On the theory of ultrafilters

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Why we need ultrafilters?

Continuity v.s. Absolute Continuity

Example 1.1.1 Let $f : X \to Y$ be a mapping between Hausdorff spaces, then the following three statements are equivalent:

1. f is continuous at $p \in X$.
2. For every neighbourhood $U_{f(p)}$, the preimage $f^{-1}(U_{f(p)})$ is a neighbourhood of p.
3. For every sequence $x_n \to p$, we also have $f(x_n) \to f(p)$.

> 💡 **Remark** We call $x_n \to x$ in the topology (X, τ), whenever U_x contains x_{k+N} for some $N \in \mathbb{N}$. Here U_x is an arbitrary open set containing x. In other words, each subset of x contains almost all elements of $\{x_n\}$ except finite lamsters.

However, somethings gets stranger as X and Y are no longer Hausdorff spaces, i.e., 2 implies 3, but it seems impossible to prove the the converse. As a result, we shall DISTINCT **continuity** and **sequential continuity** thenceforth.

We know that

> 💡 **Fact 1.1.2** Sequential compactness $\not\Rightarrow$ compactness. (See **Fact 2.2.5** for examples)

Then we might propose the following conjecture

> ❓ **Conjecture 1.1.3** Sequential continuity $\not\Rightarrow$ continuity.

The conjecture is proved true by the following example

Example 1.1.4 Let X be uncountable. Consider the example constructed by the following steps

1. Let η be the discrete topology of $X \setminus \{x_0\}$, that is, $\eta = \mathcal{P}(X \setminus \{x_0\})$.
2. Let τ be the topology of X, given by

$$\tau = \eta \cup \{A \cup \{x_0\} \mid |X \setminus A| \leq |\mathbb{N}|\}.$$

3. (X, τ) and $(X, 2^X)$ has the same set of convergent sequences. However, $i : (X, \tau) \to (X, 2^X)$ is not continuous since the latter has finer topology than the former.

▼ **Hints**

$\{x_n\}$ is convergent in either (X, τ) or $(X, 2^X)$, whenever $x_{N+1} = x_{N+1} = \cdots$ for some $N \in \mathbb{N}$. The function i is sequentially continuous everywhere but discontinuous at x_0.

Indeed, the continuity can be characterised by a kind of ***generalised sequece***, which assumes uncountable many elements. Such generalised sequence is actually a poset relation, the poset itself is called a filter when some conditions are satisfied.

From filters to ultrafilters

Definition 1.2.1 We call non-empty subset $\mathscr{F} \subset \mathcal{P}(X)$ a filter whenever

1. (Downward closed) $\forall A, B \in \mathscr{F}$, $A \cap B \in \mathscr{F}$.
2. (Upward closed) $\forall A \in \mathscr{F}$, $\{U \in \mathscr{P}(X) \mid A \subset U\} \subset \mathscr{F}$.
3. $\emptyset \notin \mathscr{F}$.

Example 1.2.2 We call $\{U \in \mathscr{P}(X) \mid x_0 \in U \subset X\}$ a principle filter generated by x_0.

Example 1.2.3 x-nieghbourhood filter is generated by all neighbourhoods containing x, which is defined on a topology space.

Example 1.2.4 The Fréchet filter is defined by the set of cofinite subsets.

Definition 1.2.5 We define **prime filters** and **ultrafilters (or maximal filters)** similar the way as prime ideals and maximal ideals:

$\mathscr{F}$ is a **prime filter** on X whenever $\mathscr{F}$ ***is generated by*** $\mathscr{F}_1$ ***and*** $\mathscr{F}_2$ implies either $\mathscr{F}_1 = \mathscr{F}$ ***or*** $\mathscr{F}_2 = \mathscr{F}$.

$\mathscr{F}$ is a **maximal filter** whenever $\mathscr{F} \subset \mathscr{F}'$ implies $\mathscr{F}' = \mathscr{F}$ if $\mathscr{F}'$ is a filter.

We also have the following propositions:

1. A maximal filter (or ultrafilter) is always prime.
2. A principle filter is always maximal.
3. For each filter $\mathscr{F}$, there exists an ultrafilter contianing $\mathscr{F}$. It depends on **the Axiom of Chioce**.

