图谱论讲稿 I(修改稿)

会议中未解决的问题

Q1 正则图补图的特征多项式?

A1 设G为度为k的正则图, 顶点数为n. 由于$\overline{G}$度为$n-k$, 故$\lambda_1(\overline{G})=n-k$, 相应的特征向量为$\mathbf{1}$. 若$\lambda_i x_i = A(G)x_i$, 则

$$(J-I-A)x_1 = \mathbf{1}\mathbf{1}^T x_i - (\lambda_i+1)x_i = -(\lambda_i+1)x_i.$$

故$P_{\overline{G}}(x) = P_G(-x-1)\cdot\dfrac{(-1)^n(x-n+k+1)}{x+k+1}$.

将结论用于join graph, 则

$$P_{G_1\triangledown G_2}(x) = \frac{P_{G_1}(x)P_{G_2}(x)}{(x-k_1)(x-k_2)}[(x-k_1)(x-k_2)-n_1n_2].$$

Q2 称G为Cartesian product可分解的, 若且仅若存在非平凡图G_1, G_2使得$G_1\square G_2 = G$. 试寻找例子以说明Cartesian product可分解的图不一定存在唯一的最小分解.

A2 等价于说明非负整系数多项式环非UFD. 注意到

$$(x^3+1)(x^2+x+1) = (x+1)(x^4+x^2+1).$$

对图G, 定义$G^1 = G$, $G^n = G\square G^{n-1}$, 则

$$(G^3\dot\cup K_1)\square(G^2\dot\cup G\dot\cup K_1) = (G\dot\cup K_1)\square(G^4\dot\cup G^2\dot\cup K_1).$$

不妨取$G=P_2=Q_1$为1维超立方体, 下证明$(G^3\dot\cup K_1)$非Cartesian product可分解: 注意到$(G^3\dot\cup K_1)$顶点数为9且不连通, 显然无法成为两个不连通3-顶点图之Cartesian product. 同理, $(G^2\dot\cup G\dot\cup K_1)$之顶点数为质数, 显然不可Cartesian product分解.

Q3 给定$A(G)$与$A(G-j)$的谱(邻接矩阵未知), 试计算$A(G)$中特征值λ_j对应的特征向量x^j中第k项分量之模长, 即x_k^j. 此处需假定$A(G)$之谱无重根.

A3 注意到$P_{G-j}(x)$为$\mathrm{adj}(xI-A)$的j,j项余子式, 设$\mathbf{1}^j$为第j项为1而其余项为0的向量, 则

$$\begin{aligned}
P_{G-j}(x) =& (\mathbf{1}^j)^T\mathrm{adj}(xI-A)\mathbf{1}^j\\
=& P_G(x)(\mathbf{1}^j)^T(xI-A)^{-1}\mathbf{1}^j\\
=& P_G(x)(\mathbf{1}^j)^T\left(\sum_i \frac{1}{x-\lambda_i}P_i\right)\mathbf{1}^j\\
=& P_G(x)e_i^T\left(\sum_i \frac{E_i}{x-\lambda_i}\right)e_i\\
=& P_G(x)\sum_i \frac{\alpha_{ij}^2}{x-\lambda_i}
\end{aligned}$$

代入$x=\lambda_k$, 则

$$P_{G-j}(\lambda_k) = \alpha_{jk}^2\prod_{i\neq k}(\lambda_k-\lambda_i).$$

此处

$$|x_k^j|^2 = \alpha_{jk}^2 = \frac{\prod_{i \neq k}(\lambda_k - \lambda_j)}{\prod_{\mu \in \mathrm{Spec}(G-j)}(\lambda_k - \mu)}.$$

Angle Matrix of a Graph

α_{ij}, the angle

For any simple graph $G(V, E)$ with neither directions, multiple edges, loops $G(V, E)$, the adjacency matrix $A = A(G)$ is symmetric. In light of the *polar decomposition*, we have

$$A = Q^T \Lambda Q = \sum_{i=1}^{r} \lambda_i \cdot P_i = \sum_{i=1}^{r} \lambda_i \cdot Q^T E_i Q$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_r$, $O_n(\mathbb{R}) \ni Q = (x_1, x_2, \ldots, x_n)$.

