : A \to B$ the morphism $\begin{bmatrix} 1 \\ g \end{bmatrix} : A \to A \oplus B$ is an admissible monic.

Lemma 8.7. *Admissible morphisms are stable under push-out along admissible monics and pull-back along admissible epics.*

Proof. Let $A \twoheadrightarrow I \rightarrowtail B$ be an admissible epic-admissible monic factorization of an admissible morphism. To prove the claim about push-outs construct the diagram

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\text{PO}"] & I \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\text{PO}"] & B \arrow[d, tail] \\ A' \arrow[r, two heads] & I' \arrow[r, tail] & B'. \end{tikzcd}$$

Proposition 2.15 yields that $A' \to I'$ is an admissible epic and the rest is clear. $\square$

Definition 8.8. A sequence of admissible morphisms

$$\begin{tikzcd}
A' \arrow[rr, "f", "\circ" marking] \arrow[dr, two heads, "e"'] & & A \arrow[rr, "f'", "\circ" marking] \arrow[dr, two heads, "e'"'] & & A'' \\
& I \arrow[ur, tail, "m"'] & & I' \arrow[ur, tail, "m'"'] &
\end{tikzcd}$$

is *exact* if $I \rightarrowtail A \twoheadrightarrow I'$ is short exact. Longer sequences of admissible morphisms are exact if the sequence given by any two consecutive morphisms is exact. Since the term “exact” is heavily overloaded, we also use the synonym *“acyclic”*, in particular in connection with chain complexes.

Lemma 8.9 (*Five lemma, II*). *If the commutative diagram*

$$\begin{tikzcd}
A_1 \arrow[r, "\circ" marking] \arrow[d, "\cong"] & A_2 \arrow[r, "\circ" marking] \arrow[d, "\cong"] & A_3 \arrow[r, "\circ" marking] \arrow[d, "f"] & A_4 \arrow[r, "\circ" marking] \arrow[d, "\cong"] & A_5 \arrow[d, "\cong"] \\
B_1 \arrow[r, "\circ" marking] & B_2 \arrow[r, "\circ" marking] & B_3 \arrow[r, "\circ" marking] & B_4 \arrow[r, "\circ" marking] & B_5
\end{tikzcd}$$

has exact rows then f is an isomorphism.

Sketch of the proof. By hypothesis there are factorizations $A_i \twoheadrightarrow I_i \rightarrowtail A_{i+1}$ of $A_i \to A_{i+1}$ and $B_i \twoheadrightarrow J_i \rightarrowtail B_{i+1}$ of $B_i \to B_{i+1}$ for $i = 1, \ldots, 4$. Using Lemma 8.4 and Exercise 3.3 there are isomorphisms $I_1 \cong J_1$ and $I_2 \cong J_2$ which one may insert into the diagram without destroying its commutativity. Dually for $I_4 \cong J_4$ and $I_3 \cong J_3$. The five lemma 3.2 then implies that f is an isomorphism. $\square$

Exercise 8.10. Assume that $\mathscr{A}$ is weakly idempotent complete (Definition 7.2).

(i) (Sharp four lemma) Consider a commutative diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\circ" marking] \arrow[d, two heads] & A_2 \arrow[r, "\circ" marking] \arrow[d, "\cong"] & A_3 \arrow[r, "\circ" marking] \arrow[d, "f"] & A_4 \arrow[d, tail] \\
B_1 \arrow[r, "\circ" marking] & B_2 \arrow[r, "\circ" marking] & B_3 \arrow[r, "\circ" marking] & B_4
\end{tikzcd}$$

with exact rows. Prove that f is an admissible monic. Dualize.

(ii) (Sharp five lemma) If the commutative diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\circ" marking] \arrow[d, two heads] & A_2 \arrow[r, "\circ" marking] \arrow[d, "\cong"] & A_3 \arrow[r, "\circ" marking] \arrow[d, "f"] & A_4 \arrow[r, "\circ" marking] \arrow[d, "\cong"] & A_5 \arrow[d, tail] \\
B_1 \arrow[r, "\circ" marking] & B_2 \arrow[r, "\circ" marking] & B_3 \arrow[r, "\circ" marking] & B_4 \arrow[r, "\circ" marking] & B_5
\end{tikzcd}$$

has exact rows then f is an isomorphism.

Hint: Use Proposition 7.6, Exercises 2.6, 3.3 as well as Corollary 3.2.

Proposition 8.11 (*Ker–Coker-sequence*). *Assume that $\mathscr{A}$ is a weakly idempotent complete exact category. Let $f : A \to B$ and $g : B \to C$ be admissible morphisms such that*

$h = gf : A \to C$ is admissible as well. There is an exact sequence

$$\operatorname{Ker} f \rightarrowtail \operatorname{Ker} h \overset{\circ}{\longrightarrow} \operatorname{Ker} g \overset{\circ}{\longrightarrow} \operatorname{Coker} f \overset{\circ}{\longrightarrow} \operatorname{Coker} h \twoheadrightarrow \operatorname{Coker} g$$

depending naturally on the diagram $h = gf$.

Proof. Observe that the morphism $\operatorname{Ker} f \rightarrowtail A$ factors over $\operatorname{Ker} h \rightarrowtail A$ via a unique morphism $\operatorname{Ker} f \to \operatorname{Ker} h$ which is an admissible monic by Proposition 7.6. Let $\operatorname{Ker} h \twoheadrightarrow X$ be a cokernel of $\operatorname{Ker} f \rightarrowtail \operatorname{Ker} h$. Dually, there is an admissible epic $\operatorname{Coker} h \twoheadrightarrow \operatorname{Coker} g$ with $Z \rightarrowtail \operatorname{Coker} h$ as kernel. The Noether isomorphism 3.5 implies that there are two commutative diagrams with exact rows and columns

$$\begin{tikzcd}
\operatorname{Ker} f \arrow[r, rightarrowtail] \arrow[d, equal] & \operatorname{Ker} h \arrow[r, twoheadrightarrow] \arrow[d, rightarrowtail] & X \arrow[d, rightarrowtail] \\
\operatorname{Ker} f \arrow[r, rightarrowtail] & A \arrow[r, twoheadrightarrow] \arrow[d, twoheadrightarrow] & \operatorname{Im} f \arrow[d, twoheadrightarrow] \\
& \operatorname{Im} h \arrow[r, equal] & \operatorname{Im} h
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\operatorname{Im} h \arrow[r, rightarrowtail] \arrow[d, equal] & \operatorname{Im} g \arrow[r, twoheadrightarrow] \arrow[d, rightarrowtail] & Z \arrow[d, rightarrowtail] \\
\operatorname{Im} h \arrow[r, rightarrowtail] & C \arrow[r, twoheadrightarrow] \arrow[d, twoheadrightarrow] & \operatorname{Coker} h \arrow[d, twoheadrightarrow] \\
& \operatorname{Coker} g \arrow[r, equal] & \operatorname{Coker} g.
\end{tikzcd}$$

It is easy to see that there is an admissible monic $X \rightarrowtail \operatorname{Ker} g$ whose cokernel we denote by $\operatorname{Ker} g \twoheadrightarrow Y$. Therefore the 3×3-lemma yields a commutative diagram with exact rows and columns

$$\begin{tikzcd}
X \arrow[r, rightarrowtail] \arrow[d, rightarrowtail] & \operatorname{Ker} g \arrow[r, twoheadrightarrow] \arrow[d, rightarrowtail] & Y \arrow[d, rightarrowtail] \\
\operatorname{Im} f \arrow[r, rightarrowtail] \arrow[d, twoheadrightarrow] & B \arrow[r, twoheadrightarrow] \arrow[d, twoheadrightarrow] & \operatorname{Coker} f \arrow[d, twoheadrightarrow] \\
\operatorname{Im} h \arrow[r, rightarrowtail] & \operatorname{Im} g \arrow[r, twoheadrightarrow] & Z.
\end{tikzcd}$$

The desired Ker–Coker-sequence is now obtained by splicing: start with the first row of the first diagram, splice it with the first row of the third diagram, and continue with the third row of the third diagram and the third row of the second diagram. The naturality assertion is obvious. □

Lemma 8.12. *Let $\mathscr{A}$ be an exact category in which each commutative triangle of admissible morphisms yields an exact* Ker–Coker-*sequence where exactness is understood in the sense of admissible morphisms. Then $\mathscr{A}$ is weakly idempotent complete.*

Proof. We check the criterion in Corollary 7.7. We need to show that in every commutative diagram of the form

$$\begin{tikzcd}
A \arrow[r, twoheadrightarrow, "f"] \arrow[dr, twoheadrightarrow, "h"'] & B \arrow[d, "g"] \\
& C
\end{tikzcd}$$

the morphism g is an admissible epic. Given such a diagram, consider the diagram

$$\begin{tikzcd}
\operatorname{Ker} f \arrow[r, tail] \arrow[dr, "\circ" marking, "0"'] & A \arrow[d, two heads, "h"] \\
 & C
\end{tikzcd}$$

whose associated Ker–Coker-sequence is

$$0 \xrightarrow{\circ} \operatorname{Ker} f \xrightarrow{\circ} \operatorname{Ker} h \xrightarrow{\circ} B \xrightarrow[\circ]{g} C \xrightarrow{\circ} 0$$

so that g is an admissible epic. $\square$

The Ker–Coker-sequence immediately yields the following version of the snake lemma, the neat proof of which was pointed out to the author by Matthias Künzer.

Corollary 8.13 (*Snake lemma, I*). *Let $\mathscr{A}$ be weakly idempotent complete. Consider a morphism of short exact sequences $A' \rightarrowtail A \twoheadrightarrow A''$ and $B' \rightarrowtail B \twoheadrightarrow B''$ with admissible components. There is a commutative diagram*

$$\begin{tikzcd}
K' \arrow[r, dotted, "k"] \arrow[d, tail] & K \arrow[r, dotted, "k'"] \arrow[d, tail] & K'' \arrow[d, tail] \\
A' \arrow[r, tail] \arrow[d, "\circ" marking] & A \arrow[r, two heads] \arrow[d, "\circ" marking] & A'' \arrow[d, "\circ" marking] \\
B' \arrow[r, tail] \arrow[d, two heads] & B \arrow[r, two heads] \arrow[d, two heads] & B'' \arrow[d, two heads] \\
C' \arrow[r, dotted, "c"] & C \arrow[r, dotted, "c'"] & C''
\end{tikzcd}$$

with exact rows and columns and there is a connecting morphism $\delta : K'' \to C'$ fitting into an exact sequence

$$K' \overset{k}{\rightarrowtail} K \xrightarrow[\circ]{k'} K'' \xrightarrow[\circ]{\delta} C' \xrightarrow[\circ]{c} C \overset{c'}{\twoheadrightarrow} C''$$

depending naturally on the morphism of short exact sequences.

Remark 8.14. Using the notations of the proof of Lemma 8.12 consider the diagram

$$\begin{tikzcd}
\operatorname{Ker} f \arrow[r, tail] \arrow[d, "\circ" marking, "0"] & A \arrow[r, two heads] \arrow[d, "\circ" marking, "h"] & B \arrow[d, "\circ" marking] \\
C \arrow[r, equal] & C \arrow[r] & 0.
\end{tikzcd}$$

The sequence $\operatorname{Ker} f \rightarrowtail \operatorname{Ker} h \xrightarrow{\circ} B \xrightarrow[\circ]{g} C \xrightarrow{\circ} 0 \twoheadrightarrow 0$ provided by the snake lemma shows that g is an admissible epic. It follows that the snake lemma can only hold if the category is weakly idempotent complete.

Proof of Corollary 8.13. By Proposition 3.1 and Lemma 8.7 we get the commutative diagram

$$\begin{tikzcd}
A' \arrow[r, tail] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\mathrm{BC}"] & A \arrow[r, two heads] \arrow[d, "\circ" marking] & A'' \arrow[d, equal] \\
B' \arrow[r, tail] \arrow[d, equal] & D \arrow[r, two heads] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\mathrm{BC}"] & A'' \arrow[d, "\circ" marking] \\
B' \arrow[r, tail] & B \arrow[r, two heads] & B''
\end{tikzcd}$$

(more explicitly, the diagram is obtained by forming the push-out $A'AB'D$). The Ker–Coker-sequence of the commutative triangle of admissible morphisms

$$\begin{tikzcd}
A \arrow[r, "\circ" marking] \arrow[dr, "\circ" marking] & D \arrow[d, "\circ" marking] \\
& B
\end{tikzcd}$$

yields the desired result by Remark 2.13. $\square$

Exercise 8.15 (*Snake lemma, II, cf. Heller [27, 4.3]*). Formulate and prove a snake lemma for a diagram of the form

$$\begin{tikzcd}
& A' \arrow[r, "\circ" marking] \arrow[d, "\circ" marking] & A \arrow[r, "\circ" marking] \arrow[d, "\circ" marking] & A'' \arrow[r] \arrow[d, "\circ" marking] & 0 \\
0 \arrow[r] & B' \arrow[r, "\circ" marking] & B \arrow[r, "\circ" marking] & B''
\end{tikzcd}$$

in weakly idempotent complete categories. Prove $\mathrm{Ker}(A' \to A) = \mathrm{Ker}(K' \to K)$ and $\mathrm{Coker}(B \to B'') = \mathrm{Coker}(C \to C'')$.

Hint: Reduce to Corollary 8.13 by using Proposition 7.6 and the Noether isomorphism 3.5.

Remark 8.16. Heller [27, 4.3] gives a direct proof of the snake lemma starting from his axioms. Using the Noether isomorphism 3.5 and the 3×3-lemma 3.6 as well as Proposition 7.6, Heller's proof is easily adapted to a proof from Quillen's axioms.

Exercise 8.17. The following assertions on an exact category $(\mathscr{A}, \mathscr{E})$ are equivalent:

(i) The category $\mathscr{A}$ is weakly idempotent complete.
(ii) If the direct sum of two morphisms $a : A' \to A$ and $b : B' \to B$ is an admissible monic then so are a and b.

Hint: Use Proposition 7.6 to prove (i) $\Rightarrow$ (ii). The morphism $\begin{bmatrix} 0 \\ s \end{bmatrix}$ appearing in the proof of Proposition 10.14 shows that (ii) cannot hold if $\mathscr{A}$ is not weakly idempotent complete.

Exercise 8.18. The following assertions on an exact category $(\mathscr{A}, \mathscr{E})$ are equivalent:

(i) The category $\mathscr{A}$ is idempotent complete.
(ii) If the direct sum of two morphisms $a : A' \to A$ and $b : B' \to B$ is an admissible morphism, then so are a and b.

Hint: In order to prove (i) $\Rightarrow$ (ii), split kernel, image and cokernel of $a \oplus b$ and apply Exercise 8.17. To prove that (ii) $\Rightarrow$ (i) let $e : A \to A$ be an idempotent, put $f = 1_A - e$ and observe that $e \oplus f$ is isomorphic to $1_A \oplus 0_A$ via $\left[\begin{smallmatrix} e & -f \\ f & e \end{smallmatrix}\right]$.

9. Chain complexes and chain homotopy

The notion of chain complexes makes sense in every additive category $\mathscr{A}$. A *(chain) complex* is a diagram $(A^\bullet, d_A^\bullet)$

$$\cdots \to A^{n-1} \xrightarrow{d_A^{n-1}} A^n \xrightarrow{d_A^n} A^{n+1} \to \cdots$$

subject to the condition that $d_A^n d_A^{n-1} = 0$ for all n and a *chain map* is a morphism of such diagrams. The category of complexes and chain maps is denoted by $\mathbf{Ch}(\mathscr{A})$. Obviously, the category $\mathbf{Ch}(\mathscr{A})$ is additive.

Lemma 9.1. *If $(\mathscr{A}, \mathscr{E})$ is an exact category then $\mathbf{Ch}(\mathscr{A})$ is an exact category with respect to the class $\mathbf{Ch}(\mathscr{E})$ of short sequences of chain maps which are exact in each degree. If $\mathscr{A}$ is abelian then so is $\mathbf{Ch}(\mathscr{A})$.*

Proof. The point is that (as in every functor category) limits and colimits of diagrams in $\mathbf{Ch}(\mathscr{A})$ are obtained by taking the limits and colimits pointwise (in each degree), in particular push-outs under admissible monics and pull-backs over admissible epics exist and yield admissible monics and epics. The rest is obvious. $\square$

Definition 9.2. The *mapping cone* of a chain map $f : A \to B$ is the complex

$$\operatorname{cone}(f)^n = A^{n+1} \oplus B^n \quad \text{with differential} \quad d_f^n = \begin{bmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{bmatrix}$$

Notice that $d_f^{n+1} d_f^n = 0$ precisely because f is a chain map. It is plain that the mapping cone defines a functor from the category of morphisms in $\mathbf{Ch}(\mathscr{A})$ to $\mathbf{Ch}(\mathscr{A})$.

The *translation functor* on $\mathbf{Ch}(\mathscr{A})$ is defined to be $\Sigma A = \operatorname{cone}(A \to 0)$. More explicitly, ΣA is the complex with components $(\Sigma A)^n = A^{n+1}$ and differentials $d_{\Sigma A}^n = -d_A^{n+1}$. If f is a chain map, its translate is given by $(\Sigma f)^n = f^{n+1}$. Clearly, Σ is an additive automorphism of $\mathbf{Ch}(\mathscr{A})$.

The *strict triangle* over the chain map $f : A \to B$ is the 3-periodic (or rather 3-helicoidal, if you insist) sequence

$$A \xrightarrow{f} B \xrightarrow{i_f} \operatorname{cone}(f) \xrightarrow{j_f} \Sigma A \xrightarrow{\Sigma f} \Sigma B \xrightarrow{\Sigma i_f} \Sigma\operatorname{cone}(f) \xrightarrow{\Sigma j_f} \cdots$$

where the chain map i_f has components $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and j_f has components $[1 \quad 0]$.