I is a **prime ideal** on R whenever I ***is generated by*** I_1 ***and*** I_2 implies ***either*** $I = I_1$ ***or*** $I = I_2$.

I is a **maximal ideal** whenever $I \subset I'$ implies $I = I'$ if I' is a nontrivial ideal.

We also have the following propositions:

1. A maximal ideal is always prime.
2. A principle ideal is always maximal.
3. For each ideal I, there exists an maximal ideal contianing I. It depending on **the Axiom of Chioce**.

Furthermore, we find that every finite subset of a filter has non-empty intersection. The following theorem strengthens the existence of ultrafilter:

Theorem 1.2.6 Let $\mathscr{C}$ be a collection of subsets with **finite intersection property (*FIP*** for short**), that is, arbitrary finite many elements in it always have nonempty intersetion**. Then there exists an ultrafilter $\mathscr{U}$ containing $\mathscr{C}$.

> Here we admit the axiom of choice.

▼ **Proof of the theorem**

Let $\mathscr{X} := \{U \supset \mathscr{C} \mid U \in \mathcal{P}(X), U \text{ has FIP}\}$, which has the **finite intersection property**.

💡 **Definition 1.2.7** Let βX denotes the space of ultrafilters, we see the **canonical embedding**

$$X \to \beta X, \quad x \mapsto \{x \mid x \in U\},$$

which resembles $X \to X^{**}$, the **canonical embedding** into double dual space.

We shall compare it with **Banach limits** in the following sections.

Space of ultrafilters

Basic properties

🚧 **Notation** The symbol $\mathscr{F}$ always stands for filters, $\mathscr{U}$ stands for ultrafilters.

Theorem 2.1.1 Each filter is contained in an ultrafilter by Zorn's lemma.

▼ **Proof of the theorem**

Consider the partially ordered set $(P, \leq)$. Here P consists the set of filters of X, $\mathscr{F}_1 \leq \mathscr{F}_2$ whenever $\mathscr{F}_1 \subset \mathscr{F}_2$. Let $\{\mathscr{F}\}_{i \in I}$ be a chain in P that contains $\mathscr{F}_1$. We claim that $\cup_{i \in I} \{\mathscr{F}_i\}$ is also a filter since it satisfies 3 principles in **Definition 1.2.1**.

Since each chains has a upper bound, there exists a maximal element by Zorn's lemma.

Theorem 2.1.2 Let $\mathscr{U}$ be an ultrafilter. If $B \cap A \neq \emptyset$ for each $A \in \mathscr{U}$, then $B \in \mathscr{U}$.

▼ **Proof of the theorem**

The set $\{B \cap A \mid A \in \mathscr{U}\}$ is equipped with the finite intersection principle, thus can be extended into an ultrafilter $\mathscr{U}'$. Therefore, $B \in \mathscr{U}'$ and $\mathscr{U}' \subset \mathscr{U}$. Thus $B \in$

$\mathscr{U} = \mathscr{U}'$.

Theorem 2.1.3 Let $\mathscr{U}$ be an ultrafilter. If $A \cup B \in \mathscr{F}$, then $(A \in \mathscr{U}) \vee (B \in \mathscr{U})$.

▼ **Proof of the theorem**

For the sake of contradiction, if $A \notin \mathscr{U}$ and $B \notin \mathscr{U}$, then there exists $C, D \in \mathscr{U}$ such that $C \cap A = D \cap B = 0$. As a result, $(A \cup B) \cap (C \cap D) = \emptyset$. Since $C \cap D$ is nonempty, $A \cup B \notin \mathscr{U}$, which leads to a contradiciton.

Theorem 2.1.4 If $\mathscr{U}_1 \neq \mathscr{U}_2$, there there exists $A \in \mathscr{U}_1$ and $B \in \mathscr{U}_2$ such that $A \cap B = \emptyset$.

▼ **Proof of the theorem**

By **Theorem 2.1.1**.

Theorem 2.1.5 If $\mathscr{U}$ is an ultrafilter and $A \in \mathscr{U}$, if $A = \dot{\bigcup}_{i=1}^{N} A_i$ is a disjoint union, then exactly one A_i is in $\mathscr{U}$.

▼ **Proof of the theorem**

Prove by induction (see **Theorem 2.1.3**).

