Brief introduction to the theory of Hopf Algebra

💡 The theory of monads (comonads) is still unshaped in this article, which we just take a glimpse of for the perception of **a kind of universal algebra/coalgebra structure**.

Algebra&coalgebra

Definition 1.1 An $\Bbbk$-algebra is a triple (A, m, u) satisfying that

- A is a $\Bbbk$-vector space;
- $m : A \otimes A \to A$ is an associative $\Bbbk$-linear map, i.e.,

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m \circ (m \otimes \mathrm{id})"] \arrow[d, "m \circ (\mathrm{id} \otimes m)"'] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"'] & A \end{tikzcd}$$

link

- the **unit** $u : \Bbbk \to A, 1 \mapsto 1_A$ is a $\Bbbk$-linear map satisfying that

$$\begin{tikzcd} & A \otimes \Bbbk \arrow[dl, "\mathrm{id} \otimes u"'] \arrow[dr, "{\cong, a \otimes 1 \mapsto a}"] & \\ A \otimes A \arrow[rr, "m"] & & A \\ & \Bbbk \otimes A \arrow[ul, "u \otimes \mathrm{id}"] \arrow[ur, "{\cong, 1 \otimes a \mapsto a}"'] & \end{tikzcd}$$

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Definition 1.2 A flip is defined as $\tau : A \otimes A \to A \otimes A, a \otimes b \mapsto b \otimes a$.

> A is a commutative algebra whenever $m \circ \tau = m$.

Definition 1.3 A coalgebra is a triple (C, Δ, ε) satisfying that

- C is a $\Bbbk$-vector space;
- $\Delta : C \to C \otimes C$ is an associative $\Bbbk$ linear map, satisfying that

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta \otimes \mathrm{id}"] \\ C \otimes C \arrow[r, "\mathrm{id} \otimes \Delta"'] & C \otimes C \otimes C \end{tikzcd}$$

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- the **counit** $\varepsilon : A \to \Bbbk$ is a $\Bbbk$-linear map satisfying that

$$\begin{tikzcd}
& \Bbbk \otimes C & \\
C \arrow[ur, "{\cong, c \mapsto 1 \otimes c}"] \arrow[rr, "\Delta"] \arrow[dr, "{\cong, c \mapsto c \otimes 1}"'] & & C \otimes C \arrow[ul, "\varepsilon \otimes \mathrm{id}"'] \arrow[dl, "\mathrm{id} \otimes \varepsilon"] \\
& C \otimes \Bbbk &
\end{tikzcd}$$

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Definition 1.4 (**Homomorphisms**) We say $f : A_1 \to A_2$ is a homomorphism of algebras, whenever

$$\begin{tikzcd}
A_1 \otimes A_1 \arrow[r, "m_1"] \arrow[d, "f \otimes f"'] & A_1 \arrow[d, "f"] & \Bbbk \arrow[r, "u_1"] \arrow[dr, "u_2"'] & A_1 \arrow[d, "f"] \\
A_2 \otimes A_2 \arrow[r, "m_2"] & A_2 & & A_2
\end{tikzcd}$$

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Similarly, we say $g : C_1 \to C_2$ is a homomorphism of coalgebras, whenever

$$\begin{tikzcd}
C_1 \arrow[r, "\Delta_1"] \arrow[d, "g"'] & C_1 \otimes C_1 \arrow[d, "g \otimes g"] & C_1 \arrow[r, "\varepsilon_1"] \arrow[d, "g"] & \Bbbk \\
C_2 \arrow[r, "\Delta_2"] & C_2 \otimes C_2 & C_2 \arrow[ur, "\varepsilon_2"'] &
\end{tikzcd}$$

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Definition 1.5 (**Sub(co)algebra**) We say

- $A' \subset A$ is a subalgebra, whenever $m : A' \otimes A' \to A'$ and $u \restriction_{A'} : \Bbbk \to A', 1 \mapsto 1_{A'} = 1_A$;
- $C' \subset C$ is a subcoalgebra, whenever $\Delta : C' \to C' \otimes C'$.

Definition 1.6 (**Ideal and coideal**) We say

- $I \subset A$ is an (two-sided) ideal of algebra, whenever $m : A \otimes I + I \otimes A \to I$;
- $I \subset C$ is an ideal of coalgebra, whenever $\Delta : C \to C \otimes I + I \otimes C, \varepsilon : I \to 0$.

Definition 1.7 (**Quotient and coquotient algebra**) Omit.

Definition 1.8 (**Sweedler's notation**) The notation

$$\sum_{(c)} c_{(1)} \otimes \cdots \otimes c_{(n)} := \Delta_{n-1}(c) := (\Delta \otimes I_{n-2}) \circ \cdots \circ (\Delta \otimes I) \circ \Delta(c).$$

is utilised by Sweedler for simplicity. Sililarly, we define

$$\sum_{(c)} c_{(1)} \otimes c_{(2)} := \sum_i c_{1i} \otimes c_{2i} = \Delta(c)$$

We can also obtain the following corollaries:

- (definition of counit) $c \otimes 1 = \sum_{(c)} c_{(1)} \otimes \varepsilon(c_{(2)}), 1 \otimes c = \sum_{(c)} \varepsilon(c_{(1)}) \otimes c_{(2)}$;
- (simple corallary) $\sum_{(c)} c_{(1)} \otimes \cdots \otimes \varepsilon(c_{(i)}) \otimes \cdots \otimes c_{(n+1)} = \sum_{(c)} c_{(1)} \otimes \cdots \otimes c_{(n)}$;
- (associativity) $\sum_{(c)} (c_{(1)})_{(1)} \otimes (c_{(1)})_{(2)} \otimes c_{(2)} = \sum_{(c)} c_{(2)} \otimes (c_{(2)})_{(1)} \otimes (c_{(2)})_{(2)}$;
- (action of τ) $\tau : \sum_{(c)} c_{(1)} \otimes c_{(2)} \to \sum_{(c)} c_{(2)} \otimes c_{(1)}$;
- (exercises from Sweedler's book)
  - $\sum_{(c)} \varepsilon(c_2) \otimes \Delta(c_{(1)}) = \sum_{(c)} \Delta(c_2) \otimes \varepsilon(c_{(1)}) = \Delta(c)$,
  - $\sum_{(c)} c_{(1)} \otimes \varepsilon(c_{(3)}) \otimes c_{(2)} = \sum_{(c)} c_{(1)} \otimes c_{(3)} \otimes \varepsilon(c_{(2)}) = \Delta(c)$,

- $\sum_{(c)} \varepsilon(c_{(1)}) \otimes c_{(3)} \otimes c_{(2)} = \sum_{(c)} c_{(2)} \otimes c_{(1)}$,
- $\sum_{(c)} \varepsilon(c_{(1)}) \otimes \varepsilon(c_{(3)}) \otimes c_{(2)} = c$.

Definition 1.9 (**Dual analog**) Let $V^* := \mathrm{Hom}_{\mathbb{k}}(V, \mathbb{k})$, then

- for any $\mathbb{k}$-coalgebra (C, Δ, ε), (C^*, m', u') is an induced algebra, where
  - $m' : C^* \otimes C^* \stackrel{\rho}{\hookrightarrow} (C \otimes C)^* \stackrel{\Delta^*}{\to} C^*$,
  - $u' : k \to C^*, k \to k^* \stackrel{\varepsilon^*}{\to} C^*$;
- for any **REFLEXIVE** (e.g., finite dimensional) $\mathbb{k}$-algebra (A, m, u), $(A^*, \Delta', \varepsilon')$ is an induced coalgebra, where
  - $\Delta' : A^* \stackrel{m^*}{\to} A^* \otimes A^* \stackrel{\rho^{-1}}{\cong} (A \otimes A)^*$,
  - $\varepsilon' : A^* \to k, A^* \stackrel{u^*}{\to} k^* \to k$.

Fact 1.10 Let C and D be coalgebras. Then we have a bialgebra $C \otimes D$, where

- $\Delta_{C\otimes D} : C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} C \otimes D \otimes C \otimes D$,
  - $\Delta_{C\otimes D} : c \otimes d \mapsto \sum_{(c),(d)} c_{(1)} \otimes d_{(1)} \otimes c_{(2)} \otimes d_{(2)}$;
- $\varepsilon_{C\otimes D} : C \otimes D \xrightarrow{\varepsilon_C \otimes \varepsilon_D} k \otimes k \cong k$,
  - $\varepsilon_{C\otimes D} : c \otimes d \mapsto \varepsilon_C(c)\varepsilon_D(d)$.

And the projections from $C \otimes D$ to C (D) is homomorphisms between coalgebras.

Fact 1.11 One can verify (via Yoneda's lemma) that

$$(\bigoplus_{\alpha\in I} C_\alpha) \otimes (\bigoplus_{\beta\in I} C_\beta) \cong \bigoplus_{\alpha,\beta\in I} (C_\alpha \otimes C_\beta).$$

Here $e_\alpha : C_\alpha \hookrightarrow C, p'_\alpha : C \twoheadrightarrow C_\alpha$ are both homomorphisms between coalgebras.

One can also prove that $\bigoplus$ is just $\coprod$, thus the direct sum of coalgebras has the same universal properties as coproduct.

Bialgebra&Hopf algebra

Definition 2.1 As $(B, m, u, \Delta, \varepsilon)$ is a bialgebra whenever the following three statements (1.-3.) holds

1. (B, m, u) is an algebra;
2. (B, Δ, ε) is a coalgebra;

> Thus $B \otimes B$ has both algebra and coalgebra structure.

3. the following equivalent statements (a.-c.) holds
   a. m and u are both homomorphisms of coalgebras,
   b. Δ and ε are both homomorphisms of algebras,
   c. all of the following four statements (i.-iv.)
      i. $\Delta(1) = 1 \otimes 1$
      ii. $\Delta(gh) = \sum_{(g),(h)} g_{(1)}h_{(1)} \otimes g_{(2)}h_{(2)}$
      iii. $\varepsilon(1) = 1$
      iv. $\varepsilon(gh) = \varepsilon(g)\varepsilon(h)$.

