Contents

1 What is Hopf algebra

$$
\begin{tikzcd}
\text{Tensor-type definitions} \arrow[d, dashed, tail] \arrow[dr, dashed, tail, bend left] & & \text{Sweedler's notation} \arrow[dl, dashed] & \\
\text{Algebra} \arrow[r, leftrightarrow, "\text{dual analog}"'] \arrow[rr, tail, bend left] & \text{Coalgebra} \arrow[r, tail] & \text{Bialgebra} \arrow[r, tail] & \text{Hopf Algebra} \\
\text{unit} \arrow[u, tail] & \text{counit} \arrow[u, tail] & \text{additional conditions} \arrow[u, tail] & \text{antipode } \mathcal{S}*\mathrm{id}_B = u\circ\epsilon \arrow[u, tail]
\end{tikzcd}
$$

$$(A, m, u) \qquad (C, \Delta, \epsilon) \qquad \left(B, \begin{matrix} m, u \\ \Delta, \epsilon \end{matrix}\right) \qquad \left(H, \begin{matrix} m, u \\ \Delta, \epsilon \end{matrix}, \mathcal{S}\right)$$

1.1 Algebra and Coalgebra

Def 1. Let k be a **commutative ring**.

Def 2. An **k-algebra** A is

1. a **ring** $(A, +, \cdot)$ with multiplicative identity e_A.
2. an **k-vector space** with k-linear structure on $(A, +)$;
3. $\forall a, b \in A$ and $\forall \lambda \in k$, we have

$$(\lambda a)b = \lambda(ab) = a(\lambda b) \quad =: \lambda ab.$$

Fact 1. As we observe that

$$a \cdot (\lambda b) = (a\lambda) \cdot b = \lambda(ab),$$

multiplication $\cdot : A \times A \to A$ can be substituted by

$$m : A \otimes A \to A, a \otimes b \mapsto c$$

when A is an k-algebra. Here $\otimes$ is a k-**bilinear** map known as tensor product.

Def 3. An ***k*-bilinear map** is a naïve map from Cartesian product (product) of two ***k*-linear spaces** $(A \times B)$ to a **Abelian group** (G), such that

1. $f(a + a', b) = f(a, b) + f(a', b)$, $\forall a', a \in A$ and $b \in B$;
2. $f(a, b + b') = f(a, b) + f(a, b')$, $\forall a \in A$ and $\forall b, b' \in B$;
3. $f(a\lambda, b) = f(a, \lambda b)$, $\forall a \in A$, $\forall b \in B$ and $\forall \lambda \in k$.

Def 4. The **tensor product** $A \otimes_k B$ is defined by universal property as follows

$$\begin{tikzcd} & G & \\ A \times B \arrow[ur, "f"] \arrow[rr, "\varphi"'] & & A \otimes_k B \arrow[ul, "\exists | g"'] \end{tikzcd} \quad (\text{bilinear}) .$$

That is, for each bilinear map from $A \times B$ to $G \in (\mathbb{A}G)_0$ (Abelian group), there exists exactly $g \in (\mathbb{A}G)_1$ (group homomorphism) such that $g \circ \varphi = f$, where $\varphi : A \times B \to A \otimes_F B$ is canonical.

Rmk 1. Here $\mathbb{A}G$ is the category of Abelian groups, where $(-)_0$ (resp. $(-)_1$) is the set of objects (resp. morphisms).

Fact 2. (**Equivalent definition of tensor product.**) It may be much more comprehensible and concise to define $A \otimes_k B$ as the **quotient *k*-space** $A \times B/\sim$, where

- $(a + a', b) \sim (a, b) + (a', b)$, $\forall a', a \in A$ and $b \in B$;
- $(a, b + b') \sim (a, b) + (a, b')$, $\forall a \in A$ and $\forall b, b' \in B$;
- $(ak, b) \sim (a, kb)$, $\forall a \in A$, $\forall b \in B$ and $\forall k \in k$.

Rmk 2. The unadorned $A \otimes B$ is over k when there is no ambiguity.

Thm 1. Tensor products are associative, i.e.,

$$U \otimes (V \otimes W) \xrightarrow{\sim} (U \otimes V) \otimes W,$$
$$u \otimes (v \otimes w) \mapsto (u \otimes v) \otimes w.$$

We omit its proof here.

Rmk 3. One can see $U \otimes V \otimes W$ is well-defined in any algebraic/coalgebraic structure.

Def 5. Here is another approach to describe algebras. An k-algebra is a triple (A, m, u) satisfying the following statements

1. A is an k-vector space;
2. $m : A \otimes A \to A$ is an associative k-linear map such that the diagram on the left hand side commutes;
3. the **unit** $u : k \to A, 1 \mapsto e_A$ is a k-linear map such that the diagram on the right hand side commutes.

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "m\circ(m\otimes \mathrm{id}_A)"] \arrow[d, "m\circ(\mathrm{id}_A\otimes m)"] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \arrow[r, "\mathrm{id}_A\otimes u"] \arrow[d, "u\otimes \mathrm{id}_A"] \arrow[dr, "m"] & A \otimes k \arrow[d, equal, "(a\otimes 1)\cong a"] \\
k \otimes A \arrow[r, equal, "(1\otimes a)\cong a"'] & A
\end{tikzcd}$$

Def 6. We say A is **reflexive** whenever the following canonical injection is an isomorphism, i.e.,

$$A \overset{\sim}{\to} \mathrm{Hom}_k(\mathrm{Hom}_k(A,k),k), a \mapsto [f \mapsto f(a)].$$

E.g. 1. Let D be an contravariant endofunctor on kAlg, such that $DA = \mathrm{Hom}_k(A,k)$. Then we obtain (DA, Dm, Du) s.t.,

1. the composition

$$\Delta' := DA \overset{Dm}{\to} D(A \otimes A) \overset{\rho}{\cong} DA \otimes DA, f \mapsto f_1 \otimes f_2$$

induces

$$(k \ni) \quad f(m(a \otimes b)) \cong_\rho f_1(a) \otimes f_2(b) \quad (\in k \otimes k);$$

2. the composition

$$\epsilon' : DA \overset{Du}{\to} Dk \overset{\psi}{\to} k, f \mapsto \lambda \,\mathrm{id}_k$$

induces

$$k \ni \quad f(u(k)) \cong_\psi \lambda \,\mathrm{id}_k(k) = \lambda k \quad \in k$$

Def 7. The triple $(Da, \rho \circ Du, \psi \circ Du)$ in the example above is actually a well defined coalgebra. In general, a **coalgebra** is a triple (C, Δ, ϵ) satisfying that

1. C is a k-vector space;
2. $\Delta : C \to C \otimes C$ is an **coassociative** k-linear map such that the diagram on the left hand side commutes;

3. the **counit** $\epsilon : C \to k$ is a k-linear map such that the diagram on the right hand side commutes.

$$
\begin{tikzcd}
C \arrow[rr, "\Delta"] \arrow[d, "\Delta"'] & & C \otimes C \arrow[d, "(\Delta \otimes \mathrm{id}_C) \circ \Delta"'] \\
C \otimes C \arrow[rr, "(\mathrm{id}_C \otimes \Delta) \circ \Delta"'] & & C \otimes C \otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, equal, "c \cong 1 \otimes c"] \arrow[d, equal, "c \cong c \otimes 1"'] \arrow[dr, "\Delta"] & k \otimes C \\
C \otimes k & C \otimes C \arrow[l, "\mathrm{id}_C \otimes \epsilon"'] \arrow[u, "\epsilon \otimes \mathrm{id}_C"']
\end{tikzcd}
$$

Rmk 4. **Sweedler's notation** is some what similar to Einstein notation or christoffel symbols to some extent. For instance, one can write

$$\Delta(c) = \sum_{c_i, c_j \in \text{ basis of } C} \lambda_{i,j}\, c_i \otimes c_j =: \sum_{(c)} c_{(1)} \otimes c_{(2)}$$

for simplicity. The following identities are direct

- (definition of counit)

$$c \otimes 1 = \sum_{(c)} c_{(1)} \otimes \epsilon(c_{(2)}), \quad 1 \otimes c = \sum_{(c)} \epsilon(c_{(1)}) \otimes c_{(2)},$$

- (coassociativity)

$$\textstyle\sum_{(c)} (c_{(1)})_{(1)} \otimes (c_{(1)})_{(2)} \otimes c_{(2)} = \sum_{(c)} c_{(2)} \otimes (c_{(2)})_{(1)} \otimes (c_{(2)})_{(2)},$$

- (flip action)

$$\tau : \sum_{(c)} c_{(1)} \otimes c_{(2)} \to \sum_{(c)} c_{(2)} \otimes c_{(1)}.$$

Rmk 5. The unit and counit are **unique**.

