Hopf Algebra

In retrospect of Abelian Group Category ($\mathbb{A}G$)

Definition 1.1 $(G, +_G)$ is an **group** whenever

- **A0**: G is closed under addition, i.e., $a +_G b \in G$ for each $a, b \in G$.
- **A1**: G satisfies the law of associativity, i.e.,

$$G \xleftarrow{+_G \circ (\mathrm{id}_G, +_G)} G \times G \times G \xrightarrow{+_G \circ (+_G, \mathrm{id}_G)} G$$

$$a + (b + c) \longleftarrow\!\shortmid (a, b, c) \longmapsto (a + b) + c$$

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> 💡 **Remark** We write $+ := +_G, \mathrm{id} := \mathrm{id}_G$ for simplicity when there is no ambiguity.

- **A3**: There exists some $0 \in G$ such that

$$0 + G \xleftarrow{\mathrm{id}} G \xrightarrow{\mathrm{id}} G + 0$$

$$0 + a \longleftarrow\!\shortmid a \longmapsto a + 0$$

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> The **uniqueness** of 0 is clear.

- **A4**: The left action $g. : G \to G, a \mapsto g + a$ is bijective for each $g \in G$.

> Note $\mathrm{inv}(g) := g.^{-1}(0) \in G$. Then $\mathrm{inv}(g). \circ g. = \mathrm{id}$, that is, $\mathrm{inv}(g) + g = 0$. Since
>
> $$\begin{aligned} g + \mathrm{inv}(g) &= [\mathrm{inv}(\mathrm{inv}(g)) + \mathrm{inv}(g)] + g + \mathrm{inv}(g) \\ &= \mathrm{inv}(\mathrm{inv}(g)) + [\mathrm{inv}(g) + g] + \mathrm{inv}(g) \\ &= \mathrm{inv}(\mathrm{inv}(g)) + \mathrm{inv}(g) \\ &= 0, \end{aligned}$$
>
> we conclude that there exists unique $\mathrm{inv}(g)$ such that $g + \mathrm{inv}(g) = \mathrm{inv}(g) + g = 0$.

💡 **Remark** We write $(-g) := \mathrm{inv}(g)$ for simplicity.

Definition 1.2 G is **Abelian** whenever $+ \circ S = + : G \times G \to G$, here

$$S : G \times G \to G \times G, \quad (a, b) \mapsto (b, a).$$

💡 **Remark** We assume G is always an additive Abelian group thenceforth.

Fact 1.3 G is Abelian group whenever G is a $\mathbb{Z}$-module, i.e. there exists an action

$$\mathbb{Z} \times G \to G, (n, g) \mapsto (-1)^{\mathrm{sgn}(n)} |n| g.$$

💡 **Remark** $G = {}_{\mathbb{Z}}G_{\mathbb{Z}}$ is bimodule.

Definition 1.4 (**Homomorphism**) $f : G \to H$ is a homomorphism between additive Abelian groups, whenever

$$f \circ +_G = +_H \circ (f, f) : G \times G \to H.$$

Note that $\mathrm{Hom}_{\mathbb{A}G}(G, H) = \{\text{Homomorphisms } G \to H\}$.

Here $\mathrm{Hom}_{\mathbb{A}G}(G, H)$ is also Abelian.

> 💡 **Remark** It is clear that $f : 0_G \to 0_H$.

Definition 1.5 The Category of Abelian groups is denoted by $\mathbb{A}G$, where $\mathsf{Ob}(\mathbb{A}G)$ are Abelian groups, $\mathsf{Mor}(\mathbb{A}G)$ are homomorphisms.

> 💡 **Remark** $\mathrm{Hom}_{\mathbb{A}G}(G, G) =: \mathrm{End}_{\mathbb{A}G}(G, G)$ has a ring structure, i.e.
>
> - $f + g : a \mapsto f(a) + g(a)$,
> - $f \circ g : a \mapsto f(g(a))$.

Definition 1.6 We have the following additive Abelian groups induced by f

$$\begin{tikzcd}
G \arrow[dr, "f"] \arrow[d, dash, "\mathrm{coim}(f) := G/\ker(f)"'] & H \arrow[d, dash, "\mathrm{coker}(f) := H/\mathrm{im}(f)"] \\
\ker(f) \arrow[dr, "f"] \arrow[d, dash] & \mathrm{im}(f) \arrow[d, dash] \\
0 & 0
\end{tikzcd}$$

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- $\mathrm{im}(f) := (f(G), +_H)$ is an additive Abelian group.
- $\ker(f) := (f^{-1}(0), +_G)$ is an additive Abelian group.
- $\mathrm{coker}(f) := (H + f(G), \tilde{+}_H)$ is an additive Abelian group, where

$$\tilde{+}_H : (h_1 + f(G), h_2 + f(G)) \mapsto (h_1 + h_2 + f(G)).$$

- $\mathrm{coim}(f) := (G + f^{-1}(0), \tilde{+}_G)$ is an additive Abelian group, where

$$\tilde{+}_G : (g_1 + f^{-1}(0), g_2 + f^{-1}(0)) \mapsto (g_1 + g_2 + f^{-1}(0)).$$

Definition 1.7 Given a family of groups $\{G_\lambda\}_{\lambda\in\Lambda}$, then we define

