Introduction to the Prime Number Theory

From Special Functions to a Simplified Proof

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January 13, 2021

Institue = None

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Intro to PNT

How prime numbers asymptotically distribute among large positive integers?

What PNT is

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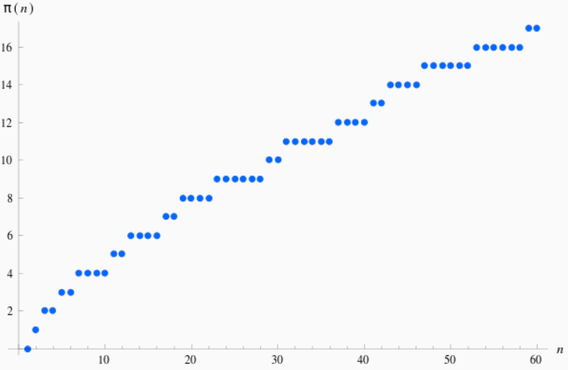

Figure 1: The values of $\pi(x)$ for the first 60 positive integers.

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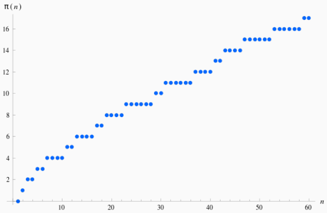

Figure 2: The values of $\pi(x)$ for the first 60 positive integers.

It was conjectured in the end of the 18th century by Gauss that $\lim_{x \to \infty} \frac{\pi(x)}{x/\log(x)} = 1$, known as the prime number theorem (PNT).

Early Proofs of PNT

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- The ”elementary” proof were found around 1948 by Atle Selberg and by Paul Erdős. (**Does not use any complex analysis.**)
- D. J. Newman found a very simple version of the Tauberian argument needed for an analytic proof of the PNT.

  We will describe the resulting proof later, which has a beautifully simple structure and uses **hardly anything beyond Cauchy’s theorem**.

In this presentation, we will introduce a simplified proof based on Newman's work.

Main Structure of a Simplified Proof

$$\left.\begin{array}{r}\Gamma(s)\text{ on }\mathbb{C}\\ \text{extension of }\zeta(s)\text{ on }\Re(s)>0\\ \text{the functional equation}\end{array}\right\}\Rightarrow {\color{blue}\zeta(s)\text{ on }\mathbb{C}}$$

$$\left.\begin{array}{r}{\color{blue}\zeta(s)\text{ on }\mathbb{C}}\\ \\ \text{zeros of }\zeta(s)\text{ when }\Re(s)=1\end{array}\right\}\Rightarrow {\color{green}\Phi(s)-\frac{1}{s-1}\text{ is analytic when }\Re(s)\geq 1}$$

$$\left.\begin{array}{r}{\color{green}\Phi(s)-\frac{1}{s-1}\text{ is analytic when }\Re(s)\geq 1}\\ \text{Analytic Theorem}\end{array}\right\}\Rightarrow \frac{\vartheta(x)}{x}\sim 1\Leftrightarrow\frac{\pi(x)}{x/\log x}\sim 1\text{ (PNT)}$$

Figure 3: Main Structure of the Simplified Proof

Propositions of $\Gamma(s)$ and $\zeta(s)$

Gamma function is one possible extension of factorial, i.e. $\Gamma(n) = (n-1)!$ when $n \in \mathbb{N}$. Furthermore, via following propositions, $\Gamma(s)$ can be extended to a meromorphic function on $\mathbb{C}$.

How $\Gamma(s)$ is defined on $\mathbb{C}$

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Therefore, $\Gamma(s)$ is analytic on $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$.

