

A glimpse of Kuratowski's 14-Set Problem

The Original Problem

Problem. Let $S \in \mathcal{P}(\mathbb{R}^n)$. Let k , I and c be unary operators such that

$$k(S) = \overline{S}, \quad I(S) = S, \quad c(S) = \mathbb{R}^n - S.$$

Prove that there at most 14 sets generated by S .

Proof Sketch.

Step I The unary operator $i(S) = S^\circ$ can be generated by k and c , i.e., $ckc = i$.

Step II Let $\langle I, i, c, k \rangle$ be the free semigroup, which consists of i^2 , $icik$, k^3i , etc.

Let M be the quotient semigroup $\langle I, i, c, k \mid \text{reduction of set operators} \rangle$. For instance, $I^k = I$, $c^3 = c$, $i^2 = i$ under such quotient relations. Then, $\{xS \mid x \in M\}$ consists of all possible sets generated by S .

Step III Utilise the relations $ic = ck$, $kc = ci$, $c^2 = I$, to move the word c to the initial of any $y \in M$. For instance, $cikci = ki^2$.

Step IV Prove that $i^2 = i$, $k^2 = k$, $ik = ikik$, $ki = kiki$.

Step V Find that $M = \{I, i, k, ik, ki, iki, kik, c, ci, ck, cik, cki, ciki, ckik\}$

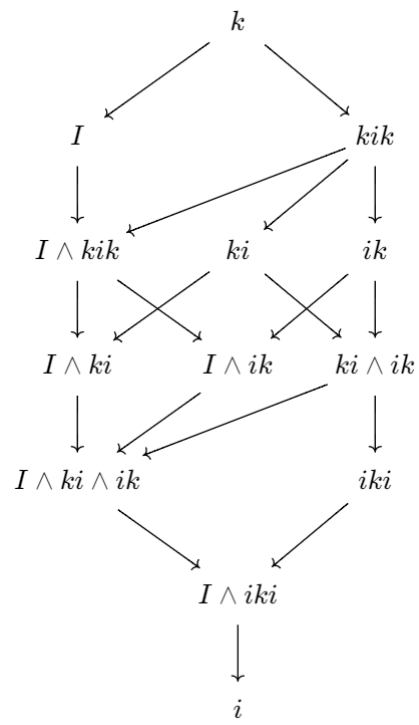
Ex1 We define $x \leq y$ for each $x, y \in M$ whenever $x(S') \subset y(S')$ for each $S \in \mathbb{R}^n$, where $x < y$ whenever $x(S') \subsetneq y(S')$. Find the partially ordered structure of M .

Step VI Find an S such that $M(S)$ consists of 14 distinct sets. For simplicity, one can just find the union of all pairwise disjoint examples which verifying the partially ordered structure.

Further Exploration

Ex2 Let $S \in \mathcal{P}(\mathbb{R}^n)$. Let k , I and c be unary operators defined above. Let \wedge be the binary operator defined as $\wedge(S_1, S_2) = S_1 \cap S_2$. State a similar problem and solve it.

Hint:



 [Website for creating commutative diagram.](#)