Remark 9.3. Let $f : A \to B$ be a chain map. Observe that the sequence of chain maps

$$B \xrightarrow{i_f} \operatorname{cone}(f) \xrightarrow{j_f} \Sigma A$$

splits in each degree, however, it need not be a split exact sequence in $\mathbf{Ch}(\mathscr{A})$, because the degreewise splitting maps need not assemble to chain maps. In fact, it is straightforward to verify that the above sequence is split exact in $\mathbf{Ch}(\mathscr{A})$ if and only if f is chain homotopic to zero in the sense of Definition 9.5.

Exercise 9.4. Assume that $\mathscr{A}$ is an abelian category. Prove that the strict triangle over the chain map $f : A \to B$ gives rise to a long exact homology sequence

$$\cdots \to H^n(A) \xrightarrow{H^n(f)} H^n(B) \xrightarrow{H^n(i_f)} H^n(\operatorname{cone}(f)) \xrightarrow{H^n(j_f)} H^{n+1}(A) \to \cdots$$

Deduce that f induces an isomorphism of $H^*(A)$ with $H^*(B)$ if and only if $\operatorname{cone}(f)$ is acyclic.

Definition 9.5. A chain map $f : A \to B$ is *chain homotopic to zero* if there exist morphisms $h^n : A^n \to B^{n-1}$ such that $f^n = d_B^{n-1} h^n + h^{n+1} d_A^n$. A chain complex A is called *null-homotopic* if 1_A is chain homotopic to zero.

Remark 9.6. The maps which are chain homotopic to zero form an ideal in $\mathbf{Ch}(\mathscr{A})$, that is to say if $h : B \to C$ is chain homotopic to zero then so are hf and gh for all morphisms $f : A \to B$ and $g : C \to D$, if h_1 and h_2 are chain homotopic to zero then so is $h_1 \oplus h_2$. The set $N(A, B)$ of chain maps $A \to B$ which are chain homotopic to zero is a subgroup of the abelian group $\operatorname{Hom}_{\mathbf{Ch}(\mathscr{A})}(A, B)$.

Definition 9.7. The *homotopy category* $\mathbf{K}(\mathscr{A})$ is the category with the chain complexes over $\mathscr{A}$ as objects and $\operatorname{Hom}_{\mathbf{K}(\mathscr{A})}(A, B) := \operatorname{Hom}_{\mathbf{Ch}(\mathscr{A})}(A, B)/N(A, B)$ as morphisms.

Remark 9.8. Notice that every null-homotopic complex is isomorphic to the zero object in $\mathbf{K}(\mathscr{A})$. It turns out that $\mathbf{K}(\mathscr{A})$ is additive, but it is very rarely abelian or exact with respect to a non-trivial exact structure (see [59, Chapter II, 1.3.6]). However, $\mathbf{K}(\mathscr{A})$ has the structure of a *triangulated category* induced by the *strict triangles* in $\mathbf{Ch}(\mathscr{A})$, see e.g. Verdier [59], Beĭlinson-Bernstein-Deligne [4], Gelfand-Manin [23], Grivel [8, Chapter I], Kashiwara-Schapira [35], Keller [39], Neeman [49] or Weibel [60].

Remark 9.9. For each object $A \in \mathscr{A}$, define $\operatorname{cone}(A) = \operatorname{cone}(1_A)$. Notice that $\operatorname{cone}(A)$ is null-homotopic with $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ as contracting homotopy.

Remark 9.10. If f and g are chain homotopy equivalent, i.e., $f - g$ is chain homotopic to zero, then $\operatorname{cone}(f)$ and $\operatorname{cone}(g)$ are isomorphic in $\mathbf{Ch}(\mathscr{A})$ but the isomorphism and its homotopy class will generally depend on the choice of a chain homotopy. In particular, the mapping cone construction does not yield a functor defined on morphisms of $\mathbf{K}(\mathscr{A})$.

Remark 9.11. A chain map $f : A \to B$ is chain homotopic to zero if and only if it factors as $hi_A = f$ over $h : \operatorname{cone}(A) \to B$, where $i_A = i_{1_A} : A \to \operatorname{cone}(A)$. Moreover, h has components $[h^{n+1} \quad f^n]$, where the family of morphisms $\{h^n\}$ is a chain homotopy of f to zero. Similarly, f is chain homotopic to zero if and only if f factors through $j_{\Sigma^{-1}B} = j_{1_{\Sigma^{-1}B}} : \operatorname{cone}(\Sigma^{-1}B) \to B$.

Remark 9.12. The mapping cone construction yields the push-out diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_A"'] \arrow[dr, phantom, "\mathrm{PO}"] & B \arrow[d, "i_f"] \\ \operatorname{cone}(A) \arrow[r, "\left[\begin{smallmatrix} 1 & 0 \\ 0 & f \end{smallmatrix}\right]"'] & \operatorname{cone}(f) \end{tikzcd}$$

in $\mathbf{Ch}(\mathscr{A})$. Now suppose that $g : B \to C$ is a chain map such that gf is chain homotopic to zero. By Remark 9.11, gf factors over i_A and using the push-out property of the above diagram it follows that g factors over i_f. This construction will depend on the choice of an explicit chain homotopy $gf \simeq 0$ in general. In particular, $\operatorname{cone}(f)$ is a *weak cokernel* in $\mathbf{K}(\mathscr{A})$ of the homotopy class of f in that it has the factorization property of a cokernel but without uniqueness. Similarly, $\Sigma^{-1}\operatorname{cone}(f)$ is a *weak kernel* of f in $\mathbf{K}(\mathscr{A})$.

10. Acyclic complexes and quasi-isomorphisms

The present section is probably only of interest to readers acquainted with triangulated categories or at least with the construction of the derived category of an abelian category. After giving the fundamental definition of acyclicity of a complex over an exact category, we may formulate the intimately connected notion of quasi-isomorphisms.

We will give an elementary proof of the fact that the homotopy category $\mathbf{Ac}(\mathscr{A})$ of acyclic complexes over an exact category $\mathscr{A}$ is a triangulated category. It turns out that $\mathbf{Ac}(\mathscr{A})$ is a strictly full subcategory of the homotopy category of chain complexes $\mathbf{K}(\mathscr{A})$ if and only if $\mathscr{A}$ is idempotent complete, and in this case $\mathbf{Ac}(\mathscr{A})$ is even thick in $\mathbf{K}(\mathscr{A})$. Since thick subcategories are strictly full by definition, $\mathbf{Ac}(\mathscr{A})$ is thick if and only if $\mathscr{A}$ is idempotent complete.

By [49, Chapter 2], the Verdier quotient $\mathbf{K}/\mathscr{T}$ is defined for any (strictly full) triangulated subcategory $\mathscr{T}$ of a triangulated category $\mathbf{K}$ and it coincides with the Verdier quotient $\mathbf{K}/\bar{\mathscr{T}}$, where $\bar{\mathscr{T}}$ is the *thick closure* of $\mathscr{T}$. The case we are interested in is $\mathbf{K} = \mathbf{K}(\mathscr{A})$ and $\mathscr{T} = \mathbf{Ac}(\mathscr{A})$. The Verdier quotient $\mathbf{D}(\mathscr{A}) = \mathbf{K}(\mathscr{A})/\mathbf{Ac}(\mathscr{A})$ is the *derived* *category* of $\mathscr{A}$. If $\mathscr{A}$ is idempotent complete then $\overline{\mathbf{Ac}(\mathscr{A})} = \mathbf{Ac}(\mathscr{A})$ and it is clear that quasi-isomorphisms are then precisely the chain maps with acyclic mapping cone. If $\mathscr{A}$ fails to be idempotent complete, it turns out that the thick closure $\overline{\mathbf{Ac}(\mathscr{A})}$ of $\mathbf{Ac}(\mathscr{A})$ is the same as the closure of $\mathbf{Ac}(\mathscr{A})$ under isomorphisms in $\mathbf{K}(\mathscr{A})$, so a chain map f is a quasi-isomorphism if and only if $\operatorname{cone}(f)$ is homotopy equivalent to an acyclic complex.

Similarly, the derived categories of bounded, left bounded or right bounded complexes are constructed as in the abelian setting. It is useful to notice that for $* \in \{+, -, b\}$ the category $\mathbf{Ac}^*(\mathscr{A})$ is thick in $\mathbf{K}^*(\mathscr{A})$ if and only if $\mathscr{A}$ is weakly idempotent complete, which leads to an easier description of quasi-isomorphisms.

If $\mathscr{B}$ is a fully exact subcategory of $\mathscr{A}$, the inclusion $\mathscr{B} \to \mathscr{A}$ yields a canonical functor $\mathbf{D}^+(\mathscr{B}) \to \mathbf{D}^+(\mathscr{A})$ and we state conditions which ensure that this functor is essentially surjective or fully faithful.

We end the section with a short discussion of Deligne's approach to total derived functors.

10.1. The homotopy category of acyclic complexes

Definition 10.1. A chain complex A over an exact category is called *acyclic* (or *exact*) if each differential factors as $A^n \twoheadrightarrow Z^{n+1}A \rightarrowtail A^{n+1}$ in such a way that each sequence $Z^n A \rightarrowtail A^n \twoheadrightarrow Z^{n+1}A$ is exact.

Remark 10.2. An acyclic complex is a complex with admissible differentials (Definition 8.1) which is exact in the sense of Definition 8.8. In particular, $Z^n A$ is a kernel of $A^n \to A^{n+1}$, an image and coimage of $A^{n-1} \to A^n$ and a cokernel of $A^{n-2} \to A^{n-1}$.

Lemma 10.3 (*Neeman [48, 1.1]*). *The mapping cone of a chain map $f : A \to B$ between acyclic complexes is acyclic.*

Proof. An easy diagram chase shows that the dotted morphisms in the diagram

$$
\begin{tikzcd}
A^{n-1} \arrow[rr, "d_A^{n-1}"] \arrow[dr, two heads, "j_A^{n-1}"] \arrow[ddd, "f^{n-1}"'] & & A^n \arrow[rr, "d_A^n"] \arrow[dr, two heads, "j_A^n"] \arrow[ddd, "f^n"'] & & A^{n+1} \arrow[ddd, "f^{n+1}"] \\
& Z^n A \arrow[ur, tail, "i_A^n"'] \arrow[d, dotted, "\exists! g^n"] & & Z^{n+1} A \arrow[ur, tail, "i_A^{n+1}"'] \arrow[d, dotted, "\exists! g^{n+1}"] & \\
& Z^n B \arrow[dr, tail, "i_B^n"] & & Z^{n+1} B \arrow[dr, tail, "i_B^{n+1}"] & \\
B^{n-1} \arrow[rr, "d_B^{n-1}"'] \arrow[ur, two heads, "j_B^{n-1}"'] & & B^n \arrow[rr, "d_B^n"'] \arrow[ur, two heads, "j_B^n"'] & & B^{n+1}
\end{tikzcd}
$$

exist and are the unique morphisms g^n making the diagram commutative.

By Proposition 3.1 we find objects $Z^n C$ fitting into a commutative diagram

$$\begin{tikzcd}
A^{n-1} \arrow[rr, "d_A^{n-1}"] \arrow[dr, two heads, "j_A^{n-1}"] \arrow[dd, "f'^{n-1}"'] & & A^n \arrow[rr, "d_A^n"] \arrow[dr, two heads, "j_A^n"] \arrow[dd, "f'^n"'] & & A^{n+1} \arrow[dd, "f'^{n+1}"] \\
 & Z^n A \arrow[ur, tail, "i_A^n"] \arrow[dd, "g^n"'] \arrow[dr, phantom, "\mathrm{BC}"] & & Z^{n+1} A \arrow[ur, tail, "i_A^{n+1}"] \arrow[dd, "g^{n+1}"'] \arrow[dr, phantom, "\mathrm{BC}"] & \\
Z^{n-1} C \arrow[ur, two heads, "h^{n-1}"'] \arrow[dd, "f''^{n-1}"'] \arrow[dr, phantom, "\mathrm{BC}"] & & Z^n C \arrow[ur, two heads, "h^n"'] \arrow[dd, "f''^n"'] \arrow[dr, phantom, "\mathrm{BC}"] & & Z^{n+1} C \arrow[dd, "f''^{n+1}"] \\
 & Z^n B \arrow[ur, tail, "k^n"] \arrow[dr, tail, "i_B^n"] & & Z^{n+1} B \arrow[ur, tail, "k^{n+1}"] \arrow[dr, tail, "i_B^{n+1}"] & \\
B^{n-1} \arrow[ur, two heads, "j_B^{n-1}"] \arrow[rr, "d_B^{n-1}"'] & & B^n \arrow[ur, two heads, "j_B^n"] \arrow[rr, "d_B^n"'] & & B^{n+1}
\end{tikzcd}$$

where $f^n = f''^n f'^n$ and the quadrilaterals marked BC are bicartesian. Recall that the objects $Z^n C$ are obtained by forming the push-outs under i_A^n and g^n (or the pull-backs over j_B^n and g^{n+1}) and that $Z^n B \rightarrowtail Z^n C \twoheadrightarrow Z^{n+1} A$ is short exact.

It follows from Corollary 2.14 that for each n the sequence

$$Z^{n-1} C \overset{\begin{bmatrix} -i_A^n h^{n-1} \\ f''^{n-1} \end{bmatrix}}{\rightarrowtail} A^n \oplus B^{n-1} \overset{\begin{bmatrix} f'^n & k^n j_B^{n-1} \end{bmatrix}}{\twoheadrightarrow} Z^n C$$

is short exact and the commutative diagram

$$\begin{tikzcd}
A^n \oplus B^{n-1} \arrow[rr, "\begin{bmatrix} -d_A^n & 0 \\ f^n & d_B^{n-1} \end{bmatrix}"] \arrow[dr, two heads, "{\begin{bmatrix} f'^n & k^n j_B^{n-1} \end{bmatrix}}"'] & & A^{n+1} \oplus B^n \arrow[rr, "\begin{bmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{bmatrix}"] \arrow[dr, two heads, "{\begin{bmatrix} f'^{n+1} & k^{n+1} j_B^n \end{bmatrix}}"'] & & A^{n+2} \oplus B^{n+1} \\
 & Z^n C \arrow[ur, tail, "\begin{bmatrix} -i_A^{n+1} h^n \\ f''^n \end{bmatrix}"'] & & Z^{n+1} C \arrow[ur, tail, "\begin{bmatrix} -i_A^{n+2} h^{n+1} \\ f''^{n+1} \end{bmatrix}"'] &
\end{tikzcd}$$

proves that $\operatorname{cone}(f)$ is acyclic. $\square$

Remark 10.4. Retaining the notations of the proof we have a short exact sequence

$$Z^n B \rightarrowtail Z^n C \twoheadrightarrow Z^{n+1} A$$

This sequence exhibits $Z^n C = \operatorname{Ker} \begin{bmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{bmatrix}$ as an extension of $Z^{n+1} A = \operatorname{Ker} d_A^{n+1}$ by $Z^n B = \operatorname{Ker} d_B^n$.

Let $\mathbf{Ac}(\mathscr{A})$ be the full subcategory of the homotopy category $\mathbf{K}(\mathscr{A})$ consisting of acyclic complexes over the exact category $\mathscr{A}$. It follows from Proposition 2.9 that the direct sum of two acyclic complexes is acyclic. Thus $\mathbf{Ac}(\mathscr{A})$ is a full additive subcategory of $\mathbf{K}(\mathscr{A})$. The previous lemma implies that even more is true:

Corollary 10.5. *The homotopy category of acyclic complexes* $\mathbf{Ac}(\mathscr{A})$ *is a triangulated subcategory of* $\mathbf{K}(\mathscr{A})$.

Remark 10.6. For reasons of convenience, many authors assume that triangulated subcategories are not only full but *strictly full*. We do not do so because $\mathbf{Ac}(\mathscr{A})$ is closed under isomorphisms in $\mathbf{K}(\mathscr{A})$ if and only if $\mathscr{A}$ is idempotent complete, see Proposition 10.9.

Lemma 10.7. *Assume that* $(\mathscr{A}, \mathscr{E})$ *is idempotent complete. Every retract in* $\mathbf{K}(\mathscr{A})$ *of an acyclic complex* A *is acyclic.*

Proof (*cf. Keller [37, 2.3, a]*). Let the chain map $f : X \to A$ be a coretraction, i.e., there is a chain map $s : A \to X$ such that $s^n f^n - 1_{X^n} = d_X^{n-1} h^n + h^{n+1} d_X^n$ for some morphisms $h^n : X^n \to X^{n-1}$. Obviously, the complex IX with components

$$(IX)^n = X^n \oplus X^{n+1} \quad \text{and differential} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is acyclic. There is a chain map $i_X : X \to IX$ given by

$$i_X^n = \begin{bmatrix} 1_{X^n} \\ d_X^n \end{bmatrix} : X^n \to X^n \oplus X^{n+1}$$

and the chain map

$$\begin{bmatrix} f \\ i_X \end{bmatrix} : X \to A \oplus IX$$

has the chain map

$$\begin{bmatrix} s^n & -d_X^{n-1} h^n & -h^{n+1} \end{bmatrix} : A^n \oplus X^n \oplus X^{n+1} \to X^n$$

as a left inverse. Hence, on replacing the acyclic complex A by the acyclic complex $A \oplus IX$, we may assume that $f : X \to A$ has s as a left inverse in $\mathbf{Ch}(\mathscr{A})$. But then the morphism $e = fs : A \to A$ is an idempotent in $\mathbf{Ch}(\mathscr{A})$ and it induces an idempotent on the exact sequences $Z^n A \rightarrowtail A^n \twoheadrightarrow Z^{n+1} A$ witnessing that A is acyclic as in the first diagram of the proof of Lemma 10.3. This means that $Z^n A \rightarrowtail A^n \twoheadrightarrow Z^{n+1} A$ decomposes as a direct sum of two short exact sequences (Corollary 2.18) since $\mathscr{A}$ is idempotent complete. Therefore the acyclic complex $A = X' \oplus Y'$ is a direct sum of the acyclic complexes X' and Y', and f induces an isomorphism from X to X' in $\mathbf{Ch}(\mathscr{A})$. The details are left to the reader. $\square$

Exercise 10.8. Prove that the sequence $X \to \operatorname{cone}(X) \to \Sigma X$ from Remark 9.3 is isomorphic to a sequence $X \to IX \to \Sigma X$ in $\mathbf{Ch}(\mathscr{A})$.

