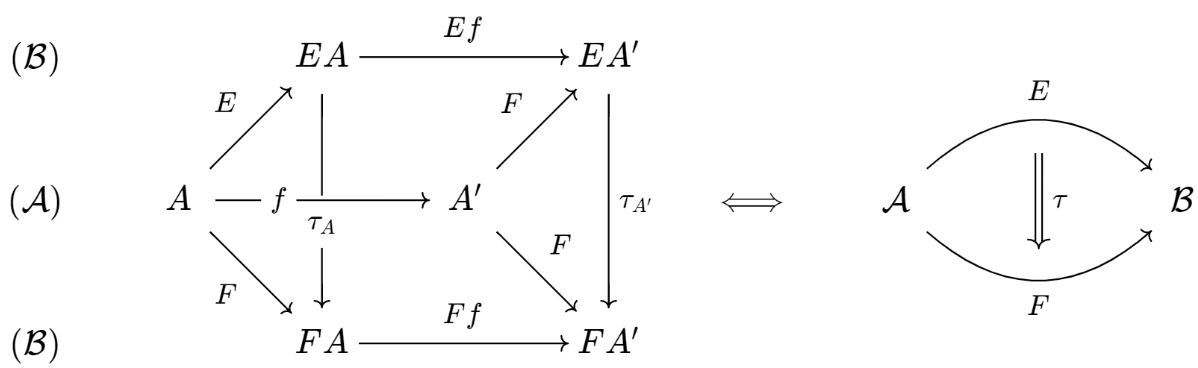
范畴论简介(2)

自然变换

Definition 1.5.1 取 $E, F : \mathcal{A} \to \mathcal{B}$ 间的函子, 自然变换 $\tau : E \to F$ 为一族映射满足 $\tau_A : EA \to FA, \forall A \in \mathsf{Ob}(\mathcal{A})$, 使得对任意 $f : A \to A'$ 总有左侧交换图(尤需注意正方形可换).



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此时, 2-胞腔(右图)自然地给出态射间变换 $\tau : Ef \to Ff$.

Definition 1.5.2 给定函子间自然变换 $\theta : F \to G$, 若存在自然变换 $\psi : G \to F$ 使得 $\theta \circ \psi = \mathrm{Id}_G, \psi \circ \theta = \mathrm{Id}_F$, 则称 ψ 与 θ **互逆**, 此时有函子间的同构 $F \cong G$. 特别地, 注意到自然变换的逆无非

$$\begin{aligned} \theta_X &: FX \xrightarrow{\sim} GX, \\ \psi_X = (\theta_X)^{-1} &: GX \xrightarrow{\sim} FX. \end{aligned}$$

从而 θ 可逆若且仅若每一 θ_X 可逆.

Example 1.5.3 函子的横合成与纵合成如下

$$
\begin{tikzcd}
\mathcal{A} \arrow[rr, bend left=50, "F"] \arrow[rr, "G" description] \arrow[rr, bend right=50, "H"'] & \arrow[phantom, u, shift left=2, "\Downarrow\tau_1"] \arrow[phantom, d, shift left=2, "\Downarrow\tau_2"] & \mathcal{B}
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{tikzcd}
\mathcal{A} \arrow[rr, bend left=50, "F", ""{name=U, below}] \arrow[rr, bend right=50, "H"', ""{name=D, above}] & & \mathcal{B}
\arrow[Rightarrow, from=U, to=D, "\tau_2 \underset{\text{纵}}{\circ} \tau_1"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=50, ""{name=U1, below}] \arrow[r, bend right=50, ""{name=D1, above}] & \mathcal{B} \arrow[r, bend left=50, ""{name=U2, below}] \arrow[r, bend right=50, ""{name=D2, above}] & \mathcal{C}
\arrow[Rightarrow, from=U1, to=D1, "\tau"]
\arrow[Rightarrow, from=U2, to=D2, "\theta"]
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{tikzcd}
\mathcal{A} \arrow[rr, bend left=30, ""{name=U, below}] \arrow[rr, bend right=30, ""{name=D, above}] & & \mathcal{C}
\arrow[Rightarrow, from=U, to=D, "\theta \underset{\text{横}}{\circ} \tau"]
\end{tikzcd}
$$

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特别地, 横合成原理如下

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=40, "F_1", ""{name=U1, below}] \arrow[r, bend right=40, "F_2"', ""{name=D1, above}] & \mathcal{B} \arrow[r, bend left=40, "G_1", ""{name=U2, below}] \arrow[r, bend right=40, "G_2"', ""{name=D2, above}] & \mathcal{C}
\arrow[Rightarrow, from=U1, to=D1, "\tau"]
\arrow[Rightarrow, from=U2, to=D2, "\psi"]
\end{tikzcd}
\qquad
\begin{tikzcd}
G_1F_1X \arrow[r, "\psi_{F_1X}"] \arrow[d, "G_1(\tau_X)"] \arrow[dr, "(\psi \underset{\text{横}}{\circ} \tau)_X" description] & G_2F_1X \arrow[d, "G_2(\tau_X)"] \\
G_1F_2X \arrow[r, "\psi_{F_2X}"] & G_2F_2X
\end{tikzcd}
$$

$$
\Longrightarrow \quad
\begin{tikzcd}
\mathcal{A} \arrow[rr, bend left=30, "G_1F_1", ""{name=U, below}] \arrow[rr, bend right=30, "G_2F_2"', ""{name=D, above}] & & \mathcal{C}
\arrow[Rightarrow, from=U, to=D, "\psi \underset{\text{横}}{\circ} \tau"]
\end{tikzcd}
$$

