Hopf Algebra

In retrospect of Abelian Group Category ($\mathbb{A}G$)
Tensor Product
Adjunctions

In retrospect of Abelian Group Category ($\mathbb{A}G$)

Definition 1.1 $(G, +_G)$ is an **group** whenever

- **A0**: G is closed under addition, i.e., $a+_Gb\in G$ for each $a,b\in G$.
- A1: G satisfies the law of associativity, i.e.,

$$G \xleftarrow{+_G \circ (\mathrm{id}_G, +_G)} G imes G imes G imes G \xrightarrow{+_G \circ (+_G, \mathrm{id}_G)} G$$



Remark We write $+ := +_G$, $\mathrm{id} := \mathrm{id}_G$ for simplicity when there is no ambiguity.

• **A3**: There exists some $0 \in G$ such that

$$0+G \longleftarrow \operatorname{id} G \longrightarrow G+0$$

$$0 + a \leftarrow a \leftarrow a + 0$$

The **uniqueness** of 0 is clear.

• A4: The left action $g.:G o G, a\mapsto g+a$ is bijective for each $g\in G.$

Note $\mathrm{inv}(g):=g.^{-1}(0)\in G.$ Then $\mathrm{inv}(g).\circ g.=\mathrm{id}$, that is, $\mathrm{inv}(g)+g=0.$ Since

$$egin{aligned} g + \mathrm{inv}(g) &= \left[\mathrm{inv}(\mathrm{inv}(g)) + \mathrm{inv}(g)
ight] + g + \mathrm{inv}(g) \ &= \mathrm{inv}(\mathrm{inv}(g)) + \left[\mathrm{inv}(g) + g
ight] + \mathrm{inv}(g) \ &= \mathrm{inv}(\mathrm{inv}(g)) + \mathrm{inv}(g) \ &= 0, \end{aligned}$$

we conclude that there exists unique $\operatorname{inv}(g)$ such that $g + \operatorname{inv}(g) = \operatorname{inv}(g) + g = 0$.



Remark We write $(-g) := \operatorname{inv}(g)$ for simplicity.

Definition 1.2 G is **Abelian** whenever $+\circ S=+:G imes G o G$, here

$$S:G imes G o G imes G,\quad (a,b)\mapsto (b,a).$$



Remark We assume G is always an additive Abelian group thenceforth.

Fact 1.3 G is Abelian group whenever G is a \mathbb{Z} -module, i.e. there exists an action

$$\mathbb{Z} imes G o G, (n,g) \mapsto (-1)^{\operatorname{sgn}(n)} |n| g.$$



Remark $G = {}_{\mathbb{Z}}G_{\mathbb{Z}}$ is bimodule.

Definition 1.4 (Homomorphism) $f:G \to H$ is a homomorphism between additive Abelian groups, whenever

$$f\circ +_G=+_H\circ (f,f):G imes G o H.$$

Note that $\operatorname{Hom}_{\mathbb{A} G}(G,H)=\{\operatorname{Homomorphisms} G o H\}.$ Here $\operatorname{Hom}_{\mathbb{A} G}(G,H)$ is also Abelian.



Remark It is clear that $f:0_G o 0_H$.

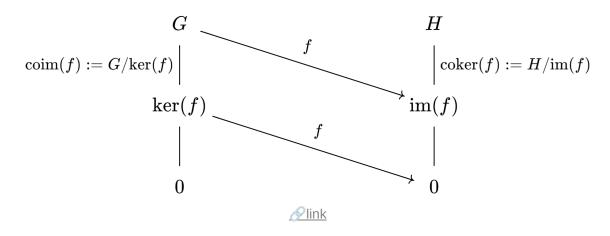
Definition 1.5 The Category of Abelian groups is denoted by $\mathbb{A}G$, where $\mathsf{Ob}(\mathbb{A}G)$ are Abelian groups, $\mathsf{Mor}(\mathbb{A}G)$ are homomorphisms.



Remark $\operatorname{Hom}_{\mathbb{A} G}(G,G)=:\operatorname{End}_{\mathbb{A} G}(G,G)$ has a ring structure, i.e.

- $f+g: a \mapsto f(a)+g(a)$,
- $f \circ g : a \mapsto f(g(a))$.

