Brief introduction to the theory of Hopf Algebra

Algebra&coalgebra
Bialgebra&Hopf algebra
Monads and comonads



The theory of monads (comonads) is still unshaped in this article, which we just take a glimpse of for the perception of **a kind of universal algebra/coalgebra structure**.

Algebra&coalgebra

Definition 1.1 An \Bbbk -algebra is a triple (A, m, u) satisfying that

- A is a k-vector space;
- $m:A\otimes A o A$ is an associative \Bbbk -linear map, i.e.,

$$A \otimes A \otimes A \xrightarrow{m \circ (m \otimes \mathrm{id})} A \otimes A$$

$$m \circ (\mathrm{id} \otimes m) \downarrow \qquad \qquad \downarrow m$$

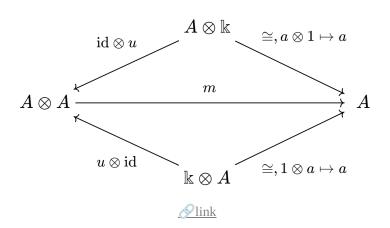
$$A \otimes A \xrightarrow{m} A$$

$$\longrightarrow A$$

$$\longrightarrow A$$

$$\longrightarrow A$$

ullet the unit $u: \mathbb{k} o A, 1 \mapsto 1_A$ is a \mathbb{k} -linear map satisfying that



Definition 1.2 A flip is defined as $au:A\otimes A\to A\otimes A, a\otimes b\mapsto b\otimes a.$

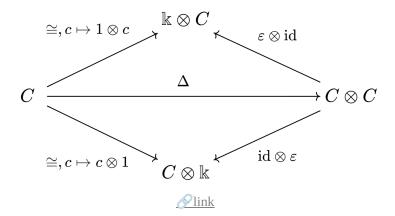
A is a commutative algebra whenever $m\circ au=m$.

Definition 1.3 A coalgebra is a triple $(C,\Delta,arepsilon)$ satisfying that

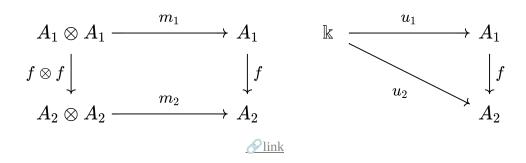
- *C* is a k-vector space;
- ullet $\Delta:C o C\otimes C$ is an associative \Bbbk linear map, satisfying that

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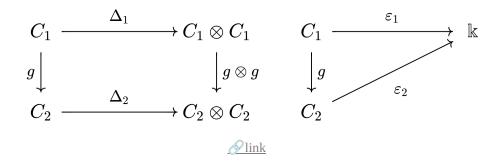
• the **counit** $arepsilon:A o \Bbbk$ is a \Bbbk -linear map satisfying that



Definition 1.4 (**Homomorphisms**) We say $f:A_1 o A_2$ is a homomorphism of algebras, whenever



Similarly, we say $g:C_1 o C_2$ is a homomorphism of coalgebras, whenever



Definition 1.5 (Sub(co)algebra) We say

- ullet $A'\subset A$ is a subalgebra, whenever $m:A'\otimes A' o A'$ and $u\restriction_{A'}\colon \Bbbk o A', 1\mapsto 1_{A'}=1_A$;
- ullet $C'\subset C$ is a subcoalgebra, whenever $\Delta:C' o C'\otimes C'$.

Definition 1.6 (Ideal and coideal) We say

- ullet $I\subset A$ is an (two-sided) ideal of algebra, whenever $m:A\otimes I+I\otimes A o I$;
- ullet $I\subset C$ is an ideal of coalgebra, whenever $\Delta:C o C\otimes I+I\otimes C$, arepsilon:I o 0 .

Definition 1.7 (Quotient and coquotient algebra) Omit.

Definition 1.8 (Sweedler's notation) The notation

$$\sum_{(c)} c_{(1)} \otimes \cdots \otimes c_{(n)} := \Delta_{n-1}(c) := (\Delta \otimes I_{n-2}) \circ \cdots \circ (\Delta \otimes I) \circ \Delta(c).$$

is utilised by Sweedler for simplicity. Sililarly, we define

$$\sum_{(c)} c_{(1)} \otimes c_{(2)} := \sum_i c_{1i} \otimes c_{2i} = \Delta(c)$$

We can also obtain the following corollaries:

