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Numerical SF and BH

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We provide the manuscript of our calculations, along with a brief introduction to the codes.

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I. MATHEMATICAL MATERIALS

A. Unitary representation of SL$(2,\mathbb{C})$

SL$(2,\mathbb{C})$ is a non-compact group of complex matrices of determine 1, that is

$$\mathrm{SL}(2,\mathbb{C}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a,\ b,\ c,\ d \in \mathbb{C},\ ad - bc = 1 \right\}. \tag{1.1}$$

It has a subgroup of SU(2), of which an element h is a matrix of the form

$$h = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \tag{1.2}$$

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with $^{-}$ representing the complex conjugate. There are 6 generators $\vec{J} = -i\vec{\sigma}/2$ and $\vec{K} = \vec{\sigma}/2$ of SL$(2,\mathbb{C})$ where $\vec{\sigma}$ are the Pauli matrices. The unitary irreducible representations of SL(2,$\mathbb{C}$) are labelled by pairs of numbers (k, p) with $k \in \mathbb{Z}/2$ and $p \in \mathbb{R}$, which are related to the two Casimirs C_1 and C_2 as

$$\begin{aligned} C_1 &= \vec{K}^2 - \vec{J}^2 = k^2 - p^2 - 1 \\ C_2 &= \vec{J} \cdot \vec{K} = kp \end{aligned} \tag{1.3}$$

The Hilbert space of the (k, p)-representation is denoted as $\mathcal{H}_{(k,p)}$. The Hermitian inner product is denoted as (ψ, χ) for vectors $\psi,\ \chi \in \mathcal{H}_{(k,p)}$.

The Hilbert space $\mathcal{H}_{(k,p)}$ comprises functions $\psi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}$ with the homogeneity property

$$\psi(\lambda z) = \lambda^{-1+ip+k}\bar{\lambda}^{-1+ip-k}\psi(z), \ \forall \lambda \in \mathbb{C} \setminus \{0\} \tag{1.4}$$

The action of an element $g \in$SL(2,$\mathbb{C}$) on ψ is

$$(g\psi)(z) = \psi(g^T z). \tag{1.5}$$

where g^T is the transpose matrix of g. There is a standard SL$(2, \mathbb{C})$-invariant 2-form on $\mathbb{C}^2 \setminus \{0\}$

$$\Omega_z = -\frac{i}{2}(z^1 \mathrm{d}z^2 - z^2 \mathrm{d}z^1) \wedge (\bar{z}^1 \mathrm{d}\bar{z}^2 - \bar{z}^2 \mathrm{d}\bar{z}^1). \tag{1.6}$$

For functions ψ and χ of type (1.4), the 2-form $\bar{\psi}\chi\Omega$ is invariant under $(z_1, z_2) =: z \to \lambda z$, $\lambda \in \mathbb{C}^2 \setminus \{0\}$. Thus, under the projection $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}\mathrm{P}^1$, $\pi_*(\bar{\psi}\chi\Omega)$ is well defined. Taking advantage of this property, we define the inner product in $\mathcal{H}_{(k,p)}$ as

$$(\psi, \chi) = \int_{\mathbb{C}\mathrm{P}^1} \pi_*(\bar{\psi}\chi\Omega). \tag{1.7}$$

The unitarity of the representation can be demonstrated due to the invariance of the inner product under SL$(2, \mathbb{C})$ transformation, which in turn is a consequence of the invariance of Ω.

B. SL$(2, \mathbb{C})$ representation in terms of SU(2)

In this section, we define

$$J_a = \sigma_a/2 \tag{1.8}$$

so that

$$[J_a, J_b] = i\epsilon_{abc}J_c. \tag{1.9}$$

To calculate the integral (5.24), let us choose the coordinate $(r, \alpha, \beta, \gamma)$ such that

$$z(r, \alpha, \beta, \gamma) = \left(re^{\frac{i\alpha}{2} - \frac{i\gamma}{2}} \sin\frac{\beta}{2}, re^{\frac{i\alpha}{2} + \frac{i\gamma}{2}} \cos\frac{\beta}{2} \right), \ 0 \le \alpha \le 2\pi, \quad 0 \le \beta \le \pi, \ 0 \le \gamma \le 2\pi, \ r > 0, \tag{1.10}$$

which gives us that

$$\Omega = \frac{r^4}{4} \sin\beta \mathrm{d}\beta \wedge \mathrm{d}\gamma. \tag{1.11}$$

According to (1.4), we have

$$\psi(z(r, \alpha, \beta, \gamma)) = r^{-2+2ip}e^{-ik\alpha}\psi(z(1, 0, \beta, \gamma)). \tag{1.12}$$

Thus, one has

$$(\bar{\psi}\chi\Omega)(z(r, \alpha, \beta, \gamma)) = \frac{1}{4}\overline{\psi(z(1, 0, \beta, \gamma))}\chi(z(1, 0, \beta, \gamma)) \sin\beta \mathrm{d}\beta \wedge \mathrm{d}\gamma. \tag{1.13}$$

which implies

$$(\psi, \chi) = \frac{1}{4}\int_{\mathbb{C}\mathrm{P}^1} \overline{\psi(z(1, 0, \beta, \gamma))}\chi(z(1, 0, \beta, \gamma)) \sin\beta \mathrm{d}\beta \mathrm{d}\gamma \tag{1.14}$$

Since SU(2) is a subgroup of SL$(2, \mathbb{C})$, the above representation leads to a unitary representation of SU(2). Thus, the Hilbert space $\mathcal{H}_{(k,p)}$ can be decomposed as the direct sum of $\mathcal{H}_j$, the j-representation space of SU(2).

Recall that SU(2) is the submanifold of $\mathbb{C}^2$ comprising the points z satisfying $\langle z, z\rangle = 1$, where the Hermitian inner product on $\mathbb{C}^2$ is defined by

$$\langle z, w\rangle = \bar{z}^1\omega^1 + \bar{z}^2\omega^2. \tag{1.15}$$

The embedding $\iota : \mathrm{SU}(2) \to \mathbb{C}^2$ results in a map $\iota^* : \mathcal{H}_{(k,p)} \to \mathrm{Fun}(\mathrm{SU}(2))$, with $\mathrm{Fun}(\mathrm{SU}(2))$ denoting the space of functions on SU(2). Due to the homogeneity (1.4), the map ι^* is one-to-one. Actually, one has

$$\psi(z) = \langle z, z\rangle^{-1+ip}(\iota^*\psi)(u(z)). \tag{1.16}$$

where we arrange the components z^1, z^2 into a matrix $u(z)$ as

$$u(z) = \frac{1}{\sqrt{\langle z, z\rangle}}\begin{pmatrix} \overline{z^2} & -\overline{z^1} \\ z^1 & z^2 \end{pmatrix} = \frac{1}{\sqrt{\langle z, z\rangle}}\begin{pmatrix} z^\dagger\epsilon \\ z^T \end{pmatrix} \tag{1.17}$$

where

$$\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.18}$$

It can be check that

$$u(e^{i\lambda/2}z) = e^{-i\lambda\sigma_3/2}u(z), \tag{1.19}$$

which implies

$$\iota^*\psi(e^{-i\lambda\sigma_3/2}u(z)) = e^{ik\lambda}\iota^*\psi(u(z)),\ \forall\psi \in \mathcal{H}_{(k,p)}. \tag{1.20}$$

We thus gets that $\iota^*\psi$ must take

$$\iota^*\psi = \sum_{j,m}\psi_{j,m}D^j_{km}, \tag{1.21}$$

i.e.,

$$\psi(z) = \langle z, z\rangle^{-1+ip}\sum_{j,m}\psi_{j,m}D^j_{km}(u(z)) \tag{1.22}$$

Under the coordinate (1.10), we have

$$u(z(1,\alpha,\beta,\gamma)) = \begin{pmatrix} e^{-\frac{i\alpha}{2}-\frac{i\gamma}{2}}\cos\left(\frac{\beta}{2}\right) & -e^{\frac{i\alpha}{2}-\frac{i\gamma}{2}}\sin\left(\frac{\beta}{2}\right) \\ e^{-\frac{i\alpha}{2}+\frac{i\gamma}{2}}\sin\left(\frac{\beta}{2}\right) & e^{\frac{i\alpha}{2}+\frac{i\gamma}{2}}\cos\left(\frac{\beta}{2}\right) \end{pmatrix} = e^{-i\alpha J_3}e^{-i\beta J_2}e^{-i\gamma J_3}. \tag{1.23}$$

Thus, the Haar measure is

$$\mathrm{d}\mu_H = \frac{1}{8\pi^2}\sin\beta\mathrm{d}\alpha\mathrm{d}\beta\mathrm{d}\gamma \tag{1.24}$$

We thus have

$$\frac{1}{2j+1}\delta_{jj'}\delta_{mm'} = \int\overline{D^j_{km}(u(z))}D^{j'}_{km'}(u(z))\mathrm{d}\mu_H = \frac{1}{4\pi}\int_{\mathbb{C}\mathrm{P}^1}\overline{D^j_{km}(u(z))}D^{j'}_{km'}(u(z))\sin\beta\mathrm{d}\beta\mathrm{d}\gamma \tag{1.25}$$

where we used that $\overline{D^j_{km}(u(z))}D^{j'}_{km'}(u(z))$ is independent of α. We thus have

$$\mathcal{H}_{(k,p)} = \bigoplus_{j-|k|\in\mathbb{Z}_{\geq 0}}\mathcal{H}_j. \tag{1.26}$$

An orthonormal basis of $\mathcal{H}_{(k,p)}$, denoted by $\psi_{k,p,j,m}$ are defined by

$$\psi_{k,p,j,m}(z) = \sqrt{\frac{2j+1}{\pi}}\langle z, z\rangle^{-1+ip}D^j_{km}(u(z)). \tag{1.27}$$

Considering the action of g on $\psi_{k,p,j,m}$, we have

$$g\psi_{k,p,j,m}(z) = \psi_{k,p,j,m}(g^Tz) = \sqrt{\frac{2j+1}{\pi}}\langle g^Tz, g^Tz\rangle^{-1+ip}D^j_{km}(u(g^Tz)) \tag{1.28}$$

C. $\mathrm{SL}(2,\mathbb{C})$ representation in terms of SU(1,1)

Given $z=(z^1,z^2)$, we can introduce a coordinate adapted to SU(1,1) as follows. First, we write the z^1 and z^2 in term of its radial and argument as

$$z^1=|z^1|e^{-\frac{i\alpha}{2}-\frac{i\gamma}{2}},\quad z^2=|z^2|e^{-\frac{i\alpha}{2}+\frac{i\gamma}{2}},\tag{1.29}$$

so that , for fixed α and β, $z=(z^1,z^2)$ is represented by a point $(|z^1|,|z^2|)$ in $\mathbb{R}^{+2}$. Then we foliate $\mathbb{R}^{+2}$ by hyperbolas. That is to say, we introduce $\beta>0$ and $x=|z^1|^2-|z^2|^2\in\mathbb{R}$ such that

$$|z^1|=\sqrt{x}\cosh\left(\frac{\beta}{2}\right),\quad |z^2|=\sqrt{x}\sinh\left(\frac{\beta}{2}\right),\forall|z^1|>|z^2|,\tag{1.30}$$

and

$$|z^2|=\sqrt{-x}\cosh\left(\frac{\beta}{2}\right),\quad |z^1|=\sqrt{-x}\sinh\left(\frac{\beta}{2}\right),\forall|z^1|<|z^2|.\tag{1.31}$$

The above procedure introduce the new coordinate (x,α,β,γ) on $\mathbb{C}^2$ such that

- in the coordinate chart U^+ with $|z^1|>|z^2|$, the coordinate are given by

$$z=\left(\sqrt{x}\cosh\left(\frac{\beta}{2}\right)e^{-\frac{i\alpha}{2}-\frac{i\gamma}{2}},\sqrt{x}\sinh\left(\frac{\beta}{2}\right)e^{-\frac{i\alpha}{2}+\frac{i\gamma}{2}}\right),\tag{1.32}$$

- in the coordinate chart U^- with $|z^1|<|z^2|$, the coordinate are given by

$$z=\left(\sqrt{-x}\sinh\left(\frac{\beta}{2}\right)e^{-\frac{i\alpha}{2}-\frac{i\gamma}{2}},\sqrt{-x}\cosh\left(\frac{\beta}{2}\right)e^{-\frac{i\alpha}{2}+\frac{i\gamma}{2}}\right).\tag{1.33}$$

Under this coordinate, the two form given in (1.6) reads

$$\Omega_z=\pm\frac{1}{4}x^2\sinh(\beta)\mathrm{d}\beta\wedge\mathrm{d}\gamma,\forall z\in U^{\pm}.\tag{1.34}$$

According to (1.4), for $z\in U^+$, we have

$$\psi(z(x,\alpha,\beta,\gamma))=x^{-1+ip}e^{-ik\alpha}\psi(z(1,0,\beta,\gamma))\tag{1.35}$$

and for $z\in U^-$, we have

$$\psi(z(x,\alpha,\beta,\gamma))=(-x)^{-1+ip}e^{-ik\alpha}\psi(z(-1,0,\beta,\gamma)).\tag{1.36}$$

Then, we have

$$\begin{aligned}(\overline{\psi}\chi\Omega)(z)=&\frac{1}{4}\overline{\psi(z(1,0,\beta,\gamma))}\chi(z(1,0,\beta,\gamma))\sinh(\beta)\mathrm{d}\beta\wedge\mathrm{d}\gamma,\forall z\in U^+\\(\overline{\psi}\chi\Omega)(z)=&-\frac{1}{4}\overline{\psi(z(-1,0,\beta,\gamma))}\chi(z(-1,0,\beta,\gamma))\sinh(\beta)\mathrm{d}\beta\wedge\mathrm{d}\gamma,\forall z\in U^-\end{aligned}\tag{1.37}$$

Then, we have

$$\begin{aligned}(\psi,\chi)=&\int_{|z^1|^2-|z^2|^2=1}\overline{\psi}\chi\Omega-\int_{|z^1|^2-|z^2|^2=-1}\overline{\psi}\chi\Omega\\=&\frac{1}{4}\int\overline{\psi(z(1,0,\beta,\gamma))}\chi(z(1,0,\beta,\gamma))\sinh(\beta)\mathrm{d}\beta\mathrm{d}\gamma+\frac{1}{4}\int\overline{\psi(z(-1,0,\beta,\gamma))}\chi(z(-1,0,\beta,\gamma))\sinh(\beta)\mathrm{d}\beta\mathrm{d}\gamma\end{aligned}\tag{1.38}$$

where the minus sign in the first line comes from the fact that the reduced orientation of $|z^1|^2-|z^2|^2=-1$ is $-\mathrm{d}\beta$ (some details on the orientation).

Given an SU(1,1) element

$$u(z^1, z^2) = \begin{pmatrix} z^1 & \overline{z^2} \\ z^2 & \overline{z^1} \end{pmatrix} = (z, \epsilon\sigma_3\overline{z}) \tag{1.39}$$

we have $\det\big(u(z^1, z^2)\big) = |z^1|^2 - |z^2|^2 = 1$. There are two possible ways to embed $u(z)$ into $\mathbb{C}^2$

$$\Pi_+ : u(z) \mapsto z,\ \ \Pi_- : u(z) \mapsto \epsilon\sigma_3\overline{z}. \tag{1.40}$$

The images $\Pi_\pm(\mathrm{SU}(1,1))$ are two disconnected hypersurface in $\mathbb{C}^2$, denoted by $\mathbb{H}_\pm$ respectively. The restriction of $\Psi \in \mathcal{H}_{(p,k)}$ gives an elements in $\mathrm{Fun}(\mathrm{SU}(1,1)) \oplus \mathrm{Fun}(\mathrm{SU}(1,1))$. This correspondence from $\Psi \in \mathcal{H}_{(p,k)}$ to $\mathrm{Fun}(\mathrm{SU}(1,1)) \oplus \mathrm{Fun}(\mathrm{SU}(1,1))$ is one-to-one.

1. *the unitary representation of SU(1,1)*

An SU(1,1) element is given by the exponential of the Lie algebra elements which are the real linear combination of

$$j_3 = i\sigma_3/2,\ \ k_1 = \sigma_1/2,\ \ k_2 = \sigma_2/2, \tag{1.41}$$

i.e.,

$$\exp(\alpha k_1 + \beta k_2 + \gamma j_3). \tag{1.42}$$

The unitary representation of SU(1,1) makes

$$\exp(\alpha k_1 + \beta k_2 + \gamma j_3)^\dagger = \exp(-\alpha k_1 - \beta k_2 - \gamma j_3), \tag{1.43}$$

which leads to

$$k_1{}^\dagger = -k_1,\ \ k_2{}^\dagger = -k_2,\ \ j^3{}_\dagger = -j_3. \tag{1.44}$$

We thus have the unitary operators

$$\hat{\mathbf{K}}_1 = -ik_1 = -i\sigma_1/2,\ \ \hat{\mathbf{K}}_2 = -ik_2 = -i\sigma_2/2,\ \ \hat{\mathbf{K}}_3 = -ij^3 = \sigma_3/2 \tag{1.45}$$

in the unitary representation space, satisfying the commutation relations

$$[\hat{\mathbf{K}}_1, \hat{\mathbf{K}}_2] = -i\hat{\mathbf{K}}_3, \quad [\hat{\mathbf{K}}_3, \hat{\mathbf{K}}_1] = i\hat{\mathbf{K}}_2, \quad [\hat{\mathbf{K}}_2, \hat{\mathbf{K}}_3] = i\hat{\mathbf{K}}^1 \tag{1.46}$$

With $\hat{\mathbf{K}}^a$, we have the Casimir operator

$$I = -(\hat{\mathbf{K}}_1)^2 - (\hat{\mathbf{K}}_2)^2 + (\hat{\mathbf{K}}_3)^2, \tag{1.47}$$

and the raising and lowering operators

$$\hat{\mathbf{K}}_\pm = \pm i(\hat{\mathbf{K}}_1 \pm i\hat{\mathbf{K}}_2). \tag{1.48}$$

Then, we have

$$[\hat{\mathbf{K}}_\pm, \hat{\mathbf{K}}_3] = \mp\hat{\mathbf{K}}_\pm \tag{1.49}$$

i.e.

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$$\hat{\mathbf{K}}_3\hat{\mathbf{K}}_\pm = \hat{\mathbf{K}}_\pm\hat{\mathbf{K}}_3 \pm \hat{\mathbf{K}}_\pm. \tag{1.50}$$

Let $|m\rangle$ be the eigenstate of the Casimir operator and $\hat{\mathbf{K}}_3$ with

$$\hat{\mathbf{K}}_3|m\rangle = m|m\rangle.$$

m must be integer or half integer because

$$e^{i\alpha\hat{\mathbf{K}}_3} = e^{i(\alpha+4\pi)\hat{\mathbf{K}}_3}. \tag{1.51}$$

In addition, (1.50) leads to

$$\begin{aligned}\hat{\mathbf{K}}_3\hat{\mathbf{K}}_-|jm\rangle &= (m-1)\hat{\mathbf{K}}_-|jm\rangle,\\ \hat{\mathbf{K}}_3\hat{\mathbf{K}}_+|jm\rangle &= (m+1)\hat{\mathbf{K}}_+|jm\rangle.\end{aligned} \tag{1.52}$$

Moreover, the operators obey

$$\begin{aligned}\hat{\mathbf{K}}_+\hat{\mathbf{K}}_- &= -I + \hat{\mathbf{K}}_3(\hat{\mathbf{K}}_3 - 1) \geq 0,\\ \hat{\mathbf{K}}_-\hat{\mathbf{K}}_+ &= -I + \hat{\mathbf{K}}_3(\hat{\mathbf{K}}_3 + 1) \geq 0.\end{aligned} \tag{1.53}$$

This inequality could always be satisfied for m whose absolute value $|m|$ is sufficiently large.

- Let us fix a value of $m > 0$ satisfying this inequality. The infinite chain of states

$$|m\rangle,\ \hat{\mathbf{K}}_+|jm\rangle, \cdots \tag{1.54}$$

all exist and are different from zero since

$$\|\hat{\mathbf{K}}_+|m\rangle\|^2 = \langle m|\hat{\mathbf{K}}_-\hat{\mathbf{K}}_+|m\rangle = 2m\||m\rangle\|^2 + \|\hat{\mathbf{K}}_-|m\rangle\| > 0. \tag{1.55}$$

Applying the lowering operator $\hat{\mathbf{K}}_-$ repeatedly to $|m\rangle$, one of the alternatives occurs:

(1) the chain terminates at m_o, i.e.,

$$\hat{\mathbf{K}}_-|m_o\rangle = 0 \tag{1.56}$$

which leads to

$$0 = \hat{\mathbf{K}}_+\hat{\mathbf{K}}_-|m_o\rangle = (-I + m_o(m_o-1))|m_o\rangle \Rightarrow I = m_o(m_o-1). \tag{1.57}$$

(2) the chain never terminates. Then, according to (1.53),

$$I \leq \min(\hat{\mathbf{K}}_3(\hat{\mathbf{K}}_3+1), \hat{\mathbf{K}}_3(\hat{\mathbf{K}}_3-1)) = \begin{cases} 0, & \text{for } m \text{ being integers},\\ -\dfrac{1}{4}, & \text{for } m \text{ being half integers}.\end{cases} \tag{1.58}$$

- For $m < 0$ with $|m|$ being sufficiently large, we have that the chain

$$\hat{\mathbf{K}}_-|m\rangle, \hat{\mathbf{K}}_-^2|m\rangle, \cdots \tag{1.59}$$

all exist. Then, once we apply $\hat{\mathbf{K}}_+$ to act on $|m\rangle$ repeatedly, we have

(1) the chain never terminates. This case is equivalent to the case (2) for $m > 0$, i.e., the case discussed in (1.58).

