Тихонов's Product Theory

A Straightforward Proof

The Main Theorem. Let $(X_i)_{i\in I}$ be a family of compact spaces. Then the product space $\prod_{i\in I} X_i$ is compact. Here **A9** is admitted when necessary.

Proof. For the sake of contradiction, assume there exists a family of open subsets $\mathcal{U}$ covers $\prod_{i\in I} X_i$ with no finite subcovers. We denote $\prod_{i\in J} X_i =: X_J$.

Step 1. Let $\pi_{I_0} : X_I \to X_{I_0}$ be a **projection map** for $I_0 \subset I$. The transitive map is defined as

$$\pi_{I_1,I_2} := \pi_{I_1}^{-1} \circ \pi_{I_2} : X_{I_1} \to X_{I_2}.$$

The transitive map is well defined when $I_2 \subset I_1 \subset I$. Let $(P, \subset)$ denotes pratially ordered set of all X_J ($J \subset I$).

Step 2. We call $p \in X_J \in P$ a **bad point** whenever for each neighbourhood U_p of p, no finite subsets of $\mathcal{U}$ cover $\pi_J^{-1}(U_p)$. When p and q are in disjoint X_{I_1} and X_{I_2} ($I_1 \cap I_2 = \emptyset$), we defind $p \vee q$ as a concatenation.

Step 3. Let $\mathbf{B}$ denote the set of all bad points, define $p, q \leqq p \vee q$ in $\mathbf{B}$. We claim that $\mathbf{B}$ is *downward closed* for $\leqq$, that is, $q \in \mathbf{B}$ implies $p \in \mathbf{B}$ for each $p \leqq q$. (It seems trivial , **Ex1**). $\mathbf{B}$ is non-empty since $\emptyset \in \mathbf{B}$ by assumption (X_I has no finite subcover).

Step 4. Let $p \in X_J \in P$ be a bad point with $J \subsetneq I$. We claim that for each $i_J \in I \setminus J$, there exists $a \in X_{\{i_J\}}$ such that $p \vee a \in \mathbf{B}$.

Assume that $p \vee a \notin \mathbf{B}$ for each $a \in X_{\{i_J\}}$. Then there exists a neighbourhood $V_{p\vee a}$ such that $\pi_{J\cup\{i_J\}}^{-1}(V_{p\vee a})$ can be covered by a finite subset $\mathcal{U}' \subset \mathcal{U}$. Without the loss of generality, let $V_{p\vee a} = O_p \times W_a \in X_J \times X_{\{i_J\}}$. Since $X_{\{i_J\}}$ is compact, one can find $\{a_k\}_{k=1}^N \in X_{\{i_J\}}$ such that $X_{\{i_J\}} = \cup_{k=1}^N W_{a_k}$, $O_p = \cap_{k=1}^N O_{p,a_k}$. Then

$$\pi_J^{-1}(O_p) = \cup_{k=1}^N \pi_{J\cup\{j_I\}}^{-1}(O_p \times W_a) \subset \cup_{i=1}^N \pi_{J\cup\{i_J\}}^{-1}(V_{a_k}).$$

As a result, $\pi_J^{-1}(O_p)$ can be covered by finite many subsets in $\mathcal{U}$, which contradicts $p \in \mathbf{B}$.

Step 5. We shall show that each chain $\mathbf{C}$ in $\mathbf{B}$ has an upper bound in $\mathbf{B}$. Indeed, $\vee\mathbf{C} := \vee\{q \mid q \in \mathbf{C}\}$ is well defined, we shall show that $(\vee\mathbf{C}) \subset \mathbf{B}$. Denote $\{p\} := \vee\mathbf{C}$.

Let V be any neighbourhood of $p \in X_J$. By definition of product topology, we assume that $V = \pi_{J,F}^{-1}(W)$ where $F \subset J$ is finite and W is open in X_F. One can find $q_0 \in \mathbf{C}$ such that $p|_F \leqq q_0$, which implies that $p|_F \in \mathbf{B}$. As a result, $\pi_F^{-1}(W) = \pi_J^{-1}(V)$ cannot be covered by finitely many subsets in $\mathcal{U}$. Therefore, $p \in \mathbf{B}$.

Step 6. In light of Zorn's lemma, there exists $p \in X_I$ such that $p \in \mathbf{B}$ by **Step 5.**, that is, p cannot be covered by finite subfamily of opensets in $\mathcal{U}$.

$\square$

Remark. Main idea of proof: define the set of bad points $\mathbf{B}$ (non-empty) and utilise the Zorn's lemma to deduce the maximum of $\mathbf{B}$, thus leads to a contradiction.

