

设旋转曲面

$$X(u, \theta) = (f(u) \cos(\theta), f(u) \sin(\theta), g(u)).$$

一方面有

$$\begin{aligned} dx &= f'(u) \cos(\theta) du - f(u) \sin(\theta) d\theta, \\ dy &= f'(u) \sin(\theta) du + f(u) \cos(\theta) d\theta, \\ dz &= g'(u) du. \end{aligned}$$

从而

$$ds^2 = dx^2 + dy^2 + dz^2 = (f'(u)^2 + g'(u)^2) du^2 + f(u)^2 d\theta^2.$$

但另一方面,

$$\begin{aligned} X_u &= (f'(u) \cos \theta, f'(u) \sin \theta, g'(u)), \\ X_\theta &= (-f(u) \sin \theta, f(u) \cos \theta, g(u)). \end{aligned}$$

从而

$$\begin{aligned} ds^2 &= Edu^2 + 2Fdu d\theta + Gd\theta^2 \\ &= (f'(u)^2 + g'(u)^2) du^2 + g(u)g'(u) du d\theta + (f(u)^2 + g(u)^2) d\theta^2 \end{aligned}$$

矛盾.