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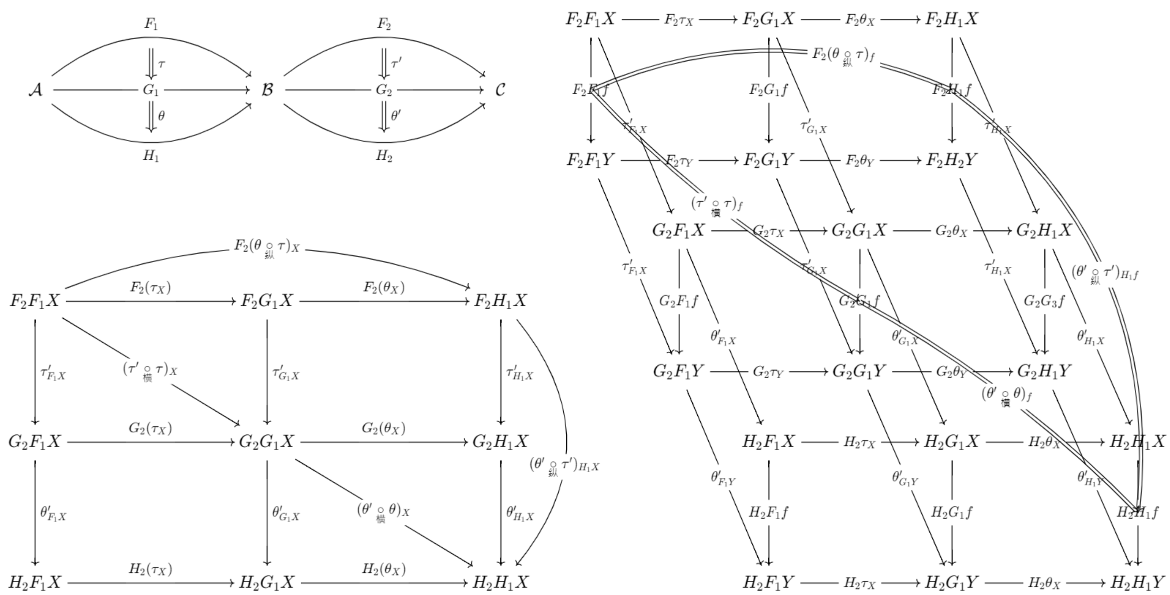
Theorem 1.5.4 以下交换图给出 $(\theta' \circ \tau') \circ (\theta \circ \tau) = (\theta' \circ \theta) \circ (\tau' \circ \tau)$. 实际上, 任取 $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$, 显然有如下元素与态射间的交换图 ($\tau, \tau', \theta, \theta'$ 均为自然变换)



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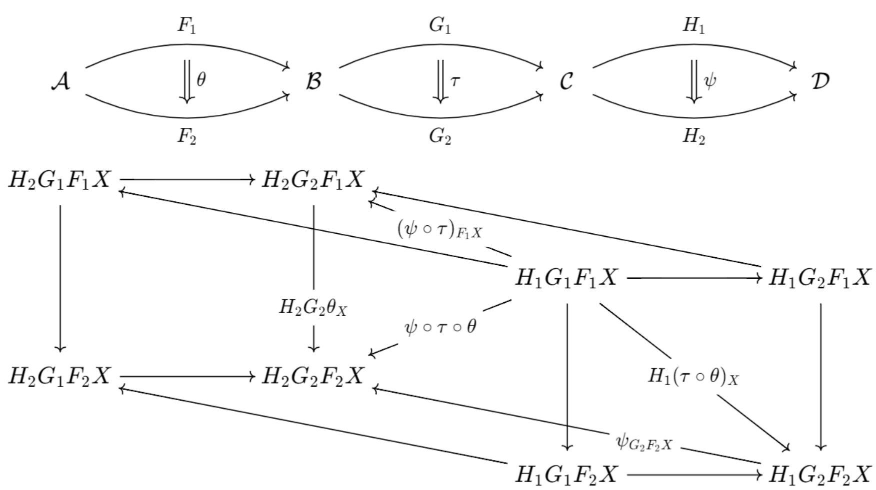
Theorem 1.5.5 有横结合律 $\psi \circ (\tau \circ \theta) = (\psi \circ \tau) \circ \theta$ $\boxed{= \psi) \circ \tau \circ (\theta}$.



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实际上, $\psi) \circ \tau \circ (\theta$ 也可定义, 即正方体另一条面对角线与棱之复合.

