Kronecker product的一些性质

Kronecker积引入

设$A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}, B=\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}, X=\begin{pmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix}, Y=\begin{pmatrix}y_{11} & y_{12}\\ y_{21} & y_{22}\end{pmatrix}$. 从而：

1. 矩阵方程$AX=Y$等价于

$$\begin{pmatrix}a_{11} & 0 & a_{12} & 0\\ 0 & a_{11} & 0 & a_{12}\\ a_{21} & 0 & a_{22} & 0\\ 0 & a_{21} & 0 & a_{22}\end{pmatrix}\cdot\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}=\begin{pmatrix}y_1\\ y_2\\ y_3\\ y_4\end{pmatrix}.$$

2. 矩阵方程$XB=Y$等价于

$$\begin{pmatrix}b_{11} & b_{21} & 0 & 0\\ b_{12} & b_{22} & 0 & 0\\ 0 & 0 & b_{11} & b_{21}\\ 0 & 0 & b_{12} & b_{22}\end{pmatrix}\cdot\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}=\begin{pmatrix}y_1\\ y_2\\ y_3\\ y_4\end{pmatrix}.$$

3. 矩阵方程$AXB=Y$等价于

$$\begin{pmatrix}a_{11}b_{11} & a_{11}b_{21} & a_{12}b_{11} & a_{12}b_{21}\\ a_{11}b_{12} & a_{11}b_{22} & a_{12}b_{12} & a_{12}b_{22}\\ a_{21}b_{11} & a_{21}b_{21} & a_{22}b_{11} & a_{22}b_{21}\\ a_{21}b_{12} & a_{21}b_{22} & a_{22}b_{12} & a_{22}b_{22}\end{pmatrix}\cdot\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}=\begin{pmatrix}y_1\\ y_2\\ y_3\\ y_4\end{pmatrix}.$$

实际上, 上式左端可写作分块矩阵形式

$$\begin{pmatrix}a_{11}b_{11} & a_{11}b_{21} & a_{12}b_{11} & a_{12}b_{21}\\ a_{11}b_{12} & a_{11}b_{22} & a_{12}b_{12} & a_{12}b_{22}\\ a_{21}b_{11} & a_{21}b_{21} & a_{22}b_{11} & a_{22}b_{21}\\ a_{21}b_{12} & a_{21}b_{22} & a_{22}b_{12} & a_{22}b_{22}\end{pmatrix}=\begin{pmatrix}a_{11}B^T & a_{12}B^T\\ a_{21}B^T & a_{22}B^T\end{pmatrix}.$$

记作$A\otimes B^T$.

Kronecker积性质

一般地, 设$P\in K^{k\times l}, Q\in K^{m\times n}$, 从而$P\otimes Q\in K^{km\times ln}$. 其形式为

$$P \otimes Q = \begin{pmatrix} p_{11}Q & p_{12}Q & \cdots & p_{1l}Q \\ p_{21}Q & p_{22}Q & \cdots & p_{2l}Q \\ \vdots & \vdots & \ddots & \vdots \\ p_{k1}Q & p_{k2}Q & \cdots & p_{kl}Q \end{pmatrix}.$$

在维度允许的前提下, 有如下性质:

1. $\otimes$为二元运算, 满足分配律, 结合律.
2. 满足双线性性, 即对$\otimes$两端自变量均线性.
3. $(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$.
4. 转置$(A \otimes B)^T = A^T \otimes B^T$.
5. 秩$\mathrm{r}(A \otimes B) = \mathrm{r}(A) \cdot \mathrm{r}(B)$.
6. 行列式$(\det A)^{\dim A}(\det B)^{\dim B} = \det{(A \otimes B)}$.
7. 迹$\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\mathrm{tr}(B)$.
8. 设A, B的特征值分别为$\{\lambda_i\}, \{\mu_i\}$, 对应的特征向量分别为$\{v_i\}, \{u_i\}$, 则$A \otimes B$的特征值为$\{\lambda_i \mu_j\}$, 对应的特征向量为$\{v_i \otimes u_j\}$.
9. 由$5 - 9$可见$A \otimes B$与$B \times A$有相同的行列式, 秩, 迹, 以及特征值.

总结

实际上, 矩阵方程$AX = Y$与$XB = Y$的本质仍为线性代数, 因此可以化作简单的线性方程组问题. Kronecker积深刻地描述了该种转化关系.