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- The product form of $\Gamma(s)$ when $s \notin \{0, -1, -2, \ldots\}$ lies that:

$$\Gamma(s) = \frac{1}{s} \prod_{k=1}^{\infty} \left[\left(1 + \frac{1}{k}\right)^{s} \left(1 + \frac{s}{k}\right)^{-1} \right] = \lim_{n \to \infty} \frac{n! n^{s}}{s(s+1) \cdots (s+n)}$$

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- (Euler's reflection formula) $\Gamma(s)\Gamma(1-s) = \dfrac{\pi}{\sin(\pi s)}$.

Some Propositions of $\Gamma(s)$

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The following propositions has been proven in the Lecture.

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3. The extension of $\zeta(s)$ on $\mathbb{C}_+$ can be written as

$$\zeta(s) = \frac{1}{\Gamma(s)} \left(\int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{s-1}\mathrm{d}t + (s-1)^{-1} + \int_1^\infty \frac{t^{s-1}e^{-t}}{1-e^{-t}}\mathrm{d}t \right)$$

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4. Via the functional equation $\zeta(1-s) = 2^{1-s} \pi^{-s} \zeta(s) \Gamma(s) \cos(s\pi/2)$, $\zeta(s)$ can be meromorphically extended on $\mathbb{C}$.

How ζ is defined on $\mathbb{C}$

Formulas/Equations

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Significant conclusions

- $\zeta(s) - \dfrac{1}{s-1}$ is analytic on $\mathbb{C}$;
- $\zeta(s)$ never vanishes when $\Re(s) > 1$.

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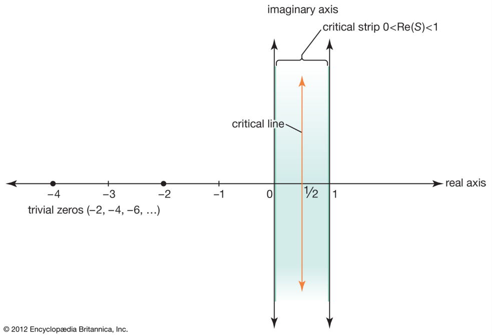

Figure 4: Critical strip of Riemann zeta function

Proof Progress

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$$\left.\begin{array}{r} \Phi(s) - \dfrac{1}{s-1} \text{ is analytic when } \Re(s) \geq 1 \\ \text{Analytic Theorem} \end{array}\right\} \Rightarrow \frac{\vartheta(x)}{x} \sim 1 \Leftrightarrow \frac{\pi(x)}{x/\log x} \sim 1 \text{ (PNT)}$$

Figure 5: Proof Progress

Prime-related Propositions of Special Functions

Definitions of some prime-related functions

- $\pi(x)$, prime-counting function, is defined as $\pi(x) = \sum_{p \leq x} 1$;
- $\Phi(s)$ denotes $\sum_{p} \frac{\log p}{p^s} \quad \Re(s) > 1$;
- Λ function is defined as $\Lambda : \mathbb{N} \to \mathbb{R}, n \mapsto \begin{cases} \log p & n = p^k, k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$;
- $\vartheta(x)$ is defined as $\vartheta(x) = \sum_{p \leq x} \log p$;
- (Well-known proposition) $\zeta(s) = \prod_{p} (1 - p^{-s})^{-1} \quad \Re(s) > 1$.

REMARK: $x \in \mathbb{R}$ and $s \in \mathbb{C}$.

Proposition: $-\dfrac{\zeta'(s)}{\zeta(s)} = \sum_{n\geq 2} \Lambda(n) e^{-s\log n} \quad \Re(s) > 1$

Proof.

Recall that $\zeta(s) = \prod_p (1-p^{-1})^{-1}$, then

$$-\frac{\zeta'(s)}{\zeta(s)} = -(\log\zeta(s))' = \sum_p \frac{e^{-s\log p}\log p}{1-e^{-s\log p}} = \sum_p \left(\log p \sum_{k=1}^{\infty} (e^{-s\log p})^k\right)$$

Since the infinite summation uniformly converges for any p,

$$\sum_p \left(\log p \sum_{k=1}^{\infty} (e^{-s\log p})^k\right) = \sum_{k=1}^{\infty}\sum_p (\log p \cdot e^{-s\log p^k}) = \sum_{n=2}^{\infty} \Lambda(n) e^{-s\log n}$$

$\square$

Note: $\Lambda : \mathbb{N} \to \mathbb{R}, n \mapsto \begin{cases} \log p & n = p^k, k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$.