Proposition 10.9 (*Keller [39, 11.2]*). *The following are equivalent*:

(i) *Every null-homotopic complex in* $\mathbf{Ch}(\mathscr{A})$ *is acyclic.*
(ii) *The category* $\mathscr{A}$ *is idempotent complete.*
(iii) *The class of acyclic complexes is closed under isomorphisms in* $\mathbf{K}(\mathscr{A})$.

Proof. (*Keller*). Let us prove that (i) implies (ii). Let $e : A \to A$ be an idempotent of $\mathscr{A}$. Consider the complex

$$\cdots \xrightarrow{1-e} A \xrightarrow{e} A \xrightarrow{1-e} A \xrightarrow{e} \cdots$$

which is null-homotopic. By (i) this complex is acyclic. This means by definition that e has a kernel and hence $\mathscr{A}$ is idempotent complete.

Let us prove that (ii) implies (iii). Assume that X is isomorphic in $\mathbf{K}(\mathscr{A})$ to an acyclic complex A. Using the construction in the proof of Lemma 10.7 one shows that X is a direct summand in $\mathbf{Ch}(\mathscr{A})$ of the acyclic complex $A \oplus IX$ and we conclude by Lemma 10.7.

That (iii) implies (i) follows from the fact that a null-homotopic complex X is isomorphic in $\mathbf{K}(\mathscr{A})$ to the (acyclic) zero complex and hence X is acyclic. $\square$

Remark 10.10. Recall that a subcategory $\mathscr{T}$ of a triangulated category $\mathbf{K}$ is called *thick* if it is strictly full and $X \oplus Y \in \mathscr{T}$ implies $X, Y \in \mathscr{T}$.

Corollary 10.11. *The triangulated subcategory* $\mathbf{Ac}(\mathscr{A})$ *of* $\mathbf{K}(\mathscr{A})$ *is thick if and only if* $\mathscr{A}$ *is idempotent complete*.

10.2. Boundedness conditions

A complex A is called *left bounded* if $A^n = 0$ for $n \ll 0$, *right bounded* if $A^n = 0$ for $n \gg 0$ and *bounded* if $A^n = 0$ for $|n| \gg 0$.

Definition 10.12. Denote by $\mathbf{K}^+(\mathscr{A})$, $\mathbf{K}^-(\mathscr{A})$ and $\mathbf{K}^b(\mathscr{A})$ the full subcategories of $\mathbf{K}(\mathscr{A})$ generated by the left bounded complexes, right bounded complexes and bounded complexes over $\mathscr{A}$.

Observe that $\mathbf{K}^b(\mathscr{A}) = \mathbf{K}^+(\mathscr{A}) \cap \mathbf{K}^-(\mathscr{A})$. Note further that $\mathbf{K}^*(\mathscr{A})$ is *not* closed under isomorphisms in $\mathbf{K}(\mathscr{A})$ for $* \in \{+, -, b\}$ unless $\mathscr{A} = 0$.

Definition 10.13. For $* \in \{+, -, b\}$ we define $\mathbf{Ac}^*(\mathscr{A}) = \mathbf{K}^*(\mathscr{A}) \cap \mathbf{Ac}(\mathscr{A})$.

Plainly, $\mathbf{K}^*(\mathscr{A})$ is a full triangulated subcategory of $\mathbf{K}(\mathscr{A})$ and $\mathbf{Ac}^*(\mathscr{A})$ is a full triangulated subcategory of $\mathbf{K}^*(\mathscr{A})$ by Lemma 10.3.

Proposition 10.14. *The following assertions are equivalent*:

(i) *The subcategories* $\mathbf{Ac}^+(\mathscr{A})$ *and* $\mathbf{Ac}^-(\mathscr{A})$ *of* $\mathbf{K}^+(\mathscr{A})$ *and* $\mathbf{K}^-(\mathscr{A})$ *are thick.*
(ii) *The subcategory* $\mathbf{Ac}^b(\mathscr{A})$ *of* $\mathbf{K}^b(\mathscr{A})$ *is thick.*
(iii) *The category* $\mathscr{A}$ *is weakly idempotent complete.*

Proof. Since $\mathbf{Ac}^b(\mathscr{A}) = \mathbf{Ac}^+(\mathscr{A}) \cap \mathbf{Ac}^-(\mathscr{A})$, we see that (i) implies (ii). Let us prove that (ii) implies (iii). Let $s : B \to A$ and $t : A \to B$ be morphisms of $\mathscr{A}$ such that $ts = 1_B$.

We need to prove that s has a cokernel and t has a kernel. The complex X given by

$$\cdots \to 0 \to B \xrightarrow{s} A \xrightarrow{1-st} A \xrightarrow{t} B \to 0 \to \cdots$$

is a direct summand of $X \oplus \Sigma X$ and the latter complex is acyclic since there is an isomorphism in $\mathbf{Ch}(\mathscr{A})$

$$\begin{tikzcd}
B \arrow[r, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"] \arrow[d, "1"'] & B \oplus A \arrow[r, "\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]"] \arrow[d, "\left[\begin{smallmatrix}0&-t\\s&1-st\end{smallmatrix}\right]"'] & A \oplus A \arrow[r, "\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]"] \arrow[d, "\left[\begin{smallmatrix}-1+st&st\\st&1-st\end{smallmatrix}\right]"'] & A \oplus B \arrow[r, "\left[\begin{smallmatrix}0&1\end{smallmatrix}\right]"] \arrow[d, "\left[\begin{smallmatrix}1-st&-s\\t&0\end{smallmatrix}\right]"'] & B \arrow[d, "1"'] \\
B \arrow[r, "\left[\begin{smallmatrix}0\\s\end{smallmatrix}\right]"'] & B \oplus A \arrow[r, "\left[\begin{smallmatrix}-s&0\\0&1-st\end{smallmatrix}\right]"'] & A \oplus A \arrow[r, "\left[\begin{smallmatrix}-1+st&0\\0&t\end{smallmatrix}\right]"'] & A \oplus B \arrow[r, "\left[\begin{smallmatrix}-t&0\end{smallmatrix}\right]"'] & B
\end{tikzcd}$$

where the upper row is obviously acyclic and the lower row is $X \oplus \Sigma X$. Since $\mathbf{Ac}^b(\mathscr{A})$ is thick, we conclude that X is acyclic, so that s has a cokernel and t has a kernel. Therefore $\mathscr{A}$ is weakly idempotent complete.

Let us prove that (iii) implies (i). Assume that X is a direct summand in $\mathbf{K}^+(\mathscr{A})$ of a complex $A \in \mathbf{Ac}^+(\mathscr{A})$. This means that we are given a chain map $f : X \to A$ for which there exists a chain map $s : A \to X$ and morphisms $h^n : X^n \to X^{n-1}$ such that $s^n f^n - 1_{X^n} = d_X^{n-1} h^n + h^{n+1} d_X^n$. On replacing A by the acyclic complex $A \oplus IX$ as in the proof of Lemma 10.7, we may assume that s is a left inverse of f in $\mathbf{Ch}^+(\mathscr{A})$. In particular, since $\mathscr{A}$ is assumed to be weakly idempotent complete, Proposition 7.6 implies that each f^n is an admissible monic and that each s^n is an admissible epic. Moreover, as both complexes X and A are left bounded, we may assume that $A^n = 0 = X^n$ for $n < 0$. It follows that $d_A^0 : A^0 \rightarrowtail A^1$ is an admissible monic since A is acyclic. But then $d_A^0 f^0 = f^1 d_X^0$ is an admissible monic, hence Proposition 7.6 implies that d_X^0 is an admissible monic as well. Let $e_X^1 : X^1 \twoheadrightarrow Z^2 X$ be a cokernel of d_X^0 and let $e_A^1 : A^1 \twoheadrightarrow Z^2 A$ be a cokernel of d_A^0. The dotted morphisms in the diagram

$$\begin{tikzcd}
X^0 \arrow[r, tail, "d_X^0"] \arrow[d, tail, "f^0"] & X^1 \arrow[r, two heads, "e_X^1"] \arrow[d, tail, "f^1"] & Z^2 X \arrow[d, tail, dotted, "g^2"] \\
A^0 \arrow[r, tail, "d_A^0"] \arrow[d, two heads, "s^0"] & A^1 \arrow[r, two heads, "e_A^1"] \arrow[d, two heads, "s^1"] & Z^2 A \arrow[d, two heads, dotted, "t^2"] \\
X^0 \arrow[r, tail, "d_X^0"] & X^1 \arrow[r, two heads, "e_X^1"] & Z^2 X
\end{tikzcd}$$

are uniquely determined by requiring the resulting diagram to be commutative. Since $s^0 f^0 = 1_{X^0}$ and $s^1 f^1 = 1_{X^1}$ it follows that $t^2 g^2 = 1_{Z^2 X}$, so t^2 is an admissible epic and g^2 is an admissible monic by Proposition 7.6.

Now since A and X are complexes, there are unique maps $m_X^2 : Z^2 X \to X^2$ and $m_A^2 : Z^2 A \to A^2$ such that $d_X^1 = m_X^2 e_X^1$ and $d_A^1 = m_A^2 e_A^1$. Note that m_A^2 is an admissible monic since A is acyclic. The upper square in the diagram

$$\begin{tikzcd} Z^2X \arrow[r, "m_X^2"] \arrow[d, tail, "g^2"] & X^2 \arrow[d, tail, "f^2"] \\ Z^2A \arrow[r, tail, "m_A^2"] \arrow[d, two heads, "t^2"] & A^2 \arrow[d, two heads, "s^2"] \\ Z^2X \arrow[r, "m_X^2"] & X^2 \end{tikzcd}$$

is commutative because e_X^1 is epic and the lower square is commutative because e_A^1 is epic. From the commutativity of the upper square it follows in particular that m_X^2 is an admissible monic by Proposition 7.6. An easy induction now shows that X is acyclic. The assertion about $\mathbf{Ac}^-(\mathscr{A})$ follows by duality. □

Remark 10.15. The isomorphism of complexes in the proof that (ii) implies (iii) appears in Neeman [48, 1.9].

10.3. Quasi-isomorphisms

In abelian categories, quasi-isomorphisms are defined to be chain maps inducing an isomorphism in homology. Taking the observation in Exercise 9.4 and Proposition 10.9 into account, one arrives at the following generalization for general exact categories:

Definition 10.16. A chain map $f : A \to B$ is called a *quasi-isomorphism* if its mapping cone is homotopy equivalent to an acyclic complex.

Remark 10.17. Assume that $\mathscr{A}$ is idempotent complete. By Proposition 10.9, a chain map f is a quasi-isomorphism if and only if cone(f) is acyclic. In particular, for abelian categories, a quasi-isomorphism is the same thing as a chain map inducing an isomorphism on homology.

Remark 10.18. If $p : A \to A$ is an idempotent in $\mathscr{A}$ which does not split, then the complex C given by

$$\cdots \xrightarrow{1-p} A \xrightarrow{p} A \xrightarrow{1-p} A \xrightarrow{p} \cdots$$

is null-homotopic but *not* acyclic. However, $f : 0 \to C$ is a chain homotopy equivalence, hence it should be a quasi-isomorphism, but cone$(f) = C$ fails to be acyclic.

10.4. The definition of the derived category

The *derived category* of the exact category $\mathscr{A}$ is defined to be the *Verdier quotient*

$$\mathbf{D}(\mathscr{A}) = \mathbf{K}(\mathscr{A})/\mathbf{Ac}(\mathscr{A})$$

as described e.g. in Neeman [49, Chapter 2] or Keller [39, §§10, 11]. For the description of derived functors given in Section 10.6 it is useful to recall that the Verdier quotient can be explicitly described by a calculus of fractions. A morphism $A \to B$ in $\mathbf{D}(\mathscr{A})$ can be represented by a *fraction* (f, s)

$$A \xrightarrow{f} B' \xleftarrow{s} B$$

where $f : A \to B$ is a morphism in $\mathbf{K}(\mathscr{A})$ and $s : B \to B'$ is a quasi-isomorphism in $\mathbf{K}(\mathscr{A})$. Two fractions (f, s) and (g, t) are equivalent if there exists a fraction (h, u) and a commutative diagram

$$\begin{tikzcd} & B' \arrow[d] & \\ A \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & B''' & B \arrow[ul, "s"'] \arrow[l, "u"'] \arrow[dl, "t"] \\ & B'' \arrow[u] & \end{tikzcd}$$

or, in words, if the fractions (f, s) and (g, t) have a common expansion (h, u). We refer to Keller [39, §§9, 10] for further details.

When dealing with the boundedness condition $* \in \{+, -, b\}$ we define

$$\mathbf{D}^*(\mathscr{A}) = \mathbf{K}^*(\mathscr{A})/\mathbf{Ac}^*(\mathscr{A})$$

It is not difficult to prove that the canonical functor $\mathbf{D}^*(\mathscr{A}) \to \mathbf{D}(\mathscr{A})$ is an equivalence between $\mathbf{D}^*(\mathscr{A})$ and the full subcategory of $\mathbf{D}(\mathscr{A})$ generated by the complexes satisfying the boundedness condition $*$, see Keller [39, 11.7].

Remark 10.19. If $\mathscr{A}$ is idempotent complete then a chain map becomes an isomorphism in $\mathbf{D}(\mathscr{A})$ if and only if its cone is acyclic by Corollary 10.11. If $\mathscr{A}$ is weakly idempotent complete then a chain map in $\mathbf{Ch}^*(\mathscr{A})$ becomes an isomorphism in $\mathbf{D}^*(\mathscr{A})$ if and only if its cone is acyclic by Proposition 10.14.

10.5. Derived categories of fully exact subcategories

The proof of the following lemma is straightforward and left to the reader as an exercise. That admissible monics and epics are closed under composition follows from the Noether isomorphism 3.5.

Lemma 10.20. *Let $\mathscr{A}$ be an exact category and suppose that $\mathscr{B}$ is a full additive subcategory of $\mathscr{A}$ which is closed under extensions in the sense that the existence of a short exact sequence $B' \rightarrowtail A \twoheadrightarrow B''$ with $B', B'' \in \mathscr{B}$ implies that A is isomorphic to an object of $\mathscr{B}$. The sequences in $\mathscr{B}$ which are exact in $\mathscr{A}$ form an exact structure on $\mathscr{B}$.*

Definition 10.21. A *fully exact subcategory* $\mathscr{B}$ of an exact category $\mathscr{A}$ is a full additive subcategory which is closed under extensions and equipped with the exact structure from the previous lemma.

Theorem 10.22 (Keller [39, 12.1]). *Let $\mathscr{B}$ be a fully exact subcategory of $\mathscr{A}$ and consider the functor $\mathbf{D}^+(\mathscr{B}) \to \mathbf{D}^+(\mathscr{A})$ induced by the inclusion $\mathscr{B} \subset \mathscr{A}$.*

(i) *Assume that for every object $A \in \mathscr{A}$ there exists an admissible monic $A \rightarrowtail B$ with $B \in \mathscr{B}$. For every left bounded complex A over $\mathscr{A}$ there exists a left bounded complex B over*

$\mathscr{B}$ and a quasi-isomorphism $A \to B$. In particular $\mathbf{D}^+(\mathscr{B}) \to \mathbf{D}^+(\mathscr{A})$ is essentially surjective.

(ii) *Assume that for every short exact sequence $B' \rightarrowtail A \to A''$* of $\mathscr{A}$ *with $B' \in \mathscr{B}$ there exists a commutative diagram with exact rows*

$$\begin{tikzcd} B' \arrow[r, tail] \arrow[d, equal] & A \arrow[r] \arrow[d] & A'' \arrow[d] \\ B' \arrow[r, tail] & B \arrow[r, two heads] & B''. \end{tikzcd}$$

For every quasi-isomorphism $s : B \to A$ in $\mathbf{K}^+(\mathscr{A})$ with B a complex over $\mathscr{B}$ there exists a morphism $t : A \to B'$ in $\mathbf{K}^+(\mathscr{A})$ such that $ts : B \to B'$ is a quasi-isomorphism. In particular, $\mathbf{D}^+(\mathscr{B}) \to \mathbf{D}^+(\mathscr{A})$ is fully faithful.

Remark 10.23. The condition in (ii) holds if condition (i) holds and, moreover, if for all short exact sequences $B' \rightarrowtail B \twoheadrightarrow A''$ with $B', B \in \mathscr{B}$ it follows that A'' is isomorphic to an object in $\mathscr{B}$. To see this, start with a short exact sequence $B' \rightarrowtail A \twoheadrightarrow A''$, then choose an admissible monic $A \rightarrowtail B$, form the push-out $AA''BB''$ and apply Proposition 2.12 and Proposition 2.15.