(2) the chain terminates at some point $-m_o$, i.e.,

$$\hat{\mathbf{K}}_+|-m_o\rangle = 0, \tag{1.60}$$

which leads to

$$\hat{\mathbf{K}}_-\hat{\mathbf{K}}_+|-m_o\rangle = -I|-m_o\rangle - m_o(-m_o+1)|-m_o\rangle \Rightarrow I = m_o(m_o-1). \tag{1.61}$$

In what follows, we will not be interested in the value of I with $-\frac{1}{4} \leq I \leq 0$, i.e., the values that the Casimir takes and we are interested in are $j(j-1)$ with

$$j \in \{k/2, k \in \mathbb{N}/2\} \cup \{\frac{1}{2} - is,\ 0 < s < \infty\}.$$

In the Hilbert space $\mathcal{H}_j$, a basis is given by the eigenstate of K^3. We have

(1) In $\mathcal{H}_j$ for $j = k/2$, the possible values of m are $m = j, j+1, j+2, \cdots$ and $m = -j, -j-1, -j-2, \cdots$, i.e., the identity operator in the Hilbert space is

$$\mathbb{1}_{\mathcal{H}_j} = \sum_{m \le -j} |jm\rangle\langle jm| + \sum_{m \ge j} |jm\rangle\langle jm|. \tag{1.62}$$

We will use $\mathcal{D}_j^{\pm}$ to denote the following subspace of $\mathcal{H}_j$

$$\mathcal{D}_j^{+} = \text{span}\{|jm\rangle, m \ge -j\}, \quad \mathcal{D}_j^{-} = \text{span}\{|jm\rangle, m \le j\}. \tag{1.63}$$

(2) In $\mathcal{H}_j$ for $j = 1/2 - is$, the possible values of m are $m = 0, \pm 1, \pm 2, \cdots$ and $m = \pm 1/2, \pm 3/2, \pm 2, \cdots$, i.e., the identity operator in the Hilbert space is

$$\mathbb{1}_{\mathcal{H}_j} = \sum_{m \in \mathbb{Z}} |jm\rangle\langle jm| + \sum_{m \in \mathbb{Z}/2, m \ne 0} |jm\rangle\langle jm|. \tag{1.64}$$

We will use $\mathcal{C}_s^{\epsilon}$ with $\epsilon = 1/2$ to designates these two possibilities. Since

$$\hat{\mathbf{K}}_+ |j, -j\rangle = 0 = \hat{\mathbf{K}}_- |j, j\rangle, \tag{1.65}$$

$\mathcal{D}^{\pm}$ are irreducible representation spaces of SU(1,1).

To relate $|jm\rangle$ with $|jj\rangle$ or $|j - j\rangle$, let us consider $[\hat{\mathbf{K}}_-, \hat{\mathbf{K}}_3^n]$. Taking advantage of $[\hat{\mathbf{K}}_-, \hat{\mathbf{K}}_3] = \hat{\mathbf{K}}_-$, we get

$$[\hat{\mathbf{K}}_-, \hat{\mathbf{K}}_3^n] = P_{n-1}(\hat{\mathbf{K}}_3)\hat{\mathbf{K}}_- \tag{1.66}$$

with P_{n-1} denoting some polynomial with degree $n-1$. Then, we have

$$\begin{aligned} P_{n-1}(\hat{\mathbf{K}}_3)\hat{\mathbf{K}}_- =& [\hat{\mathbf{K}}_-, \hat{\mathbf{K}}_3^n] = \hat{\mathbf{K}}_3^{n-1}[\hat{\mathbf{K}}_-, \hat{\mathbf{K}}_3] + [\hat{\mathbf{K}}_-, \hat{\mathbf{K}}_3^{n-1}]\hat{\mathbf{K}}_3 \\ =& \hat{\mathbf{K}}_3^{n-1}\hat{\mathbf{K}}_- + P_{n-2}(\hat{\mathbf{K}}_3)\hat{\mathbf{K}}_-\hat{\mathbf{K}}_3 \\ =& \hat{\mathbf{K}}_3^{n-1}\hat{\mathbf{K}}_- + P_{n-2}(\hat{\mathbf{K}}_3)\left([\hat{\mathbf{K}}_-, \hat{\mathbf{K}}_3] + \hat{\mathbf{K}}_3\hat{\mathbf{K}}_-\right) \\ =& \left(\hat{\mathbf{K}}_3^{n-1} + P_{n-2}(\hat{\mathbf{K}}_3)(1 + \hat{\mathbf{K}}_3)\right)\hat{\mathbf{K}}_- \end{aligned} \tag{1.67}$$

which leads to

$$P_{n-1}(\hat{\mathbf{K}}_3) = \hat{\mathbf{K}}_3^{n-1} + P_{n-2}(\hat{\mathbf{K}}_3)(1 + \hat{\mathbf{K}}_3). \tag{1.68}$$

Together with $P_0(\hat{\mathbf{K}}_3) = 1$, we get

$$P_{n-1}(\hat{\mathbf{K}}_3) = (\hat{\mathbf{K}}_3 + 1)^n - \hat{\mathbf{K}}_3^n, \tag{1.69}$$

i.e.,

$$\begin{aligned} &\hat{\mathbf{K}}_-\hat{\mathbf{K}}_3^n - \hat{\mathbf{K}}_3^n\hat{\mathbf{K}}_1 = (\hat{\mathbf{K}}_3 + 1)^n\hat{\mathbf{K}}_- - \hat{\mathbf{K}}_3^n\hat{\mathbf{K}}_- \\ \Rightarrow &\hat{\mathbf{K}}_-\hat{\mathbf{K}}_3^n = (\hat{\mathbf{K}}_3 + 1)^n\hat{\mathbf{K}}_-, \end{aligned} \tag{1.70}$$

which leads to

$$\hat{\mathbf{K}}_3^n\hat{\mathbf{K}}_- = \hat{\mathbf{K}}_-(\hat{\mathbf{K}}_3 - 1)^n. \tag{1.71}$$

Then, we have

$$\begin{aligned} \hat{\mathbf{K}}_-^n\hat{\mathbf{K}}_+^n =& \hat{\mathbf{K}}_-^{n-1}\left(-I + (\hat{\mathbf{K}}_3 + 1)\hat{\mathbf{K}}_3\right)\hat{\mathbf{K}}_+^{n-1} \\ =& \hat{\mathbf{K}}_-^{n-2}\left(-I + (\hat{\mathbf{K}}_3 + 2)(\hat{\mathbf{K}}_3 + 1)\right)\hat{\mathbf{K}}_-\hat{\mathbf{K}}_+^{n-1} \\ =& \hat{\mathbf{K}}_-^{n-2}\left(-I + (\hat{\mathbf{K}}_3 + 2)(\hat{\mathbf{K}}_3 + 1)\right)\left(-I + (\hat{\mathbf{K}}_3 + 1)(\hat{\mathbf{K}}_3 + 0)\right)\hat{\mathbf{K}}_+^{n-2} \\ =& \hat{\mathbf{K}}_-^{n-3}\left(-I + (\hat{\mathbf{K}}_3 + 3)(\hat{\mathbf{K}}_3 + 2)\right)\left(-I + (\hat{\mathbf{K}}_3 + 2)(\hat{\mathbf{K}}_3 + 1)\right)\left(-I + (\hat{\mathbf{K}}_3 + 1)(\hat{\mathbf{K}}_3 + 0)\right)\hat{\mathbf{K}}_+^{n-3} \\ =& \cdots \\ =& \prod_{m=1}^{n}(-I + (\hat{\mathbf{K}}_3 + m)(\hat{\mathbf{K}}_3 + m - 1)). \end{aligned} \tag{1.72}$$

and

$$\begin{aligned}\hat{\mathbf{K}}_+^n\hat{\mathbf{K}}_-^n =&\hat{\mathbf{K}}_+^{n-1}\left(-I+\hat{\mathbf{K}}_3(\hat{\mathbf{K}}_3-1)\right)\hat{\mathbf{K}}_-^{n-1}\\ =&\hat{\mathbf{K}}_+^{n-1}\hat{\mathbf{K}}_1\left(-I+(\hat{\mathbf{K}}_3-1)(\hat{\mathbf{K}}_3-2)\right)\hat{\mathbf{K}}_-^{n-2}\\ =&\hat{\mathbf{K}}_-^{n-2}\left(-I+\hat{\mathbf{K}}_3(\hat{\mathbf{K}}_3-1)\right)\left(-I+(\hat{\mathbf{K}}_3-1)(\hat{\mathbf{K}}_3-2)\right)\hat{\mathbf{K}}_+^{n-2}\\ =&\hat{\mathbf{K}}_-^{n-3}\left(-I+\hat{\mathbf{K}}_3(\hat{\mathbf{K}}_3-1)\right)\left(-I+(\hat{\mathbf{K}}_3-1)(\hat{\mathbf{K}}_3-2)\right)\left(-I+(\hat{\mathbf{K}}_3-2)(\hat{\mathbf{K}}_3-3)\right)\hat{\mathbf{K}}_+^{n-3}\\ =&\cdots\\ =&\prod_{m=1}^{n}(-I+(\hat{\mathbf{K}}_3-m+1)(\hat{\mathbf{K}}_3-m)).\end{aligned} \tag{1.73}$$

Therefore, we have

$$\begin{aligned}&\langle jj|\left(\hat{\mathbf{K}}_+^n\right)^\dagger\hat{\mathbf{K}}_+^n|jj\rangle\\ =&\langle jj|\hat{\mathbf{K}}_-^n\hat{\mathbf{K}}_+^n|jj\rangle=\prod_{m=1}^{n}(-j(j-1)+(j+m)(j+m-1))\\ =&\prod_{m=1}^{n}m(2j-1+m)=\frac{n!(2j-1+n)!}{(2j-1)!}=\frac{n!\Gamma(2j+n)}{\Gamma(2j)}.\end{aligned} \tag{1.74}$$

and

$$\begin{aligned}&\langle j,-j|\left(\hat{\mathbf{K}}_-^n\right)^\dagger\hat{\mathbf{K}}_-^n|j,-j\rangle\\ =&\langle j,-j|\hat{\mathbf{K}}_+^n\hat{\mathbf{K}}_-^n|j,-j\rangle=\prod_{m=1}^{n}(-j(j-1)+(-j-m+1)(-j-m))\\ =&\prod_{m=1}^{n}m(2j-1+m)=\frac{n!(2j-1+n)!}{(2j-1)!}=\frac{n!\Gamma(2j+n)}{\Gamma(2j)}.\end{aligned} \tag{1.75}$$

We thus have

$$|j,j+n\rangle=\sqrt{\frac{n!\Gamma(2j+n)}{\Gamma(2j)}}^{-1}\hat{\mathbf{K}}_+^n|jj\rangle,\ \text{and}\ |j,-j-n\rangle=\sqrt{\frac{n!\Gamma(2j+n)}{\Gamma(2j)}}^{-1}\hat{\mathbf{K}}_-^n|j,-j\rangle \tag{1.76}$$

2. *decompose $\mathcal{H}_{(k,p)}$ by SU(1,1) representation spaces*

With the representation Hilbert space $\mathcal{H}_j$, a function on SU(1,1) is given by

$$\psi_{jmn}(u)=\langle jm|u|jn\rangle. \tag{1.77}$$

An element in $\mathrm{Fun}(\mathrm{SU}(1,1))\oplus\mathrm{Fun}(\mathrm{SU}(1,1))$ takes the linear combination of

$$(\psi_{j_+m_+n_+},\psi_{j_-,m_-,n_-}).$$

We now want to find those $(\psi_{j_+m_+n_+},\psi_{j_-,m_-,n_-})$ that take the form

$$(\psi_{j_+m_+n_+},\psi_{j_-,m_-,n_-})=(\Pi_+\psi,\Pi_-\psi),\forall\psi\in\mathcal{H}_{(k,p)}. \tag{1.78}$$

Considering $\psi\in\mathcal{H}_{(p,k)}$ we have

$$\psi(e^{i\alpha/2}z)=e^{ik\alpha}\psi(z). \tag{1.79}$$

This property leads to the restriction of ψ on $\mathbb{H}_\pm \cong$SU(1,1) satisfying

$$(\Pi_+\psi)(u(z)e^{i\alpha\sigma_3/2}) = e^{ik\alpha}\psi(u(z)),\ \forall z \in \mathbb{H}_+, \tag{1.80}$$

and

$$(\Pi_-\psi)(u(z)e^{i\alpha\sigma_3/2}) = e^{-ik\alpha}\psi(u(z)),\ \forall z \in \mathbb{H}_-, \tag{1.81}$$

where we define

$$u(z) = \begin{cases} \dfrac{1}{\sqrt{\langle z,z\rangle_\sigma}}\begin{pmatrix} z^1 & \overline{z^2} \\ z^2 & \overline{z^1}\end{pmatrix} = \dfrac{1}{\sqrt{\langle z,z\rangle_\sigma}}\left(z,\epsilon\sigma_3\overline{z}\right), \forall z\in U^+, \\ \dfrac{1}{\sqrt{-\langle z,z\rangle_\sigma}}\begin{pmatrix} \overline{z^2} & z^1 \\ \overline{z^1} & z^2\end{pmatrix} = \dfrac{1}{\sqrt{-\langle z,z\rangle_\sigma}}\left(\epsilon\sigma_3 z, z\right), \forall z\in U_-. \end{cases} \tag{1.82}$$

Conversely, given two functions $\psi_\pm$ on SU(1,1) such that

$$\psi_\pm(ue^{i\alpha\sigma_3/2}) = e^{\pm i\alpha}\psi(u) \tag{1.83}$$

we can construct an element in $\psi \in \mathcal{H}_{(p,k)}$ uniquely such that

$$(\Pi_\pm\psi)(z) = \psi_\pm(u(z)). \tag{1.84}$$

Therefore, each $(\Pi_+\psi, \Pi_-\psi)$ for some $\psi\in\mathcal{H}_{(k,p)}$ is a linear combination of

$$\psi^+_{jm}(u) = (\langle jm|u|jk\rangle, 0),\ \text{and}\ \psi^-_{jm}(u) = (0, \langle j,m|u|j,-k\rangle), \tag{1.85}$$

where the values of j could be $j = 1,\cdots,k$ or $j = 1/2 - is$ (check why j cannot be 1/2). It should be noted that since $|jk\rangle \in \mathcal{D}^+$ and $\mathcal{D}^+$ is irreducible, ψ^+_{jm} is nonvanishing only for $m > 0$. The similar property is true for ψ^-_{jm}. That is to say, each element in $\mathcal{H}_{(k,p)}$ is a linear combination of those functions $\psi^\pm_{kpjm}(z)$ constructed from $\psi^\pm_{jm}$ respectively. We have

$$\begin{aligned} \psi^+_{kpjm}(z) &= \begin{cases} \langle z,z\rangle_\sigma^{-1+ip}\langle j,m|u(z)|jk\rangle, & z\in U_+ \\ 0, & z\in U_- \end{cases} \\ \psi^-_{kpjm}(z) &= \begin{cases} 0, & \forall z\in U_+ \\ (-\langle z,z\rangle_\sigma)^{-1+ip}\langle j,m|u(z)|j,-k\rangle, & \forall z\in U_- \end{cases} \end{aligned} \tag{1.86}$$

where $\langle z,w\rangle_\sigma := z^\dagger\sigma_3 w$ is the SU(1,1) invariant inner product. This identify a subspace of $\mathrm{Fun}(\mathrm{SU}(1,1)) \oplus \mathrm{Fun}(\mathrm{SU}(1,1))$ with $\mathcal{H}_{(k,p)}$. On $\mathrm{Fun}(\mathrm{SU}(1,1)) \oplus \mathrm{Fun}(\mathrm{SU}(1,1))$, we define an inner product

$$\langle (f_1,f_2)|(g_1,g_2)\rangle = \int_{\mathrm{SU}(1,1)} \mathrm{d}\mu_H(u)\overline{f_1}(u)g_1(u) + \int_{\mathrm{SU}(1,1)} \mathrm{d}\mu_H(u)\overline{f_2}(u)g_2(u) \tag{1.87}$$

with $\mathrm{d}\mu_H$ being a Haar measure on SU(1,1) with a suitable normalization factor. Then, the identification is actually isometric. We thus have

$$\mathcal{H}_{(k,p)} = \left(\bigotimes_{j>1/2}^{k}\mathcal{D}^+_j \oplus \int_0^\infty \mathrm{d}s\mathcal{C}^\epsilon_s\right) \oplus \left(\bigotimes_{j>1/2}^{k}\mathcal{D}^-_j \oplus \int_0^\infty \mathrm{d}s\mathcal{C}^\epsilon_s\right). \tag{1.88}$$

The precise meaning of this statement is encode in the completeness relation

$$\mathbb{1}_{(k,p)} = \sum_{\tau=\pm}\left\{\sum_{j>1/2}^{k}\sum_{m=\tau j}^{\tau\infty}|\psi^\tau_{kpjm}\rangle\langle\psi^\tau_{kpjm}| + \int_0^\infty \mathrm{d}s\mu_\epsilon(s)\sum_{m=\pm\epsilon}^{\infty}|\psi_{kpjm}\rangle\langle\psi_{kpjm}|\right\} \tag{1.89}$$

where $\mu_\epsilon(s)$ is some measure whose explicit form is irrelevant for the current work.

3. the SU(1,1) coherent state

The SU(1,1) coherent state is given by

$$|j,u\rangle = D^j(u)|j,\pm j\rangle. \tag{1.90}$$

To calculate, we need

$$\begin{aligned}
&\begin{pmatrix}\alpha & \bar{\beta}\\ \beta & \bar{\alpha}\end{pmatrix}\\
&= \exp\Big(\frac{\bar{\beta}}{\bar{\alpha}}\hat{\mathbf{K}}_+\Big)\exp\Big(-\ln(|\alpha|^2)\hat{\mathbf{K}}_3\Big)\exp\Big(-\frac{\beta}{\alpha}\hat{\mathbf{K}}_-\Big)\exp\Big(2i\arg(\alpha)\hat{\mathbf{K}}_3\Big)\\
&= \exp\Big(-\frac{\beta}{\alpha}\hat{\mathbf{K}}_-\Big)\exp\Big(\ln(|\alpha|^2)\hat{\mathbf{K}}_3\Big)\exp\Big(\frac{\bar{\beta}}{\bar{\alpha}}\hat{\mathbf{K}}_+\Big)\exp\Big(2i\arg(\alpha)\hat{\mathbf{K}}_3\Big)
\end{aligned} \tag{1.91}$$

where $\arg(\alpha)$ is given by $\alpha = |\alpha|\exp(i\arg(\alpha))$. We thus have

$$\begin{aligned}
D^j(u)|jj\rangle &= \exp(2i\arg(\alpha)j)\left(1-\frac{|\beta|^2}{|\alpha|^2}\right)^j\sum_{m=0}^{n}\sqrt{\frac{\Gamma(2j+m)}{m!\Gamma(2j)}}\left(\frac{\bar{\beta}}{\bar{\alpha}}\right)^m|j,j+m\rangle\\
&= \frac{1}{\bar{\alpha}^{2j}}\sum_{m=0}^{n}\sqrt{\frac{\Gamma(2j+m)}{m!\Gamma(2j)}}\left(\frac{\bar{\beta}}{\bar{\alpha}}\right)^m|j,j+m\rangle
\end{aligned} \tag{1.92}$$

eq:Dujj

and

$$\begin{aligned}
D^j(u)|j,-j\rangle &= \exp(-2i\arg(\alpha)j)\left(1-\frac{|\beta|^2}{|\alpha|^2}\right)^j\sum_{m=0}^{n}\sqrt{\frac{\Gamma(2j+m)}{m!\Gamma(2j)}}\left(-\frac{\beta}{\alpha}\right)^m|j,j+m\rangle\\
&= \frac{1}{\alpha^{2j}}\sum_{m=0}^{n}\sqrt{\frac{\Gamma(2j+m)}{m!\Gamma(2j)}}\left(-\frac{\beta}{\alpha}\right)^m|j,-j-m\rangle
\end{aligned} \tag{1.93}$$

eq:Dujmj

D. spinors

With the Pauli matrices, we define the following objects

$$\sigma_I = (1,\vec{\sigma}),\quad \tilde{\sigma}_I = (-1,\vec{\sigma}),\quad \sigma^I = \eta^{IJ}\sigma_J = (-1,\vec{\sigma}),\quad \tilde{\sigma}^I = \eta^{IJ}\tilde{\sigma}_J = (1,\vec{\sigma}). \tag{1.94}$$

We then have

eq:sigmapro

$$\sigma_I\tilde{\sigma}_J + \sigma_J\tilde{\sigma}_I = 2\eta_{IJ}\mathbb{1} = \tilde{\sigma}_I\sigma_J + \tilde{\sigma}_J\sigma_I,\ \mathrm{tr}(\sigma_I\tilde{\sigma}_J) = 2\eta_{IJ}. \tag{1.95}$$

σ_I and $\tilde{\sigma}_I$ are related by

eq:sigmatildesigma

$$\epsilon\overline{\sigma_I}\epsilon = \tilde{\sigma}_I, \tag{1.96}$$

where $\bar{\cdot}$ denote the complex conjugate and

$$\epsilon = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

The $\mathfrak{sl}(2,\mathbb{C})$ algebra, comprising traceless 2×2 complex matrices, is generated by

$$\vec{J} = \frac{i}{2}\vec{\sigma},\quad \vec{K} = \frac{1}{2}\vec{\sigma}, \tag{1.97}$$

with the commutation relation

$$[J^k,J^l] = -\epsilon^{kl}{}_m J^m,\quad [K^k,K^l] = \epsilon^{kl}{}_m J^m,\quad [K^k,J^l] = -\epsilon^{kl}{}_m K^m \tag{1.98}$$

Given $\xi \in \mathfrak{sl}(2,\mathbb{C})$, we define

$$\xi^{IJ} = \frac{1}{2}\operatorname{tr}\big(\tilde{\sigma}^I(\xi\sigma^J + \sigma^J\xi^\dagger)\big) = \frac{1}{2}\operatorname{tr}\big(\xi\sigma^J\tilde{\sigma}^I + \tilde{\sigma}^I\sigma^J\xi^\dagger\big). \tag{1.99}$$

Due to $\sigma^J\tilde{\sigma}_J = 4\mathbb{I}$ and

eq:vanishsigma2

$$\sigma^J\xi\tilde{\sigma}_J = \xi^i\sigma^J\sigma_i\tilde{\sigma}_J = \xi^i(2\eta_{iJ}\sigma^J - \sigma^J\sigma_J\tilde{\sigma}_i) = 0, \quad \forall \mathfrak{sl}(2,\mathbb{C}), \tag{1.100}$$

one has

eq:xiIJxi

$$\xi = \frac{1}{4}\xi^{IJ}\sigma_I\tilde{\sigma}_J. \tag{1.101}$$

It can be checked that

$$\xi^{IJ} + \xi^{JI} = \frac{1}{2}\operatorname{tr}\big(\xi(\sigma^J\tilde{\sigma}^I + \sigma^I\tilde{\sigma}^J) + (\tilde{\sigma}^I\sigma^J + \tilde{\sigma}^J\sigma^I)\xi^\dagger\big) = \eta^{IJ}\operatorname{tr}\big(\xi + \xi^\dagger\big) = 0 \tag{1.102}$$

which implies $\xi^{IJ} = -\xi^{JI}$, i.e., $\xi^I{}_J$ is in the Lie algebra $\mathfrak{so}(1,3)$. Moreover, we have

$$\begin{aligned}
\xi^I{}_J\eta^J{}_K &= \frac{1}{4}\operatorname{tr}\big(\xi\sigma_J\tilde{\sigma}^I + \tilde{\sigma}^I\sigma_J\xi^\dagger\big)\operatorname{tr}\big(\eta\sigma_K\tilde{\sigma}^J + \tilde{\sigma}^J\sigma_K\eta^\dagger\big)\\
=&\frac{1}{4}\operatorname{tr}\big(\xi\operatorname{tr}\big(\eta\sigma_K\tilde{\sigma}^J + \tilde{\sigma}^J\sigma_K\eta^\dagger\big)\sigma_J\tilde{\sigma}^I + \tilde{\sigma}^I\operatorname{tr}\big(\eta\sigma_K\tilde{\sigma}^J + \tilde{\sigma}^J\sigma_K\eta^\dagger\big)\sigma_J\xi^\dagger\big)\\
=&\frac{1}{2}\operatorname{tr}\big(\xi(\eta\sigma_K + \sigma_K\eta^\dagger)\tilde{\sigma}^I + \tilde{\sigma}^I(\eta\sigma_K + \sigma_K\eta^\dagger)\xi^\dagger\big)\\
=&\frac{1}{2}\operatorname{tr}\big(\xi\eta\sigma_K\tilde{\sigma}^I + \xi\sigma_K\eta^\dagger\tilde{\sigma}^I + \tilde{\sigma}^I\eta\sigma_K\xi^\dagger + \tilde{\sigma}^I\sigma_K\eta^\dagger\xi^\dagger\big).
\end{aligned} \tag{1.103}$$

This calculation gives us

$$\begin{aligned}
&\xi^I{}_J\eta^J{}_K - \eta^I{}_J\xi^J{}_K\\
=&\frac{1}{2}\operatorname{tr}\big([\xi,\eta]\sigma_K\tilde{\sigma}^I + \xi\sigma_K\eta^\dagger\tilde{\sigma}^I - \eta\sigma_K\xi^\dagger\tilde{\sigma}^I + \tilde{\sigma}^I\eta\sigma_K\xi^\dagger - \tilde{\sigma}^I\xi\sigma_K\eta^\dagger + \tilde{\sigma}^I\sigma_K[\eta^\dagger,\xi^\dagger]\big)\\
=&\frac{1}{2}\operatorname{tr}\big([\xi,\eta]\sigma_K\tilde{\sigma}^I + \tilde{\sigma}^I\sigma_K[\xi,\eta]^\dagger\big) = [\xi,\eta]^I{}_K
\end{aligned} \tag{1.104}$$

which implies that the map

eq:sl2cso13

$$\mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{so}(1,3);\ \xi \mapsto \xi^I{}_J = \frac{1}{2}\operatorname{tr}\big(\tilde{\sigma}^I(\xi\sigma_J + \sigma_J\xi^\dagger)\big) \tag{1.105}$$

is a Lie algebra homomorphism.