Proposition: $\zeta(s)$ has no zeros on $\Re(s) = 1$

Proof Sketch Suppose that $\exists t_0$ such that $\zeta(1 + it_0) = 0$.

Set $F(s) = \zeta^4(s)\zeta^5(s + it_0)\zeta^2(s + 2it_0)$. Since $s = 1$ is the zero of $F(s)$, $F(s)$ is analytic in the neighbourhood of $s = 1$. For any $x \in (1, 1 + \delta)$ where δ is small enough, we define $f(x) := F(s)|_{\mathbb{R}}$. By definition,

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$$\begin{aligned}
\Re\left(\frac{f'(x)}{f(x)}\right) =& \Re\left(4\frac{\zeta'(x)}{\zeta(x)} + 5\frac{\zeta'(x + it_0)}{\zeta(x + it_0)} + 2\frac{\zeta'(x + 2it_0)}{\zeta(x + 2it_0)}\right) \\
=& -\Re\sum_{n\geq 2}\Lambda(n)(4e^{x\log n}5e^{(x+it_0)\log n} + 2e^{(x+it_0)}) \\
=& -\sum_{n\geq 2}\Lambda(n)e^{-x\log n}(4 + 5\cos(t_0\log n) + 2\cos(2t_0\log n)) \\
=& -\sum_{n\geq 2}\Lambda(n)e^{-x\log n}((2\cos(t_0\log n) + 5/4)^2 + 7/16) < 0
\end{aligned}$$

Note: $-\dfrac{\zeta'(s)}{\zeta(s)} = \sum_{n\geq 2}\Lambda(n)e^{-s\log n}$.

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Whereas, $\lim_{x\to 1^+}(x-1)\frac{f'(x)}{f(x)} = 1$ implies that $\left(\frac{f'(x)}{f(x)}\right) \geq 0$!

NOTE: $-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\geq 2}\Lambda(n)e^{-s\log n}$.

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Figure 6: Proof Progress

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Proof.

When $\Re(s) > 1$, $-\dfrac{\zeta'(s)}{\zeta(s)} = \sum_{k=1}^{\infty} \sum p(\log p \cdot e^{-s \log p^k}) = \sum_{k=1}^{\infty} \Phi(ks)$. $\square$

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Since $\Phi(s)$ is analytic when $\Re(s) > 1$, $\Phi(s/2)$ is meromorphic when $\Re(s) > 1$, so is $\Phi(s/4)$, $\Phi(s/8)$, etc. Therefore $\Phi(s)$ is meromorphic when $\Re(s) > 0$, whose poles are all simple at the zeros of $\zeta(s)$. $\square$

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We claim that the only pole of $-\dfrac{\zeta'(s)}{\zeta(s)}$ when $\Re(s) = 1$ is at $s = 1$, since $\zeta(s)$ has neither zeros nor singularities when $\Re(s) = 1$ and $\Im(s) \neq 0$.

Hence $\Phi(s) - \dfrac{1}{s-1}$ is analytic on $\{s : \Re(s) \geq 1\}$ due to the residue of $\zeta(s)$ at $s = 1$ is 1. $\square$

Short Proof of the PNT

Declaration

This proof is simplified from Newman's original work, also refers to a Proof Sketch by Terence Tao. (see references part)

Theorem

Let $f(t)(t \geq 0)$ be a bounded and locally integrable function and suppose that the function

$$g(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t \quad \Re(s) > 0$$

extends analytically to $\Re(s) \geq 0$. Then $\int_0^\infty f(t)\mathrm{d}t$ exists (and valued $g(0)$).