- the product of $\{G_\lambda\}_{\lambda\in\Lambda}$ is defined as

$$\prod_{\lambda\in\Lambda} G_\lambda := (g_\lambda)_{\lambda\in\Lambda}, \quad g_\lambda \in G_\lambda.$$

- the coproduct of $\{G_\lambda\}_{\lambda\in\Lambda}$ is defined as

$$\coprod_{\lambda\in\Lambda} G_\lambda := (g_\lambda)_{\lambda\in\Lambda}, \quad g_\lambda \in G_\lambda, \quad |\{\lambda \mid g_\lambda \neq 0\}| < \infty.$$

Fact 1.8 For product, we see that for each $f_\lambda \in \mathrm{Hom}_{\mathbb{A}G}(X, G_\lambda)$, there exists unique f s.t.

$$\begin{tikzcd}
\prod_{\lambda\in\Lambda} G_\lambda \arrow[r, two heads, "\pi_\lambda"] & G_\lambda \\
X \arrow[u, "\exists | f"] \arrow[ur, "f_\lambda"'] &
\end{tikzcd}$$

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For coproduct, we see that for each $g_\lambda \in \mathrm{Hom}_{\mathbb{A}G}(G_\lambda, Y)$, there exists unique g s.t.

$$
\begin{tikzcd}
\coprod_{\lambda\in\Lambda} G_\lambda \arrow[d, "\exists| g"'] & G_\lambda \arrow[l, "\epsilon_\lambda"'] \arrow[dl, "g_\lambda"] \\
Y &
\end{tikzcd}
$$

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We also have the following propositions:

- each π_λ has the right inverse ϵ'_λ;
- each ϵ_λ has the left inverse π'_λ;
- there exists unique $\sigma : \coprod_{\lambda\in\Lambda} G_\lambda \to \prod_{\lambda\in\Lambda} G_\lambda$ such that

$$
\begin{tikzcd}
G_\lambda \arrow[rr, "\epsilon_\lambda"] \arrow[dr, "\epsilon'_\lambda"'] & & \coprod_{\lambda\in\Lambda} G_\lambda \arrow[dl, "\sigma"] \arrow[dr, "\pi'_\lambda"] & \\
& \prod_{\lambda\in\Lambda} G_\lambda \arrow[rr, "\pi_\lambda"'] & & G_\lambda
\end{tikzcd}
$$

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> Here σ is injective (in $\mathbb{A}G$ category).

- the following homomorphisms holds

$$
\mathrm{Hom}_{\mathbb{A}G}(\coprod_{\lambda\in\Lambda} G_\lambda, H) \cong \prod_{\lambda\in\Lambda} \mathrm{Hom}_{\mathbb{A}G}(G_\lambda, H), \quad f \mapsto (fe_\lambda)_{\lambda\in\Lambda};
$$
$$
\mathrm{Hom}_{\mathbb{A}G}(G, \prod_{\lambda\in\Lambda} H_\lambda) \cong \prod_{\lambda\in\Lambda} \mathrm{Hom}_{\mathbb{A}G}(G, H_\lambda), \quad f \mapsto (p_i f)_{\lambda\in\Lambda}.
$$

> 💡 **Remark** Let the diagram $\begin{array}{ccc} a & \xrightarrow{f} & b \\ \varphi\downarrow & & \downarrow\psi \\ c & \xrightarrow[g]{} & d \end{array}$ denote $\psi \circ a = g \circ \varphi$.

Definition 1.9

- Equaliser of $G \overset{f}{\underset{f'}{\rightrightarrows}} H$ is defined as the subgroup

$$\mathrm{Eq}(f, f') := \{a \in G \mid f(a) = f'(a)\} \quad (\overset{k}{\hookrightarrow} G).$$

- Coequaliser of $G \overset{f}{\underset{f'}{\rightrightarrows}} H$ is defined as the quotient group

$$(H \overset{c}{\twoheadrightarrow}) \quad \{h + \mathrm{im}(f) \mid h + \mathrm{im}(f) = h + \mathrm{im}(f')\} := \mathrm{Coeq}(f, f').$$

Fact 1.10 For each $g \in \mathrm{Hom}_{\mathbb{A}G}(L, G)$, there exists unique θ such that

$$\begin{tikzcd} & L \arrow[dl, "\exists|\theta"'] \arrow[d, "g"] & \\ \mathrm{Eq}(f, f') \arrow[r, hook, "k"] & G \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & H \end{tikzcd}$$

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For each $f \in \mathrm{Hom}_{\mathbb{A}G}(H, Y)$, there exists unique τ such that

$$\begin{tikzcd} G \arrow[r, shift left, "g"] \arrow[r, shift right, "g'"'] & H \arrow[r, two heads, "c"] \arrow[d, "f"'] & \mathrm{Coeq}(g, g') \arrow[dl, "\tau"] \\ & Y & \end{tikzcd}$$

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Definition 1.11 $\ker(f) := \mathrm{Eq}(f, 0)$, $\mathrm{coker}(g) := \mathrm{Coeq}(g, g')$.