Here is another definition of ultrafilter:

💡 **Definition 2.1.6** (Compare to **Definition 1.2.5**) $\mathscr{U} \subset 2^X$ is an ultrafilter whenever

1. $X \in \mathscr{U}$ and $\emptyset \notin \mathscr{U}$.
2. If $A \in \mathscr{U}$ and $B \in \mathscr{U}$, then $A \cap B \in \mathscr{U}$.
3. For all $A \subset X$, either $A \in \mathscr{U}$ or $A^c \in \mathscr{U}$.

> Exactly ***half*** of the subsets in 2^X are in $\mathscr{U}$.

One can also characterise ultrafilters by finitely-additive measure on X, i.e.,

$$\mu : 2^X \to \{0, 1\}, A \mapsto \begin{cases} 1, & A \in \mathscr{U}, \\ 0, & A \notin \mathscr{U}. \end{cases}$$

As a result, $\mu(\dot{\bigcup}_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$ for disjoint union. We obtain that each ultrafilter corresponds a finitely-additive measure.

Theorem 2.1.7 Let X be a finite set. Then each ultrafilter is principle (see **Example 1.2.2**).

▼ **Proof of the theorem**

Let $\mathscr{U}$ be an ultrafilter. First $\cap\{U \in \mathscr{U}\} \neq \emptyset$ since it is a finite intersection. If $\cap\{U \in \mathscr{U}\} \neq \emptyset$ contains two elements, i.e., $\{x_1, x_2\} \subset \cap\{U \in \mathscr{U}\}$, then $\mathscr{U}$ is contained in the principle ultrafilter generated by x_1. It contradicts our assumption.

As a result, any ultrafilter on a finite set is generated by one element, that is, a principle ultrafilter.

💡 **Fact 2.1.8** There exists a **non-principle ultrafilter** on **INFINITE** set X. For instance, the set $\{U \mid |X \setminus U| < |\mathbb{N}|\}$ is an ultrafilter yet not principle.

Space $\beta\mathbb{N}$

Ultrafilters seems trivial when we set X as a finite set, since all ultrafilters are principle. **Definition 1.2.7** provides an isomorphism.

Take $\mathbb{N}$ as an example, define the double dual of $A \subset X$ by

Definition 2.2.1

$$A^\star := \{\mathscr{U} \in \beta X \mid A \in \mathscr{U}\},$$

that is, the collection of ultrafilter contains the given set.

Theorem 2.2.2 We have the following:

1. $A^\star \cap B^\star = (A \cap B)^\star$,
2. $(A^c)^\star \dot{\cup} A^\star = \beta X$,
3. $A^\star \cup B^\star = (A \cup B)^\star$,
4. $A^\star \subset B^\star$ whenever $A \subset B$.

▼ **Proof of the theorem**

The first identity is due to

$$
\begin{aligned}
A^\star \cap B^\star &= \{\mathscr{U} \mid A, B \in \mathscr{U}\} \\
&= \{\mathscr{U} \mid A \cap B \in \mathscr{U}\} \\
&= (A \cap B)^\star.
\end{aligned}
$$

The second identity is due to

$$
\begin{aligned}
(A^c)^\star &= \{\mathscr{U} \in \beta X \mid A^c \in \mathscr{U}\} \\
&= \{\mathscr{U} \in \beta X \mid A \notin \mathscr{U}\} \\
&= \beta X \setminus \{\mathscr{U} \in \beta X \mid A \in \mathscr{U}\}.
\end{aligned}
$$

The third identity is similar to the first one.

The final identity is trivial.

Theorem 2.2.3 The following statements are equivalent:

1. (X, τ) is comact.
2. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a family of open sets of X such that $\cup_{\lambda \in \Lambda} U_\lambda = X$. Then there exists a **FINITE** subset $I \subset \Lambda$ such that $\cup_{\lambda \in I} U_\lambda = X$.
3. Let $\{F_\lambda\}_{\lambda \in \Lambda}$ be a family of closed sets such that for each finite $I \subset \Lambda$, $\cap_{\lambda \in I} F_\lambda \neq \emptyset$ (the **finite intersection property**, see **Theorem 1.2.6**). Then $\cap_{\lambda \in \Lambda} F_\lambda \neq \emptyset$.
4. Every ultrafilter converges.