Example 10.24. Let $\mathscr{I}$ be the full subcategory spanned by the *injective* objects of the exact category $(\mathscr{A}, \mathscr{E})$, see Definition 11.1. Clearly, $\mathscr{I}$ is a fully exact subcategory of $\mathscr{A}$ (the induced exact structure consists of the split exact sequences) and it satisfies condition (ii) of Theorem 10.22. If $\mathscr{I}$ satisfies condition (i) then there are *enough injectives* in $(\mathscr{A}, \mathscr{E})$, see Definition 11.9. A quasi-isomorphism of left bounded complexes of injectives is a chain homotopy equivalence, hence $\mathbf{K}^+(\mathscr{I})$ is equivalent to $\mathbf{D}^+(\mathscr{I})$. By Theorem 10.22 $\mathbf{K}^+(\mathscr{I})$ is equivalent to the full subcategory of $\mathbf{D}(\mathscr{A})$ spanned by the left bounded complexes with injective components. Moreover, if $(\mathscr{A}, \mathscr{E})$ has enough injectives, then the functor $\mathbf{K}^+(\mathscr{I}) \to \mathbf{D}^+(\mathscr{A})$ is an equivalence of triangulated categories.

10.6. Total derived functors

With these constructions at hand one can now introduce (total) derived functors in the sense of Grothendieck–Verdier and Deligne, see e.g. Keller [39, §§13–15] or any one of the references given in Remark 9.8. We follow Keller's exposition of the Deligne approach.

The problem is the following: An additive functor $F : \mathscr{A} \to \mathscr{B}$ from an exact category to another induces functors $\mathbf{Ch}(\mathscr{A}) \to \mathbf{Ch}(\mathscr{B})$ and $\mathbf{K}(\mathscr{A}) \to \mathbf{K}(\mathscr{B})$ in an obvious way. By abuse of notation we still denote these functors by F. The next question to ask is whether the functor descends to a functor of derived categories, i.e., we look for a commutative diagram

$$\begin{tikzcd} \mathbf{K}(\mathscr{A}) \arrow[r, "F"] \arrow[d, "Q_{\mathscr{A}}"'] & \mathbf{K}(\mathscr{B}) \arrow[d, "Q_{\mathscr{B}}"] \\ \mathbf{D}(\mathscr{A}) \arrow[r, dotted, "\exists ?"] & \mathbf{D}(\mathscr{B}). \end{tikzcd}$$

If the functor $F : \mathscr{A} \to \mathscr{B}$ is exact, this problem has a solution by the universal property of the derived category since then $F(\mathbf{Ac}(\mathscr{A})) \subset \mathbf{Ac}(\mathscr{B})$.

However, if F fails to be exact, it will not send acyclic complexes to acyclic complexes, or, in other words, it will not send quasi-isomorphisms to quasi-isomorphisms and our naïve question will have a negative answer. Deligne's solution consists in constructing for each $A \in \mathbf{D}(\mathscr{A})$ a functor

$$\mathbf{r}F(-, A) : (\mathbf{D}(\mathscr{B}))^{\mathrm{op}} \to \mathbf{Ab}$$

If the functor $\mathbf{r}F(-, A)$ is representable, a representing object will be denoted by $\mathbf{R}F(A)$ and $\mathbf{R}F$ is said to be *defined at* A. To be a little more specific, for $B \in \mathbf{D}(\mathscr{B})$ we define the abelian group $\mathbf{r}F(B, A)$ by the equivalence classes of diagrams

$$B \xrightarrow{f} F(A') \qquad A' \xleftarrow{s} A$$

where $f : B \to F(A')$ is a morphism of $\mathbf{D}(\mathscr{B})$ and $s : A \to A'$ is a quasi-isomorphism in $\mathbf{K}(\mathscr{A})$. Observe the analogy to the description of morphisms in $\mathbf{D}(\mathscr{A})$; it is useful to think of the diagram as "F-fractions". Accordingly, two F-fractions (f, s) and (g, t) are said to be *equivalent* if there exist commutative diagrams

$$\begin{tikzcd} & F(A') \arrow[d, "F(v)"] \\ B \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & F(A''') \\ & F(A'') \arrow[u, "F(w)"'] \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[d, "v"'] & \\ A''' & A \arrow[ul, "s"'] \arrow[l, "u"'] \arrow[dl, "t"] \\ A'' \arrow[u, "w"] & \end{tikzcd}$$

in $\mathbf{D}(\mathscr{B})$ and $\mathbf{K}(\mathscr{A})$, where (h, u) is another F-fraction. On morphisms of $\mathbf{D}(\mathscr{B})$ define $\mathbf{r}F(-, A)$ by pre-composition. By defining $\mathbf{r}F$ on morphisms of $\mathbf{D}(\mathscr{A})$ one obtains a *functor* from $\mathbf{D}(\mathscr{A})$ to the category of functors $(\mathbf{D}(\mathscr{B}))^{\mathrm{op}} \to \mathbf{Ab}$.

Let $\mathscr{T} \subset \mathbf{D}(\mathscr{A})$ be the full subcategory of objects at which $\mathbf{R}F$ is defined and choose for each $A \in \mathscr{T}$ a representing object $\mathbf{R}F(A)$ and an isomorphism

$$\mathrm{Hom}_{\mathbf{D}(\mathscr{B})}(-, \mathbf{R}F(A)) \xrightarrow{\sim} \mathbf{r}F(-, A)$$

These choices force the definition of $\mathbf{R}F$ on morphisms and thus $\mathbf{R}F : \mathscr{T} \to \mathbf{D}(\mathscr{B})$ is a functor. Even more is true:

Theorem 10.25 (*Deligne*). *Let $F : \mathbf{K}(\mathscr{A}) \to \mathbf{K}(\mathscr{B})$ be a functor and let $\mathscr{T}$ be the full subcategory of $\mathbf{D}(\mathscr{A})$ at which $\mathbf{R}F$ is defined. Let $\mathscr{S}$ be the full subcategory of $\mathbf{K}(\mathscr{A})$ spanned by the objects of $\mathscr{T}$. Denote by $I : \mathscr{S} \to \mathbf{K}(\mathscr{A})$ the inclusion.*

(i) *The category $\mathscr{T}$ is a triangulated subcategory of $\mathbf{D}(\mathscr{A})$ and $\mathscr{S}$ is a triangulated subcategory of $\mathbf{K}(\mathscr{A})$.*

(ii) *The functor* $\mathbf{R}F : \mathscr{T} \to \mathbf{D}(\mathscr{B})$ *is a triangle functor and there is a morphism of triangle functors* $Q_{\mathscr{B}}FI \Rightarrow \mathbf{R}FQ_{\mathscr{A}}I$.

(iii) *For every morphism* $\mu : F \Rightarrow F'$ *of triangle functors* $\mathbf{K}(\mathscr{A}) \to \mathbf{K}(\mathscr{B})$ *there is an induced morphism of triangle functors* $\mathbf{R}\mu : \mathbf{R}F \Rightarrow \mathbf{R}F'$ *on the intersection of the domains of* $\mathbf{R}F$ *and* $\mathbf{R}F'$.

The only subtle part of the previous theorem is the fact that $\mathscr{T}$ is triangulated. The rest is a straightforward but rather tedious verification. The essential details and references are given in Keller [39, §13].

The next question that arises is whether one can get some information on $\mathscr{T}$. A complex A is said to be F-*split* if $\mathbf{R}F$ is defined at A and the canonical morphism $F(A) \to \mathbf{R}F(A)$ is invertible. An object A of $\mathscr{A}$ is said to be F-*acyclic* if it is F-split when considered as complex concentrated in degree zero.

Lemma 10.26 (*Keller* [39, 15.1, 15.3]). *Let* $\mathscr{C}$ *be a fully exact subcategory of* $\mathscr{A}$ *satisfying hypothesis* (ii) *of Theorem* 10.22. *Assume that the restriction of* $F : \mathscr{A} \to \mathscr{B}$ *to* $\mathscr{C}$ *is exact. Then each object of* $\mathscr{C}$ *is* F-*acyclic*.

Conversely, let $\mathscr{C}$ *be the full subcategory of* $\mathscr{A}$ *consisting of the* F-*acyclic objects. Then* $\mathscr{C}$ *is a fully exact subcategory of* $\mathscr{A}$, *it satisfies condition* (ii) *of Theorem* 10.22 *and the restriction of* F *to* $\mathscr{C}$ *is exact.*

Now let $\mathscr{C}$ be a fully exact subcategory of $\mathscr{A}$ consisting of F-acyclic objects and suppose that $\mathscr{C}$ satisfies conditions (i) and (ii) of Theorem 10.22. By these assumptions, the inclusion $\mathscr{C} \to \mathscr{A}$ induces an equivalence $\mathbf{D}^+(\mathscr{C}) \to \mathbf{D}^+(\mathscr{A})$. As the restriction of F to $\mathscr{C}$ is exact, it yields a triangle functor $F : \mathbf{D}^+(\mathscr{C}) \to \mathbf{D}^+(\mathscr{B})$. To choose a quasi-inverse for the canonical functor $\mathbf{D}^+(\mathscr{C}) \to \mathbf{D}^+(\mathscr{A})$ amounts to choosing for each complex $A \in \mathbf{K}^+(\mathscr{A})$ a quasi-isomorphism $s : A \to C$ with $C \in \mathbf{K}^+(\mathscr{C})$ by [41, 1.6], a proof of which is given in [38, 6.7]. As we have just seen, the complex C is F-split, hence s yields an isomorphism $\mathbf{R}F(A) \to F(C) \cong \mathbf{R}F(C)$. Such a quasi-isomorphism $A \to C$ exists by the construction in the proof of Theorem 12.7 and our assumptions.

The admittedly concise *résumé* given here provides the basic toolkit for treating derived functors. We refer to Keller [39, §§13–15] for a much more thorough and general discussion and precise statements of the composition formula $\mathbf{R}F \circ \mathbf{R}G \cong \mathbf{R}(FG)$ and adjunction formulæ of left and right derived functors of adjoint pairs of functors.

11. Projective and injective objects

Definition 11.1. An object P of an exact category $\mathscr{A}$ is called *projective* if the represented functor $\mathrm{Hom}_{\mathscr{A}}(P, -) : \mathscr{A} \to \mathbf{Ab}$ is exact. An object I of an exact category is called *injective* if the corepresented functor $\mathrm{Hom}_{\mathscr{A}}(-, I) : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ is exact.

Remark 11.2. The concepts of projectivity and injectivity are dual to each other in the sense that P is projective in $\mathscr{A}$ if and only if P is injective in $\mathscr{A}^{\mathrm{op}}$. For our purposes it is therefore sufficient to deal with projective objects.

Proposition 11.3. *An object P of an exact category is projective if and only if any one of the following conditions holds*:

(i) *For all admissible epics $A \twoheadrightarrow A''$ and all morphisms $P \to A''$ there exists a solution to the lifting problem*

$$\begin{tikzcd} & P \arrow[dl, dotted, "\exists"'] \arrow[dr] & \\ A \arrow[rr, two heads] & & A'' \end{tikzcd}$$

making the diagram above commutative.

(ii) *The functor* $\mathrm{Hom}_{\mathscr{A}}(P, -) : \mathscr{A} \to \mathbf{Ab}$ *sends admissible epics to surjections.*

(iii) *Every admissible epic $A \twoheadrightarrow P$ splits* (*has a right inverse*).

Proof. Since $\mathrm{Hom}_{\mathscr{A}}(P, -)$ transforms exact sequences to left exact sequences in **Ab** for all objects P (see the proof of Corollary A.8), it is clear that conditions (i) and (ii) are equivalent to the projectivity of P. If P is projective and $A \twoheadrightarrow P$ is an admissible epic then $\mathrm{Hom}_{\mathscr{A}}(P, A) \twoheadrightarrow \mathrm{Hom}_{\mathscr{A}}(P, P)$ is surjective, and every pre-image of 1_P is a splitting map of $A \twoheadrightarrow P$. Conversely, let us prove that condition (iii) implies condition (i): given a lifting problem as in (i), form the following pull-back diagram

$$\begin{tikzcd} D \arrow[r, two heads, "a'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\text{PB}"] & P \arrow[d, "f"] \\ A \arrow[r, two heads, "a"] & A''. \end{tikzcd}$$

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By hypothesis, there exists a right inverse b' of a' and $f'b'$ solves the lifting problem because $af'b' = fa'b' = f$. $\square$

Corollary 11.4. *If P is projective and $P \to A$ has a right inverse then A is projective.*

Proof. This is a trivial consequence of condition (i) in Proposition 11.3. $\square$

Remark 11.5. If $\mathscr{A}$ is weakly idempotent complete, the above corollary amounts to the familiar "direct summands of projective objects are projective" in abelian categories.

Corollary 11.6. *A sum $P = P' \oplus P''$ is projective if and only if both P' and P'' are projective.*

More generally:

Corollary 11.7. *Let $\{P_i\}_{i \in I}$ be a family of objects for which the coproduct $P = \coprod_{i \in I} P_i$ exists in $\mathscr{A}$. The object P is projective if and only if each P_i is projective.*

Remark 11.8. The dual of the previous result is that a product (if it exists) is injective if and only if each of its factors is injective.

Definition 11.9. An exact category $\mathscr{A}$ is said to have *enough projectives* if for every object $A \in \mathscr{A}$ there exists a projective object P and an admissible epic $P \twoheadrightarrow A$.

Exercise 11.10 (*Heller [27, 5.6]*). Assume that $\mathscr{A}$ has enough projectives. Prove that a sequence $A' \to A \to A''$ is short exact if and only if

$$\mathrm{Hom}_{\mathscr{A}}(P, A') \rightarrowtail \mathrm{Hom}_{\mathscr{A}}(P, A) \twoheadrightarrow \mathrm{Hom}_{\mathscr{A}}(P, A'')$$

is short exact for all projective objects P.

Hint: For sufficiency prove first that $A' \to A$ is a monomorphism, then prove that it is a kernel of $A \to A''$ and finally apply the obscure axiom 2.16. In all three steps use that there are enough projectives.

Exercise 11.11 (*Heller [27, 5.6]*). Assume that $\mathscr{A}$ is weakly idempotent complete and has enough projectives. Prove that the sequence

$$A_n \to A_{n-1} \to \cdots \to A_1 \to A_0 \to 0$$

is an exact sequence of admissible morphisms if and only if for all projectives P the sequence

$$\mathrm{Hom}(P, A_n) \to \mathrm{Hom}(P, A_{n-1}) \to \cdots \to \mathrm{Hom}(P, A_1) \to \mathrm{Hom}(P, A_0) \to 0$$

is an exact sequence of abelian groups.

12. Resolutions and classical derived functors

Definition 12.1. A *projective resolution* of the object A is a positive complex $P_\bullet$ with projective components together with a morphism $P_0 \to A$ such that the *augmented complex*

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to A$$

is exact.

Proposition 12.2 (*Resolution lemma*). *If $\mathscr{A}$ has enough projectives then every object A in $\mathscr{A}$ has a projective resolution.*

Proof. This is an easy induction. Because $\mathscr{A}$ has enough projectives, there exists a projective object P_0 and an admissible epic $P_0 \twoheadrightarrow A$. Choose an admissible monic $A_0 \rightarrowtail P_0$ such that $A_0 \rightarrowtail P_0 \twoheadrightarrow A$ is exact. Now choose a projective P_1 and an admissible epic $P_1 \twoheadrightarrow A_0$. Continue with an admissible monic $A_1 \rightarrowtail P_1$ such that $A_1 \rightarrowtail P_1 \twoheadrightarrow A_0$ is exact, and so on. One thus obtains a sequence

$$\begin{tikzcd}
 & A_1 \arrow[dr, tail] & & & & \\
P_2 \arrow[ur, two heads] \arrow[rr] & & P_1 \arrow[rr] \arrow[dr, two heads] & & P_0 \arrow[r, two heads] & A \\
 & & & A_0 \arrow[ur, tail] & &
\end{tikzcd}$$

which is exact by construction, so $P_\bullet \to A$ is a projective resolution. $\square$

Remark 12.3. The defining concept of projectivity is not used in the previous proof. That is, we have proved: If $\mathscr{P}$ is a class in $\mathscr{A}$ such that for each object $A \in \mathscr{A}$ there is an admissible epic $P \twoheadrightarrow A$ with $P \in \mathscr{P}$ then each object of $\mathscr{A}$ has a $\mathscr{P}$-resolution $P_\bullet \twoheadrightarrow A$.

Consider a morphism $f : A \to B$ in $\mathscr{A}$. Let $P_\bullet$ be a complex of projectives with $P_n = 0$ for $n < 0$ and let $\alpha : P_0 \to A$ be a morphism such that the composition $P_1 \to P_0 \to A$ is zero (e.g. $P_\bullet \to A$ is a projective resolution of A). Let $Q_\bullet \xrightarrow{\beta} B$ be a resolution (not necessarily projective).

Theorem 12.4 (*Comparison theorem*). *Under the above hypotheses there exists a chain map $f_\bullet : P_\bullet \to Q_\bullet$ such that the following diagram commutes*:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "\exists f_2"'] & P_1 \arrow[r] \arrow[d, "\exists f_1"'] & P_0 \arrow[r, "\alpha"] \arrow[d, "\exists f_0"'] & A \arrow[d, "f"'] \\
\cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, two heads, "\beta"] & B .
\end{tikzcd}$$

Moreover, *the lift $f_\bullet$ of f is unique up to homotopy equivalence*.

Proof. It is convenient to put $P_{-1} = A$, $Q'_0 = Q_{-1} = B$ and $f_{-1} = f$.

Existence: The question of existence of f_0 is the lifting problem given by the morphism $f\alpha : P_0 \to B$ and the admissible epic $\beta : Q_0 \twoheadrightarrow B$. This problem has a solution by projectivity of P_0.

