Basic facts on completeness and p-adic numbers

Filter

Definition

We call the non-empty subfamily $\mathscr{F} \subset \mathcal{P}(X)$ a filter (濾子) whenever

- $A, B \in \mathscr{F}$ implies $A \cap B \in \mathscr{F}$ (downward closed),
- $A \in \mathscr{F}$ implies $\{U \in \mathcal{P}(X) \mid A \subset U\} \subset \mathscr{F}$ (upward closed),
- $\emptyset \notin \mathscr{F}$.

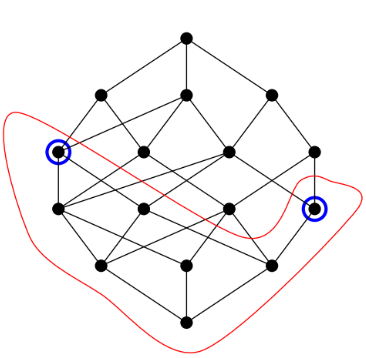
Example. Let $\{x_n\}_{n\geq 1}$ be a sequence in X. Then

$$\mathscr{F} := \{E \subset X \mid \exists N \text{ s.t. } \{x_k\}_{k\geq N} \subset E\}.$$

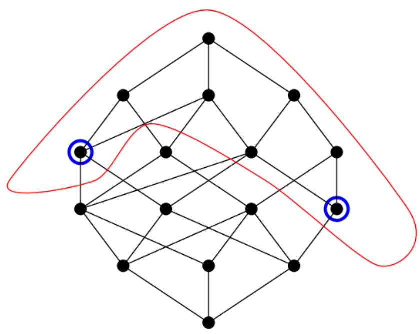
is a filter.

Ex1 $\forall (X, \tau), \forall x_0 \in X$ are given. Prove that $\{\text{neighbourhoods of } x_0\}$ is a filter.

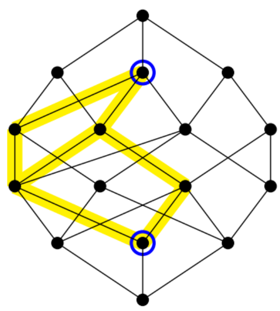
Example. For any partially ordered set $(P, \leqq)$, we define interval with endpoints $\alpha \leqq \beta$ as $[\alpha, \beta] : \{x \mid \alpha \leqq x \leqq \beta\}$.



Order ideal (generators)



Order filter (generators)



Interval (endpoints)

The topology generated by $\{[\alpha, \max]\}_{\alpha\in P}$ is called order filter.

The topology generated by $\{[\min, \alpha]\}_{\alpha\in P}$ is called order ideal.

Ex2 Is the topology generated by order ideal (or order filter) a filter?

Basis of a filter

We call $\mathscr{B} \in \mathcal{P}(X)$ a basis of filter in X whenever

- $\emptyset \notin \mathscr{B}$,
- $A, B \in \mathscr{B}$ implies $\exists C \in \mathscr{B}$ s.t. $C \subset A \cap B$.

Compare such definition with topological basis.

We say a filter converges to a point ($\mathscr{F} \to x$) whenever each neighbourhood of x contains some elements in $\mathscr{F}$.

> *Is* $\{(-1-n^{-1}, 1+n^{-1})\}_{n\geq 1}$ *a convergent filter in* $(\mathbb{R}, \tau_{\text{standard}})$*?*

We say the (X, τ) is Hausdorff whenever each filter converges to at most one point.

> *Is* $\{(-1-n^{-1}, -1+n^{-1}) \cap (1-n^{-1}, 1+n^{-1})\}_{n\geq 1}$ *a convergent filter in quotient space* $(\mathbb{R}, \tau_{\text{standard}})/(-1 \sim 1)$*? Is such quotient space Hausdorff?*

For $f : X \to Y$, $\mathscr{F}$ is a filter in X. Set

$$f\mathscr{F} := \{F \subset Y \mid \exists E \in \mathscr{F} \text{ s.t. } f(E) \subset F\}.$$

Then $f\mathscr{F}$ is also a filter in Y. The function f is continuous whenever f is continuous in filters. One can regard (convergent) filters as *generalised* (convergent) sequences.

