约定与记号 (回顾教材)

Definition 1.1 We say U is a **linear space** over the ground field $\mathbb{F}$, whenever...

Definition 1.2 Let V and W be $\mathbb{F}$-linear spaces. We say $f : W \to V$ is an $\mathbb{F}$**-linear map** whenever...

Notation 1.3 Let $\mathrm{Hom}_{\mathbb{F}}(W, V)$ denotes the space of $\mathbb{F}$-linear functions from W to V. This notation can be simplified as $\mathrm{Hom}(W, V)$ when the ground field is fixed.

Definition 1.4 Cartesian product is defined on a finite family of sets, i.e.,

$$\prod_{i \in I_0} A_i = \{(a_i)_{i \in I_0} \mid a_i \in A_i\}, \quad |I_0| < \infty.$$

> **Remark** $(1, 2) \neq (2, 1)$.

Definition 1.5 When there are infinitely many $\mathbb{F}$-linear spaces, i.e., $\{V_\lambda\}_{\lambda \in \Lambda}$ ($|\Lambda| = \infty$). Define the **direct sum** as

$$\bigoplus_{\lambda \in \Lambda} V_\lambda := \{(v_\lambda)_{\lambda \in \Lambda} \mid v_\lambda \in V_\lambda, \text{ only finite many } v_\lambda\text{'s is non-zero.}\}.$$

Definition 1.6 When there are infinitely many $\mathbb{F}$-linear spaces, i.e., $\{V_\lambda\}_{\lambda \in \Lambda}$ ($|\Lambda| = \infty$). Define the **direct product** as

$$\prod_{\lambda \in \Lambda} V_\lambda := \{(v_\lambda)_{\lambda \in \Lambda} \mid v_\lambda \in V_\lambda\}.$$

> **Remark** Let $V_\lambda = \mathbb{F} = \mathbb{R}$, then $(1, 1, 1, \dots) \in \prod_\lambda V_\lambda$, yet $(1, 1, 1, \dots) \notin \bigoplus_\lambda V_\lambda$.

Definition 1.7 Subspace...

> **Remark** A subspace requires the **same ground field** as the original space!

Fact 1.8 $\bigoplus_\lambda V_\lambda$ is a subspace of $\prod_\lambda V_\lambda$. Such subspace **is not proper whenever** $|\Lambda| = \infty$.

若干习题 (约定以下线性空间都是 $\mathbb{F}$-线性的, 即不出现第二个基域)

Ex.1 Prove **Fact 1.8**.

Ex.2 Prove that $\mathrm{Hom}(U, V)$ is $\mathbb{F}$-linear.

Ex.3 For a finite index set $|I_0| < \infty$, prove that

$$\prod_{i \in I_0} \mathrm{Hom}(U_i, V) \cong \mathrm{Hom}(\prod_{i \in I_0} U_i, V).$$

And provide a natrual isomorphism.

Ex.4 For a finite index set $|I_0| < \infty$, prove that

$$\prod_{i \in I_0} \mathrm{Hom}(U, V_i) \cong \mathrm{Hom}(U, \prod_{i \in I_0} V_i).$$

And provide a natrual isomorphism.

Ex.5 For an infinite index set $|\Lambda| \geq |\mathbb{N}|$, prove that

$$\prod_{\lambda \in \Lambda} \mathrm{Hom}(U, V_\lambda) \cong \mathrm{Hom}(U, \prod_{\lambda \in \Lambda} V_\lambda).$$

And provide a natrual isomorphism.

Ex.6 For an infinite index set $|\Lambda| \geq |\mathbb{N}|$, prove that

$$\prod_{\lambda\in\Lambda} \mathrm{Hom}(U_\lambda, V) \cong \mathrm{Hom}(\bigoplus_{\lambda\in\Lambda} U_\lambda, V).$$

And provide a natrual isomorphism.

Ex.7 For an infinite index set $|\Lambda| \geq |\mathbb{N}|$, prove that

$$\prod_{\lambda\in\Lambda} \mathrm{Hom}(U_\lambda, V) \hookleftarrow \mathrm{Hom}(\prod_{\lambda\in\Lambda} U_\lambda, V).$$

Here $A \hookrightarrow B$ means there exists a natrual isomorphism between A and a subspace of B. And provide such natrual isomorphism. Moreover, provide an example to verify $\hookrightarrow$ cannot be replaced by $\cong$.

Ex.8 For an infinite index set $|\Lambda| \geq |\mathbb{N}|$, prove that

$$\bigoplus_{\lambda\in\Lambda} \mathrm{Hom}(U_\lambda, V) \hookrightarrow \mathrm{Hom}(\prod_{\lambda\in\Lambda} U_\lambda, V).$$

Here $A \hookrightarrow B$ means there exists a natrual isomorphism between A and a subspace of B. And provide such natrual isomorphism. Moreover, provide an example to verify $\hookrightarrow$ cannot be replaced by $\cong$.

Ex.9 For an infinite index set $|\Lambda| \geq |\mathbb{N}|$, prove that

$$\bigoplus_{\lambda\in\Lambda} \mathrm{Hom}(U_\lambda, V) \hookrightarrow \mathrm{Hom}(\bigoplus_{\lambda\in\Lambda} U_\lambda, V).$$

Here $A \hookrightarrow B$ means there exists a natrual isomorphism between A and a subspace of B. And provide such natrual isomorphism. Moreover, provide an example to verify $\hookrightarrow$ cannot be replaced by $\cong$.

Ex.10 For an infinite index set $|\Lambda| \geq |\mathbb{N}|$, prove that

$$\bigoplus_{\lambda\in\Lambda} \mathrm{Hom}(U, V_\lambda) \hookrightarrow \mathrm{Hom}(U, \prod_{\lambda\in\Lambda} V_\lambda).$$

Here $A \hookrightarrow B$ means there exists a natrual isomorphism between A and a subspace of B. And provide such natrual isomorphism. Moreover, provide an example to verify $\hookrightarrow$ cannot be replaced by $\cong$.

Ex.11 For an infinite index set $|\Lambda| \geq |\mathbb{N}|$, prove that

$$\prod_{\lambda\in\Lambda} \mathrm{Hom}(U, V_\lambda) \hookleftarrow \mathrm{Hom}(U, \bigoplus_{\lambda\in\Lambda} V_\lambda).$$

Here $A \hookrightarrow B$ means there exists a natrual isomorphism between A and a subspace of B. And provide such natrual isomorphism. Moreover, provide an example to verify $\hookrightarrow$ cannot be replaced by $\cong$.

Ex.12 For an infinite index set $|\Lambda| \geq |\mathbb{N}|$, prove that

$$\bigoplus_{\lambda\in\Lambda} \mathrm{Hom}(U, V_\lambda) \hookrightarrow \mathrm{Hom}(U, \bigoplus_{\lambda\in\Lambda} V_\lambda).$$

Here $A \hookrightarrow B$ means there exists a natrual isomorphism between A and a subspace of B. And provide such natrual isomorphism. Moreover, provide an example to verify $\hookrightarrow$ cannot be replaced by $\cong$.

Ex.13 We say the $\mathbb{F}$-linear space U is finitely generated, whenever there exists finitely many $\{u_i\}_{1\leq i\leq n} \subseteq U$ such that

$$U = \left\{ \sum_{1\leq i\leq n} c_i u_i \mid c_i \in \mathbb{F} \right\}.$$

Prove that in **Ex.12**, if U is finitely generated, then the isomorphism holds.