Gauß曲率的等价描述

Gauß映射解释

考虑$N: U_p(\subset S \subset \mathbb{R}^2) \to S^2$, 则

$$\begin{aligned}
\lim_{U_p \to \{p\}} \frac{\operatorname{Area}(N_p(U_p))}{\operatorname{Area}(U_p)} &= \frac{|-\mathrm{d}N_p(X_u) \wedge -\mathrm{d}N_p(X_v)|}{|X_u \wedge X_v|} \\
&= \frac{|(b_1^1 X_u + b_1^2 X_v) \wedge (b_2^1 X_u + b_2^1 X_v)|}{|X_u \wedge X_v|} \\
&= \frac{|K| \cdot |X_u \wedge X_v|}{|X_u \wedge X_v|} \\
&= |K|
\end{aligned}$$

平行移动解释

设$\gamma: [0, l] \to S$曲面上包含p的某一具有弧长参数的闭曲线. 记$\phi: [0, l] \to S^1$为角度函数. 记正交参数网$X(u, v)$下的两个单位向量为e_1, e_2, 则不妨设γ上向量场$w = e_1 \cos\phi + e_2 \sin\phi$, 记$w' = -e_1 \sin\phi + e_2 \cos\phi$为在切平面上逆时针旋转$\pi/2$的垂直单位向量, 从而测地曲率

$$\begin{aligned}
\kappa_g =& \left\langle D_{\vec{t}} w, w' \right\rangle \\
=& \phi' \left\langle w', w' \right\rangle + \left\langle \cos\theta D_{\vec{t}}(e_1) + \sin\theta D_{\vec{t}}(e_2), w' \right\rangle \\
=& \phi' + \left\langle D_{\vec{t}} e_1, e_2 \right\rangle \\
=& \phi' + \left\langle D_{\vec{t}}(X_1/\sqrt{E}), X_2/\sqrt{G} \right\rangle \\
=& \phi' + \left\langle X_1(1/\sqrt{E})_s, X_2/\sqrt{G} \right\rangle + \frac{1}{\sqrt{EG}} \left\langle D_{\vec{t}} X_1, X_2 \right\rangle \\
=& \phi' + \frac{1}{\sqrt{EG}} \langle u_s D_1 X_1 + v_s D_2 X_1, X_2 \rangle \\
=& \phi' + \frac{1}{\sqrt{EG}} \left\langle \operatorname{proj}_{T_p S}(u_s X_{11} + v_s X_{21}), X_2 \right\rangle \\
=& \phi' + \frac{1}{\sqrt{EG}} (-u_s \cdot E_v/2 + v_s \cdot G_u/2) \\
=& \phi' - \frac{1}{2\sqrt{EG}} (G_u \cdot v_s - E_v \cdot u_s)
\end{aligned}$$

积分得

$$\begin{aligned}
\oint_\gamma \kappa_g \mathrm{d}s - (\phi(l) - \phi(0)) &= -\int_D \frac{\sqrt{G}_u}{\sqrt{E}} \mathrm{d}v - \frac{\sqrt{E}_v}{\sqrt{G}} \mathrm{d}u \\
&= \int_D \left(\frac{\sqrt{G}_u}{\sqrt{E}} \right)_u + \left(\frac{\sqrt{E}_v}{\sqrt{G}} \right)_v \mathrm{d}u \mathrm{d}v \\
&= -\int_D -\frac{1}{\sqrt{EG}} \left[\left(\frac{\sqrt{G}_u}{\sqrt{E}} \right)_u + \left(\frac{\sqrt{E}_v}{\sqrt{G}} \right)_v \right] \mathrm{d}\sigma \\
&= -\int_D K \mathrm{d}\sigma
\end{aligned}$$

特别地, 当w为γ上平行移动的向量场时, $\kappa_g = 0$. 从而

$$K=\lim_{D\to\{p\}}\frac{\phi(l)-\phi(0)}{\mathrm{Area}(D)}.$$

测地坐标解释

定义p点处测地线$\gamma_v(t)$使得$\gamma_v(0)=p,\ \gamma_v'(0)=v$. 定义指数映射$\exp_p(v)=\gamma_v(1)$, 其中记$\exp_p(0)=p$. 此处可视$G=S$为Lie群, v生成的左不变向量场对应的单参数变化群为$\{\varphi_t\}$. 指数映射满足

1. $\exp_p : T_pG\to G, v\mapsto \varphi_t(e)=\gamma(1,v)$.
2. $\exp(tX)=\varphi(t,e)=\varphi_t(e)$.
3. $\exp((t_1+t_2)X)=\varphi(t_1X)\cdot\varphi(t_2X), t\in\mathbb{R}$.
4. $\exp(-tX)=\exp(tX)^{-1}$.
5. 记$\mathfrak{g}=(T_pG,[,])$为有限维Lie代数, 则切映射

   $$\mathrm{d}\exp_p : T_0\mathfrak{g}=\mathfrak{g}\to T_pG=\mathfrak{g}$$

   为恒同映射.