Moreover, consider $\xi, \eta \in \mathfrak{sl}(2,\mathbb{C})$, we have

eq:normalsl2c

$$\begin{aligned}
-\xi^{IJ}\zeta_{IJ} = \xi^{IJ}\zeta_{JI} =&\frac{1}{2}\operatorname{tr}\big(\xi\sigma^J\tilde{\sigma}^I + \tilde{\sigma}^I\sigma^J\xi^\dagger\big)\frac{1}{2}\operatorname{tr}\big(\zeta\sigma_I\tilde{\sigma}_J + \tilde{\sigma}_J\sigma_I\zeta^\dagger\big)\\
=&\frac{1}{4}\operatorname{tr}\big(\xi\sigma^J\tilde{\sigma}^I\operatorname{tr}\big(\sigma_I\tilde{\sigma}_J\zeta + \sigma_I\zeta^\dagger\tilde{\sigma}_J\big) + \operatorname{tr}\big(\tilde{\sigma}_J\zeta\sigma_I + \zeta^\dagger\tilde{\sigma}_J\sigma_I\big)\tilde{\sigma}^I\sigma^J\xi^\dagger\big)\\
=&\frac{1}{2}\operatorname{tr}\big(\xi\sigma^J(\tilde{\sigma}_J\zeta + \zeta^\dagger\tilde{\sigma}_J) + (\tilde{\sigma}_J\zeta + \zeta^\dagger\tilde{\sigma}_J)\sigma^J\xi^\dagger\big)\\
=&\frac{1}{2}\operatorname{tr}\big(\xi\sigma^J\tilde{\sigma}_J\zeta + \xi\sigma^J\zeta^\dagger\tilde{\sigma}_J + \tilde{\sigma}_J\zeta\sigma^J\xi^\dagger + \zeta^\dagger\tilde{\sigma}_J\sigma^J\xi^\dagger\big)\\
=&2\operatorname{tr}\big(\xi\zeta + \zeta^\dagger\xi^\dagger\big),
\end{aligned} \tag{1.106}$$

where in the last step we used $\sigma^J\tilde{\sigma}_J = 4\mathbb{I}$, according to (eq:sigmapro 1.95), and (eq:vanishsigma2 1.100)

The map (eq:sl2cso13 1.105) is the generator of the following map from SL$(2,\mathbb{C})$ to SO(1,3),

eq:sl2ctoso13group

$$\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(1,3); g \mapsto L(g)^I{}_J = \frac{1}{2}\operatorname{tr}\big(\tilde{\sigma}^I g\sigma_J g^\dagger\big). \tag{1.107}$$

We have

$$L(g)^I{}_J\tilde{\sigma}^J=\frac{1}{2}\operatorname{tr}\big(g^\dagger\tilde{\sigma}^I g\sigma_J\big)\tilde{\sigma}^J=g^\dagger\tilde{\sigma}^I g,\quad L(g)^I{}_J\sigma_I=\frac{1}{2}\operatorname{tr}\big(g\sigma_J g^\dagger\tilde{\sigma}^I\big)\sigma_I=g\sigma_J g^\dagger \tag{1.108}$$

and

$$\begin{aligned}L(g)^I{}_J=&\frac{1}{2}\operatorname{tr}\Big(g^\dagger\epsilon\overline{\sigma^I}\epsilon g\epsilon\overline{\tilde{\sigma}_J}\epsilon\Big)=\frac{1}{2}\operatorname{tr}\Big(\overline{g^{-1}\sigma^I}g^{T-1}\overline{\tilde{\sigma}_J}\Big)\\=&\frac{1}{2}\operatorname{tr}\big((g^{\dagger-1})^T(\sigma^I)^T(g^{-1})^T(\tilde{\sigma}_J)^T\big)=\frac{1}{2}\operatorname{tr}\big(\tilde{\sigma}_J g^{-1}\sigma^I g^{-1\dagger}\big)=\frac{1}{2}\operatorname{tr}\big(g^{-1\dagger}\tilde{\sigma}_J g^{-1}\sigma^I\big)\\=&\eta_{JK}L(g^{-1})^K{}_L\eta^{IL}.\end{aligned} \tag{1.109}$$

where we used

$$\epsilon g^T\epsilon=-g^{-1}. \tag{1.110}$$

These formula gives us

$$L(g)^I{}_J L(g)^K{}_L\eta_{IK}=L(g)^K{}_L\eta_{JM}L(g^{-1})^M{}_K=\eta_{JL}, \tag{1.111}$$

which demonstrate that $L(g)$ belongs to the group SO(1,3), and

$$\begin{aligned}(g\xi g^{-1})^I{}_J=&\frac{1}{2}\operatorname{tr}\big(\tilde{\sigma}^I g\xi g^{-1}\sigma_J+\sigma_J g^{-1\dagger}\xi^\dagger g^\dagger\tilde{\sigma}^I\big)\\=&\frac{1}{2}\operatorname{tr}\big(g^{\dagger-1}L(g)^I{}_K\tilde{\sigma}^K\xi L(g^{-1})^N{}_J\sigma_N g^\dagger+gL(g^{-1})^N{}_J\sigma_N\xi^\dagger L(g)^I{}_K\tilde{\sigma}^K g^{-1}\big)\\=&L(g)^I{}_K L(g^{-1})^N{}_J\frac{1}{2}\operatorname{tr}\big(\tilde{\sigma}^K\xi\sigma_N+\sigma_N\xi^\dagger\tilde{\sigma}^K\big)\\=&L(g)^I{}_K L(g^{-1})^N{}_J\xi^K{}_N.\end{aligned} \tag{1.112}$$

Consider the objects J^{KL} defined as

$$J^{KL}=\frac{1}{4}\left(\sigma^K\tilde{\sigma}^L-\sigma^L\tilde{\sigma}^K\right). \tag{1.113}$$

We have

$$\begin{aligned}(J^{KL})^{IJ}=&\frac{1}{8}\operatorname{tr}\big(\sigma^K\tilde{\sigma}^L\sigma^J\tilde{\sigma}^I+\tilde{\sigma}^I\sigma^J\tilde{\sigma}^L\sigma^K-\sigma^L\tilde{\sigma}^K\sigma^J\tilde{\sigma}^I-\tilde{\sigma}^I\sigma^J\tilde{\sigma}^K\sigma^L\big)\\=&\frac{1}{8}\operatorname{tr}\Big(2\eta^{IK}\tilde{\sigma}^L\sigma^J-\tilde{\sigma}^K\sigma^I\tilde{\sigma}^L\sigma^J+2\sigma^J\tilde{\sigma}^L\eta^{KI}-\sigma^J\tilde{\sigma}^L\sigma^I\tilde{\sigma}^K-\sigma^J\tilde{\sigma}^I\sigma^L\tilde{\sigma}^K-\tilde{\sigma}^K\sigma^L\tilde{\sigma}^I\sigma^J\Big)\\=&\frac{1}{8}\operatorname{tr}\Big(4\eta^{IK}\tilde{\sigma}^L\sigma^J-2\eta^{IL}\tilde{\sigma}^K\sigma^J-2\eta^{IL}\sigma^J\tilde{\sigma}^K\Big)\\=&\frac{1}{2}\operatorname{tr}\big(\eta^{IK}\tilde{\sigma}^L\sigma^J-\eta^{IL}\tilde{\sigma}^K\sigma^J\big)\\=&\eta^{IK}\eta^{LJ}-\eta^{IL}\eta^{KJ}\end{aligned} \tag{1.114}$$

where we used $\operatorname{tr}\big(\tilde{\sigma}^J\sigma^I\big)=2\eta^{IJ}$ by (1.95). We thus have

$$(J_{KL})^{IJ}=\delta^I_K\delta^J_L-\delta^J_K\delta^I_L. \tag{1.115}$$

By definition of J_{KL}, we have

$$J_{0i}=-J_{i0}=\frac{1}{2}\sigma_i=K_i,\quad J_{kl}=-J_{lk}=\frac{1}{2}\epsilon_{kl}{}^m i\sigma_m=\epsilon_{kl}{}^m J_m, \tag{1.116}$$

that is to say J_{IJ}, equivalent to $\vec{K}$ and $\vec{J}$, form a basis of the $\mathfrak{sl}(2,\mathbb{C})$ Lie algebra. Thus, $(J_{KL})^I{}_J$, as the representation of J_{KL} on the Minkowski space, gives the corresponding basis of the $\mathfrak{so}(1,3)$ Lie algebra.

Before proceeding, we introduce the following notions. Given $z=(z^1,z^2)^T$, we define Jz as

$$Jz=\epsilon\bar{z}=(-\overline{z^2},\overline{z^1})^T. \tag{1.117}$$

It is easy to check

eq:gJ
$$\begin{aligned} gJz =& g\epsilon\bar{z} = \epsilon g^{-1T}\bar{z} = \epsilon\overline{g^{-1\dagger}z} = J(g^\dagger)^{-1}z, \\ (g^\dagger)^{-1}Jz =& (g^\dagger)^{-1}\epsilon\bar{z} = \epsilon\overline{gz} = Jgz. \end{aligned} \tag{1.118}$$

Given two spinors ξ, η, we define

eq:sl2cLiealge
$$\xi \otimes \eta^\dagger - J\eta \otimes (J\xi)^\dagger = \xi \otimes \eta^\dagger + \epsilon\bar{\eta} \otimes \xi^T\epsilon \in \mathfrak{sl}(2, \mathbb{C}) \tag{1.119}$$

where it is an Lie algebra element can be seen from

$$\operatorname{tr}\big(\xi \otimes \eta^\dagger - J\eta \otimes (J\xi)^\dagger\big) = \operatorname{tr}\big(\xi \otimes \eta^\dagger + \bar{\eta} \otimes \xi^T\epsilon^2\big) = \operatorname{tr}\big(\xi \otimes \eta^\dagger\big) - \operatorname{tr}\big((\xi \otimes \eta^\dagger)^T\big) = 0. \tag{1.120}$$

Moreover, one has

$$\begin{aligned} &\frac{1}{2}\operatorname{tr}\big(J\eta \otimes (J\xi)^\dagger\tilde{\sigma}^I\big)\tilde{\sigma}_I = -\frac{1}{2}\operatorname{tr}\big(\epsilon\bar{\eta} \otimes \xi^T\epsilon\tilde{\sigma}^I\big)\tilde{\sigma}_I = -\frac{1}{2}\operatorname{tr}\Big(\bar{\eta} \otimes \xi^T\overline{\sigma^I}\Big)\tilde{\sigma}_I \\ =& -\frac{1}{2}\operatorname{tr}\Big((\bar{\eta} \otimes \xi^T\overline{\sigma^I})^T\Big)\tilde{\sigma}_I = -\frac{1}{2}\operatorname{tr}\big(\sigma^I\xi \otimes \eta^\dagger\big)\tilde{\sigma}_I = -\xi \otimes \eta^\dagger \end{aligned} \tag{1.121}$$

which gives us the following important results

eq:spinorJspinor
$$\xi \otimes \eta^\dagger = -\frac{1}{2}\operatorname{tr}\big(J\xi \otimes (J\eta)^\dagger\tilde{\sigma}^I\big)\tilde{\sigma}_I. \tag{1.122}$$

and, thus,

eq:spinorJspinor2
$$\xi \otimes \eta^\dagger - J\eta \otimes (J\xi)^\dagger = -\frac{1}{2}\operatorname{tr}\big(J\xi \otimes (J\eta)^\dagger\tilde{\sigma}^I\big)(\tilde{\sigma}_I + \sigma_I) = -\operatorname{tr}\big(J\xi \otimes (J\eta)^\dagger\tilde{\sigma}^i\big)\sigma_i \tag{1.123}$$

For a give spinor ζ, an associated null vector $\iota(\zeta)$ is given by

$$\zeta \otimes \zeta^\dagger = \frac{1}{\sqrt{2}}\iota(\zeta)^I\sigma_I, \text{ with } \sigma_I = (1, \vec{\sigma}), \tag{1.124}$$

i.e.,

eq:iotazeta
$$\iota(\zeta)^I = \frac{1}{\sqrt{2}}\operatorname{tr}\big((\zeta \otimes \zeta^\dagger)\tilde{\sigma}^I\big). \tag{1.125}$$

Moreover, we have

$$(g\zeta) \otimes (g\zeta)^\dagger = g(\zeta \otimes \zeta^\dagger)g^\dagger = \frac{1}{\sqrt{2}}\sigma_J L(g)^J{}_I\iota(\zeta)^I \tag{1.126}$$

where we used (eq:gsigma 1.108). This equation leads to

eq:iotag
$$\iota(g\zeta)^J = L(g)^J{}_I\iota(\zeta)^I. \tag{1.127}$$

It is straightforward to check that for $\zeta^\dagger\zeta = 1$

eq:unitn
$$\iota(\zeta) = \frac{1}{\sqrt{2}}\left(n_0, \vec{n}_\zeta\right), \tag{1.128}$$

where n_0 and $\vec{n}_\zeta$ are given by

$$n_0 = \zeta^\dagger\zeta, \ \vec{n}_\zeta = (\zeta^2\overline{\zeta^1} + \zeta^1\overline{\zeta^2}, i(\zeta^1\overline{\zeta^2} - \zeta^2\overline{\zeta^1}), \zeta^1\overline{\zeta^1} - \zeta^2\overline{\zeta^2}) \tag{1.129}$$

and

$$\iota(J\zeta) = \frac{1}{\sqrt{2}}(n_0, -\vec{n}_\zeta). \tag{1.130}$$

In what follows, we are concerned with ζ satisfying $\zeta^\dagger\zeta = 1$. We follows (1.119) to construct the following Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ element

$$\xi_\zeta = (\zeta\otimes\zeta^\dagger - J\zeta\otimes(J\zeta)^\dagger) \in \mathfrak{sl}(2,\mathbb{C}). \tag{1.131}$$

We have

$$\begin{aligned}\xi_\zeta =&(\zeta\otimes\zeta^\dagger - J\zeta\otimes(J\zeta)^\dagger) = \frac{1}{\sqrt{2}}\left(-\iota(\zeta)^I\sigma_I\tilde\sigma_0 + \sigma_0\iota(\zeta)^I\tilde\sigma_I\right)\\ =&\frac{1}{\sqrt{2}}u^J\iota(\zeta)^I\left(\sigma_J\tilde\sigma_I - \sigma_I\tilde\sigma_J\right) = 2\sqrt{2}u^K\iota(\zeta)^L J_{KL}\end{aligned} \tag{1.132}$$

where $u = (1,0,0,0)^T$ and we used, according to (1.122), $\iota(J\zeta)^I\sigma_I = -\iota(\zeta)^I\tilde\sigma_I$. Since $\iota(\xi)^I$ are real, we have, according to (1.105)

$$(\xi_\zeta)^{IJ} = 2u^K\iota(\zeta)^L(\delta^I_K\delta^J_L - \delta^J_K\delta^I_L) = 2\sqrt{2}\left(u^I\iota(\zeta)^J - u^J\iota(\zeta)^I\right) = 2\sqrt{2}u^I\wedge\iota(\zeta)^J. \tag{1.133}$$

II. SPINFOAM TRANSITION AMPLITUDE

A. Y_γ map

Y_γ map is a map from the SU(2) or SU(1,1) representation space to the unitary representation space of $\mathrm{SL}(2,\mathbb{C})$. The image of Y_γ map in the unitary representation space of $\mathrm{SL}(2,\mathbb{C})$ is solve the simplicity constraint at least weakly.

For the SU(2) canse ,we Y_γ map

$$Y_\gamma : \bigoplus \mathcal{H}_j \to \bigoplus_{(k,p)}\mathcal{H}_{(k,p)}; |jm\rangle \mapsto \psi_{j,\gamma j,j,m}. \tag{2.1}$$

For SU(1,1) case, we have

$$Y_\gamma : \bigoplus_{j\in\mathbb{N}/2}\mathcal{D}^\pm_j \to \bigoplus_{(k,p)}\mathcal{H}_{(k,p)}; |jm\rangle \mapsto \psi^\pm_{j,\gamma j,j,m}. \tag{2.2}$$

and

$$Y_\gamma : \int \mathrm{d}s\mathcal{C}^\epsilon_s \to \bigoplus_{(k,p)}\mathcal{H}_{(k,p)}; |jm\rangle \mapsto \psi^+_{\gamma\sqrt{s^2+1/4},-\sqrt{4s^2+1},j,m} \text{ or } \psi^-_{\gamma\sqrt{s^2+1/4},-\sqrt{4s^2+1},j,m} \tag{2.3}$$

It should be noted that $\gamma\sqrt{s^2+1/4}\in\mathbb{N}/2$ implies that Y_γ is well-defined only for $\mathcal{C}^\epsilon_s$ with those s making $2\gamma\sqrt{s^2+1/4}$ integers.

B. vertex amplitude for SU(2) case

A spin foam vertex is a vertex dual to a 4-simplex of a spacetime triangularisation. Thus, it is connected with 5 edges and 10 faces (as shown in Fig. 1). Consider a small region surrounding the vertex. The intersection of the faces and the sphere gives a boundary graph Γ. The intersection lines of the faces with the sphere are called links and the intersection of points with the sphere are call nodes. On Γ, we have the LQG Hilbert space

$$\mathcal{H}_\Gamma = L^2(\mathrm{SU}(2)^{10}). \tag{2.4}$$

The vertex amplitude A_v is defines as a state in $\mathcal{H}^*_\Gamma$ which will map each element in $\mathcal{H}_\Gamma$ to a complex number.

To give the vertex amplitude, let us consider the basis in $\mathcal{H}_\Gamma$ constructed as follows. We assign to each face a spin j_f. For each face f, we then assign to its starting edge e and ending edge e' a magnetic number m_{fe} and $m_{fe'}$ respectively (see Fig. 1). Then, we have the basis

$$\left|\{j_f\}_f,\{m_{fe}\}_{fe}\right\rangle = \bigotimes_f |j_f,m_{fe'}\rangle\otimes\langle j_f,m_{fe}|. \tag{2.5}$$

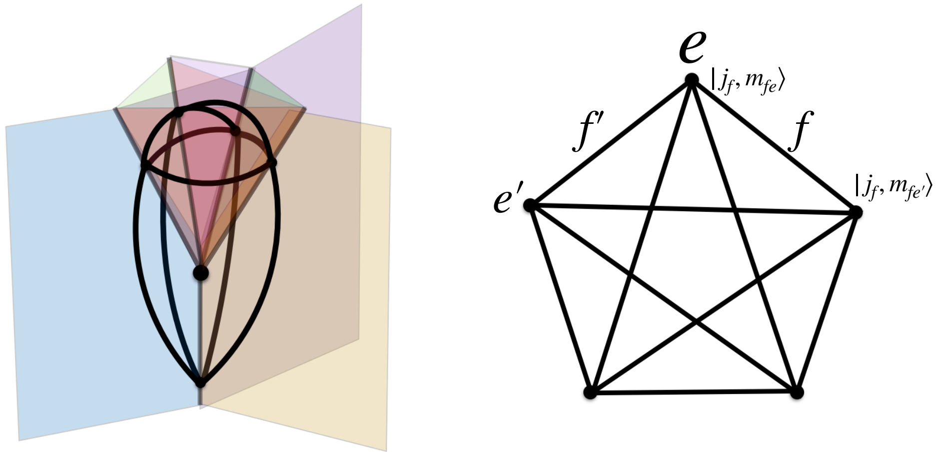


FIG. 1: A spin foam vertex and the boundary graph. fig:v

The vertex amplitude is defined by giving its inner product with the basis,

eq:SFamplitude

$$
\begin{aligned}
\langle A_v | \{j_f\}_f, \{m_{fe}\}_{fe} \rangle =& A_v(\{j_f\}_f, \{m_{fe}\}_{fe}) \\
=& \int_{\mathrm{SL}(2,\mathbb{C})^5} \prod_e \mathrm{d}g_{ve} \delta(g_{ve_1}) \prod_f \sqrt{d_{j_f}} \langle j_f m_{fe} | Y_\gamma^\dagger g_{ev} g_{ve'} Y_\gamma | j_f m_{fe'} \rangle
\end{aligned}
\tag{2.6}
$$

According to (2.6), the group element $g_{ve'} Y_\gamma$ will act on $\bigotimes_{f \text{ sharing } e'} |j_f m_{fe'}\rangle$. Noticing that there are 4 faces sharing each edge, we interpret $\bigotimes_{f \text{ sharing } e'} |j_f m_{fe'}\rangle$ as a quantum tetrahedron. Thus, in the integrand in the action (2.6), $g_{ve'} Y_\gamma \bigotimes_{f \text{ sharing } e'} |j_f m_{fe'}\rangle$ can be interpreted as that we parallel transport the quantum tetrahedron from the the edge e' to the vertex v. The inner product $\langle j_f m_{fe} | Y_\gamma^\dagger g_{ev} g_{ve'} Y_\gamma | j_f m_{fe'} \rangle$ means that we glue those tetrahedra parallel transported from different edges to v. This interpretation is illustrated in Fig. 2.

