偏微分方程复习

特征理论

一阶线性方程

特征线法

灵活运用特征线法可有效地转化PDE问题为ODE问题, 例如:

$$\begin{cases} -yu_x + xu_y = u \\ u(x,0) = \psi(x) \end{cases}$$

每条特征线可由以s为自变量的函数给定. 同时考虑转化一条特征线上的PDE问题为ODE问题

$$\begin{cases} u = u(x^s(t), y^s(t)) \\ u_0 = u(x^s(t_0), y^s(t_0)) \end{cases}$$

考虑$-yu_x + xu_y$为u_t, 则有ODE问题

$$\begin{cases} x_t^s = -y & x^s(0) = s \\ y_t^s = x & y^s(0) = 0 \\ u_t^s = u & u^s(0) = \psi(s) \end{cases}$$

解得

$$\begin{cases} x^s = s\cos t \\ y^s = s\sin t \\ u^s = e^t\psi(s) \end{cases}$$

由于每条特征线经过x正半轴与负半轴(实际上特征线为同心圆族), 换元得

$$u^s = \exp\left(\arctan\frac{y^s}{x^s}\right)\psi(\sqrt{(x^s)^2+(y^s)^2})$$

由于表达式对于s一致, 故$u=\exp\left(\arctan\left(y/x\right)\right)\psi(\sqrt{x^2+y^2})$.

半平面内的一阶线性偏微分方程

考虑方程

$$\begin{cases} u_t + a(t,x)u_x = f(t,x), \quad x>0, t>0 \\ t=0: u=\varphi(x) \\ x=0: u=\mu(t) \end{cases}$$

当特征线沿t^-方向与t轴无交点时, 解得

$$u(t,x) = \varphi(x^t(0)) + \int_0^t f(\tau, x^t(\tau))\mathrm{d}\tau.$$

其中$x^t(\tau)$为经过(t,x)的特征线在$t=\tau$时x的取值. 反之, 当特征线沿t^-方向与t轴有交点时, 设交点为t_x, 则

$$u(t,x) = \mu(t_x) + \int_{t_x}^t f(\tau, x^t(\tau))\mathrm{d}\tau.$$

此处, 一切特征线即向量场$a(t,x)$之积分曲线.

含有两个自变量的一阶线性方程组

对方程组$U=(u_1,u_2,\ldots,u_n)$, 并设A可对角化的常系数矩阵. 考虑方程

$$\begin{cases} \partial_t U + A\partial_x U = F, t>0, x>0 \\ t=0: U=\varphi(x) \\ x=0: BU=\mu(t) \end{cases}$$

其中$B_{l\times n}$为常系数矩阵. 不妨设对角化结果为$A=P\Lambda P^{-1}$, 其中$\Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_n)$, 且

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le 0 < \lambda_{k+1} \le \cdots \lambda_n$$

记$V := P^{-1}U$, 则原PDE化为

$$\begin{cases} \partial_t V + \Lambda\partial_x V = P^{-1}F, t>0, x>0 \\ t=0: V=P^{-1}\varphi(x) \\ x=0: (BP)V=\mu(t) \end{cases}$$

因此设$V=(V^I,V^{II}),BP=(Q_1\quad Q_2)$, 则$Q_1V^I+Q_2V^{II}=\mu(t)$. 注意到仅$V^{II}$需$x=0$时的边值条件, 故$Q_2$应可逆. 从而$\mathrm{rank}(B)\geq n-k$.

二阶线性齐次方程分类

设a_{ij},b_k,c,f均为连续可微函数, 且$\det\left(a_{ij}\right)\neq 0$(约定$a_{12}=a_{21}$), 若以下方程

$$a_{11}u_{xx}+2a_{12}u_{xy}+a_{22}u_{yy}+b_1u_x+b_2u_y+cu=f$$

在某点处满足

- $\Delta>0$, 则为双曲型方程, 例如弦振动方程;
- $\Delta=0$, 则为抛物型, 例如热传导方程;
- $\Delta<0$, 则为椭圆型, 例如调和方程.

记$\xi=\xi(x,y),\eta=\eta(x,y)$为非退化换元, 即$\det\dfrac{\partial(\xi,\eta)}{\partial(x,y)}\neq 0$, 则

$$\begin{cases}a_{11}:u_{xx}=u_{\xi\xi}\xi_x^2+2u_{\xi\eta}\xi_x\eta_x+u_{\eta\eta}\eta_x^2+u_\xi\xi_{xx}+u_\eta\eta_{xx}\\ a_{12}:u_{xy}=u_{\xi\xi}\xi_x\xi_y+u_{\xi\eta}(\xi_x\eta_y+\xi_y\eta_x)+u_{\eta\eta}\eta_x\eta_y+u_\xi\xi_{xy}+u_\eta\eta_{xy}\\ a_{22}:u_{yy}=u_{\xi\xi}\xi_y^2+2u_{\xi\eta}\xi_y\eta_y+u_{\eta\eta}\eta_y^2+u_\xi\xi_{yy}+u_\eta\eta_{yy}\end{cases}$$

从而原方程可化作

$$\tilde{a}_{11}u_{\xi\xi}+2\tilde{a}_{12}u_{\xi\eta}+\tilde{a}_{22}u_{\eta\eta}+\tilde{b}_1u_\xi+\tilde{b}_2u_\eta+\tilde{c}u=\tilde{f}.$$

其中

$$\begin{cases}\tilde{a}_{11}=a_{11}\xi_x^2+2a_{12}\xi_x\xi_y+a_{22}\xi_y^2\\ \tilde{a}_{12}=a_{11}\xi_x\eta_x+a_{12}(\xi_x\eta_y+\xi_y\eta_x)+a_{22}\xi_x\eta_x\\ \tilde{a}_{22}=a_{11}\eta_x^2+2a_{12}\eta_x\eta_y+a_{22}\eta_y^2\end{cases}$$

注意到$\tilde{a}_{11}$与$\tilde{a}_{22}$形式相同, 考虑方程

$$a_{11}\varphi_x^2+a_{12}\varphi_x\varphi_y+a_{22}\varphi_y^2=0.$$

特征线满足$a_{11}\mathrm{d}y^2-2a_{12}\mathrm{d}x\mathrm{d}y+a_{22}\mathrm{d}x^2$.

- $\Delta>0$时, 特征线$y-\lambda_ix=c_i,i=1,2$. 令$\xi=(y-\lambda_1x),\eta=y-\lambda_2x$, 则$\tilde{a}_{11}=\tilde{a}_{12}=0$. 从而的双曲型方程的第一标准型

  $$u_{\xi\eta}=A_1u_\xi+B_1u_\eta+C_1u+D_1.$$

  若再令$r=\xi+\eta,s=\xi-\eta$, 则得第二标准型

$$u_{rr} - u_{xx} = A_1^* u_r + B_1^* u_s + C_1^* u + D_1^*.$$

- $\Delta = 0$时, 特征线为$y - \lambda_{1,2} x = c$. 令$\xi = y - \lambda_i x$, η为某一与ξ线性无关之量即可得抛物型方程的标准型

$$u_{\eta\eta} = A_2 u_\xi + B_2 u_\eta + C_2 u + D_2.$$

- $\Delta < 0$时, 令$r = -\Re[\lambda] x + y$, $s = -\Im[\lambda]$即得椭圆型标准型

$$u_{rr} + u_{ss} = A_3 u_r + B_3 u_s + C_3 u + D_3.$$

可令$v = ue^{-ar-bs}$以消去一次项.

例: 探究方程$yu_{xx} + 2xyu_{xy} + u_{yy} + u_x + 2u_y + u = 0$在$x = y^{-2}$上的双曲区段, 并近似之为标准双曲型方程

判别式$4x^2y^2 - 4y = 4y(x^2y - 1)$. 故$y < 0$或$x^{-2} < y$时为双曲型. 令$\xi, \eta = y - \lambda_i x$, 其中$\lambda = -x \pm \sqrt{\dfrac{x^2y-1}{y}}$. 换元得

$$u_{\xi\eta} + u_\xi \left(3 + \frac{xy^2 - 1}{y\sqrt{y}\sqrt{x^2y-1}}\right) + u_\eta \left(3 + \frac{xy^2 - 1}{y\sqrt{y}\sqrt{x^2y-1}}\right) + u = 0$$

$x = y^{-2}$时有$u_{\xi\eta} + 3u_\xi + 3u_\eta + u = 0$. 换元$v = ue^{3(\xi+\eta)}$, 则

$$v_{\xi\eta=}(u_{\xi\eta} + 3u_\xi + 3u_\eta + 9u)e^{3(\xi+\eta)} = 8v$$

从而得双曲型方程之标准型.