- (definition of counit) $c\otimes 1=\sum_{(c)}c_{(1)}\otimes \varepsilon(c_{(2)})$, $1\otimes c=\sum_{(c)}\varepsilon(c_{(1)})\otimes c_{(2)}$;
- (simple corallary) $\sum_{(c)} c_{(1)} \otimes \cdots \otimes \varepsilon(c_{(i)}) \otimes \cdots \otimes c_{(n+1)} = \sum_{(c)} c_{(1)} \otimes \cdots \otimes c_{(n)};$
- (associativity) $\sum_{(c)} (c_{(1)})_{(1)} \otimes (c_{(1)})_{(2)} \otimes c_{(2)} = \sum_{(c)} c_{(2)} \otimes (c_{(2)})_{(1)} \otimes (c_{(2)})_{(2)};$
- (action of au) $au: \sum_{(c)} c_{(1)} \otimes c_{(2)} o \sum_{(c)} c_{(2)} \otimes c_{(1)};$
- (exercises from Sweedler's book)
 - $\circ \ \sum_{(c)} arepsilon(c_2) \otimes \Delta(c_{(1)}) = \sum_{(c)} \Delta(c_2) \otimes arepsilon(c_{(1)}) = \Delta(c),$
 - $\circ \ \sum_{(c)} c_{(1)} \otimes arepsilon(c_{(3)}) \otimes c_{(2)} = \sum_{(c)} c_{(1)} \otimes c_{(3)} \otimes arepsilon(c_{(2)}) = \Delta(c),$

$$\circ \sum_{(c)} arepsilon(c_{(1)}) \otimes c_{(3)} \otimes c_{(2)} = \sum_{(c)} c_{(2)} \otimes c_{(1)},$$

$$\circ \ \sum_{(c)} arepsilon(c_{(1)}) \otimes arepsilon(c_{(3)}) \otimes c_{(2)} = c.$$

Definition 1.9 (Dual analog) Let $V^* := \operatorname{Hom}_{\Bbbk}(V, \Bbbk)$, then

ullet for any \Bbbk -coalgebra $(C,\Delta,arepsilon)$, (C^*,m',u') is an induced algebra, where

$$\circ m': C^* \otimes C^* \stackrel{\rho}{\hookrightarrow} (C \otimes C)^* \stackrel{\Delta^*}{\rightarrow} C^*,$$

$$\circ~u':k o C^*, k o k^*\stackrel{arepsilon^*}{ o} C^*;$$

• for any **REFLEXIVE** (e.g., finite dimensional) \Bbbk -algebra (A,m,u), $(A^*,\Delta',\varepsilon')$ is an induced coalgebra, where

$$\circ \ \Delta':A^*\stackrel{m^*}{
ightarrow}A^*\otimes A^*\stackrel{
ho^{-1}}{\cong}(A\otimes A)^*$$
 ,

$$\circ \ arepsilon': A^* o k, A^* \overset{u^*}{ o} k^* o k.$$

Fact 1.10 Let C and D be coalgebras. Then we have a bialgebra $C\otimes D$, where

$$\bullet \ \ \Delta_{C\otimes D}: C\otimes D\stackrel{\Delta_C\otimes \Delta_D}{\longrightarrow} C\otimes C\otimes D\otimes D\stackrel{\mathrm{id}\otimes \tau\otimes \mathrm{id}}{\longrightarrow} C\otimes D\otimes C\otimes D,$$

$$\circ \ \Delta_{C\otimes D}: c\otimes d\mapsto \sum_{(c),(d)}c_{(1)}\otimes d_{(1)}\otimes c_{(2)}\otimes d_{(2)};$$

$$ullet \ arepsilon_{C\otimes D}:C\otimes D\stackrel{arepsilon_C\otimesarepsilon_D}{\longrightarrow} k\otimes k\cong k,$$

$$\circ \ arepsilon_{C\otimes D}: c\otimes d\mapsto arepsilon_C(c)arepsilon_D(d).$$

And the projections from $C\otimes D$ to $C\left(D\right)$ is homomorphisms between coalgebras.

Fact 1.11 One can verify (via Yoneda's lemma) that

$$(igoplus_{lpha\in I} C_lpha)\otimes (igoplus_{eta\in I} C_eta)\congigoplus_{lpha,eta\in I} (C_lpha\otimes C_eta).$$

Here $e_{lpha}:C_{lpha}\hookrightarrow C$, $p'_{lpha}:C woheadrightarrow C_{lpha}$ are both homomorphisms between coalgebras.

One can also prove that \bigoplus is just \coprod , thus the direct sum of coalgebras has the same universal properties as coproduct.

Bialgebra&Hopf algebra

Definition 2.1 As $(B,m,u,\Delta,arepsilon)$ is a bialgebra whenever the following three statements (1.-3.) holds

- 1. (B, m, u) is an algebra;
- 2. (B, Δ, ε) is a coalgebra;

Thus $B\otimes B$ has both algebra and coalgebra structure.

