等温参数曲面的复形式

前置结论

等温参数坐标存在性

考虑$\mathrm{d}s^2 = E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2$, 令$z = u + iv$, 则

$$\begin{aligned}\mathrm{d}s^2 =&\left(\sqrt{E}\mathrm{d}u - \frac{F+\sqrt{EG-F^2}}{\sqrt{E}}\mathrm{d}v\right)\left(\sqrt{E}\mathrm{d}u - \frac{F-\sqrt{EG-F^2}}{\sqrt{E}}\mathrm{d}v\right)\\ =&\frac{E+G+2\sqrt{EG-F^2}}{4}\cdot\left|\mathrm{d}z + \frac{E+2iF-G}{E+G+2\sqrt{EG-F^2}}\cdot\mathrm{d}\overline{z}\right|^2\end{aligned}$$

在等温参数$\mathrm{d}s^2 = \lambda^2(\mathrm{d}x^2 + \mathrm{d}y^2)$中令$w = x + iy$得

$$\mathrm{d}s^2 = |\lambda w_z|^2\cdot\left|\mathrm{d}z + \frac{w_{\overline{z}}}{w_z}\mathrm{d}\overline{z}\right|^2.$$

比较得Beltrami方程

$$\frac{E+2iF-G}{E+G+2\sqrt{EG-F^2}}\cdot\frac{\partial w}{\partial z} = \frac{\partial w}{\partial\overline{z}}.$$

其中左侧系数模长不超过1. 据复分析知识, 方程在邻域内有唯一解, 从而正则曲面存在局部等温坐标.

转换映射

设$X(x,y)$与$Y(u,v)$为曲面S上某一邻域的两种参数表示, 则

$$\begin{aligned}\mathrm{d}s^2 =&\lambda^2(\mathrm{d}u^2 + \mathrm{d}v^2)\\ =&\lambda^2[(u_x\mathrm{d}x + u_y\mathrm{d}y)^2 + (v_x\mathrm{d}x + v_y\mathrm{d}y)^2]\\ =&\lambda^2[(u_x^2+v_x^2)\mathrm{d}x^2 + (u_y^2+v_y^2)\mathrm{d}y^2 + 2(u_xu_y + v_xv_y)\mathrm{d}x\mathrm{d}y]\end{aligned}$$

从而转换映射满足Cauchy Riemann方程

$$\begin{aligned}u_x = v_y, u_y = -v_x,\\ \text{or } u_y = v_x, u_x = -v_y.\end{aligned}$$

曲率

Gauss曲率为

$$K=-\frac{1}{\sqrt{EG}}\left[\left(\frac{\sqrt{E}_v}{\sqrt{G}}\right)_v+\left(\frac{\sqrt{G}_u}{\sqrt{E}}\right)_u\right]=-\frac{\Delta\ln\lambda}{\lambda}.$$

平均曲率

$$H=\frac{Eg+eG-2Ff}{2(EG-F^2)}=\frac{e+g}{2\lambda}=\frac{\langle X_{uu}+X_{vv},N\rangle}{2\lambda}.$$

其中注意到$\langle X_{uu}+X_{vv},X_u\rangle=\frac{E_u}{2}-\frac{G_u}{2}=0$, 同理得$X_{uu}+X_{vv}$与$N$平行. 因此

$$2\lambda HN=X_{uu}+X_{vv}.$$

等温参数下曲面

Weierstrass表示

考虑$\varphi(u,v)=(\varphi^1(u,v),\varphi^2(u,v),\varphi^3(u,v))$, 令$z=u+iv$. 当且仅当曲面为等温参数时有

$$\sum_{k=1}^3(\varphi_z^k)^2=\frac{1}{4}\sum_{k=1}^3[(\varphi_u^k)^2+2i\varphi_u^k\varphi_v^k-(\varphi_v^k)^2]=\frac{E+2iF-G}{4}=0.$$