As proposed by canonical loop quantum gravity, the boundary state $|\{j_f\}_f, \{m_{fe}\}_{fe}\rangle$ can be represented by wave functions on holonomy. We identify

$$
\langle \{h_{\ell_f}\}_{\ell_f} | \{j_f\}_f, \{m_{fe}\}_{fe} \rangle = \prod_f \sqrt{d_{j_f}} D^{j_f}_{m_{fe}, m_{fe'}}(h_{\ell_f}). \tag{2.7}
$$

where ℓ_f denotes the boundary link associated with the face f. We do the identification to get the usual expression of the amplitude A_v under the holonomy representation as shown below. Rewriting the inner product between $\langle A_v|$ and $|\{j_f\}_f, \{m_{fe}\}_{fe}\rangle$ under the holonomy representation, we have

$$
\begin{aligned}
\langle A_v | \{j_f\}_f, \{m_{fe}\}_{fe} \rangle &= \int \mathrm{d}\mu_h \, \langle A_v | \{h_{\ell_f}\}_{\ell_f} \rangle \langle \{h_{\ell_f}\}_{\ell_f} | \{j_f\}_f, \{m_{fe}\}_{fe} \rangle \\
&= \int \mathrm{d}\mu_h \, \langle A_v | \{h_{\ell_f}\}_{\ell_f} \rangle \prod_f \sqrt{d_{j_f}} D^{j_f}_{m_{fe}, m_{fe'}}(h_{\ell_f}),
\end{aligned}
\tag{2.8}
$$

leading to

$$
\langle A_v | \{h_{\ell_f}\}_{\ell_f} \rangle = \sum_{\{j_f\}_f, \{m_{fe}\}_{fe}} \langle A_v | \{j_f\}_f, \{m_{fe}\}_{fe} \rangle \prod_f \sqrt{d_{j_f}} \, \overline{D^{j_f}_{m_{fe}, m_{fe'}}(h_{\ell_f})} \tag{2.9}
$$

Substituting the expression of $\langle A_v | \{j_f\}_f, \{m_{fe}\}_{fe} \rangle$ from (2.6), we have

eq:amplitudeInSingleVertex

$$
\begin{aligned}
\langle A_v | \{h_{\ell_f}\}_{\ell_f} \rangle =& \sum_{\{j_f\}_f, \{m_{fe}\}_{fe}} \int_{\mathrm{SL}(2,\mathbb{C})^5} \prod_e \mathrm{d}g_{ve} \delta(g_{ve_1}) \prod_f \langle j_f m_{fe} | Y_\gamma^\dagger g_{ev} g_{ve'} Y_\gamma | j_f m_{fe'} \rangle \sqrt{d_{j_f}} \, \overline{D^{j_f}_{m_{fe}, m_{fe'}}(h_{\ell_f})} \\
=& \sum_{\{j_f\}_f, \{m_{fe}\}_{fe}} \int_{\mathrm{SL}(2,\mathbb{C})^5} \prod_e \mathrm{d}g_{ve} \delta(g_{ve_1}) \prod_f D^{j_f}_{m_{fe}, m_{m_{fe'}}}(Y_\gamma^\dagger g_{ev} g_{ve'} Y_\gamma) d_{j_f} D^{j_f}_{m_{fe'}, m_{fe}}(h^{-1}_{\ell_f}) \\
=& \sum_{\{j_f\}_f} \int_{\mathrm{SL}(2,\mathbb{C})^5} \prod_e \mathrm{d}g_{ve} \delta(g_{ve_1}) \prod_f d_{j_f} \, \mathrm{tr}_{j_f}(Y_\gamma^\dagger g_{ev} g_{ve'} Y_\gamma h^{-1}_{\ell_f})
\end{aligned}
\tag{2.10}
$$

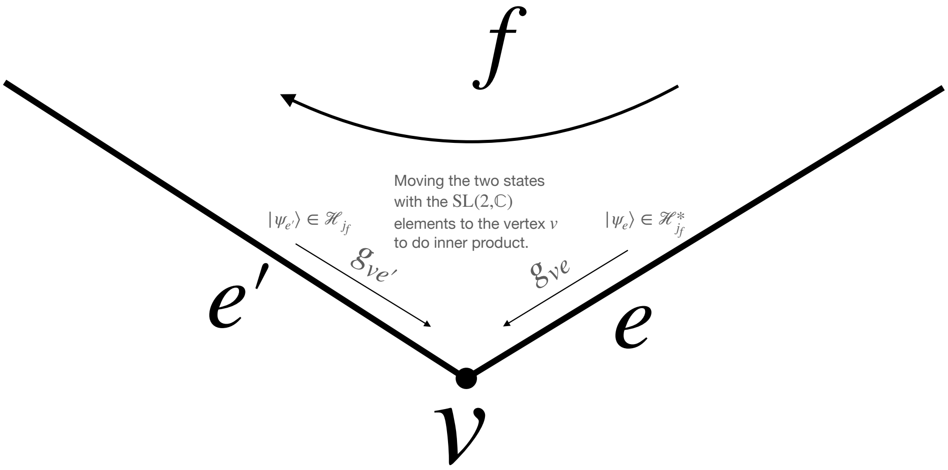


FIG. 2: Spinfoam vertex amplitude. `fig:v`

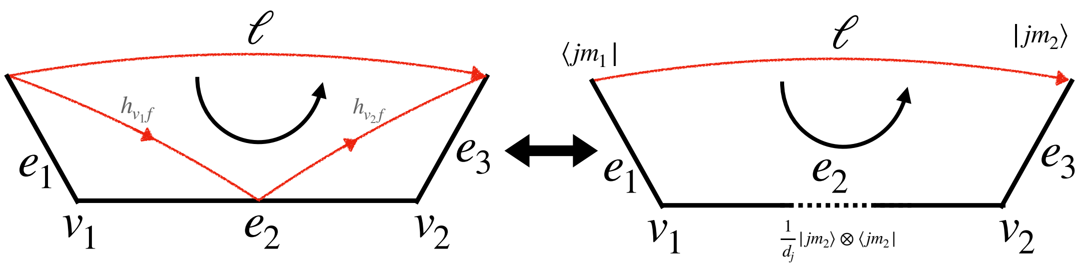


FIG. 3: Glue the faces. In the right figure, we insert $\frac{1}{d_j}\sum_{m_2}|jm_2\rangle\langle jm_2| = d_j^{-1}\mathbb{1}_{\mathcal{H}_j}$. The factor $1/d_j$ is obtained from the results of spin foam amplitude. `fig:g`

For a general boundary state $\Psi \in \mathcal{H}_\Gamma$, we have

$$\langle A_v|\Psi\rangle = \int \mathrm{d}\mu_h \left\langle A_v \middle| \{h_{\ell_f}\}_{\ell_f} \right\rangle \Psi(\{h_{\ell_f}\}_{\ell_f}). \tag{2.11}$$

1. *glue the face amplitude*

Consider a triangularisation of a closed region $\mathcal{R}$ in the spacetime. The dual graph, a 2 simplicial complex, is denoted by $\mathcal{K}$. The intersection of the boundary $\partial\mathcal{R}$ and $\mathcal{K}$ gives a boundary graph $\Gamma_{\partial\mathcal{R}}$. The transition amplitude is a state of the Hilbert space $\mathcal{H}^*_{\partial\mathcal{R}}$. It is given by gluing the vertex amplitude. We could apply the glue face by face.

To start with, consider the face amplitude obtained from (2.10), which is

$$
\begin{aligned}
A_{v,f}(\psi) :=& \sum_{j_f} \int \mathrm{d}\mu_h d_{j_f}\, \mathrm{tr}_{j_f}(Y_\gamma^\dagger g_{ev} g_{ve'} Y_\gamma h_{\ell_f}^{-1}) \psi(h_{\ell_f}) \\
=& \sum_{j_f} \int \mathrm{d}\mu_h d_{j_f}\, \mathrm{tr}_{j_f}(Y_\gamma^\dagger g_{ev} g_{ve'} Y_\gamma h_{\ell_f}^{-1}) \sum_{j,m,m'} \psi_{jm'm} \sqrt{d_j} D^j_{mm'}(h_{\ell_f}) \\
=& \sum_{j_f} \int \mathrm{d}\mu_h d_{j_f} \sum_{n,n'} D^{j_f}_{nn'}(Y_\gamma^\dagger g_{ev} g_{ve'} Y_\gamma)\, \overline{D^{j_f}_{nn'}(h_{\ell_f})} \sum_{j,m,m'} \sqrt{d_j} \psi_{jm'm} D^j_{mm'}(h_{\ell_f}) \\
=& \sum_{j_f} \sum_{n,n'} D^{j_f}_{nn'}(Y_\gamma^\dagger g_{ev} g_{ve'} Y_\gamma) \sum_{j,m,m'} \sqrt{d_j} \psi_{jm'm} \delta_{j,j_f} \delta_{mn} \delta_{m'n'} \\
=& \sum_{j_f,n,n'} \sqrt{d_{j_f}} D^{j_f}_{nn'}(Y_\gamma^\dagger g_{ev} g_{ve'} Y_\gamma) \psi_{j_f n' n} \\
=& \psi(Y_\gamma^\dagger g_{ev} g_{ve'} Y_\gamma).
\end{aligned}
\tag{2.12}
$$

where ψ is a state located at ℓ_f. As shown in Fig. 3, we introduce two links $\ell_{v_1 f}$ and $\ell_{v_2 f}$ where the holonomy $h_{v_1 f}$ and $h_{v_2 f}$ are defined. View them as the boundary link associated to vertices v_1 and v_2 respectively. The face amplitude is then

$$
\begin{aligned}
A_f(h_\ell) =& \int \mathrm{d}h_{v_1 f} \mathrm{d}h_{v_2 f} \delta(h_{v_1 f} h_{v_2 f} h_\ell^{-1}) A_{v_1,f}(h_{v_1 f}) A_{v_2,f}(h_{v_2 f}) \\
=& \int \mathrm{d}h_{v_2 f} A_{v_1,f}(h_\ell h_{v_2 f}^{-1}) A_{v_2,f}(h_{v_2 f}) \\
=& \sum_{j_f} d_{j_f}\, \mathrm{tr}\big(Y_\gamma^\dagger g_{e_1 v_1} g_{v_1 e_2} Y_\gamma Y_\gamma^\dagger g_{e_2 v_2} g_{v_2 e_3} Y h_\ell^{-1}\big)
\end{aligned}
\tag{2.13}
$$

where in the third step (2.12) is applied.

How can we understand the gluing procedure shown in (2.13). To understand it, let us expand $A_{v,f}(h_\ell)$ as

$$
A_{v,f}(h_\ell) = \sum_{j,m,n} \sqrt{d_j} \langle jm | Y_\gamma^\dagger g_{ev} g_{ve'} Y_\gamma | jn \rangle \sqrt{d_j} D^j_{nm}(h_\ell^{-1}) \tag{2.14}
$$

Then, expanding the δ-function in (2.13), we have

$$
\begin{aligned}
A_f(h_\ell) =& \int \mathrm{d}h_{v_1 f} \mathrm{d}h_{v_2 f} \sum_j d_j\, \mathrm{tr}\big(h_{v_1 f} h_{v_2 f} h_\ell^{-1}\big) A_{v_1,f}(h_{v_1 f}) A_{v_2,f}(h_{v_2 f}) \\
=& \int \mathrm{d}h_{v_1 f} \mathrm{d}h_{v_2 f} \sum_{j m_1 m_2 m_3} d_j D^j_{m_1 m_2}(h_{v_1 f}) D^j_{m_2 m_3}(h_{v_2 f}) D^j_{m_3 m_1}(h_\ell^{-1}) A_{v_1,f}(h_{v_1 f}) A_{v_2,f}(h_{v_2 f}) \\
=& \sum_{\substack{j m_1 m_2 m_3 \\ j_1 k_1 l_1 \\ j_2 k_2 l_2}} \sqrt{d_{j_1}} \langle j_1 k_1 | Y_\gamma^\dagger g_{e_1 v_1} g_{v_1 e_2} Y_\gamma | j_1 l_1 \rangle \delta_{m_1,k_1} \delta_{m_2,l_1} \delta_{j j_1} \sqrt{d_{j_2}} \langle j_2 k_2 | Y_\gamma^\dagger g_{e_2 v_2} g_{v_2 e_3} Y_\gamma | j_2 l_2 \rangle \delta_{m_2,k_2} \delta_{m_3,l_2} \delta_{j j_2} D^j_{m_3 m_1}(h_\ell^{-1}) \\
=& \sum_{j m_1 m_2 m_3} \sqrt{d_j} \langle j, m_1 | Y_\gamma^\dagger g_{e_1 v_1} g_{v_1 e_2} Y_\gamma | j m_2 \rangle \langle j, m_2 | Y_\gamma^\dagger g_{e_2 v_2} g_{v_2 e_3} Y_\gamma | j_2 m_3 \rangle \sqrt{d_j} D^j_{m_3 m_1}(h_\ell^{-1})
\end{aligned}
\tag{2.15}
$$

Identifying $\langle j m_3 | \otimes | j, m_1 \rangle$ with $\sqrt{d_j} D^j_{m_3 m_1}(h_\ell)$, we get

$$
\langle A_f | j m_3 m_1 \rangle = \sum_{m_2} \sqrt{d_j} \langle j, m_1 | Y_\gamma^\dagger g_{e_1 v_1} g_{v_1 e_2} Y_\gamma | j m_2 \rangle \langle j, m_2 | Y_\gamma^\dagger g_{e_2 v_2} g_{v_2 e_3} Y_\gamma | j_2 m_3 \rangle. \tag{2.16}
$$

This result can be understand as follows. As shown in Fig. 3, v_1 and v_2 are connected by the edge e_2. In each boundary of the 4-simplex dual to v_1 and v_2, there is a tetrahedron dual to e_2, denoted by $\tau(e_2 v_1)$ and $\tau(e_2 v_2)$. In the 2-simplicial complex $\mathcal{K}$, the edge e_2 connecting v_1 and v_2, meaning the identification of the two tetrahedra $\tau(e_2 v_1)$ and $\tau(e_2 v_2)$.

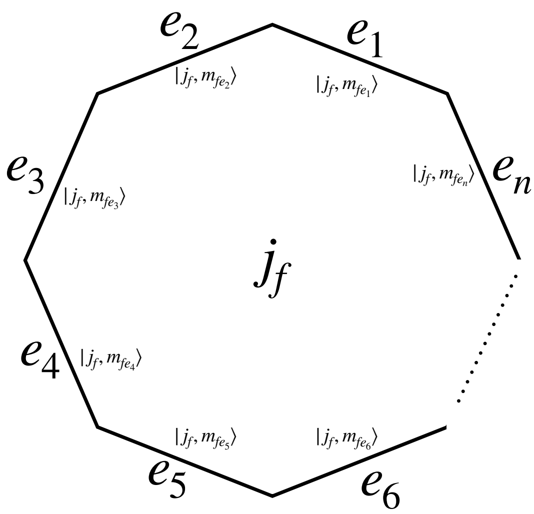


FIG. 4: A face in $\mathcal{K}$

fig:f

In the path integral formation, when gluing two paths, we need to first identify the final state of the first path and initial states of the substantial path, and, then, sum over all possibilities of the identified states. It is the same here that we first identify $\tau(e_2v_1)$ and $\tau(e_2v_2)$ and, then, sum over all possibilities of the identified tetrahedron. The possibilities of the tetrahedron are given by intertwiners at the edge e_2, or, at the corresponding vertices of the boundary graphs. Here, rather than summing over all possible intertwiners, we sum over the magnetic numbers m. This explanation is shown in Fig. fig:gluing 3.

In general, we have

$$A_f(h_{\ell_f}) = \sum_{j_f} d_{j_f}\, \mathrm{tr}\Big(Y_\gamma^\dagger g_{e_1v_1}g_{v_1e_2}Y\cdot Y^\dagger g_{e_2v_2}g_{v_2e_3}Y_\gamma\cdots Y^\dagger g_{e_{n-1}v_n}g_{v_{n-1}e_n}Y_\gamma h_{\ell_f}^{-1}\Big). \tag{2.17}$$

With A_f, we have

$$Z_{\mathcal{K}} = \int \prod{}'\mathrm{d}g_{ev}\prod_f A_f. \tag{2.18}$$

2. face amplitude for face without boundary

Consider a face amplitude for a face without boundary as shown in Fig. fig:faceamp 4. We have

$$A_f = \sum_{j_f}(2j_f+1)\,\mathrm{tr}_{j_f}\Big(\prod_{v\in\partial f} Y_\gamma^\dagger g_{ev}g_{ve'}Y_\gamma\Big). \tag{2.19}$$

According to (eq:gactingonjm 1.28), we have

$$\begin{aligned}
&\langle j,m|Y_\gamma^\dagger g_1^{-1}g_2Y_\gamma|j,m'\rangle\\
=&(2j+1)\int_{\mathbb{CP}^1}\langle g_1^T\zeta,g_1^T\zeta\rangle^{-1-i\gamma j}\langle g_2^T\zeta,g_2^T\zeta\rangle^{-1+i\gamma j}\overline{D^j_{jm}(u(g_1^T\zeta))}D^j_{jm'}(u(g_2^T\zeta))\mathrm{d}^2\zeta\\
=&(2j+1)\int_{\mathbb{CP}^1}\langle g_1^T\zeta,g_1^T\zeta\rangle^{-1-i\gamma j}\langle g_2^T\zeta,g_2^T\zeta\rangle^{-1+i\gamma j}D^j_{mj}(u(g_1^T\zeta)^\dagger)D^j_{jm'}(u(g_2^T\zeta))\mathrm{d}^2\zeta\\
=&(2j+1)\int_{\mathbb{CP}^1}\langle g_1^T\zeta,g_1^T\zeta\rangle^{-1-i\gamma j}\langle g_2^T\zeta,g_2^T\zeta\rangle^{-1+i\gamma j}\langle jm|u(g_1^T\zeta)^\dagger|jj\rangle\langle jj|u(g_2^T\zeta)|jm'\rangle\mathrm{d}^2\zeta
\end{aligned} \tag{2.20}$$

eq:matrixElementYGY

where the measure, as discussed in (eq:integralexplicitly 1.13), is given by $\mathrm{d}^2\zeta = \frac{1}{4\pi}\sin\beta\mathrm{d}\beta\mathrm{d}\gamma$, with the factor $1/\pi$ coming from the normalization of $\psi_{k,p,j,m}$ given in (eq:integralexplicitly 1.13). Then, we have

$$
\begin{aligned}
&\operatorname{tr}_{j_f}\Big(\prod_{v\in\partial f} Y_\gamma^\dagger g_{ev} g_{ve'} Y_\gamma\Big) = \operatorname{tr}_{j_f}\Big(\prod_{v\in\partial f} Y_\gamma^\dagger g_{ve}^{-1} g_{ve'} Y_\gamma\Big)\\
=&(2j_f+1)^{|V|}\int_{(\mathbb{C}\mathrm{P}^1)^{|V|}}\prod_v \mathrm{d}^2\zeta_v \prod_{v\in\partial f}\Big[\langle g_{ve}^T\zeta_v, g_{ve}^T\zeta_v\rangle^{-1-i\gamma j}\langle g_{ve'}^T\zeta_v, g_{ve'}^T\zeta_v\rangle^{-1+i\gamma j}\Big]\times\\
&\prod_{e_m}\langle j_f j_f|u(g^T_{v_{m-1}e_m}\zeta_{v_{m-1}})u(g^T_{v_m e_m}\zeta_{v_m})^\dagger|j_f j_f\rangle
\end{aligned}
\tag{2.21}
$$

where $|V|$ denotes the number of vertices in ∂f, v_{m-1} is the starting point of e_m and v_m is the ending point of e_m. It is easy to verify that

$$
\langle jj|u(g^T_{ve'}\zeta_v)u(g^T_{v'e'}\zeta_{v'})^\dagger|jj\rangle = \langle\frac{1}{2}\frac{1}{2}|u(g^T_{ve'}\zeta_v)u(g^T_{v'e'}\zeta_{v'})^\dagger|\frac{1}{2}\frac{1}{2}\rangle^{2j},
\tag{2.22}
$$

eq:uunderjj

where v and v' are the ending vertices of e. By definition (eq:uz 1.17), we have

$$
u(g^T\zeta) = \frac{1}{\sqrt{\langle g^T\zeta, g^T\zeta\rangle}}\begin{pmatrix}\zeta^\dagger\overline{g}\epsilon\\ \zeta^T g\end{pmatrix}
\tag{2.23}
$$

We thus have

$$
u(g^T_{ve'}\zeta_v)u(g^T_{v'e'}\zeta_{v'})^\dagger = \frac{1}{\sqrt{\langle g^T_{ve'}\zeta_v, g^T_{ve'}\zeta_v\rangle\langle g^T_{v'e'}\zeta_{v'}, g^T_{v'e'}\zeta_{v'}\rangle}}\begin{pmatrix}\zeta_v^\dagger\overline{g_{ve'}}\epsilon\\ -\zeta_v^T g_{v'e'}\end{pmatrix}\begin{pmatrix}-\epsilon g^T_{v'e'}\zeta_{v'} & g^\dagger_{v'e'}\overline{\zeta_{v'}}\end{pmatrix}
\tag{2.24}
$$

eq:ugzdaggerugz

which leads to

$$
\begin{aligned}
\langle\frac{1}{2}\frac{1}{2}|u(g^T_{ve'}\zeta_v)u(g^T_{v'e'}\zeta_{v'})^\dagger|\frac{1}{2}\frac{1}{2}\rangle =&\frac{1}{\sqrt{\langle g^T_{ve'}\zeta_v, g^T_{ve'}\zeta_v\rangle\langle g^T_{v'e'}\zeta_{v'}, g^T_{v'e'}\zeta_{v'}\rangle}}\zeta_v^\dagger\overline{g_{ve'}}g^T_{v'e'}\zeta_{v'}\\
=&\frac{1}{\sqrt{\langle g^T_{ve'}\zeta_v, g^T_{ve'}\zeta_v\rangle\langle g^T_{v'e'}\zeta_{v'}, g^T_{v'e'}\zeta_{v'}\rangle}}\langle g^T_{ve'}\zeta_v, g^T_{v'e'}\zeta_{v'}\rangle.
\end{aligned}
\tag{2.25}
$$

eq:upupelement

This equation leads to

$$
\begin{aligned}
&\operatorname{tr}_{j_f}\Big(\prod_{v\in\partial f} Y_\gamma^\dagger g_{ev} g_{ve'} Y_\gamma\Big)\\
=&(2j_f+1)^{|V|}\int_{(\mathbb{C}\mathrm{P}^1)^{|V|}}\prod_v \mathrm{d}^2\zeta_v \prod_{v\in\partial f}\Big[\langle g_{ve}^T\zeta_v, g_{ve}^T\zeta_v\rangle^{-1-(i\gamma+1)j_f}\langle g_{ve'}^T\zeta_v, g_{ve'}^T\zeta_v\rangle^{-1+(i\gamma-1)j_f}\Big]\times\\
&\left(\prod_{e'\in\partial f}\langle g^T_{ve'}\zeta_v, g^T_{v'e'}\zeta_{v'}\rangle\right)^{2j_f}
\end{aligned}
\tag{2.26}
$$