例如记$GL(n,\mathbb{R})$的切空间为$gl(n,\mathbb{R})$, 记$A\in gl(n,\mathbb{R})$的左不变向量场为X_A, 曲线$\gamma(t,A)$从I_n出发. 从而据定义得:

$$X_A(B)=\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}(B\circ\gamma(t,A))=B\cdot A.$$

从而X_A的积分曲线$\xi_A(t)$满足$\xi_A(0)=I_n,\ \xi'_A(0)=\xi_A(t)\cdot A$. 解得$\xi_A(t)=e^{tA}$. 从而指数映照为$\exp : gl(n,\mathbb{R})\to GL(n,\mathbb{R}), A\mapsto e^A$. 相应的Lie代数$\mathfrak{g}=(gl(n,\mathbb{R}),[,])$可如下计算(其中$\varphi_t(P)=P\cdot e^{tA}$为$A$生成的单参数变换群):

$$\begin{aligned}[A,B]=L_AB=&\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathrm{d}\varphi_{-t}(X_B(\varphi_t(A)))\\ =&\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}(e^{tA}\cdot B\cdot e^{-tA})\\ =&AB-BA\end{aligned}$$

对一般的二维正则曲面S, 记p点测地凸邻域由指数映射$\exp_p(\rho,\theta)=\exp_p(\rho e^{i\theta})$给出. 据定义, p点邻域处

$$\partial_\rho\exp_p(\rho,\theta)=\partial_\rho\gamma_{\rho e^{i\theta}}(1)=\partial_\rho\gamma_{e^{i\theta}}(\rho)$$

为切向量(单位方向向量), 从而$E=1$. 而测地线关于ρ的二阶导数与平面垂直, 因此$\Gamma_{11}^2=0$. 因此

$$0=E_\rho=2(\Gamma_{11}^1E+\Gamma_{11}^2F)=2\Gamma_{11}^1=0.$$

进而$F_\rho=\Gamma_{11}^1F+\Gamma_{11}^2G+\dfrac{1}{2}E_\rho=0$. 由于$\rho\to0$时$F=0$, 故$F\equiv0$.

从几何角度而言

$$\|\partial_\theta\exp_p(\rho,\theta)\|=\|\partial_\theta\gamma_{e^{i\theta}}(\rho)\|=\lim_{\Delta\theta\to0}\frac{\|\rho e^{i(\theta_0+\Delta\theta)}-\rho e^{i\theta_0}\|}{|\Delta\theta|}=\rho.$$

从而$\lim_{\rho\to0}\dfrac{\sqrt{G}}{\rho}=1$.

测地坐标下Gauß曲率

$$K=-\frac{1}{\sqrt{EG}}\left[\left(\frac{\sqrt{G}_u}{\sqrt{E}}\right)_u+\left(\frac{\sqrt{E}_v}{\sqrt{G}}\right)_v\right]=\frac{-\sqrt{G}_{\rho\rho}}{\sqrt{G}}.$$

从而常Gauß曲率曲面满足微分方程$\sqrt{G}_{\rho\rho}+K\sqrt{G}=0$, 列举如下:

1. $K=0$时, $\sqrt{G}=\rho f_1(\theta)+f_2(\theta)$. 由$\lim_{\rho\to 0}\dfrac{\sqrt{G}}{\rho}=1$知$G=\rho^2$.
2. $K>0$时, $\sqrt{G}=A_1(\theta)\cos(\rho\sqrt{K})+A_2(\theta)\sin(\rho\sqrt{K})$. 由$\lim_{\rho\to 0}\dfrac{\sqrt{G}}{\rho}=1$知 $G=\dfrac{1}{K}\sin^2(\sqrt{K}\rho)$.
3. $K<0$时, 同理解得$G=-\dfrac{1}{K}\sinh^2(\sqrt{-K}\rho)$.

另一方面, 当p点处Gauß曲率不为0时, 记$\sqrt{G}=\sum_{k=0}^{\infty}a_k\rho^k$, 由$\sqrt{G}_{\rho\rho}+K\sqrt{G}=0$得

$$\sum_{k=0}^{\infty}((k+1)(k+2)a_{k+2}+Ka_k)\rho^k\equiv 0.$$

因此

$$\sqrt{G}=\frac{1}{\sqrt{K}}\sum_{k\geq 0}(-1)^k\frac{(\sqrt{K}\rho)^{2k+1}}{(2k+1)!}.$$

局部来看, 记半径为$\rho=r\ll 1$的测地圆弧长为L, 从而

$$L=\lim_{\varepsilon\to 0}\int_{S^1\backslash(-\varepsilon,\varepsilon)}\sqrt{G}\mathrm{d}\theta=2\pi r-\frac{\pi}{3}r^3K(p)+O(r^5).$$

从而

$$K(p)=\lim_{r\to 0}\frac{3}{\pi}\frac{2\pi r-L}{r^3}.$$

同理, 记半径为$\rho=r\ll 1$的测地圆面积为A, 则

$$K(p)=\lim_{r\to 0}\frac{12}{\pi}\frac{\pi r^2-A}{r^4}.$$

Gauß-Bonnet公式解释

整体的Gauß曲率可用以下公式计算.

Gauß-Bonnet公式叙述如是: 边缘分段可微的曲面D满足

$$2\pi\chi(D)=\int_D K\mathrm{d}\sigma+\int_{\partial D}\kappa_g\mathrm{d}s+\sum\alpha.$$

Euler-Poincaré定义叙述如是:

$$2\pi\sum_{p\in D}I_p=\int_D K\mathrm{d}\sigma.$$