- In category of $k\mathrm{Alg}$, the initial object k together with $\mathrm{Hom}_{k\mathrm{Alg}}(k, -)$ defines a unit. The uniqueness is clear.

- In category of kCoAlg, the counit is given by definiton. For uniqueness, if ϵ and ϵ' are both counit, then

$$\epsilon(c) = \epsilon\left(\sum_{(c)} c_{(1)}\epsilon'(c_{(2)})\right) = \sum_{(c)} \epsilon(c_{(1)})\epsilon'(c_{(2)}).$$

Thm 2. For **ANY** coalgebra (C, u, ϵ), the canonical injection induces

$$m' : DC \otimes DC \overset{\rho}{\hookrightarrow} D(C \otimes C) \xrightarrow{D\Delta} DC.$$

The unit is given by $u' : k \xrightarrow{\psi} Dk \xrightarrow{D\epsilon} DC, \lambda \mapsto \lambda\epsilon$.

The dual of algebra A induces an coalgebraic structure whenever $(Du \otimes Dv) \cong D(u \otimes v)$ is canonical isomorphic (e.g., A is reflexive).

Def 8. The **algebraic (resp. coalgebraic) homomorphisms** are morphisms in kAlg (resp. kCoAlg), i.e.,

- $f \in \mathrm{Hom}_{kA}(A_1, A_2)$ whence the following diagrams commutes;

$$\begin{tikzcd} A_1 \otimes A_1 \arrow[r, "m_1"] \arrow[d, "f\otimes f"'] & A_1 \arrow[d, "f"] \\ A_2 \otimes A_2 \arrow[r, "m_2"] & A_2 \end{tikzcd} \qquad \begin{tikzcd} k \arrow[r, "u_1"] \arrow[dr, "u_2"'] & A_1 \arrow[d, "f"] \\ & A_2 \end{tikzcd}$$

- $g \in \mathrm{Hom}_{kC}(C_1, C_2)$ whence the following diagrams commutes.

$$\begin{tikzcd} C_1 \arrow[r, "\Delta_1"] \arrow[d, "g"'] & C_1 \otimes C_1 \arrow[d, "g\otimes g"] \\ C_2 \arrow[r, "\Delta_2"] & C_2 \otimes C_2 \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, "\epsilon_1"] \arrow[d, "g"] & k \\ C_2 \arrow[ur, "\epsilon_2"'] & \end{tikzcd}$$

Def 9. **Subalgebra** and **subcoalgebra** are defined as follows:

- (A', m', u') is a subalgebra of (A, m, u) whenever
  1. A' is an k-linear subspace of A,
  2. m' equals m on A',
  3. u' equals u on k, thus $e_A = e_{A'}$;
- (C', Δ', ϵ') is a subalgebra of (C, Δ, ϵ) whenever
  1. C' is an k-linear subspace of C,
  2. Δ' equals Δ on C',
  3. ϵ' equals ϵ on k.

Def 10. The definition of **ideal** and **coideal** are as follows

- I is an (two-sided) ideal of an algebra A, whenever $m : A \otimes I + I \otimes A \to I$;
- $I \subset C$ is an coideal of an algebra C, whenever $\Delta : C \to C \otimes I + I \otimes C, \quad \epsilon : I \to 0.$

1.2 Bialgebra and Hopf algebra

Def 11. We say a pentuple $(B, m, u, \Delta, \epsilon)$ is a **bialgebra** whenever the following three statements holds

- (B, m, u) is an algebra, and (B, Δ, ϵ) is a coalgebra; Thus $B \otimes B$ has both algebra and coalgebra structure;
- one of the following equivalent statements holds:
  1. m and u are both homomorphisms between coalgebras,
  2. Δ and ϵ are both homomorphisms between algebras.

  Here $1 \Leftrightarrow 2$ is shown as follows.

E.g. 2. If m and u are coalgebraic homomorphisms, then Δ and ϵ are homomorphisms between coalgebras.

- $u \in \mathrm{Hom}_{k\mathrm{CoAlg}}(B \otimes B, B)$ since

$$\begin{tikzcd} k \arrow[r, "\mathrm{id}\,(\Delta_k)"] \arrow[d, "u"'] & k \otimes k \arrow[d, "u \otimes u"] \\ B \arrow[r, "\Delta"] & B \otimes B \end{tikzcd} \qquad \begin{tikzcd} k \arrow[r, "\mathrm{id}\,(\epsilon_k)"] \arrow[d, "u"'] & k \\ B \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

- $m \in \mathrm{Hom}_{k\mathrm{CoAlg}}(B \otimes B, B)$, since

$$
\begin{tikzcd}
B \arrow[rr, "\Delta_B"] & & B \otimes B \\
B \otimes B \arrow[u, "m_B"] \arrow[rr, "\Delta_{B\otimes B}"] \arrow[drr, "\Delta_B \otimes \Delta_B"'] & & B\otimes {\color{red}B} \otimes {\color{blue}B} \otimes B \arrow[u, "m_B \otimes m_B"'] \\
& & B\otimes {\color{blue}B} \otimes {\color{red}B} \otimes B \arrow[u, "\mathrm{id}_B \otimes {\color{teal}\tau} \otimes \mathrm{id}_B"']
\end{tikzcd}
\qquad
\begin{tikzcd}
k \arrow[r, "\mathrm{id}\,(\Delta_k)"] \arrow[d, "u"'] & k \otimes k \arrow[d, "u \otimes u"] \\
B \arrow[r, "\Delta_B"] & B \otimes B
\end{tikzcd}
$$

$$
\begin{tikzcd}
b \arrow[rr, mapsto, "\Delta_B"] & & b_{(1)} \otimes b_{(2)} \\
b^{(1)} \otimes b^{(2)} \arrow[u, mapsto, "m_B"] \arrow[rr, mapsto, "\Delta_{B\otimes B}"] \arrow[dr, mapsto, "\Delta_B \otimes \Delta_B"'] & & b^{(1)}_{(1)} \otimes {\color{red}b^{(2)}_{(1)}} \otimes {\color{blue}b^{(1)}_{(2)}} \otimes b^{(2)}_{(2)} \arrow[u, mapsto, "m_B \otimes m_B"'] \\
& b^{(1)}_{(1)} \otimes {\color{blue}b^{(1)}_{(2)}} \otimes {\color{red}b^{(2)}_{(1)}} \otimes b^{(2)}_{(2)} \arrow[ur, mapsto, "\mathrm{id}_B \otimes {\color{teal}\tau} \otimes \mathrm{id}_B"'] &
\end{tikzcd}
$$

Def 12. $\varphi : B \to B'$ is a homomorphism between bialgebras whenever φ is homomorphism for both algebra and coalgebra.