- 3. the following equivalent statements (a.-c.) holds
 - a. m and u are both homomorphisms of coalgebras,
 - b. Δ and ε are both homomorphisms of algebras,
 - c. all of the following four statements (i.-iv.)

i.
$$\Delta(1)=1\otimes 1$$

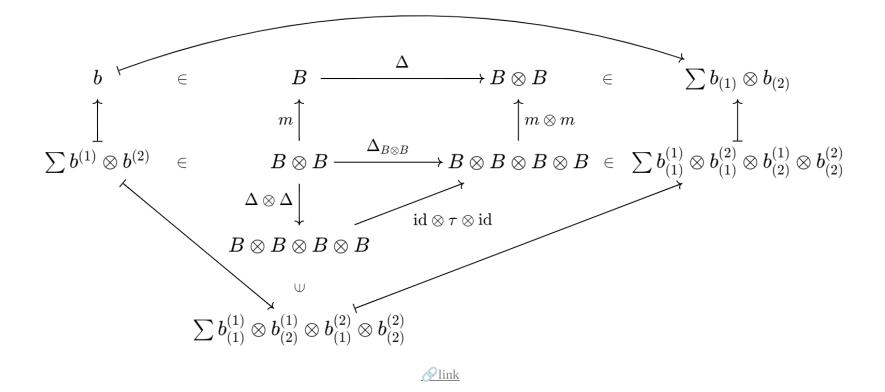
ii.
$$\Delta(gh) = \sum_{(g),(h)} g_{(1)} h_{(1)} \otimes g_{(2)} h_{(2)}$$

iii.
$$\varepsilon(1)=1$$

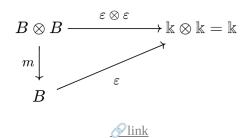
iv.
$$\varepsilon(gh) = \varepsilon(g)\varepsilon(h)$$
.

Here the equivalence of b. and c. are clear. The equivalence between a. and c. is seen as follows

1. the middle rectangle shows that m is a coalgebra homomorphism



2. it suggests that m perserves the counit



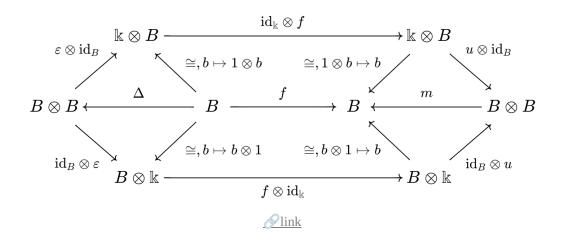
3. u is also a homomorphism of coalgebras, that is,

Fact 2.2 $\operatorname{End}_{\Bbbk}(B) = \operatorname{Hom}_{\Bbbk}(B)$ has a ring structure, i.e.,

- $(\operatorname{End}_{\Bbbk}(B),+)$ is an Abelian group, i.e., $+:(f,g)\mapsto [b\mapsto f(b)+g(b)];$
- ullet (End,st) is an semigroup, i.e., $st:(f,g)\mapsto m\circ (f\otimes g)\circ \Delta$, where

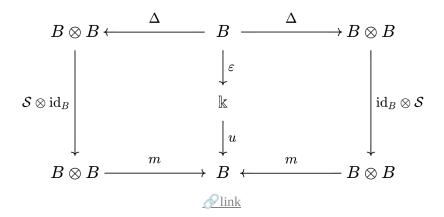
$$(f*g)*h=f*(g*h):B\stackrel{\Delta_2}{\longrightarrow}B\otimes B\otimes B\stackrel{f\otimes g\otimes h}{\longrightarrow}B\otimes B\otimes B\stackrel{m_2}{\longrightarrow}B;$$

• $u\circarepsilon:B o \Bbbk o B$ is the unit of the ring, i.e., f*uarepsilon=uarepsilon*f=f



Definition 2.3 We say $\mathcal{S}\in \mathrm{Hom}_{\Bbbk}(B)$ is an **antipode** iff $\mathcal{S}*\mathrm{id}_B=u\circ \varepsilon=\mathrm{id}_B*\mathcal{S}$, i.e.,

$$\sum_{(b)} \mathcal{S}(b_{(1)}) b_{(2)} = u arepsilon(b) = \sum_{(b)} b_{(1)} \mathcal{S}(b_{(2)}).$$

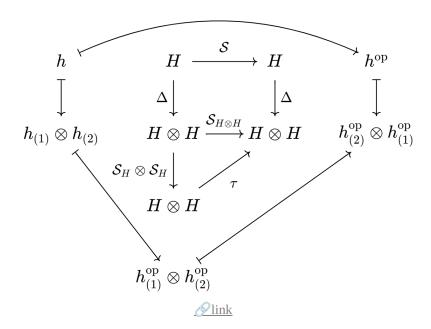


Here the antipode is also known as $(\mathrm{id}_B)^{-1}$, the uniqueness is obvious.

Definition 2.4 A bialgebra with an antipode is a **Hopf algebra**, denoted by H.

