Statement of the original problem

For any hypercube $\mathbb{Q}_n$ $(n \geq 2)$, there exists some **symmetric flips** of some 1's in the adjacency matrix $A(\mathbb{Q}_n)$ such that the spectral of $A(\mathbb{Q}_n)$ are $\pm\sqrt{n}$, that is, $\text{spec}(A(\mathbb{Q}_n)) = (\sqrt{n}^{(2^{n-1})}, -\sqrt{n}^{(2^{n-1})})$ since $\text{trace}(A) = 0$.

In 2019, H. Huang defined a sequence of symmetric square matrices iteratively as follows, which is highlighted in his paper:

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_{n+1} := \begin{pmatrix} A_n & I \\ I & -A_n \end{pmatrix}.$$

Notice that the method of iterating above resembles the how Sylvester constructed the Hadamard matrix in 1867; whereas the number of flips in $A(\mathbb{Q}_n)$ still has room for reduction under such construction. For instance, via Huang's construction we have

$$A_3 = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \end{pmatrix}.$$

Here 4 pairs of 1's flip, here 4 is defined as the **flip index**. Actually, the flip index of signed matrix

$$A_3' = \begin{pmatrix} 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

is only 3 (the **minimum flip index** of $\mathbb{Q}_3$ actually).

In the main part of the manuscript, we shall discuss that:

- the equivalent statement of the original problem
- the minimum flip index of $\mathbb{Q}_n$

Since any signed matrix of $A(\mathbb{Q}_n)$ with column vectors $(q_1, q_2, \ldots, q_{2^n})$ is symmetric, the following statements are equivalent.

$$\text{spec}(\text{sign}(A^2)) = n \Leftrightarrow \langle q_i, q_j \rangle = \delta_i^j \Leftrightarrow \frac{1}{\sqrt{n}} A \in O_{2^n}(\mathbb{R}).$$

A short discussion of the main problem

For the case that $n \geq 3$, we shall prove the existence of signed matrix $\mathrm{sign}A(\mathbb{Q}_n)$ such that $\mathrm{sign}(A(\mathbb{Q}_n))$ consists of orthogonal column vectors. Let

- $v_1, \ldots, v_{2^n}$ be verteces of $\mathbb{Q}_n$.
- $u_1, \ldots, u_{2^n}$ be column vectors of $A(\mathbb{Q}_n)$ w.r.t. v_i's.
- $w_1, \ldots, w_{2^n}$ be column vectors of some valid $\mathrm{sign}A(\mathbb{Q}_n)$ w.r.t. v_i's.

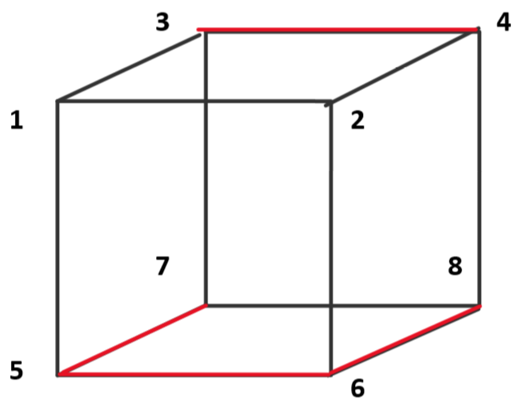
Since $\langle u_i, u_j \rangle \neq 0$ if and only if $N(u_i) \cap N(u_j) \neq 0$. When $i \neq j$, the following statements are equivalent:

- $N(u_i) \cap N(u_j) \neq 0$.
- $|N(u_i) \cap N(u_j)| = 2$.
- u_i and u_j are adjoint in some $\mathbb{Q}_2$-subgraph in $\mathbb{Q}_n$.

For each $\mathbb{Q}_2$-subgraph in $\mathbb{Q}_n$, the induced adjacency matrix is either $\pm A_2$ in the sense of *congruence*. As some -1 elements in $\mathrm{sign}(A)$ are signed, we *colour* the corresponding edges in $\mathbb{Q}_n$ (the remains are called *uncoloured*). We only need to prove that:

> There exists a colouring of edges for $\mathbb{Q}_n$ such that every $\mathbb{Q}_2$**-subgraph** has odd number of edges coloured.

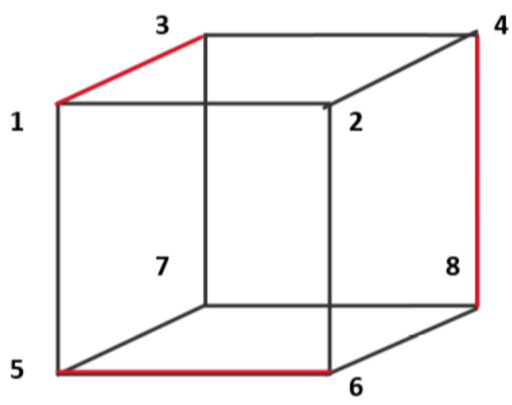
Here, Huang's construction for $\mathrm{sign}(A(\mathbb{Q}_3))$ is given as follows:



In order to seek for the minimal flip index, we only need to prove that:

> There exists a colouring of edges for $\mathbb{Q}_n$ such that every $\mathbb{Q}_2$-subgraph has **only one** edge coloured.

For instance, one possible colouring of edges in $\mathbb{Q}_3$ with minimum flip index is given as follows.



We define the **dimer** as the graph congruent to the *Chinese character* "日". The statement above equals that all $\mathbb{Q}_2$ subgraphs in $\mathbb{Q}_n$ has a perfect dimer covering, that is, a *complete* and *non-overlapping* covering. Therefore, the *mid-edges* of all dimers in a perfect covering establish an one-to-one correspondence to all coloured edges. For instance, the dimers 2437512, 2687342, 1268751 perfectly covers the $\mathbb{Q}_3$ cube.

We shall prove that such a covering exists for all $\mathbb{Q}_n$ for $n \geq 3$ via *Mathematical Induction*.

1. There is a perfect dimer covering for $\mathbb{Q}_3$.
2. Assume that the statement is true for $n = k$. In order to prove that $\mathbb{Q}_{k+1}$ has a perfect dimer covering, we first perfectly cover the *inside and outside* $\mathbb{Q}_k$ *cubes* by assumption. Since the rest of $\mathbb{Q}_2$-subgraphs correspond the edges of $\mathbb{Q}_k$, we only need to prove that $\mathbb{Q}_k$ has a perfect P_2 *covering* on its edges, which equals that the **line graph** $L(\mathbb{Q}_n)$ has a perfect matching. Since every line graph is *claw-free*, the line graph of a connected graph with an even size has at least one perfect matching.

Hence the minimal flip index of $\mathbb{Q}_n$ is number of dimers in any perfect covering, that is, $n(n-1)(n-2)2^{n-4}$ for $n \geq 3$.