Let $n \geq 0$ and suppose by induction that there are morphisms $f_n : P_n \to Q_n$ and $f_{n-1} : P_{n-1} \to Q_{n-1}$ such that $df_n = f_{n-1}d$. Consider the following diagram:

$$\begin{tikzcd}
P_{n+1} \arrow[rr, "d"] \arrow[dr, dotted, "\exists! f'_{n+1}"] \arrow[dd, dotted, "\exists f_{n+1}"'] & & P_n \arrow[rr, "d"] \arrow[dd, "f_n"] & & P_{n-1} \arrow[dd, "f_{n-1}"] \\
 & Q'_{n+1} \arrow[dr, tail] & & & \\
Q_{n+1} \arrow[ur, two heads] \arrow[rr, "d"] & & Q_n \arrow[rr, "d"] \arrow[dr, two heads] & & Q_{n-1} \\
 & & & Q'_n \arrow[ur, tail] &
\end{tikzcd}$$

By induction the right-hand square is commutative, so the morphism $P_{n+1} \to Q_{n-1}$ is zero because the morphism $P_{n+1} \to P_{n-1}$ is zero. The morphism $P_{n+1} \to Q'_n$ is zero as well because $Q'_n \rightarrowtail Q_{n-1}$ is monic. Since $Q'_{n+1} \rightarrowtail Q_n \twoheadrightarrow Q'_n$ is exact, there exists a unique morphism $f'_{n+1} : P_{n+1} \to Q'_{n+1}$ making the upper right triangle in the left-hand square commute. Because P_{n+1} is projective and $Q_{n+1} \twoheadrightarrow Q'_{n+1}$ is an admissible epic, there is a morphism $f_{n+1} : P_{n+1} \to Q_{n+1}$ such that the left-hand square commutes. This settles the existence of $f_\bullet$.

Uniqueness: Let $g_\bullet : P_\bullet \to Q_\bullet$ be another lift of f and put $h_\bullet = f_\bullet - g_\bullet$. We will construct by induction a chain contraction $s_n : P_{n-1} \to Q_n$ for h. For $n \leqslant 0$ we put $s_n = 0$. For $n \geq 0$ assume by induction that there are morphisms s_{n-1}, s_n such that $h_{n-1} = ds_n + s_{n-1}d$. Because h is a chain map, we have $d(h_n - s_n d) = h_{n-1}d - (h_{n-1} - s_{n-1}d)d = 0$, so the following diagram commutes:

$$\begin{tikzcd}
 & & P_n \arrow[dl, dotted, "\exists! s'_{n+1}"'] \arrow[dd, "h_n - s_n d"] \arrow[ddrr, "0"] \arrow[ddll, dotted, bend right, "\exists s_{n+1}"'] & & \\
 & Q'_{n+1} \arrow[dr, tail] & & & \\
Q_{n+1} \arrow[ur, two heads] \arrow[rr, "d"] & & Q_n \arrow[rr, "d"] \arrow[dr, two heads] & & Q_{n-1} \\
 & & & Q'_n \arrow[ur, tail] &
\end{tikzcd}$$

and we get a morphism $s_{n+1} : P_n \to Q_{n+1}$ such that $ds_{n+1} = h_n - s_n d$ as in the existence proof. $\square$

Corollary 12.5. *Any two projective resolutions of an object A are chain homotopy equivalent.*

Corollary 12.6. *Let $P_\bullet$ be a right bounded complex of projectives and let $A_\bullet$ be an acyclic complex. Then* $\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P_\bullet, A_\bullet) = 0$.

In order to deal with derived functors on the level of the derived category, one needs to sharpen both the resolution lemma and the comparison theorem.

Theorem 12.7 (*Keller [37, 4.1, Lemma, b)]*). *Let $\mathscr{A}$ be an exact category with enough projectives. For every right bounded complex A over $\mathscr{A}$ exists a right bounded complex with projective components P and a quasi-isomorphism $P \overset{\alpha}{\to} A$.*

Proof. Renumbering if necessary, we may suppose $A_n = 0$ for $n < 0$. The complex P will be constructed by induction. For the inductive formulation it is convenient to define $P_n = B_n = 0$ for $n < 0$. Put $B_0 = A_0$, choose an admissible epic $p'_0 : P_0 \twoheadrightarrow B_0$ from a projective P_0 and define $p''_0 = d_0^A$. Let B_1 be the pull-back over p'_0 and p''_0. Consider the following commutative diagram:

$$\begin{tikzcd}
 & & & & P_0 \arrow[dr, two heads, "p'_0"] & & 0 \arrow[dr, equal] & \\
 & & & B_1 \arrow[ur, "i'_0"] \arrow[dr, "i''_0"'] \arrow[rr, phantom, "\mathrm{PB}"] & & B_0 \arrow[ur] \arrow[dr, equal] \arrow[rr, phantom, "\mathrm{PB}"] & & 0 \\
A_3 \arrow[rr, "d_2^A"'] \arrow[uurrrr, bend left, "0"] & & A_2 \arrow[rr, "d_1^A"'] \arrow[ur, dotted, "\exists! p''_1"'] \arrow[uurr, bend left, "0"] & & A_1 \arrow[ur, "p''_0"'] \arrow[rr, "d_0^A"'] & & A_0 \arrow[ur] &
\end{tikzcd}$$

The morphism p''_1 exists by the universal property of the pull-back and moreover $p''_1 d_2^A = 0$ because $d_1^A d_2^A = 0$.

Suppose by induction that in the following diagram everything is constructed except B_{n+1} and the morphisms terminating or issuing from there. Assume further that P_n is projective and that $p''_n d^A_{n+1} = 0$.

$$\begin{tikzcd}
 & & & P_n \arrow[dr, two heads, "p'_n"] & & P_{n-1} \arrow[dr, two heads, "p'_{n-1}"] & \\
 & & B_{n+1} \arrow[ur, tail, "i'_n"] \arrow[dr, two heads, "i''_n"'] \arrow[rr, phantom, "\mathrm{PB}"] & & B_n \arrow[ur, tail, "i'_{n-1}"] \arrow[dr, two heads, "i''_{n-1}"'] \arrow[rr, phantom, "\mathrm{PB}"] & & B_{n-1} \\
A_{n+3} \arrow[r, "d^A_{n+2}"'] & A_{n+2} \arrow[ur, dotted, "\exists! p''_{n+1}"'] \arrow[rr, "d^A_{n+1}"'] & & A_{n+1} \arrow[ur, "p''_n"'] \arrow[rr, "d^A_n"'] & & A_n \arrow[ur, "p''_{n-1}"'] &
\end{tikzcd}$$

As indicated in the diagram, we obtain B_{n+1} by forming the pull-back over p'_n and p''_n. We complete the induction by choosing an admissible epic $p'_{n+1} : P_{n+1} \twoheadrightarrow B_{n+1}$ from a projective P_{n+1}, constructing p''_{n+1} as in the first paragraph and finally noticing that $p''_{n+1} d^A_{n+2} = 0$.

The projective complex is given by the P_n's and the differential $d^P_{n-1} = i'_{n-1} p'_n$, which satisfies $(d^P)^2 = 0$ by construction.

Let α be given by $\alpha_n = i''_{n-1} p'_n$ in degree n, manifestly a chain map. We claim that α is a quasi-isomorphism. The mapping cone of α is seen to be exact using Proposition 2.12: For each n there is an exact sequence

$$B_{n+1} \xrightarrow{i_n = \begin{bmatrix} -i'_n \\ i''_n \end{bmatrix}} P_n \oplus A_{n+1} \xrightarrow{p_n = \begin{bmatrix} p'_n & p''_n \end{bmatrix}} B_n .$$

We thus obtain an exact complex C with $C_n = P_n \oplus A_{n+1}$ in degree n and differential

$$d^C_{n-1} = i_{n-1} p_n = \begin{bmatrix} -i'_{n-1} p'_n & -i'_{n-1} p''_n \\ i''_{n-1} p'_n & i''_{n-1} p''_n \end{bmatrix} = \begin{bmatrix} -d^P_{n-1} & 0 \\ \alpha_n & d^A_n \end{bmatrix}$$

which shows that $C = \mathrm{cone}(\alpha)$. $\square$

Theorem 12.8 (*Horseshoe lemma*). *A horseshoe can be filled in*: *Suppose we are given a horseshoe diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & P'_2 \arrow[r] & P'_1 \arrow[r] & P'_0 \arrow[r, two heads] & A' \arrow[d, tail] \\
 & & & & A \arrow[d, two heads] \\
\cdots \arrow[r] & P''_2 \arrow[r] & P''_1 \arrow[r] & P''_0 \arrow[r, two heads] & A''
\end{tikzcd}$$

that is to say, *the column is short exact and the horizontal rows are projective resolutions of A' and A''. Then the direct sums $P_n = P'_n \oplus P''_n$ assemble to a projective resolution of A in such a way that the horseshoe can be embedded into a commutative diagram with* *exact rows and columns*

$$\begin{tikzcd}
\cdots \arrow[r] & P_2' \arrow[r] \arrow[d, tail] & P_1' \arrow[r] \arrow[d, tail] & P_0' \arrow[r, two heads] \arrow[d, tail] & A' \arrow[d, tail] \\
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, two heads] & P_1 \arrow[r] \arrow[d, two heads] & P_0 \arrow[r, two heads] \arrow[d, two heads] & A \arrow[d, two heads] \\
\cdots \arrow[r] & P_2'' \arrow[r] & P_1'' \arrow[r] & P_0'' \arrow[r, two heads] & A''.
\end{tikzcd}$$

Remark 12.9. All the columns except the rightmost one are split exact. However, the morphisms $P_{n+1} \to P_n$ are *not* the sums of the morphisms $P'_{n+1} \to P'_n$ and $P''_{n+1} \to P''_n$. This only happens in the trivial case that the sequence $A' \rightarrowtail A \twoheadrightarrow A''$ is already split exact.

Proof. This is an easy application of the five lemma 3.2 and the 3×3-lemma 3.6. By lifting the morphism $\varepsilon'' : P''_0 \to A''$ over the admissible epic $A \twoheadrightarrow A''$ we obtain a morphism $\varepsilon : P_0 \to A$ and a commutative diagram

$$\begin{tikzcd}
\operatorname{Ker}\varepsilon' \arrow[r, tail] \arrow[d, dotted] & P_0' \arrow[r, two heads, "\varepsilon'"] \arrow[d, tail, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] & A' \arrow[d, tail] \\
\operatorname{Ker}\varepsilon \arrow[r, tail, dotted] \arrow[d, dotted] & P_0 \arrow[r, "\varepsilon"] \arrow[d, two heads, "{[0\ 1]}"] & A \arrow[d, two heads] \\
\operatorname{Ker}\varepsilon'' \arrow[r, tail] & P_0'' \arrow[r, two heads, "\varepsilon''"] & A''.
\end{tikzcd}$$

It follows from the five lemma that ε is actually an admissible epic, so its kernel exists. The two vertical dotted morphisms exist since the second and the third columns are short exact. Now the 3×3-lemma implies that the dotted column is short exact. Finally note that $P'_1 \to P'_0$ and $P''_1 \to P''_0$ factor over admissible epics to $\operatorname{Ker}\varepsilon'$ and $\operatorname{Ker}\varepsilon''$ and proceed by induction. □

Remark 12.10. In concrete situations it may be useful to remember that only the projectivity of P''_n is used in the proof.

Remark 12.11 (*Classical derived functors*). Using the results of this section, the theory of classical derived functors, see e.g. Cartan–Eilenberg [14], Mac Lane [44], Hilton–Stammbach [29] or Weibel [60], is easily adapted to the following situation:

Let $(\mathscr{A}, \mathscr{E})$ be an exact category with enough projectives and let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor to an abelian category. By the resolution lemma 12.2 a projective resolution $P_\bullet \twoheadrightarrow A$ exists for every object $A \in \mathscr{A}$ and is well-defined up to homotopy equivalence by the comparison theorem (Corollary 12.5). It follows that for two projective resolutions $P_\bullet \twoheadrightarrow A$ and $Q_\bullet \twoheadrightarrow A$ the complexes $F(P_\bullet)$ and $F(Q_\bullet)$ are chain homotopy equivalent. Therefore it makes sense to define the *left derived functors* of F as

$$L_i F(A) := H_i(F(P_\bullet))$$

Let us indicate why $L_i F(A)$ is a functor. First observe that a morphism $f : A \to A'$ extends uniquely up to chain homotopy equivalence to a chain map $f_\bullet : P_\bullet \to P'_\bullet$ if $P_\bullet \twoheadrightarrow A$ and $P'_\bullet \twoheadrightarrow A'$ are projective resolutions of A and A'. From this uniqueness it follows easily that $L_i F(fg) = L_i F(f) L_i F(g)$ and $L_i F(1_A) = 1_{L_i F(A)}$ as desired. Using the horseshoe lemma 12.8 one proves that a short exact sequence $A' \rightarrowtail A \twoheadrightarrow A''$ yields a long exact sequence

$$\cdots \to L_{i+1} F(A'') \to L_i F(A') \to L_i F(A) \to L_i F(A'') \to L_{i-1} F(A') \to \cdots$$

and that $L_0 F$ sends exact sequences to right exact sequences in $\mathscr{B}$ so that the $L_i F$ are a universal δ-functor. Moreover, $L_0 F$ is characterized by being the best right exact approximation to F and the $L_i F$ measure the failure of $L_0 F$ to be exact. In particular, if F sends exact sequences to right exact sequences then $L_0 F \cong F$ and if F is exact, then in addition $L_i F = 0$ if $i > 0$.

Remark 12.12. By the discussion in Section 10.6, the assumption that $(\mathscr{A}, \mathscr{E})$ has enough projectives is unnecessarily restrictive. In order for the classical left derived functor of $F : \mathscr{A} \to \mathscr{B}$ to exist, it suffices to assume that there is a fully exact subcategory $\mathscr{C} \subset \mathscr{A}$ satisfying the duals of the conditions in Theorem 10.22 with the additional property that F restricted to $\mathscr{C}$ is exact (see Lemma 10.26). These conditions ensure that the total derived functor $\mathbf{L}F : \mathbf{D}^-(\mathscr{A}) \to \mathbf{D}^-(\mathscr{B})$ exists and thus it makes sense to define $L_i F(A) = H_i(\mathbf{L}F(A))$, where the object $A \in \mathscr{A}$ is considered as a complex concentrated in degree zero. More explicitly, choose a $\mathscr{C}$-resolution $C_\bullet \to A$ and let $L_i F(A) := H_i(F(C_\bullet))$. It is not difficult to check that the $L_i F$ are a universal δ-functor: They form a δ-functor as $\mathbf{L}F$ is a triangle functor and $H_* : \mathbf{D}^-(\mathscr{B}) \to \mathscr{B}$ sends distinguished triangles to long exact sequences; this δ-functor is universal because it is effaçable, as $L_i F(C) = 0$ for $i > 0$.

Exercise 12.13 (*Heller [27, 6.3, 6.5]*). Let $(\mathscr{A}, \mathscr{E})$ be an exact category and consider the exact category $(\mathscr{E}, \mathscr{F})$ as described in Exercise 3.9. Prove that an exact sequence $P' \rightarrowtail P \twoheadrightarrow P''$ is projective in $(\mathscr{E}, \mathscr{F})$ if P' and P'' (and hence P) are projective in $(\mathscr{A}, \mathscr{E})$. If $(\mathscr{A}, \mathscr{E})$ has enough projectives then so has $(\mathscr{E}, \mathscr{F})$ and every projective is of the form described before.

13. Examples and applications

It is of course impossible to give an exhaustive list of examples. We simply list some of the popular ones.

13.1. Additive categories

Every additive category $\mathscr{A}$ is exact with respect to the class $\mathscr{E}_{\min}$ of split exact sequences, i.e., the sequences isomorphic to

$$A \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} A \oplus B \xrightarrow{[0\ 1]} B$$

for $A, B \in \mathscr{A}$. Every object $A \in \mathscr{A}$ is both projective and injective with respect to this exact structure.

13.2. Quasi-abelian categories

We have seen in Section 4 that quasi-abelian categories are exact with respect to the class $\mathscr{E}_{\max}$ of all kernel–cokernel pairs. Evidently, this class of examples includes in particular all abelian categories. There is an abundance of non-abelian quasi-abelian categories arising in functional analysis:

Example 13.1 (*Cf. e.g. Bühler [11, IV.2]*). Let **Ban** be the category of Banach spaces and bounded linear maps over the field k of real or complex numbers. It has kernels and cokernels – the cokernel of a morphism $f : A \to B$ is given by $B/\overline{\operatorname{Im} f}$. It is an easy consequence of the open mapping theorem that **Ban** is quasi-abelian. Notice that the forgetful functor **Ban** $\to$ **Ab** is exact and reflects exactness, it preserves monics but fails to preserve epics (morphisms with dense range). The ground field k is projective and by Hahn–Banach it also is injective. More generally, it is easy to see that for each set S the space $\ell^1(S)$ is projective and $\ell^\infty(S)$ is injective. Since every Banach space A is isometrically isomorphic to a quotient of $\ell^1(B_{\leqslant 1}A)$ and to a subspace of $\ell^\infty(B_{\leqslant 1}A^*)$ there are enough of both, projective and injective objects in **Ban**.

Example 13.2. Let **Fre** be the category of completely metrizable topological vector spaces (Fréchet spaces) and continuous linear maps. Again, **Fre** is quasi-abelian by the open mapping theorem (the proof of Theorem 2.3.3 in Chapter IV.2 of [11] applies *mutatis mutandis*), and there are exact functors **Ban** $\to$ **Fre** and **Fre** $\to$ **Ab**. It is still true that k is projective, but k fails to be injective (Hahn–Banach breaks down).

Example 13.3. Consider the category **Pol** of polish abelian groups (i.e., second countable and completely metrizable topological groups) and continuous homomorphisms. From the open mapping theorem – which is a standard consequence of Pettis' theorem (cf. e.g. [36, (9.9), p. 61]) stating that for a non-meager set A in G the set $A^{-1}A$ is a neighborhood of the identity – it follows that **Pol** is quasi-abelian (again one easily adapts the proof of Theorem 2.3.3 in Chapter IV.2 of [11]).

Further functional analytic examples are discussed in detail e.g. in Rump [50] and Schneiders [55]. Rump [53] gives a rather long list of examples.