Fact 2.5 Here are some propositions on Hopf algebra H:

- $\mathcal{S}: H o H^{\mathrm{op}}, gh \mapsto \mathcal{S}(h)\mathcal{S}(g);$
- $\varepsilon \circ \mathcal{S} = \varepsilon : \mathbb{k} \to H, \mathcal{S} \circ u = u : H \to \mathbb{k}, \mathcal{S}(1) = 1;$
- $\mathcal{S}_{H\otimes H}= au\circ(S_H\otimes\mathcal{S}_H)$



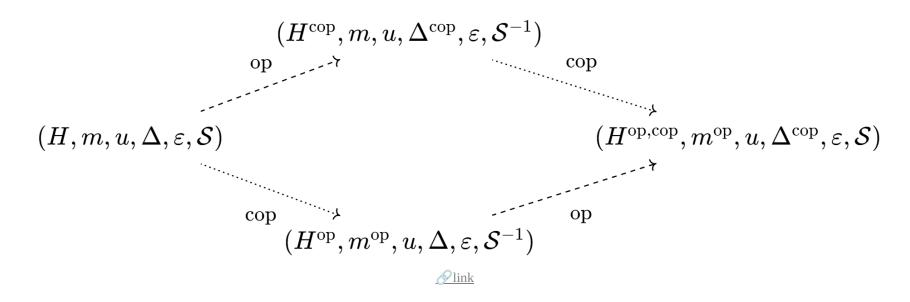
Review that $\Delta_{H\otimes H}=(\mathrm{id}\otimes au\otimes\mathrm{id})\circ(\Delta_H\otimes\Delta_H)$. One can prove that

$$\mathcal{S}: H
ightarrow H^{ ext{cop}}; \quad \Delta(S(h)) = \sum_{(h)} \mathcal{S}(h_{(2)}) \otimes \mathcal{S}(h_{(1)}), \mathcal{S}(arepsilon(h)) \mapsto arepsilon(h)$$

is a coalgebra homomorphism.

- ullet if H is commutative or cocommutative, then the following equivalent statements hold
 - $\circ \mathcal{S} \circ \mathcal{S} = \mathrm{id}_H$
 - $\circ \ \sum_{(h)} \mathcal{S}(h_{(2)}) h_{(1)} = u arepsilon(h), orall h \in H,$
 - $\circ \ \sum_{(h)} h_{(2)} \mathcal{S}(h_{(1)}) = u arepsilon(h), orall h \in H.$

Fact 2.6 As $\mathcal{S}:H o H^{\operatorname{cop}}$ and $\mathcal{S}:H o H^{\operatorname{op}}$, the following are also Hopf algebras:



Fact 2.7 The following are copy from the book of Sweedler

1.
$$\sum_{(h)} h_{(1)} S(h_{(2)}) \otimes h_{(3)} = h \text{ (or } 1 \otimes h);$$

2.
$$\sum_{(h)} S(h_{(1)})h_{(2)} \otimes h_{(3)} = h;$$

3.
$$\sum_{(h)} h_{(1)} \otimes h_{(2)} S(h_{(3)}) = h;$$

4.
$$\sum_{(h)} h_{(1)} \otimes S(h_{(2)}) h_{(3)} = h;$$

5.
$$\sum_{(g),(h)} h_{(1)}S(g_{(1)}fh_{(2)})g_{(2)} = \varepsilon(gh)S(f);$$

6.
$$\sum_{(h)} (1 \otimes S(h_{(1)})h_{(2)}) \Delta S(h_{(3)}) = \Delta S(h);$$

7.
$$\sum_{(h)} (1 \otimes S(h_{(3)})h_{(1)}) = \Delta S(h_{(2)}) = (S \otimes S)\Delta(h);$$

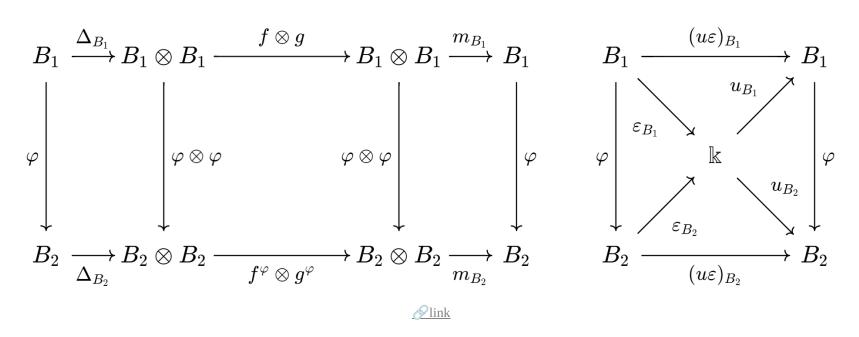
8.
$$\sum_{(h)} h_{(1)} \otimes \ldots \otimes h_{(i-1)} \otimes h_{(i)} S(h_{(i+1)}) \otimes h_{(i+2)} \otimes \ldots \otimes h_{(n)} = \sum_{(h)} h_{(1)} \otimes \ldots \otimes h_{(n-2)};$$

9.
$$\sum_{(h)} h_{(1)} \otimes \ldots \otimes h_{(i-1)} \otimes S(h_{(i)}) h_{(i+1)} \otimes h_{(i+2)} \otimes \ldots \otimes h_{(n)} = \sum_{(h)} h_{(1)} \otimes \ldots \otimes h_{(i-1)} \otimes \sum_{(h)} h_{(i)} \otimes \ldots \otimes h_{(i-1)} \otimes \Delta S(h_{(i)}) \otimes h_{(i+1)} \otimes \ldots \otimes h_{(n-1)}$$

$$= \sum_{(h)} h_{(1)} \otimes \ldots \otimes h_{(i-1)} \otimes S(h_{(i+1)}) \otimes S(h_{(i)}) \otimes h_{(i+2)} \otimes \ldots \otimes h_{(n)}$$

Definition 2.8 (Homomorphisms) We say

ullet $arphi:B_1 o B_2$ is a homomorphism of algebras, whenever



• $\varphi: H_1 \to H_2$ is a homomorphism of Hopf algebras, whenever φ is a homomorphism of bialgebras, and $\varphi \circ \mathcal{S}_{H_1} = \mathcal{S}_{H_2} \circ \varphi$.