Definition 1.12 We say $0 \to G \xrightarrow{f} H \xrightarrow{g} L \to 0$ is a short exact sequence, whenever

$$\begin{tikzcd}
G \arrow[d, dash] \arrow[dr, "f"] & H \arrow[d, dash] \arrow[dr, "g"] & L \arrow[d, dash] \\
\ker(f) \arrow[d, dash] \arrow[dr, "f"] & \substack{\mathrm{im}(f) \\ \| \\ \ker(g)} \arrow[d, dash] \arrow[dr, "g"] & \mathrm{im}(g) \arrow[d, dash] \\
0 & 0 & 0
\end{tikzcd}$$

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Definition 1.13 Consider $f_i \in \mathrm{Hom}_{\mathbb{A}G}(G_i, H)$ ($i \in \{1, 2\}$), define **pullback of homomorphism** as

$$p^* := f_1 \circ \pi_1 - f_2 \circ \pi_2 : G_1 \times G_2 \to H.$$

$$\begin{tikzcd}
 & & G_1 \arrow[dr, "f_1"] & & \\
0 \arrow[r, dashed] & P \arrow[r, dashed] \arrow[ur, "\pi_1 \restriction_P"] \arrow[dr, "\pi_2 \restriction_P"'] & G_1 \times G_2 \arrow[r, dashed, "p^*"] \arrow[u, dashed, "\pi_1"'] \arrow[d, dashed, "\pi_2"] & (G_1 \times G_2)/P \arrow[r, dashed] & 0 \\
 & & G_2 \arrow[ur, "f_2"'] & &
\end{tikzcd}$$

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Here the middle row is a short exact sequence.

Consider $g_i \in \mathrm{Hom}_{\mathbb{A}G}(G, H_i)$ ($i \in \{1, 2\}$), define the **pushback of homomorphism** as

$$q^* := \epsilon_1 \circ g_1 - \epsilon_2 \circ g_2 : G \to H_1 \oplus H_2.$$

$$
\begin{tikzcd}
 & & H_1 \arrow[d, dashed, "\epsilon_1"] \arrow[dr, "\tilde{\epsilon}_1"] & & \\
0 \arrow[r, dashed] & G \arrow[ur, "g_1"] \arrow[r, dashed, "q^*"] \arrow[dr, "g_2"'] & H_1 \oplus H_2 \arrow[r, dashed] & (H_1 \oplus H_2)/G \arrow[r, dashed] & 0 \\
 & & H_2 \arrow[u, dashed, "\epsilon_2"'] \arrow[ur, "\tilde{\epsilon}_2"'] & &
\end{tikzcd}
$$

link

Here the middle row is a short exact sequence.

Fact 1.14 The pullback of homomorphisms satisfy

$$
\begin{tikzcd}
 & G_1 \arrow[dr, "f_1"] & \\
P \arrow[ur, "\pi_1 \restriction_P"] \arrow[dr, "\pi_2 \restriction_P"'] & X \arrow[l, "\exists | g"'] \arrow[u, "g_1"'] \arrow[d, "g_2"] & H \\
 & G_2 \arrow[ur, "f_2"'] &
\end{tikzcd}
$$

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> $\forall g_1, g_2 \ (f_1 \circ g_1 = f_2 \circ g_2)$, there exists unique $g \in \mathrm{Hom}_{\mathbb{A}G}(X, P)$ s.t. above holds.

The pushout of homomorphisms satisfy

$$
\begin{tikzcd}
 & H_1 \arrow[d, "h_1"'] \arrow[dr, "\tilde{\epsilon}_1"] & \\
G \arrow[ur, "g_1"] \arrow[dr, "g_2"'] & Y & (H_1 \oplus H_2)/G \arrow[l, "h"'] \\
 & H_2 \arrow[u, "h_2"] \arrow[ur, "\tilde{\epsilon}_2"'] &
\end{tikzcd}
$$

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> $\forall g_1, g_2\ (f_1 \circ g_1 = f_2 \circ g_2)$, there exists unique $g \in \mathrm{Hom}_{\mathbb{A}G}(X, P)$ s.t. above holds.

Fact 1.15 Characterisations of surjective and injective homomorphisms.

- We say $f \in \mathrm{Hom}_{\mathbb{A}G}(G, H)$ is injective, whenever one of the following holds:
  1. f is a monomorphism, i.e.,

     $$(f \circ g_1 = f \circ g_2) \Leftrightarrow (g_1 = g_2), \quad \forall g_1, g_2 \in \mathrm{Hom}_{\mathbb{A}_G}(L, G).$$

  2. f is the kernel of $H \twoheadrightarrow H/f(G)$.
- We say $f \in \mathrm{Hom}_{\mathbb{A}G}(G, H)$ is surjective, whenever one of the following holds:
  1. f is a epimorphism, i.e.,

     $$(g_1 \circ f = g_2 \circ f) \Leftrightarrow (g_1 = g_2), \quad \forall g_1, g_2 \in \mathrm{Hom}_{\mathbb{A}_G}(H, L).$$

  2. f is the cokernel of $\ker(f) \hookrightarrow G$.

Tensor Product

Definition 2.1 A category $\mathcal{C}$ consists of the class of objects $\mathsf{Ob}(\mathcal{C})$ and the class of morphisms

$$\mathsf{Mor}(\mathcal{C}) := \bigcup_{A, B \in \mathsf{Ob}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(A, B),$$

and satisfies the following statements.