13.3. Fully exact subcategories

Recall from Section 10.5 that a *fully exact subcategory* $\mathscr{B}$ of an exact category $\mathscr{A}$ is a full subcategory $\mathscr{B}$ which is closed under extensions and equipped with the exact structure formed by the sequences which are exact in $\mathscr{A}$ (see Lemma 10.20).

Example 13.4. By the embedding Theorem A.1, every small exact category is a fully exact subcategory of an abelian category.

Example 13.5. The full subcategories of projective or injective objects of an exact category $\mathscr{A}$ are fully exact. The induced exact structures are the split exact structures.

Example 13.6. Let $\widehat{\otimes}$ be the projective tensor product of Banach spaces. A Banach space F is *flat* if $F\,\widehat{\otimes} -$ is exact. It is well-known that the flat Banach spaces are precisely the $\mathscr{L}_1$-spaces of Lindenstrauss–Pełczyński. The category of flat Banach spaces is a fully exact subcategory of **Ban**. The exact structure is the pure exact structure consisting of the short sequences whose Banach dual sequences are split exact, see [11, Chapter IV.2] for further information and references.

13.4. Frobenius categories

An exact category is said to be *Frobenius* provided that it has enough projectives and injectives and, moreover, the classes of projectives and injectives coincide [28]. Frobenius categories $\mathscr{A}$ give rise to *algebraic triangulated categories* (see [40, 3.6]) by passing to the *stable category* $\underline{\mathscr{A}}$ of $\mathscr{A}$. By definition, $\underline{\mathscr{A}}$ is the category consisting of the same objects as $\mathscr{A}$ and in which a morphism of $\mathscr{A}$ is identified with zero if it factors over an injective object. It is not hard to prove that $\underline{\mathscr{A}}$ is additive and it has the structure of a triangulated category as follows:

The translation functor is obtained by choosing for each object A a short exact sequence $A \rightarrowtail I(A) \twoheadrightarrow \Sigma(A)$ where $I(A)$ is injective. The assignment $A \mapsto \Sigma(A)$ induces an auto-equivalence of $\underline{\mathscr{A}}$. Given a morphism $f : A \to B$ in $\mathscr{A}$ consider the push-out diagram

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, "f"'] \arrow[dr, phantom, "\text{PO}"] & I(A) \arrow[r, two heads] \arrow[d] & \Sigma(A) \arrow[d, equal] \\ B \arrow[r, tail] & C(f) \arrow[r, two heads] & \Sigma(A) \end{tikzcd}$$

and call the sequence $A \to B \to C(f) \to \Sigma(A)$ a *standard triangle*. The class Δ of *distinguished triangles* consists of the triangles which are isomorphic in $\underline{\mathscr{A}}$ to (the image of) a standard triangle.

Theorem 13.7 (*Happel [25, 2.6, p.16]*). *The triple* $(\underline{\mathscr{A}}, \Sigma, \Delta)$ *is a triangulated category.*

Example 13.8. Consider the category $\mathbf{Ch}(\mathscr{A})$ of complexes over the additive category $\mathscr{A}$ equipped with the degreewise split exact sequences. It turns out that $\mathbf{Ch}(\mathscr{A})$ is a *Frobenius category*. The complex $I(A)$ introduced in the proof of Lemma 10.7 is injective. It is not hard to verify that the stable category $\underline{\mathbf{Ch}(\mathscr{A})}$ coincides with the homotopy category $\mathbf{K}(\mathscr{A})$ and that the triangulated structure provided by Happel's Theorem 13.7 is the same as the one mentioned in Remark 9.8 (see also Exercise 10.8).

The reader may consult Happel [25] for further information, examples and applications.

13.5. Further examples

Example 13.9 (*Vector bundles*). Let X be a scheme. The category of algebraic vector bundles over X, i.e., the category of locally free and coherent $\mathscr{O}_X$-modules, is an exact category with the exact structure consisting of the locally split short exact sequences.

Example 13.10 (*Chain complexes*). If $(\mathscr{A}, \mathscr{E})$ is an exact category then the category of chain complexes $\mathbf{Ch}(\mathscr{A})$ is an exact category with respect to the exact structure $\mathbf{Ch}(\mathscr{E})$ of short sequences of complexes which are exact in each degree, see Lemma 9.1.

Example 13.11 (*Diagram categories*). Let $(\mathscr{A}, \mathscr{E})$ be an exact category and let $\mathscr{D}$ be a small category. The category $\mathscr{A}^{\mathscr{D}}$ of functors $\mathscr{D} \to \mathscr{A}$ is an exact category with the exact structure $\mathscr{E}^{\mathscr{D}}$. The verification of the axioms of an exact category for $(\mathscr{A}^{\mathscr{D}}, \mathscr{E}^{\mathscr{D}})$ is straightforward, as limits and colimits in $\mathscr{A}^{\mathscr{D}}$ are formed pointwise, see e.g. Borceux [5, 2.15.1, p. 87].

Example 13.12 (*Filtered objects*). Let $(\mathscr{A}, \mathscr{E})$ be an exact category. A (bounded) *filtered object* A in $\mathscr{A}$ is a sequence of admissible monics in $\mathscr{A}$

$$A = (\cdots \rightarrowtail A^n \xrightarrow{i_A^n} A^{n+1} \rightarrowtail \cdots)$$

such that $A^n = 0$ for $n \ll 0$ and that i_A^n is an isomorphism for $n \gg 0$. A *morphism* f from the filtered object A to the filtered object B is a collection of morphisms $f^n : A^n \to B^n$ in $\mathscr{A}$ satisfying $f^{n+1} i_A^n = i_B^n f^n$. Thus there is a category $\mathscr{F}\mathscr{A}$ of filtered objects. It follows from Proposition 2.9 that $\mathscr{F}\mathscr{A}$ is additive. The 3×3-lemma 3.6 implies that the class $\mathscr{F}\mathscr{E}$ consisting of the pairs of morphisms (i, p) of $\mathscr{F}\mathscr{A}$ such that (i^n, p^n) is in $\mathscr{E}$ for each n is an exact structure on $\mathscr{F}\mathscr{A}$. Notice that for a nonzero abelian category $\mathscr{A}$ the category of filtered objects $\mathscr{F}\mathscr{A}$ is *not* abelian.

Example 13.13. Paul Balmer [2] (following Knebusch) gives the following definition: An *exact category with duality* is a triple $(\mathscr{A}, *, \varpi)$ consisting of an exact category $\mathscr{A}$, a contravariant and exact endofunctor $*$ on $\mathscr{A}$ together with a natural isomorphism $\varpi : \mathrm{id}_{\mathscr{A}} \Rightarrow * \circ *$ satisfying $\varpi_M^* \varpi_{M^*} = \mathrm{id}_{M^*}$ for all $M \in \mathscr{A}$. There are natural notions of symmetric spaces and isometries of symmetric spaces, (admissible) Lagrangians of a symmetric space and hence of metabolic spaces. If $\mathscr{A}$ is essentially small it makes sense to speak of the set $\mathrm{MW}(\mathscr{A}, *, \varpi)$ of isometry classes of symmetric spaces and the subset $\mathrm{NW}(\mathscr{A}, *, \varpi)$ of isometry classes of metabolic spaces and both turn out to be abelian monoids with respect to the *orthogonal sum* of symmetric spaces. The *Witt group* is $\mathrm{W}(\mathscr{A}, *, \varpi) = \mathrm{MW}(\mathscr{A}, *, \varpi)/\mathrm{NW}(\mathscr{A}, *, \varpi)$. In case $\mathscr{A}$ is the category of vector bundles over a scheme $(X, \mathscr{O}_X)$ and $* = \mathrm{Hom}_{\mathscr{O}_X}(-, \mathscr{O}_X)$ is the usual duality functor, one obtains the classical Witt group of a scheme.

Extending these considerations to the level of the derived category leads to *Balmer's triangular Witt groups* which had a number of striking applications to the theory of quadratic forms and K-theory, we refer the interested reader to Balmer's survey [2]. For a beautiful link to algebraic K-theory we refer to Schlichting [54].

13.6. Higher algebraic K-theory

Let $(\mathscr{A}, \mathscr{E})$ be a small exact category. The *Grothendieck group* $K_0(\mathscr{A}, \mathscr{E})$ is defined to be the quotient of the free (abelian) group generated by the isomorphism classes of objects of $\mathscr{A}$ modulo the relations $[A] = [A'][A'']$ for each short exact sequence $A' \rightarrowtail A \twoheadrightarrow A''$ in $\mathscr{E}$. This generalizes the K-theory of a ring, where $(\mathscr{A}, \mathscr{E})$ is taken to be the category of finitely generated projective modules over R with the split exact structure. If $(\mathscr{A}, \mathscr{E})$ is the category of algebraic vector bundles over a scheme X then by definition $K_0(\mathscr{A}, \mathscr{E})$ is the (naïve) Grothendieck group $K_0(X)$ of the scheme (cf. [58, 3.2, p. 313]).

Quillen's landmark paper [51] introduces today's definition of higher algebraic K-theory and proves its basic properties. Exact categories enter via the *Q-construction*, which we outline briefly. Given a small exact category $(\mathscr{A}, \mathscr{E})$ one forms a new category $Q\mathscr{A}$: The objects of $Q\mathscr{A}$ are the objects of $\mathscr{A}$ and $\operatorname{Hom}_{Q\mathscr{A}}(A, A')$ is defined to be the set of equivalence classes of diagrams

$$A \overset{p}{\twoheadleftarrow} B \overset{i}{\rightarrowtail} A',$$

in which p is an admissible epic and i is an admissible monic, where two diagrams are considered equivalent if there is an isomorphism of such diagrams inducing the identity on A and A'. The composition of two morphisms (p, i), (p', i') is given by the following construction: form the pull-back over p' and i so that by Proposition 2.15 there is a diagram

$$\begin{tikzcd}[column sep=small]
 & & & B'' \arrow[dl, two heads, dotted, "q"'] \arrow[dr, tail, dotted, "j'"] \arrow[dd, phantom, "\mathrm{PB}"] & & & \\
 & B \arrow[dl, two heads, "p"'] \arrow[dr, tail, "i"] & & & B' \arrow[dl, two heads, "p'"'] \arrow[dr, tail, "i'"] & & \\
A & & A' & & & A'' &
\end{tikzcd}$$

and put $(p', i') \circ (p, i) = (pq, i'j')$. This is easily checked to yield a category and it is not hard to make sense of the statement that the morphisms $A \to A'$ in $Q\mathscr{A}$ correspond to the different ways that A arises as an admissible subquotient of A'.

Now any small category $\mathscr{C}$ gives rise to a simplicial set $N\mathscr{C}$, called the *nerve of $\mathscr{C}$* whose n-simplices are given by sequences of composable morphisms

$$C_0 \to C_1 \to \cdots \to C_n$$

where the i-th face map is obtained by deleting the object C_i and the i-th degeneracy map is obtained by replacing C_i by $1_{C_i} : C_i \to C_i$. The *classifying space $B\mathscr{C}$* of $\mathscr{C}$ is the geometric realization of the nerve $N\mathscr{C}$.

Quillen proves the fundamental result that

$$K_0(\mathscr{A}, \mathscr{E}) \cong \pi_1(B(Q\mathscr{A}), 0)$$

which motivates the definition

$$K_n(\mathscr{A}, \mathscr{E}) := \pi_{n+1}(B(Q\mathscr{A}), 0).$$

Obviously, an exact functor $F : (\mathscr{A}, \mathscr{E}) \to (\mathscr{A}', \mathscr{E}')$ yields a functor $Q\mathscr{A} \to Q\mathscr{A}'$ and hence a homomorphism $F_* : K_n(\mathscr{A}, \mathscr{E}) \to K_n(\mathscr{A}', \mathscr{E}')$ which is easily seen to depend only on the isomorphism class of F.

We do not discuss K-theory any further and recommend the lecture of Quillen's original article [51] and Srinivas's book [57] expanding on Quillen's article. For a good overview over many topics of current interest we refer to the handbook of K-theory [20].

Appendix A. The embedding theorem

For abelian categories, one has the Freyd–Mitchell embedding theorem, see [17,47], allowing one to prove diagram lemmas in abelian categories "by chasing elements". In order to prove diagram lemmas in exact categories, a similar technique works. More precisely, one has:

Theorem A.1 (*Thomason [58, A.7.1, A.7.16]*). *Let* $(\mathscr{A}, \mathscr{E})$ *be a small exact category.*

(i) *There is an abelian category* $\mathscr{B}$ *and a fully faithful exact functor* $i : \mathscr{A} \to \mathscr{B}$ *that reflects exactness. Moreover,* $\mathscr{A}$ *is closed under extensions in* $\mathscr{B}$.
(ii) *The category* $\mathscr{B}$ *may canonically be chosen to be the category of left exact functors* $\mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ *and* i *to be the Yoneda embedding* $i(A) = \mathrm{Hom}_{\mathscr{A}}(-, A)$.
(iii) *Assume moreover that* $\mathscr{A}$ *is weakly idempotent complete. If* f *is a morphism in* $\mathscr{A}$ *and* $i(f)$ *is epic in* $\mathscr{B}$ *then* f *is an admissible epic.*

Remark A.2. In order for (iii) to hold it is necessary to assume weak idempotent completeness of $\mathscr{A}$. Indeed, if $\mathscr{A}$ fails to be weakly idempotent complete, there must be a retraction r without kernel. By definition there exists s such that $rs = 1$, but then $i(rs)$ is epic, so $i(r)$ is epic as well.

Remark A.3. Let $\mathscr{B}$ be an abelian category and assume that $\mathscr{A}$ is a full subcategory which is closed under extensions, i.e., $\mathscr{A}$ is fully exact subcategory of $\mathscr{B}$ in the sense of Definition 10.21. Then, by Lemma 10.20, $\mathscr{A}$ is an exact category with respect to the class $\mathscr{E}$ of short sequences in $\mathscr{A}$ which are exact in $\mathscr{B}$. This is a basic recognition principle of exact categories, for many examples arise in this way. The embedding theorem provides a partial converse to this recognition principle.

Remark A.4. Quillen states in [51, p. "92/16/100"]:

> Now suppose given an exact category $\mathscr{M}$. Let $\mathscr{A}$ be the additive category of additive contravariant functors from $\mathscr{M}$ to abelian groups which are left exact, i.e., carry [an exact sequence $M' \rightarrowtail M \twoheadrightarrow M''$] to an exact sequence
>
> $$0 \to F(M'') \to F(M) \to F(M')$$
>
> (Precisely, choose a universe containing $\mathscr{M}$, and let $\mathscr{A}$ be the category of left exact functors whose values are abelian groups in the universe.) Following well-known ideas (e.g. [22]), one can prove $\mathscr{A}$ is an abelian category, that the Yoneda functor h embeds $\mathscr{M}$ as a full subcategory of $\mathscr{A}$ closed under extensions, and finally that a [short] sequence [...] is in $\mathscr{E}$ if and only if h carries it into an exact sequence in $\mathscr{A}$. The details will be omitted, as they are not really important for the sequel.

Freyd stated a similar theorem in [16], again without proof, and with the additional assumption of idempotent completeness, since he uses Heller's axioms. The first proof published is in Laumon [43, 1.0.3], relying on the Grothendieck–Verdier theory of sheafification [56]. The sheafification approach was also used and further refined by Thomason [58,

Appendix A]. A quite detailed sketch of the proof alluded to by Quillen is given in Keller [37, A.3].

The proof given here is due to Thomason [58, A.7] amalgamated with the proof in Laumon [43, 1.0.3]. We also take the opportunity to fix a slight gap in Thomason's argument (our Lemma A.10, compare with the first sentence after [58, (A.7.10)]). Since Thomason fails to spell out the nice sheaf-theoretic interpretations of his construction and since referring to SGA 4 seems rather brutal, we use the terminology of the more lightweight Mac Lane–Moerdijk [46, Chapter III]. Other good introductions to the theory of sheaves may be found in Artin [1] or Borceux [7], for example.

A.7. Separated presheaves and sheaves

Let $(\mathscr{A}, \mathscr{E})$ be a small exact category. For each object $A \in \mathscr{A}$, let

$$\mathscr{C}_A = \{(p' : A' \twoheadrightarrow A) : A' \in \mathscr{A}\}$$

be the set of admissible epics onto A. The elements of $\mathscr{C}_A$ are the *coverings* of A.

Lemma A.5. *The family* $\{\mathscr{C}_A\}_{A\in\mathscr{A}}$ *is a* basis for a Grothendieck topology J *on* $\mathscr{A}$:

(i) *If* $f : A \to B$ *is an isomorphism then* $f \in \mathscr{C}_B$.
(ii) *If* $g : A \to B$ *is arbitrary and* $(q' : B' \twoheadrightarrow B) \in \mathscr{C}_B$ *then the pull-back*

$$\begin{tikzcd} A' \arrow[r] \arrow[d, two heads, "p'"'] \arrow[dr, phantom, "\text{PB}"] & B' \arrow[d, two heads, "q'"] \\ A \arrow[r, "g"] & B \end{tikzcd}$$

yields a morphism $p' \in \mathscr{C}_A$. ("Stability under base-change").
(iii) *If* $(p : B \twoheadrightarrow A) \in \mathscr{C}_A$ *and* $(q : C \twoheadrightarrow B) \in \mathscr{C}_B$ *then* $pq \in \mathscr{C}_A$. ("Transitivity").

In particular, $(\mathscr{A}, J)$ *is a* site.

Proof. This is obvious from the definition, see [46, Definition 2, p. 111]. □

The Yoneda functor $y : \mathscr{A} \to \mathbf{Ab}^{\mathscr{A}^{\mathrm{op}}}$ associates to each object $A \in \mathscr{A}$ the presheaf (of abelian groups) $y(A) = \mathrm{Hom}_{\mathscr{A}}(-, A)$. In general, a *presheaf* is the same thing as a functor $G : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$, which we will assume to be additive except in the next lemma. We will see shortly that $y(A)$ is in fact a *sheaf* on the site $(\mathscr{A}, J)$.

