Numerical Methods to PDEs

Homework#3

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Problem 1

<u>*Some explanation and formulation of the steps:*</u>

Step 2:

The Jacobian can be written and formulated as:

Jacobian = $\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$, After exploring on the shape functions and coordinates, we have:

Jacobian = $\begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix}$

Step 3:

The gradient of the basis function on the reference triangle can be formulated as:

$\nabla \lambda_r = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} \end{bmatrix}$, We have the relation: $\begin{matrix} \frac{\partial N_1}{\partial \xi} = -1 & \frac{\partial N_2}{\partial \xi} = 1 & \frac{\partial N_3}{\partial \xi} = 0 \\ \frac{\partial N_1}{\partial \eta} = -1 & \frac{\partial N_2}{\partial \eta} = 0 & \frac{\partial N_3}{\partial \eta} = 1 \end{matrix}$

Step 4:

The gradient of the basis function on the element triangle can be formulated as:

$\nabla \Phi_r = J^{-1} \cdot \nabla \lambda_r$ (Since the Jacobian I'm using is the transpose of what we talked about in class, therefore there is no transpose here)

Step 5:

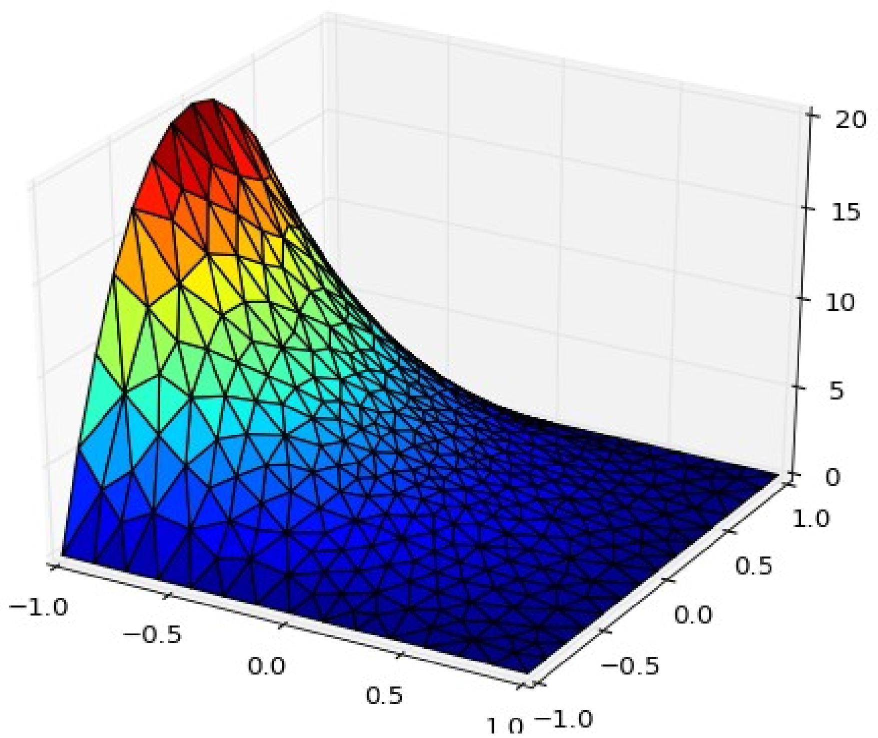
Aelem can be calculated easily with the formulation above . Also , the integration $\int_S \kappa(T(\alpha))\,d\alpha$ is trivial since $\kappa(x)$ is constant value in the three cases (In case 3 we only need to determine if the element lies in the center circle).