- For any pair $\mathrm{Hom}_{\mathcal{C}}(A,B)$ and $\mathrm{Hom}_{\mathcal{C}}(C,D)$ in $\mathrm{Mor}(\mathcal{C})$,

$$\mathrm{Hom}_{\mathcal{C}}(A,B) \cap \mathrm{Hom}_{\mathcal{C}}(C,D) = \emptyset \Leftrightarrow (A,B) \neq (C,D).$$

- The composition of morphisms satisfies the path structure, i.e.,
    - $\circ : \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{C}}(A,C), \quad (f,g) \mapsto f \circ g$;
    - $\mathrm{id}_B \in \mathrm{Hom}_{\mathcal{C}}(B,B)$ exists, i.e., we have $\mathrm{id}_B \circ g = g$ and $f \circ \mathrm{id}_B = f$.

💡 **Remark** $\mathcal{S} := \{U \mid U \text{ is a set.}\}$ is no longer a set, since it is contained in $\mathcal{S} \dot\cup \{\mathcal{S}\}$, thus the concept of **class** is introduced for describing those are larger than sets. Whereas, we assume the category are small.

Definition 2.2 A subcategory $\mathcal{C}' \subset \mathcal{C}$ satisfies ($A, B \in \mathsf{Ob}(\mathcal{C}') \subset \mathsf{Ob}(\mathcal{C})$)

$$\begin{array}{ccccccc} \mathcal{C} & : & \mathsf{Ob}(\mathcal{C}) & \mathsf{Mor}(\mathcal{C}) & \in & \mathrm{Hom}_{\mathcal{C}}(A,B) \\ \cup & & \cup & \cup & & \cup \\ \mathcal{C}' & : & \mathsf{Ob}(\mathcal{C}') & \mathsf{Mor}(\mathcal{C}') & \in & \mathrm{Hom}_{\mathcal{C}'}(A,B) \end{array}$$

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Definition 2.3 Full subcategory $\mathcal{C}' \subset \mathcal{C}$ satisfies ($A, B \in \mathsf{Ob}(\mathcal{C}') \subset \mathsf{Ob}(\mathcal{C})$)

$$\begin{array}{ccccccc} \mathcal{C} & : & \mathsf{Ob}(\mathcal{C}) & \mathsf{Mor}(\mathcal{C}) & \in & \mathrm{Hom}_{\mathcal{C}}(A,B) \\ \cup & & \cup & \cup & & \parallel \\ \mathcal{C}' & : & \mathsf{Ob}(\mathcal{C}') & \mathsf{Mor}(\mathcal{C}') & \in & \mathrm{Hom}_{\mathcal{C}'}(A,B) \end{array}$$

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Definition 2.4 A **contravariant functor** $F : \mathcal{C} \to \mathcal{D}, X \mapsto FX, f \mapsto Ff$ satisfies

$$\begin{tikzcd}
X \arrow[rr, "f \circ g"] \arrow[d, "F"'] & {} \arrow[d, phantom, "\Downarrow F"] & Z \arrow[r, "\mathrm{id}_Z"] \arrow[d, "F"'] & Z \arrow[d, "F"] \\
FX \arrow[rr, "Fg \circ Ff"] & {} & FZ \arrow[r, "\mathrm{id}_{FZ}"] & FZ
\end{tikzcd}$$

link

Definition 2.5 The **opposite category** $\mathcal{C}^{\mathrm{op}}$ is defined by reversing arrows of morphisms in $\mathcal{C}$, i.e.,

$$X \xrightarrow{f} Y \Leftrightarrow X^{\mathrm{op}} \xleftarrow{f^{\mathrm{op}}} Y^{\mathrm{op}}.$$

A **contravariant functor** is the composition of a contravariant functor with $^{\mathrm{op}}$ functor. Here $^{\mathrm{op}}$ is also a contravariant functor.

Definition 2.6 A natural transformation between $F, G : \mathcal{C} \to \mathcal{D}$ is defined as $\{\Phi_X\}_{X \in \mathsf{Ob}(\mathcal{C})}$, i.e.,

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left=60, "F", ""{name=U, below}] \arrow[rr, bend right=60, "G"', ""{name=D, above}] & & \mathcal{D}
\arrow[Rightarrow, from=U, to=D, "\Phi"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, "\Phi_X"] & FY \arrow[d, "\Phi_Y"] \\
GX \arrow[r, "Gf"] & GY
\end{tikzcd}$$

link

> Denote $\mathrm{Hom}_{(\mathcal{C},\mathcal{D})}(F, G)$ as the morphisms of natural transformations between F and G.

Definition 2.7 We say

- $\mathcal{C}$ and $\mathcal{D}$ are isomorphic, whenever there exists $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $FG = \mathrm{id}_{\mathcal{D}}$ and $GF = \mathrm{id}_{\mathcal{C}}$.
- $\mathcal{C}$ and $\mathcal{D}$ are equivalent, whenever there exists $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $\theta : FG \xrightarrow{\sim} \mathrm{id}_{\mathcal{D}}$ and $\tau : GF \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}}$. Here both θ and τ are natural transformations.

