Furstenberg's Proof.

Assume you two are both familiar with *Ch 1* in 尤's book. Here is a super cool stuff I'd like to show: Furstenberg's proof of the infinitude of primes.

Problem. Try to prove there are infinite primes in a super cool style.

Furstenberg's proof. Define the topology on $\mathbb{Z}$, that is, open sets are generated by arithmetic sequences (in the form of $a\mathbb{Z}+b$). We denote

$$S(a,b) := a\mathbb{Z}+b := \{an+b \mid n \in \mathbb{Z}\}.$$

Then $S(a,b)$ are both open and closed.

Assume that prime numbers are finite. Then $\cup_{p \text{ is prime}} S(p,0)$ is a finite union of closed sets, thus closed. As a result, $\{\pm 1\} = \mathbb{Z} - \cup_{p \text{ is prime}} S(p,0)$ is open. This would be a contradiction!

Here is another thought-provoking problem we shall discuss in several days.

> Definition 1-5 seems trivial yet referenced for the sake of rigor.

Definiton 1. We call $G = G(V,E)$ a **simple graph** whenenver V is a set and $E \subset \{\{x,y\} \mid x,y \in V, x \neq y\}$. Here V (or E) is the set of vertices (or edges).

Simple graph is always *unweighted, undirected, without self-loops and multi-edges*.

Definition 2. Let G be a simple graph. G is **k-colourable on vertices** whenever there exists a function $f \in \{1,2,\dots,k\}^V$ s.t. $f(x) \neq f(y)$ when $\{x,y\} \in E$.

Definition 3. The **minimal number for vertex colouring** is the minimal positive integer $k_{\min} =: \chi(G)$, such that G is $k_{\min}$-colourable on vertices.

Definition 4. The **vertex-deleted graphs** of G are in the form of

$$G_{x_0}(V',E') := G_{x_0}(\{x \in V \mid x \neq x_0\}, \{\{x,y\} \in E \mid x,y \neq x_0\}).$$

Informally speaking, G_{x_0} is obtained from G by deleting x_0 and all edges connecting to it.

Definition 5. We call a simple graph G **critical** whenever $\chi(G) < \infty$ and

$$\sup_{x \in G} \chi(G_x) = \max_{x \in G} \chi(G_x) < \chi(G).$$

Problem. Proves that **all critical graph is finite**, that is, $|V| < \infty$.

The *axiom of chioce* is required when necessary.