Here the equivalence of b. and c. are clear. The equivalence between a. and c. is seen as follows

1. the middle rectangle shows that m is a coalgebra homomorphism

$$\begin{tikzcd}
b \arrow[rrrrr, mapsto, bend left] & \in & B \arrow[r, "\Delta"] & B \otimes B & \in & \sum b_{(1)} \otimes b_{(2)} \\
\sum b^{(1)} \otimes b^{(2)} \arrow[u, mapsto] \arrow[ddrr, mapsto] & \in & B \otimes B \arrow[u, "m"] \arrow[r, "\Delta_{B\otimes B}"] \arrow[d, "\Delta\otimes\Delta"'] & B \otimes B \otimes B \otimes B \arrow[u, "m \otimes m"'] & \in & \sum b^{(1)}_{(1)} \otimes b^{(2)}_{(1)} \otimes b^{(1)}_{(2)} \otimes b^{(2)}_{(2)} \arrow[u, mapsto] \\
 & & B \otimes B \otimes B \otimes B \arrow[ur, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"'] & & & \\
 & & \cup & & & \\
 & & \sum b^{(1)}_{(1)} \otimes b^{(1)}_{(2)} \otimes b^{(2)}_{(1)} \otimes b^{(2)}_{(2)} \arrow[uuurrr, mapsto] & & &
\end{tikzcd}$$

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2. it suggests that m perserves the counit

$$\begin{tikzcd}
B \otimes B \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "m"'] & \Bbbk \otimes \Bbbk = \Bbbk \\
B \arrow[ur, "\varepsilon"'] &
\end{tikzcd}$$

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3. u is also a homomorphism of coalgebras, that is,

$$\begin{tikzcd}
\Bbbk \arrow[r, "\mathrm{id}\,(\Delta_{\Bbbk})"] \arrow[d, "u"'] & \Bbbk \otimes \Bbbk \arrow[d, "u\otimes u"] \\
B \arrow[r, "\Delta"] & B \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \arrow[r, "\mathrm{id}\,(\varepsilon_{\Bbbk})"] \arrow[d, "u"'] & \Bbbk \\
B \arrow[ur, "\varepsilon"'] &
\end{tikzcd}$$

link

Fact 2.2 $\mathrm{End}_{\Bbbk}(B) = \mathrm{Hom}_{\Bbbk}(B)$ has a ring structure, i.e.,

- $(\mathrm{End}_{\Bbbk}(B), +)$ is an Abelian group, i.e., $+ : (f, g) \mapsto [b \mapsto f(b) + g(b)]$;
- $(\mathrm{End}, *)$ is an semigroup, i.e., $* : (f, g) \mapsto m \circ (f \otimes g) \circ \Delta$, where

$$(f * g) * h = f * (g * h) : B \xrightarrow{\Delta_2} B \otimes B \otimes B \xrightarrow{f\otimes g\otimes h} B \otimes B \otimes B \xrightarrow{m_2} B;$$

- $u \circ \varepsilon : B \to \Bbbk \to B$ is the unit of the ring, i.e., $f * u\varepsilon = u\varepsilon * f = f$

$$\begin{tikzcd}
 & \Bbbk \otimes B \arrow[rrr, "\mathrm{id}_{\Bbbk}\otimes f"] & & & \Bbbk \otimes B \arrow[dl, "{\cong, 1\otimes b\mapsto b}"'] \arrow[dr, "u\otimes\mathrm{id}_B"] & \\
B \otimes B \arrow[ur, "\varepsilon\otimes\mathrm{id}_B"] \arrow[dr, "\mathrm{id}_B\otimes\varepsilon"'] & & B \arrow[ll, "\Delta"'] \arrow[ul, "{\cong, b\mapsto 1\otimes b}"'] \arrow[dl, "{\cong, b\mapsto b\otimes 1}"] \arrow[r, "f"] & B & & B \otimes B \arrow[ll, "m"'] \\
 & B \otimes \Bbbk \arrow[rrr, "f\otimes\mathrm{id}_{\Bbbk}"'] & & & B \otimes \Bbbk \arrow[ul, "{\cong, b\otimes 1\mapsto b}"] \arrow[ur, "\mathrm{id}_B\otimes u"'] &
\end{tikzcd}$$

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Definition 2.3 We say $\mathcal{S} \in \mathrm{Hom}_{\Bbbk}(B)$ is an **antipode** iff $\mathcal{S} * \mathrm{id}_B = u \circ \varepsilon = \mathrm{id}_B * \mathcal{S}$, i.e.,

$$\sum_{(b)} \mathcal{S}(b_{(1)}) b_{(2)} = u\varepsilon(b) = \sum_{(b)} b_{(1)} \mathcal{S}(b_{(2)}).$$

$$\begin{tikzcd}
B \otimes B \arrow[dd, "\mathcal{S} \otimes \mathrm{id}_B"'] & B \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "\varepsilon"] & B \otimes B \arrow[dd, "\mathrm{id}_B \otimes \mathcal{S}"] \\
 & \Bbbk \arrow[d, "u"] & \\
B \otimes B \arrow[r, "m"] & B & B \otimes B \arrow[l, "m"']
\end{tikzcd}$$

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Here the antipode is also known as $(\mathrm{id}_B)^{-1}$, the uniqueness is obvious.

Definition 2.4 A bialgebra with an antipode is a **Hopf algebra**, denoted by H.