Let us rewrite the variable as

$$
\zeta_v\mapsto z_v,\;\; g_{ve}\mapsto\overline{g}_{ve}
\tag{2.27}
$$

The second one defines a transformation on $\mathrm{SL}(2,\mathbb{C})$ and the measure is invariant under the transformation (refer to *ArXiv:1304.5626, footnote 1*). We thus have

$$
\begin{aligned}
A_f =&\sum_{j_f}(2j_f+1)^{|V|+1}\int_{(\mathbb{C}\mathrm{P}^1)^{|V|}}\prod_v \mathrm{d}^2 z_v \prod_{v\in\partial f}\Big[\langle g_{ve}^\dagger z_v, g_{ve}^\dagger z_v\rangle^{-1-(i\gamma+1)j_f}\langle g_{ve'}^\dagger z_v, g_{ve'}^\dagger z_v\rangle^{-1+(i\gamma-1)j_f}\Big]\times\\
&\left(\prod_{e'\in\partial f}\langle g^\dagger_{ve'}z_v, g^\dagger_{v'e'}z_{v'}\rangle\right)^{2j_f}.
\end{aligned}
\tag{2.28}
$$

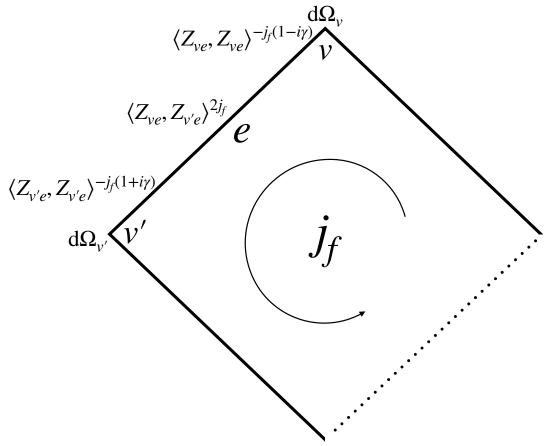


FIG. 5: A face in $\mathcal{K}$ with data

Finally, let us introduce

$$Z_{ve} = g_{ve}^{\dagger} z_v, \quad \mathrm{d}\Omega_v = \frac{\mathrm{d}^2 z_v}{\langle Z_{ve}, Z_{ve}\rangle\langle Z_{ve'}, Z_{ve'}\rangle}, \tag{2.29}$$

so that $\mathrm{d}\Omega_v$ is rescaling invariant. We have

$$\begin{aligned} A_f &= \sum_{j_f} (2j_f+1)^{|V|+1} \int_{(\mathbb{CP}^1)^{|V|}} \prod_v \mathrm{d}^2\Omega_v \prod_{v\in\partial f} \frac{1}{\langle Z_{ve}, Z_{ve}\rangle^{(1+i\gamma)j_f}\langle Z_{ve'}, Z_{ve'}\rangle^{(1-i\gamma)j_f}} \prod_{e'\in\partial f} \langle Z_{ve'}, Z_{v'e'}\rangle^{2j_f} \\ &= \sum_{j_f} (2j_f+1)^{|V|+1} \int_{(\mathbb{CP}^1)^{|V|}} \prod_v \mathrm{d}^2\Omega_v \prod_{e\in\partial f} \frac{\langle Z_{s_e e}, Z_{t_e e}\rangle^{2j_f}}{\langle Z_{s_e e}, Z_{s_e e}\rangle^{(1-i\gamma)j_f}\langle Z_{t_e e}, Z_{t_e e}\rangle^{(1+i\gamma)j_f}} \end{aligned} \tag{2.30}$$

where in the first line, e' is the edge starting from v and ending at v', the vertex next to v, and in the second line, we use s_e and t_e to denote the starting and target vertex of e.

According to the formula (2.30), we get the following picture. Given a face f, at each vertex v of in ∂f there is a measure $\mathrm{d}\Omega_v$; On each edge $e\subset\partial f$, there is Z_{ve} and $Z_{v'e}$ located at the starting point v and the ending point v' respectively (see Fig. 5). The face amplitude A_f is given in term of the data. More importantly, the face amplitude A_f is a function of $\{g_{ve}\}_{ve}$ which is the boundary data surrounding the face.

Given a simplicial complex $\mathcal{K}$ whose dual skeleton is a 2-complex $\mathcal{C}$ made of vertices V, edges E and faces F. Let us assume $\partial\mathcal{K}=\emptyset$ for the moment. As mentioned before, the transition amplitude associated with $\mathcal{K}$ is obtained by gluing the faces, i.e., identifying the boundary data surrounding the faces and doing integration over these boundary data. Then, we have

$$A(\mathcal{K}) = \int_{\mathrm{SL}(2,\mathbb{C})} \prod_{ve} \mathrm{d}g_{ve} \prod_f A_f(g_{ve}) \tag{2.31}$$

with

$$A_f = \sum_{j_f} (2j_f+1)^{|V_f|+1} \int_{(\mathbb{CP}^1)^{|V_f|}} \prod_v \mathrm{d}^2\Omega_v e^{S_f(j_f, g_{ve}, z_{vf})} \tag{2.32}$$

and

$$S_f = \sum_{e\in\partial f} j_f \left[\ln\left(\frac{\langle Z_{s_e ef}, Z_{t_e ef}\rangle^2}{\langle Z_{s_e ef}, Z_{s_e ef}\rangle\langle Z_{t_e ef}, Z_{t_e ef}\rangle}\right) + i\gamma \ln\left(\frac{\langle Z_{s_e ef}, Z_{s_e ef}\rangle}{\langle Z_{t_e ef}, Z_{t_e ef}\rangle}\right)\right] \tag{2.33}$$

where $|V_f|$ is the number of vertices of f, and we add the sub-index f to indicate the variables belonging the face f.

It is convenient to introduce the partial amplitude $A_{\{j_f\}}(\mathcal{K})$ by expanding the summation over j_f, i.e.,

$$A_j(\mathcal{K}) = \int \prod_{ve} \mathrm{d}g_{ve} \prod_v \mathrm{d}^2\Omega_v \exp\Big[S\Big(\{j_f\}, \{g_{ve}\}, \{z_{vf}\}\Big)\Big] \tag{2.34}$$

with $j=\{j_f\}_f$ and

$$S(\{j_f\},\{g_{ev},z_{fv}\})=\sum_f S_f(j_f,g_{ev},z_{fv})$$

for abbreviation, so that

$$A(\mathcal{K})=\sum_{\{j_f\}}\left(\prod_f d_{j_f}^{|V_f|+1}\right)A_{\{j_f\}}(\mathcal{K}) \tag{2.35}$$

3. SF amplitude for faces with boundary

Given a face with boundary, we have the amplitude

$$A_f=\mathrm{tr}\Big(Y_\gamma^\dagger g_{e_0v_1}g_{v_1e_1}Y_\gamma\cdot Y_\gamma^\dagger g_{e_1v_2}g_{v_2e_2}Y_\gamma\cdot\dots\cdot Y_\gamma^\dagger g_{e_{n-1}v_n}g_{v_ne_n}Y_\gamma h_{\ell_f}^{-1}\Big) \tag{2.36}$$

Due to (2.20), we have

$$\begin{aligned}A_f=&(2j_f+1)^{|V|+1}\int_{\mathbb{C}P^1}\prod_{v_m}\|g^T_{v_me_{m-1}}\zeta_{v_n}\|^{2(-1-i\gamma j_f)}\|g^T_{v_me_m}z_{v_m}\|^{2(-1+i\gamma j_f)}\\ &\sum_{m_o,m_n}\langle j_fm_o|u(g^T_{v_1e_0}\zeta_{v_1})^\dagger|j_fj_f\rangle\left(\prod_{e_k}\langle j_fj_f|u(g^T_{v_ke_k}\zeta_{v_k})u(g^T_{v_{k+1}e_k}\zeta_{v_{k+1}})^\dagger|j_fj_f\rangle\right)\langle j_fj_f|u(g^T_{v_ne_n}\zeta_{v_n})|j_fm_n\rangle D^{j_f}_{m_nm_o}(h_{\ell_f}^{-1}).\end{aligned} \tag{2.37}$$

Applying (2.22) and (2.25), we have leading to

$$\begin{aligned}A_f(h_{\ell_f})=&(2j_f+1)^{|V|+1}\int_{\mathbb{C}P^1}\prod_{v_m}\|g^T_{v_me_{m-1}}\zeta_{v_m}\|^{2(-1-i\gamma j_f)}\|g^T_{v_me_m}\zeta_{v_m}\|^{2(-1+i\gamma j_f)}\\ &\sum_{m_o,m_n}\langle j_fm_o|u(g^T_{v_1e_0}\zeta_{v_1})^\dagger|j_fj_f\rangle\left(\prod_{e_k}\langle\frac{1}{2}\frac{1}{2}|u(g^T_{v_ke_k}\zeta_{v_k})u(g^T_{v_{k+1}e_k}\zeta_{v_{k+1}})^\dagger|\frac{1}{2}\frac{1}{2}\rangle\right)^{2j}\langle j_fj_f|u(g^T_{v_ne_n}\zeta_{v_n})|j_fm_n\rangle D^{j_f}_{m_nm_o}(h_{\ell_f}^{-1})\\ =&(2j_f+1)^{|V|+1}\int_{\mathbb{C}P^1}\prod_{v_m}\|g^T_{v_me_{m-1}}\zeta_{v_m}\|^{2(-1-i\gamma j_f)}\|g^T_{v_me_m}\zeta_{v_m}\|^{2(-1+i\gamma j_f)}\prod_{e_k}\|g^T_{v_ke_k}\zeta_{e_k}\|^{-2j_f}\|g^T_{v_{k+1}e_k}\zeta_{v_{k+1}}\|^{-2j_f}\\ &\left(\prod_{e_k}\langle g^T_{v_ke_k}\zeta_{v_k},g^T_{v_{k+1}e_k}\zeta_{v_{k+1}}\rangle\right)^{2j}\sum_{m_o,m_n}\langle j_fm_o|u(g^T_{v_1e_0}\zeta_{v_1})^\dagger|j_fj_f\rangle\langle j_fj_f|u(g^T_{v_ne_n}\zeta_{v_n})|j_fm_n\rangle D^{j_f}_{m_nm_o}(h_{\ell_f}^{-1})\\ =&(2j_f+1)^{|V|+1}\int_{\mathbb{C}P^1}\prod_{\text{internal edges }e}\|g^T_{s_ee}\zeta_{s_e}\|^{2(-1+j_f(i\gamma-1))}\|g^T_{t_ee}\zeta_{t_e}\|^{2(-1-j_f(i\gamma+1))}\left(\langle g^T_{s_ee}\zeta_{s_e},g^T_{t_ee}\zeta_{t_e}\rangle\right)^{2j_f}\\ &\|g^T_{v_1e_0}\zeta_{v_1}\|^{2(-1-i\gamma j_f)}\|g^T_{v_ne_n}\zeta_{v_n}\|^{2(-1+i\gamma j_f)}\sum_{m_o,m_n}\langle j_fm_o|u(g^T_{v_1e_0}\zeta_{v_1})^\dagger|j_fj_f\rangle\langle j_fj_f|u(g^T_{v_ne_n}\zeta_{v_n})|j_fm_n\rangle D^{j_f}_{m_nm_o}(h_{\ell_f}^{-1}).\end{aligned} \tag{2.38}$$

In this work, we are concerned with the coherent boundary state given by

$$\psi_{g_\ell}(h_\ell)=\sum_{j_\ell}d_{j_\ell}e^{-\frac{t}{2}j_\ell(j_\ell+1)}\sum_m D^{j_\ell}_{mm}(g_\ell^{-1}h_\ell), \tag{2.39}$$

with $g_\ell\in\mathrm{SL}(2,\mathbb{C})$. Then, we have

$$\begin{aligned}&\langle A_f|\psi_{g_{\ell_f}}\rangle=\int\mathrm{d}\mu_H(h_\ell)A_f(h_\ell)\psi_{g_{\ell_f}}\\ =&(2j_f+1)^{|V|+1}\int_{\mathbb{C}P^1}\prod_{\text{internal edges }e}\|g^T_{s_ee}\zeta_{s_e}\|^{2(-1+j_f(i\gamma-1))}\|g^T_{t_ee}\zeta_{t_e}\|^{2(-1-j_f(i\gamma+1))}\left(\langle g^T_{s_ee}\zeta_{s_e},g^T_{t_ee}\zeta_{t_e}\rangle\right)^{2j_f}\\ &\|g^T_{v_1e_0}\zeta_{v_1}\|^{2(-1-i\gamma j_f)}\|g^T_{v_ne_n}\zeta_{v_n}\|^{2(-1+i\gamma j_f)}\sum_{m_o,m_n}\langle j_fm_o|u(g^T_{v_1e_0}\zeta_{v_1})^\dagger|j_fj_f\rangle\langle j_fj_f|u(g^T_{v_ne_n}\zeta_{v_n})|j_fm_n\rangle e^{-\frac{t}{2}j_f(j_f+1)}D^{j_f}_{m_n,m_o}(g_\ell^{-1}).\end{aligned} \tag{2.40}$$

Then, we have

$$
\begin{aligned}
&\sum_{m_o,m_n} \langle j_f m_o | u(g^T_{v_1e_0}\zeta_{v_1})^\dagger | j_f j_f\rangle \langle j_f j_f | u(g^T_{v_ne_n}\zeta_{v_n}) | j_f m_n\rangle D^{j_f}_{m_n,m_o}(g_\ell)\\
=&\langle j_f, j_f | u(g^T_{v_ne_n}\zeta_{v_n}) g_\ell^{-1} u(g^T_{v_1e_0}\zeta_{v_1})^\dagger | j_f, j_f\rangle\\
=&\langle \frac{1}{2},\frac{1}{2} | u(g^T_{v_ne_n}\zeta_{v_n}) g_\ell^{-1} u(g^T_{v_1e_0}\zeta_{v_1})^\dagger | \frac{1}{2},\frac{1}{2}\rangle^{2j_f}
\end{aligned}
\tag{2.41}
$$

According to (eq:ugzdaggerugz 2.42), we have

$$
u(g^T_{v_ne_n}\zeta_{v_n}) g_\ell^{-1} u(g^T_{v_1e_0}\zeta_{v_1})^\dagger = \frac{1}{\sqrt{\|g^T_{v_ne_n}\zeta_{v_n}\|\|g^T_{v_1e_0}\zeta_{v_1}\|}} \begin{pmatrix} \zeta^\dagger_{v_n}\overline{g_{v_ne_n}}\epsilon \\ \zeta^T_{v_n} g_{v_ne_n}\end{pmatrix} g_\ell^{-1} \begin{pmatrix} -\epsilon g^T_{v_0e_0}\zeta_{v_1} & g^\dagger_{v_1e_0}\overline{\zeta_{v_1}}\end{pmatrix} \quad \text{eq:ugzdaggerugz}
\tag{2.42}
$$

leading to

$$
\begin{aligned}
&\langle \frac{1}{2},\frac{1}{2} | u(g^T_{v_ne_n}\zeta_{v_n}) g_\ell^{-1} u(g^T_{v_1e_0}\zeta_{v_1})^\dagger | \frac{1}{2},\frac{1}{2}\rangle\\
=&-\left(\|g^T_{v_ne_n}\zeta_{v_n}\|\|g^T_{v_1e_0}\zeta_{v_1}\|\right)^{-j_f} \zeta^\dagger_{v_n}\overline{g_{v_ne_n}}\epsilon g_\ell^{-1}\epsilon g^T_{v_1e_0}\zeta_{v_1}\\
=&\left(\|g^T_{v_ne_n}\zeta_{v_n}\|\|g^T_{v_1e_0}\zeta_{v_1}\|\right)^{-j_f} \langle \epsilon g^T_{v_ne_n}\zeta_{v_n}, g_\ell^{-1}\epsilon g^T_{v_1e_0}\zeta_{v_1}\rangle.
\end{aligned}
\tag{2.43}
$$

We could redefine the variables as

$$
\zeta_v \mapsto z_v,\ \ g_{ve} \mapsto \overline{g_{ve}}\epsilon.
\tag{2.44}
$$

Since $\epsilon = \exp(-i\pi/2\sigma_2) \in \mathrm{SU}(2)$, the $\mathrm{SL}(2,\mathbb{C})$ measure is invariant under the transformation. Moreover, $\epsilon \in \mathrm{SU}(2)$ ensure that $\langle \epsilon\cdot, \cdot\epsilon\rangle = \langle\cdot,\cdot\rangle$. In other words, this transformation can be freely done for the terms involving the inner products and normals. With the new variables, we have

$$
\begin{aligned}
&\langle A_f | \psi_{g_{\ell_f}}\rangle = \int \mathrm{d}\mu_H(h_\ell) A_f(h_\ell)\psi_{g_{\ell_f}}\\
=&(2j_f+1)^{|V|+1}\int_{\mathbb{C}P^1} \prod_{\text{internal edges } e} \|g^\dagger_{s_ee}z_{s_e}\|^{2(-1+j_f(i\gamma-1))}\|g^\dagger_{t_ee}z_{t_e}\|^{2(-1-j_f(i\gamma+1))}\left(\langle g^\dagger_{s_ee}z_{s_e}, g^\dagger_{t_ee}z_{t_e}\rangle\right)^{2j_f}\\
&\|g^\dagger_{v_1e_0}z_{v_1}\|^{2(-1-(1+i\gamma)j_f)}\|g^\dagger_{v_ne_n}z_{v_n}\|^{2(-1+(i\gamma-1)j_f)}\langle g^\dagger_{v_ne_n}z_{v_n}, g_\ell^{-1}g^\dagger_{v_1e_0}z_{v_1}\rangle^{2j_f}\\
=&(2j_f+1)^{|V|+1}\int_{\mathbb{C}P^1} \prod_{\text{internal edges } e} \|g^\dagger_{s_ee}z_{s_e}\|^{2(-1+j_f(i\gamma-1))}\|g^\dagger_{t_ee}z_{t_e}\|^{2(-1-j_f(i\gamma+1))}\left(\langle g^\dagger_{s_ee}z_{s_e}, g^\dagger_{t_ee}z_{t_e}\rangle\right)^{2j_f}\\
&\|g^\dagger_{t_{e_0}e_0}z_{v_1}\|^{2(-1-(1+i\gamma)j_f)}\|g^\dagger_{s_{e_n}e_n}z_{v_n}\|^{2(-1+(i\gamma-1)j_f)}\langle g^\dagger_{v_ne_n}z_{v_n}, g_\ell^{-1}g^\dagger_{v_1e_0}z_{v_1}\rangle^{2j_f}\\
=&(2j_f+1)^{|V|+1}\int_{\mathbb{C}P^1} \|g^\dagger_{t_{e_0}e_0}z_{v_1}\|^{2(-1-(1+i\gamma)j_f)}\|g^\dagger_{s_{e_n}e_n}z_{v_n}\|^{2(-1+(i\gamma-1)j_f)}\\
&\prod_{\text{internal edges } e} \|g^\dagger_{s_ee}z_{s_e}\|^{2(-1+j_f(i\gamma-1))}\|g^\dagger_{t_ee}z_{t_e}\|^{2(-1-j_f(i\gamma+1))}\\
&\left(\prod_{\text{internal edges } e}\langle g^\dagger_{s_ee}z_{s_e}, g^\dagger_{t_ee}z_{t_e}\rangle\right)^{2j_f}\quad \langle g^\dagger_{v_ne_n}z_{v_n}, g_\ell^{-1}g^\dagger_{v_1e_0}z_{v_1}\rangle^{2j_f} e^{-\frac{t}{2}j_f(j_f+1)}
\end{aligned}
\tag{2.45}
$$

Similarly as the discussion for (eq:action 2.33), we get the action for a boundary face f_b

$$
\begin{aligned}
S_{f_b} =& j_f\Bigg[\sum_{\text{internal } e\in\partial f} i\gamma\ln\left(\frac{\langle Z_{s_eef}, Z_{s_eef}\rangle}{\langle Z_{t_eef}, Z_{t_eef}\rangle}\right) + \sum_{\text{internal edges } e}\ln\left(\frac{\langle Z_{s_eef}, Z_{t_eef}\rangle^2}{\langle Z_{s_eef}, Z_{s_eef}\rangle\langle Z_{t_eef}, Z_{t_eef}\rangle}\right) +\\
& i\gamma\ln\left(\frac{\langle Z_{s_{e_n}e_nf}, Z_{s_{e_n}e_nf}\rangle}{\langle Z_{t_{e_0}e_0f}, Z_{t_{e_0}e_0f}\rangle}\right) + \ln\left(\frac{\langle Z_{s_{e_n}e_nf}, g_\ell^{-1}Z_{t_{e_0}e_0f}\rangle^2}{\langle Z_{s_{e_n}e_nf}, Z_{s_{e_n}e_nf}\rangle\langle Z_{t_{e_0}e_0f}, Z_{t_{e_0}e_0f}\rangle}\right)\Bigg] - \frac{t}{2}j_f(j_f+1)
\end{aligned}
\quad \text{eq:actionBoundary}
\tag{2.46}
$$