Ex3 Why convergence of filters seems "better" than convergence of sequences? (Hint: sequences can ill afford to discribe "uncountable cases". One may encounter something like net convergence in the study of completeness and sequencially completeness in functional analysis.)

Cauchy filter

We call $(G, +)$ an Abelian group whenever

1. G is closed under the binary operation $+$, that is,

$$\forall a, b \in G, a + b \in G.$$

2. G admits associativity, that is,

$$\forall a, b, c \in G, (a+b)+c = a+(b+c) =: a+b+c.$$

3. G has a (additive) unit, that is,

$$\exists e \in G \text{ s.t. } \forall g \in G, g+e=e+g=g.$$

4. $\forall g \in G$ has an (additive) inverse, that is,

$$\forall g \in G, \exists \tilde{g} \in G \text{ s.t. } g+\tilde{g}=\tilde{g}+g=e.$$

5. G (additively) commutes, that is,

$$\forall a, b \in G, a+b=b+a.$$

Ex4 Verify the uniqueness of e and $\tilde{g}$. We write the inverse $\tilde{g}$ as $-g$, $a+(-b)$ as $a-b$ thenceforth.

Write $\mathscr{N}_x$ as the set of all neighbourhoods of x. For $x \in G$ and $S \subset G$, define $x+S$ as $\{x+s \mid s \in S\}$.

Let $(X,+)$ be Abelian group and also a Hausdorff space (X, τ) (e.g., $(\mathbb{R}, \tau_{\text{standard}})$). Such X is a object in the category $\mathbf{TopAb}$. Then

- $\exists e \in X$ s.t. $\mathscr{N}_e = -\mathscr{N}_e$.
- $\mathscr{N}_x = x + \mathscr{N}_0$.

We call $\mathscr{F}$ a Cauchy filter in $\mathscr{F}$ whenever $\forall U \in \mathscr{N}_0, \exists E \in \mathscr{F}$ such that $E-E \subset U$. For instance, $\mathscr{N}_x$ is a Cauchy filter for arbitrary fixed $x \in X$.

One can also regard Cauchy filters as *generalised* Cauchy sequences.

Ex5 Verity that each convergent filter in above X is always a Cauchy filter.

Completeness

Definition

Consider the injection $i: \mathbb{Q} \to \mathbb{R}, r \mapsto r$. Then

- $\mathbb{Q} \to i(\mathbb{Q})$ is a homeomorphism.
- $i(\mathbb{Q})$ is dense in $\mathbb{R}$.
- $\mathbb{R}$ is complete in sense of Cauchy sequences, that is, each Cauchy sequence in $i(\mathbb{Q})$ converges in $\mathbb{R}$.

Completeness in general

The completeness of $A \in \mathrm{Obj}(\mathbf{TopAb})$ is a morphism $f \in \mathrm{Mor}(\mathbf{TopAb})$ such that

- $f : A \to f(A)$ is a homeomorphism.
- $f(A)$ is dense in $\tilde{A}$.
- $\tilde{A}$ is complete in sense of Cauchy filters (either in A or $\tilde{A}$), that is, each Cauchy filter converges to exactly one point in $\tilde{A}$.

One may observe that for each continuous function $f \in C(\mathbb{Q})$, there exists a unique continuous function $\tilde{f} \in C(\mathbb{R})$ such that $\tilde{f}|_{\mathbb{Q}} = f$. This is due to the universal property (the translation 泛性质 is often heard, e.g. 拥有学习能力系学生之泛性质) of completeness, i.e., for each morphism $f : A \to B$ in the category $\mathbf{TopAb}$ (or $\mathbf{TopRing}$), there exists unique $\tilde{f}$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "f"'] & \tilde{A} \arrow[d, "\exists | \tilde{f}"] \\ & B \end{tikzcd}$$

Ex6 Verify the completeness is unique in sense of homeomorphism. (It is quite hard).

