高维波动方程的快速解法

例题

例**1** (数学物理方法P34-1-1)

$$
\begin{aligned}
&u_{tt}=a^2(u_{xx}+u_{yy}+u_{zz})\\
&t=0:u=0,u_t=x^2+yz
\end{aligned}
$$

解 解具有一般形式$u=t(x^2+yz)+t^2\cdot(\cdots)$, 注意到

$$
\begin{aligned}
\partial_{tt}-a^2\Delta:&t(x^2+yz)\mapsto -2a^2t\\
&\frac{t^3a^2}{3}\mapsto 2a^2t
\end{aligned}
$$

从而$u=t(x^2+yz)+\dfrac{a^2t^3}{3}$.

例**2** (数学物理方法P34-3)

$$
\begin{aligned}
&u_{tt}=a^2(u_{xx}+u_{yy})\\
&t=0:u=x^2(x+y),u_t=0
\end{aligned}
$$

解 解具有一般形式$u=x^2(x+y)+t^2\cdot(\cdots)$, 注意到

$$
\begin{aligned}
\partial_{tt}-a^2\Delta:&x^2(x+y)\mapsto -6a^2x-2a^2y\\
&\frac{t^2}{2}(6a^2x+2a^2y)\mapsto 6a^2x+2a^2y
\end{aligned}
$$

从而$u=x^2(x+y)+a^2t^2(3x+y)$.

例**3** (数学物理方法P34-8)

$$
\begin{aligned}
&u_{tt}=u_{xx}+u_{yy}+u_{zz}+2(y-t)\\
&t=0:u=0,u_t=x^2+yz
\end{aligned}
$$

解 注意到

$$
\begin{aligned}
\partial_{tt}-\Delta:&-\frac{t^3}{3}\mapsto -2t\\
&t^2y\mapsto 2y
\end{aligned}
$$

从而设$v=u-t^2y+\dfrac{t^3}{3}$, 则v满足方程

$$\begin{aligned}
&v_{tt} = v_{xx} + v_{yy} + v_{zz}\\
&t = 0 : v = 0; v_t = x^2 + yz
\end{aligned}$$

可口算得$v = t(x^2 + yz) + \dfrac{t^3}{3}$, 从而$u = t(x^2 + yz) + t^2 y$.

一般理论

一般地, 有

$$\begin{aligned}
&u_{tt} - \sum_{i=1}^{p} u_{x_i x_i} = f(t, x)\\
&t = 0 : u = \varphi(x), u_t = \psi(x)
\end{aligned}$$

且$f(t, x)$, $\varphi(x)$与$\psi(x)$均为t, x_i与相关之有限多项式(暂定之). 考虑算子$P := \partial_{tt} - \sum_{i=1}^{p} \partial_{x_i x_i}$, 并注意到:

$$\begin{aligned}
P : &\frac{t^{m+2} x^\alpha}{(m+2)(m+1)} \mapsto t^m x^\alpha - \frac{t^{m+2} \Delta x^\alpha}{(m+2)(m+1)}\\
&\frac{t^{m+4} \Delta x^\alpha}{A_{m+4}^4} \mapsto \frac{t^{m+2} \Delta x^\alpha}{A_{m+2}^2} - \frac{t^{m+4} \Delta^2 x^\alpha}{A_{m+4}^4}\\
&\cdots
\end{aligned}$$

从而$P : \sum_{n \ge 1} \dfrac{t^{m+2n} \Delta^{n-1} x^\alpha}{A_{m+2n}^{2n}} \mapsto t^m x^\alpha$. 令

$$v = u - \sum_{\alpha} \sum_{n \ge 1} \frac{t^{m+2n} \Delta^{n-1} x^\alpha}{A_{m+2n}^{2n}}.$$

则v满足以下PDE系统(实际上已完成齐次化)

$$\begin{aligned}
&v_{tt} - \sum_{i=1}^{p} v_{x_i x_i} = f(t, x)\\
&t = 0 : v = \varphi(x), v_t = \psi(x)
\end{aligned}$$

依照先前递推式, 解得

$$v(t, x) = \sum_{n \ge 0} \left(\frac{t^{2n} \Delta^n \varphi(x)}{(2n)!} + \frac{t^{2n+1} \Delta^n \psi(x)}{(2n+1)!} \right).$$

综上,

$$u(t, x) = \sum_{n \ge 0} \left(\frac{t^{2n} \Delta^n \varphi(x)}{(2n)!} + \frac{t^{2n+1} \Delta^n \psi(x)}{(2n+1)!} \right) + \sum_{\alpha} \sum_{n \ge 1} \frac{t^{m+2n} \Delta^{n-1} x^\alpha}{A_{m+2n}^{2n}}.$$

f项也可采用Duhamel原理叙述, 即

$$P : \int_0^t \sum_{n \geq 0} \frac{\tau^{2n+1} \Delta_x^n f(\tau, x)}{(2n+1)!} \mathrm{d}\tau \mapsto f(t, x).$$