Ex.1 使用球平均法解决Cauchy问题

$$\begin{cases}\partial_{tt}u-\sum_{i=1}^{2n+3}\partial_{x_jx_j}u=0\\ t=0:u=\varphi(x),u_t=\psi(x)\end{cases}$$

解: 记

$$M_u(t,x,r)=\int_{\partial B_{2n+3}(x,r)}u(t,y)\mathrm{d}S_y$$

由于$\mathbb{R}^{2n+3}$中径向函数之Laplacian满足$\Delta=r^{-(2n+2)}\partial_r(r^{2n+2}\partial_r)$, 故原问题转化为

$$\begin{cases}\partial_{tt}M_u=r^{-(2n+2)}\partial_r(r^{2n+2}\partial_r)M_u\\ t=0:M_u=\int_{\partial B_{2n+3}(x,r)}\varphi(y)\mathrm{d}S_y\\ t=0:\partial_tM_u=\int_{\partial B_{2n+3}(x,r)}\psi(y)\mathrm{d}S_y\end{cases}$$

下仅需证明, 存在算子$\mathcal{A}_n(r,\partial_r)$使得$\mathcal{A}_nr^{-(2n+2)}\partial_r(r^{2n+2}\partial_r)\mathcal{A}_n^{-1}=\partial_{rr}$.

设$\mathcal{B}_nr^{2n+1}=\mathcal{A}_n$, 则$\mathcal{B}_n(\partial_{rr}-2n/r\cdot\partial_r)\mathcal{B}_n^{-1}=\partial_{rr}$. 注意到递推式

$$(r^{-1}\partial_r)(\partial_{rr}-2(n+2)/r\cdot\partial_r)=(\partial_{rr}-2n/r\cdot\partial_r)(r^{-1}\partial_r)$$

从而

$$[(r^{-1}\partial_r)^nr^{2n+1}](r^{-(2n+2)}\partial_r(r^{2n+2}\partial_r))[(r^{-1}\partial_r)^nr^{2n+1}]^{-1}=\partial_{rr}.$$

令$[(r^{-1}\partial_r)^nr^{2n+1}]M_u=v$, 则PDE化为

$$\begin{cases}\partial_{tt}v=\partial_{rr}v\\ t=0:v=[(r^{-1}\partial_r)^nr^{2n+1}]\int_{\partial B_{2n+3}(x,r)}\varphi(y)\mathrm{d}S_y\\ t=0:\partial_tv=[(r^{-1}\partial_r)^nr^{2n+1}]\int_{\partial B_{2n+3}(x,r)}\psi(y)\mathrm{d}S_y\end{cases}$$

解得(不妨限定$r<t$)

$$
\begin{aligned}
v =&\frac{[((t+r)^{-1}\partial_{t+r})^n(t+r)^{2n+1}]\int_{\partial B_{2n+3}(x,r+t)}\varphi(y)\mathrm{d}S_y}{2}\\
&-\frac{[((t-r)^{-1}\partial_{t-r})^n(t-r)^{2n+1}]\int_{\partial B_{2n+3}(x,r-t)}\varphi(y)\mathrm{d}S_y}{2}\\
&+\frac{1}{2}\int_{t-r}^{t+r}(\xi^{-1}\partial_\xi)^n\xi^{2n+1}\int_{\partial B_{2n+3}(x,\xi)}\psi(y)\mathrm{d}S_y\mathrm{d}\xi
\end{aligned}
$$

当$r \ll 1$时有

$$
v = r^{2n+1}\partial_t((t^{-1}\partial_t)^n t^{2n+1}\int_{\partial B_{2n+3}(x,t)}\varphi(y)\mathrm{d}S_y) + r^{2n+1}(t^{-1}\partial_t)^n t^{2n+1}\int_{\partial B_{2n+3}(x,t)}\psi(y)\mathrm{d}S_y.
$$

注意到在小范围内, $v\sim kr^{2n+1}$, 且$(r^{-1}\partial_r)^n r^{2n+1}:\dfrac{k}{(2n+1)!!}\mapsto kr^{2n+1}$. 从而

$$
\begin{aligned}
u =&\lim_{r\to 0}\frac{1}{|\partial B_{2n+3}(x,r)|}M_u(t,x,r)\\
=&\lim_{r\to 0}\frac{1}{|\omega_{2n+3}|r^{2n+2}}\cdot\frac{1}{(2n+1)!!}\cdot\partial_t((t^{-1}\partial_t)^n t^{2n+1}\int_{\partial B_{2n+3}(x,r)}\varphi(y)\mathrm{d}S_y)\\
&+\frac{1}{|\omega_{2n+3}|r^{2n+2}}\cdot\frac{1}{(2n+1)!!}\cdot(t^{-1}\partial_t)^n t^{2n+1}\int_{\partial B_{2n+3}(x,r)}\psi(y)\mathrm{d}S_y\\
=&\frac{1}{(2n+1)!!|\omega_{2n+3}|}\partial_t((t^{-1}\partial_t)^n t^{2n+1}\int_{\partial B_{2n+3}(0,1)}\varphi(x+t\xi)\mathrm{d}S_\xi)\\
&+\frac{1}{(2n+1)!!|\omega_{2n+3}|}(t^{-1}\partial_t)^n t^{2n+1}\int_{\partial B_{2n+3}(0,1)}\psi(x+t\xi)\mathrm{d}S_\xi
\end{aligned}
$$

其中$(2n+1)!!|\omega_{2n+3}| = 2^{n+2}\pi^{n+1}$. 综上

$$
\begin{aligned}
u =&\frac{1}{2^{n+2}\pi^{n+1}}\Bigg[\partial_t((t^{-1}\partial_t)^n t^{2n+1}\int_{\partial B_{2n+3}(0,1)}\varphi(x+t\xi)\mathrm{d}S_\xi)\\
&+(t^{-1}\partial_t)^n t^{2n+1}\int_{\partial B_{2n+3}(0,1)}\psi(x+t\xi)\mathrm{d}S_\xi\Bigg].
\end{aligned}
$$