Thm 3. $\mathrm{End}_k(B) = \mathrm{Hom}_k(B)$ has a **ring structure (with multiplicative identity)**, that is,

- $(\mathrm{End}_k(B), +)$ is Abelian group, where
$$+ : (f, g) \mapsto [b \mapsto f(b) + g(b)];$$
- $(\mathrm{End}, *)$ is an semigroup, where
$$* : (f, g) \mapsto m \circ (f \otimes g) \circ \Delta,$$

moreover, the associativity is given by

$$\begin{tikzcd}
B & (B\otimes B)\otimes B \arrow[l, "m\circ(m\circ\mathrm{id})"'] \arrow[r, equal, "\cong"] & B\otimes(B\otimes B) \arrow[r, "m\circ(\mathrm{id}\circ m)"] & B \\
B \arrow[u, "(f*g)*h"] \arrow[r, "(\Delta\otimes\mathrm{id})\circ\Delta"'] & (B\otimes B)\otimes B \arrow[u, "(f\otimes g)\otimes h"'] \arrow[r, equal, "\cong"'] & B\otimes(B\otimes B) \arrow[u, "f\otimes(g\otimes h)"] & B \arrow[u, "(f*g)*h"'] \arrow[l, "(\mathrm{id}\otimes\Delta)\circ\Delta"]
\end{tikzcd}$$

- $u \circ \epsilon : B \to k \to B$ is the multiplicative identity, i.e.,

$$\begin{tikzcd}
& k\otimes B \arrow[rr, "\mathrm{id}_k\otimes f"] & & k\otimes B \arrow[d, "u\otimes\mathrm{id}_B"'] \arrow[dr, equal, "\cong"] & \\
B \arrow[ur, equal, "\cong"] \arrow[dr, equal, "\cong"'] \arrow[r, "\Delta"'] \arrow[rrrr, bend left, dashed, "f"] & B\otimes B \arrow[u, "\epsilon\otimes\mathrm{id}_B"] \arrow[d, "\mathrm{id}_B\otimes\epsilon"] & & B\otimes B \arrow[r, "m"'] & B \\
& B\otimes k \arrow[rr, "f\otimes\mathrm{id}_k"] & & B\otimes k \arrow[u, "\mathrm{id}_B\otimes u"'] \arrow[ur, equal, "\cong"'] &
\end{tikzcd}$$

Therefore, $f * (u \circ \epsilon) = (u \circ \epsilon) * f = f$.

Thm 4. $(f_1 \otimes f_2) \circ (g_1 \otimes g_2) = (f_1 \circ g_1) \otimes (f_2 \circ g_2)$ when both sides are defined, proved by the universal properties in the commutative diagram below

$$\begin{tikzcd}
A\times B \arrow[rrr, "\otimes"] \arrow[dr, "f\times g"] \arrow[dd, "(f'\circ f)\times(g'\circ g)"'] & & & A\otimes B \arrow[dl, "\exists|f\otimes g"'] \arrow[dd, dashed, "\exists|(f'\circ f)\otimes(g'\circ g)"] \\
& A'\times B' \arrow[r, "\otimes"] \arrow[dl, "f'\times g'"] & A'\otimes B' \arrow[dr, "\exists|f'\otimes g'"'] & \\
A''\times B'' \arrow[rrr, "\otimes"] & & & A''\otimes B''
\end{tikzcd}$$

Def 13. $\mathcal{S}$ is an **antipode** whenever

$$\mathcal{S} * \mathrm{id}_B = u \circ \epsilon = \mathrm{id}_B * \mathcal{S}.$$

Here $\mathcal{S}$ is the multiplicative inverse of id_B, i.e.,

$$\begin{tikzcd}
& B \otimes B \arrow[rr, "\mathrm{id}_B\otimes\mathcal{S}"] & & B \otimes B \arrow[dr, "m"] & \\
B \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rrrr, bend left, "\mathrm{id}_B*\mathcal{S}"'] \arrow[rrrr, bend right, "\mathcal{S}*\mathrm{id}_B"] \arrow[rr, "\epsilon"] & & k \arrow[rr, "u"] & & B \\
& B \otimes B \arrow[rr, "\mathcal{S}\otimes\mathrm{id}_B"'] & & B \otimes B \arrow[ur, "m"'] &
\end{tikzcd}$$

The uniqueness is due to

$$\mathcal{S}' = (\mathcal{S} * \mathrm{id}_B) * \mathcal{S}' = \mathcal{S} * (\mathrm{id}_B * \mathcal{S}') = \mathcal{S}.$$

Def 14. A **Hopf algebra** is usually written as a hextuple $(H, m, u, \Delta, \epsilon, \mathcal{S})$, where $(H, m, u, \Delta, \epsilon)$ is a **bialgebra with antipode** $\mathcal{S}$.

Fact 3. Both opposite and coopposite of a Hopf algebra are also a Hopf algebras, where the coopposite one is given by

- $\mathcal{S} : H \to H^{\mathrm{op}}, g \cdot h \mapsto \mathcal{S}(g) \cdot_{\mathrm{op}} \mathcal{S}(h) \mapsto \mathcal{S}(h) \cdot \mathcal{S}(g)$;
- $\mathcal{S}(1) = 1$, $\mathcal{S}(e_H) = \mathrm{id}_H^{-1}(e_H) = e_H$;
- Recall $\Delta_{H\otimes H} = (\mathrm{id}_H \otimes \tau \otimes \mathrm{id}_H) \circ (\Delta_H \otimes \Delta_H)$, we have an $\mathcal{S}$-analog as follows:

$$
\begin{tikzcd}
h \arrow[rr, mapsto, "\Delta_H"] \arrow[dd, mapsto, "\mathcal{S}_H"'] & & h_{(1)} \otimes h_{(2)} \arrow[dl, "\mathcal{S}_{H\otimes H}"'] \arrow[dd, mapsto, "\mathcal{S}_H\otimes\mathcal{S}_H"] \\
& h^{\mathrm{cop}}_{(2)} \otimes h^{\mathrm{cop}}_{(1)} \arrow[dr, mapsto, "\tau"] & \\
h^{\mathrm{cop}} \arrow[ur, mapsto, "\Delta_H"] \arrow[rr, "\Delta_H^{\mathrm{cop}}"] & & h^{\mathrm{cop}}_{(1)} \otimes h^{\mathrm{cop}}_{(2)}
\end{tikzcd}
\qquad
\begin{tikzcd}
H \arrow[rr, "\Delta"] \arrow[dd, "\mathcal{S}_H"'] & & H \otimes H \arrow[dl, "\mathcal{S}_{H\otimes H}"'] \arrow[dd, "\mathcal{S}_H\otimes\mathcal{S}_H"] \\
& H \otimes H \arrow[dr, "\tau"] & \\
H \arrow[ur, "\Delta_H"] \arrow[rr, "\Delta_H^{\mathrm{cop}}"] & & H \otimes H
\end{tikzcd}
$$

Thus $(H^{\mathrm{cop}}, m, u, \Delta^{\mathrm{cop}}, \epsilon, \mathcal{S}^{-1})$ is a well defined Hopf algebra.

$$
\begin{tikzcd}
(H, m, u, \Delta, \epsilon, \mathcal{S}) \arrow[r, dashed, "\mathrm{op}"] \arrow[d, dotted, "\mathrm{cop}"'] & (H^{\mathrm{cop}}, m, u, \Delta^{\mathrm{cop}}, \epsilon, \mathcal{S}^{-1}) \arrow[d, dotted, "\mathrm{cop}"] \\
(H^{\mathrm{op}}, m^{\mathrm{op}}, u, \Delta, \epsilon, \mathcal{S}^{-1}) \arrow[r, dashed, "\mathrm{op}"] & (H^{\mathrm{op,cop}}, m^{\mathrm{op}}, u, \Delta^{\mathrm{cop}}, \epsilon, \mathcal{S})
\end{tikzcd}
$$

Rmk 6. As $(H^{\mathrm{op}}, m^{\mathrm{op}}, u, \Delta, \epsilon, \mathcal{S}^{-1})$ is a Hopf algebra,

$$(H^{\mathrm{op,cop}}, m^{\mathrm{op}}, u, \Delta^{\mathrm{cop}}, \epsilon, \mathcal{S})$$

is also a Hopf algebra.