> 💡 We say $\mathscr{U}$ **converges** to x on (X, τ), whenever each neighbourhood of x contains some elements in $\mathscr{U}$. Especially, every convergent filter **converges to exactly one point on Hausdorff space**.

5. $\cap\{\overline{A} \mid A \in \mathscr{F}\} \neq \emptyset$, $\mathscr{F}$ is arbitrary ultrafilter.

▼ **Proof of the theorem**

2️⃣ and 3️⃣ are equivalent definition of compactness. An equivalent statement of 2️⃣ is

> Let $\{F_\lambda\}_{\lambda\in\Lambda}$ be a family of closed sets of X such that $\cap_{\lambda\in\Lambda}F_\lambda = \emptyset$. Then there exists a **FINITE** subset $I \subset \Lambda$ such that $\cap_{\lambda\in I}F_\lambda = \emptyset$.

Thus 2️⃣ and 3️⃣ are equivalent. We mainly focus on

2️⃣➡️4️⃣ If there exists a non-convergent ultrafilter $\mathscr{U}$, then for eahc $x \in X$ there exists a neighbour $V_x \notin \mathscr{U}$. Since the open conver $\{V_x\}_{x\in X}$ has finite subcover $\{V_{x_i}\}_{i=1}^N$, we have $V_{x_i}^c \in \mathscr{U}$ and $\cap_{i=1}^N V_{x_i}^c = \emptyset$, which contradicts our assumption.

4️⃣➡️5️⃣ For arbitrary filter $\mathscr{F}$, there exists an ultrafilter $\mathscr{U} \supset \mathscr{F}$. Let x be one of the convergent points that $\mathscr{U}$ converges to. $\forall A \in \mathscr{F}$, $\forall V_x$, we have $V_x \cap A = \emptyset$. As a result, $x \in \overline{A}$.

5️⃣➡️2️⃣ For the sack of contradiction, suppose that (X, τ) is not compact. Then there exists an open cover $\{O_i\}_{i\in I}$ of X without finite subcovers. Construct a basis of filter by

$$\mathscr{B} := \{\cap_{i\in\Lambda} O_i^c \mid n \geq 1, \Lambda \subset O, |\Lambda| < |\mathbb{N}|\}$$

Then $\mathscr{B}$ is equipped with **FIP** (See **Theorem 1.2.6**), thus can be extended to a filter $\mathscr{F}$. However, $\cap\mathscr{F} = \emptyset$, which leads to a contradiction.

Theorem 2.2.4 The set $\mathscr{B} := \{A^\star \mid A \subset \mathbb{N}\}$ is a basis of **compact Hausdorff space** on $\beta\mathbb{N}$.

▼ **Proof of the theorem**

We shall prove 1️⃣ $\mathscr{B}$ is a valid basis, 2️⃣ the compactness of $\beta\mathbb{N}$, as well as 3️⃣ $\beta\mathbb{N}$ is a Hausdorff space.

1️⃣ For each $\mathscr{U} \in \beta\mathbb{N}$, there exists A such that $\mathscr{U} \in A^\star$. If $\mathscr{U} \in A^\star \cap B^\star$, then $\mathscr{U} \in (A \cap B)^\star$ by **Theorem 2.2.2**. Therefore $\mathscr{B}$ is a well-defined topological basis.

2️⃣ In fact each element in $\mathscr{B}$ is bath open and closed. Let Y be closed set in $\beta\mathbb{N}$, then there exists a set of $A_\lambda \subset \mathbb{N}$ such that

$$Y = (\cup A_\lambda^*)^c = \cap(A_\lambda^c)^*.$$

Hence each closed set is a intersection of sets in $\mathscr{B}$. Let $\{[U_\lambda]\}_{\lambda\in\lambda}$ be arbitrary collection of open sets covering βX. For the sake of contradiction we assume that

βX has no finite subcover. Then $\cap_{k=1}^{N} U_{\lambda_k}^c \neq \emptyset$. **Theorem 1.2.6** proves the existence of ultrafilter $\mathscr{U}$ containing $\{U_\lambda^c\}_{\lambda\in\Lambda}$. There exists U_{λ_0} such that $\mathscr{U} \in [U_{\lambda_0}]$. Therefore both U_{λ_0} and $U_{\lambda_0}^c$ are contained in $\mathscr{U}_0$, which leads to a contradictory!