一维波动方程解法

一维全/半空间上的解

达朗贝尔公式

方程

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0 \quad x \in \mathbb{R}, t > 0 \\ t = 0 : u = \varphi(x), u_t = \psi(x) \end{cases}$$

解为

$$u(t,x)=\frac{\varphi(x-at)+\varphi(x+at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\mathrm{d}\xi.$$

有源方程的齐次化解

$$\begin{cases}u_{tt}-a^2u_{xx}=f \quad x\in\mathbb{R},t>0\\ t=0:u=0,u_t=0\end{cases}$$

考虑齐次化方程

$$\begin{cases}W_{tt}-a^2W_{xx}=0 \quad x\in\mathbb{R},t>\tau\\ W=0:W=0,W_t=f(x,\tau)\end{cases}$$

则

$$u(t,x)=\int_0^t W(t,x;\tau)\mathrm{d}\tau.$$

回代得

$$u(t,x)=\frac{1}{2a}\int_0^t\int_{x-a\tau}^{x+a\tau}f(\tau,x)\mathrm{d}\tau\mathrm{d}t.$$

半空间上的解

$$\begin{cases}u_{tt}-a^2u_{xx}=f(t,x) \quad x\in\mathbb{R}_+,t>0\\ t=0:u=\varphi(x),u_t=\psi(x)\\ x=0:u=\mu(t)\end{cases}$$

当$x\geq at$时, 同上; $x\leq at$时, 考虑$v=u-\mu(t)$, 则

$$\begin{cases}v_{tt}-a^2u_{xx}=\tilde{f}(t,x):=f(t,x)-\mu''(t) \quad x\in\mathbb{R}_+,t>0\\ t=0:v=\tilde{\varphi}(x):=\varphi(x)-\mu(0),v_t=\tilde{\psi}(x):=\psi(x)-\mu'(0)\\ x=0:v\equiv 0\end{cases}$$

考虑奇延拓, 则

$$
\begin{aligned}
v=&\frac{\tilde{\varphi}(x-at)+\tilde{\varphi}(x+at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\tilde{\psi}(\xi)\mathrm{d}\xi \\
&+\frac{1}{2a}\int_0^t\int_{x-a\tau}^{x+a\tau}\tilde{f}(\tau,x)\mathrm{d}\tau\mathrm{d}t \\
=&\frac{\varphi(at+x)-\varphi(at-x)}{2}+\frac{1}{2a}\int_{at-x}^{at+x}\psi(\xi)\mathrm{d}\xi \\
&+\frac{1}{2a}\int_0^t\int_{|a\tau-x|}^{a\tau+x}f(\tau,x)\mathrm{d}\tau\mathrm{d}t+\mu(t-x/a)
\end{aligned}
$$

齐次化原理

记ODE问题$u'(t)+Au(t)=0$的解为$u=u(t)$. 记$u(t)=S(t)u(0)$, 则方程$u'(t)+Au(t)=f(t)$的解为

$$
u(t)=\int_0^t S(\tau)f(t-\tau)\mathrm{d}\tau+S(t)u(0).
$$

从而转化方程

$$
\begin{cases}u_{tt}-a^2u_{xx}=0 \quad x\in\mathbb{R},t>0\\ t=0:u=\varphi(x),u_t=\psi(x).\end{cases}
$$

为ODE问题, i.e.

$$
\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}u\\u_t\end{pmatrix}=\begin{pmatrix}0&1\\a^2\partial_{xx}&0\end{pmatrix}\begin{pmatrix}u\\u_t\end{pmatrix} \quad x\in\mathbb{R},t>0\\ t=0:\begin{pmatrix}u\\u_t\end{pmatrix}=\begin{pmatrix}\varphi(x)\\\psi(x)\end{pmatrix}\end{cases}
$$

设其解为$[u(t,x),u_t(t,x)]=S(t)[u(0,x),v(0,x)]$, 此处$S(t)$应当理解为某一与$t$相关之算子而非分离变量. 今考虑方程

$$
\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}u\\u_t\end{pmatrix}=\begin{pmatrix}0&1\\a^2\partial_{xx}&0\end{pmatrix}\begin{pmatrix}u\\u_t\end{pmatrix}+\begin{pmatrix}0\\f(t,x)\end{pmatrix} \quad x\in\mathbb{R},t>0\\ t=0:\begin{pmatrix}u\\u_t\end{pmatrix}=\begin{pmatrix}\varphi(x)\\\psi(x)\end{pmatrix}\end{cases}
$$

则解为$\begin{pmatrix}u\\u_t\end{pmatrix}=\int_0^t S(\tau)[0,f(t-\tau,x)]\mathrm{d}\tau+S(t)[u(0),u_t(0)]$. 由于

$$
S(t)\begin{pmatrix}0\\\psi(x)\end{pmatrix}=\begin{pmatrix}\frac{1}{2a}\int_{x-at}^{x+at}\psi(w)\mathrm{d}w\\ *\end{pmatrix}.
$$

从而

$$
\begin{aligned}
u &= \int_0^t \frac{1}{2a}\int_{x-a\tau}^{x+a\tau} f(t-\tau,w)\mathrm{d}w\mathrm{d}t + S(t)u(0) \\
&= \frac{1}{2a}\int_{G(t,x)} f(\tau,w)\mathrm{d}\tau\mathrm{d}w + u(t)
\end{aligned}
$$

此处$u(t)$具有含参数x的表达, 即

$$
u(t) = \frac{\varphi(x+at)+\varphi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\psi(w)\mathrm{d}w.
$$

相容性条件

以方程

$$
\begin{cases}
u_{tt}-u_{xx}=0, \quad 0<t<kx, \\
u|_{t=kx}=\phi(x), \\
t=0: u=\psi_1(x), u_t=\psi_2(x).
\end{cases}
$$

为例, 当$0<t<x$时, 直接解得

$$
u(t,x) = \frac{\psi_1(x-t)+\psi_1(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t}\psi_2(s)\mathrm{d}s.
$$

$0<x<t$时, 注意到解具有形式$u=F(t-x)+G(x+t)$. 故

$$
\begin{aligned}
&F(kx-x)+G(kx+x)=\phi(x), \\
&F(-x)+G(x)=\psi_1(x), \\
&-F(-x)+G(x)=\int_0^x \psi_2(x) - F(0)+G(0).
\end{aligned}
$$

从而$G(x)=\dfrac{1}{2}\big(\psi_1(x)+\int_0^x\psi_2(x)+2G(0)-\phi(0)\big)$. 因此

$$
\begin{aligned}
u(t,x) &= G(x+t)+\phi\left(\frac{t-x}{k-1}\right) - G\left(\frac{k+1}{k-1}(t-x)\right) \\
&= \frac{1}{2}\left(\psi_1(x+t)-\psi_1\left(\frac{k+1}{k-1}(t-x)\right)+\int_{\frac{k+1}{k-1}(t-x)}^{x+t}\psi_2(x)\right) \\
&\quad +\phi\left(\frac{t-x}{k-1}\right)
\end{aligned}
$$

二阶相容性条件分别为:

1. $\phi(0)=\psi_1(0)$.

2. 考虑$u(k\mathrm{d}x,\mathrm{d}x)-u(0,0)$, 则$\phi'(0)=\psi_1'(0)+k\psi_2'(0)$.

3. 同上, 考虑二阶差分(或Taylor级数第二项)得

$$u(k\mathrm{d}x,\mathrm{d}x)=A+B\mathrm{d}x+\frac{1}{2}u_{tt}k^2\mathrm{d}x^2+u_{tx}k\mathrm{d}x+\frac{1}{2}u_{xx}\mathrm{d}x^2.$$

因此二阶相容性条件为

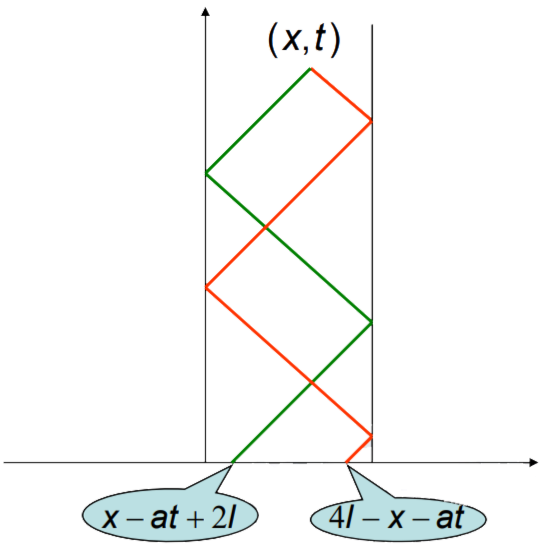
$$\begin{aligned}\phi''(0)&=\frac{k^2}{2}u_{tt}(0)+ku_{tx}(0)+\frac{1}{2}u_{xx}(0)\\&=\frac{k^2+1}{2}\psi_1''(0)+k\psi_2'(0)\end{aligned}$$

奇延拓法推广(有限空间上的解)

考虑方程

$$\begin{cases}u_{tt}-a^2u_{xx}=f(t,x) & x\in(0,l),t>0\\ t=0:u=\varphi(x),u_t=\psi(x)\\ x=0:u=\mu_1(t)\\ x=l:u=\mu_2(t)\end{cases}$$

由于可设$u=v+\mu_1(t)+\dfrac{x}{l}(\mu_2(t)-\mu_1(t))$以转化边值条件, 故不妨假定$\mu_i\equiv 0$. 奇延拓区域至$(-l,l)$后周期延拓之即可. 例如下图所示的区域中解为



$$\begin{aligned}u(t,x)&=\frac{\varphi(x-at+2l)-\varphi(4l-x-at)}{2}\\&=\frac{1}{2a}\int_{x-at+2l}^{4l-x-at}\psi(\xi)\mathrm{d}\xi+\int_\Gamma f(\tau,w)\mathrm{d}\tau\mathrm{d}w.\end{aligned}$$

其中Γ为一切矩形区域(包括一条边位于底部的四边形), 且上至下第i块的面积符号为$(-1)^{i+1}$.