Lemma A.6. *Consider the site* $(\mathscr{A}, J)$ *and let* $G : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ *be a functor.*

(i) *The presheaf* G *is separated if and only if for each admissible epic* p *the morphism* $G(p)$ *is monic.*
(ii) *The presheaf* G *is a sheaf if and only if for each admissible epic* $p : A \twoheadrightarrow B$ *the diagram*

$$G(B) \xrightarrow{G(p)} G(A) \underset{d^0 = G(p_0)}{\overset{d^1 = G(p_1)}{\rightrightarrows}} G(A \times_B A)$$

is an equalizer (*difference kernel*), *where* $p_0, p_1 : A\times_B A \twoheadrightarrow A$ *denote the two projections*. *In other words*, *the presheaf* G *is a sheaf if and only if for all admissible epics* $p : A \twoheadrightarrow B$ *the diagram*

$$\begin{array}{ccc} G(B) & \xrightarrow{G(p)} & G(A) \\ \Big\downarrow{\scriptstyle G(p)} & & \Big\downarrow{\scriptstyle d^1} \\ G(A) & \xrightarrow{d^0} & G(A \times_B A) \end{array}$$

is a pull-back.

Proof. Again, this is obtained by making the definitions explicit. Point (i) is the definition, [46 p. 129], and point (ii) is [46 Proposition 1 [bis], p. 123]. □

The following lemma shows that the sheaves on the site $(\mathscr{A}, J)$ are quite familiar gadgets.

Lemma A.7. *Let* $G : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ *be an additive functor*. *The following are equivalent*:

(i) *The presheaf* G *is a sheaf on the site* $(\mathscr{A}, J)$.
(ii) *For each admissible epic* $p : B \twoheadrightarrow C$ *the sequence*

$$0 \to G(C) \xrightarrow{G(p)} G(B) \xrightarrow{d^0-d^1} G(B\times_C B)$$

is exact.
(iii) *For each short exact sequence* $A \rightarrowtail B \twoheadrightarrow C$ *in* $\mathscr{A}$ *the sequence*

$$0 \to G(C) \to G(B) \to G(A)$$

is exact, *i.e.*, G *is* left exact.

Proof. By Lemma A.6(ii) we have that G is a sheaf if and only if the sequence

$$0 \to G(C) \xrightarrow{\left[\begin{smallmatrix} G(p) \\ G(p) \end{smallmatrix}\right]} G(B) \oplus G(B) \xrightarrow{\left[\begin{smallmatrix} G(p_0) & -G(p_1) \end{smallmatrix}\right]} G(B \times_C B)$$

is exact. Since $p_1 : B\times_C B \twoheadrightarrow B$ is a split epic with kernel A, there is an isomorphism $B\times_C B \to A \oplus B$ and it is easy to check that the above sequence is isomorphic to

$$0 \to G(C) \to G(B) \oplus G(B) \to G(A) \oplus G(B)$$

Because left exact sequences are stable under taking direct sums and passing to direct summands, (i) is equivalent to (iii). That (i) is equivalent to (ii) is obvious by Lemma A.6(ii). □

Corollary A.8 (*Thomason [58, A.7.6]*). *The represented functor* $y(A) = \mathrm{Hom}_{\mathscr{A}}(-, A)$ *is a sheaf for every object* A *of* $\mathscr{A}$.

Proof. Given an exact sequence $B' \rightarrowtail B \twoheadrightarrow B''$ we need to prove that

$$0 \to \mathrm{Hom}_{\mathscr{A}}(B'', A) \to \mathrm{Hom}_{\mathscr{A}}(B, A) \to \mathrm{Hom}_{\mathscr{A}}(B', A)$$

is exact. That the sequence is exact at $\mathrm{Hom}_{\mathscr{A}}(B, A)$ follows from the fact that $B \twoheadrightarrow B''$ is a cokernel of $B' \rightarrowtail B$. That the sequence is exact at $\mathrm{Hom}_{\mathscr{A}}(B'', A)$ follows from the fact that $B \twoheadrightarrow B''$ is epic. □

A.8. Outline of the proof

Let now $\mathscr{Y}$ be the category of additive functors $\mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ and let $\mathscr{B}$ be the category of (additive) sheaves on the site $(\mathscr{A}, J)$. Let $j_* : \mathscr{B} \to \mathscr{Y}$ be the inclusion. By Corollary A.8, the Yoneda functor y factors as

$$\begin{tikzcd} \mathscr{A} \arrow[r, "i"] \arrow[dr, "y"'] & \mathscr{B} \arrow[d, "j_*"] \\ & \mathscr{Y} \end{tikzcd}$$

via a functor $i : \mathscr{A} \to \mathscr{B}$. We will prove that the category $\mathscr{B} = \mathrm{Sheaves}(A, J)$ is abelian and we will check that the functor i has the properties asserted in the embedding theorem.

The category $\mathscr{Y}$ is a Grothendieck abelian category (there is a generator, small products and coproducts exist and filtered colimits are exact) – as a functor category, these properties are inherited from $\mathbf{Ab}$, as limits and colimits are taken pointwise. The crux of the proof of the embedding theorem is to show that j_* has an exact left adjoint such that $j^* j_* \cong \mathrm{id}_{\mathscr{B}}$, namely sheafification. As soon as this is established, it follows that $\mathscr{B}$ is abelian, and the rest will be relatively painless.

A.9. Sheafification

The goal of this section is to construct the sheafification functor on the site $(\mathscr{A}, J)$ and to prove its basic properties. We will construct an endofunctor $L : \mathscr{Y} \to \mathscr{Y}$ which associates to each presheaf a separated presheaf and to each separated presheaf a sheaf. The sheafification functor will then be given by $j^* = LL$.

We need one more concept from the theory of sites:

Lemma A.9. *Let* $A \in \mathscr{A}$. *A covering* $p'' : A'' \twoheadrightarrow A$ *is a* refinement *of the covering* $p' : A' \twoheadrightarrow A$ *if and only if there exists a morphism* $a : A'' \to A'$ *such that* $p'a = p''$.

Proof. This is the specialization of a *matching family* as given in [46, p. 121] in the present context. □

By definition, refinement gives the structure of a filtered category on $\mathscr{C}_A$ for each $A \in \mathscr{A}$. More precisely, let $\mathscr{D}_A$ be the following category: the objects are the coverings $(p' : A' \twoheadrightarrow A)$ and there exists at most one morphism between any two objects of $\mathscr{D}_A$: there exists a morphism $(p' : A' \twoheadrightarrow A) \to (p'' : A'' \twoheadrightarrow A)$ in $\mathscr{D}_A$ if and only if there exists $a : A'' \to A'$ such that $p'a = p''$. To see that $\mathscr{D}_A$ is filtered, let $(p' : A' \twoheadrightarrow A)$ and $(p'' : A'' \twoheadrightarrow A)$ be two objects and put $A''' = A' \times_A A''$, so there is a pull-back diagram

$$\begin{tikzcd} A''' \arrow[r, two heads, "a'"] \arrow[d, two heads, "a"'] \arrow[dr, phantom, "\mathrm{PB}"] & A'' \arrow[d, two heads, "p''"] \\ A' \arrow[r, two heads, "p'"] & A. \end{tikzcd}$$

Put $p''' = p'a = p''a'$, so the object $(p''' : A''' \twoheadrightarrow A)$ of $\mathscr{D}_A$ is a common refinement of $(p' : A' \twoheadrightarrow A)$ and $(p'' : A'' \twoheadrightarrow A)$.

Lemma A.10. *Let $A_1, A_2 \in \mathscr{A}$ be any two objects.*

(i) *There is a functor $Q : \mathscr{D}_{A_1} \times \mathscr{D}_{A_2} \to \mathscr{D}_{A_1 \oplus A_2}, (p'_1, p'_2) \mapsto (p'_1 \oplus p'_2)$.*

(ii) *Let $(p' : A' \twoheadrightarrow A_1 \oplus A_2)$ be an object of $\mathscr{D}_{A_1 \oplus A_2}$ and for $i = 1, 2$ let*

$$\begin{tikzcd} A'_i \arrow[r] \arrow[d, two heads, "p'_i"] \arrow[dr, phantom, "\mathrm{PB}"] & A' \arrow[d, two heads, "p'"] \\ A_i \arrow[r] & A_1 \oplus A_2 \end{tikzcd}$$

be a pull-back diagram in which the bottom arrow is the inclusion. This construction defines a functor

$$P : \mathscr{D}_{A_1 \oplus A_2} \longrightarrow \mathscr{D}_{A_1} \times \mathscr{D}_{A_2}, \quad p' \longmapsto (p'_1, p'_2)$$

(iii) *There are a natural transformation $\mathrm{id}_{\mathscr{D}_{A_1 \oplus A_2}} \Rightarrow PQ$ and a natural isomorphism $QP \cong \mathrm{id}_{\mathscr{D}_{A_1} \times \mathscr{D}_{A_2}}$. In particular, the images of P and Q are cofinal.*

Proof. That P is a functor follows from its construction and the universal property of pull-back diagrams in conjunction with axiom [E2$^{\mathrm{op}}$]. That Q is well-defined follows from Proposition 2.9 and that $PQ \cong \mathrm{id}_{\mathscr{D}_{A_1} \times \mathscr{D}_{A_2}}$ is easy to check. That there is a natural transformation $\mathrm{id}_{\mathscr{D}_{A_1 \oplus A_2}} \Rightarrow QP$ follows from the universal property of products. $\square$

Let $(p'' : A'' \twoheadrightarrow A)$ be a refinement of $(p' : A' \twoheadrightarrow A)$, and let $a : A'' \to A'$ be such that $p'a = p''$. By the universal property of pull-backs, a yields a unique morphism $A'' \times_A A'' \to A' \times_A A'$ which we denote by $a \times_A a$. Hence, for every additive functor $G : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$, we obtain a commutative diagram in $\mathbf{Ab}$:

$$\begin{tikzcd} \operatorname{Ker}(d^0 - d^1) \arrow[r] \arrow[d, dotted, "\exists!"] & G(A') \arrow[r, "d^0 - d^1"] \arrow[d, "G(a)"] & G(A' \times_A A') \arrow[d, "G(a \times_A a)"] \\ \operatorname{Ker}(d^0 - d^1) \arrow[r] & G(A'') \arrow[r, "d^0 - d^1"] & G(A'' \times_A A''). \end{tikzcd}$$

The next thing to observe is that the dotted morphism does not depend on the choice of a. Indeed, if $\tilde{a}$ is another morphism such that $p'\tilde{a} = p''$, consider the diagram

$$\begin{tikzcd}
A'' \arrow[dr, dotted, "\exists! b"] \arrow[drr, bend left, "\tilde{a}"] \arrow[ddr, bend right, "a"'] & & \\
& A' \times_A A' \arrow[r, "p_1'"] \arrow[d, "p_0'"] \arrow[dr, phantom, "\mathrm{PB}"] & A' \arrow[d, "p'"] \\
& A' \arrow[r, "p'"] & A
\end{tikzcd}$$

and $b : A'' \to A' \times_A A'$ is such that

$$G(b)(d^0 - d^1) = G(b)G(p_0') - G(b)G(p_1') = G(a) - G(\tilde{a})$$

so $G(a) - G(\tilde{a}) = 0$ on $\operatorname{Ker}(d^0 - d^1)$.

For $G : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$, we put $\ell G(p' : A' \twoheadrightarrow A) := \operatorname{Ker}(G(A') \overset{d^0-d^1}{\to} G(A' \times_A A'))$ and we have just seen that this defines a functor $\ell G : \mathscr{D}_A \to \mathbf{Ab}$.

Lemma A.11. *Define*

$$LG(A) = \varinjlim_{\mathscr{D}_A} \ell G(p' : A' \twoheadrightarrow A).$$

(i) *LG is an additive contravariant functor in A.*
(ii) *L is a covariant functor in G.*

Proof. This is immediate from going through the definitions:

To prove (i), let $f : A \to B$ be an arbitrary morphism. Lemma A.5(ii) shows that by taking pull-backs we obtain a functor

$$\mathscr{D}_B \overset{f^*}{\to} \mathscr{D}_A$$

which, by passing to the colimit, induces a unique morphism $LG(B) \xrightarrow{LG(f)} LG(A)$ compatible with f^*. From this uniqueness, we deduce $LG(fg) = LG(g)LG(f)$. The additivity of LG is a consequence of Lemma A.10.

To prove (ii), let $\alpha : F \Rightarrow G$ be a natural transformation between two (additive) presheaves. Given an object $A \in \mathscr{A}$, we obtain a morphism between the colimit diagrams defining $LF(A)$ and $LG(A)$ and we denote the unique resulting map by $L(\alpha)_A$. Given a morphism $f : A \to B$, there is a commutative diagram

$$\begin{tikzcd}
LF(B) \arrow[r, "LF(f)"] \arrow[d, "L(\alpha)_B"] & LF(A) \arrow[d, "L(\alpha)_A"] \\
LG(B) \arrow[r, "LG(f)"] & LG(A),
\end{tikzcd}$$

as is easily checked. The uniqueness in the definition of $L(\alpha)_A$ implies that for each $A \in \mathscr{A}$ the equation

$$L(\alpha \circ \beta)_A = L(\alpha)_A \circ L(\beta)_A$$

holds. The reader in need of more details may consult [7, p. 206f]. □

Lemma A.12 (*Thomason [58, A.7.8]*). *The functor $L : \mathscr{Y} \to \mathscr{Y}$ has the following properties*:

(i) *It is additive and preserves finite limits.*
(ii) *There is a natural transformation $\eta : \mathrm{id}_{\mathscr{Y}} \Rightarrow L$.*

Proof. That L preserves finite limits follows from the fact that filtered colimits and kernels in **Ab** commute with finite limits, as limits in $\mathscr{Y}$ are formed pointwise, see also [7, Lemma 3.3.1]. Since L preserves finite limits, it preserves in particular finite products, hence it is additive. This settles point (i).

For each $(p' : A' \twoheadrightarrow A) \in \mathscr{D}_A$ the morphism $G(p') : G(A) \to G(A')$ factors uniquely over

$$\tilde{\eta}_{p'} : G(A) \to \mathrm{Ker}(G(A') \to G(A' \times_A A'))$$

By passing to the colimit over $\mathscr{D}_A$, this induces a morphism $\tilde{\eta}_A : G(A) \to LG(A)$ which is clearly natural in A. In other words, the $\tilde{\eta}_A$ yield a natural transformation $\eta_G : G \Rightarrow LG$, i.e., a morphism in $\mathscr{Y}$. We leave it to the reader to check that the construction of η_G is compatible with natural transformations $\alpha : G \Rightarrow F$ so that the η_G assemble to yield a natural transformation $\eta : \mathrm{id}_{\mathscr{Y}} \Rightarrow L$, as claimed in point (ii). □

Lemma A.13 (*Thomason [58, A.7.11, (a)–(c)]*). *Let $G \in \mathscr{Y}$ and let $A \in \mathscr{A}$.*

(i) *For all $x \in LG(A)$ there exists an admissible epic $p' : A' \twoheadrightarrow A$ and $y \in G(A')$ such that $\eta(y) = LG(p')(x)$ in $LG(A')$.*
(ii) *For all $x \in G(A)$, we have $\eta(x) = 0$ in $LG(A)$ if and only if there exists an admissible epic $p' : A' \twoheadrightarrow A$ such that $G(p')(x) = 0$ in $G(A')$.*
(iii) *We have $LG = 0$ if and only if for all $A \in \mathscr{A}$ and all $x \in G(A)$ there exists an admissible epic $p' : A' \twoheadrightarrow A$ such that $G(p')(x) = 0$.*

Proof. Points (i) and (ii) are immediate from the definitions. Point (iii) follows from (i) and (ii). □

Lemma A.14 (*Mac Lane–Moerdijk [46, Lemma 2, p. 131], Thomason [58, A.7.11, (d), (e)]*). *Let $G \in \mathscr{Y}$.*

(i) *The presheaf G is separated if and only if $\eta_G : G \to LG$ is monic.*
(ii) *The presheaf G is a sheaf if and only if $\eta_G : G \to LG$ is an isomorphism.*

Proof. Point (i) follows from Lemma A.13 (ii) and point (ii) follows from the definitions. □

Proposition A.15 (*Thomason [58, A.7.12]*). *Let $G \in \mathscr{Y}$.*

(i) *The presheaf LG is separated.*
(ii) *If G is separated then LG is a sheaf.*

Proof. Let us prove (i) by applying Lemma A.6 (i), so let $x \in LG(A)$ and consider an admissible epic $b : B \twoheadrightarrow A$ for which $LG(b)(x) = 0$. We have to prove that then $x = 0$ in $LG(A)$. By the definition of $LG(A)$, we know that x is represented by some $y \in \operatorname{Ker}\left(G(A') \xrightarrow{d^0-d^1} G(A' \times_A A')\right)$ for some admissible epic $(p' : A' \twoheadrightarrow A)$ in $\mathscr{D}_A$. Since $LG(b)(x) = 0$ in $LG(B)$, we know that the image of y in

$$\operatorname{Ker}\left(G(A' \times_A B) \xrightarrow{d^0-d^1} G((A' \times_A B) \times_B (A' \times_A B))\right)$$

is equivalent to zero in the filtered colimit over $\mathscr{D}_B$ defining $LG(B)$. Therefore there exists a morphism $D \to A' \times_A B$ in $\mathscr{A}$ such that its composite with the projection onto B is an admissible epic $D \twoheadrightarrow B$. By Lemma A.13 (ii), it follows that y maps to zero in $G(D)$. Now the composite $D \twoheadrightarrow B \twoheadrightarrow A$ is in $\mathscr{D}_A$ and hence y is equivalent to zero in the filtered colimit over $\mathscr{D}_A$ defining $LG(A)$. Thus, $x = 0$ in $LG(A)$, as required.