Critical graphs are finite graphs

Definiton 1. We call $G = G(V, E)$ a **simple graph** whenenver V is a set and $E \subset \{\{x, y\} \mid x, y \in V, x \neq y\}$. Here V (or E) is the set of vertices (or edges).

Simple graph is always *unweighted, undirected, without self-loops and multi-edges*.

Definition 2. Let G be a simple graph. G is **k-colourable on vertices** whenever there exists a function $f \in \{1, 2, \ldots, k\}^V$ s.t. $f(x) \neq f(y)$ when $\{x, y\} \in E$.

Definition 3. The **minimal number for vertex colouring** is the minimal positive integer $k_{\min} =: \chi(G)$, such that G is $k_{\min}$-colourable on vertices.

Definition 4. The **vertex-deleted graphs** of G are in the form of

$$G_{x_0}(V', E') := G_{x_0}(\{x \in V \mid x \neq x_0\}, \{\{x, y\} \in E \mid x, y \neq x_0\}).$$

Informally speaking, G_{x_0} is obtained from G by deleting x_0 and all edges connecting to it.

Definition 5. We call a simple graph G **critical** whenever $\chi(G) < \infty$ and

$$\sup_{x \in G} \chi(G_x) = \max_{x \in G} \chi(G_x) < \chi(G).$$

Main problem. Proves that **all critical graphs are finite graphs**, that is, $|V| < \infty$ for each critical graph $G(V, E)$.

> *The axiom of chioce is required when necessary.*

The main problem is a straight corollary of the following **marvelous theorem**.

De Bruijn–Erdős theorem. Let $G(V, E)$ be a infinite graph, that is, $|V| \geq |\mathbb{N}|$. If $\chi(F) \leq k$ for each finite subgraph F in G, then $\chi(G) \leq k$.

Proof. Let $S = \{1, 2, \ldots, k\}$, $X = k^{V(G)}$ be the product topology space (X, τ). Let $\mathcal{F} \subset \tau$ denotes all possible schemes of colouring of every finite subgraphs in G. Then $\mathcal{F}$ consists of closed sets in (X, τ) and each finite subset in $\mathcal{F}$ has non-empty intersection. Since $k^{V(G)}$ is compact, $\cap \mathcal{F}$ is nonempty (by **FICC.**).

$\square$

Finite intersection criterion of compact sets (FICC). Let X be a compact topology space, $\mathcal{F}$ be a familiy of closed subsets of X. Then $\cap\mathcal{F} \neq \emptyset$ whenever each finite subfamily of $\mathcal{F}$ has non-empty intersection.

Proof. It is just the opposite of Heine-Borel covering theorem, which is trivial.

□

Hall's marriage theorem

Hall's marriage theorem (oral edition). V_1 表示 A 性别的光棍, V_2 表示 B 性别的光棍, 然而只有特定的人才有结婚的可能性, 例如 $u \in V_1$ 与 $v \in V_2$ 可结婚若且仅若 u 与 v 存在连边. 记 V_1 与 V_2 之间的连边构成集合 E. 实际上, 常称简单图 $G(V_1 \dot\cup V_2, E)$ 为二部图 (bipartite graph). 对任意 $S \subset V_1$, 记

$$\Gamma(S) = \{v \in V_2 \mid uv \in E \text{ for some } u \in V_1\} \subset V_2.$$

即所有留有 S 中 A 性别的光棍可生成的结婚证之最大集合 (商掉时间与效力等属性, 例如一个人可以有多张证). 则 A 性别的光棍可以脱单若且仅若 $|\Gamma(S)| \geq |S|$ 对一切 $S \in \mathcal{P}(V_1)$ 恒成立.

Proof. Finite case. When V_1 is finite, we assume S_2 is finite without the loss of generality. Then we shall analysis the following two cases.

Case I. Suppose that for all proper subspace $S \neq \emptyset$ of V_1, $|\Gamma(S)| \geq |S| + 1$. Then for arbitrary $e = uv \in E$, the vertex-deleted graph $G_{u,v}$ still satisfies Hall's condition.

Case II Suppose that for some proper subset $S_0 \neq \emptyset$ of V_1, $|\Gamma(S_0)| = |S_0|$. Then the induces subgraph $G(S_0 \dot\cup \Gamma(S_0))$ and $G((V_1 \setminus S_0) \dot\cup (V_1 \setminus \Gamma(S_0)))$ satisfies Hall's condition.

> *Either is trivial, or is trivial.*

Trivial by mathematical induction.

Infinite case (The Тихонов's Product Theory is required, left as **Ex2**).

□