1.3 An example, $\mathcal{O}(\mathrm{SL}_2)$

E.g. 3. The following example provides a simple but nontrivial commutative Hopf algebra. One can generalise it to $\mathcal{O}(\mathrm{SL}_n)$.

- Consider the **commutative** polynomial ring

  $$\tilde{H} = k[X_{11}, X_{12}, X_{21}, X_{22}], \quad X_{i,j} : k^{2\times 2} \to k, \quad A \mapsto a_{i,j};$$

  Here m and u is defined as usual.

- $\Delta(X_{i,j}) = \sum_t X_{i,t} \otimes X_{t,j}$ is the comultiplication;
  One can verify $\Delta(\det(X_{ij})) = (\det(X_{ij}))\otimes(\det(X_{ij}))$.

- $\epsilon(X_{i,j}) = \delta_{i,j}$ is the counit;
  One can verify $\epsilon(\det(X_{ij})) = 1$.

- one can verify that $\mathcal{S}$ is **undefined**, i.e., there is no $\mathcal{S}$ such that

$$\begin{tikzcd}
& \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \otimes \begin{pmatrix} \mathcal{S}(X_{11}) & \mathcal{S}(X_{12}) \\ \mathcal{S}(X_{21}) & \mathcal{S}(X_{22}) \end{pmatrix} \arrow[d, "m"] \\
\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \arrow[ur, "(\mathcal{S}\otimes\mathrm{id})\circ\Delta"] \arrow[r, "u\circ\varepsilon"] \arrow[dr, "(\mathrm{id}\otimes\mathcal{S})\circ\Delta"'] & \begin{pmatrix} u\circ\varepsilon(X_{11}) & u\circ\varepsilon(X_{12}) \\ u\circ\varepsilon(X_{21}) & u\circ\varepsilon(X_{22}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
& \begin{pmatrix} \mathcal{S}(X_{11}) & \mathcal{S}(X_{12}) \\ \mathcal{S}(X_{21}) & \mathcal{S}(X_{22}) \end{pmatrix} \otimes \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \arrow[u, "m"']
\end{tikzcd}$$

However, if $I := X_{11}X_{22} - X_{12}X_{21} - 1 = 0$ is assumed, then $H := \tilde{H}/I$ is a well-defined Hopf alegbra.

- (m, u) for $\tilde{H}/I$ is naturally defined.

- the comultiplication is also well defined, or equivalently, $I \subset \ker[(\pi \otimes \pi) \circ \Delta]$;

$$\begin{array}{ccc} \tilde{H} & \xrightarrow{\quad \Delta_{\tilde{H}} \quad} & \tilde{H} \otimes \tilde{H} \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi \otimes \pi} \\ \tilde{H}/I & \overset{\Delta_{\tilde{H}/I}}{\dashrightarrow} & \tilde{H}/I \otimes \tilde{H}/I \end{array}$$

Let $t := \det(X_{ij})$. Then $h(t-1) \in I$ for each $h \in \tilde{H}$. It yields that

$$\begin{aligned} \Delta(h(t-1)) &= \sum_{(h)} (h_{(1)} \otimes h_{(2)})(\Delta(t) - 1 \otimes 1) \\ &= \sum_{(h)} (h_{(1)} \otimes h_{(2)})(t \otimes t - 1 \otimes 1) \\ &= \sum_{(h)} (h_{(1)} \otimes h_{(2)})(t \otimes (t-1) + (t-1) \otimes 1) \\ &\in \tilde{H} \otimes I + I \otimes \tilde{H}. \end{aligned}$$

- The counit maps $h(t-1)$ to zero, thus $\epsilon_{\tilde{H}/I}$ is well defined.

The antipode maps

$$\begin{pmatrix} \mathcal{S}(X_{11}) & \mathcal{S}(X_{12}) \\ \mathcal{S}(X_{21}) & \mathcal{S}(X_{22}) \end{pmatrix} = \begin{pmatrix} X_{22} & -X_{12} \\ -X_{21} & X_{11} \end{pmatrix};$$

Rmk 7. One can substitute $k[X_{ij}]$ by $R[X_{ij}]$ for generalisations. Here R is a k-algebra. The commutative case is discussed in the next section; the non-commutative case refers to quantum groups.

2 Group structures induced by commutative Hopf algebras

$$
\begin{tikzcd}
 & \text{affine group} & & \text{Comm. Hopf algebra} \\
\text{Representative functor} \arrow[ur, bend left, "\text{Yoneda}"] & G : k\mathrm{Alg} \to \mathrm{Sets} \arrow[rr, "\cong", "\alpha"'] & & \mathrm{Hom}_{k\mathrm{Alg}}(A, -) \\
\text{Group structure} \arrow[uur, bend left] & (G, m, e, \mathrm{inv}) \arrow[rr, Leftrightarrow, "\sim"] & & (A, \Delta, \epsilon, \mathcal{S})
\end{tikzcd}
$$

2.1 Affine groups

E.g. 4. The example above might provoke some motivational discussion.

- $\mathrm{SL}_2 : k\mathrm{Alg} \to \mathrm{Grp}/\mathrm{Sets}$ is a **covariant functor**.
- SL_2 is **defined by the polynomial** $X_{11}X_{22} - X_{12}X_{21} - 1$.
- A Hopf algebra is somewhat isomorphic to a group.

Def 15. Let $k\mathrm{Alg}$ denote the **category of commutative k-algebras** thenceforth.

Def 16. Take S as a subset of $k[X_1, \ldots, X_n]$ (the set of polynomials). The **functor of points** $S(-)$ is defined as

$$
\begin{aligned}
& S(-) : k\mathrm{Alg} \to \mathrm{Sets}; \\
& \text{(Ob)} \quad R \mapsto \{a \in R^n : f(a) = 0, \forall f \in S\}, \\
& \text{(Mor)} \quad \mathrm{Hom}_{k\mathrm{Alg}}(R, R') \mapsto \text{set transformations.}
\end{aligned}
$$

Rmk 8. The $\mathrm{Hom}_{k\mathrm{Alg}}(k[X_1, \ldots, X_n]/(S), R)$ consists of those in $k[X_1, \ldots, X_n]$ taking zero on $S(R)$. Here (S) is the ideal generated by S.

Fact 4. Let $\mathfrak{a} := (S)$ be an ideal generated by S. $A =: k[X_1, \ldots, X_n]/\mathfrak{a}$ is defined as a **coordinate k-algebra**. The following two functors

$$\begin{array}{ll} k\mathrm{Alg} \to \mathrm{Sets},\ R \mapsto \mathfrak{a}(R) & \{a = (a_1, \ldots, a_n)\}, \\ k\mathrm{Alg} \to \mathrm{Sets},\ R \mapsto \mathrm{Hom}_{k\mathrm{Alg}}(A, R) & \{f/\mathfrak{a} \mapsto f(a)\}. \end{array}$$

are canonically isomorphic.

Rmk 9. It has been a stock issue that what kinds of $F : k\mathrm{Alg} \to \mathrm{Sets}$ are isomorphic to $\mathrm{Hom}_{k\mathrm{Alg}}(A, -)$ for some commutative k-algebra A.

Def 17. (**Not recommended**) An **affine scheme** (over k) is a functor $X : k\mathrm{Alg} \to \mathrm{Sets}$, such that $X(-) \cong \mathrm{Hom}_{k\mathrm{Alg}}(A, -)$ for some A. Here A is the **coordinate ring**, denoted by $\mathcal{O}(X) := A$.