Definition 2.9 (Ideals, subalgebra, quotient algebra) Omit.

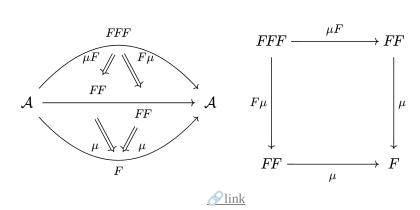
Monads and comonads

Definition 3.1 Let ${\mathcal A}$ be any category, $F:{\mathcal A} o {\mathcal A}$ be an endofunctor with action

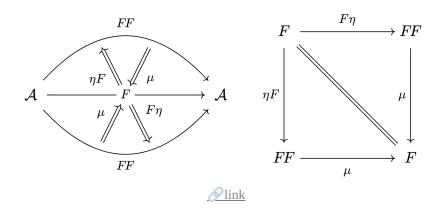
$$arrho_A: F(A) o A, \quad orall A \in \mathsf{Ob}(\mathcal{A}).$$

A monand on ${\mathcal A}$ is defined as a triple (F,μ,η) , where

- $F: \mathcal{A}
 ightarrow \mathcal{A}$ is an endomorphism;
- $\mu:FF o F$ is a natural transformation s.t. $\mu\circ(\mu F)=\mu\circ(F\mu):FFF o F$;



• $\eta: \mathrm{id}_\mathcal{A} o F$ is also a natural transformation s.t. $\mu \circ (\eta F) = \mu \circ (F\eta): F o F$.



One can see that

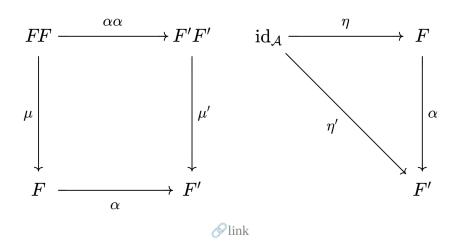
- F as $\mathfrak{A}\otimes_R -: {}_R\mathcal{M} o {}_R\mathcal{M}$ (\mathfrak{A} is an R-algebra);
- $arrho_M: \mathfrak{A}\otimes_R M o M$ is an action on left R module;
- $\mu:\mathfrak{A}\otimes_R\mathfrak{A}\otimes_R- o\mathfrak{A}\otimes_R-$ satisfies the associativity of algebra;
- $\eta: \mathrm{id} \cong R \otimes_R o \mathfrak{A} \otimes_R -$ is the unit of the monad.



Remark Here we analysis the functor category

$$\mathrm{Fuc}_{(\mathcal{A},\mathcal{A})} = (\mathsf{Ob}(\mathrm{Fuc}_{(\mathcal{A},\mathcal{A})}) = \mathsf{Mor}_{(\mathcal{A},\mathcal{A})},\, \mathsf{Mor}(\mathrm{Fuc}_{(\mathcal{A},\mathcal{A})}) = \mathsf{Nat}_{(\mathcal{A},\mathcal{A})}).$$

Definition 3.2 We say the natural transformation lpha:F o F' is a morphism of monads whenever



Fact 3.3 The free functor is always a left adjoint of forgetful functor, e.g.,

• the natural transformation $\mu: FF(-) \to F(-)$ ($\mu_A: FF(A) \to F(A)$) induces an F-module structure, where the free functor is

$$F.: \mathcal{A}
ightarrow \mathcal{A}_F, \quad A \mapsto (F(A), \mu_A), [X \stackrel{f}{
ightarrow} Y] \mapsto [(F(X), \mu_X) \stackrel{(Ff, \mu_f)}{
ightarrow} (F(Y), \mu_Y)];$$

ullet the right adjoint of F. is $U_F: {\mathcal A}_F o {\mathcal A}$, defined by

$$\mathsf{Mor}_{\mathcal{A}_F}(F.(A),B)\stackrel{\sim}{ o} \mathsf{Mor}_{\mathcal{A}}(A,U_F(B)), \quad f\mapsto f\circ \eta_A.$$

For a more trivial example, let R be a ring and $X\in \mathsf{Ob}(\mathbb{A} G)$, then $R\otimes X$ is a left R module defined by the action

$$m \otimes \mathrm{id} : R \otimes (R \otimes X) o R \otimes X, r_1 \otimes r_2 \otimes X \mapsto r_1 r_2 \otimes X.$$