Definition 2.8 (**Characterstics of a functor**) Let $T : \mathcal{C} \to \mathcal{D}$ be a contravariant functor, then

- the image $T(\mathcal{C})$ is always a full subcategory of $\mathcal{D}$, that is,

$$\mathrm{Hom}_{\mathcal{D}}(TA, TB) = \mathrm{Hom}_{T(\mathcal{C})}(TA, TB), \quad \forall A, B \in \mathsf{Ob}(\mathcal{C}).$$

- the image $T(\mathcal{C})$ is **essential**, whenever $\mathsf{Ob}(T(\mathcal{C})) = \{Y \mid \exists X \in \mathsf{Ob}(\mathcal{C}) : Y \cong T(X)\}$.
- T is **faithful** whenever the following map is always an **injection** for arbitrary $A, B \in \mathsf{Ob}(\mathcal{C})$:

$$T : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{T(\mathcal{C})}(TA, TB), \quad [A \xrightarrow{f} B] \mapsto (TA \xrightarrow{Tf} TB).$$

- T is **full**, whenever the following map is always a **surjection** for arbitrary $A, B \in \mathsf{Ob}(\mathcal{C})$:

$$T : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{T(\mathcal{C})}(TA, TB), \quad [A \xrightarrow{f} B] \mapsto (TA \xrightarrow{Tf} TB).$$

> 💡 **Remark** T stands for equivalent relation whenever it is full, faithful, essential and surjective.

Definition 2.9 Consider the contravariant functor

$$
\begin{aligned}
h_X\ : \mathcal{C} &\to \mathbb{S}ets; \\
Y &\mapsto \mathrm{Hom}_{\mathcal{C}}(Y, X), \\
[Z \xrightarrow{f} Y] &\mapsto \mathrm{Hom}_{\mathcal{C}}(f, X) : \mathrm{Hom}_{\mathcal{C}}(Y, X) \to \mathrm{Hom}_{\mathcal{C}}(Z, X), \\
&\qquad\qquad\qquad\qquad [(Y \xrightarrow{g} X) \mapsto (Z \xrightarrow{f\circ g} X)].
\end{aligned}
$$

Definition 2.10 Consider the contravariant functor $F : \mathcal{C} \to \mathbb{S}ets$, then there exists a natural transform

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "h_X"{name=U}] \arrow[r, bend right=50, "F"'{name=D}] & \mathbb{S}ets
\arrow[Rightarrow, from=U, to=D, "\Phi"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \arrow[d, "Ff"] & h_X(X) \arrow[l, "\Phi_X"'] \arrow[d, "h_X(f)"] & = & \mathrm{Hom}_{\mathcal{C}}(X, X) \arrow[d, phantom, "\mathrm{Hom}_X(f, X)"] & \ni & \mathrm{id}_X \arrow[d, mapsto] \\
FY & h_X(Y) \arrow[l, "\Phi_Y"] & = & \mathrm{Hom}_{\mathcal{C}}(Y, X) & \ni & f
\end{tikzcd}
$$

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Theorem 2.11 (**米田**(よねだ) **lemma**) There exists a bijection

$$
\begin{tikzcd}
\mathrm{Hom}_{(\mathcal{C},\mathcal{D})}(h_X, F) \arrow[rr] & & F(X) \\
\Phi \arrow[rr, mapsto] & & \Phi_X(\mathrm{id}_X)
\end{tikzcd}
$$

$$
\Phi \quad : \quad \mathrm{Mor}_{\mathcal{C}}(-, X) \longrightarrow F
$$

$$
\mathrm{Hom}_{\mathcal{C}}(X, X) \longmapsto F(X) \qquad \Phi_X \quad : \quad \mathrm{Hom}_{\mathcal{C}}(X, X) \longrightarrow F(X)
$$

$$
\mathrm{id}_X \longmapsto \Phi_X(\mathrm{id}_X)
$$

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Definition 2.12 The tensor product of G and H is the quotient group

$$\otimes_{\mathbb{Z}} : G \times H \to G \times H / \sim$$

Here

- $(ng, h) \sim (g, nh) =: n(g, h)$;
- $(n_1 + n_2, m) \sim (n_1, m) + (n_2, m)$;
- $(n, m_1 + m_2) \sim (n, m_1) + (n, m_2)$.

Remark $G \otimes H$ is isomorphic to a quotient group of some free abelian group, i.e.,

$$G \otimes H \cong \left(\prod_{(g,h) \in (G \times H)} (\mathbb{Z}^{(g,h)}) \right) / \sim' .$$

Moreover, $\mathbb{Z} \otimes G \cong \ G \cong \ G \otimes \mathbb{Z}$, proved by universal properties.

Fact 2.13 For each $\mathbb{Z}$-bilinear function $\tilde{f} : G \times H \to L$, there exists a unique $f \in \mathrm{Hom}_{\mathbb{A}G}(G \otimes H, L)$ such that

$$\begin{tikzcd}
G \times H \arrow[r, "\tilde{f} \in \mathrm{Bil}_{\mathbb{A}G}(G, H; L)"] \arrow[d, "\otimes"'] & L \\
G \otimes H \arrow[ur, "\exists | f \in \mathrm{Hom}_{\mathbb{A}G}(G \times H, L)"'] &
\end{tikzcd}$$

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Remark Tensor product is also **defined by such universal property**.