Fact 2.5 Here are some propositions on Hopf algebra H:

- $\mathcal{S} : H \to H^{\mathrm{op}}, gh \mapsto \mathcal{S}(h)\mathcal{S}(g)$;
- $\varepsilon \circ \mathcal{S} = \varepsilon : \Bbbk \to H, \mathcal{S} \circ u = u : H \to \Bbbk, \mathcal{S}(1) = 1$;
- $\mathcal{S}_{H \otimes H} = \tau \circ (\mathcal{S}_H \otimes \mathcal{S}_H)$

$$\begin{tikzcd}
h \arrow[rrr, mapsto, bend left] \arrow[d, mapsto] & H \arrow[r, "\mathcal{S}"] \arrow[d, "\Delta"'] & H \arrow[d, "\Delta"] & h^{\mathrm{op}} \arrow[d, mapsto] \\
h_{(1)} \otimes h_{(2)} \arrow[ddr, mapsto] & H \otimes H \arrow[r, "\mathcal{S}_{H \otimes H}"] \arrow[d, "\mathcal{S}_H \otimes \mathcal{S}_H"'] & H \otimes H & h^{\mathrm{op}}_{(2)} \otimes h^{\mathrm{op}}_{(1)} \\
 & H \otimes H \arrow[ur, "\tau"'] & & \\
 & h^{\mathrm{op}}_{(1)} \otimes h^{\mathrm{op}}_{(2)} \arrow[uurr, mapsto] & &
\end{tikzcd}$$

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> Review that $\Delta_{H \otimes H} = (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta_H \otimes \Delta_H)$. One can prove that
>
> $$\mathcal{S} : H \to H^{\mathrm{cop}}; \quad \Delta(S(h)) = \sum_{(h)} \mathcal{S}(h_{(2)}) \otimes \mathcal{S}(h_{(1)}), \mathcal{S}(\varepsilon(h)) \mapsto \varepsilon(h)$$
>
> is a coalgebra homomorphism.

- if H is commutative or cocommutative, then the following equivalent statements hold
  - $\mathcal{S} \circ \mathcal{S} = \mathrm{id}_H$,
  - $\sum_{(h)} \mathcal{S}(h_{(2)})h_{(1)} = u\varepsilon(h), \forall h \in H$,
  - $\sum_{(h)} h_{(2)}\mathcal{S}(h_{(1)}) = u\varepsilon(h), \forall h \in H$.

Fact 2.6 As $\mathcal{S} : H \to H^{\mathrm{cop}}$ and $\mathcal{S} : H \to H^{\mathrm{op}}$, the following are also Hopf algebras:

$$\begin{tikzcd}
 & (H^{\mathrm{cop}}, m, u, \Delta^{\mathrm{cop}}, \varepsilon, \mathcal{S}^{-1}) \arrow[dr, dotted, "\mathrm{cop}"] & \\
(H, m, u, \Delta, \varepsilon, \mathcal{S}) \arrow[ur, dashed, "\mathrm{op}"] \arrow[dr, dotted, "\mathrm{cop}"'] & & (H^{\mathrm{op,cop}}, m^{\mathrm{op}}, u, \Delta^{\mathrm{cop}}, \varepsilon, \mathcal{S}) \\
 & (H^{\mathrm{op}}, m^{\mathrm{op}}, u, \Delta, \varepsilon, \mathcal{S}^{-1}) \arrow[ur, dashed, "\mathrm{op}"'] &
\end{tikzcd}$$

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Fact 2.7 The following are copy from the book of Sweedler

1. $\sum_{(h)} h_{(1)} S(h_{(2)}) \otimes h_{(3)} = h$ (or $1 \otimes h$);
2. $\sum_{(h)} S(h_{(1)}) h_{(2)} \otimes h_{(3)} = h$;
3. $\sum_{(h)} h_{(1)} \otimes h_{(2)} S(h_{(3)}) = h$;
4. $\sum_{(h)} h_{(1)} \otimes S(h_{(2)}) h_{(3)} = h$;
5. $\sum_{(g),(h)} h_{(1)} S(g_{(1)} f h_{(2)}) g_{(2)} = \varepsilon(gh) S(f)$;
6. $\sum_{(h)} (1 \otimes S(h_{(1)}) h_{(2)}) \Delta S(h_{(3)}) = \Delta S(h)$;
7. $\sum_{(h)} (1 \otimes S(h_{(3)}) h_{(1)}) = \Delta S(h_{(2)}) = (S \otimes S)\Delta(h)$;
8. $\sum_{(h)} h_{(1)} \otimes \ldots \otimes h_{(i-1)} \otimes h_{(i)} S(h_{(i+1)}) \otimes h_{(i+2)} \otimes \ldots \otimes h_{(n)} = \sum_{(h)} h_{(1)} \otimes \ldots \otimes h_{(n-2)}$;
9. $\sum_{(h)} h_{(1)} \otimes \ldots \otimes h_{(i-1)} \otimes S(h_{(i)}) h_{(i+1)} \otimes h_{(i+2)} \otimes \ldots \otimes h_{(n)} = \sum_{(h)} h_{(1)} \otimes \ldots \otimes h_{(n-2)}$;
10. $$\begin{aligned} &\sum_{(h)} h_{(1)} \otimes \ldots \otimes h_{(i-1)} \otimes \Delta S(h_{(i)}) \otimes h_{(i+1)} \otimes \ldots \otimes h_{(n-1)} \\ =&\sum_{(h)} h_{(1)} \otimes \ldots \otimes h_{(i-1)} \otimes S(h_{(i+1)}) \otimes S(h_{(i)}) \otimes h_{(i+2)} \otimes \ldots \otimes h_{(n)} \end{aligned}.$$

