A glimpse of Kuratowski's 14-Set Problem

The Original Problem

Problem. Let $S \in \mathcal{P}(\mathbb{R}^n)$. Let k, I and c be unary operators such that

$$k(S) = \overline{S}, \quad I(S) = S, \quad c(S) = \mathbb{R}^n - S.$$

Prove that there at most 14 sets generated by S.

Proof Sketch.

Step I The unary operator $i(S) = S^\circ$ can be generated by k and c, i.e., $ckc = i$.

Step II Let $\langle I, i, c, k \rangle$ be the free semigroup, which consists of i^2, $icik$, k^3i, etc.

Let M be the quotient semigroup $\langle I, i, c, k \mid \text{reduction of set operators} \rangle$. For instance, $I^k = I, c^3 = c, i^2 = i$ under such quotient relations. Then, $\{xS \mid x \in M\}$ consists of all possible sets generated by S.

Step III Utilise the relations $ic = ck$, $kc = ci$, $c^2 = I$, to move the word c to the initial of any $y \in M$. For instance, $cikci = ki^2$.

Step IV Prove that $i^2 = i$, $k^2 = k$, $ik = ikik$, $ki = kiki$.

Step V Find that $M = \{I, i, k, ik, ki, iki, kik, c, ci, ck, cik, cki, ciki, ckik\}$

Ex1 We define $x \leq y$ for each $x, y \in M$ whenever $x(S') \subset y(S')$ for each $S \in \mathbb{R}^n$, where $x < y$ whenever $x(S') \subsetneq y(S')$. Find the partially ordered structure of M.

Step VI Find an S such that $M(S)$ consists of 14 distinct sets. For simplicity, one can just find the union of all pairwise disjoint examples which verifying the partially ordered structure.

Further Exploration

Ex2 Let $S \in \mathcal{P}(\mathbb{R}^n)$. Let k, I and c be unary operators defined above. Let $\wedge$ be the binary operator defined as $\wedge(S_1, S_2) = S_1 \cap S_2$. State a similar problem and solve it.

Hint:

$$
\begin{tikzcd}
 & k \arrow[dl] \arrow[dr] & \\
I \arrow[d] & & kik \arrow[dll] \arrow[dl] \arrow[d] \\
I \wedge kik \arrow[d] \arrow[dr] & ki \arrow[dl] \arrow[dr] & ik \arrow[dl] \arrow[d] \\
I \wedge ki \arrow[d] & I \wedge ik \arrow[dl] & ki \wedge ik \arrow[dll] \arrow[d] \\
I \wedge ki \wedge ik \arrow[dr] & & iki \arrow[dl] \\
 & I \wedge iki \arrow[d] & \\
 & i &
\end{tikzcd}
$$

Website for creating commutative diagram.