C. timelike spinfoam amplitude: SU(1,1) discrete series

Given a wedge as shown in 2. Let us assume that the triangle dual to the face f belongs to a timelike tetrahedron and the triangle itself is spacelike. The amplitude associated to this wedge, called wedge amplitude, is a function of the pair $g_{ve'}$, g_{ve} and the boundary state $\psi \in L^2(\mathrm{SU}(1,1))$. A general element in $\psi \in L^2(\mathrm{SU}(1,1))$ can be expanded as a linear combination of these functions $u \mapsto \langle\psi_1|u|\psi_2\rangle$ for $|\psi_1\rangle, |\psi_2\rangle$ being states in the same unitary irrep space of SU(1,1). Then, we will distribute $\langle\psi_1|$ and $|\psi_2\rangle$ to e and e', the two boundary edges of the wedge, respectively, and use subscript fe and fe' to relabel them, i.e., $\psi_{fe} \equiv \psi_1$ and $\psi_{fe'} \equiv \psi_2$. Then, similar to the case of SU(2), we calculate the spin foam amplitude by transporting ψ_{fe} and $\psi_{fe'}$ from the edges to the central vertex v and taking their inner product, i.e.,

$$A = \langle\psi_{fe}|Y_\gamma^\dagger g_{ve}^{-1} g_{ve'} Y_\gamma|\psi_{fe'}\rangle. \tag{2.47}$$

We are concerned with the coherent boundary states in $\mathcal{D}_j^\pm$ defined by

$$|j,u\rangle = D^j(u)|j,\alpha j\rangle, \quad \text{with } \alpha = \pm 1 \text{ corresponding to } \mathcal{D}_j^\pm. \tag{2.48}$$

Then, we have

$$A = \langle j_{fv}, \alpha_{fv} j_{fv}|D^j(u_{fe}^\dagger) Y_\gamma^\dagger g_{ve}^{-1} g_{ve'} Y_\gamma D^j(u_{fe'})|j_{fv}, \alpha_{fv} j_{fv}\rangle \tag{2.49}$$

By definition of Y_γ map, the function on $\mathbb{C}^2$ corresponding to $Y_\gamma D^j(u_o)|j,\alpha j\rangle$, denoted in the usual fashion by $\langle z|Y_\gamma D^j(u_o)|j,\alpha j\rangle$, is

$$\langle z|Y_\gamma D^j(u_o)|j,\alpha j\rangle = \Theta(\alpha\langle z,z\rangle_\sigma)(\alpha\langle z,z\rangle_\sigma)^{-1+ip}\langle j,\alpha j|D^j(u_o^\dagger u(z))|j,\alpha j\rangle \tag{2.50}$$

where Θ is the step function with $\Theta(x) = (\mathrm{sgn}(x)+1)/2$. By definition of $u(z)$, for

$$u_o = \left(\xi, \epsilon\sigma_3\bar{\xi}\right)$$

we have

$$u_o^\dagger u(z) = \frac{1}{\sqrt{\langle z,z\rangle_\sigma}}\begin{pmatrix}\xi^\dagger \\ -\xi^T\sigma_3\epsilon\end{pmatrix}\begin{pmatrix}z & \epsilon\sigma_3\bar{z}\end{pmatrix} = \frac{1}{\sqrt{\langle z,z\rangle_\sigma}}\begin{pmatrix}\xi^\dagger z & \xi^\dagger\epsilon\sigma_3\bar{z} \\ -\xi^T\sigma_3\epsilon z & \xi^T\bar{z}\end{pmatrix}, \quad \langle z,z\rangle_\sigma > 0 \tag{2.51}$$

and

$$u_o^\dagger u(z) = \frac{1}{\sqrt{-\langle z,z\rangle_\sigma}}\begin{pmatrix}\xi^\dagger \\ -\xi^T\sigma_3\epsilon\end{pmatrix}\begin{pmatrix}\epsilon\sigma_3\bar{z} & z\end{pmatrix} = \frac{1}{\sqrt{-\langle z,z\rangle_\sigma}}\begin{pmatrix}\xi^\dagger\epsilon\sigma_3\bar{z} & \xi^\dagger z \\ \xi^T\bar{z} & -\xi^T\sigma_3\epsilon z\end{pmatrix}, \quad \langle z,z\rangle_\sigma < 0. \tag{2.52}$$

Then, applying the results (1.92) and (1.93), we get the following calculations.

(1) For $\langle z,z\rangle_\sigma > 0$, $\Theta(\alpha\langle z,z\rangle_\sigma)$ ensure that it is enough to do calculation for $|j,\alpha j\rangle = |jj\rangle$

$$\langle j,j|D^j(u_o^\dagger u(z))|j,j\rangle = \langle z,z\rangle_\sigma^j(\xi^T\bar{z})^{-2j} = \langle z,z\rangle_\sigma^j\langle z,\sigma_3\xi\rangle_\sigma^{-2j} = \langle z,z\rangle_\sigma^j\langle z,\xi^+\rangle_\sigma^{-2j} \tag{2.53}$$

where we apply

$$u_o^{-1} = \begin{pmatrix}\xi^\dagger\sigma_3 \\ -\xi^T\epsilon\end{pmatrix} \tag{2.54}$$

to derive

$$\xi^+ = \sigma_3\xi = u_o^{-1\dagger}\begin{pmatrix}1\\0\end{pmatrix}.$$

(2) For $\langle z,z\rangle_\sigma < 0$, $\Theta(\alpha\langle z,z\rangle_\sigma)$ ensure that it is enough to do calculation for $|j,\alpha j\rangle = |j,-j\rangle$

$$\langle j,-j|D^j(u_o^\dagger u(z))|j,-j\rangle = (-\langle z,z\rangle_\sigma)^j(\xi^\dagger\epsilon\sigma_3\bar{z})^{-2j} = (-\langle z,z\rangle_\sigma)^j(-z^\dagger\sigma_3\epsilon\bar{\xi})^{-2j} = (-\langle z,z\rangle_\sigma)^j(-\langle z,\xi^-\rangle)^{-2j} \tag{2.55}$$

where we apply (2.54) again to get

$$\xi^- = u_o^{-1\dagger}\begin{pmatrix}0\\1\end{pmatrix}.$$

We therefore get

$$\begin{aligned}\langle z|Y_\gamma D^j(u_o)|j,\alpha j\rangle =&\Theta(\alpha\langle z,z\rangle_\sigma)(\alpha\langle z,z\rangle_\sigma)^{-1+ip}\langle j,\alpha j|D^j(u(z)u_o)|j,\alpha j\rangle\\ =&\Theta(\alpha\langle z,z\rangle_\sigma)(\alpha\langle z,z\rangle_\sigma)^{-1+ip+j}(\alpha\langle z,\xi^\alpha\rangle_\sigma)^{2j}\end{aligned} \tag{2.56}$$

Since we consider the unitary representation of $\mathrm{SL}(2,\mathbb{C})$, we have that the amplitude A is actually the inner product between the two $\mathrm{SL}(2,\mathbb{C})$ states $g_{ve}Y_\gamma D^j(u_{fe})|j_{fv},\alpha_{fv}j_{fv}\rangle$ and $g_{ve'}Y_\gamma D^j(u_{fe'})|j_{fv},\alpha_{fv}j_{fv}\rangle$. We thus have

$$\begin{aligned}A=&\int_{\mathbb{CP}^1}\Omega_z\Theta(\alpha_{fv}\langle g_{ve}^Tz,g_{ve}^Tz\rangle_\sigma)(\alpha_{fv}\langle g_{ve}^Tz,g_{ve}^Tz\rangle_\sigma)^{-1-i\gamma j_{fv}+j_{fv}}\overline{(\alpha_{fv}\langle g_{ve}^Tz,\xi_{ef}^{\alpha_{fv}}\rangle_\sigma)}^{-2j_{fv}}\\ &\times\Theta(\alpha_{fv}\langle g_{ve'}^Tz,g_{ve'}^Tz\rangle_\sigma)(\alpha_{fv}\langle g_{ve'}^Tz,g_{ve'}^Tz\rangle_\sigma)^{-1+i\gamma j_{fv}+j_{fv}}(\alpha_{fv}\langle g_{ve'}^Tz,\xi_{e'f}^{\alpha_{fv}}\rangle_\sigma)^{-2j_{fv}}\\ =&\int_{\mathbb{CP}^1}\frac{\Theta(\alpha_{fv}\langle g_{ve}^Tz,g_{ve}^Tz\rangle_\sigma)\Theta(\alpha_{fv}\langle g_{ve'}^Tz,g_{ve'}^Tz\rangle_\sigma)\Omega_z}{\langle g_{ve}^Tz,g_{ve}^Tz\rangle_\sigma\langle g_{ve'}^Tz,g_{ve'}^Tz\rangle_\sigma}\frac{\langle g_{ve}^Tz,g_{ve}^Tz\rangle_\sigma^{j_{fv}(1-i\gamma)}\langle g_{ve'}^Tz,g_{ve'}^Tz\rangle_\sigma^{j_{fv}(1+i\gamma)}}{\langle\xi_{ef}^{\alpha_{fv}},g_{ve}^Tz\rangle_\sigma^{2j_{fv}}\langle g_{ve'}^Tz,\xi_{e'f}^{\alpha_{fv}}\rangle_\sigma^{2j_{fv}}}\end{aligned} \tag{2.57}$$

where we used $\alpha_{fv}^2=1$. The wedge amplitude is expressed in terms of g^T. In order to match the format of the spacelike amplitude, we need to substitute g_{ve}^T with $g_{ve}^\dagger$. This substitution is possible since we eventually need to integrate over the elements of $\mathrm{SL}(2,\mathbb{C})$. The measure $\mathrm{d}g_{ve}$ is invariant under the transformation $g_{ve}\to\overline{g}_{ve}$. Indeed, according to *arXiv: 2007.01998*, the $\mathrm{SL}(2,\mathbb{C})$ Haar measure reads

$$\mathrm{d}g=\frac{1}{16\pi^4\times 2^3}\frac{\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_3\mathrm{d}y_3}{|1+\frac{x_1+iy_1}{\sqrt{2}}|} \tag{2.58}$$

where x_a,y_a are the coordinate on $\mathrm{SL}(2,\mathbb{C})$ given by

$$g=\begin{pmatrix}1+\frac{x_1+iy_1}{\sqrt{2}} & \frac{x_2+iy_2}{\sqrt{2}}\\ \frac{x_3+iy_3}{\sqrt{2}} & \frac{1+\frac{x_2+iy_2}{\sqrt{2}}\frac{x_3+iy_3}{\sqrt{2}}}{1+\frac{x_1+iy_1}{\sqrt{2}}}\end{pmatrix}. \tag{2.59}$$

Clearly the measure is invariant under the transformation $g\to\overline{g}$, i.e., $x_a\to x_a$, $y_a\to -y_a$. After the substitution, we have

$$A=\int_{\mathbb{CP}^1}\frac{\Theta(\alpha_{fv}\langle Z_{vef},Z_{vef}\rangle_\sigma)\Theta(\alpha_{fv}\langle Z_{ve'f},Z_{ve'f}\rangle_\sigma)\Omega_z}{\langle Z_{vef},Z_{vef}\rangle_\sigma\langle Z_{ve'f},Z_{ve'f}\rangle_\sigma}\frac{\langle Z_{vef},Z_{vef}\rangle_\sigma^{j_{fv}(1-i\gamma)}\langle Z_{ve'f},Z_{ve'f}\rangle_\sigma^{j_{fv}(1+i\gamma)}}{\langle\xi_{ef}^{\alpha_{fv}},Z_{vef}\rangle_\sigma^{2j_{fv}}\langle Z_{ve'f},\xi_{e'f}^{\alpha_{fv}}\rangle_\sigma^{2j_{fv}}} \tag{2.60}$$

with

$$Z_{vef}=g_{ve}^\dagger z_{vf}. \tag{2.61}$$

We thus have the action associated to the wedge

$$S_{fv}=j_{fv}\left(i\gamma\ln\left(\frac{\langle Z_{ve'f},Z_{ve'f}\rangle_\sigma}{\langle Z_{vef},Z_{vef}\rangle_\sigma}\right)-\ln\left(\frac{\langle\xi_{ef}^{\alpha_{fv}},Z_{vef}\rangle_\sigma^2\langle Z_{ve'f},\xi_{e'f}^{\alpha_{fv}}\rangle_\sigma^2}{\langle Z_{vef},Z_{vef}\rangle_\sigma\langle Z_{ve'f},Z_{ve'f}\rangle_\sigma}\right)\right) \tag{2.62}$$

eq:timelikeS

III. GEOMETRIC INTERPRETATION OF THE AMPLITUDE

A. Spacelike SF model

In the spacelike model, we have the amplitude taking the integration of the form

$$A=\sum_j\int\mathrm{d}g\mathrm{d}\Omega e^{S(j,g,z)} \tag{3.1}$$

Applying the Poisson summation formula, we have

$$A=\sum_{k=-\infty}^{\infty}\int\mathrm{d}g\mathrm{d}\Omega\mathrm{d}x' e^{S(x',g,z)-2\pi ikx'}. \tag{3.2}$$

If $k=0$ dominate the summation over k, we have

$$A \cong \int \mathrm{d}g \mathrm{d}\Omega \mathrm{d}x' e^{S(x',g,z)}. \tag{3.3}$$

In the spacelike model, the action S takes the form

$$S = \sum_{\text{internal face } f} j_f S_f + \sum_{\text{boundary face } f} \left(j_f S'_f - \frac{t}{2} j_f (j_f+1) \right) \tag{3.4}$$

We thus need to consider the integral over x' for the following function

$$S = \sum_{\text{internal face } f} x'_f S_f + \sum_{\text{boundary face } f} \left(x'_f S'_f - \frac{t}{2} x'_f (x'_f+1) \right) \tag{3.5}$$

To analysis the asymptotic behavior for $t \to 0$, we rescale x' by introducing $x = tx'$ so that we have

$$S = \frac{1}{t} \left(\sum_{\text{internal face } f} x_f S_f + \sum_{\text{boundary face } f} \left(x_f S'_f - \frac{1}{2} x_f^2 \right) \right) - \frac{x_f}{2} \tag{3.6}$$

Thus, the asymptotic behavior for $t \to 0$ can be analyzed by the stationary phase approximation by defining the effective action as

eq:effectiveactiont20

$$S_{\text{eff}} = \sum_{\text{internal face } f} x_f S_f + \sum_{\text{boundary face } f} \left(x_f S'_f - \frac{1}{2} x_f^2 \right) \tag{3.7}$$

1. *the critical equation for internal faces*

According to the discussion for (3.7), the action for an internal face is just (2.33). The critical equations $S'=0$ and $\mathrm{Re}(S)=0$ leads to

eq:criticaleqs

$$\begin{aligned}
\mathrm{Re}(S) = 0 \Rightarrow & \frac{Z_{vef}}{\|Z_{vef}\|} = e^{i\alpha_{ef}} \frac{Z_{v'ef}}{\|Z_{v'ef}\|} \\
\delta_{z_{vf}} S = 0 \Rightarrow & \frac{g_{ve} Z_{vef}}{\langle Z_{vef}, Z_{vef} \rangle} = \frac{g_{ve'} Z_{ve'f}}{\langle Z_{ve'f}, Z_{ve'f} \rangle} \\
\delta_{g_{ve}} S = 0 \Rightarrow & \sum_{f \text{ at } e} j_f \epsilon_{ef}(v) \frac{\langle Z_{vef}, \vec{\sigma} Z_{vef} \rangle}{\langle Z_{vef}, Z_{vef} \rangle} = 0
\end{aligned} \tag{3.8}$$

where $v = s_e$ and $v' = t_e$, the first equation implies that Z_{vef} is parallel to $Z_{v'ef}$, the second equation involves two edges e and e' satisfying $e \cap e' = v$ and $e, e' \in \partial f$, and $\epsilon_{ef}(v)$ is define by

eq:orientationep

$$\epsilon_{ef}(v) = \begin{cases} 1 & v = t_e \\ -1 & v = s_e \end{cases} \tag{3.9}$$

According to the first equation of (3.8), we can define an auxiliary normalized spinor ζ_{ef} by

eq:phiev

$$\zeta_{ef} = e^{i\phi^f_{ev}} \frac{Z_{vef}}{\|Z_{vef}\|} = e^{i\phi^f_{ev'}} \frac{Z_{v'ef}}{\|Z_{v'ef}\|}, \quad \phi^f_{ev'} - \phi^f_{ev} = \alpha_{ef}. \tag{3.10}$$

By this definition, we have

eq:parallel1

$$\begin{aligned}
& z_{vf} = (g^\dagger_{ve})^{-1} \zeta_{ef} e^{-i\phi^f_{ev}} \|Z_{vef}\| = (g^\dagger_{ve'})^{-1} \zeta_{e'f} e^{-i\phi^f_{e'v}} \|Z_{ve'f}\| \\
\Rightarrow & e^{i\phi^f_{ev}} \|Z_{vef}\| g_{ve} J \zeta_{ef} = e^{i\phi^f_{e'v}} \|Z_{ve'f}\| g_{ve'} J \zeta_{e'f}.
\end{aligned} \tag{3.11}$$

where we used the first equation of (1.118). By the second equation of (3.8), we have

$$\frac{g_{ve}\zeta_{ef}e^{-i\phi^f_{ev}}}{\|Z_{vef}\|}=\frac{g_{ve'}\zeta_{e'f}e^{-i\phi^f_{e'v}}}{\|Z_{ve'f}\|},$$
$$\Rightarrow\frac{(g^\dagger_{ve})^{-1}J\zeta_{ef}e^{i\phi^f_{ev}}}{\|Z_{vef}\|}=\frac{(g^\dagger_{ve'})^{-1}J\zeta_{e'f}e^{i\phi^f_{e'v}}}{\|Z_{ve'f}\|} \tag{3.12}$$

where we used the second equation of (1.118).

The equations (3.11) and (3.12) both take the form $g_{ve}\xi_{ef}=g_{ve'}\xi_{e'f}$ for some spinors ξ_{ef} and $\xi_{e'f}$. This equation means that the $\mathrm{SL}(2,\mathbb{C})$ element makes the spinors ξ_{ef} and $\xi_{e'f}$ equation to each other after transporting them to the vertex v from the edges e and e'. Putting the two equations (3.11) and (3.12) together, we have

$$g_{ve}\zeta_{ef}\otimes\zeta^\dagger_{ef}g^{-1}_{ve}=g_{ve'}\zeta_{e'f}\otimes\zeta^\dagger_{e'f}g^{-1}_{ve'},$$
$$g_{ve}J\zeta_{ef}\otimes(J\zeta_{ef})^\dagger g^{-1}_{ve}=g_{ve'}J\zeta_{e'f}\otimes(J\zeta_{e'f})^\dagger g^{-1}_{ve'} \tag{3.13}$$

which leads to

$$g_{ve}X_{ef}g^{-1}_{ve}=g_{ve'}X_{e'f}g^{-1}_{ve'} \tag{3.14}$$

with

$$X_{ef}:=\gamma j_f(\zeta_{ef}\otimes\zeta^\dagger_{ef}-J\zeta_{ef}\otimes(J\zeta_{ef})^\dagger). \tag{3.15}$$

According to (1.133), one has that

$$X^{IJ}_{ef}=2\sqrt{2}\gamma j_f u^I\wedge\iota(\zeta_{ef})^J=2\gamma j_f u^I\wedge n^J_{\zeta_{ef}} \tag{3.16}$$

is a bivector, where $n^J_\zeta=(0,\vec{n}_\zeta)$. Then, (3.14) means

$$L(g_{ve})^M{}_J L(g_{ve})^N{}_K X^{JK}_{ef}=L(g_{ve'})^M{}_J L(g_{ve'})^N{}_K X^{JK}_{e'f}. \tag{3.17}$$

This equation tells us that the $\mathrm{SL}(2,\mathbb{C})$ elements g_{ve} satisfying the critical point equation transport a bi-vector X_{ef} from segment ef of the face f to the central vertex v. This transformation makes the bivectors X_{ef} and $X_{e'f}$, for the edges e,e' belonging to the same face f, identical after the bivectors are transformed. Actually from the following discussion, we will see that the bivector X_{ef} is actually the normal to a boundary triangle of a tetrahedron.

According to the last equation of (3.8), we have

$$0=\sum_f j_f\epsilon_{ef}(v)\frac{\langle Z_{vef},\vec{\sigma}Z_{vef}\rangle}{\langle Z_{vef},Z_{vef}\rangle}=\sum_f j_f\epsilon_{ef}(v)\left\langle\frac{Z_{vef}}{\|Z_{vef}\|},\vec{\sigma}\frac{Z_{vef}}{\|Z_{vef}\|}\right\rangle$$
$$=\sum_f j_f\epsilon_{ef}(v)\langle\zeta_{ef},\vec{\sigma}\zeta_{ef}\rangle \tag{3.18}$$

It is noticed that ζ_{ef}, as a variable belonging to ef, can be given either by Z_{vef} or by $Z_{v'ef}$ with v' being the other vertex of the edge e. Then the above equation leads to

$$0=\sum_f j_f\epsilon_{ef}(v)\langle\zeta_{ef},\vec{\sigma}\zeta_{ef}\rangle=\sum_f j_f\epsilon_{ef}(v)\left\langle\frac{Z_{v'ef}}{\|Z_{v'ef}\|},\vec{\sigma}\frac{Z_{v'ef}}{\|Z_{v'ef}\|}\right\rangle=-\sum_f j_f\epsilon_{ef}(v')\left\langle\frac{Z_{v'ef}}{\|Z_{v'ef}\|},\vec{\sigma}\frac{Z_{v'ef}}{\|Z_{v'ef}\|}\right\rangle \tag{3.19}$$

where we used $\epsilon_{ef}(v)=-\epsilon_{ef}(v')$. The last equation of (3.8), when applied to v', is compatible with this equation. For $\langle\zeta_{ef},\vec{\sigma}\zeta_{ef}\rangle$, we have

$$\langle\zeta_{ef},\vec{\sigma}\zeta_{ef}\rangle=\mathrm{tr}\Big((\vec{\sigma}\zeta_{ef})\otimes\zeta^\dagger_{ef}\Big)=\mathrm{tr}\Big(\vec{\sigma}(\zeta_{ef}\otimes\zeta^\dagger_{ef})\Big)=\sqrt{2}\vec{n}_{\zeta_{ef}}, \tag{3.20}$$

where n_ζ is the unit 3-vector defined in (1.128). Thus, the equation (3.18) leads to

$$0=\sum_{f\text{ at }e}2\gamma j_f\epsilon_{ef}(v)\vec{n}_{\zeta_{ef}} \tag{3.21}$$

which results in a tetrahedron whose boundary triangle t_f, the one dual to the face f at e, takes the area $\vec{A}_f=2\gamma j_f\epsilon_{ef}(v)\vec{n}_{\zeta_{ef}}$. We could embed the tetrahedron into the Minkowski space such that its normal is u^I. Then, the bivector normal of t_f is just $X^{IJ}_{ef}=2\gamma j_f u^I\wedge n^I_{\zeta_{ef}}$.