分离变量法求解波动方程

无源且满足**Dirichlet**条件之情形

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0 \quad x \in (0,l), t > 0 \\ t = 0 : u = \varphi(x), u_t = \psi(x) \\ x \in \{0, l\} : u = 0 \end{cases}$$

分离变量得特征值$\sqrt{\lambda_k} = \dfrac{k\pi}{l}$, 从而解具有一般形式

$$u(t,x) = \sum_{k \geq 1} (A_k \cos \frac{k\pi at}{l} + B_k \sin \frac{k\pi at}{l}) \sin \frac{k\pi x}{l}.$$

此处

$$\begin{cases} \varphi(x) = \displaystyle\sum_{k \geq 1} A_k \sin \frac{k\pi x}{l} \\ \psi(x) = \displaystyle\sum_{k \geq 1} \frac{k\pi a}{l} B_k \sin \frac{k\pi x}{l} \end{cases}$$

取标准正交系$\{e_k(x)\}_{k \geq 1}$为$e_k = \sqrt{\dfrac{2}{l}} \sin \dfrac{k\pi x}{l}$即可. 可验证$e_k$满足$e_k''(x) + \lambda e_k(x) = 0$与$e_k(0) = e_k(l) = 0$.

从而

$$\begin{cases} A_k = \displaystyle\frac{2}{l} \int_0^l \varphi(x) \sin \frac{k\pi x}{l} \mathrm{d}x \\ B_k = \displaystyle\frac{2}{k\pi a} \int_0^l \psi(x) \sin \frac{k\pi x}{l} \mathrm{d}x \end{cases}$$

特殊边值条件之情形

对一端固定, 一段自由之情形:

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0 \quad x \in (0,l), t > 0 \\ t = 0 : u = \varphi(x), u_t = \psi(x) \\ x = 0 : u = 0;\ x = l : u_x = 0 \end{cases}$$

则特征函数需满足$e_k(0)=0, e_k'(l)=0$, 即$e_k=\sin\dfrac{(k-\frac{1}{2})\pi at}{l}$.

$$u(t,x)=\sum_{k\geq 1}(A_k\cos\frac{(k-\frac{1}{2})\pi at}{l}+B_k\sin\frac{(k-\frac{1}{2})\pi at}{l})\sin\frac{(k-\frac{1}{2})\pi x}{l}.$$

对两端自由之情形, 即

$$\begin{cases}u_{tt}-a^2u_{xx}=0 \quad x\in(0,l), t>0\\ t=0: u=\varphi(x), u_t=\psi(x)\\ x\in\{0,l\}: u_x=0\end{cases}$$

此时特征函数满足$e_k''(x)+\lambda_k e_k(x)=0, e_k'(0)=e_k'(l)=0$, 从而$e_k(x)=C_0$或$e_k(x)=\cos(\lambda_k x)$, 其中$\lambda_k=k\pi/l$. 因此

$$u(t,x)=A_0+B_0t+\sum_{k\geq 1}(A_k\cos\frac{k\pi at}{l}+B_k\sin\frac{k\pi at}{l})\cos\frac{k\pi x}{l}.$$

对方程

$$\begin{cases}u_{tt}-a^2u_{xx}=0 \quad x\in(0,l), t>0\\ t=0: u=\varphi(x), u_t=\psi(x)\\ x=0: u=0; \ x=l: u_x+\sigma u=0\end{cases}$$

特征方程$e_k''(x)+\lambda e_k(x)=0, e_k(0)=0, e_k'(l)+\sigma e_k(l)=0$. 从而$\lambda_k$为方程

$$\tan\left(\sqrt{\lambda_k}l\right)=-\sqrt{\lambda_k}/\sigma$$

的根. 注意到$\sqrt{\lambda_k}\geq 0$, 故解具有一般形式

$$u(t,x)=\sum_{k\geq 1}(A_k\cos\sqrt{\lambda_k}at+B_k\sin\sqrt{\lambda_k}at)\cos\sqrt{\lambda_k}x.$$

可验证$\{e_k\}$仍为$(0,1)$上的正交基. 对该类方程, 若改写$x=0$时边值条件为

- $u_t=0$, 则λ_k满足$\cot\left(\sqrt{\lambda_k}l\right)=\sqrt{\lambda_k}/\sigma$. $\sqrt{\lambda_k}>0$.
- $u_x-\sigma' u=0$, 则λ_k满足$\tan\left(l\sqrt{\lambda_k}\right)=\dfrac{\sqrt{\lambda_k}(\sigma+\sigma')}{\lambda_k-\sigma\sigma'}$.

  此处不存在λ_k使得$\lambda_k=\sigma\sigma'$且$\tan\left(l\sqrt{\lambda_k}\right)=\infty$: 因为此时未定义式$(\lambda_k-\sigma\sigma')\tan\left(l\sqrt{\lambda_k}\right)$与$o(\sqrt{\lambda_k-\sigma\sigma'})$为等价无穷小.

有源情形

考虑方程

$$\begin{cases} u_{tt} - u_{xx} = f(t,x) \quad x \in (0,l),\, t > 0 \\ t = 0 : u = u_t = 0 \\ x \in \{0, l\} : u = 0 \end{cases}$$

采用齐次化方法, 考虑方程

$$\begin{cases} W_{tt} - W_{xx} = 0 \quad x \in (0,l),\, t > \tau \\ t = 0 : W = 0, W_t = f(x,\tau) \\ x \in \{0, l\} : W = 0 \end{cases}$$

解得

$$u(t,x) = \int_0^t \sum_{k\ge 1} B_k(\tau) \sin \frac{k\pi a(t-\tau)}{l} \sin \frac{k\pi x}{l} \mathrm{d}\tau$$

其中

$$B_k(\tau) = \frac{2}{k\pi a} \int_0^l f(\xi,\tau) \mathrm{d}\xi.$$

若$f(t,x) = \Phi(x)$, 则考虑$\Phi(x)$在$(0,l)$上的Fourier级数展开

$$\Phi(x) = \sum_{k\ge 1} C_k \sin \frac{2k\pi x}{l}.$$

注意到u解具有一般形式

$$u(t,x) = \sum_{k\ge 1} (A_k \cos \frac{k\pi at}{l} + B_k \sin \frac{k\pi at}{l}) \sin \frac{k\pi x}{l} = \sum_{k\ge 1} T_n(t) \sin \frac{k\pi x}{l}.$$

从而有常微分方程

$$\begin{cases} T_n''(t) + \frac{(ak\pi)^2}{l^2} T_n(t) = C_k \\ t = 0 : T_n = \partial_t T_n = 0. \end{cases}$$

解之得

$$T_n(t) = \frac{l^2 C_k}{(ak\pi)^2} \cdot [1 - \cos{(k\pi at/l)}].$$

$(0,l)$上一般情形之换元

考虑方程

$$
\begin{cases}
u_{tt}-u_{xx}=f(t,x) \quad x\in(0,l),\, t>0\\
t=0: u=\varphi(x),\, u_t=\psi(x)\\
x=0: u=\mu_1(t)\\
x=l: u=\mu_2(t)
\end{cases}
$$

先做代换$v=u-\mu_1(t)-\dfrac{x}{l}(\mu_2(t)-\mu_1(t))$, 则得方程

$$
\begin{cases}
v_{tt}-v_{xx}=\tilde{f}(t,x) \quad x\in(0,l),\, t>0\\
t=0: v=\tilde{\varphi}(x),\, v_t=\tilde{\psi}(x)\\
x=0: v\equiv 0\\
x=l: v\equiv 0
\end{cases}
$$

解之即可. 若换以不同的边值条件, 对应换元法如下(不唯一, 取$u-\tilde{u}=v$):

- $u(t,0)=\mu_1(t), u_x(t,l)=\mu_2(t)$, 则令$\tilde{u}=\mu_1(t)+x\mu_2(t)$.
- $u(t,0)=\mu_1(t)$, $(u_x+\sigma u)(t,l)=\mu_2(t)$. 则令

$$\tilde{u}=\mu_1+\frac{x(\mu_2-\sigma\mu_1)}{(1+\sigma l)l}$$

- $u_x(t,0)=\mu_1(t), u_x(t,l)=\mu_2(t)$. 则令

$$\tilde{u}=x\mu_1+\frac{x^2}{2l}(\mu_2-\mu_1)+F(t).$$

- $u_x(t,0)=\mu_1(t), (u_x+\sigma u)(t,l)=\mu_2(t)$. 则令

$$\tilde{u}=x\mu_1-\frac{l\sigma+1}{\sigma}\mu_1+\frac{1}{\sigma}\mu_2$$

- $(u_x-\sigma_1 u)(t,0)=\mu_1(t), (u_x+\sigma_2 u)(t,l)=\mu_2(t)$. 则令

$$\tilde{u}=-\frac{1}{\sigma_1}\mu_1+x^2\left(\frac{\sigma_2\mu_1+\mu_2}{\sigma_1(\sigma_2 l^2+2l)}\right)$$

高维波动方程解法

全空间的波动方程一般解

奇数维情形

通常采用使用球平均法解决Cauchy问题

$$
\begin{cases}
\partial_{tt}u - a^2 \sum_{i=1}^{2n+3} \partial_{x_j x_j} u = 0 \\
t = 0 : u = \varphi(x), u_t = \psi(x)
\end{cases}
$$

记平均函数

$$
M_u(t,x,r) = \int_{\partial B_{2n+3}(x,r)} u(t,y) \mathrm{d}S_y
$$

由于$\mathbb{R}^{2n+3}$中径向函数之Laplacian满足$\Delta = r^{-(2n+2)}\partial_r(r^{2n+2}\partial_r)$, 故原问题转化为

$$
\begin{cases}
\partial_{tt}M_u = a^2 r^{-(2n+2)}\partial_r(r^{2n+2}\partial_r)M_u \\
t = 0 : M_u = \int_{\partial B_{2n+3}(x,r)} \varphi(y)\mathrm{d}S_y \\
t = 0 : \partial_t M_u = \int_{\partial B_{2n+3}(x,r)} \psi(y)\mathrm{d}S_y
\end{cases}
$$