Definition 2.8 (**Homomorphisms**) We say

- $\varphi : B_1 \to B_2$ is a homomorphism of algebras, whenever

$$\begin{tikzcd} B_1 \arrow[r, "\Delta_{B_1}"] \arrow[d, "\varphi"'] & B_1 \otimes B_1 \arrow[r, "f \otimes g"] \arrow[d, "\varphi \otimes \varphi"] & B_1 \otimes B_1 \arrow[r, "m_{B_1}"] \arrow[d, "\varphi \otimes \varphi"'] & B_1 \arrow[d, "\varphi"] \\ B_2 \arrow[r, "\Delta_{B_2}"'] & B_2 \otimes B_2 \arrow[r, "f^\varphi \otimes g^\varphi"'] & B_2 \otimes B_2 \arrow[r, "m_{B_2}"'] & B_2 \end{tikzcd} \qquad \begin{tikzcd} B_1 \arrow[rr, "(u\varepsilon)_{B_1}"] \arrow[dr, "\varepsilon_{B_1}"'] \arrow[dd, "\varphi"'] & & B_1 \arrow[dd, "\varphi"] \\ & \Bbbk \arrow[ur, "u_{B_1}"'] \arrow[dr, "u_{B_2}"] & \\ B_2 \arrow[rr, "(u\varepsilon)_{B_2}"'] \arrow[ur, "\varepsilon_{B_2}"] & & B_2 \end{tikzcd}$$

🔗link

- $\varphi : H_1 \to H_2$ is a homomorphism of Hopf algebras, whenever φ is a homomorphism of bialgebras, and $\varphi \circ \mathcal{S}_{H_1} = \mathcal{S}_{H_2} \circ \varphi$.

Definition 2.9 (**Ideals, subalgebra, quotient algebra**) Omit.

Monads and comonads

Definition 3.1 Let $\mathcal{A}$ be any categoty, $F : \mathcal{A} \to \mathcal{A}$ be an endofunctor with action

$$\varrho_A : F(A) \to A, \quad \forall A \in \mathsf{Ob}(\mathcal{A}).$$

A monand on $\mathcal{A}$ is defined as a triple (F, μ, η), where

- $F : \mathcal{A} \to \mathcal{A}$ is an endomorphism;
- $\mu : FF \to F$ is a natural transformation s.t. $\mu \circ (\mu F) = \mu \circ (F\mu) : FFF \to F$;

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=60, "FFF"] \arrow[r, "FF"] \arrow[r, bend right=60, "F"'] & \mathcal{A} \end{tikzcd} \qquad \begin{tikzcd} FFF \arrow[r, "\mu F"] \arrow[d, "F\mu"'] & FF \arrow[d, "\mu"] \\ FF \arrow[r, "\mu"'] & F \end{tikzcd}$$

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- $\eta : \mathrm{id}_{\mathcal{A}} \to F$ is also a natural transformation s.t. $\mu \circ (\eta F) = \mu \circ (F\eta) : F \to F$.

$$\begin{tikzcd}[column sep=huge]
\mathcal{A} \arrow[rr, bend left=60, "FF"] \arrow[rr, "F" description] \arrow[rr, bend right=60, "FF"'] \arrow[rr, phantom, shift left=5, "\overset{\eta F}{\Uparrow}\;\overset{\mu}{\Downarrow}"] \arrow[rr, phantom, shift right=5, "\underset{\mu}{\Uparrow}\;\underset{F\eta}{\Downarrow}"] & & \mathcal{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
F \arrow[r, "F\eta"] \arrow[d, "\eta F"'] \arrow[dr, equal] & FF \arrow[d, "\mu"] \\
FF \arrow[r, "\mu"'] & F
\end{tikzcd}$$

link

> One can see that
>
> - F as $\mathfrak{A} \otimes_R - : {}_R\mathcal{M} \to {}_R\mathcal{M}$ ($\mathfrak{A}$ is an R-algebra);
> - $\varrho_M : \mathfrak{A} \otimes_R M \to M$ is an action on left R module;
> - $\mu : \mathfrak{A} \otimes_R \mathfrak{A} \otimes_R - \to \mathfrak{A} \otimes_R -$ satisfies the associativity of algebra;
> - $\eta : \mathrm{id} \cong R \otimes_R - \to \mathfrak{A} \otimes_R -$ is the unit of the monad.

💡 **Remark** Here we analysis the functor category

$$\mathrm{Fuc}_{(\mathcal{A},\mathcal{A})} = (\mathsf{Ob}(\mathrm{Fuc}_{(\mathcal{A},\mathcal{A})}) = \mathsf{Mor}_{(\mathcal{A},\mathcal{A})}, \mathsf{Mor}(\mathrm{Fuc}_{(\mathcal{A},\mathcal{A})}) = \mathsf{Nat}_{(\mathcal{A},\mathcal{A})}).$$

Definition 3.2 We say the natural transformation $\alpha : F \to F'$ is a morphism of monads whenever

$$\begin{tikzcd}
FF \arrow[r, "\alpha\alpha"] \arrow[d, "\mu"'] & F'F' \arrow[d, "\mu'"] \\
F \arrow[r, "\alpha"'] & F'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{id}_{\mathcal{A}} \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & F \arrow[d, "\alpha"] \\
& F'
\end{tikzcd}$$

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Fact 3.3 The free functor is always a left adjoint of forgetful functor, e.g.,

- the natural transformation $\mu : FF(-) \to F(-)$ ($\mu_A : FF(A) \to F(A)$) induces an F-module structure, where the free functor is

$$F. : \mathcal{A} \to \mathcal{A}_F, \quad A \mapsto (F(A), \mu_A), [X \xrightarrow{f} Y] \mapsto [(F(X), \mu_X) \xrightarrow{(Ff, \mu_f)} (F(Y), \mu_Y)];$$