The functor $R\otimes -: \mathbb{A}G o {}_R\mathcal{M}$ is free with right adjoint $U_R: {}_R\mathcal{M} o \mathbb{A}G$, given by

$$\operatorname{Hom}_R(R\otimes X,{_RM})\stackrel{\sim}{ o} \operatorname{Hom}_{\mathbb{A} G}(X,U_F({_RM})),\quad f\mapsto f\circ (\eta\otimes \operatorname{id}).$$

The inverse of such isomorphism is given by

$$(X \stackrel{h}{ o} M) \mapsto (R \otimes X \stackrel{\operatorname{id} \otimes h}{\longrightarrow} R \otimes M \stackrel{arrho_M}{ o} M, \quad r \otimes x \mapsto rh(x)).$$

Definition 3.4 The **Kleisli category of ring** R is ${}_R \tilde{\mathcal{M}}$, defined as follows

$$\mathsf{Ob} = \mathsf{Ob}(\mathbb{A}G), \quad \mathsf{Mor} = \bigcup_{A,B \in \mathsf{Ob}(_R \, ilde{\mathcal{M}})} \mathrm{Hom}_{_R \, ilde{\mathcal{M}}}(A,B) := \mathrm{Hom}_{\mathbb{A}G}(A,R \otimes B).$$

Here the composition $(\circ_{_{R}\mathcal{\tilde{M}}}$, or simply $\circ)$ is defined by

$$g\circ_{_R ilde{\mathcal{M}}}f:X\stackrel{f}{
ightarrow}R\otimes Y\stackrel{\mathrm{id}_R\otimes g}{\longrightarrow}R\otimes R\otimes Z\stackrel{m\otimes\mathrm{id}_Z}{\longrightarrow}R\otimes Z.$$

The identical map is $\mathrm{id}_X:X o R\otimes X,x\mapsto 1\otimes x$ (1 is the unit of R), where

$$\operatorname{id}_{R\otimes Y}\circ f:X\stackrel{f}{ o} R\otimes Y\stackrel{\operatorname{id}_{R}\otimes\operatorname{id}_{R\otimes Y}}{\longrightarrow} R\otimes R\otimes Y\stackrel{m\otimes\operatorname{id}_{Y}}{\longrightarrow} R\otimes Y,\quad x\mapsto f(x); \ f\circ\operatorname{id}_{X}:X\stackrel{\operatorname{id}_{X}}{ o} R\otimes X\stackrel{\operatorname{id}_{R}\otimes f}{\longrightarrow} R\otimes R\otimes Y\stackrel{m\otimes\operatorname{id}_{Y}}{ o} R\otimes Y,\qquad x\mapsto f(x).$$

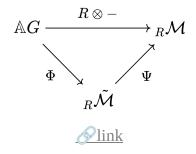
Indeed, the following three functors

$$ullet R\otimes -: \mathbb{A}G o _R \mathcal{M}, \quad X\mapsto R\otimes X, [X\stackrel{f}{ o}Y]\mapsto [R\otimes X\stackrel{\mathrm{id}_R\otimes f}{\longrightarrow} R\otimes Y],$$

$$ullet \Phi: \mathbb{A}G o _R ilde{\mathcal{M}}, X \mapsto X, \quad [X \stackrel{f}{ o} Y] \mapsto [X \stackrel{\operatorname{id}_X}{ o} R \otimes X \stackrel{\operatorname{id}_R \otimes f}{ o} R \otimes Y],$$

$$\Psi:_R ilde{\mathcal{M}}
ightarrow_R \mathcal{M}, \quad X \mapsto R \otimes X, \ [X \stackrel{f}{
ightarrow} R \otimes Y] \mapsto [R \otimes X \stackrel{\operatorname{id}_R \otimes f}{\longrightarrow} R \otimes R \otimes Y \stackrel{m \otimes \operatorname{id}_Y}{\longrightarrow} R \otimes Y],$$

satisfies the commutative diagram



Definition 3.5 The Kleisli category of a monad is similar, i.e.,

$$\bullet \ \ \Phi: \mathcal{A} \to \tilde{\mathcal{A}}_F, \quad X \mapsto (X_F, \mu_X), [X \xrightarrow{f} Y] \mapsto [X \xrightarrow{\eta_X} F(X) \xrightarrow{Ff} F(Y)],$$

$$\bullet \ \ \Psi: \tilde{\mathcal{A}}_F \to \mathcal{A}_F, \quad X \mapsto FX, [X \xrightarrow{f} FY] \mapsto [F(X) \overset{Ff}{\to} FFY \overset{\mu_Y}{\to} FY],$$

• $F.: \mathcal{A}
ightarrow \mathcal{A}_F$ such that $\Psi \Phi = F.$.

Definition 3.6 rewrite **Definition 3.1-Definition 3.5** for coalgebra.