Definition 1.6 We have the following additive Abelian groups induced by f



- $\operatorname{im}(f) := (f(G), +_H)$ is an additive Abelian group.
- $\ker(f) := (f^{-1}(0), +_G)$ is an additive Abelian group.
- ullet $\operatorname{coker}(f) := (H + f(G), ilde{+}_H)$ is an additive Abelian group, where

$$ilde{+}_H: (h_1+f(G),h_2+f(G)) \mapsto (h_1+h_2+f(G)).$$

ullet $\operatorname{coim}(f):=(G+f^{-1}(0), ilde{+}_G)$ is an additive Abelian group, where

$$ilde{+}_G: (g_1+f^{-1}(0),g_2+f^{-1}(0)) \mapsto (g_1+g_2+f^{-1}(0)).$$

Definition 1.7 Given a family of groups $\{G_{\lambda}\}_{\lambda\in\Lambda}$, then we define

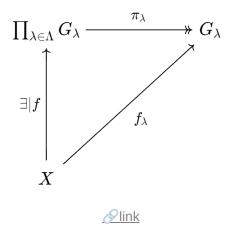
- the product of $\{G_{\lambda}\}_{\lambda\in\Lambda}$ is defined as

$$\prod_{\lambda\in\Lambda}G_\lambda:=(g_\lambda)_{\lambda\in\Lambda},\quad g_\lambda\in G_\lambda.$$

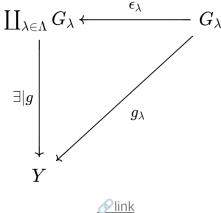
- the coproduct of $\{G_{\lambda}\}_{\lambda\in\Lambda}$ is defined as

$$\coprod_{\lambda\in\Lambda}G_\lambda:=(g_\lambda)_{\lambda\in\Lambda},\quad g_\lambda\in G_\lambda,\quad |\{\lambda\mid g_\lambda
eq 0\}|<\infty.$$

Fact 1.8 For product, we see that for each $f_\lambda\in \operatorname{Hom}_{\mathbb{A} G}(X,G_\lambda)$, there exists unique f s.t.

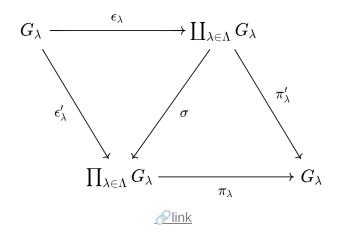


For coproduct, we see that for each $g_\lambda\in \operatorname{Hom}_{\mathbb{A} G}(G_\lambda,Y)$, there exists unique g s.t.



We also have the following propositions:

- each π_{λ} has the right inverse ϵ'_{λ} ;
- each ϵ_{λ} has the left inverse π'_{λ} ;
- there exists unique $\sigma:\coprod_{\lambda\in\Lambda}G_\lambda\to\prod_{\lambda\in\Lambda}G_\lambda$ such that



Here σ is injective (in $\mathbb{A}G$ category).

the following homomorphisms holds

$$egin{aligned} \operatorname{Hom}_{\mathbb{A} G}(\coprod_{\lambda \in \Lambda} G_{\lambda}, H) &\cong \prod_{\lambda \in \Lambda} \operatorname{Hom}_{\mathbb{A} G}(G_{\lambda}, H), \quad f \mapsto (fe_{\lambda})_{\lambda \in \Lambda}; \ \operatorname{Hom}_{\mathbb{A} G}(G, \prod_{\lambda \in \Lambda} H_{\lambda}) &\cong \prod_{\lambda \in \Lambda} \operatorname{Hom}_{\mathbb{A} G}(G, H_{\lambda}), \quad f \mapsto (p_i f)_{\lambda \in \Lambda}. \end{aligned}$$

Remark Let the diagram $\varphi \downarrow \begin{subarray}{c} a \longrightarrow b \\ c \longrightarrow d \end{subarray} \psi \text{ denote } \psi \circ a = g \circ \varphi.$

Definition 1.9

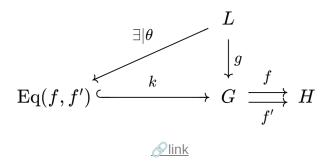
ullet Equaliser of $G \overset{f}{\underset{f'}{\longrightarrow}} H$ is defined as the subgroup

$$\operatorname{Eq}(f,f') := \{ a \in G \mid f(a) = f'(a) \} \quad (\stackrel{k}{\hookrightarrow} G).$$

- Coequaliser of $G \overset{f}{\underset{f'}{\longrightarrow}} H$ is defined as the quotient group

$$(H\stackrel{c}{ wo}) \quad \{h+\operatorname{im}(f)\mid h+\operatorname{im}(f)=h+\operatorname{im}(f')\}:=\operatorname{Coeq}(f,f').$$

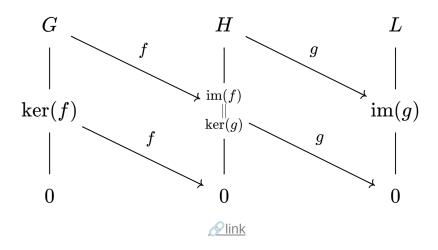
Fact 1.10 For each $g \in \operatorname{Hom}_{\mathbb{A} G}(L,G)$, there exists unique heta such that



For each $f\in \operatorname{Hom}_{\mathbb{A} G}(H,Y)$, there exists unique au such that

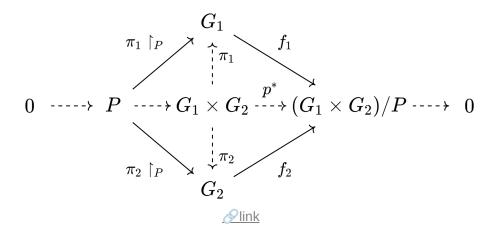
Definition 1.11 $\ker(f) := \operatorname{Eq}(f, 0)$, $\operatorname{coker}(g) := \operatorname{Coeq}(g, g')$.