- the right adjoint of $F.$ is $U_F : \mathcal{A}_F \to \mathcal{A}$, defined by

$$\mathsf{Mor}_{\mathcal{A}_F}(F.(A), B) \xrightarrow{\sim} \mathsf{Mor}_{\mathcal{A}}(A, U_F(B)), \quad f \mapsto f \circ \eta_A.$$

> For a more trivial example, let R be a ring and $X \in \mathsf{Ob}(\mathbb{A}G)$, then $R \otimes X$ is a left R module defined by the action
>
> $$m \otimes \mathrm{id} : R \otimes (R \otimes X) \to R \otimes X, r_1 \otimes r_2 \otimes X \mapsto r_1 r_2 \otimes X.$$
>
> The functor $R \otimes - : \mathbb{A}G \to {}_R\mathcal{M}$ is free with right adjoint $U_R : {}_R\mathcal{M} \to \mathbb{A}G$, given by
>
> $$\mathrm{Hom}_R(R \otimes X, {}_RM) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{A}G}(X, U_F({}_RM)), \quad f \mapsto f \circ (\eta \otimes \mathrm{id}).$$
>
> The inverse of such isomorphism is given by

$$(X \xrightarrow{h} M) \mapsto (R \otimes X \xrightarrow{\mathrm{id}\otimes h} R \otimes M \xrightarrow{\varrho_M} M, \quad r \otimes x \mapsto rh(x)).$$

Definition 3.4 The **Kleisli category of ring** R is ${}_R\tilde{\mathcal{M}}$, defined as follows

$$\mathsf{Ob} = \mathsf{Ob}(\mathbb{A}G), \quad \mathsf{Mor} = \bigcup_{A,B\in\mathsf{Ob}({}_R\tilde{\mathcal{M}})} \mathrm{Hom}_{{}_R\tilde{\mathcal{M}}}(A,B) := \mathrm{Hom}_{\mathbb{A}G}(A, R\otimes B).$$

Here the composition ($\circ_{{}_R\tilde{\mathcal{M}}}$, or simply $\circ$) is defined by

$$g \circ_{{}_R\tilde{\mathcal{M}}} f : X \xrightarrow{f} R \otimes Y \xrightarrow{\mathrm{id}_R \otimes g} R \otimes R \otimes Z \xrightarrow{m\otimes \mathrm{id}_Z} R \otimes Z.$$

The identical map is $\mathrm{id}_X : X \to R \otimes X, x \mapsto 1 \otimes x$ (1 is the unit of R), where

$$\mathrm{id}_{R\otimes Y} \circ f : X \xrightarrow{f} R \otimes Y \xrightarrow{\mathrm{id}_R \otimes \mathrm{id}_{R\otimes Y}} R \otimes R \otimes Y \xrightarrow{m\otimes \mathrm{id}_Y} R \otimes Y, \quad x \mapsto f(x);$$
$$f \circ \mathrm{id}_X : X \xrightarrow{\mathrm{id}_X} R \otimes X \xrightarrow{\mathrm{id}_R \otimes f} R \otimes R \otimes Y \xrightarrow{m\otimes \mathrm{id}_Y} R \otimes Y, \quad x \mapsto f(x).$$

Indeed, the following three functors

- $R \otimes - : \mathbb{A}G \to {}_R\mathcal{M}, \quad X \mapsto R \otimes X, [X \xrightarrow{f} Y] \mapsto [R \otimes X \xrightarrow{\mathrm{id}_R \otimes f} R \otimes Y],$
- $\Phi : \mathbb{A}G \to {}_R\tilde{\mathcal{M}}, X \mapsto X, \quad [X \xrightarrow{f} Y] \mapsto [X \xrightarrow{\mathrm{id}_X} R \otimes X \xrightarrow{\mathrm{id}_R \otimes f} R \otimes Y],$
- $\Psi : {}_R\tilde{\mathcal{M}} \to {}_R\mathcal{M}, \quad X \mapsto R\otimes X,$ $[X \xrightarrow{f} R \otimes Y] \mapsto [R\otimes X \xrightarrow{\mathrm{id}_R \otimes f} R\otimes R \otimes Y \xrightarrow{m \otimes \mathrm{id}_Y} R \otimes Y]$,

satisfies the commutative diagram

$$\begin{tikzcd} \mathbb{A}G \arrow[rr, "R\otimes -"] \arrow[dr, "\Phi"'] & & {}_R\mathcal{M} \\ & {}_R\tilde{\mathcal{M}} \arrow[ur, "\Psi"'] & \end{tikzcd}$$

🔗link

Definition 3.5 The Kleisli category of a monad is similar, i.e.,

- $\Phi : \mathcal{A} \to \tilde{\mathcal{A}}_F, \quad X \mapsto (X_F, \mu_X), [X \xrightarrow{f} Y] \mapsto [X \xrightarrow{\eta_X} F(X) \xrightarrow{Ff} F(Y)],$
- $\Psi : \tilde{\mathcal{A}}_F \to \mathcal{A}_F, \quad X \mapsto FX, [X \xrightarrow{f} FY] \mapsto [F(X) \xrightarrow{Ff} FFY \xrightarrow{\mu_Y} FY],$
- $F. : \mathcal{A} \to \mathcal{A}_F$ such that $\Psi\Phi = F.$.