Theorem 1.5.6 有纵结合律 $\psi \circ (\tau \circ \theta) = (\psi \circ \tau) \circ \theta$. 图中, 中间方块可交换, 从而纵结合律成立.

$$\begin{tikzcd} \mathcal{A} \arrow[rr, bend left=60, "F"] \arrow[rr, bend left=20, "G"] \arrow[rr, bend right=20, "H"'] \arrow[rr, bend right=60, "K"'] \arrow[rr, phantom, shift left=4ex, "\theta\Downarrow"] \arrow[rr, phantom, "\tau\Downarrow"] \arrow[rr, phantom, shift right=4ex, "\psi\Downarrow"] & & \mathcal{B} \end{tikzcd}
\qquad
\begin{tikzcd} FX \arrow[r, "\theta_X"] \arrow[d, "Ff"] & GX \arrow[r, "\tau_X"] \arrow[d, "Gf"] & HX \arrow[r, "\psi_X"] \arrow[d, "Hf"] & KX \arrow[d, "Kf"] \\ FY \arrow[r, "\theta_Y"] & GY \arrow[r, "\tau_Y"] & HY \arrow[r, "\psi_Y"] & KY \end{tikzcd}$$

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Definition 1.5.7 若函子 $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{A}$ 满足同构 $\psi : FG \xrightarrow{\sim} \mathrm{Id}_{\mathcal{A}}$ 与 $\theta : GF \xrightarrow{\sim} \mathrm{Id}_{\mathcal{B}}$, 则称 G 为 F 的拟逆函子.

$$\begin{tikzcd}
(\mathcal{B}) & & FGFA \arrow[rr, "FGFf"] \arrow[dd, "F\theta_{GFA}"] & & FGFA' \arrow[dd, "F\theta_{GFA'}"] \\
(\mathcal{A}) & GFA \arrow[ur, "F"] \arrow[rr, "GFf"] \arrow[dd, "\psi_{GFA}"'] & & GFA' \arrow[ur, "F"] \arrow[dd, "\psi_{GFA'}"] & \\
(\mathcal{B}) & & FA \arrow[ul, "G"] \arrow[rr, "Ff"] & & FA' \arrow[ul, "G"'] \\
(\mathcal{A}) & A \arrow[ur, "F"] \arrow[rr, "f"'] & & A' \arrow[ur, "F"'] &
\end{tikzcd}$$

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特别地, 若 $FG = \mathrm{Id}_{\mathcal{A}}$ 且 $GF = \mathrm{Id}_{\mathcal{B}}$, 则称 F 与 G 是范畴间的同构.

Proposition 1.5.8 若函子 $G, G' : \mathcal{B} \to \mathcal{A}$ 均为 $F : \mathcal{A} \to \mathcal{B}$ 的拟逆, 则 $G \cong G'$. 实际上, 显然有

$$G \xrightarrow{\sim} G(FG') \xrightarrow{\sim} (GF)G' \xrightarrow{\sim} G'.$$

Example 1.5.9 记 $\mathcal{V}$ 为域 k 上线性空间所成之范畴, $\forall V \in \mathsf{Ob}(\mathcal{V})$, 记 $V^* := \mathrm{Hom}_k(A, k)$ 为对偶, 同理有 V^{**}. 定义共变函子 $F : \mathcal{V} \to \mathcal{V}$ 满足

- $FV = V^{**}, \forall V \in \mathsf{Ob}(\mathcal{V})$.
- $Ff = f^{**} := (f^*)^*, \forall f \in \mathrm{Hom}_k(V_1, V_2)$.

定义自然变换 $\tau_V : V \to V^{**}$ 为

$$\tau_V(x)(\theta) =: \theta(x), \quad \forall x \in V, \theta \in V^*.$$

容易验证右侧交换图. 从而 τ 为 $\mathrm{Id}_{\mathcal{V}}$ 到 F 的自然变换.

$$\begin{tikzcd}
V_1 \arrow[r, "f"] \arrow[d, "\tau_{V_1}"] & V_2 \arrow[d, "\tau_{V_2}"] \\
V_1^{**} \arrow[r, "Ff = f^{**}"] & V_2^{**}
\end{tikzcd}$$

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▼ **Proof of the theorem**

实际上, 对任意 $x \in V_1, \theta \in V_2^*$, 总有 $\tau_{V_2}(1_{\mathcal{V}} f)(x)(\theta) = \theta f(x)$. 注意到 f^* 诱导映射

$$f^* : V_2^* \to V_1^*, (\theta : V_2 \to k) \mapsto \theta f.$$

从而 $(f^*)^* \tau_{V_1}(x)(\theta) = \tau_{V_1}(x) f^* \theta = (f^* \theta) x = \theta f(x)$.

从而根据余直积之定义, 存在唯一的 $h_i : X \to X$ 使得 $hq_i = q_i$, 从而 $h_i = 1_X = \sum_{j=1}^n q_j p_j$.

正合列

Definition 1.6.1 记 $\{M_i\}$ 为一族左 R-模, $\{f_i\}$ 为一族模同态. 称

$$\cdots \to M_k \xrightarrow{f_k} M_{k+1} \xrightarrow{f_{k+1}} M_{k+2} \xrightarrow{f_{k+2}} M_{k+3} \xrightarrow{f_{k+3}} M_{k+4} \to \cdots$$

正合, 当且仅当 $\ker f_{k+1} = \mathrm{Im} f_k$ 对一切 k 恒成立, 如下图所示.