- $\alpha_{ij} : \|E_i x_j\| = \sqrt{x_j^T E_i x_j}$ denotes the ij-th *angle*, that is, the i-th index of eigenvector x_j. The *angle matrix* is given by $(\alpha_{ij})_{r \times n}$.
- $\sum_{i=1}^{r} \alpha_{ij}^2 = \sum_i x_j^T E_i x_j = x_j^T I x_j = 1$.
- $\sum_{j=1}^{n} \alpha_{ij}^2 = \sum_j x_j^T E_i x_j = \mathrm{trace}(Q^T E_i Q) = \dim \mathscr{E}_{\lambda_i}$. Here $\mathscr{E}_{\lambda_i}$ denotes the root space of λ_i.

β_i, the main angle

The *main angle* β_i is defined as $\dfrac{\|P_i \cdot \mathbf{1}_n\|}{\sqrt{n}}$. Hence

$$\sum_i \beta_i^2 = \frac{\mathbf{1}^T I \mathbf{1}}{n} = 1.$$

For any polynomial $P(x)$ which is well-defined on the ring R and matrix ring $\mathscr{M}(n, R)$, we notice that $P_i P_j = O$ for any $i \neq j$. Hence we deduce that

$$P(A) = \sum_i P(\lambda_i) P_i$$

with the corollary

$$\mathbf{1}^T P(A) \mathbf{1} = n \sum_i P(\lambda_i) \beta_i^2.$$

Further more, the polynomial P can be replaced by *some function* once its convergence is determined.

Operation, Modification&Compositions of Graphs

Basic Graph Operations

Simple operations:

- The *disjoint union* $G_1 \dot\cup G_2$.
- The *complement* $\overline{G}$.
- The *join* $G_1 \bigtriangledown G_2 := \overline{\overline{G_1} \dot\cup \overline{G_2}}$.
- The *vertex-deleted graph* $G - j$.
- The *pendant-added* graph G_j.

- The *coalescence* $G_u \cdot H_v$, or $G \cdot H$ for short, w.r.t. $u \in V(G), v \in V(H)$.
- The *bridged graph* $GuvH$.

Some special operations:

- The *corona* $G \circ H$.
- The *subdivision* graph $S(G)$.
- The *line graph* $L(G)$.

*NEPS(*non-complete extended p-sum) operations.

Characteristic Polynomials under the Modifications

For *disjoint union*, we have

$$P_{G_1 \dot\cup G_2}(x) = P_{G_1}(x) P_{G_2}(x).$$

Further more, for any finite graph set $\{G_i\}_{i=1}^m$ we have

$$P_{\dot\cup \{G_i\}_{i=1}^m}(x) = \prod_{i=1}^m P_{G_i}(x).$$

For *complement*, we have

$$\begin{aligned}
P_{\overline{G}}(x) =& \det(xI - J + I + A) \\
=& \det((x+1)I + A) - \mathbf{1}^T \operatorname{adj}((x+1)I + A)\mathbf{1} \\
=& (-1)^n P_G(-x-1)(1 - \mathbf{1}^T((x+1)I + A)^{-1}\mathbf{1}) \\
=& (-1)^n P_G(-x-1) \left(1 - n \sum_{i=1}^r \frac{\beta_i^2}{x + 1 + \lambda_i}\right)
\end{aligned}$$

> Lemma (UNNECESSARY). $\det(A + uv^T) = \det(A) + v^T \operatorname{adj}(A) u$.
>
> Proof of the lemma. Since
>
> $$\begin{vmatrix} 1 & v^T \\ \mathbf{0} & A + uv^T \end{vmatrix} = \begin{vmatrix} 1 & v^T \\ -u & A \end{vmatrix} = \begin{vmatrix} 1 + v^T A^{-1} u & v^T \\ 0 & A \end{vmatrix}$$
>
> Q.E.D.

For *join*, we have

$$\begin{aligned}
P_{\overline{G_1 \dot\cup G_2}}(x) =& (-1)^{n_1+n_2} P_{G_1 \dot\cup G_2}(-x-1) \left(1 - (n_1 + n_2) \sum_{i=1}^s \frac{\beta_i^2}{x + 1 + \lambda_i}\right) \\
& (-1)^{n_1+n_2} P_{G_1 \dot\cup G_2}(-x-1) \left(1 - n_1 \sum_{i=1}^s \frac{(\beta_i^{(1)})^2}{x + 1 + \lambda_i} - n_2 \sum_{i=1}^s \frac{(\beta_i^{(2)})^2}{x + 1 + \lambda_i}\right) \\
=& -(-1)^{n_1+n_2} P_{G_1 \dot\cup G_2}(-x-1) \\
& + (-1)^{n_1+n_2} P_{G_1 \dot\cup G_2}(-x-1) \left(1 - n_1 \sum_{i=1}^s \frac{(\beta_i^{(1)})^2}{x + 1 + \lambda_i}\right) \\
& + (-1)^{n_1+n_2} P_{G_1 \dot\cup G_2}(-x-1) \left(1 - n_2 \sum_{i=1}^s \frac{(\beta_i^{(2)})^2}{x + 1 + \lambda_i}\right) \\
=& (-1)^{n_2} P_{\overline{G_1}}(x) P_{G_2}(-x-1) + (-1)^{n_1} P_{\overline{G_2}}(x) P_{G_1}(-x-1) \\
& - (-1)^{n_1+n_2} P_{G_1 \dot\cup G_2}(-x-1)
\end{aligned}$$