注意到

$$
[(r^{-1}\partial_r)^n r^{2n+1}](r^{-(2n+2)}\partial_r(r^{2n+2}\partial_r))[(r^{-1}\partial_r)^n r^{2n+1}]^{-1} = \partial_{rr}.
$$

令$[(r^{-1}\partial_r)^n r^{2n+1}]M_u = v$, 则PDE化为

$$
\begin{cases}
\partial_{tt}v = a^2\partial_{rr}v \\
t = 0 : v = [(r^{-1}\partial_r)^n r^{2n+1}] \int_{\partial B_{2n+3}(x,r)} \varphi(y)\mathrm{d}S_y \\
t = 0 : \partial_t v = [(r^{-1}\partial_r)^n r^{2n+1}] \int_{\partial B_{2n+3}(x,r)} \psi(y)\mathrm{d}S_y
\end{cases}
$$

解得(不妨限定$r < at$)

$$
\begin{aligned}
v = & \frac{[((at+r)^{-1}\partial_{at+r})^n (at+r)^{2n+1}] \int_{\partial B_{2n+3}(x,at+r)} \varphi(y)\mathrm{d}S_y}{2} \\
& - \frac{[((at-r)^{-1}\partial_{at-r})^n (at-r)^{2n+1}] \int_{\partial B_{2n+3}(x,at-r)} \varphi(y)\mathrm{d}S_y}{2} \\
& + \frac{1}{2a}\int_{at-r}^{at+r} (\xi^{-1}\partial_\xi)^n \xi^{2n+1} \int_{\partial B_{2n+3}(x,\xi)} \psi(y)\mathrm{d}S_y \mathrm{d}\xi
\end{aligned}
$$

当$r \ll 1$时有

$$
\begin{aligned}
v =& r\partial_t \left[(t^{-1}\partial_t)^n t^{2n+1} \int_{\partial B_{2n+3}(x,at)} \varphi(y) \mathrm{d}S_y \right] \\
&+ r(t^{-1}\partial_t)^n t^{2n+1} \int_{\partial B_{2n+3}(x,at)} \psi(y) \mathrm{d}S_y.
\end{aligned}
$$

注意到在小范围内, $v \sim kr^{2n+1}$, 且$(r^{-1}\partial_r)^n r^{2n+1} : \dfrac{k}{(2n+1)!!} \mapsto kr$. 从而

$$
\begin{aligned}
u =& \lim_{r\to 0} \frac{1}{|\partial B_{2n+3}(x,r)|} M_u(t,x,r) \\
=& \frac{1}{(2n+1)!!|\omega_{2n+2}|a^{2n+2}} \cdot \partial_t \left[(t^{-1}\partial_t)^n t^{2n+1} \int_{\partial B_{2n+3}(x,at)} \varphi(y) \mathrm{d}S_y \right] \\
&+ \frac{1}{(2n+1)!!|\omega_{2n+2}|a^{2n+2}} \cdot (t^{-1}\partial_t)^n t^{2n+1} \int_{\partial B_{2n+3}(x,at)} \psi(y) \mathrm{d}S_y
\end{aligned}
$$

其中$(2n+1)!!|\omega_{2n+2}| = 2^{n+2}\pi^{n+1}$.

偶数维情形

对偶数维情形, 不妨扩充PDE

$$
\begin{cases}
\partial_{tt}u - a^2 \sum_{i=1}^{2n+2} \partial_{x_j x_j} u = 0 \\
t = 0 : u = \varphi(x), u_t = \psi(x)
\end{cases}
$$

至

$$
\begin{cases}
\partial_{tt}u - a^2 \sum_{i=1}^{2n+3} \partial_{x_j x_j} u = 0 \\
t = 0 : u = \varphi(x), u_t = \psi(x)
\end{cases}
$$

其中假定φ, ψ与x_{2n+3}无关, 因此

$$
\begin{aligned}
\int_{\partial B_{2n+3}(x,at)} h(y) \mathrm{d}S_y &= 2\int_{B_{2n+2}(x,at)} h(y_1, \ldots, y_{2n+2}) \cdot \frac{\mathrm{d}\sigma}{\cos\gamma} \\
&= 2\int_{B_{2n+2}(x,at)} h(y_1, \ldots, y_{2n+2}) \cdot \frac{at\mathrm{d}\sigma}{\sqrt{(at)^2 - |y-x|^2}} \\
&= 2at\int_{B_{2n+2}(x,at)} \frac{h(y)\mathrm{d}S_y}{\sqrt{(at)^2 - |y-x|^2}}
\end{aligned}
$$

从而

$$
\begin{aligned}
u =& \frac{1}{(2n+1)!!|\omega_{2n+2}|a^{2n+2}} \cdot \partial_t \left[(t^{-1}\partial_t)^n t^{2n} \int_{\partial B_{2n+3}(x,at)} \varphi(y) \mathrm{d}S_y \right] \\
&+ \frac{1}{(2n+1)!!|\omega_{2n+2}|a^{2n+2}} \cdot (t^{-1}\partial_t)^n t^{2n} \int_{\partial B_{2n+3}(x,at)} \psi(y) \mathrm{d}S_y \\
=& \frac{1}{(2\pi)^{n+1}a^{2n+1}} \cdot \partial_t \left[(t^{-1}\partial_t)^n t^{2n} \int_{B_{2n+2}(x,at)} \frac{\varphi(y)\mathrm{d}S_y}{\sqrt{(at)^2 - |y-x|^2}} \right] \\
&+ \frac{1}{(2\pi)^{n+1}a^{2n+1}} \cdot (t^{-1}\partial_t)^n t^{2n} \int_{B_{2n+2}(x,at)} \frac{\psi(y)\mathrm{d}S_y}{\sqrt{(at)^2 - |x-y|^2}}
\end{aligned}
$$

特别地, 二维极坐标解为

$$
\begin{aligned}
u(t,x,y) &= \frac{1}{2\pi a} \partial_t \left[\int_0^{at} \int_{S^1} \frac{\varphi(x + r\cos\theta, y + r\sin\theta)}{\sqrt{(at)^2 - r^2}} r \mathrm{d}r \mathrm{d}\theta \right] \\
&= \frac{1}{2\pi a} \int_0^{at} \int_{S^1} \frac{\psi(x + r\cos\theta, y + r\sin\theta)}{\sqrt{(at)^2 - r^2}} r \mathrm{d}r \mathrm{d}\theta
\end{aligned}
$$

径向解可参考此处.

有源情形

对非齐次波动方程

$$
\begin{cases}
\partial_{tt} u - a^2 \sum_{i=1}^{m} \partial_{x_j x_j} u = f(t,x) \\
t = 0 : u = 0, u_t = 0
\end{cases}
$$

转化为

$$
\begin{cases}
\partial_{tt} W - a^2 \sum_{i=1}^{m} \partial_{x_j x_j} W = 0, t > \tau \\
t = \tau : u = 0, u_t = f(x,\tau)
\end{cases}
$$

则解为

$$
u(t,x) = \int_0^t W(\tau, x) \mathrm{d}\tau.
$$

二维与三维波动方程

解与极坐标换元

$n=3$时, 方程

$$\begin{cases}\partial_{tt}u-a^2\sum_{i=1}^{3}\partial_{x_jx_j}u=f(t,x)\\ t=0:u=\varphi(x),u_t=\psi(x)\end{cases}$$

解为

$$\begin{aligned}u(x,t)=&\partial_t\left[\frac{1}{4\pi a^2t}\int_{\partial B_3(x,at)}\varphi(x')\mathrm{d}S_{x'}\right]+\frac{1}{4\pi a^2t}\int_{\partial B_3(x,at)}\psi(x')\mathrm{d}S_{x'}\\&+\int_0^t\frac{1}{4\pi a^2\tau}\int_{\partial B_3(x,a\tau)}f(\tau,x'')\mathrm{d}S_{x''}\end{aligned}$$

极坐标换元得

$$\begin{cases}x=at\cos\theta\cos\alpha\\ y=at\sin\theta\cos\alpha\\ z=at\sin\alpha\\ \mathrm{d}S_y=(at)^2\mathrm{d}\sin\alpha\mathrm{d}\theta\end{cases}$$

$n=2$时, 方程

$$\begin{cases}\partial_{tt}u-a^2\sum_{i=1}^{2}\partial_{x_jx_j}u=f(t,x)\\ t=0:u=\varphi(x),u_t=\psi(x)\end{cases}$$

解为

$$\begin{aligned}u(x,t)=&\partial_t\left[\frac{1}{2\pi a}\int_{\partial B_2(x,at)}\frac{\varphi(x')}{\sqrt{(at)^2-|x-x'|^2}}\mathrm{d}S_{x'}\right]\\&+\frac{1}{2\pi a}\int_{\partial B_2(x,at)}\frac{\psi(x')}{\sqrt{(at)^2-|x-x'|^2}}\mathrm{d}S_{x'}\\&+\int_0^t\frac{1}{2\pi a}\int_{\partial B_2(x,a\tau)}\frac{f(\tau,x'')}{\sqrt{(at)^2-|x-x''|^2}}\mathrm{d}S_{x''}\end{aligned}$$

递推法

例1 (数学物理方法P34-1-1)

$$\begin{cases} u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz}) \\ t = 0 : u = 0, u_t = x^2 + yz \end{cases}$$

解具有一般形式$u = t(x^2 + yz) + t^2 \cdot (\cdots)$, 注意到

$$\begin{aligned} \partial_{tt} - a^2\Delta : & t(x^2 + yz) \mapsto -2a^2 t \\ & \frac{t^3 a^2}{3} \mapsto 2a^2 t \end{aligned}$$

从而$u = t(x^2 + yz) + \dfrac{a^2 t^3}{3}$.