Definition 1.12 We say $0 o G \overset{f}{ o} H \overset{g}{ o} L o 0$ is a short exact sequence, whenever



Definition 1.13 Consider $f_i\in \operatorname{Hom}_{\mathbb{A} G}(G_i,H)$ ($i\in\{1,2\}$), define **pullback of homomorphism** as

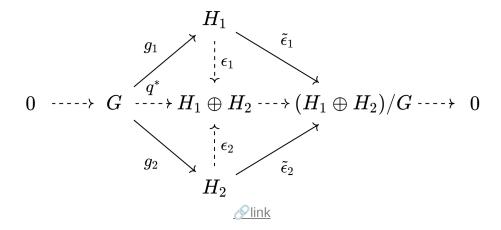
$$p^*:=f_1\circ\pi_1-f_2\circ\pi_2:G_1 imes G_2 o H.$$



Here the middle row is a short exact sequence.

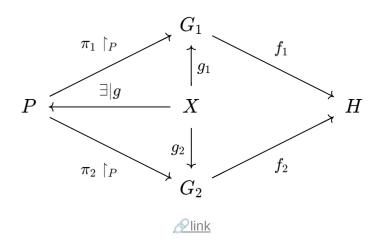
Consider $g_i \in \operatorname{Hom}_{\mathbb{A} G}(G,H_i)$ ($i \in \{1,2\}$), define the **pushback of homomorphism** as

$$q^*:=\epsilon_1\circ g_1-\epsilon_2\circ g_2:G o H_1\oplus H_2.$$



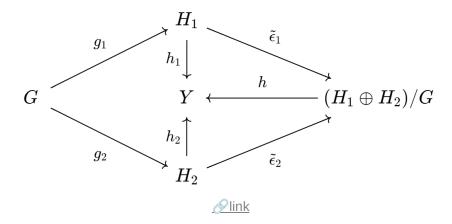
Here the middle row is a short exact sequence.

Fact 1.14 The pullback of homomorphisms satisfy



 $orall g_1,g_2$ $(f_1\circ g_1=f_2\circ g_2)$, there exists unique $g\in {
m Hom}_{\mathbb{A} G}(X,P)$ s.t. above holds.

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m Hom}_{\mathbb{A} G}(X,P)$ s.t. above holds.

Fact 1.15 Characterisations of surjective and injective homomorphisms.

- We say $f \in \operatorname{Hom}_{\mathbb{A} G}(G,H)$ is injective, whenever one of the following holds:
 - 1. f is a monomorphism, i.e.,

$$(f\circ g_1=f\circ g_2)\Leftrightarrow (g_1=g_2),\quad orall g_1,g_2\in \operatorname{Hom}_{\mathbb{A}_G}(L,G).$$

- 2. f is the kernel of H woheadrightarrow H/f(G).
- ullet We say $f\in \operatorname{Hom}_{\mathbb{A} G}(G,H)$ is surjective, whenever one of the following holds:
 - 1. f is a epimorphism, i.e.,

$$(g_1\circ f=g_2\circ f)\Leftrightarrow (g_1=g_2),\quad orall g_1,g_2\in \operatorname{Hom}_{\mathbb{A}_G}(H,L).$$

2. f is the cokernel of $\ker(f) \hookrightarrow G$.

Tensor Product

Definition 2.1 A category $\mathcal C$ consists of the class of objects $\mathsf{Ob}(\mathcal C)$ and the class of morphisms

$$\mathsf{Mor}(\mathcal{C}) := \bigcup_{A,B \in \mathsf{Ob}(\mathcal{C})} \mathsf{Hom}_{\mathcal{C}}(A,B),$$

and satisfies the following statements.