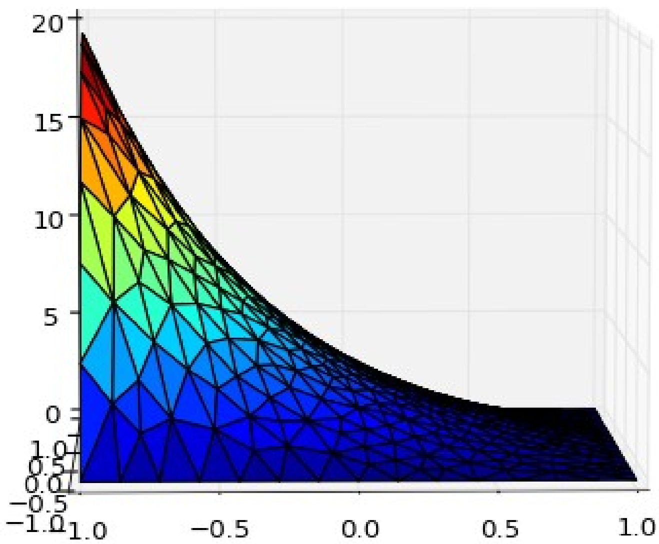
Step 6:

The integration $|J|\int_S f(T(\alpha))\lambda_r(\alpha)d\alpha$ is simply $|J|\,f(T(\alpha))\int_S \lambda_r(\alpha)d\alpha$ since $f(T(\alpha))$ is constant value (Only that in case 2 and 3 we need to determine the location of the element). It is easy to calculate that $\int_S \lambda_r(\alpha)d\alpha = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{bmatrix}$.

<u>The Results are plotted as below:</u>

Case a):




20
15
10
5
0
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
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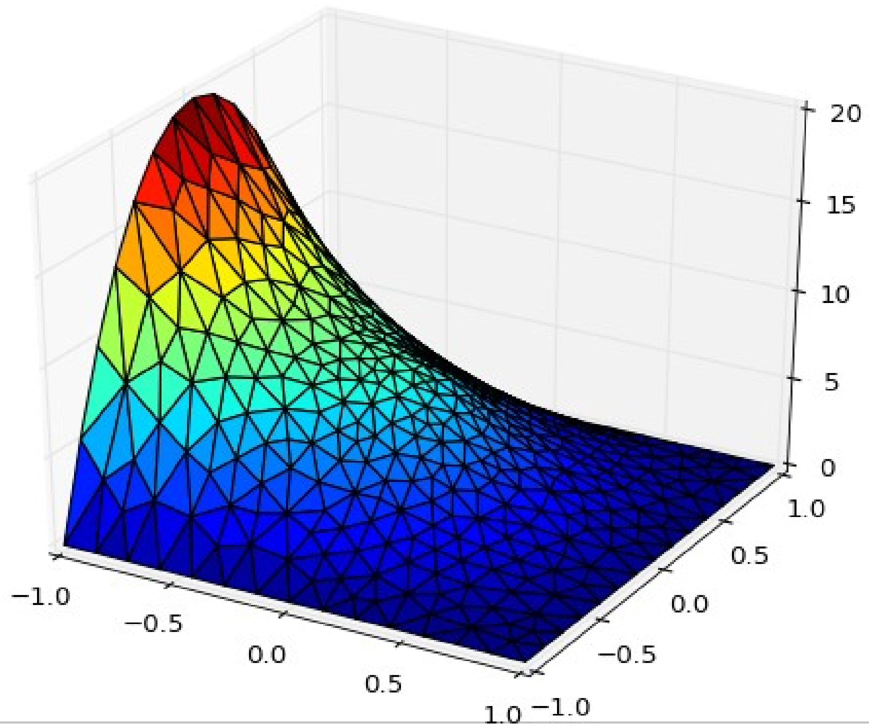
0.5
1.0

Case b):