Introduction of p-adic valuation

p-adic number

The standard absolute value $|\cdot|_\infty$ (or $|\cdot|$ for simplicity) is induced by the standard topology for $\mathbb{R}$ (and its subspaces). We say x is close enough to y whence $|x-y|_\infty \ll 1$.

We shall define a new absolute value $|\cdot|_p$ on $\mathbb{Q}$, which is called p-adic absolute value, as follows.

The p-valuation of $d \in \mathbb{Z}$ is defined by the powers of p it contains, i.e.,

$$v_p(d) := \sup\{n \in \mathbb{Z} \mid d \cdot p^{-n} \in \mathbb{Z}\}.$$

One may observe $v_p(d_1 d_2) = v_p(d_1) + v_p(d_2)$, thus the domain of v_p can be extended to $\mathbb{Q}$.

> *Here $v_p(0) = v_p(0) + v_p(d)$ is well-defined since one can set $\infty + k = \infty$.*

Define $|r|_p := p^{-v_p(r)}$ for $r \in \mathbb{Q}$. Here $|0|_p = p^{-\infty} = 0$.

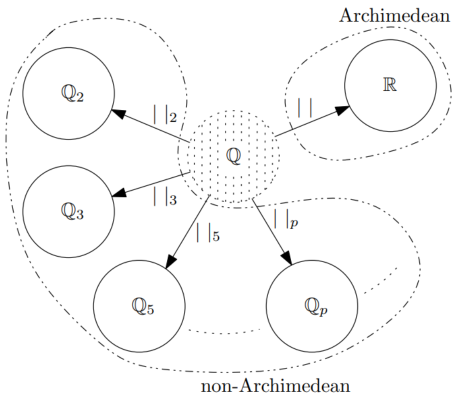
For each $x \in \mathbb{Q}$, there exists a unique $k_0 \in \mathbb{Z}$ and unique factorisation

$$x = \sum_{k \geq k_0} a_{k_0} p^{k_0}, \quad a_k \in \{0, 1, \dots, p-1\}.$$

Thus $v_p(x) = -k_0$ and $|x|_p = p^{k_0}$. Such factorisation is called the standard representation of p-adic numbers.

For instance, $|9|_3 = 3^{-2}$, $|28|_7 = 7^{-1}$.

Ex7 Prove that $|x+y|_p \leq \max(|x|_p, |y|_p) \leq |x|_p + |y|_p$ and determine when equality holds. As a corollary, the absolute value $|\cdot|_p$ is non-Archimedean.



Ex8 Define a family valuations for the field of rational functions such that we can determine the multiplicity of zeros and ∞'s on each point of $\mathbb{R}$, (or $\mathbb{R} \cup \{\infty\}$, $\mathbb{C}$, $\mathbb{C} \cup \{\infty\}$, whatever you like).

> *Rational function takes the form of* $\dfrac{f(x)}{g(x)}$, *f and g are (finite) polynomials. e.g.,* $\dfrac{x^2+x-1}{x^3+2}$ *is a rational function.*

The p-adic valuation on $\mathbb{R}$

There exists an extension of p-adic valuation on $\mathbb{R}$. We omit the proof since it requires the knowledge of abstract algebra. It requires the axiom of choice.

Topology of p-adic number

Let $\mathbb{Q}_p$ denotes the topology of $\mathbb{Q}$ induced by $|\cdot|_p$. Since the metric is discrete, each open ball in $\mathbb{Q}_p$ is also closed.

Ex9 Prove that

- $\mathbb{Q}_p$ is totally disconnected, whose only connected subspaces are singletons.
- $p\mathbb{Z}$ is a compact subspace in $\mathbb{Q}_p$, thus $\mathbb{Q}_p$ is locally compact.

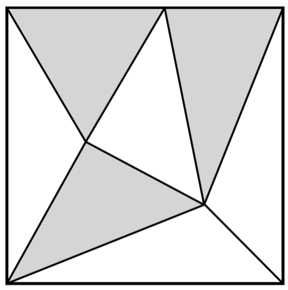
- At most p distinct points in $\mathbb{Q}_p$ are equidistant from each other.

📎 This thesis on p-adic numbers is easy to read for beginners.