$$\begin{tikzcd}
\cdots \arrow[dr] & M_{k-2} \arrow[d] \arrow[dr] & M_{k-1} \arrow[d] \arrow[dr] & M_k \arrow[d] \arrow[dr, "f_k"] & M_{k+1} \arrow[d] \arrow[dr, "f_{k+1}"] & M_{k+2} \arrow[d] \arrow[dr] & \cdots \arrow[d] \arrow[dr] & & \\
& \cdots \arrow[dr] & \ker f_{k-1} \arrow[d] \arrow[dr] & \ker f_k \arrow[d] \arrow[dr, "f_k"] & \ker f_{k+1} \arrow[d] \arrow[dr, "f_{k+1}"] & \ker f_{k+2} \arrow[d] \arrow[dr] & \ker f_{k+3} \arrow[d] \arrow[dr] & \cdots \arrow[d] \arrow[dr] & \\
& & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots
\end{tikzcd}$$

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Proposition 1.6.2 在同一模范畴内, 自然有以下结论:

- $0 \to \ker f \overset{i}{\hookrightarrow} M \xrightarrow{f} N$ 正合, 特别地, $0 \to M \xrightarrow{f} N$ 正合当且仅当 f 为单同态;
- $M \xrightarrow{f} N \overset{\pi}{\twoheadrightarrow} \mathrm{coker} f \to 0$ 正合, 特别地, $M \xrightarrow{f} N \to 0$ 正合当且仅当 f 为满同态;
- $0 \to M \xrightarrow{f} N \to 0$ 正合当且仅当 f 为同构.

Theorem 1.6.3 (**五引理**) 选取横正合列间的的交换图如下

$$
\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "t_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "t_2"] & A_3 \arrow[r, "f_3"] \arrow[d, "t_3"] & A_4 \arrow[r, "f_4"] \arrow[d, "t_4"] & A_5 \arrow[d, "t_5"] \\
B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & B_3 \arrow[r, "g_3"] & B_4 \arrow[r, "g_4"] & B_5
\end{tikzcd}
$$

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- 若 t_4 与 t_2 均为满射, t_5 为单射, 则 t_3 为满射. $\begin{pmatrix} * & * & * & * & 0 \\ * & 0 & * & 0 & * \end{pmatrix} \implies \begin{pmatrix} * & * & * & * & 0 \\ * & 0 & \boxed{0} & 0 & * \end{pmatrix}$.
- 若 t_4 与 t_2 均为单射, t_1 为满射, 则 t_3 为单射. $\begin{pmatrix} * & 0 & * & 0 & * \\ 0 & * & * & * & * \end{pmatrix} \implies \begin{pmatrix} * & 0 & \boxed{0} & 0 & * \\ 0 & * & * & * & * \end{pmatrix}$
- 若 t_2, t_4 均为同构, t_5 为单射, t_1 为满射, 则 t_3 为同构.

▼ **Proof of the theorem**

- 若 t_4 与 t_2 均为满射, t_5 为单射. 任取 $b_3 \in B_3$, 则存在 $a_4 \in A_4$ 使得 $t_4(a_4) = g_3(b_3)$. 由于

$$0 = g_4 g_3(b_3) = g_4 t_4(a_4) = t_5 f_4(a_4),$$

加之 t_5 为单射, $f_4(a_4) = 0$. 因此存在 $a_3 \in A_3$ 使得 $f_3(a_3) = a_4$. 再注意到

$$
\begin{aligned}
g_3(t_3(a_3) - b_3) &= g_3 t_3(a_3) - g_3 b_3 \\
&= t_4 f_3(a_3) - t_4 a_4 \\
&= t_4(a_4) - t_4(a_4) \\
&= 0.
\end{aligned}
$$

从而存在 $b_2 \in B_2$ 使得 $g_2(b_2) = t_3(a_3) - b_3$. 由于 f_2 为满射, 故存在 a_2 使得 $t_2(a_2) = b_2$.

可发现 b_3 的某一原像大致与 $f_2(a_2)$ 以及 a_3 有关. 计算得

$$\begin{aligned} t_3 f_2(a_2) &= g_2 t_2(a_2) \\ &= g_2(b_2) \\ &= t_3(a_3) - b_3. \end{aligned}$$

因此 b_3 的某一原像为 $a_3 - f_2(a_2)$, 从而 t_3 为满射.

- 若 t_4 与 t_2 均为单射, t_1 为满射. 任取 $a_3 \in A_3$ 使得 $t_3(a_3) = 0$, 下验证 a_3 只能为 0. 根据交换图以及 t_4 为单的,

$$0 = g_3 t_3(a_3) = t_4 f_3(a_3) = f_3(a_3).$$

从而存在 $a_2 \in A_2$ 使得 $f_2(a_2) = a_3$. 注意到

$$g_2 t_2(a_2) = t_3 f_2(a_2) = t_3(a_3) = 0,$$

则存在 $b_1 \in B_1$ 使得 $g_1(b_1) = t_2(a_2)$. 由于 t_1 为满射, 取 $a_1 \in A_1$ 使得 $t_1(a_1) = b_1$. 再注意到

$$t_2(a_2) = g_1(b_1) = g_1 t_1(a_1) = t_2 f_1(a_1).$$

从而 $f_1(a_1) = a_2$. 故 $a_3 = f_2 f_1(a_1) = 0$.

- 最后一则是显然的.