3️⃣ For arbitrary ultrafilter $\mathscr{U}$ and $\mathscr{U}'$, there exists $A \in \mathscr{U}$ and $B \in \mathscr{U}'$ such that $A \cap B = \emptyset$ by definition of ultrafilter. Therefore, we have $\mathscr{U} \in A^\star$, $\mathscr{U}' \in B^\star$, and $A^\star \cap B^\star = \emptyset$.

💡 **Fact 2.2.4** Let $\mathscr{P}(X)$ be the product topology space $(\{0,1\}^X, \tau)$, where τ is generated by

$$V(\alpha,\beta) := \{(u_i)_{i\in X} \in \mathcal{P}(X) \mid u_{\alpha_j} = 1, u_{\beta_k} = 0\}.$$

That is, $V(\alpha,\beta) = (1)_\alpha \times (0)_\beta \times \{0,1\}_{X\setminus(\alpha\dot{\cup}\beta)}$. We equal $\{0,1\}^X$ and $\mathcal{P}(X)$ here.

> For instance, an open set in $\mathscr{P}(\{1,2,3\})$ is given by
>
> $$V(\{1\},\{2\}) = \{\{(1,0,1)\},\{(1,0,0)\}\} = \{\{1,3\},\{1\}\}.$$

We claim that $\mathscr{P}(X)$ is finer than βX.

Fact 2.2.5 (**Stone-Čech compactification**) The topology $\beta\mathbb{N}$ is called **Stone-Čech compactification of** $\mathbb{N}$. Since no sequences converge to a point in $\beta\mathbb{N}\setminus\mathbb{N}$ and $\mathbb{N}$ is not sequentially compact, we deduce that $\beta\mathbb{N}$ is not sequentially compact. However, $\beta\mathbb{N}$ is indeed compact!

> As we mentioned in **Example 1.1.4**, sequentially properties no longer imply absolute properties.

Theorem 2.2.6 We set J as **canonical embedding** in **Definition 1.2.7**. Then $J(x)$ is isolated in βX, $J(X)$ is dense in βX.

▼ **Proof of the theorem**

We claim that $J(x)$ is isolated in βX since $\{x\}^\star = \{J(x)\}$ is both close and open.

For each $\mathscr{U} \in \beta X$, there exists $A \subset X$ such that $\mathscr{U} \in A^\star$ by definition of topological spaces. Take $x \in A$, then $J(x) \in A^\star$. It reveals that each neighbourhood of $\mathscr{U}$ has non empty intersection with $J(X)$, thus $J(X)$ is dense in $\beta\mathbb{N}$.

Algebratical properties of $\beta\mathbb{N}$ space

Definition 2.3.1 We define the binary operation on $\beta\mathbb{N}$ as follows

$$\begin{aligned} + :& \beta\mathbb{N} \times \beta\mathbb{N} \to \beta\mathbb{N}, \\ & (\mathscr{U}_1, \mathscr{U}_2) \mapsto \{A \subset X \mid \{n \in \mathbb{N} \mid A - n \in \mathscr{U}_2\} \in \mathscr{U}_1\}. \end{aligned}$$

▼ Proof that $+$ is closed on $\beta\mathbb{N}$.

For each $\mathscr{U}_1$ and $\mathscr{U}_2$ in $\beta\mathbb{N}$, it is easy to show $X \in (\mathscr{U}_1 + \mathscr{U}_2)$ and $\emptyset \notin \mathscr{U}_1 + \mathscr{U}_2$.

Suppose that $A, B \in (\mathscr{U}_1 + \mathscr{U}_2)$, we claim that $(A \cap B) \in (\mathscr{U}_1 + \mathscr{U}_2)$. This is due to

$$\{n \mid A - n \in \mathscr{U}_2\} \cap \{n \mid B - n \in \mathscr{U}_2\} \subset \{n \mid A \cap B - n \in \mathscr{U}_2\}.$$

Suppose $A \notin (\mathscr{U}_1 + \mathscr{U}_2)$. Then $\{n \mid A - n \in \mathscr{U}_2\} \notin \mathscr{U}_1$ and $\{n \mid A - n \notin \mathscr{U}_2\} \in \mathscr{U}_1$. Since $A - n \notin \mathscr{U}_2$ whenever $A^c - n \in \mathscr{U}_2$. Thus $A^c \in (\mathscr{U}_1 + \mathscr{U}_2)$.