20
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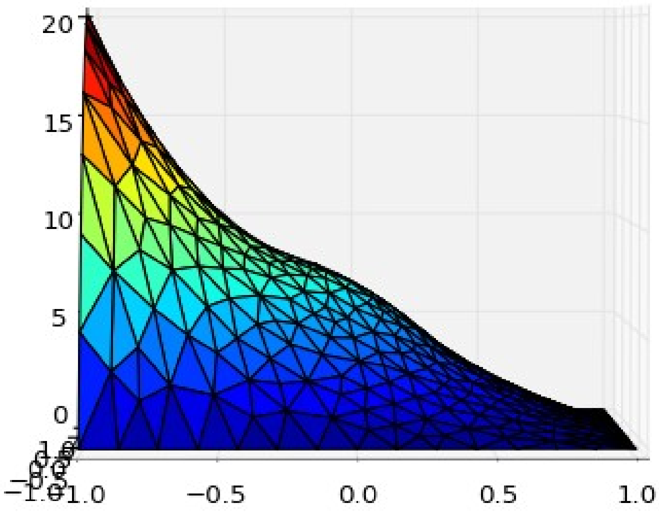
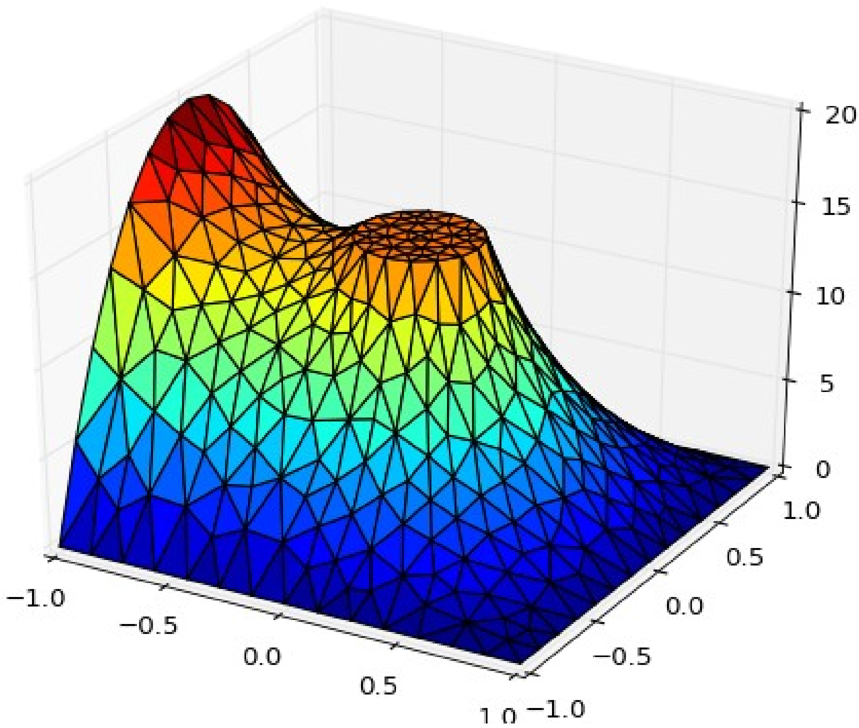
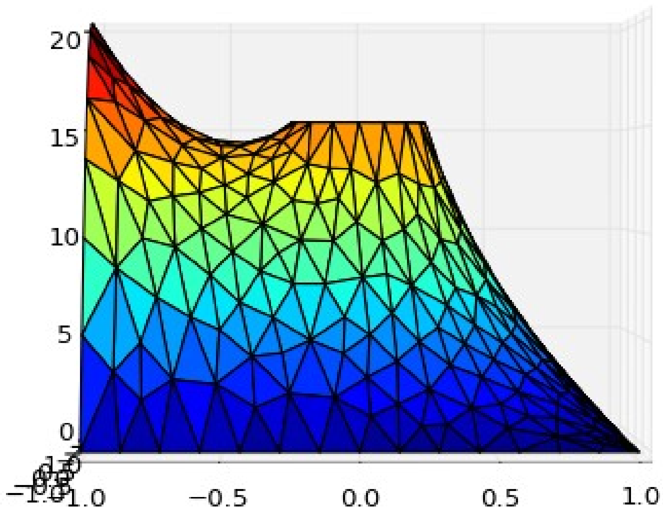
1.0



Case C):