Theorem 1.6.4 (同调列中的蛇引理) 以笔者怠惰故，此段摘抄自同调代数笔记，读者可将一切复形视作模同一范畴的对象，将相应的链复形同态视作模同态. 对链复形与短正合列 $0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$, 交换图

$$
\begin{tikzcd}
 & \bullet \arrow[d, "\partial_{q+2}"] & \bullet \arrow[d, "\partial_{q+2}"] & \bullet \arrow[d, "\partial_{q+2}"] & \\
0 \arrow[r] & C_{q+1} \arrow[r, "f_{q+1}"] \arrow[d, "\partial_{q+1}"] & D_{q+1} \arrow[r, "g_{q+1}"] \arrow[d, "\partial_{q+1}"] & E_{q+1} \arrow[r] \arrow[d, "\partial_{q+1}"] & 0 \\
0 \arrow[r] & C_q \arrow[r, "f_q"] \arrow[d, "\partial_q"] & D_q \arrow[r, "g_q"] \arrow[d, "\partial_q"] & E_q \arrow[r] \arrow[d, "\partial_q"] & 0 \\
0 \arrow[r] & C_{q-1} \arrow[r, "f_{q-1}"] \arrow[d, "\partial_{q-1}"] & D_{q-1} \arrow[r, "g_{q-1}"] \arrow[d, "\partial_{q-1}"] & E_{q-1} \arrow[r] \arrow[d, "\partial_{q-1}"] & 0 \\
 & \bullet & \bullet & \bullet &
\end{tikzcd}
$$

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中横行均正合. 对 $e_q \in Z_q(E)$ 定义**边缘同态**

$$\partial_* : H_q(E) \to H_{q-1}(C), \quad [e_q] \mapsto [f_{q-1}^{-1} \partial_q g_q^{-1}(e_q)].$$

从而可良定义长正合列

$$\cdots \xrightarrow{\partial_*} H_{q+1}(C) \xrightarrow{f_*} H_{q+1}(D) \xrightarrow{g_*} \boxed{H_{q+1}(E) \xrightarrow{\partial_*} H_q(C)} \xrightarrow{f_*} H_q(D) \xrightarrow{g_*} H_q(E) \xrightarrow{\partial_*} \cdots.$$

▼ **Proof of the theorem**

实际上, 只需证明以下正合横列给出同态 $\ker(\partial_q'') \xrightarrow{\delta} \operatorname{coker}(\partial_q')$

$$
\begin{tikzcd}
 & C_q \arrow[r, "f_q"] \arrow[d, "\partial_q'"] & D_q \arrow[r, "g_q"] \arrow[d, "\partial_q"] & E_q \arrow[r] \arrow[d, "\partial_q''"] & 0 \\
0 \arrow[r] & C_{q-1} \arrow[r, "f_{q-1}"] & D_{q-1} \arrow[r, "g_{q-1}"] & E_{q-1} &
\end{tikzcd}
$$

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补全纵列作短正合列, 得右
图.

下依次证明:

1️⃣ 蓝线 (连接框的路径) 给出

$\delta := \overline{\partial'_q}\, f_{q-1}^{-1}\, \partial_q\, g_q^{-1}\, \partial''_{q0}$: $\ker(\partial''_q) \to \operatorname{coker}(\partial'_1)$;

2️⃣ δ 为良定义的同态;

3️⃣ $\operatorname{im}(g_{q0}) = \ker(\delta)$;

4️⃣ $\ker(\overline{f_q}) = \operatorname{im}(\delta)$.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \ker(\partial'_q) \arrow[r, "f_{q0}"] \arrow[d, "\partial'_{q0}"'] & \ker(\partial_q) \arrow[r, "g_{q0}"] \arrow[d, "\partial_{q0}"'] & \boxed{\ker(\partial''_q)} \arrow[d, "\partial''_{q0}"'] \arrow[dddll, dashed, bend right, "\delta"'] & \\
 & C_q \arrow[r, "f_q"] \arrow[d, "\partial'_q"'] & \boxed{D_q} \arrow[r, "g_q"] \arrow[d, "\partial_q"'] & \boxed{E_q} \arrow[r] \arrow[d, "\partial''_q"'] & 0 \\
0 \arrow[r] & \boxed{C_{q-1}} \arrow[r, "f_{q-1}"] \arrow[d, "\overline{\partial'_q}"'] & \boxed{D_{q-1}} \arrow[r, "g_{q-1}"] \arrow[d, "\overline{\partial_q}"'] & E_{q-1} \arrow[d, "\overline{\partial''_q}"'] & \\
 & \boxed{\operatorname{coker}(\partial'_q)} \arrow[r, "\overline{f_q}"] \arrow[d] & \operatorname{coker}(\partial_q) \arrow[r, "\overline{g_q}"] \arrow[d] & \operatorname{coker}(\partial''_q) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

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1️⃣ 证明 δ 为映射 (即无法一对多):

1. $\forall e \in \ker(\partial''_q)$, 由于 g_q 为满射, 固存在 $d \in D_q$ 使得 $g_q(d) = \partial''_{q0}(e)$.
2. 注意到 $\partial''_q \circ \partial''_{q0} \equiv 0$, 从而 $\partial''_q g_q(d) = g_{q-1}\partial_q(d) = 0$.
3. 根据短正合列, $\partial_q(d) \in \ker(g_{q-1}) = \operatorname{im}(f_{q-1})$, 因此存在 $c' \in C_{q-1}$ 使得 $f_{q-1}(c') = \partial_q(d)$. 此处 f_{q-1} 为单的, c' 的选取仅取决于 d.
4. 对任意 d_1 与 d_2 使得 $g_q(d_1) = g_q(d_2) = \partial''_{q0}(e)$, $(d_1 - d_2) \in \ker(g_q) = \operatorname{im}(f_q)$. 取 $c \in C_q$ 使得 $f_q(c) = d_1 - d_2$,
5. 当 d_1 变为 d_2 时, c'_1 变为 c'_2, 其间相差 $\partial'_q(c) \in \ker(\overline{\partial'_q})$, 从而 $\overline{\partial'_q}(c'_1) = \overline{\partial'_q}(c'_2)$. 可见 δ 为良定义的映射.