Rmk 10. Meticulously speaking, we should clarify that affine schemes (resp. affine group schemes) and representable functors (resp. affine groups) is defined via a fully faithful functor $A \mapsto \mathrm{Spec}(A)$ (resp. $A \mapsto h^A$), although such two definitions are almost the same.

$$
\begin{array}{ccccc}
h^A & \longleftarrow & A & \longmapsto & \mathrm{Spec}(A) \\
\mathrm{Funct}(k\mathrm{Alg}, \mathrm{Sets}) & \longleftarrow & k\mathrm{Alg}^{\mathrm{op}} & \longrightarrow & \mathrm{Sch}/k \\
\text{representable functors} & \xleftarrow{\sim} & k\mathrm{Alg}^{\mathrm{op}} & \xrightarrow{\sim} & \text{affine schemes} \\
\text{affine groups} & \xleftarrow{\sim} & k\mathrm{CommHopfAlg}^{\mathrm{op}} & \xrightarrow{\sim} & \text{affine group schemes}
\end{array}
$$

E.g. 5. Here are some examples of affine schemes

X	X **as a functor**	**coordinate algebra** A
$\mathbb{A}^n$	$R \mapsto R^n$ as a set	$k[X_1, \ldots, X_n]$
$\mathbb{G}_m$	$R \mapsto R^\times$	$k[X, 1/X] \cong k[x, y]/(XY - 1)$
GL_n	$R \mapsto \mathrm{GL}_n(R)$	$k[X_{ij}, 1/\det((X_{ij})_{n\times n})]$
SL_n	$R \mapsto \mathrm{SL}_n(R)$	$k[X_{ij}]/(\det((X_{ij})_{n\times n}) - 1)$
μ_n	$R \mapsto \{a \in R : a^n = 1\}$	$k[X]/(X^n - 1)$

Qn 1. Some questions occur here.

- Is $\mathrm{Hom}_{k\mathrm{Alg}}(A, A')$ canonically isomorphic to the morphism between $\mathrm{Hom}_{k\mathrm{Alg}}(A, -)$ and $\mathrm{Hom}_{k\mathrm{Alg}}(A', -)$? Is it natural for each A and A'?
- When does X has an instinct group structure?
- What categories should we select for further analysis?

Def 18. We observe from the example above that all of $\mathbb{G}_m$, SL_n, GL_n and μ_n are equipped with an additional group structure in comparison of $\mathbb{A}^n$. An **affine group** is a representative functor (affine scheme) with group structure, i.e.,

$$k\mathrm{Alg} \to \mathrm{Grp}.$$

2.2 A categorical explanation

Def 19. A category $\mathcal{C}$ consists of the class of objects $(\mathcal{C})_0$ and morphisms $(\mathcal{C})_1$. Here we use the notations of quivers for simplicity. One can write a morphism as $s(f) \xrightarrow{f} t(f)$.

E.g. 6. Given category $\mathcal{C}$, with

- $(\mathcal{C})_0$ as its objects,
- $(\mathcal{C})_1$ as its morphisms,

the category of morphisms is given by

- $(\mathcal{C})_1$ as its objects,
- $\forall f, g \in (\mathcal{C})_1$, the morphism is a pair $(\alpha, \alpha') \in (\mathcal{C})_1 \times (\mathcal{C})_1$ such that

$$\begin{pmatrix} s(f) = s(\alpha) & s(g) = t(\alpha) \\ t(f) = s(\alpha') & t(g) = t(\alpha') \end{pmatrix}, \quad \alpha' \circ f = g \circ \alpha.$$

The category of functors $\mathrm{Funct}(\mathcal{C}, \mathrm{Sets})$ is given by

- functors as its objects,
- natural transformations as its morphisms.

Def 20. We say $h^\bullet : \mathcal{C} \to \mathrm{Funct}(\mathcal{C}, \mathrm{Sets})$ is an covariant **Yoneda embedding**, where $h^A := \mathrm{Hom}_{\mathcal{C}}(A, -)$.

Thm 5. Yoneda lemma gives the following bijection for both F and A, i.e.,

$$\mathsf{Nat}(h^A, F) \xrightarrow{1:1} F(A), \qquad \Phi \mapsto \Phi_A(\mathrm{id}_A) =: a_\Phi;$$
$$F(A) \xrightarrow{1:1} \mathsf{Nat}(h^A, F), \qquad a \mapsto \Phi_a.$$

Rmk 11. For fix $A \in (\mathcal{C})_0$,

- $\Phi \to a$ is given by $\Phi_A(\mathrm{id}_A)$;
- $a \to \Phi$ is given by

$$\Phi : \mathrm{Hom}_{\mathcal{C}}(A, \bullet) \to F(\bullet), \quad [A \xrightarrow{f} \bullet] \mapsto (Ff)(a_\Phi).$$

$$\begin{tikzcd}
h^X(X) \arrow[rr, "h^X(f)"] \arrow[ddd, bend right, "\Phi_X"'] & & h^X(Y) \arrow[ddd, bend left, "\Phi_Y"] \\
\mathrm{id}_X \arrow[r, mapsto, "h^X(f)"] \arrow[d, mapsto, "\Phi_X"'] & h^X(f)(\mathrm{id}_X) \arrow[r, equal] \arrow[d, mapsto, dashed, "\Phi_Y"] & f \arrow[d, mapsto, "\Phi_Y"] \\
a_\Phi \arrow[r, mapsto, "F(f)"'] & F(f)(a_\Phi) \arrow[r, equal] & \Phi_Y(f) \\
F(X) \arrow[rr, "F(f)"'] & & F(Y)
\end{tikzcd}$$

Def 21. The covariant Yoneda embedding are similarly defined, i.e.,

$$(\mathrm{Funct}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}))_0 \ni h_A := \mathrm{Hom}_{\mathcal{C}}(-, A).$$

Rmk 12. Let $F = h^B$ (resp. h_B). We see from Yoneda lemma that $h_\bullet$ and $h^\bullet$ are fully faithful. Moreover, we have the following isomorphisms are natural for each $A, B \in (\mathcal{C})_0$

$$\mathsf{Nat}(h_A, h_B) \cong \mathrm{Hom}_{\mathcal{C}}(A, B) \cong \mathsf{Nat}(h^B, h^A).$$

Def 22. A $\begin{array}{c}\text{contravariant}\\ \text{covariant}\end{array}$ functor $\begin{array}{c}F\in(\mathrm{Funct}(\mathcal{C},\mathrm{Sets}))_0\\ F\in(\mathrm{Funct}(\mathcal{C}^{\mathrm{op}},\mathrm{Sets}))_0\end{array}$ is **representable**, whenever there exists $A \in (\mathcal{C})_0$ and an isomorphism s.t. $\begin{array}{c}\alpha{:}F\xrightarrow{\sim}h^A\\ \alpha{:}F\xrightarrow{\sim}h_A\end{array}$.

The category of representable functors $\mathrm{Rep}(\mathcal{C}, \mathrm{Sets})$ is a full-subcategory of $\mathrm{Funct}(\mathcal{C}, \mathrm{Sets})$.

Fact 5. Each category is isomorphic to a (full-sub)category of functors, for instance,

- a set is uniquely defined by how it maps to/from;
- a tensor product is uniquely determined by all bilinear maps defined on its Cartesian product;
- every person is uniquely determined by $\begin{array}{c}\text{his}\\ \text{hers}\end{array}$ relationship with every single person in the universe, whereas a given collection of relationships to every single person in the universe determines an individual whenever it is representable.

Def 23. Recall that affine groups are defined as representable functors with group structure. A **group object** in a given category is described as follows.