Hence

$$\begin{aligned}&P_{G_1\bigtriangledown G_2}(x)+(-1)^{n_1+n_2}P_{G_1\dot\cup G_2}(-x-1)\\=&(-1)^{n_2}P_{\overline{G_1}}(x)P_{G_2}(-x-1)+(-1)^{n_1}P_{\overline{G_2}}(x)P_{G_1}(-x-1)\end{aligned}.$$

We deduce that

$$P_{G_1\bigtriangledown G_2}(x)=P_{G_1}(x)P_{G_2}(x)\left(1-n_1n_2\sum_{i=1}^{r_1}\sum_{j=1}^{r_2}\frac{(\beta_i^{(1)}\beta_j^{(2)})^2}{(x-\lambda_i^{(1)})(x-\lambda_j^{(2)})}\right).$$

Characteristic Polynomials under the Compositions

For *vertex-deleted graph* $G-j$, $P_{G-j}(x)$ is the j-th diagonal element of $\mathrm{adj}(xI-A)$. Since

$$\mathrm{adj}(xI-A)=P_G(x)(xI-A)^{-1}=P_G(x)\sum_{i=1}^{r}\frac{P_i}{x-\lambda_i}$$

we deduce that $P_{G-j}(x)=P_G(x)\displaystyle\sum_{i=1}^{r}\frac{\alpha_{ij}^2}{x-\lambda_i}$.

For the *pendant-added graph* G_j, we notice that

$$P_{G_j}(x)=xP_G(x)-P_{G-j}(x).$$

Hence $P_{G_j}(x)=P_G(x)\left(x-\displaystyle\sum_{i=1}^{r}\frac{\alpha_{ij}^2}{x-\lambda_i}\right)$.

For *coalescence* $G\cdot H$ in which $u\in G$ and $v\in H$, we have

$$\begin{aligned}P_{G\cdot H}(x)=&\begin{vmatrix}xI-A' & -r & O\\ -r^T & x & -s^T\\ O & -s & xI-B'\end{vmatrix}\\=&\begin{vmatrix}xI-A' & -r & O\\ -r^T & x & -s^T\\ O & \mathbf{0} & xI-B'\end{vmatrix}+\begin{vmatrix}xI-A' & \mathbf{0} & O\\ -r^T & x & -s^T\\ O & -s & xI-B'\end{vmatrix}\\&-\begin{vmatrix}xI-A' & \mathbf{0} & O\\ -r^T & x & -s^T\\ O & \mathbf{0} & xI-B'\end{vmatrix}\\=&P_G(x)P_{H-v}(x)+P_{G-u}(x)P_H(x)-xP_{G-u}(x)P_{H-v}(x)\end{aligned}.$$

For the *bridged graph* $GuvH$, we have

$$\begin{aligned}P_{GuvH}(x)&=P_{G_u}(x)P_{H-v}(x)+P_G(x)P_H(x)-xP_G(x)P_{H-v}(x)\\&=P_G(x)P_H(x)-P_{G-u}(x)P_{H-v}(x)\end{aligned}$$

For *corona graph* $G\circ H$, we have

$$
\begin{aligned}
P_{G\circ H}(x) =& \det\begin{vmatrix} xI-A & -J_1 & -J_2 & \cdots & -J_n \\ -J_1^T & xI-B & & & \\ -J_2^T & & xI-B & & \\ \vdots & & & \ddots & \\ -J_n^T & & & & xI-B \end{vmatrix} \\
=& \det\begin{vmatrix} xI-A-\dfrac{m}{x-r}I & -J_1 & -J_2 & \cdots & -J_n \\ \mathbf{0} & xI-B & & & \\ \mathbf{0} & & xI-B & & \\ \vdots & & & \ddots & \\ \mathbf{0} & & & & xI-B \end{vmatrix} \\
=& P_G\left(x-\frac{m}{x-r}\right)(P_H(x))^n
\end{aligned}
$$

where $n=|V(G)|$.