例2 (数学物理方法P34-3)

$$\begin{cases} u_{tt} = a^2(u_{xx} + u_{yy}) \\ t = 0 : u = x^2(x + y), u_t = 0 \end{cases}$$

解具有一般形式$u = x^2(x + y) + t^2 \cdot (\cdots)$, 注意到

$$\begin{aligned} \partial_{tt} - a^2\Delta : & x^2(x + y) \mapsto -6a^2 x - 2a^2 y \\ & \frac{t^2}{2}(6a^2 x + 2a^2 y) \mapsto 6a^2 x + 2a^2 y \end{aligned}$$

从而$u = x^2(x + y) + a^2 t^2(3x + y)$.

例3 (数学物理方法P34-8)

$$\begin{cases} u_{tt} = u_{xx} + u_{yy} + u_{zz} + 2(y - t) \\ t = 0 : u = 0, u_t = x^2 + yz \end{cases}$$

注意到

$$\begin{aligned} \partial_{tt} - \Delta : & -\frac{t^3}{3} \mapsto -2t \\ & t^2 y \mapsto 2y \end{aligned}$$

从而设$v = u - t^2 y + \dfrac{t^3}{3}$, 则$v$满足方程

$$\begin{cases} v_{tt} = v_{xx} + v_{yy} + v_{zz} \\ t = 0 : v = 0; v_t = x^2 + yz \end{cases}$$

可口算得$v=t(x^2+yz)+\dfrac{t^3}{3}$, 从而$u=t(x^2+yz)+t^2y$.

一般地, 有

$$\begin{cases} u_{tt}-\displaystyle\sum_{i=1}^{p}u_{x_ix_i}=f(t,x) \\ t=0:u=\varphi(x),u_t=\psi(x) \end{cases}$$

且$f(t,x)$, $\varphi(x)$与$\psi(x)$均为t,x_i与相关之有限多项式(暂定之). 考虑算子$P:=\partial_{tt}-\sum_{i=1}^{p}\partial_{x_ix_i}$, 并注意到:

$$\begin{aligned} P:&\frac{t^{m+2}x^\alpha}{(m+2)(m+1)}\mapsto t^m x^\alpha-\frac{t^{m+2}\Delta x^\alpha}{(m+2)(m+1)} \\ &\frac{t^{m+4}\Delta x^\alpha}{A_{m+4}^4}\mapsto\frac{t^{m+2}\Delta x^\alpha}{A_{m+2}^2}-\frac{t^{m+4}\Delta^2x^\alpha}{A_{m+4}^4} \\ &\cdots \end{aligned}$$

从而$P:\sum_{n\geq1}\dfrac{t^{m+2n}\Delta^{n-1}x^\alpha}{A_{m+2n}^{2n}}\mapsto t^m x^\alpha$. 令

$$v=u-\sum_{\alpha}\sum_{n\geq1}\frac{t^{m+2n}\Delta^{n-1}x^\alpha}{A_{m+2n}^{2n}}.$$

则v满足以下PDE系统(实际上已完成齐次化)

$$\begin{cases} v_{tt}-\displaystyle\sum_{i=1}^{p}v_{x_ix_i}=f(t,x) \\ t=0:v=\varphi(x),v_t=\psi(x) \end{cases}$$

依照先前递推式, 解得

$$v(t,x)=\sum_{n\geq0}\left(\frac{t^{2n}\Delta^n\varphi(x)}{(2n)!}+\frac{t^{2n+1}\Delta^n\psi(x)}{(2n+1)!}\right).$$

综上,

$$u(t,x)=\sum_{n\geq0}\left(\frac{t^{2n}\Delta^n\varphi(x)}{(2n)!}+\frac{t^{2n+1}\Delta^n\psi(x)}{(2n+1)!}\right)+\sum_{\alpha}\sum_{n\geq1}\frac{t^{m+2n}\Delta^{n-1}x^\alpha}{A_{m+2n}^{2n}}.$$

f项也可采用Duhamel原理叙述, 即

$$P:\int_0^t\sum_{n\geq0}\frac{\tau^{2n+1}\Delta_x^n f(\tau,x)}{(2n+1)!}\mathrm{d}\tau\mapsto f(t,x).$$

非全空间的波动方程

径向对称情形

以如下方程为例:

$$\begin{cases}\partial_{tt}u-\Delta u=0 \quad t>0,r>1\\ t=0:u=\varphi(r),u_t=\psi(r)\\ r=1:\dfrac{\partial u}{\partial n}=0\end{cases}$$

换元$v=ru$, 降维得

$$\begin{cases}\partial_{tt}v-\partial_{rr}v=0 \quad t>0,r>1\\ t=0:u=r\varphi(r),u_t=r\psi(r)\\ r=1:w_r-w=0\end{cases}$$

故当$r\geq t+1$时

$$w=\frac{(r+t)\varphi(r+t)+(r-t)\varphi(r-t)}{2}+\frac{1}{2}\int_{r-t}^{r+t}\omega\psi(\omega)\mathrm{d}\omega.$$

当$1\leq r\leq t+1$时, 记$w=F(r-t)+G(r_t)$. 代入$r=t+1,1$得

$$\begin{cases}F(1)+G(2t+1)=\dfrac{(2t+1)\varphi(2t+1)+\varphi(1)}{2}+\dfrac{1}{2}\displaystyle\int_1^{2t+1}\omega\psi(\omega)\mathrm{d}\omega\\ F'(1-t)+G'(1+t)=F(1-t)+G(1+t)\end{cases}$$

解得

$$\begin{cases}G(\xi)=G(1)-\varphi(1)+\dfrac{\xi\varphi(\xi)+\varphi(1)}{2}+\dfrac{1}{2}\displaystyle\int_1^t\omega\psi(\omega)\mathrm{d}\omega\\ F(\eta)=e^{\eta-1}(\varphi(1)-2G(1))+G(2-\eta)+2e^{\eta}\displaystyle\int_1^{\eta}e^{-\omega}G(2-\omega)\mathrm{d}\omega\end{cases}$$

从而

$$u=\begin{cases}\dfrac{(r+t)\varphi(r+t)+(2-r-t)\varphi(2-r-t)}{2r}+\dfrac{1}{2}\displaystyle\int_{2-r-t}^{r+t}\omega\psi(\omega)\mathrm{d}\omega\\ \quad-\dfrac{e^{r-t-2}}{r}\displaystyle\int_1^{2-r-t}\omega e^{\omega}(\varphi(\omega)-\psi(\omega)\mathrm{d}\omega), \quad 1\leq r\leq t+1\\ \dfrac{(r+t)\varphi(r+t)+(r-t)\varphi(r-t)}{2r}+\dfrac{1}{2r}\displaystyle\int_{r-t}^{r+t}\omega\psi(\omega)\mathrm{d}\omega\end{cases}$$

一般情形

对方程

$$\begin{cases}\partial_{tt}u-a^2\sum_{i=1}^m\partial_{x_jx_j}u=f(t,x) \quad ,x\in\Omega,t>0\\ t=0:u=\varphi(x),u_t=\psi(x)\\ u\text{ saitisfies certain boundary conditions on }\partial\Omega\end{cases}$$

设$u(t,x)=T(t)X(x)$, 考虑特征函数X_k满足

- 在区域Ω上, $\Delta X_k=\lambda_kX_k$,
- X_k满足同样的边界条件.

从而$T_k''(t)+a^2\lambda_kT_k(t)=0$. 解得:

- 当$\lambda_k>0$时, $T_k=A_k\cos\left(\sqrt{\lambda_k}at\right)+B_k\sin\left(\sqrt{\lambda_k}at\right)$.
- 当$\lambda_k=0$时, $T_k=A_k+B_kt$.
- 当$\lambda_k<0$时, $T_k=A_k\cosh\left(\sqrt{-\lambda_k}at\right)+B_k\sinh\left(\sqrt{-\lambda_k}at\right)$.

从而

$$u(t,x)=\sum_kT_k(t)X_k(x).$$

其中

$$\begin{cases}\varphi(x)=\sum_k\frac{\int_\Omega X_k(x)\varphi(x)\mathrm{d}x}{\int_\Omega X_k(x)^2\mathrm{d}x}X_k(x)\\ \psi(x)=\sum_{k,\lambda\neq0}\frac{1}{a\sqrt{|\lambda_k|}}\frac{\int_\Omega X_k(x)\psi(x)\mathrm{d}x}{\int_\Omega X_k(x)^2\mathrm{d}x}X_k(x)\\ \qquad\qquad+\sum_{k,\lambda=0}\frac{\int_\Omega X_k(x)\psi(x)\mathrm{d}x}{\int_\Omega X_k(x)^2\mathrm{d}x}X_k(x)\end{cases}$$

Cauchy问题补充

关于波的衰减:

- 对一切x, 一致地有$u(t,x)=O(t^{-\frac{n-1}{2}})$.
- 对任意给定的x, $|u(t,x)|\leq C(1+t)^{-\frac{n-1}{2}}$.