- Let $\mathcal{C}$ be a category equipped with
  1. **finite product** (usually written as $\times$),
  2. a **final/terminal object** (usually written as $\mathbf{1}$).
- the final object $\mathbf{1}$ leads to canonical isomorphisms

$$G \times \mathbf{1} \overset{\sim}{\to} G \overset{\sim}{\leftarrow} \mathbf{1} \times G.$$

- A group in $\mathcal{C}$ is a triple (G, m, e) s.t.
  1. $G \in (\mathcal{C})_0$, $m, u \in (\mathcal{C})_1$;
  2. the **associativity** is given by

     $m \circ (\mathrm{id} \times m) = m \circ (m \times \mathrm{id}) : G \times G \times G \to G$;
  3. the identity is $\mathbf{1} \overset{e}{\to} G$, such that

$$G \times \mathbf{1} \overset{e\times\mathrm{id}}{\longrightarrow} G \times G \overset{m}{\to} G,$$
$$\mathbf{1} \times G \overset{e\times\mathrm{id}}{\longrightarrow} G \times G \overset{m}{\to} G,$$

     are canonical isomorphisms;
  4. there is an $\mathrm{inv} : G \to G$ such that

$$\begin{array}{ccccc} G & \xrightarrow{(\mathrm{inv},\mathrm{id})} & G \times G & \xleftarrow{(\mathrm{id},\mathrm{inv})} & G \\ \downarrow & & \downarrow m & & \downarrow \\ \mathbf{1} & \xrightarrow{\quad e \quad} & G & \xleftarrow{\quad e \quad} & \mathbf{1} \end{array}$$

E.g. 7. In order to prove that $\mathrm{Rep}(k\mathrm{Alg}, \mathrm{Sets})$ assumes finite product an contains a terminal object, we consider the category $h^\bullet(k\mathrm{Alg})$ for simplicity. Then it is clear that

- $h^\bullet(k\mathrm{Alg})$ has finite product induced by tensor product,

$$\begin{tikzcd} A \arrow[r, "f"] & \bullet & A \otimes B \arrow[l, "f\otimes g"'] \\ B \arrow[ru, "g"'] \end{tikzcd}$$

$$h^A \times h^B \xrightarrow{\qquad \sim \qquad} h^{A\otimes B}$$

- $h^\bullet(k\mathrm{Alg})$ has a terminal object h^k, since $\mathrm{Hom}_{h^\bullet(k\mathrm{Alg})}(h^A, h^k) \cong \mathrm{Hom}_{k\mathrm{Alg}}(k, A) = \{[1 \to e_A]\}$.

Rmk 13. Via Yoneda's lemma, G has a group structure whenever $G(S) := h_G(S)$ has a group structure for each object S.

Category	$\mathrm{Rep}(k\mathrm{Alg}, \mathrm{Sets})$	$h^\bullet(k\mathrm{Alg})$
Objects	rep. functor G	$h^A = \mathrm{Hom}_{k\mathrm{Alg}}(A, -)$
Morphisms	$G \to H$	$\mathrm{Hom}_{h^\bullet(k\mathrm{Alg})}(h^A, h^B)$ $= \mathsf{Nat}(h^A, h^B)$ $\cong \mathrm{Hom}_{k\mathrm{Alg}}(B, A)$
finite prod.	$G \times G$	$h^A \times h^B$
Multiplicative structure	$h_G(S) \times h_G(S) \xrightarrow{\sim} h_{G\times G}(S)$	$h^A(R) \times h^A(R) \xrightarrow{\sim} h^{A\otimes A}(R)$
Final object	$h_G(\mathbf{1}) = \{[G \to \mathbf{1}]\}$	$\mathrm{Hom}_{h^\bullet(k\mathrm{Alg})}(h^A, h^k)$ $\cong \mathrm{Hom}_{k\mathrm{Alg}}(k, A)$ $= \{[1 \mapsto e_A]\}$

Def 24. The **homomorphism between group elements** $G \xrightarrow{\alpha} H$ in $\mathcal{C}$ is a natural transformation

$h_\alpha \in \mathsf{Nat}(h_G, h_H)$, such that the following diagram commutes

$$\begin{tikzcd} h_G(R) \arrow[r, "h_\alpha(R)"] \arrow[d, "h_G(\phi)"'] & h_H(R) \arrow[d, "h_H(\phi)"] \\ h_G(R') \arrow[r, "h_\alpha(R')"] & h_H(R') \end{tikzcd}$$

Qn 2. Given affine group (as a functor) $G : k\mathrm{Alg} \to \mathrm{Grp}$, we know that G is a representable functor with group structure. One may ask

- how h^A represents G?
- what additional structure does the k-algebra A (coordinate algebra) requires?

Rmk 14. The representation $\alpha : G \to \mathrm{Hom}_{k\mathrm{Alg}}(A, -)$ is determined by $h_G(S)$ for each $S \in (\mathrm{Rep}(k\mathrm{Alg}, \mathrm{Sets}))_0$.

- The group structure $(G, m) : R \mapsto (G(R), m(R))$ is given by

  $$\mathrm{Hom}_{k\mathrm{Alg}}(A, R) \xrightarrow{\sim} G(R) \cong G(h^R) = \mathrm{Hom}_{\mathrm{Rep}}(h^R, G),$$
  $$f \mapsto f(a) \cong \Phi_a(f).$$

  Remember that $a \in G(R)$ is a universal element, i.e.,

  $$\Phi : \mathrm{Hom}_{\mathcal{C}}(A, \bullet) \to F(\bullet), \quad f \mapsto f(a_\Phi).$$

- It is $(A, a) \in ((k\mathrm{Alg})_0, (k\mathrm{Alg})_1)$ that represents G.

- The group structure of G induces the natural transformation
$$h_m : h_{G\times G}\,(\cong h_G \times h_G) \to h_G.$$
Thus (G, m) is a group whenever $(h_G(S), h_m(S))$ is a group for each $S \in (\mathrm{Funct}(\mathcal{C}, \mathrm{Sets}))_0$.

Qn 3. How can we find (A, a) together with an isomorphism $\alpha : G \to h^A$?

E.g. 8. For $\mathrm{GL}_n : k\mathrm{Alg} \to \mathrm{Grp}$ (affine group over k), one can verify its representation
$$\alpha : \mathrm{GL}_n \to \mathrm{Hom}_{k\mathrm{Alg}}(\mathcal{O}(\mathrm{GL}_n), -)$$
where
$$A = \mathcal{O}(\mathrm{GL}_n) = \frac{k[X_{1,1}, \ldots, X_{n,n}, Y]}{(Y \cdot \det(X_{i,j}) - 1)}.$$

Thm 6. We claim that $\mathcal{O}(G) \cong \mathsf{Nat}(G_0, \mathbb{A}^1)$, as one can verify the following isomorphisms
$$\mathcal{O}(G) \cong \mathrm{Hom}_{k\mathrm{Alg}}(k[t], \mathcal{O}(G)) \underset{\text{Yoneda}}{\cong} \mathsf{Nat}(G_0, \mathbb{A}^1).$$
Here $\mathcal{O}(G)$ is a **canonical coordinate ring**. One can also equal
$$A = \mathcal{O}(G) = \mathsf{Nat}(G_0, \mathbb{A}^1), \quad a = \Phi$$
for simplicity.

Thm 7. When $\alpha : G \to h^A$ is a representation, the coordinate ring A is commutative k-alegbra. Given

$\Phi \in \mathsf{Nat}(G_0, \mathbb{A}^1)$, for each $\phi \in \mathrm{Hom}_{k\mathrm{Alg}}(R, R')$, the following diagram commutes

$$\begin{tikzcd}
G_0(R) \arrow[r, "\Phi_R"] \arrow[d, "G_0(\phi)"'] & R \arrow[d, "\phi"] \\
G_0(R') \arrow[r, "\Phi_{R'}"] & R'
\end{tikzcd}$$

The ring structure of $\mathsf{Nat}(G_0, \mathbb{A}^1)$ is given by

- $(\Phi + \Phi')_R : G_0(R) \to R, g \mapsto \Phi_R(g) + \Phi'_R(g)$,
- $(\Phi \cdot \Phi')_R : G_0(R) \to R, g \mapsto \Phi_R(g) \cdot \Phi'_R(g)$.