Fact 2.14 For each $L \in \mathsf{Ob}(\mathbb{A}G)$, we have

$$\begin{aligned}
&\operatorname{Hom}_{\mathbb{A}G}\left(\left(\bigoplus_{i\in I} G^i\right)\otimes\left(\bigoplus_{j\in J} H^j\right), L\right)\\
&\cong \operatorname{Bil}_{\mathbb{A}G}\left(\left(\bigoplus_{i\in I} G^i\right), \left(\bigoplus_{j\in J} H^j\right); L\right)\\
&\cong \prod_{i\in I, j\in J} \operatorname{Bil}_{\mathbb{A}G}\left(G^i, H^j; L\right)\\
&\cong \prod_{i\in I, j\in J} \operatorname{Hom}_{\mathbb{A}G}\left(G^i \otimes H^j, L\right)\\
&\cong \operatorname{Hom}_{\mathbb{A}G}\left(\bigoplus_{i\in I, j\in J} G^i \otimes H^j, L\right).
\end{aligned}$$

Thus $(\bigoplus_{i\in I} G^i)\otimes(\bigoplus_{j\in J} H^j) \cong \bigoplus_{i\in I, j\in J} G^i \otimes H^j$.

> (Via **Theorem 2.11**) $\forall X, Y \in \mathsf{Ob}(\mathcal{C})$, $X = Y$ whenever $\operatorname{Id}_{\mathcal{C}}(X) = \operatorname{Id}_{\mathcal{C}}(Y)$, whenever
>
> $$\operatorname{Hom}_{(\mathcal{C},\mathcal{C})}(\mathsf{Mor}_{\mathcal{C}}(-, X), \operatorname{Id}_{\mathcal{C}}) = \operatorname{Hom}_{(\mathcal{C},\mathcal{C})}(\mathsf{Mor}_{\mathcal{C}}(-, Y), \operatorname{Id}_{\mathcal{C}}).$$
>
> In contravariant case, one only need to show
>
> $$\operatorname{Hom}_{\mathcal{C}}(X, L) = \operatorname{Hom}_{\mathcal{C}}(Y, L), \quad \forall L \in \mathsf{Ob}(\mathcal{C}).$$

Definition 2.15 For any $f \in \operatorname{Hom}_{\mathbb{A}G}(G, H)$ and $g \in \operatorname{Hom}_{\mathbb{A}G}(M, N)$, we have

$$
\begin{tikzcd}
G \times M \arrow[r, "f \times g"] \arrow[d, "\otimes"'] \arrow[dr, "\otimes \circ (f \times g)"] & H \times N \arrow[d, "\otimes"] \\
G \otimes M \arrow[r, "\exists|(f \otimes g)"'] & H \otimes N
\end{tikzcd}
\qquad
\begin{tikzcd}
f \times g \arrow[r, "\Phi"] \arrow[d, "\otimes"'] & h \\
f \otimes g \arrow[ur, "\exists|\Phi^{\otimes} := \Phi \circ \otimes \circ (-) \circ \otimes^{-1}"'] &
\end{tikzcd}
$$

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Here $\Phi^{\otimes}$ is well defined.

Fact 2.16 For each $G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3$, $H_1 \xrightarrow{g_1} H_2 \xrightarrow{g_2} H_3$, we have

$$
\begin{tikzcd}
G_1 \times H_1 \arrow[r, "f_1 \times g_1"] \arrow[dd, "\otimes"] & G_2 \times H_2 \arrow[r, "f_2 \times g_2"] \arrow[d, "\otimes"] & G_3 \times H_3 \arrow[dd, "\otimes"] \\
 & G_2 \otimes H_2 \arrow[dr, "f_2 \otimes g_2"] & \\
G_1 \otimes H_1 \arrow[ur, "f_1 \otimes g_1"] \arrow[rr, "\exists|(f_2 \circ f_1) \otimes (g_2 \circ g_1)"'] & & G_3 \otimes H_3
\end{tikzcd}
$$

link

Thus $(f_2 \circ f_1) \otimes (g_2 \circ g_1) = (f_2 \otimes g_2) \circ (f_1 \otimes g_1)$.

Fact 2.17 $L \otimes (M \otimes N) \xrightarrow{\sim} (L \otimes M) \otimes N$ is natural.

Adjunctions

Definition 3.1 We say $F : \mathcal{C} \to \mathbb{S}ets$ is a **representable** (contravariant, for simplicity) functor, if

$$\rho \quad : \quad h_X \longrightarrow F$$

$$\mathrm{Hom}_{\mathcal{C}}(X,X) \longmapsto F(X) \qquad \rho_X \quad : \quad \mathrm{Hom}_{\mathcal{C}}(X,X) \longrightarrow F(X)$$

$$\mathrm{id}_X \longmapsto u_F$$

for some $X \in \mathsf{Ob}(\mathcal{C})$. Here u_F is iamge of natural transform of ρ in 米田(よねだ) bijection. Moreover,

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X,X) \longmapsto F(X) \\ \rho \quad : \quad h_X \longrightarrow F \\ \mathrm{Hom}_{\mathcal{C}}(Y,X) \longmapsto F(Y) \end{array}$$

$$\begin{tikzcd}
\mathrm{id}_X \arrow[rrr, mapsto, "\rho_X"] \arrow[ddd, mapsto, "h_X(f)"'] & & & u_F \arrow[ddd, mapsto, "Ff"] \\
& \mathrm{Hom}_{\mathcal{C}}(X,X) \arrow[r, "\rho_X"] \arrow[d, "h_X(f)"] & F(X) \arrow[d, "Ff"] & \\
& \mathrm{Hom}_{\mathcal{C}}(Y,X) \arrow[r, "\rho_Y"] & F(Y) & \\
& f \arrow[r, mapsto, "\rho_Y"] \arrow[dl, equal] & c \arrow[dr, equal] & \\
h_X(f)(\mathrm{id}_X) \arrow[rrr, mapsto] & & & \rho_Y h_X(f)(\mathrm{id}_X)
\end{tikzcd}$$

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We see that $\rho_\bullet : \mathsf{Mor}_{\mathcal{C}}(\bullet, X) \to F(\bullet)$, $f \mapsto \rho_\bullet(f) = Ff(u_F)$.