Application: Monskey's theorem

It is not possible to dissect a square into an odd number of triangles of equal area.

> *Here quadrilaterals with vetrex angled 180° are NOT triangles.*



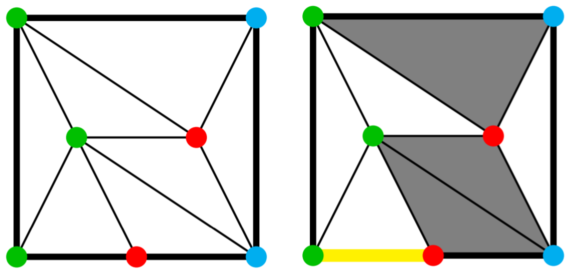
Proof.

Step I. We shall first prove **Sperner's lemma** in dimension 2, saying

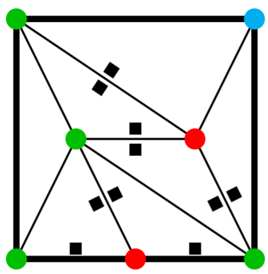
When colouring the vertices of a square among blue, red and green colours, then

$$\begin{aligned} &\text{number of red-green coloured edges on the boundary} \\ \equiv\ &\text{number of red-green-blue coloured squares} \mod 2. \end{aligned}$$

For instance:



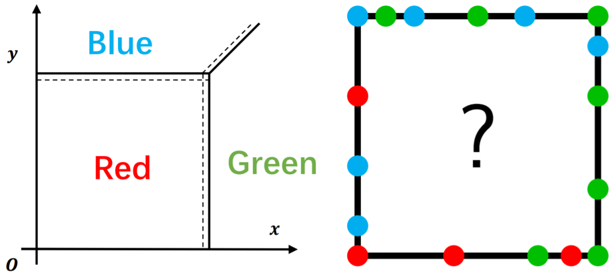
Ex10 Proof Sperner's lemma by counting the ■ by edges and by triangles.



Step II. Consider the valuation $|\cdot|_2$ on $[0,1]^2 \subset \mathbb{R}^2$. Colour $(x,y) \in [0,1]^2$ with

1. Red whence $|x|_2 < 1 \wedge |y|_2 < 1$,
2. Gre whence $|x|_2 \geq 1 \wedge |x|_2 \geq |y|_2$,
3. Blu whence $|y|_2 \geq 1 \wedge |x|_2 < |y|_2$,

shown as follows:



Step III. For the sake of contradiction, we assume the existence of *the partition*. Then we coloured the vertices in accordance with **Step II.** As a result, there exists a 3-coloured triangle (in light of Sperner's lemma).

Step IV. Let S denotes the area of each triangle. Let $\{(x_i, y_i)\}_{i=1,2,3}$ be vertices of any 3-coloured triangle. Then

$$|S|_2 = \left|\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}\right|_2 = 2\left|\begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix}\right|_2.$$

Without the loss of generality, let $x_1 = y_1 = 0$, and colour (x_1, y_2), (x_2, y_2) and (x_3, y_3) with Red , Blue, Green respectively. Therefore,

$$|S|_2 = 2|x_2y_3 - x_3y_3|_2 = 2\max\{|x_2y_3|_2, |x_3y_2|_2\} \geq 2.$$

> $|a+b|_p = \max\{|a|_p, |b|_p\}$ *whenever* $|a_p| \neq |b_p|$ *when* $a, b \in \mathbb{Q}$. *Such equality still holds as we extend the valuation onto* $\mathbb{R}$.

When $S = \dfrac{1}{n}$ for odd n, $|S|_2 = 1 < 2$, which leads to a contradiction.

□

Ex11 It is possible to dissect $[0,1]^n$ a into m of triangles of equal area iff $n! \mid m$.

Ex12 It is possible to dissect regular n-polygon $(n \geq 5)$ into m triangle of equal area iff $n \mid m$.

Ex13 It is impossible to dissect the convex hull of $\{(0,0),(\pi,0),(0,1),(1,1)\}$ in to any number of triangles of equal area.