能量积分法

Grönwall不等式

能量法旨在证明解的唯一性与稳定性. 能量函数常用Grönwall不等式控制, 即对有界的非负连续函数$u, v : [0, T] \to \mathbb{R}_+$与一致有界的函数$K$使得

$$u(t) \leq K(t) + \int_0^t u(s)v(s)\mathrm{d}s.$$

则$u(t) \leq \|K\|_\infty \cdot \exp\int_0^t v(s)\mathrm{d}s$. 令$v(s) = \lambda_0, u(s) = p'(s)$, 则

$$p'(t) \leq K(s) + \lambda_0 p(t) \implies p(t) \leq \|K(s)\|_\infty \cdot e^{\lambda_0 t}.$$

波动方程

有界波动方程

以满足给定边值条件的一维波动方程为例:

$$\begin{cases} u_{tt} - a^2 u_{xx} = f(t, x), \quad t > 0, 0 < x < l \\ t = 0 : u = \varphi(x), u_t = \psi(x) \\ x = 0 : \alpha_1 u_x - \beta_1 u = \lambda(t) \\ x = l : \alpha_2 u_x + \beta_2 u = \mu(t) \\ \alpha_i, \beta_i \geq 0, \alpha_i^2 + \beta_i^2 > 0 \end{cases}$$

为证明解至多唯一, 只需取任意两解u_1, u_2, 记$\tilde{u} = u_1 - u_2$. 故

$$\begin{aligned} 0 &= \int_0^l f(t, x)\tilde{u}_t(x)\mathrm{d}x \\ &= \int_0^l (\tilde{u}_{tt} - a^2\tilde{u}_{xx})\tilde{u}_t(x)\mathrm{d}x \\ &= \int_0^l \left(\frac{\tilde{u}_t^2}{2}\right)_t + \left(\frac{a^2\tilde{u}_x^2}{2}\right)_t \mathrm{d}x - a^2[\tilde{u}_x\tilde{u}_t]_0^l \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^l \frac{\tilde{u}_t^2 + a^2\tilde{u}_x^2}{2}\mathrm{d}x - a^2[\tilde{u}_x\tilde{u}_t]_0^l \end{aligned}$$

对$\alpha_i \neq 0$之情形(即robin条件), 有

$$-[\tilde{u}_x\tilde{u}_t]_0^l = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\beta_1}{2\alpha_1}\tilde{u}^2|_{x=0} + \frac{\beta_2}{2\alpha_2}\tilde{u}^2|_{x=l}\right).$$

对某侧$\alpha_i = 0$之情形(此时$\beta_i \neq 0$), 则对应的$\tilde{u}_t = 0$. 记能量函数

$$E(t) = \frac{1}{2}\int_0^l \frac{\tilde{u}_t^2 + a^2\tilde{u}_x^2}{2}\mathrm{d}x + \frac{\beta_1}{2\alpha_1}\tilde{u}^2|_{x=0} + \frac{\beta_2}{2\alpha_2}\tilde{u}^2|_{x=l}$$

从而$E(t) \equiv E(0) = 0$, 即$\tilde{u}_t = \tilde{u}_x \equiv 0$. 即$\tilde{u} = \tilde{u}(0,0) = 0$.

为证明稳定性, 由于存在v满足$v_{tt} - a^2 v_{xx} = 0$与边值条件, 则只需考虑$w = u - v$之稳定性, 即以下方程解稳定:

$$\begin{cases} w_{tt} - a^2 w_{xx} = f, \quad t > 0, 0 < x < l \\ t = 0 : w = \varphi(x), w_t = \psi(x) \\ x = 0 : \alpha_1 w_x - \beta_1 w = 0 \\ x = l : \alpha_2 w_x + \beta_2 w = 0 \\ \alpha_i, \beta_i \geq 0, \alpha_i^2 + \beta_i^2 > 0 \end{cases}$$

同上令$E(t)$, 对任意给定的$T > 0$, 取$t \in (0, T)$总有

$$E'(t) \leq \int_0^l f^2 \mathrm{d}x + \int_0^l u_t^2 \mathrm{d}x = E(t) + \int_0^l f^2 \mathrm{d}x.$$

从而$E(t) \leq (E(0) + \int_0^T e^{-t}\int_0^l f^2 \mathrm{d}x\mathrm{d}\tau)e^t$. 即存在常数$C^*(T)$使得

$$E(t) \leq C^*(T)(E(0) + \int_0^t \int_0^l f^2 \mathrm{d}x\mathrm{d}\tau).$$

记$E_0 = \int_0^l u^2(t,x)\mathrm{d}x$, 则$E_0'(t) \leq E_0(t) + E(t)$. 可解得, $E_0(t)$亦被限制于某一常数内. 从而存在常数$C(T)$使得

$$E(t) + E_0(t) \leq C(T)(E_0(0) + E(0) + \int_0^T \int_0^l f^2 \mathrm{d}x\mathrm{d}\tau).$$

对高维情形, 以判定下列方程解之至多唯一性于稳定性为例:

$$\begin{cases} u_{tt} - a^2 \Delta_x u = f(x,t) \quad x \in \Omega \subset \mathbb{R}^n, t > 0 \\ t = 0 : u = \varphi(x), u_t = \psi(x) \\ \text{some given boundary conditions} \end{cases}$$

Step I: 对适当的给定区域(如Ω), 记能量函数

$$E(t) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega u_t^2 + a^2|\nabla u|^2 \mathrm{d}x \quad \text{iff Dirichlet/Newman} \\ \dfrac{1}{2}\displaystyle\int_\Omega u_t^2 + a^2|\nabla u|^2 \mathrm{d}x + \dfrac{a^2}{2}\displaystyle\int_{\partial\Omega} u^2\sigma(S)\mathrm{d}S \quad \text{Robin} \end{cases}$$

Step II: 当$f = 0$时, 能量函数$E'(t) = 0$. 因此任意两解之差恒为零, 即解至多唯一.

Step III: 给时间区间$(0,T)$, 能量函数

$$E'(t)=\int_\Omega fu_t\mathrm{d}x\leq\frac{1}{2}\int_\Omega f^2\mathrm{d}x+E(t).$$

考虑对$e^{-t}E(t)$求导, 则解得

$$E(t)\leq C_1(T)\left(E(0)+\int_0^T\int_\Omega f^2\mathrm{d}x\mathrm{d}\tau\right)$$

Step IV: 记$E_0(t)=\dfrac{1}{2}\displaystyle\int_0^l u^2\mathrm{d}x$, 则$E_0'(t)\leq E_0(t)+E(t)$. 代入$E(t)$可解得

$$E(t)\leq C_2(T)\left(E(0)+E_0(0)+\int_0^T\int_\Omega f^2\mathrm{d}x\mathrm{d}\tau\right).$$

从而

$$E_0(t)+E(t)\leq C(T)\left(E_0(0)+E(0)+\int_0^T\int_\Omega f^2\mathrm{d}x\mathrm{d}s\right).$$

半无界波动方程的能量积分

对半无界区域而言, 作依赖区域所在的锥体与$\mathbb{R}_+\times\Omega$之解之相交部分即可. 为证明方便故, 常将相交部分扩充至圆台使得各个t对应的区域有统一且相似的表达式. 若Ω为有界区域, 则取柱体分析即可. 下以半区域的n维波动方程为例

考虑方程(证明解至多唯一, 且关于初值/源稳定)

$$\begin{cases}u_{tt}-a^2\Delta_x u=f, \quad t>0,x\in\Omega:=\mathbb{R}^{n-1}\times\mathbb{H}\\ t=0:u=\varphi(x),y_t=\psi(x)\\ x_n=0:\alpha u_{x_n}-\beta u=\mu(t)\end{cases}$$

Step I: 任取$p\in\Omega$, 取合适的区域(例如半圆台形), 记$\Omega_t=\{x:|x-r|\leq a(|r|-at)\}$, 其中$p\in\{t_p\}\times\Omega_{t_p}$, $r=\{0\}^{n-1}\times|r|$为给定值. 记能量函数

$$E(t)=\frac{1}{2}\int_{\Omega_t}|u_t|^2+a^2|\nabla u|^2\mathrm{d}x.$$

Step II: Robin条件下考虑对区域

$$V_T:=\cup_{0<t<T}(\{t\}\times\Omega_t)$$

积分, 其中$T<|r|$. 故

$$
\begin{aligned}
\int_{V_T} u_t f \mathrm{d}x\mathrm{d}t =& \int_{V_T} u_t(u_{tt} - a^2\Delta u)\mathrm{d}x\mathrm{d}t \\
=& \int_{V_T} (\frac{u_t^2 + |a\nabla u|^2}{2})_t - \sum_i (u_{x_i}u_t)_{x_i}\mathrm{d}x\mathrm{d}t \\
=& \int_{\partial V_T} \frac{u_t^2 + |a\nabla u|^2}{2} n_t - \sum_i (u_{x_i}u_t)n_{x_i}\mathrm{d}S \\
=& E(t) - E(0) + \int_{\partial V_T, x_n=0} u_{x_n}u_t\mathrm{d}S \\
&+ \frac{1}{\sqrt{1+a^2}} \int_{\partial V_T, |x|+t=T} (u_t + \alpha\cdot a\nabla u)^2\mathrm{d}S \\
\geq& E(t) - E(0) + \frac{1}{2}\int_{\partial V_T, x_n=0} \sigma(u^2)_t\mathrm{d}S \\
=& E(t) - E(0) + \frac{1}{2}\int_{\partial(\partial V_T, x_n=0)} \sigma u^2 n_t\mathrm{d}s \\
\geq& E(t) - E(0) - \int_{V_T:t=x_n=0} \frac{\sigma u^2}{2}\mathrm{d}s \\
&+ \int_{V_T:t=t,x_n=0} \frac{\sigma u^2}{2}\mathrm{d}s
\end{aligned}
$$

记能量函数

$$
E_3(t) = \int_{V_T\cap(\{t\}\times\Omega)} \frac{u_t^2 + |a\nabla u|^2}{2}\mathrm{d}x + \int_{V_T\cap(\{t\}\times\partial\Omega)} \frac{\sigma u^2}{2}\mathrm{d}s
$$

明所欲证.