• For any pair $\operatorname{Hom}_{\mathcal{C}}(A,B)$ and $\operatorname{Hom}_{\mathcal{C}}(C,D)$ in $\operatorname{Mor}(\mathcal{C})$,

$$\operatorname{Hom}_{\mathcal{C}}(A,B)\cap\operatorname{Hom}_{\mathcal{C}}(C,D)=\emptyset\Leftrightarrow (A,B)
eq (C,D).$$

• The composition of morphisms satisfies the path structure, i.e.,

$$\circ \circ : \operatorname{Hom}_{\mathcal{C}}(B,C) \times \operatorname{Hom}_{\mathcal{C}}(A,B) \to \operatorname{Hom}_{\mathcal{C}}(A,C), \quad (f,g) \mapsto f \circ g;$$

$$\circ \ \operatorname{id}_B \in \operatorname{Hom}_{\mathcal{C}}(B,B)$$
 exists, i.e., we have $\operatorname{id}_B \circ g = g$ and $f \circ \operatorname{id}_B = f$.



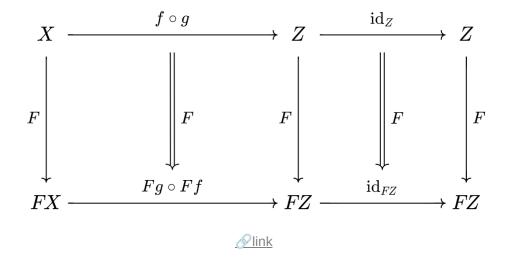
Remark $\mathcal{S} := \{U \mid U \text{ is a set.}\}$ is no longer a set, since it is contained in $\mathcal{S}\dot{\cup}\{\mathcal{S}\}$, thus the concept of **class** is introduced for describing those are larger than sets. Whereas, we assume the category are small.

Definition 2.2 A subcategory $\mathcal{C}' \subset \mathcal{C}$ satisfies $(A, B \in \mathsf{Ob}(\mathcal{C}') \subset \mathsf{Ob}(\mathcal{C}))$

Definition 2.3 Full subcategory $\mathcal{C}' \subset \mathcal{C}$ satisfies $(A, B \in \mathsf{Ob}(\mathcal{C}') \subset \mathsf{Ob}(\mathcal{C}))$

$$\mathcal{C}$$
 : $\mathsf{Ob}(\mathcal{C})$ $\mathsf{Mor}(\mathcal{C}) \in \mathsf{Hom}_{\mathcal{C}}(A,B)$
 \cup \cup \cup \parallel
 \mathcal{C}' : $\mathsf{Ob}(\mathcal{C}')$ $\mathsf{Mor}(\mathcal{C}') \in \mathsf{Hom}_{\mathcal{C}'}(A,B)$

Definition 2.4 A contravariant functor $F:\mathcal{C} o \mathcal{D}, X \mapsto FX, f \mapsto Ff$ satisfies

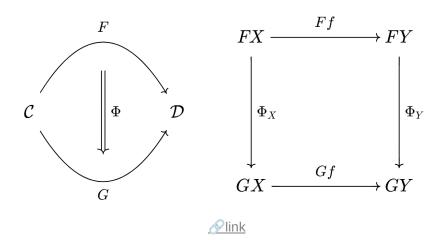


Definition 2.5 The **opposite category** \mathcal{C}^{op} is defined by reversing arrows of morphisms in \mathcal{C} , i.e.,

$$X \stackrel{f}{\longrightarrow} Y \Leftrightarrow X^{\mathrm{op}} \stackrel{f^{\mathrm{op}}}{\longleftarrow} Y^{\mathrm{op}}.$$

A **contravariant functor** is the composition of a contravariant functor with $^{\rm op}$ functor. Here $^{\rm op}$ is also a contravariant functor.

Definition 2.6 A natural transformation between $F,G:\mathcal{C}\to\mathcal{D}$ is defined as $\{\Phi_X\}_{X\in\mathsf{Ob}(\mathcal{C})}$, i.e.,



Denote $\mathrm{Hom}_{(\mathcal{C},\mathcal{D})}(F,G)$ as the morphisms of natural transformations between F and G.

Definition 2.7 We say

- $\mathcal C$ and $\mathcal D$ are isomorphic, whenever there exists $F:\mathcal C\to\mathcal D$ and $G:\mathcal D\to\mathcal C$ such that $FG=\mathrm{id}_{\mathcal D}$ and $GF=\mathrm{id}_{\mathcal C}$.
- $\mathcal C$ and $\mathcal D$ are equivalent, whenever there exists $F:\mathcal C\to\mathcal D$ and $G:\mathcal D\to\mathcal C$ such that $\theta:FG\overset{\sim}{\to}\mathrm{id}_{\mathcal D}$ and $\tau:GF\overset{\sim}{\to}\mathrm{id}_{\mathcal C}$. Here both θ and τ are natural transformations.