此处曲面正则性要求

$$\frac{E+G}{4}=\sum_{k=1}^3|\varphi_u^k+i\varphi_v^k|^2=\|\varphi_z\|^2\neq 0.$$

等温参数下, 曲面极小若且仅若$H=0$, 即

$$(\varphi_{uu}^1+\varphi_{vv}^1,\varphi_{uu}^2+\varphi_{vv}^2,\varphi_{uu}^3+\varphi_{vv}^3)=0.$$

从而每一φ^k均满足Cauchy Riemann方程, 即调和. 从而$\partial_{\bar{z}}(\varphi_z^k)=0$, 即$\varphi_z^k$全纯.

相反地, 对一切满足$\psi_1^2+\psi_2^2+\psi_3^2=0$的全纯函数$\{\psi_k\}_{k=1}^3$. 若$\psi_k$满足正则性条件(即$\frac{E+G}{4}=|\psi_1|^2+|\psi_2|^2+|\psi_3|^2\neq 0$), 则$\varphi^k=\mathrm{Re}\int\psi_k\mathrm{d}z$, 导出了极小曲面$\varphi(u,x)$.

Weierstrass表示定理给出了一般的$\{\psi_k\}_{k=1}^3$的构造方式. 以下讨论暂时剔除$\psi_3=0$时的显然情形. 令$\omega=\psi_1\mathrm{d}z+i\psi_2\mathrm{d}z$为全纯1-形式, 则

$$
\begin{aligned}
0 = \sum_{k=1}^{3} \psi_k^2 \mathrm{d}z^2 = & (\psi_1 \mathrm{d}z - i\psi_2 \mathrm{d}z)(\psi_1 \mathrm{d}z + i\psi_2 \mathrm{d}z) + \psi_3^2 \mathrm{d}z \\
= & (\psi_1 \mathrm{d}z - i\psi_2 \mathrm{d}z)\omega + g^2\omega^2
\end{aligned}
$$

其中g为某一亚纯函数. 因此$\psi_1 \mathrm{d}z - i\psi_2 \mathrm{d}z = -g^2\omega$. 反解得

$$
\begin{cases}
\psi_1 \mathrm{d}z = & \dfrac{1}{2}(1 - g(z)^2)\omega, \\
\psi_2 \mathrm{d}z = & \dfrac{i}{2}(1 + g(z)^2)\omega, \\
\psi_3 \mathrm{d}z = & g(z)\omega.
\end{cases}
$$

因此

$$
\begin{cases}
\varphi^1 = & \mathrm{Re} \displaystyle\int \frac{1}{2}(1 - g(z)^2)\omega, \\
\varphi^2 = & \mathrm{Re} \displaystyle\int \frac{i}{2}(1 + g(z)^2)\omega, \\
\varphi^3 = & \mathrm{Re} \displaystyle\int g(z)\omega.
\end{cases}
$$

曲面基本形式

第一基本形式为

$$
\mathrm{d}s^2 = 4\left(\sum_{k=1}^{3} |\psi_k|^2\right)(\mathrm{d}u^2 + \mathrm{d}v^2) = (1 + |g|^2)|\omega|^2.
$$

面积元

$$
\mathrm{d}\sigma = \sqrt{EG}\mathrm{d}u \wedge \mathrm{d}v = \frac{E}{-2i}\mathrm{d}z \wedge \mathrm{d}\overline{z} = \frac{i}{2}(1 + |g|^2)|\omega|^2.
$$

法向量即g在球极投影$\pi : S^2 \to \overline{\mathbb{C}}$下的原相, 即

$$
N = \pi^{-1}(g) = \frac{1}{1 + |g|^2}(2\mathrm{Re}(g), 2\mathrm{Im}(g), |g|^2 - 1).
$$

实际上, g即Gauß映射, $g(u + iv)$即$X(u, v)$法向量(的球极投影).

第二基本形式为(可通过与密切平面的距离定义, 过程略)$-2\mathrm{Im}(\omega \mathrm{d}g)$.