Thus A is a commutative k-algebra.

Fact 6. The group structure of G induces the comultiplication of A, i.e., $\Delta : A \to A \otimes A$ is given by

$$\begin{tikzcd}
& G(R) \times G(R) \arrow[r, "m(R)"] \arrow[d, "\alpha_R \times \alpha_R"] & G(R) \arrow[d, "\alpha_R"] & \\
f_1, f_2 & h^A(R) \times h^A(R) \arrow[r, "\alpha_{m(R)}"] \arrow[d, "\cong"'] & h^A(R) & f_1 \cdot f_2 \\
f_1 \otimes f_2 & h^{A\otimes A}(R) \arrow[ur, "\mathrm{Hom}(\Delta, R)"'] & &
\end{tikzcd}$$

Here

- $h^{A_1} \times h^{A_2} \cong h^{A_1 \otimes A_2}$ is a canonical isomorphism given by the universal property of tensor product.
- $f_1 \cdot f_2$ is defined as $m_{\alpha(R)}(f_1, f_2)$.
- **If** $\Delta : A \to A \otimes A, a \mapsto a_1 \otimes a_2$ **is defined**, then

$$(f_1 \cdot f_2)(a) = (f_1 \otimes f_2)(a_1 \otimes a_2).$$

Fact 7. It is essencially the same to provide

1. an **affine group** over k, that is, (G, m) is both a group structure and a representable functor
$$\text{forget} \circ G : k\text{Alg} \to \text{Sets}.$$
2. a k-algebra A together with a comultiplication Δ s.t. $h^A(-)$ has a **group structure** given by
$$f_1 \cdot f_2 = (f_1, f_2) \circ \Delta.$$

$2 \implies 1$: Take $G = h^A$ endowed with the multiplication $m : G \times G \to G$ defined by Δ.
$1 \implies 2$: Consider $A := \mathsf{Nat}(G_0, \mathbb{A}^1)$.

Qn 4. When Δ is defines the group structure of G?

2.3 Affine groups $\Longleftrightarrow$ commutative Hopf algebras

Qn 5. Let A represents G. According to our intuitions,

- Δ (comultiplication) of A corresponds to m (multiplication) of G,
- ϵ (counit) of A corresponds to $(\mathbf{1}, e)$ (identity) of G,
- $\mathcal{S}$ (antipode) of A corresponds to ${}^{-1}$ (inversion) of G.

Thm 8. Given k-algebra A together with homomorphisms $\Delta : A \to A \otimes A$ and $\epsilon : A \to k$, there exists natural transformations m and e defined by Δ and ϵ. Moreover (h^A, m, e) **is an affine monoid whenever** (A, Δ, ϵ) **is a commutative bialgebra**.

Write $M := h^A$. Then

- $e : \mathbf{1} \to M$ is a natural transformation defined by ϵ, i.e.,

$$\begin{tikzcd}
\mathbf{1}(R) \arrow[rr, "e(R)"] \arrow[d, "\mathbf{1}_R"'] & & M(R) \arrow[d, "\alpha_R"] \\
\mathrm{Hom}_{k\mathrm{Alg}}(k, R) \arrow[rr, "\alpha_{e(R)}", "\mathrm{Hom}_{k\mathrm{Alg}}(\epsilon, R)"'] & & \mathrm{Hom}_{k\mathrm{Alg}}(A, R)
\end{tikzcd}$$

$$h^k(R) \xrightarrow{\quad h^{\varepsilon}(R) \quad} h^A(R)$$

- $m : M \times M \to M$ is a natural transformation defined by Δ, i.e.,

$$\begin{tikzcd}
M(R)\times M(R) \arrow[rr, "m(R)"] \arrow[d, "\alpha_R\times\alpha_R"'] & & M(R) \arrow[dd, "\alpha_R"] \\
\mathrm{Hom}_{k\mathrm{Alg}}(A,R)\times\mathrm{Hom}_{k\mathrm{Alg}}(A,R) \arrow[d, "\cong"'] \arrow[drr, "\alpha_{m(R)}"] & & \\
\mathrm{Hom}_{k\mathrm{Alg}}(A\otimes A,R) \arrow[rr, "\mathrm{Hom}_{k\mathrm{Alg}}(\Delta,R)"'] & & \mathrm{Hom}_{k\mathrm{Alg}}(A,R)
\end{tikzcd}$$

$$h^{A\otimes A}(R) \xrightarrow{h^{\Delta}(R)} h^A(R)$$

We claim that A is an bialgebra whenever M (or equivalently, $\alpha(M)=h^A$) has a monoid structure.

$$\begin{tikzcd}
M^3 \arrow[r, "\mathrm{id}_M\times m"] \arrow[d, "m\times\mathrm{id}_M"'] & M^2 \arrow[d, "m"] \\
M^2 \arrow[r, "m"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
M\times\mathbf{1} \arrow[r, "\mathrm{id}_M\times e"] \arrow[dr, "\cong"'] & M\times M \arrow[d, "m"] & \mathbf{1}\times M \arrow[l, "e\times\mathrm{id}_M"'] \arrow[dl, "\cong"] \\
& M &
\end{tikzcd}$$

$$\begin{tikzcd}
(h^A)^3 \arrow[r, "h^{\mathrm{id}_A}\times h^{\Delta'}"] \arrow[d, "h^{\Delta'}\times h^{\mathrm{id}_A}"'] & (h^A)^2 \arrow[d, "h^{\Delta'}"] \\
(h^A)^2 \arrow[r, "h^{\Delta'}"] & h^A
\end{tikzcd}
\qquad
\begin{tikzcd}
h^A\times h^k \arrow[r, "h^{\mathrm{id}_A}\times h^{\epsilon}"] \arrow[dr, "\cong"'] & h^A\times h^A \arrow[d, "h^{\Delta'}"] & h^k\times h^A \arrow[l, "h^{\varepsilon}\times h^{\mathrm{id}_A}"'] \arrow[dl, "\cong"] \\
& h^A &
\end{tikzcd}$$

$$\begin{tikzcd}
h^{A\otimes A\otimes A} \arrow[r, "h^{\mathrm{id}_A\otimes\Delta}"] \arrow[d, "h^{\Delta\otimes\mathrm{id}_A}"'] & h^{A\otimes A} \arrow[d, "h^{\Delta}"] \\
h^{A\otimes A} \arrow[r, "h^{\Delta}"] & h^A
\end{tikzcd}
\qquad
\begin{tikzcd}
h^{A\otimes k} \arrow[r, "h^{\mathrm{id}_A\otimes\epsilon}"] \arrow[dr, "\cong"'] & h^{A\otimes A} \arrow[d, "h^{\Delta}"] & h^{k\otimes A} \arrow[l, "h^{\epsilon\otimes\mathrm{id}_A}"'] \arrow[dl, "\cong"] \\
& h^A &
\end{tikzcd}$$

$$\begin{tikzcd}
A\otimes A\otimes A & A\otimes A \arrow[l, "\mathrm{id}_A\otimes\Delta"'] \\
A\otimes A \arrow[u, "\Delta\otimes\mathrm{id}_A"] & A \arrow[l, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes k & A\otimes A \arrow[l, "\mathrm{id}_A\otimes\epsilon"'] \arrow[r, "\epsilon\otimes\mathrm{id}_A"] & k\otimes A \\
& A \arrow[ul, "\cong"] \arrow[u, "\Delta"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

Here, all (four) commutitive diagram in each column are canonically equivalent.