Definition 2.8 (Characterstics of a functor) Let $T:\mathcal{C}\to\mathcal{D}$ be a contravariant functor, then

• the image $T(\mathcal{C})$ is always a full subcategory of \mathcal{D} , that is,

$$\operatorname{Hom}_{\mathcal{D}}(TA,TB)=\operatorname{Hom}_{T(\mathcal{C})}(TA,TB), \quad \forall A,B\in\operatorname{\mathsf{Ob}}(\mathcal{C}).$$

- the image $T(\mathcal{C})$ is **essential**, whenever $\mathsf{Ob}(T(\mathcal{C})) = \{Y \mid \exists X \in \mathsf{Ob}(\mathcal{C}) : Y \cong T(X)\}.$
- T is **faithful** whenever the following map is always an **injection** for arbitrary $A,B\in \mathsf{Ob}(\mathcal{C})$:

$$T: \operatorname{Hom}_{\mathcal{C}}(A,B) o \operatorname{Hom}_{T(\mathcal{C})}(TA,TB), \quad [A \stackrel{f}{ o} B] \mapsto (TA \stackrel{Tf}{ o} TB).$$

• T is **full**, whenever the following map is always a **surjection** for arbitrary $A,B\in \mathsf{Ob}(\mathcal{C})$:

$$T: \operatorname{Hom}_{\mathcal{C}}(A,B) o \operatorname{Hom}_{T(\mathcal{C})}(TA,TB), \quad [A \stackrel{f}{ o} B] \mapsto (TA \stackrel{Tf}{ o} TB).$$

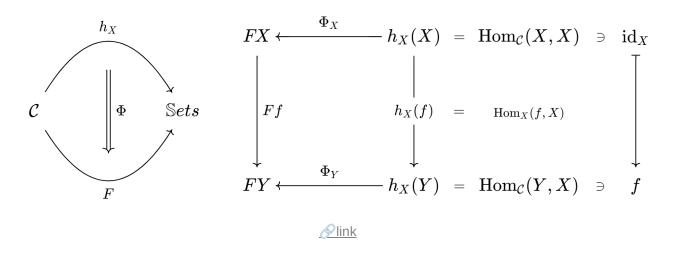


 $\mbox{\bf Remark}\ T$ stands for equivalent relation whenever it is full, faithful, essential and surjective.

Definition 2.9 Consider the contravariant functor

$$egin{aligned} h_X: \mathcal{C} & o \mathbb{S}ets; \ Y & \mapsto \operatorname{Hom}_{\mathcal{C}}(Y,X), \ [Z \overset{f}{ o} Y] & \mapsto \operatorname{Hom}_{\mathcal{C}}(f,X) : \operatorname{Hom}_{\mathcal{C}}(Y,X) & \to \operatorname{Hom}_{\mathcal{C}}(Z,X), \ [(Y \overset{g}{ o} X) & \mapsto (Z \overset{f \circ g}{ o} X)]. \end{aligned}$$

Definition 2.10 Consider the contravariant functor $F:\mathcal{C}\to \mathbb{S}ets$, then there exists a natural transform



Theorem 2.11 (米田 lemma) There exists a bijection

Definition 2.12 The tensor product of G and H is the quotient group

$$\otimes_{\mathbb{Z}}:G imes H o G imes H/\sim$$

Here

- $(ng,h) \sim (g,nh) =: n(g,h);$
- $(n_1+n_2,m)\sim (n_1,m)+(n_2,m)$;
- $(n, m_1 + m_2) \sim (n, m_1) + (n, m_2)$.

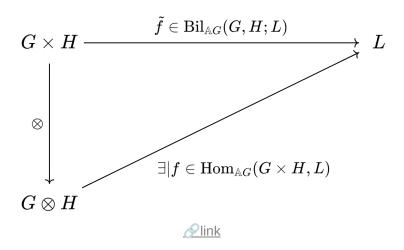
W

Remark $G\otimes H$ is isomorphic to a quotient group of some free abelian group, i.e.,

$$G\otimes H\cong \left(\prod_{(g,h)\in (G imes H)}(\mathbb{Z}^{(g,h)})
ight)/\sim'.$$

Moreover, $\mathbb{Z}\otimes G\cong \ G\cong \ G\otimes \mathbb{Z}$, proved by universal properties.

Fact 2.13 For each $\mathbb Z$ -bilinear function $ilde f:G imes H\to L$, there exists a unique $f\in \mathrm{Hom}_{\mathbb A G}(G\otimes H,L)$ such that





Remark Tensor product is also defined by such universal property.