2️⃣ δ 显然为良定义的同态, 就 1️⃣ 中各步骤逐一验证即可.

3️⃣ 分两步证明 $\operatorname{im}(g_{q0}) = \ker(\delta)$:

1. 任取 $d \in \ker(\partial_q)$, 则 $g_{q0}(d) \in \ker(\partial''_q)$, 从而依照交换图有 (六步变四步)

$$\delta(g_{q0}(d)) = \overline{\partial'_q}\, f_{q-1}^{-1}\, \partial_q\, g_q^{-1}\, \partial''_{q0}\, g_{q0}(d) = \overline{\partial'_q}\, f_{q-1}^{-1}\, \partial_q\, \partial_{q0}(d) = 0.$$

   因此得 $\operatorname{im}(g_{q0}) \subseteq \ker(\delta)$.

2. 另一方面, 沿用 1️⃣ 中符号. 任取 $\forall e \in \ker(\delta)$, 则 $c' \in \ker(\overline{\partial'_q}) = \operatorname{im}(\partial'_q)$. 取 c 使得 $\partial'_q(c) = c'$, 从而 $f_{q-1}(c') = \partial_q f_q(c)$, 即 $d - f_q(c) \in \ker(\partial_q) = \operatorname{im}(\partial_{q0})$. 故存在 $d' \in \ker(\partial_q)$ 使得 $\partial_{q0}(d') = d - f_q(c)$. 因此

$$\partial''_{q0} g_{q0}(d') = g_q \partial_{q0}(d') = g_q(d) = \partial''_{q0}(e).$$

由于 ∂''_{q0} 为单的, 故 $\mathrm{im}(g_{q0}) \supseteq \ker(\delta)$.

得证.

4️⃣ 类比 3️⃣.

至此, 我们证明了 🐍 引理. 短正合列 $0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$ 导出长正合同调列.

Theorem 1.6.5 (**强形式蛇引理**) 同一模范畴中, 横正合列间的交换图

$$\begin{tikzcd}
& \ker(f) \arrow[r, "a'"] & \ker(g) \arrow[r, "b'"] & \ker(h) \arrow[dddll, bend right, "\partial"'] & \\
M_1 \arrow[r] \arrow[d, two heads, "满同态"] & M_2 \arrow[r, "a"] \arrow[d, "f"] & M_3 \arrow[r, "b"] \arrow[d, "g"] & M_4 \arrow[r] \arrow[d, "h"] & M_5 \arrow[d, hook, "单同态"] \\
N_1 \arrow[r] & N_2 \arrow[r, "c"] & N_3 \arrow[r, "d"] & N_4 \arrow[r] & N_5 \\
& \mathrm{coker}(f) \arrow[r, "c'"] & \mathrm{coker}(g) \arrow[r, "d'"] & \mathrm{coker}(h) &
\end{tikzcd}$$

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给出正合列 $\ker(f) \xrightarrow{a'} \ker(g) \xrightarrow{b'} \ker(h) \xrightarrow{\partial} \mathrm{coker}(f) \xrightarrow{c'} \mathrm{coker}(g) \xrightarrow{d'} \mathrm{coker}(h)$.

Proposition 1.6.6 ∂ 是自然的, i.e., 上方交换图导出下方交换图

$$
\begin{tikzcd}[column sep=small]
 & K_1 \arrow[rr] \arrow[d, two heads, "满同态"] \arrow[ddl] & & K_2 \arrow[rr, "x"] \arrow[d, "\lambda"] \arrow[ddl, "1"] & & K_3 \arrow[rr, "y"] \arrow[d, "\mu"] \arrow[ddl, "2"] & & K_4 \arrow[rr] \arrow[d, "\nu"] \arrow[ddl, "3"] & & K_5 \arrow[d, hook, "单同态"] \arrow[ddl] \\
 & L_1 \arrow[rr] \arrow[ddl] & & L_2 \arrow[rr, "z"] \arrow[ddl, "4"] & & L_3 \arrow[rr, "w"] \arrow[ddl, "5"] & & L_4 \arrow[rr] \arrow[ddl, "6"] & & L_5 \arrow[ddl] \\
M_1 \arrow[rr] \arrow[d, two heads, "满同态"] & & M_2 \arrow[rr, "a"] \arrow[d, "f"] & & M_3 \arrow[rr, "b"] \arrow[d, "g"] & & M_4 \arrow[rr] \arrow[d, "h"] & & M_5 \arrow[d, hook, "单同态"] & \\
N_1 \arrow[rr] & & N_2 \arrow[rr, "c"] & & N_3 \arrow[rr, "d"] & & N_4 \arrow[rr] & & N_5 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\ker(f) \arrow[r, "a'"] \arrow[d, "1'"] & \ker(g) \arrow[r, "b'"] \arrow[d, "2'"] & \ker(h) \arrow[r, bend left, "\partial"] \arrow[d, "3'"] & \operatorname{coker}(f) \arrow[r, "c'"] \arrow[d, "4'"] & \operatorname{coker}(g) \arrow[r, "d'"] \arrow[d, "5'"] & \operatorname{coker}(h) \arrow[d, "6'"] \\
\ker(\lambda) \arrow[r, "x'"'] & \ker(\mu) \arrow[r, "y'"'] & \ker(\nu) \arrow[r, bend left, "\partial"] & \operatorname{coker}(\lambda) \arrow[r, "z'"'] & \operatorname{coker}(\mu) \arrow[r, "w'"'] & \operatorname{coker}(\nu)
\end{tikzcd}
$$