Definition 3.6 rewrite **Definition 3.1-Definition 3.5** for coalgebra.

Definition 3.7 ($\mathcal{A}$ and $\mathcal{B}$ are categories) For $L : \mathcal{B} \to \mathcal{A}$ and $R : \mathcal{A} \to \mathcal{B}$, we say $L \dashv R$ (L is a **left adjoint** of R), whence

$$\rho : \mathrm{Hom}_{\mathcal{A}}(LX, Y) \cong \mathrm{Hom}_{\mathcal{B}}(X, RY), \quad \forall X \in \mathsf{Ob}(\mathcal{B}), Y \in \mathsf{Ob}(\mathcal{A}).$$

Here we also have $L \vdash R$, i.e., R is a **right adjoint** of L. Here there exists a natural transformation

$$\begin{tikzcd} \mathsf{Mor}_{\mathcal{B}}(B, R(-)) \arrow[r, "\Phi_B"] \arrow[d, "\mathsf{Mor}_{\mathcal{B}}(f, R(-))"'] & \mathsf{Mor}_{\mathcal{A}}(LB, -) \arrow[d, "\mathsf{Mor}_{\mathcal{A}}(Lf, -)"] \\ \mathsf{Mor}_{\mathcal{B}}(B', R(-)) \arrow[r, "\Phi_{B'}"] & \mathsf{Mor}_{\mathcal{A}}(LB', -) \end{tikzcd}$$

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Theorem 3.8 R has a left adjoint whenever for each $B \in \mathsf{Ob}(\mathcal{B})$, $\mathsf{Mor}_{\mathcal{B}}(B, R(-))$ is always representable (by LB), i.e.,

$$\rho : \mathsf{Mor}_{\mathcal{A}}(LB, -) \xrightarrow{\sim} \mathsf{Mor}_{\mathcal{B}}(B, R(-)).$$

Moreover, if R has a left adjoint L which is not an isomorphism, then L is unique.

Such isomorphism is also natural for A and B, i.e.,

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(B, RA') \arrow[rrr, "\eta_{B,A'}"'] \arrow[ddd, "\mathrm{Hom}_{\mathcal{B}}(f, RA')"] & & & \mathrm{Hom}_{\mathcal{A}}(LB, A') \arrow[ddd, "\mathrm{Hom}_{\mathcal{A}}(Lf, A')"'] \\
& \mathrm{Hom}_{\mathcal{B}}(B, RA) \arrow[r, "\eta_{B,A}"'] \arrow[ul, "\mathrm{Hom}_{\mathcal{B}}(B, Rg)"] \arrow[d, "\mathrm{Hom}_{\mathcal{B}}(f, RA)"'] & \mathrm{Hom}_{\mathcal{A}}(LB, A) \arrow[ur, "\mathrm{Hom}_{\mathcal{A}}(LB, g)"'] \arrow[d, "\mathrm{Hom}_{\mathcal{A}}(Lf, A)"] & \\
& \mathrm{Hom}_{\mathcal{B}}(B', RA) \arrow[r, "\eta_{B',A}"] \arrow[dl, "\mathrm{Hom}_{\mathcal{B}}(B', Rg)"'] & \mathrm{Hom}_{\mathcal{A}}(LB', A) \arrow[dr, "\mathrm{Hom}_{\mathcal{A}}(LB', g)"] & \\
\mathrm{Hom}_{\mathcal{B}}(B', RA') \arrow[rrr, "\eta_{B',A'}"] & & & \mathrm{Hom}_{\mathcal{A}}(LB', A')
\end{tikzcd}$$

link

> The pair $(\otimes, \mathrm{Hom})$ induces an adjoint relation, i.e.,
>
> $$\mathrm{Hom}_{\mathbb{A}G}(L \otimes M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{A}G}(M, \mathrm{Hom}_{\mathbb{A}G}(L, N)),$$
> $$f \mapsto (m \mapsto f(- \otimes m));$$
>
> $$\mathrm{Hom}_{\mathbb{A}G}(L \otimes M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{A}G}(L, \mathrm{Hom}_{\mathbb{A}G}(M, N)),$$
> $$f \mapsto (l \mapsto f(l \otimes -)).$$
>
> Then we define
>
> - $\varepsilon_G : H \otimes \mathrm{Hom}_{\mathbb{A}G}(H, G) \to G, h \otimes f \mapsto f(h)$;
> - $\eta_G : G \to \mathrm{Hom}_{\mathbb{A}G}(H, H \otimes G), g \mapsto - \otimes g$,
>
> to obtain that
>
> - $\mathrm{id}_{H\otimes G} : h \otimes g \xrightarrow{\mathrm{id}_H \otimes \eta_G} h \otimes (-\otimes g) \xrightarrow{\varepsilon_{H\otimes G}} h \otimes g$, that is, $\mathrm{id}_{H\otimes G} = \varepsilon_{H\otimes G} \circ (\mathrm{id}_H \otimes \eta_G)$;
> - $\mathrm{id}_{\mathrm{Hom}_{\mathbb{A}G}(H,G)} : f \xrightarrow{\eta_{\mathrm{Hom}_{\mathbb{A}G}(H,G)}} (-\otimes f) \xrightarrow{\varepsilon_G} f$, that is, $\mathrm{id}_{\mathrm{Hom}_{\mathbb{A}G}(H,G)} = \varepsilon_G \circ (\eta_{\mathrm{Hom}_{\mathbb{A}G}(H,G)})$.