Fact 2.14 For each $L \in \mathsf{Ob}(\mathbb{A} G)$, we have

$$egin{aligned} \operatorname{Hom}_{\mathbb{A}G} \left(\left(igoplus_{i \in I} G^i
ight) \otimes \left(igoplus_{j \in J} H^j
ight), L
ight) \ &\cong \operatorname{Bil}_{\mathbb{A}G} \left(\left(igoplus_{i \in I} G^i
ight), \left(igoplus_{j \in J} H^j
ight); L
ight) \ &\cong \prod_{i \in I, j \in J} \operatorname{Bil}_{\mathbb{A}G} \left(G^i, H^j; L
ight) \ &\cong \prod_{i \in I, j \in J} \operatorname{Hom}_{\mathbb{A}G} \left(G^i \otimes H^j, L
ight) \ &\cong \operatorname{Hom}_{\mathbb{A}G} \left(igoplus_{i \in I, j \in J} G^i \otimes H^j, L
ight). \end{aligned}$$

Thus
$$(igoplus_{i\in I}G^i)\otimes (igoplus_{j\in J}H^j)\cong igoplus_{i\in I,j\in J}G^i\otimes H^j$$
 .

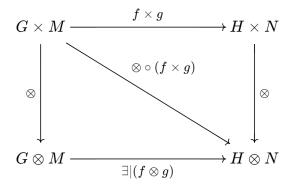
(Via **Theorem 2.11**) $orall X,Y\in \mathsf{Ob}(\mathcal{C})$, X=Y whenever $\mathrm{Id}_{\mathcal{C}}(X)=\mathrm{Id}_{\mathcal{C}}(Y)$, whenever

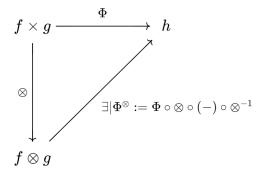
$$\operatorname{Hom}_{(\mathcal{C},\mathcal{C})}(\operatorname{\mathsf{Mor}}_{\mathcal{C}}(-,X),\operatorname{Id}_{\mathcal{C}})=\operatorname{Hom}_{(\mathcal{C},\mathcal{C})}(\operatorname{\mathsf{Mor}}_{\mathcal{C}}(-,Y),\operatorname{Id}_{\mathcal{C}}).$$

In contravariant case, one only need to show

$$\operatorname{Hom}_{\mathcal{C}}(X,L)=\operatorname{Hom}_{\mathcal{C}}(Y,L),\quad orall L\in \operatorname{\mathsf{Ob}}(\mathcal{C}).$$

Definition 2.15 For any $f\in \operatorname{Hom}_{\mathbb{A} G}(G,H)$ and $g\in \operatorname{Hom}_{\mathbb{A} G}(M,N)$, we have





Olink

Here Φ^{\otimes} is well defined.

Fact 2.16 For each $G_1\stackrel{f_1}{\to}G_2\stackrel{f_2}{\to}G_3$, $H_1\stackrel{g_1}{\to}H_2\stackrel{g_2}{\to}H_3$, we have

Thus $(f_2 \circ f_1) \otimes (g_2 \circ g_1) = (f_2 \otimes g_2) \circ (f_1 \otimes g_1)$.

Fact 2.17 $L\otimes (M\otimes N)\stackrel{\sim}{ o} (L\otimes M)\otimes N$ is natural.

Adjunctions

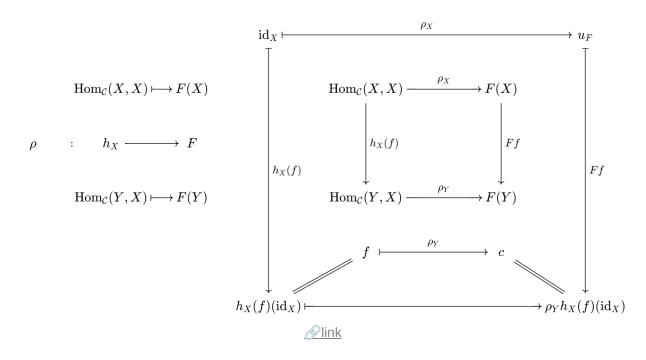
Definition 3.1 We say $F:\mathcal{C}\to\mathbb{S}ets$ is a **representable** (contravariant, for simplicity) functor, if

$$\rho \quad : \quad h_X \longrightarrow F$$

$$\operatorname{Hom}_{\mathcal{C}}(X,X) \longmapsto F(X) \qquad \rho_X \quad : \quad \operatorname{Hom}_{\mathcal{C}}(X,X) \longrightarrow F(X)$$

$$\operatorname{id}_X \longmapsto u_F$$

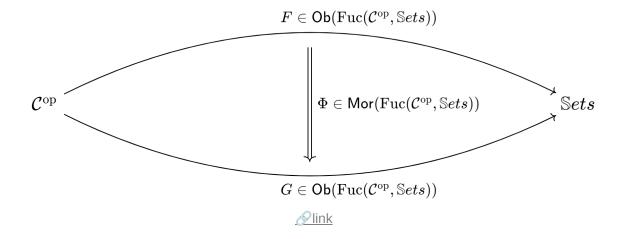
for some $X\in \mathsf{Ob}(\mathcal{C})$. Here u_F is iamge of natural transform of ρ in 光田 bijection. Moreover,



We see that $\rho_{ullet}: \mathsf{Mor}_{\mathcal{C}}(ullet,X) o F(ullet)$, $f \mapsto \rho_{ullet}(f) = Ff(u_F)$.