Let us prove (ii). If G is a separated presheaf, we have to check that for every admissible epic $B \twoheadrightarrow A$ the diagram

$$LG(A) \longrightarrow LG(B) \underset{d^0=G(p_0)}{\overset{d^1=G(p_1)}{\rightrightarrows}} LG(B \times_A B)$$

is a difference kernel. By (i) LG is separated, so $LG(A) \to LG(B)$ is monic, and it remains to prove that every element $x \in LG(B)$ with $(d^0 - d^1)x = 0$ is in the image of $LG(A)$. By Lemma A.13(i) there is an admissible epic $q : C \twoheadrightarrow B$ and $y \in G(C)$ such that $\eta(y) = LG(q)(x)$. It follows that $\eta G(p_0)(y) = \eta G(p_1)(y)$ in $LG(C \times_A C)$. Now, G is separated, so $\eta : G \Rightarrow LG$ is monic by Lemma A.14(i), and we conclude from this that $G(p_0)(y) = G(p_1)(y)$ in $G(C \times_A C)$. In other words, $y \in \operatorname{Ker}(G(C) \overset{d^0-d^1}{\longrightarrow} G(C \times_A C))$ yields a class in $LG(A)$ representing x. □

Corollary A.16. *For a presheaf $G \in \mathscr{Y}$ we have $LG = 0$ if and only if $LLG = 0$.*

Proof. Obviously $LG = 0$ entails $LLG = 0$ as L is additive by Lemma A.12. Conversely, as LG is separated by Proposition A.15(i), it follows that the morphism $\eta_{LG} : LG \to LLG$ is monic by Lemma A.14(i), so if $LLG = 0$ we must have $LG = 0$. □

Definition A.17. The *sheafification functor* is $j^* = LL : \mathscr{Y} \to \mathscr{B}$.

Lemma A.18. *The sheafification functor $j^* : \mathscr{Y} \to \mathscr{B}$ is left adjoint to the inclusion functor $j_* : \mathscr{B} \to \mathscr{Y}$ and satisfies $j^* j_* \cong \mathrm{id}_{\mathscr{B}}$. Moreover, sheafification is exact.*

Proof. By Lemma A.14(ii) the morphism $\eta_G : G \to LG$ is an isomorphism if and only if G is a sheaf, so it follows that $j^* j_* \cong \mathrm{id}_{\mathscr{B}}$.

Let $Y \in \mathscr{Y}$ and $B \in \mathscr{B}$. The natural transformation $\eta : \mathrm{id}_{\mathscr{Y}} \Rightarrow L$ gives us on the one hand a natural transformation

$$\varrho_Y = \eta_{LY}\eta_Y : Y \longrightarrow LLY = j_* j^* Y$$

and on the other hand a natural isomorphism

$$\lambda_B = (\eta_{LB}\eta_B)^{-1} : j^* j_* B = LLB \longrightarrow B.$$

Now the compositions

$$j_* B \xrightarrow{\varrho_{j_* B}} j_* j^* j_* B \xrightarrow{j_* \lambda_B} j_* B \qquad \text{and} \qquad j^* Y \xrightarrow{j^* \varrho_Y} j^* j_* j^* Y \xrightarrow{\lambda_{j^* Y}} j^* Y$$

are manifestly equal to $\mathrm{id}_{j_* B}$ and $\mathrm{id}_{j^* Y}$ so that j^* is indeed left adjoint to j_*. In particular j^* preserves cokernels. That j^* preserves kernels follows from the fact that $L : \mathscr{Y} \to \mathscr{Y}$ has this property by Lemma A.12(i) and the fact that $\mathscr{B}$ is a full subcategory of $\mathscr{Y}$. Therefore j^* is exact. $\square$

Remark A.19. It is an illuminating exercise to prove exactness of j^* directly by going through the definitions.

Lemma A.20. *The category $\mathscr{B}$ is abelian.*

Proof. It is clear that $\mathscr{B}$ is additive. The sheafification functor $j^* = LL$ preserves kernels by Lemma A.12(i) and as a left adjoint it preserves cokernels. To prove $\mathscr{B}$ abelian, it suffices to check that every morphism $f : A \to B$ has an analysis

$$\begin{tikzcd}
 & A \arrow[rrrr, "f"] \arrow[dr, two heads] & & & & B \arrow[dr, two heads] & \\
\mathrm{Ker}(f) \arrow[ur, tail] & & \mathrm{Coim}(f) \arrow[r, "\cong"] & \mathrm{Im}(f) \arrow[urr, tail] & & & \mathrm{Coker}(f).
\end{tikzcd}$$

Since j^* preserves kernels and cokernels and $j^* j_* \cong \mathrm{id}_{\mathscr{B}}$ such an analysis can be obtained by applying j^* to an analysis of $j_* f$ in $\mathscr{Y}$. $\square$

A.10. Proof of the embedding theorem

Let us recapitulate: one half of the axioms of an exact structure yields that a small exact category $\mathscr{A}$ becomes a *site* $(\mathscr{A}, J)$. We denoted the Yoneda category of contravariant functors $\mathscr{A} \to \mathbf{Ab}$ by $\mathscr{Y}$ and the Yoneda embedding $A \mapsto \mathrm{Hom}(-, A)$ by $y : \mathscr{A} \to \mathscr{Y}$. We have shown that the category $\mathscr{B}$ of sheaves on the site (A, J) is abelian, being a full reflective subcategory of $\mathscr{Y}$ with sheafification $j^* : \mathscr{Y} \to \mathscr{B}$ as an exact reflector (left adjoint). Following Thomason, we denoted the inclusion $\mathscr{B} \to \mathscr{Y}$ by j_*. Moreover, we have shown that the Yoneda embedding takes its image in $\mathscr{B}$, so we obtained a commutative diagram of categories

$$\begin{tikzcd} \mathscr{A} \arrow[r, "i"] \arrow[dr, "y"'] & \mathscr{B} \arrow[d, "j_*"] \\ & \mathscr{Y}, \end{tikzcd}$$

in other words $y = j_* i$. By the Yoneda lemma, y is fully faithful and j_* is fully faithful, hence i is fully faithful as well. This settles the first part of the following lemma:

Lemma A.21. *The functor $i : \mathscr{A} \to \mathscr{B}$ is fully faithful and exact.*

Proof. By the above discussion, it remains to prove exactness.

Clearly, the Yoneda embedding sends exact sequences in $\mathscr{A}$ to left exact sequences in $\mathscr{Y}$. Sheafification j^* is exact and since $j^* j_* \cong \mathrm{id}_{\mathscr{B}}$, we have that $j^* y = j^* j_* i \cong i$ is left exact as well. It remains to prove that for each admissible epic $p : B \twoheadrightarrow C$ the morphism $i(p)$ is epic. By Corollary A.16, it suffices to prove that $G = \operatorname{Coker} y(p)$ satisfies $LG = 0$, because $\operatorname{Coker} i(p) = j^* \operatorname{Coker} y(p) = LLG = 0$ then implies that $i(p)$ is epic. To this end we use the criterion in Lemma A.13(iii), so let $A \in \mathscr{A}$ be any object and $x \in G(A)$. We have an exact sequence $\operatorname{Hom}(A, B) \xrightarrow{y(p)_A} \operatorname{Hom}(A, C) \xrightarrow{q_A} G(A) \to 0$, so $x = q_A(f)$ for some morphism $f : A \to C$. Now form the pull-back

$$\begin{tikzcd} A' \arrow[r, two heads, "p'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\mathrm{PB}"] & A \arrow[d, "f"] \\ B \arrow[r, two heads, "p"] & C \end{tikzcd}$$

and observe that $G(p')(x) = G(p')(q_A(f)) = q_{A'}(fp') = q_{A'}(pf') = 0$. $\square$

Lemma A.22 (*Thomason [58, A.7.15]*). *Let $A \in \mathscr{A}$ and $B \in \mathscr{B}$ and suppose there is an epic $e : B \twoheadrightarrow i(A)$. There exist $A' \in \mathscr{A}$ and $k : i(A') \to B$ such that $ek : A' \to A$ is an admissible epic.*

Proof. Let G be the cokernel of $j_* e$ in $\mathscr{Y}$. Then we have $0 = j^* G = LLG$ because $j^* j_* e \cong e$ is epic. By Corollary A.16 it follows that $LG = 0$ as well. Now observe that $G(A) \cong \operatorname{Hom}(A, A)/\operatorname{Hom}(i(A), B)$ and let $x \in G(A)$ be the class of 1_A. From Lemma A.13(iii) we conclude that there is an admissible epic $p' : A' \twoheadrightarrow A$ such that $G(p')(x) = 0$ in $G(A') \cong \operatorname{Hom}(A', A)/\operatorname{Hom}(i(A'), B)$. But this means that the admissible epic p' factors as ek for some $k \in \operatorname{Hom}(i(A'), B)$ as claimed. $\square$

Lemma A.23. *The functor i reflects exactness.*

Proof. Suppose $A \xrightarrow{m} B \xrightarrow{e} C$ is a sequence in $\mathscr{A}$ such that $i(A) \xrightarrow{i(m)} i(B) \xrightarrow{i(e)} i(C)$ is short exact in $\mathscr{B}$. In particular, $i(m)$ is a kernel of $i(e)$. Since i is fully faithful, it follows that m is a kernel of e in $\mathscr{A}$, hence we are done as soon as we can show that e is an admissible epic. Because $i(e)$ is epic, Lemma A.22 allows us to find $A' \in \mathscr{A}$ and $k : i(A') \to i(B)$

such that ek is an admissible epic and since e has a kernel we conclude by the dual of Proposition 2.16. $\square$

Lemma A.24. *The essential image of $i : \mathscr{A} \to \mathscr{B}$ is closed under extensions.*

Proof. Consider a short exact sequence $i(A) \rightarrowtail G \twoheadrightarrow i(B)$ in $\mathscr{B}$, where $A, B \in \mathscr{A}$. By Lemma A.22 we find an admissible epic $p : C \twoheadrightarrow B$ such that $i(p)$ factors over G. Now consider the pull-back diagram

$$\begin{tikzcd} D \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\mathrm{PB}"] & G \arrow[d, two heads] \\ i(C) \arrow[r, two heads, "i(p)"] & i(B) \end{tikzcd}$$

and observe that $D \twoheadrightarrow i(C)$ is a split epic because $i(p)$ factors over G. Therefore we have isomorphisms $D \cong i(A) \oplus i(C) \cong i(A \oplus C)$. If K is a kernel of p then $i(K)$ is a kernel of $D \twoheadrightarrow G$, so we obtain an exact sequence

$$i(K) \xrightarrow{\left[\begin{smallmatrix} i(a) \\ i(c) \end{smallmatrix}\right]} i(A) \oplus i(C) \twoheadrightarrow G,$$

where $c = \ker p$, which shows that G is the push-out

$$\begin{tikzcd} i(K) \arrow[r, tail, "i(c)"] \arrow[d, "i(a)"'] \arrow[dr, phantom, "\mathrm{PO}"] & i(C) \arrow[d] \\ i(A) \arrow[r, tail] & G. \end{tikzcd}$$

Now i is exact by Lemma A.21 and hence preserves push-outs along admissible monics by Proposition 5.2, so i preserves the push-out $G' = A \cup_K C$ of a along the admissible monic c and thus G is isomorphic to $i(G')$. $\square$

Proof of the Embedding Theorem A.1. Let us summarize what we know: the embedding $i : \mathscr{A} \to \mathscr{B}$ is fully faithful and exact by Lemma A.21. It reflects exactness by Lemma A.23 and its image is closed under extensions in $\mathscr{B}$ by Lemma A.24. This settles point (i) of the theorem.

Point (ii) is taken care of by Lemma A.7 and Corollary A.8.

It remains to prove (iii). Assume that $\mathscr{A}$ is weakly idempotent complete. We claim that every morphism $f : B \to C$ such that $i(f)$ is epic is in fact an admissible epic. Indeed, by Lemma A.22 we find a morphism $k : A \to B$ such that $fk : A \twoheadrightarrow C$ is an admissible epic and we conclude by Proposition 7.6. $\square$

Appendix B. Heller's axioms

Proposition B.1 (*Quillen*). *Let $\mathscr{A}$ be an additive category and let $\mathscr{E}$ be a class of kernel–cokernel pairs in $\mathscr{A}$. The pair $(\mathscr{A}, \mathscr{E})$ is a weakly idempotent complete exact category if* *and only if $\mathcal{E}$ satisfies Heller's axioms*:

(i) *Identity morphisms are both admissible monics and admissible epics.*
(ii) *The class of admissible monics and the class of admissible epics are closed under composition.*
(iii) *Let f and g be composable morphisms. If gf is an admissible monic then so is f and if gf is an admissible epic then so is g.*
(iv) *If all the rows and the second two columns of the commutative diagram*

$$\begin{tikzcd} A' \arrow[r, tail, "f'"] \arrow[d, "a"] & B' \arrow[r, two heads, "g'"] \arrow[d, tail, "b"] & C' \arrow[d, tail, "c"] \\ A \arrow[r, tail, "f"] \arrow[d, "a'"] & B \arrow[r, two heads, "g"] \arrow[d, two heads, "b'"] & C \arrow[d, two heads, "c'"] \\ A'' \arrow[r, tail, "f''"] & B'' \arrow[r, two heads, "g''"] & C'' \end{tikzcd}$$

are in $\mathcal{E}$ then the first column is also in $\mathcal{E}$.

Proof. Note that (i) and (ii) are just axioms [E0], [E1] and their duals.

For a weakly idempotent complete exact category $(\mathcal{A}, \mathcal{E})$, point (iii) is proved in Proposition 7.6 and point (iv) follows from the 3×3-lemma 3.6.

Conversely, assume that $\mathcal{E}$ has properties (i)–(iv) and let us check that $\mathcal{E}$ is an exact structure.

By properties (i) and (iii) an isomorphism is both an admissible monic and an admissible epic since by definition $f^{-1}f = 1$ and $ff^{-1} = 1$. If the short sequence $\sigma = (A' \to A \to A'')$ is isomorphic to the short exact sequence $B' \rightarrowtail B \twoheadrightarrow B''$ then property (iv) tells us that σ is short exact. Thus, $\mathcal{E}$ is closed under isomorphisms.

Heller proves [27, Proposition 4.1] that (iv) implies its dual, that is: if the commutative diagram in (iv) has exact rows and both (a, a') and (b, b') belong to $\mathcal{E}$ then so does (c, c').[2] It follows that Heller's axioms are self-dual.

Let us prove that [E2] holds – the remaining axiom [E2$^{\mathrm{op}}$] will follow from the dual argument. Given the diagram

$$\begin{tikzcd} A' \arrow[r, tail, "f'"] \arrow[d, "a"] & B' \\ A & \end{tikzcd}$$

we want to construct its push-out B and prove that the morphism $A \to B$ is an admissible monic. Observe that $\left[\begin{smallmatrix} a \\ f' \end{smallmatrix}\right] : A' \to A \oplus B'$ is the composition

$$A' \xrightarrow{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} A \oplus A' \xrightarrow[\cong]{\left[\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right]} A \oplus A' \xrightarrow{\left[\begin{smallmatrix} 1 & 0 \\ 0 & f' \end{smallmatrix}\right]} A \oplus B'.$$

[2] Indeed, by (iii) c' is an admissible epic and so it has a kernel D. Because $c'gb = 0$, there is a morphism $B' \to D$ and replacing C' by D in the diagram of (iv) we see that $A' \rightarrowtail B' \twoheadrightarrow D$ is short exact. Therefore $C' \cong D$ and we conclude by the fact that $\mathcal{E}$ is closed under isomorphisms.

By (iii) split exact sequences belong to $\mathscr{E}$, and the proof of Proposition 2.9 shows that the direct sum of two sequences in $\mathscr{E}$ also belongs to $\mathscr{E}$. Therefore $\left[\begin{smallmatrix} a \\ f' \end{smallmatrix}\right]$ is an admissible monic and it has a cokernel $[-f\ b] : A \oplus B' \twoheadrightarrow B$. It follows that the left-hand square in the diagram

$$\begin{tikzcd} A' \arrow[r, tail, "f'"] \arrow[d, "a"] \arrow[dr, phantom, "\mathrm{BC}"] & B' \arrow[r, two heads, "g'"] \arrow[d, "b"] & C' \arrow[d, equal] \\ A \arrow[r, "f"] & B \arrow[r, "g"] & C' \end{tikzcd}$$

is bicartesian. Let $g' : B' \twoheadrightarrow C'$ be a cokernel of f' and let g be the morphism $B \to C'$ such that $gf = 0$ and $gb = g'$. Now consider the commutative diagram

$$\begin{tikzcd} A' \arrow[r, tail, "\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]"] \arrow[d, equal] & A \oplus A' \arrow[r, two heads, "{[-1\ 0]}"] \arrow[d, tail, "\left[\begin{smallmatrix} 1 & a \\ 0 & f' \end{smallmatrix}\right]"] & A \arrow[d, "f"] \\ A' \arrow[r, tail, "\left[\begin{smallmatrix} a \\ f \end{smallmatrix}\right]"] & A \oplus B' \arrow[r, two heads, "{[-f\ b]}"] \arrow[d, two heads, "{[0\ g']}"] & B \arrow[d, "g"] \\ & C' \arrow[r, equal] & C' \end{tikzcd}$$

in which the rows are exact and the first two columns are exact. It follows that the third column is exact and hence f is an admissible monic.

Now that we know that $(\mathscr{A}, \mathscr{E})$ is an exact category, we conclude from (iii) and Proposition 7.6 that $\mathscr{A}$ must be weakly idempotent complete. $\square$

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