空间曲线的小结论

Frenet框架

$\{t,n,b\}$满足$\gamma'(s)=t$, 且

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}t\\n\\b\end{pmatrix}=\begin{pmatrix}0&\kappa&0\\-\kappa&0&-\tau\\0&\tau&0\end{pmatrix}\cdot\begin{pmatrix}t\\n\\b\end{pmatrix}.$$

注:

1. Frenet矩阵具有反对称性. 记$F(s)$为Frenet矩阵, $v=(t,n,b)$, 则$v(s)=F(s)v(0)$. 由正交性知

   $$0=(v(s)^2)'=[(F(s)+F^T(s))v]\cdot v.$$

   从而$F(s)+F^T(s)\equiv 0$.

局部Taylor展开: 对弧长参数的曲线$\gamma(s)$, $\gamma(0)$附近展开得

$$\begin{aligned}\gamma(s)&=\gamma(0)+s\gamma'(0)+\frac{s^2}{3}\gamma''(0)+\frac{1}{6}\gamma'''(s)+o(s^2)\\&=\gamma_0+(s-\frac{\kappa_0^2s^3}{6})t+(\frac{\kappa_0s^2}{2}+\frac{\kappa_0's^3}{6})n-\frac{\kappa_0\tau_0s^3}{6}b\\&\quad o(s^3)\end{aligned}$$

密切圆与密切球

对弧长参数且曲率挠率均非零的曲线γ, 其密切圆显然为n向半径为κ^{-1}的元.

γ落在球面上的充要条件为$\frac{1}{\kappa^2}+\left(\frac{1}{\kappa}\right)_s^2\frac{1}{\tau^2}=R^2$, 其中$R$为对应的球面半径.

下求$\gamma(s)$处的近似球面半径. 设球心p, 则$\gamma(s)-p(s)=\lambda(s)n+\mu(s)b$. 求导得

$$t=\lambda'n+\mu'b-\lambda(\kappa t+\tau b)+\mu\tau n.$$

因此

$$\begin{aligned}\lambda\kappa+1=0\\\mu\tau+\lambda'=0\\\mu'-\lambda\tau=0\end{aligned}$$

解得$\lambda=\frac{1}{-\kappa}$, $\mu=\frac{(1/\kappa)_s}{\tau}$. 因此密切球面$R=\sqrt{\frac{1}{\kappa^2}+\left(\frac{1}{\kappa}\right)_s^2\frac{1}{\tau^2}}$, 朝向$n$.

显然密切圆于密切球上.

曲率与挠率公式

对具有正则参数的曲线$\gamma(t)$, 有

$$\begin{aligned}\gamma'(t)&=s_t\cdot t\\\gamma''(t)&=\kappa s_t^2\cdot n+s_{tt}\cdot t\\\gamma'''(t)&=(\kappa s_t)_t\cdot n-\kappa(\kappa t+\tau b)s_t^3+(s_{tt}\cdot t)_t\end{aligned}$$

从而

$$\kappa = \frac{|\gamma'(t) \times \gamma''(t)|}{|\gamma'(t)|^3}$$
$$\tau = \frac{-[\gamma'(t), \gamma''(t), \gamma'''(t)]}{|\gamma'(t)|^6 \kappa^2} = \frac{[\gamma'''(t), \gamma''(t), \gamma'(t)]}{|\gamma'(t) \times \gamma''(t)|^2}$$

对平面曲线$\gamma'(t)$, 曲率κ带符号($n = b \times t$). 当$\gamma(t) = (x(t), y(t))$时, 有

$$|k| = \frac{|(x', y') \times (x'', y'')|}{|(x', y')|^2} = \frac{|x'y'' - x''y'|}{|x'^2 + y'^2|^{3/2}}.$$

对极坐标$\rho = r\theta$, 有

$$\rho' = r'\hat{r} + r\theta'\hat{\theta}$$
$$\rho'' = r''\hat{r} + r'\theta'\hat{\theta} + r'\theta'\hat{\theta} + r\theta''\hat{\theta} - r\theta'\theta'\hat{r}$$

从而

$$|\kappa| = \frac{|r'(2r'\theta' + r\theta'') - r\theta'(r'' - r\theta'\theta')|}{|r'^2 + r^2\theta'^2|^{3/2}}$$
$$= \frac{|2r'^2\theta' + rr'\theta'' - rr''\theta' + r^2\theta'^3|}{|r'^2 + r^2\theta'^2|^{3/2}}$$

当θ为参数时, $\kappa = \dfrac{|2r'^2 - rr'' + r^2|}{|r'^2 + r^2|^{3/2}}$.

渐屈线与焦曲面

为法线的包络线, 即两点间法线收敛于渐屈线上的一点. 设$\gamma : I \to \mathbb{R}^3$为有弧长参数的曲线, 则$\gamma(s)$对应的渐屈线上的点$\alpha(s)$满足$\alpha(s) = \gamma(s) + \lambda(s)n(s)$. 显然$\gamma(s)$之法线总为渐屈线之切线, 故

$$\alpha'(s) = t + \lambda' n + \lambda(-\kappa t) \parallel n.$$

从而$\lambda(s) = \dfrac{1}{k(s)}$. 故$\alpha(s) = \gamma(s) + \dfrac{1}{\kappa(s)}n(s)$. 当$\kappa \neq 0$时, 渐屈线正则.

记$X : U \to S$为没有抛物点或脐点的正则曲面, 则曲率线坐标下, 参数曲面

$$Y(u, v) := X(u, v) + \frac{1}{\kappa_1}N(u, v)$$
$$Z(u, v) := X(u, v) + \frac{1}{\kappa_2}N(u, v)$$

称作焦曲面. 实际上

$$Y_u \wedge Y_v = (X_u + \frac{1}{\kappa_1}N_u + (\kappa_1^{-1})_u N) \wedge (X_v + \frac{1}{\kappa_1}N_v + (\kappa_1^{-1})_v N)$$
$$= (\kappa_1^{-1})_u N \wedge (X_v - \frac{\kappa_2}{\kappa_1}X_v)$$

故κ_i关于u, v的一阶导数不为零时, Y, Z均正则.