Theorem 3.2 The functor category of $\mathcal C$ is usually defined as $\operatorname{Fuc}(\mathcal C^{\operatorname{op}},\mathbb Sets)$, where

$$igcup_{ ext{Functors}} igcup_{ ext{Functors}} igcup_{ ext{Func}(\mathcal{C}^{ ext{op}}, \mathbb{S}ets)) := \{F \mid F : \mathcal{C}^{op}
ightarrow \mathbb{S}ets\}, \ ext{Mor}(ext{Fuc}(\mathcal{C}^{ ext{op}}, \mathbb{S}ets)) := igcup_{F,G \in \mathsf{Ob}(ext{Fuc}(\mathcal{C}^{ ext{op}}, \mathbb{S}ets))} igcup_{F,G \in \mathsf{Ob}(ext{Fuc}(\mathcal{C}^{ ext{op}}, \mathbb{S}ets))} igcup_{ ext{Natural}} igcup_{ ext{Transformations}} igcup_{F,G \in \mathsf{Ob}(ext{Fuc}(\mathcal{C}^{ ext{op}}, \mathbb{S}ets))} igcup_{F,G \in \mathsf{Ob}(\mathsf{Fuc}(\mathcal{C}^{ ext{op}}, \mathbb{S}ets))} igcup_{F,G \in \mathsf{Ob}(\mathsf{Fuc}(\mathcal{C}^{\mathsf{op}}, \mathbb{S}ets))} igcup_{F,G \in \mathsf{Ob}(\mathsf{Fuc}(\mathcal{C}^{\mathsf{op}}, \mathbb{S}ets))} igcup_{F,G \in \mathsf{Ob}(\mathsf{Fuc}(\mathcal{C}^{\mathsf{op}}, \mathbb{S}ets))} igcup_{F,G \in \mathsf{Ob}(\mathsf{Ob}(\mathsf{Fuc}(\mathcal{C}^{\mathsf{op}}, \mathbb{S}ets)))} igcup_{F,G \in \mathsf{Ob}(\mathsf{Ob}(\mathsf{Fuc}(\mathcal{C}^{\mathsf{op}}, \mathbb{S}ets)))} igcup_{F,G \in \mathsf{Ob}(\mathsf{Ob}(\mathsf{C}^{\mathsf{op}}, \mathbb{S}ets))} igcup_{F,G \in \mathsf{Ob}(\mathsf{C}^{\mathsf{op}}, \mathbb{S}ets))} igcup_{F,G \in \mathsf{Ob}(\mathsf{C}^{\mathsf{op}}$$



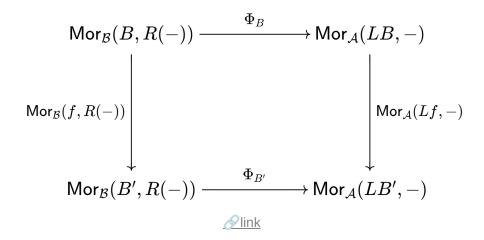
Theorem 3.3 Each category is a full subcategory of functor category. Consider

Hence $\mathcal{C}\cong \text{ subcategory of representable functors, a full subcategory of } Fuc(\mathcal{C}^{op},\mathbb{S}ets).$

Definition 3.4 For $L: \mathcal{B} \to \mathcal{A}$ and $R: \mathcal{A} \to \mathcal{B}$, we say $L \dashv R$ (L is a **left adjoint** of R), whence

$$ho: \operatorname{Hom}_{\mathcal{A}}(LX,Y) \cong \ \operatorname{Hom}_{\mathcal{B}}(X,RY), \quad orall X \in \operatorname{\mathsf{Ob}}(\mathcal{B}), Y \in \operatorname{\mathsf{Ob}}(\mathcal{A}).$$

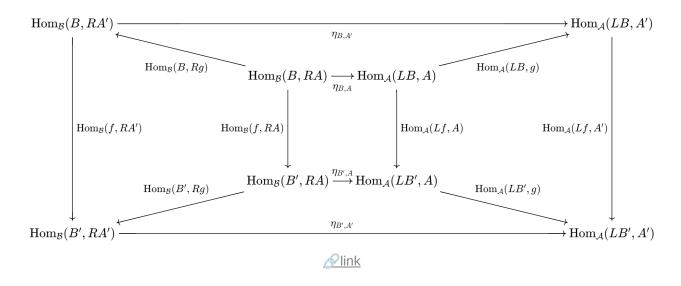
Here we also have $L \vdash R$, i.e., R is a **right adjoint** of L. Here there exists a natural transformation