Theorem 3.2 The functor category of $\mathcal{C}$ is usually defined as $\mathrm{Fuc}(\mathcal{C}^{\mathrm{op}}, \mathbb{S}ets)$, where

$$\downarrow \text{Functors}$$

$$\begin{aligned}
\mathsf{Ob}(\mathrm{Fuc}(\mathcal{C}^{\mathrm{op}}, \mathbb{S}ets)) &:= \{F \mid F : \mathcal{C}^{op} \to \mathbb{S}ets\}, \\
\mathsf{Mor}(\mathrm{Fuc}(\mathcal{C}^{\mathrm{op}}, \mathbb{S}ets)) &:= \bigcup_{F,G \in \mathsf{Ob}(\mathrm{Fuc}(\mathcal{C}^{\mathrm{op}}, \mathbb{S}ets))} \mathsf{Mor}_{\mathcal{C}^{\mathrm{op}}, \mathbb{S}ets}(F, G).
\end{aligned}$$

$$\uparrow \begin{array}{c}\text{Natural} \\ \text{transformations}\end{array}$$

$$
\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[rr, bend left, "F \in \mathsf{Ob}(\mathrm{Fuc}(\mathcal{C}^{\mathrm{op}}, \mathbb{S}ets))"] \arrow[rr, bend right, "G \in \mathsf{Ob}(\mathrm{Fuc}(\mathcal{C}^{\mathrm{op}}, \mathbb{S}ets))"'] \arrow[rr, phantom, "\Downarrow \Phi \in \mathsf{Mor}(\mathrm{Fuc}(\mathcal{C}^{\mathrm{op}}, \mathbb{S}ets))"] & & \mathbb{S}ets
\end{tikzcd}
$$

link

Theorem 3.3 Each category is a full subcategory of functor category. Consider

$$
\begin{array}{llcll}
h: & \mathcal{C} & \longrightarrow & \mathrm{Fuc}(\mathcal{C}^{\mathrm{op}}, \mathbb{S}ets) & \\
 & X & \longmapsto & \mathsf{Mor}_{\mathcal{C}}(-, X) = & h_X \\
 & \mathrm{Hom}_{\mathcal{C}}(Y, Z) & \longmapsto & \mathrm{Hom}_{\mathcal{C}}(h_Y, h_Z) & \\
 & \Psi & & \Psi & \\
 & Y \xrightarrow{f} Z & \longrightarrow & (P \xrightarrow{g} Y) \mapsto (P \xrightarrow{f \circ g} Z) &
\end{array}
$$

link

Hence $\mathcal{C} \cong$ subcategory of representable functors, a full subcategory of $\mathrm{Fuc}(\mathcal{C}^{\mathrm{op}}, \mathbb{S}ets)$.

Definition 3.4 For $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$, we say $L \dashv R$ (L is a **left adjoint** of R), whence

$$\rho : \mathrm{Hom}_{\mathcal{A}}(LX, Y) \cong \mathrm{Hom}_{\mathcal{B}}(X, RY), \quad \forall X \in \mathsf{Ob}(\mathcal{B}), Y \in \mathsf{Ob}(\mathcal{A}).$$

Here we also have $L \vdash R$, i.e., R is a **right adjoint** of L. Here there exists a natural transformation

$$\begin{tikzcd}
\mathsf{Mor}_{\mathcal{B}}(B, R(-)) \arrow[r, "\Phi_B"] \arrow[d, "\mathsf{Mor}_{\mathcal{B}}(f, R(-))"'] & \mathsf{Mor}_{\mathcal{A}}(LB, -) \arrow[d, "\mathsf{Mor}_{\mathcal{A}}(Lf, -)"] \\
\mathsf{Mor}_{\mathcal{B}}(B', R(-)) \arrow[r, "\Phi_{B'}"] & \mathsf{Mor}_{\mathcal{A}}(LB', -)
\end{tikzcd}$$

link

Theorem 3.5 R has a left adjoint whenever $\mathsf{Mor}_{\mathcal{B}}(B, R(-))$ is always representable for each $B \in \mathsf{Ob}(\mathcal{B})$, i.e.,

$$\rho : \mathsf{Mor}_{\mathcal{A}}(LB, -) \xrightarrow{\sim} \mathsf{Mor}_{\mathcal{B}}(B, R(-)).$$

Moreover, if R has a left adjoint L which is not an isomorphism, then L is unique.