2. the critical equation for boundary face with Thiemann's coherent state

According to (2.46) and the discussion for (3.7), for a boundary face f_b, the asymptitics need us to consider the effective action The effective action becomes

$$\begin{aligned} S_{f_b} =& j_f \left[\sum_{\text{internal } e \in \partial f} i\gamma \ln\left(\frac{\langle Z_{s_e e f}, Z_{s_e e f}\rangle}{\langle Z_{t_e e f}, Z_{t_e e f}\rangle} \right) + \sum_{\text{internal edges } e} \ln\left(\frac{\langle Z_{s_e e f}, Z_{t_e e f}\rangle^2}{\langle Z_{s_e e f}, Z_{s_e e f}\rangle \langle Z_{t_e e f}, Z_{t_e e f}\rangle} \right) + \right. \\ & \left. i\gamma \ln\left(\frac{\langle Z_{s_{e_n} e_n f}, Z_{s_{e_n} e_n f}\rangle}{\langle Z_{t_{e_0} e_0 f}, Z_{t_{e_0} e_0 f}\rangle} \right) + \ln\left(\frac{\langle Z_{s_{e_n} e_n f}, g_\ell^{-1} Z_{t_{e_0} e_0 f}\rangle^2}{\langle Z_{s_{e_n} e_n f}, Z_{s_{e_n} e_n f}\rangle \langle Z_{t_{e_0} e_0 f}, Z_{t_{e_0} e_0 f}\rangle} \right) - \frac{t}{2} j_f^2 \right] \end{aligned} \tag{3.22}$$

so that (3.7) is a result of the replacement $j_f \mapsto x/t$.

Now j_f, or equivalently x, is an integration variable, we thus need to consider $\partial_{j_f} S = 0$, that means its real and imaginary part are both 0. Let us begin with its real part. We have

$$\sum_{\text{internal edges } e} \mathrm{Re}\ln\left(\frac{\langle Z_{s_e e f}, Z_{t_e e f}\rangle^2}{\langle Z_{s_e e f}, Z_{s_e e f}\rangle \langle Z_{t_e e f}, Z_{t_e e f}\rangle} \right) + \mathrm{Re}\ln\left(\frac{e^{-t j_f} \langle Z_{s_{e_n} e_n f}, g_\ell^{-1} Z_{t_{e_0} e_0 f}\rangle^2}{\langle Z_{s_{e_n} e_n f}, Z_{s_{e_n} e_n f}\rangle \langle Z_{t_{e_0} e_0 f}, Z_{t_{e_0} e_0 f}\rangle} \right) = 0 \tag{3.23}$$

For g_ℓ, we impose the decomposition used in canonical LQG

$$g_\ell = n_{s_\ell} e^{(-ip+\phi)\tau_3} n_{t_\ell}^{-1}, \tag{3.24}$$

where $n_{e_\ell}, n_{t_\ell} \in \mathrm{SU}(2)$, $p, \phi \in \mathbb{R}$. Then, we could define new variables

$$\tilde{Z}_{s_{e_n} e_n f} = n_{t_\ell}^{-1} Z_{s_{e_n} e_n f}, \quad \tilde{Z}_{t_{e_0} e_0 f} = n_{s_\ell}^{-1} Z_{t_{e_0} e_0 f} \tag{3.25}$$

and rewrite (3.23) as

$$\sum_{\text{internal edges } e} \mathrm{Re}\ln\left(\frac{\langle Z_{s_e e f}, Z_{t_e e f}\rangle^2}{\langle Z_{s_e e f}, Z_{s_e e f}\rangle \langle Z_{t_e e f}, Z_{t_e e f}\rangle} \right) + \mathrm{Re}\ln\left(\frac{e^{-t j_f} \langle \tilde{Z}_{s_{e_n} e_n f}, e^{(ip-\phi)\tau_3} \tilde{Z}_{t_{e_0} e_0 f}\rangle^2}{\langle \tilde{Z}_{s_{e_n} e_n f}, \tilde{Z}_{s_{e_n} e_n f}\rangle \langle \tilde{Z}_{t_{e_0} e_0 f}, \tilde{Z}_{t_{e_0} e_0 f}\rangle} \right) = 0 \tag{3.26}$$

We also want to get the solution which makes the real part of the action vanishing. To get this, we need to consider the normalization factor of the coherent state ψ_g^t whose norm takes

$$\|\psi_g^t\| = \frac{\sqrt{2\sqrt{\pi}} e^{t/8} \sqrt{|p|} e^{\frac{|p|^2}{2t}}}{t^{3/4} \sqrt{\sinh(|p|)}}. \tag{3.27}$$

Instead of the transition amplitude $\langle A_f | \psi_g^t \rangle$, we need to study $\langle A_f | \psi_g^t \rangle / \|\psi_g^t\|$ and the norm of the state will provide an addition term $-p^2/(2t)$, so that vanishing the real part of effective action results int

$$\begin{aligned} & \sum_{\text{internal edges } e} \mathrm{Re}\ln\left(\frac{\langle Z_{s_e e f}, Z_{t_e e f}\rangle^2}{\langle Z_{s_e e f}, Z_{s_e e f}\rangle \langle Z_{t_e e f}, Z_{t_e e f}\rangle} \right) + \mathrm{Re}\ln\left(\frac{\langle \tilde{Z}_{s_{e_n} e_n f}, e^{(ip-\phi)\tau_3} \tilde{Z}_{t_{e_0} e_0 f}\rangle^2}{\langle \tilde{Z}_{s_{e_n} e_n f}, \tilde{Z}_{s_{e_n} e_n f}\rangle \langle \tilde{Z}_{t_{e_0} e_0 f}, \tilde{Z}_{t_{e_0} e_0 f}\rangle} \right) \\ & - \frac{|p|^2}{2 j_f t} - \frac{t}{2} j_f = 0 \end{aligned} \tag{3.28}$$

Comparing this equation with (3.28), we get

$$-\frac{|p|^2}{2 j_f t} - \frac{t}{2} j_f = -t j_f. \tag{3.29}$$

This relation leads to the area matching condition

$$j_f = \frac{|p|}{t}. \tag{3.30}$$

With this precondition, to solve (eq:realpartp 3.28), we introduce the parametrization

$$
\begin{aligned}
\frac{\tilde{Z}_{s_{e_0}e_0f}}{\|\tilde{Z}_{s_{e_0}e_0f}\|} &= \left\{\sin\left(\frac{\beta_0}{2}\right)\exp\left(\frac{i\alpha_0}{2}-\frac{i\gamma_0}{2}\right),\cos\left(\frac{\beta_0}{2}\right)\exp\left(\frac{i\alpha_0}{2}+\frac{i\gamma_0}{2}\right)\right\}\\
\frac{\tilde{Z}_{s_{e_n}e_nf}}{\|\tilde{Z}_{s_{e_n}e_nf}\|} &= \left\{\sin\left(\frac{\beta_n}{2}\right)\exp\left(\frac{i\alpha_n}{2}-\frac{i\gamma_n}{2}\right),\cos\left(\frac{\beta_n}{2}\right)\exp\left(\frac{i\alpha_n}{2}+\frac{i\gamma_n}{2}\right)\right\}
\end{aligned}
\tag{3.31}
$$

Then, we have

$$
\begin{aligned}
&\left\langle \frac{\tilde{Z}_{s_{e_n}e_nf}}{\|\tilde{Z}_{s_{e_n}e_nf}\|}, e^{(ip-\phi)\tau_3}\frac{\tilde{Z}_{s_{e_0}e_0f}}{\|\tilde{Z}_{s_{e_0}e_0f}\|}\right\rangle\\
=&e^{\frac{i}{2}(\alpha_0-\alpha_n)}\left[\sin\left(\frac{\beta_0}{2}\right)\sin\left(\frac{\beta_n}{2}\right)e^{\frac{p}{2}-\frac{i}{2}(\gamma_0-\gamma_n-\phi)}+\cos\left(\frac{\beta_0}{2}\right)\cos\left(\frac{\beta_n}{2}\right)e^{-\frac{p}{2}+\frac{i}{2}(\gamma_0-\gamma_n-\phi)}\right].
\end{aligned}
\tag{3.32}
$$

The real part of $\ln(\cdots)$ depends on its absolution value. We then need to calculate

$$
\begin{aligned}
&\left|\left\langle \frac{\tilde{Z}_{s_{e_n}e_nf}}{\|\tilde{Z}_{s_{e_n}e_nf}\|}, e^{(ip-\phi)\tau_3}\frac{\tilde{Z}_{s_{e_0}e_0f}}{\|\tilde{Z}_{s_{e_0}e_0f}\|}\right\rangle^2\right|\\
=&\left\langle \frac{\tilde{Z}_{s_{e_n}e_nf}}{\|\tilde{Z}_{s_{e_n}e_nf}\|}, e^{(ip-\phi)\tau_3}\frac{\tilde{Z}_{s_{e_0}e_0f}}{\|\tilde{Z}_{s_{e_0}e_0f}\|}\right\rangle\overline{\left\langle \frac{\tilde{Z}_{s_{e_n}e_nf}}{\|\tilde{Z}_{s_{e_n}e_nf}\|}, e^{(ip-\phi)\tau_3}\frac{\tilde{Z}_{s_{e_0}e_0f}}{\|\tilde{Z}_{s_{e_0}e_0f}\|}\right\rangle}\\
=&\frac{1}{2}(\sin(\beta_0)\sin(\beta_n)\cos(\gamma_0-\gamma_n-\phi)-\sinh(p)(\cos(\beta_0)+\cos(\beta_n))+\cos(\beta_0)\cos(\beta_n)\cosh(p)+\cosh(p))\\
=&\sin^2\left(\frac{\beta_0}{2}\right)\sin^2\left(\frac{\beta_n}{2}\right)e^{p}+\frac{1}{2}\sin(\beta_0)\sin(\beta_n)\cos(\gamma_0-\gamma_n-\phi)+\cos^2\left(\frac{\beta_0}{2}\right)\cos^2\left(\frac{\beta_n}{2}\right)e^{-p}\\
=&\left(\sin\left(\frac{\beta_0}{2}\right)\sin\left(\frac{\beta_n}{2}\right)e^{\frac{p}{2}}+\cos\left(\frac{\beta_0}{2}\right)\cos\left(\frac{\beta_n}{2}\right)e^{-\frac{p}{2}}\right)^2-\frac{1}{2}\sin(\beta_0)\sin(\beta_n)[1-\cos(\gamma_0-\gamma_n-\phi)]\\
\leq&\left(\sin\left(\frac{\beta_0}{2}\right)\sin\left(\frac{\beta_n}{2}\right)e^{\frac{p}{2}}+\cos\left(\frac{\beta_0}{2}\right)\cos\left(\frac{\beta_n}{2}\right)e^{-\frac{p}{2}}\right)^2,
\end{aligned}
\tag{3.33}
$$

where the last inequality used $\sin(\beta_0),\sin(\beta_n)>0$ due to $\beta_0,\beta_n\in(0,\pi)$. Substituting the area matching condition, we have

$$
\begin{aligned}
&e^{-tj_f}\left|\left\langle \frac{\tilde{Z}_{s_{e_n}e_nf}}{\|\tilde{Z}_{s_{e_n}e_nf}\|}, e^{(ip-\phi)\tau_3}\frac{\tilde{Z}_{s_{e_0}e_0f}}{\|\tilde{Z}_{s_{e_0}e_0f}\|}\right\rangle^2\right|\\
=&e^{-|p|}\left(\sin\left(\frac{\beta_0}{2}\right)\sin\left(\frac{\beta_n}{2}\right)e^{\frac{p}{2}}+\cos\left(\frac{\beta_0}{2}\right)\cos\left(\frac{\beta_n}{2}\right)e^{-\frac{p}{2}}\right)^2-e^{-|p|}\frac{1}{2}\sin(\beta_0)\sin(\beta_n)[1-\cos(\gamma_0-\gamma_n-\phi)]\\
=&\left(\sin\left(\frac{\beta_0}{2}\right)\sin\left(\frac{\beta_n}{2}\right)e^{\frac{p-|p|}{2}}+\cos\left(\frac{\beta_0}{2}\right)\cos\left(\frac{\beta_n}{2}\right)e^{\frac{-|p|-p}{2}}\right)^2-e^{-|p|}\frac{1}{2}\sin(\beta_0)\sin(\beta_n)[1-\cos(\gamma_0-\gamma_n-\phi)]\\
\leq&\left(\sin\left(\frac{\beta_0}{2}\right)\sin\left(\frac{\beta_n}{2}\right)e^{\frac{p-|p|}{2}}+\cos\left(\frac{\beta_0}{2}\right)\cos\left(\frac{\beta_n}{2}\right)e^{\frac{-|p|-p}{2}}\right)^2\\
\leq&\left(\sin\left(\frac{\beta_0}{2}\right)\sin\left(\frac{\beta_n}{2}\right)+\cos\left(\frac{\beta_0}{2}\right)\cos\left(\frac{\beta_n}{2}\right)\right)^2\\
=&\cos^2\left(\frac{\beta_n-\beta_0}{2}\right)\leq 1.
\end{aligned}
\tag{3.34}
$$

eqlessthanone

where the fourth step used $\pm p-|p|\leq 0$. This result with the fact

$$
\frac{\langle Z_{s_eef},Z_{t_eef}\rangle^2}{\|Z_{s_eef}\|^2\|Z_{t_eef}\|^2}\leq 1 \tag{3.35}
$$

result in the solution to (3.28) as

$$
\begin{aligned}
\frac{Z_{s_e ef}}{\|Z_{s_e ef}\|} =& e^{i\alpha_{ef}}\frac{Z_{t_e ef}}{\|Z_{t_e ef}\|},\\
\frac{\tilde{Z}_{t_{e_0} s_0 f}}{\|\tilde{Z}_{t_{e_0} s_0 f}\|} =& \begin{cases} e^{i\frac{\alpha_0-\gamma_0}{2}}(1,0), & p>0\\ e^{i\frac{\alpha_0+\gamma_0}{2}}(0,1), & p<0,\end{cases}\\
\frac{\tilde{Z}_{s_{e_n} s_n f}}{\tilde{Z}_{s_{e_n} s_n f}} =& \begin{cases} e^{i\frac{\alpha_n-\gamma_n}{2}}(1,0), & p>0\\ e^{i\frac{\alpha_n+\gamma_n}{2}}(0,1), & p<0,\end{cases}
\end{aligned}
\tag{3.36}
$$

eq:realpartsol

Now let us go to the imaginary part of $\partial_{j_f} S$. When (3.36) is satisfied, we have

$$
\begin{aligned}
&\text{phase part of }\left\langle \frac{\tilde{Z}_{s_{e_n} e_n f}}{\|\tilde{Z}_{s_{e_n} e_n f}\|}, e^{(ip-\phi)\tau_3}\frac{\tilde{Z}_{s_{e_0} e_0 f}}{\|\tilde{Z}_{s_{e_0} e_0 f}\|}\right\rangle^2\\
=&\frac{\left\langle \frac{\tilde{Z}_{s_{e_n} e_n f}}{\|\tilde{Z}_{s_{e_n} e_n f}\|}, e^{(ip-\phi)\tau_3}\frac{\tilde{Z}_{s_{e_0} e_0 f}}{\|\tilde{Z}_{s_{e_0} e_0 f}\|}\right\rangle}{\overline{\left\langle \frac{\tilde{Z}_{s_{e_n} e_n f}}{\|\tilde{Z}_{s_{e_n} e_n f}\|}, e^{(ip-\phi)\tau_3}\frac{\tilde{Z}_{s_{e_0} e_0 f}}{\|\tilde{Z}_{s_{e_0} e_0 f}\|}\right\rangle}}\\
=&\frac{e^{i(\alpha_0-\alpha_n)}\left(e^{i(\gamma_0-\gamma_n-\phi)}\cos\left(\frac{\beta_0}{2}\right)\cos\left(\frac{\beta_n}{2}\right)+e^p\sin\left(\frac{\beta_0}{2}\right)\sin\left(\frac{\beta_n}{2}\right)\right)}{\cos\left(\frac{\beta_0}{2}\right)\cos\left(\frac{\beta_n}{2}\right)+\sin\left(\frac{\beta_0}{2}\right)\sin\left(\frac{\beta_n}{2}\right)e^{p+i(\gamma_0-\gamma_n-\phi)}}
\end{aligned}
\tag{3.37}
$$

leading to

$$
\text{phase part of }\left\langle \frac{\tilde{Z}_{s_{e_n} e_n f}}{\|\tilde{Z}_{s_{e_n} e_n f}\|}, e^{(ip-\phi)\tau_3}\frac{\tilde{Z}_{s_{e_0} e_0 f}}{\|\tilde{Z}_{s_{e_0} e_0 f}\|}\right\rangle^2\Bigg|_{\text{critical point}} = e^{i(\alpha_0-\alpha_n)-\mathrm{sgn}(p)(\gamma_0-\gamma_n-\phi)}
\tag{3.38}
$$

Moreover, by definition of $\tilde{Z}$ in (3.25), (3.36) implies

$$
\frac{Z_{t_{e_0} e_0 f}}{\|Z_{t_{e_0} e_0 f}\|} = n_{s_\ell} e^{i\frac{\alpha_0-\mathrm{sgn}(p)\gamma_0}{2}}\mathring{Z}(p),\qquad \frac{Z_{s_{e_n} e_n f}}{\|Z_{s_{e_n} e_n f}\|} = n_{t_\ell} e^{i\frac{\alpha_n-\mathrm{sgn}(p)\gamma_n}{2}}\mathring{Z}(p).
\tag{3.39}
$$

where $\mathring{Z}(p)=(1,0)$ for $p>0$ and $\mathring{Z}(p)=(0,1)$ for $p<0$.

- In summary, given a boundary state

  $$
  g_\ell = n_{s_\ell} e^{(-ip+\phi)\tau_3} n_{t_\ell}^{-1}
  $$

  and the area matching condition

  $$
  j_f = \frac{|p|}{t},
  \tag{3.40}
  $$

  eq:areamatching

  $\partial_{j_f} S = 0$ for boundary face lead to

  $$
  \begin{aligned}
  &\frac{Z_{s_e ef}}{\|Z_{s_e ef}\|} = e^{i\alpha_{ef}}\frac{Z_{t_e ef}}{\|Z_{t_e ef}\|},\\
  &\frac{Z_{t_{e_0} e_0 f}}{\|Z_{t_{e_0} e_0 f}\|} = e^{i\alpha_{e_0 f}} n_{s_\ell}\mathring{Z}(p)\\
  &\frac{Z_{s_{e_n} e_n f}}{\|Z_{s_{e_n} e_n f}\|} = e^{i\alpha_{e_n f}} n_{t_\ell}\mathring{Z}(p)\\
  &\sum_{\text{internal } e\in\partial f}\gamma\ln\left(\frac{\langle Z_{s_e ef}, Z_{s_e ef}\rangle}{\langle Z_{t_e ef}, Z_{t_e ef}\rangle}\right)+\sum_{\text{internal edges } e}\mathrm{Im}\ln\left(\frac{\langle Z_{s_e ef}, Z_{t_e ef}\rangle^2}{\langle Z_{s_e ef}, Z_{s_e ef}\rangle\langle Z_{t_e ef}, Z_{t_e ef}\rangle}\right)+\\
  &\gamma\ln\left(\frac{\langle Z_{s_{e_n} e_n f}, Z_{s_{e_n} e_n f}\rangle}{\langle Z_{t_{e_0} e_0 f}, Z_{t_{e_0} e_0 f}\rangle}\right)+2(\alpha_{e_0 f}-\alpha_{e_n f})+\mathrm{sgn}(p)\phi+2k\pi=0,
  \end{aligned}
  \tag{3.41}
  $$

for $\alpha_{ef} \in \mathbb{R}$ and $k \in \mathbb{Z}$. In order to compare with the result of SF model with the SU(2) coherent state as the boundary state, we define

eq:bdyspinor

$$\xi_{s_{\ell_f}} = n_{s_\ell} \mathring{Z}(p), \;\; \xi_{t_{\ell_f}} = n_{t_\ell} \mathring{Z}(p). \tag{3.42}$$

Then, the last equation becomes

eq:djsvanishingFinal

$$\begin{aligned}
&\frac{Z_{s_e ef}}{\|Z_{s_e ef}\|} = e^{i\alpha_{ef}} \frac{Z_{t_e ef}}{\|Z_{t_e ef}\|},\\
&\frac{Z_{t_{e_0} e_0 f}}{\|Z_{t_{e_0} e_0 f}\|} = e^{i\alpha_{e_0 f}} \xi_{s_{\ell_f}}\\
&\frac{Z_{s_{e_n} e_n f}}{\|Z_{s_{e_n} e_n f}\|} = e^{i\alpha_{e_n f}} \xi_{t_{\ell_f}}\\
&\sum_{\text{internal } e\in\partial f} \gamma \ln\left(\frac{\langle Z_{s_e ef}, Z_{s_e ef}\rangle}{\langle Z_{t_e ef}, Z_{t_e ef}\rangle}\right) + \sum_{\text{internal edges } e} \operatorname{Im}\ln\left(\frac{\langle Z_{s_e ef}, Z_{t_e ef}\rangle^2}{\langle Z_{s_e ef}, Z_{s_e ef}\rangle \langle Z_{t_e ef}, Z_{t_e ef}\rangle}\right) +\\
&\gamma \ln\left(\frac{\langle Z_{s_{e_n} e_n f}, Z_{s_{e_n} e_n f}\rangle}{\langle Z_{t_{e_0} e_0 f}, Z_{t_{e_0} e_0 f}\rangle}\right) + \operatorname{Im}\ln\left(\frac{\langle \xi_{s_{\ell_f}}, Z_{t_{e_0} e_0 f}\rangle^2}{\|\xi_{s_{\ell_f}}\| \|Z_{t_{e_0} e_0 f}\|^2}\right) + \operatorname{Im}\ln\left(\frac{\langle Z_{t_{e_n} e_n f}, \xi_{t_{\ell_f}}\rangle^2}{\|Z_{t_{e_n} e_n f}\|^2 \|\xi_{t_{\ell_f}}\|^2}\right) + \operatorname{sgn}(p)\phi = 0,
\end{aligned} \tag{3.43}$$