The full proof has been presented in the LECTURE.

Analytic Theorem

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NOTE: $\vartheta(x) = \sum_{p \leq x} \log p$.

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Proof.

For $\Re(s) > 1$, we have

$$\Phi(s) = \sum_p \frac{\log p}{p^s} = s \sum_p \int_p^\infty \frac{\log p}{x^{s+1}} \mathrm{d}x = s \int_1^\infty \frac{\vartheta(x)}{x^{s+1}} \mathrm{d}x = s \int_0^\infty e^{-st} \vartheta(e^t) \mathrm{d}t$$

Set $g(s) := \frac{\Phi(s+1)}{s+1} - \frac{1}{s} = \int_0^\infty e^{-st} \left(\frac{\vartheta(e^t)}{e^t} - 1 \right) \mathrm{d}t$, $f(x) = \frac{\vartheta(e^t)}{e^t} - 1$.

Since

$$\vartheta(x) = \sum_{n=0}^\infty \left(\vartheta\left(\frac{x}{2^n}\right) - \vartheta\left(\frac{x}{2^{n+1}}\right)\right) \le \sum_{n=0}^\infty \left(\log \binom{[x/2^n]}{[x/2^{n+1}]} \right)$$

$$\le \sum_{n=1}^\infty \log 2^{[x/2^n]} \le \sum_{n=1}^\infty \frac{x \log 2}{2^n} \le (2 \log 2) x$$

$f(x)$ is bounded when $x \in [0, \infty)$. Hence $\int_0^\infty \frac{\vartheta(e^t)}{e^t} - 1 \mathrm{d}t = \int_0^\infty \frac{\vartheta(x) - x}{x^2} \mathrm{d}x$ exists.

Theorem

$$\lim_{x\to\infty}\frac{\vartheta(x)}{x}=1$$

Proof.

We claim that $\vartheta(x)\sim x$ when x is large enough. Otherwise, we have $\limsup_{x\to\infty}\frac{\vartheta(x)}{x}>1$ or $\liminf_{x\to\infty}\frac{\vartheta(x)}{x}<1$.

Without the loss of generality, suppose that $\limsup_{x\to\infty}\frac{\vartheta(x)}{x}>1$. Equivalently speaking, there exists $\lambda>1$ and an increasing sequence $\{x_n\}\stackrel{n\to\infty}{\to}\infty$ such that $\frac{\vartheta(x)}{x}>\lambda$. For any $x\in\{x_n\}$, we have

$$\int_x^{\lambda x}\frac{\vartheta(t)-t}{t^2}\mathrm{d}t\geq\int_x^{\lambda x}\frac{\lambda x-t}{t^2}\mathrm{d}t=\int_1^{\lambda}\frac{\lambda-t}{t^2}\mathrm{d}t>0$$

As a result, it contradicts to the existence of $\int_1^{\infty}\frac{\vartheta(t)-t}{t^2}\mathrm{d}t$. $\square$

The Final Theorem (PNT)

$$\lim_{n\to\infty} \frac{\pi(x)}{x} = 1$$

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Proof.

$\forall \varepsilon \in (0,1)$, we have

$$\vartheta(x) \leq \sum_{p\leq x} \log x = \pi(x)\log x$$

and

$$\vartheta(x) \geq \sum_{x^{1-\varepsilon}<p\leq x} \log p \geq (1-\varepsilon)(\pi(x) - \pi(x^{1-\varepsilon}))\log x$$

Therefore $(1-\varepsilon)[\pi(x) - \pi(x^{1-\varepsilon})] \leq \dfrac{\vartheta(x)}{\log x} \leq \pi(x)$.

The PNT follows easily since $\lim\limits_{x\to\infty} \dfrac{\vartheta(x)}{x}$. $\square$

Thanks for Listening!

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