Definition 3.7 ($\mathcal A$ and $\mathcal B$ are categories) For $L:\mathcal B\to\mathcal A$ and $R:\mathcal A\to\mathcal B$, we say $L\dashv R$ (L is a **left adjoint** of R), whence

$$ho: \operatorname{Hom}_{\mathcal{A}}(LX,Y) \cong \operatorname{Hom}_{\mathcal{B}}(X,RY), \quad orall X \in \operatorname{\mathsf{Ob}}(\mathcal{B}), Y \in \operatorname{\mathsf{Ob}}(\mathcal{A}).$$

Here we also have $L \vdash R$, i.e., R is a **right adjoint** of L. Here there exists a natural transformation

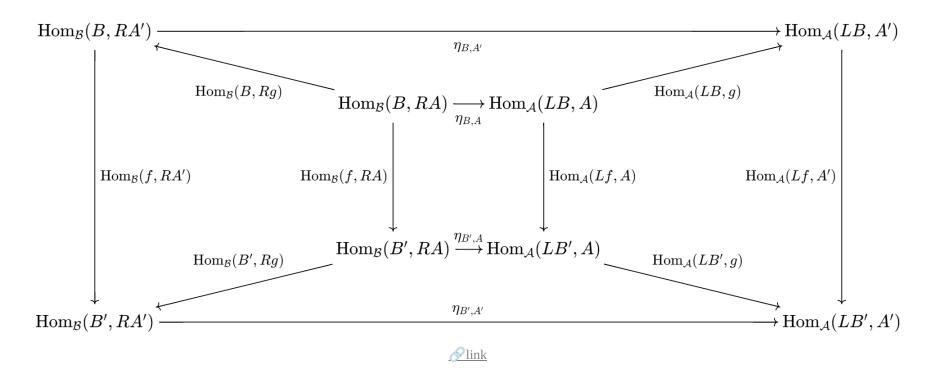
$$\begin{array}{c|c} \mathsf{Mor}_{\mathcal{B}}(B,R(-)) & \xrightarrow{\Phi_B} & \mathsf{Mor}_{\mathcal{A}}(LB,-) \\ \\ \mathsf{Mor}_{\mathcal{B}}(f,R(-)) & & & & \mathsf{Mor}_{\mathcal{A}}(Lf,-) \\ \\ \mathsf{Mor}_{\mathcal{B}}(B',R(-)) & \xrightarrow{\Phi_{B'}} & \mathsf{Mor}_{\mathcal{A}}(LB',-) \\ \\ & & & & & & & & & & & & & & \\ \end{array}$$

Theorem 3.8 R has a left adjoint whenever for each $B\in\mathsf{Ob}(\mathcal{B})$, $\mathsf{Mor}_\mathcal{B}(B,R(-))$ is always representable (by LB), i.e.,

$$ho: \mathsf{Mor}_{\mathcal{A}}(LB,-) \overset{\sim}{ o} \mathsf{Mor}_{\mathcal{B}}(B,R(-)).$$

Moreover, if R has a left adjoint L which is not an isomorphism, then L is unique.

Such isomorphism is also natural for A and B, i.e.,



The pair $(\otimes, \operatorname{Hom})$ induces an adjoint relation, i.e.,

$$\operatorname{Hom}_{\mathbb{A} G}(L\otimes M,N)\stackrel{\sim}{ o} \operatorname{Hom}_{\mathbb{A} G}(M,\operatorname{Hom}_{\mathbb{A} G}(L,N)), \ f\mapsto (m\mapsto f(-\otimes m));$$

$$\operatorname{Hom}_{\mathbb{A} G}(L\otimes M,N)\stackrel{\sim}{ o} \operatorname{Hom}_{\mathbb{A} G}(L,\operatorname{Hom}_{\mathbb{A} G}(M,N)), \ f\mapsto (l\mapsto f(l\otimes -)).$$

Then we define

- $arepsilon_G: H \otimes \operatorname{Hom}_{\mathbb{A} G}(H,G) o G, h \otimes f \mapsto f(h);$
- $ullet \ \eta_G:G o \operatorname{Hom}_{\mathbb{A} G}(H,H\otimes G), g\mapsto -\otimes g,$

to obtain that

- $\bullet \ \, \mathrm{id}_{H\otimes G}: h\otimes g \stackrel{\mathrm{id}_{H}\otimes \eta_{G}}{\longrightarrow} \ \, h\otimes (-\otimes g) \stackrel{\varepsilon_{H\otimes G}}{\longrightarrow} \ \, h\otimes g \text{, that is, } \mathrm{id}_{H\otimes G} = \varepsilon_{H\otimes G}\circ (\mathrm{id}_{H}\otimes \eta_{G});$
- $\bullet \quad \mathrm{id}_{\mathrm{Hom}_{\mathbb{A}G}(H,G)}: f \stackrel{\eta_{\mathrm{Hom}_{\mathbb{A}G}(H,G)}}{\longrightarrow} (-\otimes f) \stackrel{\varepsilon_G}{\longrightarrow} f \text{, that is, } \mathrm{id}_{\mathrm{Hom}_{\mathbb{A}G}(H,G)} = \varepsilon_G \circ (\eta_{\mathrm{Hom}_{\mathbb{A}G}(H,G)}).$