Definition 3.9 In **Fact 3.8**, $\eta : \mathrm{id}_{\mathcal{A}} \to RL$ and $\varepsilon : LR \to \mathrm{id}_{\mathcal{B}}$ are called unit and counit, i.e.,

$$\begin{tikzcd}
R & RLR \arrow[l, "R\varepsilon"'] \\
& R \arrow[u, "\eta R"'] \arrow[ul, "="]
\end{tikzcd}
\qquad
\begin{tikzcd}
LRL \arrow[d, "\varepsilon L"] & L \arrow[l, "L\eta"'] \arrow[dl, "="] \\
L &
\end{tikzcd}$$

$$\begin{tikzcd}
LRA \arrow[dr, "\varepsilon_A"'] & LB \arrow[l, "\exists ! Lg"'] \arrow[d, "f"'] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
RA & RLB \arrow[l, "\exists ! Rf"'] \\
B \arrow[u, "g"] \arrow[ur, "\eta_B"'] &
\end{tikzcd}$$

link

Then for each adjoint pair (L, R) ($L \dashv R$), it yields that

- L is full and faithful, whenever $RL \xrightarrow{\sim} \mathrm{id}_{\mathcal{B}}$;
- R is full and faithful, whenever $LR \xrightarrow{\sim} \mathrm{id}_{\mathcal{A}}$.

Theorem 3.10 For adjunctions (L, R, η, ϵ) and $(\tilde{L}, \tilde{R}, \tilde{\eta}, \tilde{\epsilon})$ (between categories $\mathcal{A}$ and $\mathcal{B}$), there is an isomorphism between natural transforms

$$h : \mathsf{Nat}_{(L,\tilde{L})} \xrightarrow{\sim} \mathsf{Nat}_{(\tilde{R},R)}, \quad \alpha \mapsto \hat{\alpha} := R\tilde{\varepsilon} \circ R\alpha\tilde{R} \circ \eta\tilde{R}.$$

The inverse is given by $\alpha = \varepsilon\tilde{L} \circ L\hat{\alpha}\tilde{L} \circ L\tilde{\eta}$.

Theorem 3.11 Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be quasi-inverse pair, i.e., there exists

- $\eta : \mathrm{id} \xrightarrow{\sim} GF$, which is a natural transfromation;
- $\varepsilon : FG \xrightarrow{\sim} \mathrm{id}$, which is a natural transformation.

Then we claim that (F, G) is an adjoint pair. Let η be unit without the loss of generality, then

$$\varepsilon' := FG \xrightarrow{FG\varepsilon^{-1}} FGFG \xrightarrow{F\eta^{-1}G} FG \xrightarrow{\varepsilon} \mathrm{id}$$

is a well defined counit. One can verity $\mathrm{id} : F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon' F} F$ and $\mathrm{id} : G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon'} G$.

Fact 3.11 For adjoint pair (F, G), F-module $\mathcal{A}$ is isomorphic to some induced G-comodule $\mathcal{A}$, i.e., $\mathcal{A}_F \cong \mathcal{A}^G$, since for each $A \in \mathsf{Ob}(\mathcal{A}_F)$, we have

$$A \xrightarrow{\eta_A} GFA \xrightarrow{G\varrho_A} GA; \quad FA \xrightarrow{F\varrho^A} FGA \xrightarrow{\varepsilon_A} A.$$

As for morphisms, we have,

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "\alpha"] & GFA \arrow[r, "G\varrho_A"] \arrow[d, "GF\alpha"] & GA \arrow[d, "G\alpha"] \\
A' \arrow[r, "\eta_{A'}"] & GFA' \arrow[r, "G\varrho_{A'}"] & GA'
\end{tikzcd}$$

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> The converse (comonand $\mathcal{A}^G$ implies monand $\mathcal{A}_F$) also holds (by simple verifying), thus each adjoint pair produce a pair of isomorphic monand and comonand, i.e., $\mathcal{A}_F \cong \mathcal{A}^G$.

> However, on can only obtian the equivalence between $\tilde{\mathcal{A}}_G$ and $\tilde{\mathcal{A}}^F$. In comparision of **Definition 3.4**, we introduce
>
> $$G. : \mathcal{A} \to \mathcal{A}^G, \quad A \mapsto (G(A), \delta_A), [X \xrightarrow{f} Y] \mapsto [(F(X), \delta_X) \xrightarrow{(Ff, \delta_f)} (F(Y), \delta_Y)].$$
>
> As $\mathsf{Ob}(\tilde{\mathcal{A}}_G) \cong \mathsf{Ob}(\tilde{\mathcal{A}}^F)$, we only need to verify the morphisms. $\forall A, A' \in \mathsf{Ob}(\mathcal{A})$, i.e.,
>
> $$\begin{aligned} & \mathsf{Mor}_{\tilde{\mathcal{A}}_G}(A, A') \cong \mathsf{Mor}_{\mathcal{A}_G}(G.A, G.A') \cong \mathsf{Mor}_{\mathcal{A}}(A, GA') \\ \cong \; & \mathsf{Mor}_{\mathcal{A}}(FA, A') \cong \mathsf{Mor}_{\mathcal{A}^F}(F.A, F.A') \cong \mathsf{Mor}_{\tilde{\mathcal{A}}^F}(A, A')^{\bullet} \end{aligned}$$