Thm 9. Similarly, A **is an Hopf algebra whenever** M **has a group structure**. Since the following commutative diagram are canonically equivalent

$$
\begin{tikzcd}
M \arrow[r, "{(\mathrm{inv}_M, \mathrm{id}_M)}"] \arrow[d, two heads] & M \times M \arrow[d, "m"'] & M \arrow[l, "{(\mathrm{id}_M, \mathrm{inv}_M)}"'] \arrow[d, two heads] \\
\mathbf{1} \arrow[r, hook, "e_M"] & M & \mathbf{1} \arrow[l, hook', "e_M"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \otimes A \arrow[dl, bend right, "m \circ (\mathcal{S} \otimes \mathrm{id}_A)"'] \arrow[dr, bend left, "m \circ (\mathrm{id}_A \otimes \mathcal{S})"] & \\
A & & A \\
k \arrow[u, hook, "u"] & A \arrow[uu, "\Delta"] \arrow[l, two heads, "\epsilon"'] \arrow[r, two heads, "\epsilon"] & k \arrow[u, hook', "u"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
h^A \arrow[r, "{(h^{\mathcal{S}}, h^{\mathrm{id}_A})}"] \arrow[d, two heads, "h^u"'] & h^A \times h^A \arrow[d, "h^{\Delta'}"'] & h^A \arrow[l, "{(h^{\mathrm{id}_A}, h^{\mathcal{S}})}"'] \arrow[d, two heads, "h^u"] \\
h^k \arrow[r, hook, "h^\epsilon"] & h^A & h^k \arrow[l, hook', "h^\epsilon"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
h^A \arrow[r, "h^{\mathcal{S} \otimes \mathrm{id}}"] \arrow[d, two heads, "h^u"'] & h^{A \otimes A} \arrow[d, "h^{\Delta}"'] & h^A \arrow[l, "h^{\mathrm{id} \otimes \mathcal{S}}"'] \arrow[d, two heads, "h^u"] \\
h^k \arrow[r, hook, "h^\epsilon"] & h^A & h^k \arrow[l, hook', "h^\epsilon"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \arrow[dd, "\mathcal{S} \otimes \mathrm{id}_A"'] & A \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "\epsilon"'] & A \otimes A \arrow[dd, "\mathrm{id}_A \otimes \mathcal{S}"] \\
& k \arrow[d, "u"'] & \\
A \otimes A \arrow[r, "m"] & A & A \otimes A \arrow[l, "m"']
\end{tikzcd}
$$

Thm 10. Similarly, G is an affine Abelian group whenever A is cocommutative, i.e., $\Delta = \tau \circ \Delta$.

Fact 8. Correspondence between affine group G and commutative Hopf algebra $A = \mathcal{O}(G)$.

affine group G	**Hopf algebra** A
$m : G \times G \to G$	$\Delta : A \to A \otimes A$
$e : \mathbf{1} \to G$	$\epsilon : A \to k$
$\mathrm{inv} : G \to G$	$\mathcal{S} : A \to A$
$m = m \circ \tau$	$\tau \circ \Delta = \Delta$
monoid in $k\mathrm{Alg}^{\mathrm{op}}$	comm. bialgebra
group in $k\mathrm{Alg}^{\mathrm{op}}$	comm. Hopf algebra
$\vdots$	$\vdots$

2.4 Examples

Here are some examples of affine groups and their coordinate algebra (as an commutative Hopf algebra).

E.g. 9. $\mathbb{G}_a : R \mapsto (R, +)$ is an affine group called **additive group**. Thus $\mathcal{O}(\mathbb{G}_a)$ equals k with a free element, that is, the polynomial ring $k[X]$.

$$\mathbb{G}_a(R) \xrightarrow{\sim} \mathrm{Hom}_{k\mathrm{Alg}}(k[X], R),$$
$$r \mapsto [X \mapsto r].$$

The group structure of $\mathbb{G}_a(R)$ is given by

$$\mathbb{G}_a(R) \times \mathbb{G}_a(R), \quad (a, b) \mapsto a + b.$$

Hence $\forall f \in k[X]$, $(\Delta f)_R(a, b) = f_R(a + b)$. Consider ΔX to obtain

$$\Delta : k[X] \to k[X] \otimes k[X], \quad X \mapsto 1 \otimes X + X \otimes 1.$$

One can also verify that

$$(1 \otimes X + X \otimes 1)^n = \sum_{s=0}^{n} \binom{n}{s} X^s \otimes X^{n-s}.$$

By definition, $(\epsilon \otimes \mathrm{id})(1 \otimes X + X \otimes 1) = 1 \otimes X \cong X$. Hence $\epsilon : f \mapsto f(0)$. One can verify that the antipode is $\mathcal{S} : f(X) \mapsto f(-X)$.

E.g. 10. For **additive group** $\mathbb{G}_a$, we have

- $$\mathbb{G}_a(R) \xrightarrow{\sim} \mathrm{Hom}_{k\mathrm{Alg}}(k[X], R),$$
  $$r \mapsto [X \mapsto r].$$
- $(\Delta f)_R(a, b) = f_R(a + b)$, $\Delta : X \mapsto 1 \otimes X + X \otimes 1$,
- $\epsilon : f \mapsto f(0)$,
- $\mathcal{S} : f \mapsto f \circ (-)$.

E.g. 11. For **multiplicative group** $\mathbb{G}_m(R) = (R^\times, \cdot)$, we have

- $$\mathbb{G}_m(R) \xrightarrow{\sim} \mathrm{Hom}_{k\mathrm{Alg}}(k[X, X^{-1}], R),$$
  $$r \mapsto [X \mapsto r].$$
- $(\Delta f)_R(a, b) = f_R(ab)$, $\Delta : X \mapsto X \otimes X$,
- $\epsilon : f \mapsto f(1)$,
- $\mathcal{S} : f \mapsto f \circ (1/-)$.

E.g. 12. For **trivial group** $\mathbb{G}_e(R) = \{e\}$, we have

- $$\mathbb{G}_e(R) \xrightarrow{\sim} \mathrm{Hom}_{k\mathrm{Alg}}(k, R),$$
$$e \mapsto [1 \mapsto 1].$$
- $(\Delta f)_R(\lambda, \mu) = f(\lambda\mu)$, $\Delta : 1 \mapsto 1 \otimes 1$,
- $\epsilon : f \mapsto f(1)$,
- $\mathcal{S} : f \mapsto f$.

E.g. 13. For **general linear group** GL_n, we have

- $$\mathrm{GL}_n(R) \xrightarrow{\sim} \mathrm{Hom}_{k\mathrm{Alg}}(k[X_{ij}, Y]/(Y\det(X_{ij}) - 1), R),$$
$$(r_{ij}) \mapsto [(X_{ij}, Y) \mapsto f(r_{ij}, \det(r_{ij})^{-1})]$$
- (the i, j-th entry of $A \cdot B$)
$$(\Delta X_{ij})_R(A, B) = \left(\sum_s (X_{is} \otimes X_{sj})\right)_R (A \otimes B)$$
$$= \sum_s a_{is} b_{sj} = (A \cdot B)_{ij},$$
$\Delta(X_{ij}) = \sum_s X_{is} \otimes X_{sj}$, thus $\Delta Y = Y \otimes Y$,
- $\epsilon(X_{ij}) = \delta_{ij}$, $\epsilon(Y) = 1$,
- $\mathcal{S}(X) = Y(\mathrm{Adj}X)$, $\mathcal{S}(Y) = \det(X_{ij})$.

2.5 An example of Quantum groups

Fact 9. Until the mid-1980s, the only Hopf algebras seriously studied were either commutative or cocommutative. NON-COMMUTATIVE Hopf algebras are discovered by Drinfeld and Jimbo independently in the work of physicists. The following example is a q-analog of SL(2).

E.g. 14. $\mathrm{SL}_q(2, R)$ defines on an non-commutative k algebra $k[X_{11}, X_{12}, X_{21}, X_{22}]$, providing that ($q \in k^\times$))

$$ba = qab, \quad ca = qac, \quad dc = qcd, \quad db = qbd,$$
$$bc = cb, \quad ad - da = (q^{-1} - q)bc.$$

The counit and comultiplication is the same as $\mathrm{SL}(2, R)$.