Dirichlet或Newman条件下的能量函数为

$$
E_{1,2}(t) = \int_{V_T\cap(\{t\}\times\Omega)} \frac{u_t^2 + |a\nabla u|^2}{2}\mathrm{d}x.
$$

Step III: 给时间区间$(0, T)$, 能量函数

$$
E'(t) = \int_\Omega f u_t\mathrm{d}x \leq \frac{1}{2}\int_{\Omega_T} f^2\mathrm{d}x + E(t).
$$

考虑对$e^{-t}E(t)$求导, 则解得

$$
E(t) \leq C_1(T)\left(E(0) + \int_0^T \int_{\Omega_T} f^2\mathrm{d}x\mathrm{d}\tau\right)
$$

Step IV: 记$E_0(t) = \dfrac{1}{2}\displaystyle\int_{\Omega_t} u^2\mathrm{d}x$, 则$E_0'(t) \leq E_0(t) + E(t)$. 代入$E(t)$可解得

$$E(t) \leq C_2(T)\left(E(0)+E_0(0)+\int_0^T\int_{\Omega_T} f^2 \mathrm{d}x\mathrm{d}\tau\right).$$

从而

$$E_0(t)+E(t) \leq C(T)\left(E_0(0)+E(0)+\int_0^T\int_{\Omega_T} f^2 \mathrm{d}x\mathrm{d}s\right).$$

从而解关于初值$(E_0(0), E(0))$与源f稳定.

例题

(数学物理方法 P46-1) 证明含阻尼项的有界$x \in (0, l)$振动方程($c > 0$)

$$\begin{cases} u_{tt} = a^2 u_{xx} - cu_t + f(t,x) \\ u(0,t) = \mu_1(t), u(l,t) = \mu_2(t) \\ t = 0 : u = \varphi(x), u_t = \psi(x) \end{cases}$$

解至多唯一, 且关于初边值稳定.

证明: 证明唯一性. 即证明

$$\begin{cases} v_{tt} = a^2 v_{xx} - cv_t \\ v(0,t) = 0, v(l,t) = 0 \\ t = 0 : v = 0, v_t = 0 \end{cases}$$

的唯一解为零解. 注意到

$$\begin{aligned} 0 &= \int_0^l v_t(v_{tt} + cv_t - a^2 v_{xx}) \mathrm{d}x \\ &= \int_0^l \left(\frac{v_t^2}{2}\right)_t + cv_t^2 + a^2\left(\frac{v_x^2}{2}\right)_t \mathrm{d}x \\ &\geq \int_0^l \left(\frac{v_t^2 + a^2 v_x^2}{2}\right)_t \mathrm{d}x \end{aligned}$$

记能量函数

$$E(t) = \int_0^l \frac{v_t^2 + a^2 v_x^2}{2} \mathrm{d}x.$$

从而$v_t \equiv 0$, $v_x \equiv 0$. 再由$v_0 = 0$知$v \equiv 0$.

再证明稳定性. 令$E_0(t) = \int_0^l u^2 \mathrm{d}x$, 从而对任意$T > 0, t \in (0, T)$均有

$$E_0'(t)\le\int_0^l u^2+u_t^2\mathrm{d}x\le E_0(t)+E(t)\le E_0(t)+E(0).$$

由Grönwall不等式知

$$E_0(t)\le E(0)(e^t-1)+E_0(0)e^t.$$

从而解关于初始值稳定.

有外力时, 定解问题转化为

$$\begin{cases}v_{tt}=a^2v_{xx}-cv_t+f\\ v(0,t)=0,v(l,t)=0\\ t=0:v=0,v_t=0\end{cases}$$

从而$E(0)=E_0(0)=0$. 注意到

$$E'(t)\le\int_0^l -cu_t^2+fu_t\mathrm{d}x\le E(t)+\int_0^l f^2\mathrm{d}u.$$

从而

$$E(t)\le E(0)e^t+e^t\int_0^t e^{-\tau}\int_0^l f^2\mathrm{d}x\mathrm{d}\tau\le e^T\int_0^T\int_0^l f^2\mathrm{d}x\mathrm{d}t.$$

故解关于初值及扰动项稳定.

一阶线性偏微分方程组

考虑方程

$$\begin{cases}\partial_tU+A\partial_xU+BU=F,x\in\mathbb{R},t>0\\ U(0,x)=\varphi(x)\end{cases}$$

为证明解至多唯一, 只需令$U(0,x)=0,F=0$, 并证明零解为唯一解.

任取$T>0$, 在$0<t<T$区间内取$\lambda:=\inf_{t\in[0,T]}\lambda_{\min}(A)$, $\mu:=\sup_{t\in[0,T]}\lambda_{\max}(A)$. 做区域$\Omega_t=\{x:t+\dfrac{x}{\mu}\le T,t-\dfrac{x}{\lambda}\le T,t>0\}$. 从而

$$
\begin{aligned}
0 &= \int_{\Omega_t} U^T(U_t + AU_x + BU)\mathrm{d}x \\
&= \int_{\Omega_t} \frac{(|U|^2)_t}{2} + U^T(B - A_x)U + \frac{(U^TAU)_x}{2}\mathrm{d}x \\
&= \int_{\Omega_t} \frac{(|U|^2)_t}{2} + U^T(B - A_x)U\mathrm{d}x + \frac{U^TAU}{2}|_{\lambda(T-t)}^{\mu(T-t)} \\
&\geq \int_{\Omega_t} U^T(B - A_x)U\mathrm{d}x
\end{aligned}
$$

记能量函数为

$$
E(t) = \int_{\Omega_t} \frac{(|U|^2)_t}{2}\mathrm{d}x + \frac{U^TAU}{2}|_{\lambda(T-t)}^{\mu(T-t)}
$$

故

$$
\begin{aligned}
E'(t) &= \partial_t \int_{\Omega_t} U^T(B - A_x)U\mathrm{d}x \\
&= U^T(B - A_x)U|_{\lambda(T-t)}^{\mu(T-t)} + \int_{\Omega_t} \partial_t(U^T(B - A_x)U)\mathrm{d}x \\
&\leq C(T)E(t)
\end{aligned}
$$

从而$E(t) \leq E(0)e^{C(T)t}$. 初值为0时唯一解即零解.

若考虑F项, 则可同上解得以下不等式

$$
E'(t) \leq \tilde{C}(T)E(t) + \int_{\Omega_t} |F|^2\mathrm{d}x.
$$

从而

$$
E(t) \leq e^{\tilde{C}(T)t}\left(E(0) + \int_0^t \int_{\Omega_t} |F|^2\right).
$$

记$E_0(t) = \int_{\Omega_t} U^TU\mathrm{d}x$. 故$E_0'(t) \leq E_0(t) + E(t)$. 从而

$$
E_0(t) + E(t) \leq (1 + e^{\tilde{C}(T)t})\left(E_0(0) + E(0) + \int_0^T \int_{\Omega_T} f^2\mathrm{d}x\mathrm{d}s\right).
$$

从而解关于初值$(E_0(0), E(0))$与源f稳定.

Fourier法与基本解

Fourier变换简介

记$\mathbb{R}^n$上的Fourier变换(有处定义不采用$(2\pi)^{-n/2}$)为

$$\mathscr{F}: f(x) \mapsto \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-i\xi\cdot x} \mathrm{d}x.$$

相应地逆变换为

$$\mathscr{F}^{-1}: f(x) \mapsto \check{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{i\xi\cdot x} \mathrm{d}x.$$

对速降空间(Schwarz space)$\mathcal{S}$, $\mathscr{F}: \mathcal{S} \to \mathcal{S}$为双射. 同时, $\mathscr{F}: L^2(\Omega) \to L^2(\Omega)$亦为双射(设函数在相差零测集的意义下相同). 一般地, 对任意$p \in (1, \infty)$, 有双射关系

$$\mathscr{F}: L^p(\Omega) \to L^{p^*}(\Omega).$$

其中共轭指标满足$p^{-1} + (p^*)^{-1} = 1$. 该定理为Riesz-Thorin定理.