Such isomorphism is also natural for A and B, i.e.,

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(B, RA') \arrow[rrr, "\eta_{B,A'}"'] \arrow[ddd, "\mathrm{Hom}_{\mathcal{B}}(f, RA')"] & & & \mathrm{Hom}_{\mathcal{A}}(LB, A') \arrow[ddd, "\mathrm{Hom}_{\mathcal{A}}(Lf, A')"'] \\
& \mathrm{Hom}_{\mathcal{B}}(B, RA) \arrow[r, "\eta_{B,A}"'] \arrow[ul, "\mathrm{Hom}_{\mathcal{B}}(B, Rg)"] \arrow[d, "\mathrm{Hom}_{\mathcal{B}}(f, RA)"'] & \mathrm{Hom}_{\mathcal{A}}(LB, A) \arrow[ur, "\mathrm{Hom}_{\mathcal{A}}(LB, g)"'] \arrow[d, "\mathrm{Hom}_{\mathcal{A}}(Lf, A)"] & \\
& \mathrm{Hom}_{\mathcal{B}}(B', RA) \arrow[r, "\eta_{B',A}"] \arrow[dl, "\mathrm{Hom}_{\mathcal{B}}(B', Rg)"'] & \mathrm{Hom}_{\mathcal{A}}(LB', A) \arrow[dr, "\mathrm{Hom}_{\mathcal{A}}(LB', g)"] & \\
\mathrm{Hom}_{\mathcal{B}}(B', RA') \arrow[rrr, "\eta_{B',A'}"] & & & \mathrm{Hom}_{\mathcal{A}}(LB', A')
\end{tikzcd}$$

link

Fact 3.6 The pair $(\otimes, \mathrm{Hom})$ induces an adjoint relation, i.e.,

$$
\begin{aligned}
\mathrm{Hom}_{\mathbb{A}G}(L\otimes M,N) &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{A}G}(M,\mathrm{Hom}_{\mathbb{A}G}(L,N)),\\
f &\mapsto (m\mapsto f(-\otimes m));
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Hom}_{\mathbb{A}G}(L\otimes M,N) &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{A}G}(L,\mathrm{Hom}_{\mathbb{A}G}(M,N)),\\
f &\mapsto (l\mapsto f(l\otimes -)).
\end{aligned}
$$

Then we define

- $\varepsilon_G : H\otimes \mathrm{Hom}_{\mathbb{A}G}(H,G)\to G, h\otimes f\mapsto f(h)$;
- $\eta_G : G\to \mathrm{Hom}_{\mathbb{A}G}(H,H\otimes G), g\mapsto -\otimes g$,

to obtain that

- $\mathrm{id}_{H\otimes G} : h\otimes g \xrightarrow{\mathrm{id}_H\otimes\eta_G} h\otimes(-\otimes g)\xrightarrow{\varepsilon_{H\otimes G}} h\otimes g$, that is, $\mathrm{id}_{H\otimes G}=\varepsilon_{H\otimes G}\circ(\mathrm{id}_H\otimes\eta_G)$;
- $\mathrm{id}_{\mathrm{Hom}_{\mathbb{A}G}(H,G)} : f \xrightarrow{\eta_{\mathrm{Hom}_{\mathbb{A}G}(H,G)}} (-\otimes f)\xrightarrow{\varepsilon_G} f$, that is, $\mathrm{id}_{\mathrm{Hom}_{\mathbb{A}G}(H,G)}=\varepsilon_G\circ(\eta_{\mathrm{Hom}_{\mathbb{A}G}(H,G)})$.

Definition 3.7 In **Fact 3.6**, $\eta:\mathrm{id}_{\mathcal{A}}\to RL$ and $\varepsilon: LR\to\mathrm{id}_{\mathcal{B}}$ are called unit and counit, i.e.,

$$
\begin{tikzcd}
R & RLR \arrow[l, "R\varepsilon"'] & LRL \arrow[d, "\varepsilon L"] & L \arrow[l, "L\eta"'] \arrow[dl, "="] \\
& R \arrow[u, "\eta R"'] \arrow[ul, "="] & L &
\end{tikzcd}
$$

$$
\begin{tikzcd}
LRA \arrow[dr, "\varepsilon_A"'] & LB \arrow[l, "\exists|Lg"'] \arrow[d, "f"'] & RA & RLB \arrow[l, "\exists|Rf"'] \\
& A & B \arrow[u, "g"] \arrow[ur, "\eta_B"'] &
\end{tikzcd}
$$

link

Then for each adjoint pair (L,R) ($L\dashv R$), it yields that

- L is full and faithful, whenever $RL \xrightarrow{\sim} \mathrm{id}$;
- R is full and faithful, whenever $LR \xrightarrow{\sim} \mathrm{id}$.

Theorem 3.8 Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be quasi-inverse pair, i.e., there exists

- $\eta : GF \xrightarrow{\sim} \mathrm{id}$, which is a natural transfromation;
- $\theta^{-1} : FG \xrightarrow{\sim} \mathrm{id}$, which is a natural transformation.

Then we claim that (F, G) is an adjoint pair. Let η be unit without the loss of generality, then

$$\theta' := FG \xrightarrow{FG\theta} FGFG \xrightarrow{F\eta G} FG \xrightarrow{\theta^{-1}} \mathrm{id}$$

is a well defined counit. One can verity $\mathrm{id} : F \xrightarrow{\eta F} FGF \xrightarrow{F\theta} F$ and $\mathrm{id} : G \xrightarrow{G\eta} GFG \xrightarrow{\theta G} G$.