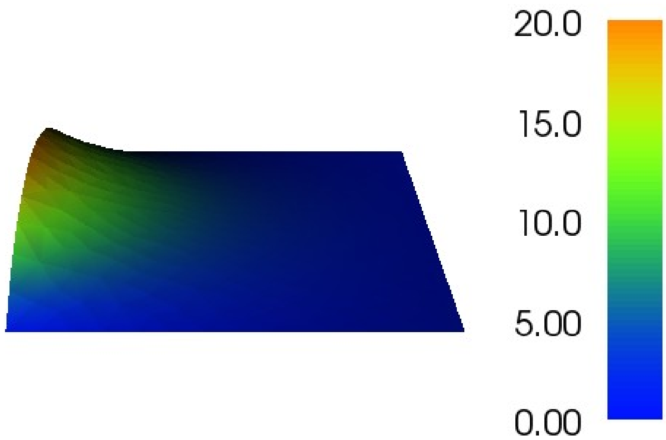
Problem 2

Some explanation and formulation of the steps:

Please refer to the code

The Results are plotted as below:

Case a):



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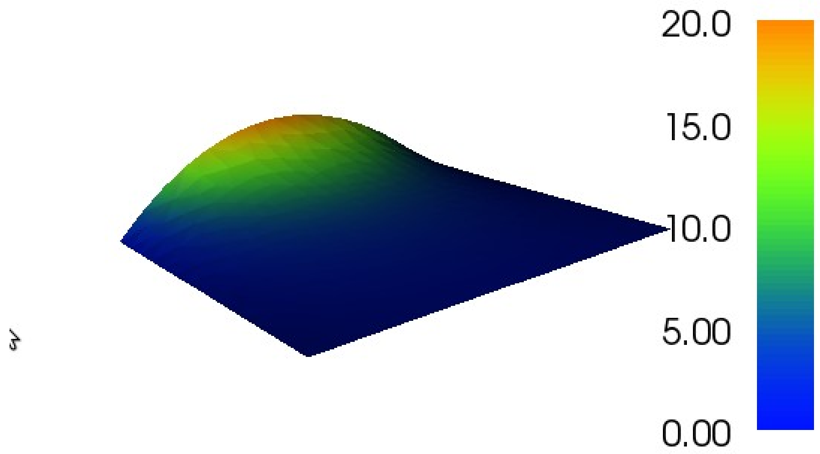
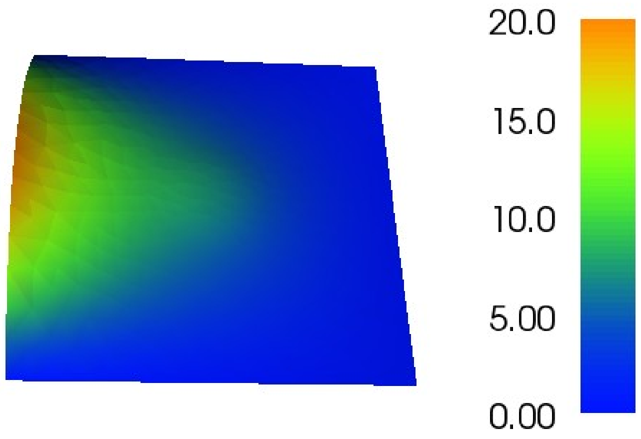
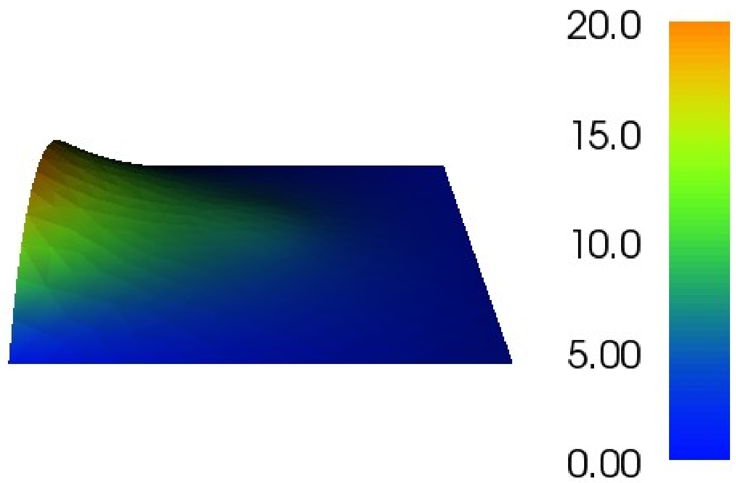


Case b):





Case c):