当$p = p^* = \dfrac{1}{2}$时$\mathscr{F}$保距, 即对任意$f, g \in L^2(\Omega)$均有

$$\langle f, g \rangle = \langle \mathscr{F}[f], \mathscr{F}[g] \rangle.$$

简单的Fourier变换

考虑$\mathscr{F}: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, 则

- $\mathscr{F}$保持线性, 即保持自变量的加和与数乘.
- $\mathscr{F}[f \circ (-x_0)](\xi) = \hat{f}(\xi) \cdot e^{-ix_0\cdot\xi}$.
- $\mathscr{F}[f \circ (c\cdot)](\xi) = c^{-n} \hat{f}(c^{-1}\xi)$.
- (接上条) 对非奇异常矩阵A, $\mathscr{F}[f \circ (A\cdot)] = (\det A)^{-1} \hat{f}({}^tA^{-1}\xi)$.
- $\mathscr{F}[f * g] = (2\pi)^{n/2} \mathscr{F}[f] \cdot \mathscr{F}[g]$.
- $\mathscr{F}[f \cdot g] = (2\pi)^{-n/2} \mathscr{F}[f](\xi) * \mathscr{F}[g](\xi)$.
- $\mathscr{F}[\partial_{x_j} f] = i\xi_j \cdot \mathscr{F}[f]$. 常以方便故记$\mathcal{D}_{x_j} := \dfrac{\partial_{x_j}}{i}$.
- $\mathscr{F}[(\prod_\alpha i^{-k}\xi_k) \cdot f](\xi) = \partial^\alpha \mathscr{F}[f]$.
- (接上条) 设$\alpha = (\alpha(1), \alpha(2), \ldots, \alpha(n))$为指标, 并定义$\mathcal{D}^\alpha = \prod_k \mathcal{D}_{x_k}^{\alpha(k)}$, $\xi^\alpha = \prod_k \xi_k^{\alpha(k)}$. 则

$$\mathscr{F}[\mathcal{D}^\alpha f]=\xi^\alpha\mathscr{F}[f].$$

同理, 对关于若干α的多项式$P(\Lambda)=P(\alpha,\beta,\cdots,\gamma)$, 有

$$\mathscr{F}[\mathcal{D}^{P(\Lambda)}f]=\xi^{P(\Lambda)}\mathscr{F}[f].$$

- $\mathscr{F}^2: f(x)\mapsto f(-x)$. $\mathscr{F}^4$恒等.

Fourier变换法应用

对以下方程

$$\begin{cases}u_{tt}+a^2u_{xxxx}=0\\ t=0: u=\varphi(x), u_t=a\psi''(x)\end{cases}$$

关于x做Fourier变换得

$$\begin{cases}\hat{u}_{tt}+a^2\xi^4\hat{u}=0\\ t=0: \hat{u}=\hat{\varphi}(\xi), \hat{u}_t=-a\xi^2\hat{\psi}(\xi)\end{cases}$$

解得$\hat{u}(t,\xi)=\hat{\varphi}(\xi)\cos a\xi^2t-\hat{\psi}(\xi)\sin a\xi^2t$. 从而

$$\begin{aligned}u(t,x)=&\mathscr{F}^{-1}[\hat{u}(t,\cdot)](\xi)\\ =&\mathscr{F}^{-1}[\hat{\varphi}(\xi)\cdot\cos at\xi^2]-\mathscr{F}^{-1}[\hat{\psi}(\xi)\cdot\sin at\xi^2]\\ =&\frac{1}{\sqrt{2\pi}}\left(\varphi*\mathscr{F}^{-1}[\cos at\xi^2]-\psi*\mathscr{F}^{-1}[\sin at\xi^2]\right)\\ =&\frac{1}{2\sqrt{2at}}\int_{\mathbb{R}}\varphi(\xi)\left[\cos\frac{(\xi-x)^2}{4at}+\sin\frac{(\xi-x)^2}{4at}\right]\mathrm{d}\xi\\ &+\frac{1}{2\sqrt{2at}}\int_{\mathbb{R}}\psi(\xi)\left[\cos\frac{(\xi-x)^2}{4at}-\sin\frac{(\xi-x)^2}{4at}\right]\mathrm{d}\xi\\ =&\frac{1}{2\sqrt{at}}\int_{\mathbb{R}}\varphi(\xi)\left[\cos\frac{(\xi-x)^2-at\pi}{4at}\right]\mathrm{d}\xi\\ &+\frac{1}{2\sqrt{at}}\int_{\mathbb{R}}\psi(\xi)\left[\cos\frac{(\xi-x)^2+at\pi}{4at}\right]\mathrm{d}\xi\end{aligned}$$

热传导方程

全空间上的热传导方程

对方程

$$\begin{cases}u_t-a^2\Delta u=f(t,x),t>0,x\in\mathbb{R}^n\\ t=0:u=\varphi(x)\end{cases}$$

考虑对x做Fourier变化所得的PDE问题

$$\begin{cases}\partial_t\hat{u}(t,\xi)+a^2|\xi|^2\hat{u}(t,\xi)=\hat{f}(t,\xi)\\ t=0:\hat{u}(t,\xi)=\hat{\varphi}(\xi)\end{cases}$$

解得ODE问题

$$\begin{aligned}u(t,x)=&(2a\sqrt{\pi t})^{-n}\int_{\mathbb{R}^n}e^{-|x-y|^2/4at}\varphi(y)\mathrm{d}y\\ &+(2a\sqrt{\pi})^{-n}\int_0^t\int_{\mathbb{R}^n}\frac{e^{-|x-y|^2/4a(t-\tau)}}{\sqrt{t-\tau}^n}\mathrm{d}y\mathrm{d}\tau\end{aligned}$$

设基本解$E(t,x)=\dfrac{\exp\dfrac{-|x|^2}{4at}}{(2a\sqrt{\pi t})^n}$, 从而

$$u(t,x)=[E(t,\cdot)*\varphi](x)+\int_0^t[E(t-\tau,\cdot)*f(\tau,\cdot)](x)\mathrm{d}\tau.$$

基本解关于$t\to 0$为光滑的good kernel, 即满足如下性质:

- $E(t,x)\in C^\infty(\{t>0\})$.
- $t>0$时, $\partial_tE(t,x)=a^2\Delta_xE(t,x)$.
- $\int_{\mathbb{R}^n}E(t,x)\mathrm{d}x=1$. 注意到$E(t,x)$恒正, 故绝对积分一致有界.
- 对任意$\delta>0$, $\lim_{t\to 0^+}\int_{\mathbb{R}^n-B_n(0,\delta)}|E(t,x)|\mathrm{d}x=0$.

从物理角度而言, 热方程之解应当具有以下性质(不难验证):

- 齐次热传导方程之解满足$u(t,x)\in[\inf\varphi(x),\sup\varphi(x)]$.

再论迭代法

就以下方程为例

$$\begin{cases}u_t-a^2(u_{xx}+4u_{yy})=y^2t^2\\ t=0:u=x^2y\end{cases}$$

记算子$P: u\mapsto \partial_t u - a^2(\partial_{xx}+4\partial_{yy})u$. 注意到

$$
\begin{aligned}
\frac{t^3}{3}y^2 &\mapsto y^2t^2-\frac{8a^2t^3}{3}\\
\frac{2a^2t^4}{3} &\mapsto \frac{8a^2t^3}{3}\\
x^2y &\mapsto -2a^2y\\
2a^2ty &\mapsto 2a^2y
\end{aligned}
$$

从而$u=x^2y+2a^2ty+\dfrac{t^3}{3}y^2+\dfrac{2a^2t^4}{3}$.

分离变量法

对热传导方程

$$
\begin{cases}
u_t-a^2u_{xx}=0, \quad 0<x<l, t>0\\
t=0: u=\varphi(x)\\
\text{some given boundary conditions}
\end{cases}
$$

StepI: 寻找一个仅满足边值条件的函数v, 下考虑$w=u-v$. 分离变量得特征方程$\dfrac{X''}{X}=\dfrac{T'}{a^2T}=-\lambda_k$, 考虑正交基$\{e_k\}_{k\geq 0}$使得$e_k(x)$满足边值条件, 且$e_k''(x)+\lambda_k e_k(x)=0$. 注意: 当满足Newman条件时应补上0特征值.

Step II: 设解具有一般形式($u(t,x)=0$时$\theta_k\equiv 0$):

$$
\sum_{\exists\lambda=0}\varphi(0)+\sum_{k\geq 1}A_k e^{-\lambda_k t}\sin\left(\sqrt{-\lambda}x+\theta_k\right).
$$

其中

$$
A_k=\frac{2}{l}\int_0^l\varphi(x)\sin\left(\sqrt{-\lambda}x+\theta_k\right)\mathrm{d}x.
$$

热稳态

(数学物理方法P56-6) 半径为a的半圆形平板, 其表面绝热, 在板的周围边界上保持常温u_0, 而在直径边界上保持常温u_1, 求板的稳恒状态.

解: 稳恒时, 温度分布函数u满足$\partial_t u=0$, 从而$\Delta u=0$. 定解问题为

$$\begin{cases} \partial_{rr}u + \dfrac{\partial_r}{r}u + \dfrac{\partial_{\theta\theta}}{r^2}u = 0 \\ u(a,\theta) = u_0, \quad 0 < \theta < \pi \\ u(r,0) = u(r,\pi) = u_1, \quad 0 \le r \le a \end{cases}$$

令$v = R(r)\Theta(\theta) + u_1$, 从而

$$r^2\frac{R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

由$\Theta'' + \lambda_k\Theta = 0$及$\Theta(0) = \Theta(\pi) = 0$知$\lambda_k = k^2$. 解Euler方程

$$r^2R_k'' + rR_k' - \lambda_kR_k = 0$$

得

$$\begin{cases} R_k = B_kr^k + C_kr^{-k} & k > 0 \\ R_0 = C_0 + D_0\ln r & k - 0 \end{cases}$$

实际上, 由有界性知$C_k = 0$. 从而解具有形式

$$u = u_1 + \sum_{k\ge1} B_kr^k\sin(k\theta).$$

故

$$\frac{2}{\pi}\int_0^\pi \sin(k\theta)(u_0 - u_1)\mathrm{d}\theta = B_ka^k.$$

解得$B_k = \dfrac{2(u_0 - u_1)}{a^kk\pi}[1 - (-1)^k]$. 故

$$u(r,\theta) = u_1 + \frac{4(u_0 - u_1)}{\pi}\sum_{n\ge1}\frac{\sin[(2n-1)\theta]}{2n-1}\cdot\left(\frac{r}{a}\right)^{2n-1}.$$

未完待续...