Theorem 3.5 R has a left adjoint whenever $\mathrm{Mor}_{\mathcal{B}}(B,R(-))$ is always representable for each $B\in \mathrm{Ob}(\mathcal{B})$, i.e.,

$$ho: \mathsf{Mor}_{\mathcal{A}}(LB,-) \overset{\sim}{ o} \mathsf{Mor}_{\mathcal{B}}(B,R(-)).$$

Moreover, if R has a left adjoint L which is not an isomorphism, then L is unique. Such isomorphism is also natural for A and B, i.e.,



Fact 3.6 The pair (\otimes, Hom) induces an adjoint relation, i.e.,

$$\operatorname{Hom}_{\mathbb{A} G}(L\otimes M,N)\stackrel{\sim}{ o} \operatorname{Hom}_{\mathbb{A} G}(M,\operatorname{Hom}_{\mathbb{A} G}(L,N)), \ f\mapsto (m\mapsto f(-\otimes m));$$

$$\operatorname{Hom}_{\mathbb{A} G}(L\otimes M,N)\stackrel{\sim}{ o} \operatorname{Hom}_{\mathbb{A} G}(L,\operatorname{Hom}_{\mathbb{A} G}(M,N)), \ f\mapsto (l\mapsto f(l\otimes -)).$$

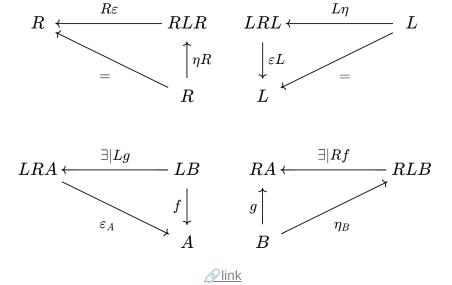
Then we define

- $arepsilon_G: H \otimes \operatorname{Hom}_{\mathbb{A} G}(H,G) o G, h \otimes f \mapsto f(h);$
- $ullet \ \eta_G:G o \operatorname{Hom}_{\mathbb{A} G}(H,H\otimes G),g\mapsto -\otimes g_{\mathbb{A} G}(H,H\otimes G)$

to obtain that

- $\operatorname{id}_{H\otimes G}:h\otimes g\overset{\operatorname{id}_{H}\otimes \eta_{G}}{\longrightarrow}h\otimes (-\otimes g)\overset{\varepsilon_{H\otimes G}}{\longrightarrow}h\otimes g$, that is, $\operatorname{id}_{H\otimes G}=\varepsilon_{H\otimes G}\circ (\operatorname{id}_{H}\otimes \eta_{G});$
- $\begin{array}{c} \bullet \ \ \mathrm{id}_{\mathrm{Hom}_{\mathbb{A} G}(H,G)}: f \stackrel{\eta_{\mathrm{Hom}_{\mathbb{A} G}(H,G)}}{\longrightarrow} (-\otimes f) \stackrel{\varepsilon_G}{\longrightarrow} f \text{, that is, } \mathrm{id}_{\mathrm{Hom}_{\mathbb{A} G}(H,G)} = \varepsilon_G \circ \\ (\eta_{\mathrm{Hom}_{\mathbb{A} G}(H,G)}). \end{array}$

Definition 3.7 In Fact 3.6, $\eta:\mathrm{id}_{\mathcal{A}}\to RL$ and $\varepsilon:LR\to\mathrm{id}_{\mathcal{B}}$ are called unit and counit, i.e.,



Then for each adjoint pair (L,R) $(L\dashv R)$, it yields that

- ullet L is full and faithful, whenever $RL\stackrel{\sim}{ o} \mathrm{id};$
- ullet R is full and faithful, whenever $LR\stackrel{\sim}{ o} \mathrm{id}.$

Theorem 3.8 Let $F:\mathcal{C} o \mathcal{D}$ and $G:\mathcal{D} o \mathcal{C}$ be quasi-inverse pair, i.e., there exists

- $\eta: GF \stackrel{\sim}{ o} \mathrm{id}$, which is a natural transfromation;
- $heta^{-1}: FG \overset{\sim}{ o} \operatorname{id}$, which is a natural transformation.

Then we claim that $({\cal F},{\cal G})$ is an adjoint pair. Let η be unit without the loss of generality, then

$$heta' := \mathit{FG} \overset{\mathit{FG} heta}{\longrightarrow} \mathit{FGFG} \overset{\mathit{F}\eta\mathit{G}}{\longrightarrow} \mathit{FG} \overset{ heta^{-1}}{\longrightarrow} \mathrm{id}$$

is a well defined counit. One can verity $\mathrm{id}: F \xrightarrow{\eta F} FGF \xrightarrow{F\theta} F$ and $\mathrm{id}: G \xrightarrow{G\eta} GFG \xrightarrow{\theta G} G$.