Theorem 2.3.2 Let $\mathscr{U}_i$ be the principle ultrafilter generated by x_i, $i = 1, 2$. Then $\mathscr{U}_1 + \mathscr{U}_2$ equals the principle filter generated by $x_1 + x_2$.

▼ **Proof of the theorem**

This is due to

$$\begin{aligned} & A \in (\mathscr{U}_1 + \mathscr{U}_2) \\ \Leftrightarrow & \{n \in \mathbb{N} \mid A - n \in \mathscr{U}_2\} \in \mathscr{U}_1 \\ \Leftrightarrow & x_1 \in \{n \in \mathbb{N} \mid A - n \in \mathscr{U}_1\} \\ \Leftrightarrow & (A - x_1) \in \mathscr{U}_2 \\ \Leftrightarrow & x_1 \in (A - x_1) \\ \Leftrightarrow & (x_1 + x_2) \in A. \end{aligned}$$

Theorem 2.3.3 $+$ is is associative.

▼ **Proof of the theorem**

We shall prove that $(\mathscr{U}_1+\mathscr{U}_2)+\mathscr{U}_3=\mathscr{U}_1+(\mathscr{U}_2+\mathscr{U}_3)$. Since

$$\begin{aligned}&A\in\mathscr{U}_1+(\mathscr{U}_2+\mathscr{U}_3)\\ \Leftrightarrow&\{n\mid\{m\mid A-m-n\in\mathscr{U}_3\}\in\mathscr{U}_2\}\in\mathscr{U}_1\\ \Leftrightarrow&\{n\mid\{m\mid A-n\in\mathscr{U}_3\}-m\in\mathscr{U}_2\}\in\mathscr{U}_1\\ \Leftrightarrow&(U_1+\mathscr{U}_2)+\mathscr{U}_3.\end{aligned}$$

Theorem 2.3.4 $+$ is left continuous, that is, $\mathscr{U}\mapsto\mathscr{U}+\mathscr{U}_0$ is continuous for arbitrary fixed $\mathscr{U}_0$.

▼ **Proof of the theorem**

For each basic open set $A^\star$, the preimage of $A^\star$ under the mapping $\mathscr{U}\mapsto\mathscr{U}+\mathscr{U}_0$ is

$$\begin{aligned}&\{\mathscr{U}\mid\mathscr{U}+\mathscr{U}_0\in A^\star\}\\ \Leftrightarrow&\{\mathscr{U}\mid A\in(\mathscr{U}+\mathscr{U}_0)\}\\ \Leftrightarrow&\{\mathscr{U}\mid\{n\mid A-n\in\mathscr{U}_0\}\in\mathscr{U}\}\\ \Leftrightarrow&\{n\mid A-n\in\mathscr{U}_0\}^\star.\end{aligned}$$

Theorem 2.3.5 There exists $\mathscr{I}\in\beta\mathbb{N}$ such that $\mathscr{I}+\mathscr{I}=\mathscr{I}$, if Axiom of Chioce is admitted.

▼ **Proof of the theorem**

Let $\mathscr{A}:=\{U\subset\beta\mathbb{N}\mid U\text{ is a compact semigroup}\}$. Then $\beta\mathbb{N}\in\mathscr{A}$ and $\mathscr{A}$ is partially ordered by inclusion. Each chain in $\mathscr{A}$ has a non-empty and compact. By Zorn's lemma we deduce the existence of minimal element A in $\mathscr{A}$.

We claim that each $\mathscr{U}\in A$ is the desired $\mathscr{I}$. By left continuity we deduce that $A+\mathscr{U}=\{\mathscr{U}'+\mathscr{U}\mid\mathscr{U}'\in A\}$ is also a compact semigroup, and $A+\mathscr{U}=A$ by minimality.

Let $B:=\{\mathscr{U}'\in A\mid\mathscr{U}'+\mathscr{U}=\mathscr{U}\}$. Since $A+\mathscr{U}=A$, B is non-empty. We notice that B is also a compact semigroup by continuity. Thus $B=A$ by minimality.