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Theorem 1.6.7 观察如下 R-模同态图

$$
\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "a"] \arrow[d, "f"] & Y \arrow[r, "b"] \arrow[d, "g"] & Z \arrow[r] & 0 \\
0 \arrow[r] & X' \arrow[r, "a'"] & Y' \arrow[r, "b'"] & Z' \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "a"] & Y \arrow[r, "b"] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & X' \arrow[r, "a'"] & Y' \arrow[r, "b'"] & Z' \arrow[r] & 0
\end{tikzcd}
$$

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其中左(右)图上下两行均为短正合列且左(右)侧方块交换, 则存在唯一的模同态 $h : Z \to Z'$($f : X \to X'$) 使得右(左)边方块也交换.

并且若 f 是满同态, 则有正合列

$$0 \to \ker(f) \xrightarrow{\tilde{a}} \ker(g) \xrightarrow{\tilde{b}} \ker(h) \to 0.$$

若 h 是单同态, 则有正合列

$$0 \to \operatorname{coker}(f) \xrightarrow{\tilde{a}'} \operatorname{coker}(g) \xrightarrow{\tilde{b}'} \operatorname{coker}(h) \to 0.$$

▼ **Proof of the proposition**

对左图而言, 由于 $(b'g)a = 0$. 根据 $\ker b \ \ (= a(X))$ 之泛性质, 存在唯一的 $h : Z \to Z'$ 使得 $hb = (b'g)$; 右图同理, 利用 $\operatorname{coker}(a) \ \ (= b(Z))$ 之泛性质即可. 端详蛇引理给出的长正合列

$$0 \to \ker(f) \xrightarrow{\tilde{a}} \ker(g) \xrightarrow{\tilde{b}} \ker(h) \to \operatorname{coker}(f) \xrightarrow{\tilde{a}'} \operatorname{coker}(g) \xrightarrow{\tilde{b}'} \operatorname{coker}(h) \to 0.$$

- f 满若且仅若 $\operatorname{coker}(f) = 0$;
- h 单若且仅若 $\ker(h) = 0$.

明所欲证.

Definition 1.6.8 称 Abel 范畴 $\mathcal{C}$ 与 $\mathcal{D}$ 间的加性共变函子 $F : \mathcal{C} \to \mathcal{D}$ 为

- 半正合的, 若且仅若 $\mathcal{C}$ 中正合列 $(0 \to)A \to B \to C(\to 0)$ 推出正合列 $FA \to FB \to FC$.
- 左正合的, 若且仅若 $\mathcal{C}$ 中正合列 $0 \to A \to B \to C(\to 0)$ 推出正合列 $0 \to FA \to FB \to FC$.
- 右正合的, 若且仅若 $\mathcal{C}$ 中正合列 $(0 \to)A \to B \to C \to 0$ 推出正合列 $FA \to FB \to FC \to 0$.
- 正合的, 若且仅若 $\mathcal{C}$ 中正合列 $0 \to A \to B \to C \to 0$ 推出正合列 $0 \to FA \to FB \to FC \to 0$.

此处考虑或忽视括号中内容均可, 同为正合性之等价定义.关于 Abel 范畴上加性反变函子的正合性之序数同理, 此处从略.

Definition 1.6.9 称 Abel 范畴 $\mathcal{C}$ 与 $\mathcal{D}$ 间的加性反变函子 $F : \mathcal{C} \to \mathcal{D}$ 为

- 半正合的, 若且仅若 $\mathcal{C}$ 中正合列 $(0 \to)A \to B \to C(\to 0)$ 推出正合列 $FC \to FB \to FA$.
- 左正合的, 若且仅若 $\mathcal{C}$ 中正合列 $(0 \to)A \to B \to C \to 0$ 推出正合列 $0 \to FC \to FB \to FA$.
- 右正合的, 若且仅若 $\mathcal{C}$ 中正合列 $0 \to A \to B \to C(\to 0)$ 推出正合列 $FC \to FB \to FA \to 0$.

- 正合的, 若且仅若 $\mathcal{C}$ 中正合列 $0 \to A \to B \to C \to 0$ 推出正合列 $0 \to FC \to FB \to FA \to 0$.