Definition 3.9 In **Fact 3.8**, $\eta:\mathrm{id}_\mathcal{A} o RL$ and $\varepsilon:LR o\mathrm{id}_\mathcal{B}$ are called unit and counit, i.e.,

$$R \leftarrow RE$$
 RLR
 $LRL \leftarrow L\eta$
 L
 $\uparrow \eta R$
 $\downarrow \varepsilon L$
 $\downarrow \varepsilon L$

Then for each adjoint pair (L,R) $(L\dashv R)$, it yields that

- ullet L is full and faithful, whenever $RL\stackrel{\sim}{ o} \mathrm{id}_{\mathcal{B}};$
- ullet R is full and faithful, whenever $LR\stackrel{\sim}{ o} \mathrm{id}_{\mathcal{A}}$

Theorem 3.10 For adjunctions (L, R, η, ϵ) and $(\tilde{L}, \tilde{R}, \tilde{\eta}, \tilde{\epsilon})$ (between categories \mathcal{A} and \mathcal{B}), there is an isomorphism between natural transforms

$$h: \mathsf{Nat}_{(L, ilde{L})} \overset{\sim}{ o} \mathsf{Nat}_{(ilde{R}, R)}, \quad lpha \mapsto \hat{lpha} := R ilde{arepsilon} \circ R lpha ilde{R} \circ \eta ilde{R}.$$

The inverse is given by $lpha=arepsilon ilde{L}\circ L\hat{lpha} ilde{L}\circ L ilde{\eta}.$

Theorem 3.11 Let $F:\mathcal{C} o\mathcal{D}$ and $G:\mathcal{D} o\mathcal{C}$ be quasi-inverse pair, i.e., there exists

- $\eta: \operatorname{id} \stackrel{\sim}{ o} GF$, which is a natural transfromation;
- $arepsilon: FG \overset{\sim}{ o} \mathrm{id}$, which is a natural transformation.

Then we claim that (F,G) is an adjoint pair. Let η be unit without the loss of generality, then

$$arepsilon' := \mathit{FG} \overset{\mathit{FG}arepsilon^{-1}}{\longrightarrow} \mathit{FGFG} \overset{\mathit{F}\eta^{-1}G}{\longrightarrow} \mathit{FG} \overset{arepsilon}{\to} \mathrm{id}$$

is a well defined counit. One can verity $\operatorname{id}: F \overset{F\eta}{\to} FGF \overset{\varepsilon'F}{\to} F$ and $\operatorname{id}: G \overset{\eta G}{\longrightarrow} GFG \overset{G\varepsilon'}{\longrightarrow} G$.

Fact 3.11 For adjoint pair (F,G), F-module $\mathcal A$ is isomorphic to some induced G-comodule $\mathcal A$, i.e., $\mathcal A_F\cong\mathcal A^G$, since for each $A\in\mathsf{Ob}(\mathcal A_F)$, we have

$$A \overset{\eta_A}{
ightarrow} GFA \overset{G_{arrho_A}}{\longrightarrow} GA; \quad FA \overset{F_{arrho}^A}{\longrightarrow} FGA \overset{arepsilon_A}{
ightarrow} A.$$

As for morphisms, we have,

The converse (comonand \mathcal{A}^G implies monand \mathcal{A}_F) also holds (by simple verifying), thus each adjoint pair produce a pair of isomorphic monand and comonand, i.e., $\mathcal{A}_F \cong \mathcal{A}^G$.

However, on can only obtian the equivalence between $\tilde{\mathcal{A}}_G$ and $\tilde{\mathcal{A}}^F$. In comparision of **Definition 3.4**, we introduce

$$G.: \mathcal{A}
ightarrow \mathcal{A}^G, \quad A \mapsto (G(A), \delta_A), [X \stackrel{f}{
ightarrow} Y] \mapsto [(F(X), \delta_X) \stackrel{(Ff, \delta_f)}{
ightarrow} (F(Y), \delta_Y)].$$

As $\mathsf{Ob}(ilde{\mathcal{A}}_G)\cong \;\mathsf{Ob}(ilde{\mathcal{A}}^F)$, we only need to verify the morphisms. $orall A,A'\in\mathsf{Ob}(\mathcal{A})$, i.e.,

$$\mathsf{Mor}_{ ilde{\mathcal{A}}_G}(A,A') \cong \mathsf{Mor}_{\mathcal{A}_G}(G.A,G.A') \cong \mathsf{Mor}_{\mathcal{A}}(A,GA') \ \cong \mathsf{Mor}_{\mathcal{A}}(FA,A') \cong \mathsf{Mor}_{\mathcal{A}^F}(F.A,F.A') \cong \mathsf{Mor}_{ ilde{\mathcal{A}}^F}(A,A')$$