For *subdivision graph*, we can notice that $A(S(G))=\begin{pmatrix} O & B^T \\ B & O \end{pmatrix}$. Here B denotes the *induced graph* (from edges to vertices). Hence

$$P_{S(G)}(x)=x^{|E|-|V|}Q_G(x^2)$$

where $Q_G(x)$ denotes the characteristic polynomial of *signless Laplacian* $BB^T=A+D$.

For *line graph* $L(G)$, the characteristic polynomial is given by

$$P_{L(G)}(x)=(x+2)^{|E|-|V|}(x)Q_G(x+2).$$

NEPS(non-complete extended p-sum)

The *NEPS* is a multiple operation of a set of graph via a given tuple $\mathcal{B}\subset\{0,1\}^n\setminus\{0\}^n$, e.g. $\{G_i\}_{i=1}^n$. It outcomes $\mathrm{NEPS}(\{G_i\}_{i=1}^n,\mathcal{B})$ satisfying:

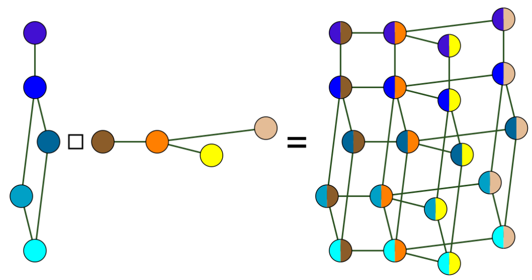
- The vertices of the the outcome is the (classical) *Cartesian product*

$$\prod_{i=1}^n V(G_i)=G_1\times\cdots\times G_n.$$

- The vertices $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_n)$ are adjacent if and only if there exists a $\beta\subset\mathcal{B}$ such that $\beta_i=0\Leftrightarrow x_i=y_i$ and $x_j\sim y_j\Leftrightarrow\beta_j=1$.

As for *Cartesian product* $G\square H$, $V(G\square H)\cong V(G)\times V(G)$, $(g_i,h_i)\sim(g_j,h_j)$ if either

- $g_1=g_2$ and $h_1\sim h_2$ in H, w.r.t. $(0,1)\in\mathcal{B}$.
- $h_1=h_2$ and $g_1\sim g_2$ in G, w.r.t. $(1,0)\in\mathcal{B}$.

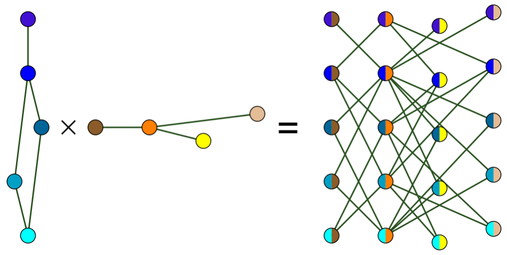


It is trivial to verify that:

- $\square$ is commutative.
- $\square$ is associative.

that is, the associativity and commutativity.

As for *tensor product* $G \times H$, $V(G \times H) \cong V(G) \times V(H)$, $(g_i, h_i) \sim (g_j, h_j)$ if and only if $(g_i \sim g_j) \wedge (h_i \sim h_j)$, w.r.t. $\mathcal{B} = \{(1,1)\}$.



The strong product $G \boxtimes H$, w.r.t. $\mathcal{B} = \{0,1\}^2 - \{0,0\}$, etc.

The adjacency matrix (trivial proof by definition) is

$$A(\text{NEPS}(\{G_i\}_{i=1}^n; \mathcal{B})) = \sum_{\beta \in \mathcal{B}} \otimes_{i=1}^n A(G_i)^{\beta_i}.$$

Since $(A \otimes C)(B \otimes D) = (AC) \otimes (BD)$, the spectral outcome

$$\sum_{\beta \in \mathcal{B}} \left(\prod_{i=1}^n \lambda_i^{i_k \beta_i} \right)$$

where $\lambda_i^{i_k}$ is the k-th eigenvalue of G_i with corresponding eigenvalue $x_i^{i_k}$. Thus the eigen vector of $A(\text{NEPS}(\{G_i\}_{i=1}^n; \mathcal{B}))$ is

$$\sum_{\beta \in \mathcal{B}} \otimes_{i=1}^n x_i^{i_k}.$$