Proposition 1.6.10 Hom 函子给出模到 Abel 群

- $\mathrm{Hom}_R({}_RM, -) : {}_R\mathcal{M} \to \mathbb{A}G$ 为左正合共变函子;
- $\mathrm{Hom}_R(-, {}_RM) : {}_R\mathcal{M} \to \mathbb{A}G$ 为左正合反变函子;
- $\mathrm{Hom}_R(-, {}_RM) : {}_R\mathcal{M} \to \mathbb{A}G$ **并非**右正合反变函子;
- $\mathrm{Hom}_R({}_RM, -) : {}_R\mathcal{M} \to \mathbb{A}G$ **并非**右正合共变函子.

▼ **Proof of the proposition**

1️⃣ 考虑 R-正合列 $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$, 只需证明以下为正合列

$$0 \to \mathrm{Hom}_R(M,X) \xrightarrow{\mathrm{Hom}_R(M,f)} \mathrm{Hom}_R(M,Y) \xrightarrow{\mathrm{Hom}_R(M,g)} \mathrm{Hom}_R(M,Z).$$

- 兹有断言: $\mathrm{Hom}_R(M,f)$ 为单射. 实际上, 任选 $\varphi \in \mathrm{Hom}_R(M,X)$ 使得 $\mathrm{Hom}_R(M,f)(\varphi) = f\varphi = 0$, 由于 f 为单射, 故 $\varphi = 0$.
- 再有断言: $\mathrm{im}(\mathrm{Hom}_R(M,f)) = \ker(\mathrm{Hom}_R(M,g))$. 任取 $\psi \in \mathrm{im}(\mathrm{Hom}_R(M,f))$, 总有 $\varphi \in \mathrm{Hom}_R(M,f)$ 使得 $f\varphi = \psi$. 故

  $$\mathrm{Hom}_R(M,g)(\psi) = g\psi = gf\varphi = 0,$$

  从而 $\mathrm{im}(\mathrm{Hom}_R(M,f)) \subseteq \ker(\mathrm{Hom}_R(M,g))$.

  另一方面, 取 $\psi \in \ker(\mathrm{Hom}_R(M,g)) \subseteq \mathrm{Hom}_R(M,Y)$, 则 $g\psi = 0$. 下只需证明存在 $h \in \mathrm{Hom}_R(M,X)$ 使得 $fh = \psi$. 注意到 $\mathrm{im}(\psi) \subseteq \ker(g) = \mathrm{im}(f)$, 故有映射

  $$h : f^{-1} \circ \psi : m \mapsto x(\in X' \subseteq X) \quad (f(x) = \psi(m)).$$

  此处 $\psi(M) \subseteq f(M)$ 作为子模, 故存在 X 的子模 X' 使得 $f : X' \xrightarrow{\sim} \psi(M)$ 为同构. 从而 $\mathrm{im}(\mathrm{Hom}_R(M,f)) \supseteq \ker(\mathrm{Hom}_R(M,g))$.

2️⃣ 考虑 R-正合列 $X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$, 只需证明以下为正合列

$$0 \to \mathrm{Hom}_R(Z,M) \xrightarrow{\mathrm{Hom}_R(g,M)} \mathrm{Hom}_R(Y,M) \xrightarrow{\mathrm{Hom}_R(f,M)} \mathrm{Hom}_R(X,M).$$

- 兹有断言: $\mathrm{Hom}_R(g,M)$ 为单射. 实际上, 任选 $\varphi\in\mathrm{Hom}_R(Z,M)$ 使得 $\mathrm{Hom}_R(g,M)(\varphi)=\varphi g=0$, 由于 g 为满射, 故 $\varphi=0$.
- 再有断言: $\mathrm{im}(\mathrm{Hom}_R(g,M))=\ker(\mathrm{Hom}_R(f,M))$. 任取 $\psi\in\mathrm{im}(\mathrm{Hom}_R(g,M))$, 总有 $\varphi\in\mathrm{Hom}_R(g,M)$ 使得 $\varphi g=\psi$. 故

  $$\mathrm{Hom}_R(f,M)(\psi)=\psi f=\varphi gf=0,$$

  从而 $\mathrm{im}(\mathrm{Hom}_R(g,M))\subseteq\ker(\mathrm{Hom}_R(f,M))$.

  另一方面, 取 $\psi\in\ker(\mathrm{Hom}_R(f,M))\subseteq\mathrm{Hom}_R(Y,M)$, 则 $\psi f=0$. 下只需证明存在 $h\in\mathrm{Hom}_R(Z,M)$ 使得 $hg=\psi$. 注意到 $\ker(g)=\mathrm{im}(f)\subseteq\ker(\psi)$, 故有映射

  $$h:=\psi\circ g^{-1}:Z\to M,z\mapsto\psi(y)\quad(g(y)=z).$$

  从而 $\mathrm{im}(\mathrm{Hom}_R(g,M))\supseteq\ker(\mathrm{Hom}_R(f,M))$.

3️⃣ 与 4️⃣ 同理. 例如正合列 $0\to m\mathbb{Z}\overset{i}{\hookrightarrow}\mathbb{Z}$ 在 $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z})$ 下非反变正合函子. 端详链

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})\xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(i,\mathbb{Z})}\mathrm{Hom}_{\mathbb{Z}}(m\mathbb{Z},\mathbb{Z})\to 0$$

可知 $\mathrm{Hom}_{\mathbb{Z}}(i,\mathbb{Z})$ 并非满射 ($m\mapsto 1$ 没有原像).