The equations given by $\partial_z S = 0$ and $\partial_g S = 0$ for the boundary face will be the same as those for the internal faces. We will take $\partial_{z_{v_1 f}} S_{f_b}$ and $\partial_{z^\dagger_{v_1 f}} S_{f_b}$ as an example to show this. We have

$$\begin{aligned}
\partial_{z_{v_1 f}} S_{f_b} =& -(1+i\gamma)\frac{Z^\dagger_{v_1 e_0 f} g^\dagger_{v_1 e_0}}{\langle Z_{v_1 e_0 f}, Z_{v_1 e_0 f}\rangle} - (1-i\gamma)\frac{Z^\dagger_{v_1 e_1 f} g^\dagger_{v_1 e_1}}{\langle Z_{v_1 e_1 f}, Z_{v_1 e_1 f}\rangle} + 2\frac{Z^\dagger_{v_n e_n f} g_\ell^{-1} g^\dagger_{v_1 e_0}}{\langle Z_{v_n e_n f}, g_\ell Z_{v_1 e_0 f}\rangle}\\
\partial_{z^\dagger_{v_1 f}} S_{f_b} =& -(1-i\gamma)\frac{g_{v_1 e_1 f} Z_{v_1 e_1 f}}{\langle Z_{v_1 e_1 f}, Z_{v_1 e_1 f}\rangle} - (1+i\gamma)\frac{g_{v_1 e_0 f} Z_{v_1 e_0 f}}{\langle Z_{v_1 e_0 f}, Z_{v_1 e_0 f}\rangle} + 2\frac{g_{v_1 e_1} Z_{v_2 e_1 f}}{\langle Z_{v_1 e_1 f}, Z_{v_2 e_1 f}\rangle}
\end{aligned} \tag{3.44}$$

Substituting (3.43), we get

$$\begin{aligned}
\partial_{z_{v_1 f}} S_{f_b} =& -(1+i\gamma)\frac{Z^\dagger_{v_1 e_0 f} g^\dagger_{v_1 e_0}}{\|Z_{v_1 e_0 f}\|^2} - (1-i\gamma)\frac{Z^\dagger_{v_1 e_1 f} g^\dagger_{v_1 e_1}}{\|Z_{v_1 e_1 f}\|^2} + 2\frac{\|Z_{v_n e_n f}\| \xi^\dagger_{t_{\ell_f}} e^{-i\alpha_{e_n f}} n_{t_\ell} e^{(ip+\phi)\tau_3} n_{s_\ell}^{-1} g^\dagger_{v_1 e_0}}{\langle \|Z_{v_n e_n f}\| \xi_{t_{\ell_f}} e^{i\alpha_{e_n f}}, n_{t_\ell} e^{(ip+\phi)\tau_3} n_{s_\ell}^{-1} Z_{v_1 e_0 f}\rangle}\\
=& -(1+i\gamma)\frac{Z^\dagger_{v_1 e_0 f} g^\dagger_{v_1 e_0}}{\|Z_{v_1 e_0 f}\|^2} - (1-i\gamma)\frac{Z^\dagger_{v_1 e_1 f} g^\dagger_{v_1 e_1}}{\|Z_{v_1 e_1 f}\|^2} + 2\frac{\xi^\dagger_{s_{\ell_f}} g^\dagger_{v_1 e_0}}{\langle \xi_{s_{\ell_f}}, Z_{v_1 e_0 f}\rangle}\\
=& -(1+i\gamma)\frac{Z^\dagger_{v_1 e_0 f} g^\dagger_{v_1 e_0}}{\|Z_{v_1 e_0 f}\|^2} - (1-i\gamma)\frac{Z^\dagger_{v_1 e_1 f} g^\dagger_{v_1 e_1}}{\|Z_{v_1 e_1 f}\|^2} + 2\frac{Z^\dagger_{v_1 e_0 f} g^\dagger_{v_1 e_0}}{\langle Z_{v_1 e_0 f}, Z_{v_1 e_0 f}\rangle}\\
=&(1-i\gamma)\left(\frac{Z^\dagger_{v_1 e_0 f} g^\dagger_{v_1 e_0}}{\|Z_{v_1 e_0 f}\|^2} - \frac{Z^\dagger_{v_1 e_1 f} g^\dagger_{v_1 e_1}}{\|Z_{v_1 e_1 f}\|^2}\right)
\end{aligned} \tag{3.45}$$

This result leads to

eq:parallelBdyEdge

$$\frac{Z^\dagger_{v_1 e_0 f} g^\dagger_{v_1 e_0}}{\|Z_{v_1 e_0 f}\|^2} = \frac{Z^\dagger_{v_1 e_1 f} g^\dagger_{v_1 e_1}}{\|Z_{v_1 e_1 f}\|^2}. \tag{3.46}$$

The same as the result obtained by treat the face as an internal face (see the result shown below in (3.8)).

Remark: According to (3.43), ignoring the last equation therein and choose the boundary spinor as the one defined in (3.42), we come back the critical equations obtained in the model where one choose $|j_f \xi_{t_{\ell_f}}\rangle \otimes \langle j_f \xi_{s_{\ell_f}}|$ as the boundary state. Then, the results given in that model, such as the value of the action taken at the critical points, can be employed directly to our model. Let $S_f^{(0)}$ denote the value of the action taken at the critical points in the model where $|j_f \xi_{t_{\ell_f}}\rangle \otimes \langle j_f \xi_{s_{\ell_f}}|$ are chosen as the boundary state. The last equation in (3.43) can be expressed as

$$S_f^{(0)} + i j_f \operatorname{sgn}(p)\phi = 0. \tag{3.47}$$

In addition, we could also choose the boundary state as

$$\psi_\ell = \sum_{j_f} \exp\left(-\frac{t}{2}\left(j_f - \frac{|p|}{t}\right)^2 + i j_f \mathrm{sgn}(p)\phi\right) |j_f \xi_{t_{\ell_f}}\rangle \otimes \langle j_f \xi_{s_{\ell_f}}|. \tag{3.48}$$

With this coherent state and requiring $\mathrm{Re}(S_f)=0$, we will get the same critical equation as (3.43).

Now let us consider the parallel transport equation (3.46). Substituting (3.43) into it, we get

$$e^{i\alpha_{e_0 f}} \frac{g_{v_1 e_0}\xi_{s_{\ell_f}}}{\|Z_{v_1 e_0 f}\|} = e^{-i\phi^f_{e_1 v_1}} \frac{g_{v_1 e_1}\zeta_{e_1 f}}{\|Z_{v_1 e_1 f}\|}. \tag{3.49}$$

where as in (3.10) we considered the first equation in (3.43) for e_1 and define

$$\zeta_{e_1 f} = e^{i\phi^f_{e_1 v_1}} \frac{Z_{v_1 e_1 f}}{\|Z_{v_1 e_1 f}\|} = e^{i\phi^f_{e_1 v_2}} \frac{Z_{v_2 e_1 f}}{\|Z_{v_2 e_1 f}\|} \tag{3.50}$$

Moreover, we have

$$z_{v_1 f} = \|Z_{v_1 e_0 f}\| e^{i\alpha_{e_0 f}} (g^\dagger_{v_1 e_0})^{-1} \xi_{s_{\ell_f}} = \|Z_{v_1 e_1 f}\| e^{-i\phi_{e_1 v_1}} (g^\dagger_{v_1 e_0})^{-1} \zeta_{e_1 f} \tag{3.51}$$

Combining (3.49) and (3.51), we get

$$g_{v_1 e_0}\left(\xi_{s\ell_{s_f}} \otimes \xi^\dagger_{s_{\ell_f}} - J\xi_{s\ell_{s_f}} \otimes (J\xi_{s_{\ell_f}})^\dagger\right) g^{-1}_{v_1 e_0} = g_{v_1 e_1}\left(\zeta_{e_1 f} \otimes \zeta^\dagger_{e_1 f} - J\zeta_{e_1 f} \otimes (J\zeta_{e_1 f})^\dagger\right) g^{-1}_{v_1 e_1}. \tag{3.52}$$

Defining $X_{e_1 f}$ as in (3.15) and $X_{e_0 f}$ as

$$X_{e_0 f} = \gamma j_f \left(\xi_{s_{\ell_f}} \otimes \xi^\dagger_{s_{\ell_f}} - J\xi_{s_{\ell_f}} \otimes (J\xi_{s_{\ell_f}})^\dagger\right) \tag{3.53}$$

we get

$$g_{v_1 e_0} X_{e_0 f} g^{-1}_{v_1 e_0} = g_{v_1 e_1} X_{e_1 f} g^{-1}_{v_1 e_1}. \tag{3.54}$$

Similarly, the bivector at e_n should be defined as

$$X_{e_n f} = \gamma j_f \left(\xi_{t_{\ell_f}} \otimes \xi^\dagger_{t_{\ell_f}} - J\xi_{t_{\ell_f}} \otimes (J\xi_{t_{\ell_f}})^\dagger\right) \tag{3.55}$$

and we have the parallel transport equation

$$g_{v_n e_{n-1}} X_{e_{n-1} f} g^{-1}_{v_n e_{n-1}} = g_{v_n e_n} X_{e_n f} g^{-1}_{v_n e_n}. \tag{3.56}$$

In (3.54) and (3.56), the bivectors $X_{e_0 f}$ and $X_{e_n f}$ are now known variables. In other words, these two equations define the boundary condition for the bivectors. Actually, the different in dealing with the boundary edge and an internal edge is that for a boundary edge, saying e_0, we did not define $\zeta_{e_0 f}$ corresponding to it, and we also do not have the phase $\phi^f_{e_0 v_1}$ for it. However, as seen from (3.49), the spinor $\xi_{s_{\ell_f}}$ plays a role of $\zeta_{e_0 f}$ and, correspondingly $-\alpha_{e_0 f}$ plays a role of $\phi^f_{e_0 v_1}$. That is to say, we define

$$\zeta_{e_0 f} = \xi_{s_{\ell_f}},\ \zeta_{e_n f} = \xi_{t_{\ell_f}},\ \phi^f_{e_0 v_1} = -\alpha_{e_0 f},\ \phi^f_{e_n v_n} = -\alpha_{e_n f}.$$

3. *reconstruct the 4-simplex from the critical point*

According to the above discussion, the equation (3.14) can be interpreted as gluing two tetrahedra together along a common face, to finally get a 4-simplex. Now let us sketch how to get the 4-simplex explicitly. One can go to *arXiv: 1109.0499* for the details.

Given a vertex v, the construction need an order of the 5 edges connected to v. Given an order of the edges, we will use e_k to denote the kth edge in the order. The 4-simplex can be constructed from the normals

$$N_{ve} = L(g_{ve})^I{}_J u^J. \tag{3.57}$$

Here, we will assume the following nondegeneracy condition

$$\det(N_{ve'},N_{ve''},N_{ve'''},N_{ve''''})\neq 0,\quad \forall e',e'',e''',e''''\subset\{e_1,e_2,e_3,e_4,e_5\}. \tag{3.58}$$

Let us embed the vectors N_{ve} into $\mathbb{R}^5$ to define

$$\tilde{N}_{ve}=(N_{ve},1)\equiv(\tilde{N}^0_{ve},\cdots\tilde{N}^3_{ve},\tilde{N}^4_{ve}) \tag{3.59}$$

Due to the identity

$$\sum_{e,e',e'',e''',e''''}\varepsilon_{ee'e''e'''e''''}N^{\tilde{I}}_{ve}N^{\tilde{J}}_{ve'}N^{\tilde{K}}_{ve''}N^{\tilde{L}}_{ve'''}N^{\tilde{M}}_{ve''''}=\varepsilon^{\tilde{I}\tilde{J}\tilde{K}\tilde{L}\tilde{M}}. \tag{3.60}$$

where we define $\varepsilon_{e_1e_2e_3e_4e_5}=1$ and $\varepsilon^{01234}=1$, we have

$$\sum_{e,e',e'',e''',e''''}\varepsilon_{ee'e''e'''e''''}N^{\tilde{I}}_{ve}N^{\tilde{J}}_{ve'}N^{\tilde{K}}_{ve''}N^{\tilde{L}}_{ve'''}N^{\tilde{M}}_{ve''''}\varepsilon_{\tilde{N}\tilde{J}\tilde{K}\tilde{L}\tilde{M}}=-4!\delta^{\tilde{I}}_{\tilde{N}} \tag{3.61}$$

where we used the fact that $\varepsilon_{01234}=\eta_{00}\varepsilon^{01234}=-1$. Using the convention $I,J,K,L,M\in\{0,1,2,3\}$, we get

$$\sum_{e,e',e'',e''',e''''}\varepsilon_{ee'e''e'''e''''}N^{I}_{ve}N^{\tilde{J}}_{ve'}N^{\tilde{K}}_{ve''}N^{\tilde{L}}_{ve'''}N^{\tilde{M}}_{ve''''}\varepsilon_{4\tilde{J}\tilde{K}\tilde{L}\tilde{M}}=0. \tag{3.62}$$

Due to the factor $\varepsilon_{4\tilde{J}\tilde{K}\tilde{L}\tilde{M}}$, $\tilde{J},\tilde{K},\tilde{L},\tilde{M}$ can only be $0,1,2,3$. Consequently, we get

$$\sum_{e,e',e'',e''',e''''}\varepsilon_{ee'e''e'''e''''}N^{I}_{ve}N^{J}_{ve'}N^{K}_{ve''}N^{L}_{ve'''}N^{M}_{ve''''}\varepsilon_{IKLM}=0. \tag{3.63}$$

leading to the solution to

eq:closureinSimplex

$$\sum_e\tilde{\beta}_e(v)N_{ve}=0. \tag{3.64}$$

as

$$\tilde{\beta}_e(v)=\frac{\alpha}{4!}\sum_{e'e''e'''e''''}\varepsilon_{ee'e''e'''e''''}\det(N_{ve'},N_{ve''},N_{ve'''},N_{ve''''}),\ \forall\alpha\in\mathbb{R}. \tag{3.65}$$

Due to (3.14), we could define $X_{vf_{ee'}}$ for f spanned by e and e' as

eq:Xvf

$$(X_{vf})^{IJ}=L(g_{ve})^I{}_KL(g_{ve})^J{}_L(X_{ef})^{KL}=L(g_{ve'})^I{}_KL(g_{ve'})^J{}_L(X_{e'f})^{KL}. \tag{3.66}$$

According to (3.16), one gets that $X_{vf_{ee'}}$ takes the form

$$X_{vf_{ee'}}=N_{ef}\wedge Y_{ef}=N_{e'f}\wedge Y_{e'f}\Rightarrow X_{vf_{ee'}}\propto N_{ef}\wedge N_{e'f}\propto\tilde{\beta}_e(v)N_{ve}\wedge\tilde{\beta}_{e'}(v)N_{ve'}. \tag{3.67}$$

We thus could define the coefficients $C_{ee'}$ so that

eq:coefficientNNX

$$\tilde{\beta}_e(v)N_{ve}\wedge\tilde{\beta}_e(v)N_{ve}=C_{ee'}X_{vf_{ee'}}. \tag{3.68}$$

Given an edge e_o, (3.64) leads to

$$\sum_{e\neq e_o}\tilde{\beta}_{e_o}(v)N_{ve_o}\wedge\tilde{\beta}_e(v)N_{ve}=0. \tag{3.69}$$

Substituting (3.68), we get

$$0=\sum_{e\neq e_o}C_{e_oe}X_{vf_{e_oe}}. \tag{3.70}$$

The closure condition (3.21) that tells us

$$\sum_{f\text{ at }e_o}\epsilon_{e_of}(v)X_{vf}=0 \tag{3.71}$$

we thus get

$$C_{e_o e} = \tilde{C}_{e_o} \epsilon_{e_o f}(v). \tag{3.72}$$

Substituting this result into (3.68), we get

$$\tilde{\beta}_e(v) N_{ve} \wedge \tilde{\beta}_e(v) N_{ve} = \tilde{C}_e \epsilon_{e f_{ee'}}(v) X_{v f_{ee'}} \Rightarrow \tilde{C}_e = \frac{\epsilon_{e f_{ee'}}(v) (X_{v f_{ee'}})_{IJ} [\tilde{\beta}_e(v) N_{ve}^J \wedge \tilde{\beta}_e(v) N_{ve}^J]}{X_{v f_{ee'}}^{IJ} (X_{v f_{ee'}})_{IJ}}. \tag{3.73}$$

From the the above discussion, we conclude that the RHS of the second equation is independent of the choice of e'. Furthermore, a further investigation on that RHS could gives us the conclusion that C_e is independent of the choice of e, and $\tilde{C} \equiv \tilde{C}_e = C_o \alpha^2$, where α is the constant involved in $\tilde{\beta}_e(e)$. We thus reach the conclusion

$$\tilde{\beta}_e(v) N_{ve} \wedge \tilde{\beta}_e(v) N_{ve} = C_o \alpha^2 \epsilon_{e f_{ee'}}(v) X_{v f_{ee'}}. \tag{3.74}$$

Divided by $|C_o|\alpha^2$ for both side, we have

$$\begin{aligned} &\beta_e(v) N_{ve} \wedge \beta_{e'}(v) N_{ve'} = \tilde{\varepsilon}(v) \epsilon_{e f_{ee'}}(v) X_{v f_{ee'}}, \forall e, e'. \\ &\tilde{\varepsilon}(v) = \operatorname{sgn}(C_o), \quad \beta_e(v) = \frac{1}{4! \sqrt{|C_o|}} \sum_{e' e'' e''' e''''} \varepsilon_{e e' e'' e''' e''''} \det(N_{ve'}, N_{ve''}, N_{ve'''}, N_{ve''''}). \end{aligned} \tag{3.75}$$

In summary, from the solution to the critical equation, we get a set of "observables", $\beta_e(v)$ for each edge e and $\tilde{\varepsilon}(v)$, defined by (3.75). The associated 4-simplex will be construct from these observables. Specifically, we need to find the 5 vertices p_a with $a = 1, 2, 3, 4, 5$ of the 4-simplex. In the 4-simplex, either the boundary tetrahedra τ_a takes $\beta_{e_a}(v) N_{ve_a}$ as their outwards normals, or the boundary tetrahedra τ_a takes $\beta_{e_a}(v) N_{ve_a}$ as their inwards normals.

Give a boundary tetrahedron, saying, $\tau_5 = p_1 p_2 p_3 p_4$, let us choose the corresponding basis in $\mathbb{R}^4$ as

$$E_{12} = p_2 - p_1, \ E_{13} = p_3 - p_1, \ E_{14} = p_4 - p_1, F^I = \varepsilon^I{}_{JKL} E_{12}^J E_{13}^K E_{14}^L. \tag{3.76}$$

Then, any point $p \in \mathbb{R}^4$ can be decomposed as

$$p = p_1 + w^1 E_{12} + w^2 E_{13} + w^3 E_{14} + w^0 F. \tag{3.77}$$

The points in the hypersurface of the tetrahedron τ_5 are those with $w^0 = 0$. The metric under the w-coordinate is

$$g_{IJ} = \eta_{KL} \frac{\partial x^K}{\partial w_I} \frac{\partial x^L}{\partial w_J} = \begin{pmatrix} F^T \\ E_{12}^T \\ E_{13}^T \\ E_{14}^T \end{pmatrix} \eta(F, E_{12}, E_{13}, E_{14}) = \begin{pmatrix} \eta_{IJ} F^I F^J & 0 & 0 & 0 \\ 0 & \eta_{IJ} E_{12}^I E_{12}^J & \eta_{IJ} E_{12}^I E_{13}^J & \eta_{IJ} E_{12}^I E_{14}^J \\ 0 & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots \end{pmatrix} \tag{3.78}$$

The reduced volume element under the w-coordinate is

$$\frac{1}{\sqrt{|\eta^{IJ} (\mathrm{d}w^0)_I (\mathrm{d}w^0)_J|}} \sqrt{|\det(g)|} \mathrm{d}w^1 \mathrm{d}w^2 \mathrm{d}w^3 = \frac{\sqrt{|\det(g)|}}{\sqrt{|g^{00}|}} \mathrm{d}w^1 \mathrm{d}w^2 \mathrm{d}w^3 \tag{3.79}$$

Recall that the volume of the tetrahedron is 1/6 of the volume of the parallelepiped formed by E_{12}, E_{13} and E_{14}. In the w-coordinate, the parallelepiped is given by $w^0 = 0, 0 \leq w^i \leq 1$. We thus have the volume of τ_5 as

$$\begin{aligned} V_{\tau_5} =& \frac{1}{6} \int_{[0,1]^3} \frac{1}{\sqrt{|g^{00}|}} \sqrt{\det(g)} \mathrm{d}w^1 \mathrm{d}w^2 \mathrm{d}w^3 = \frac{1}{6} \frac{1}{\sqrt{|g^{00}|}} \sqrt{|\det(g)|} \\ =& \frac{1}{6} \frac{1}{\sqrt{|g^{-1}|^{00}}} \sqrt{|\det(g)|} = \sqrt{|\eta_{|J} F^I F^J}^{-1} |\det(\partial x / \partial w)| \\ =& \frac{\left|\epsilon_{IJKL} F^I E_{12}^J E_{13}^K E_{14}^L\right|}{6 \sqrt{|\eta_{IJ} F^I F^J|}} = \frac{\left|\epsilon_{IJKL} \frac{F^I}{\|F\|} E_{12}^J E_{13}^K E_{14}^L\right|}{6}. \end{aligned} \tag{3.80}$$

where we used the formula for a block matrix to get g^{-1}. With this formula, we define the volume vector as

$$V_{\tau_5}^I = \frac{1}{6} \epsilon^I{}_{JKL} E_{12}^J E_{13}^K E_{14}^L = \frac{1}{6} (* E_{12} \wedge E_{13} \wedge E_{14})^I. \tag{3.81}$$

Note that $V^I_{\tau_5}$ could be either inwards or outwards, depending on the position of the 5th vertex p_5 in the 4-simplex.

According to this discussion, we find that it is more convenient to solve the edges vectors $E_{e_ae_b} = p_b - p_a$ from the critical point data. By definition, $E_{e_ae_b}$ satisfies

$$\begin{aligned} E_{e_ae_b} + E_{e_be_a} &= 0 \\ E_{e_ae_b} + E_{e_be_c} + E_{e_ce_a} &= 0 \end{aligned} \tag{3.82}$$

According to the second equation, we have

$$\begin{aligned} &E_{e_2e_3} \wedge E_{e_2e_4} \wedge E_{e_2e_5} = (E_{e_2e_1} + E_{e_1e_3}) \wedge (E_{e_2e_1} + E_{e_1e_4}) \wedge (E_{e_2e_1} + E_{e_1e_5}) \\ =&E_{e_1e_3} \wedge E_{e_1e_4} \wedge E_{e_1e_5} + E_{e_2e_1} \wedge E_{e_1e_4} \wedge E_{e_1e_5} + E_{e_1e_3} \wedge E_{e_2e_1} \wedge E_{e_1e_5} + E_{e_1e_3} \wedge E_{e_1e_4} \wedge E_{e_2e_1} \end{aligned} \tag{3.83}$$

leading to

$$\begin{aligned} &- E_{e_1e_3} \wedge E_{e_1e_4} \wedge E_{e_1e_5} + E_{e_1e_2} \wedge E_{e_1e_4} \wedge E_{e_1e_5} - E_{e_1e_2} \wedge E_{e_1e_3} \wedge E_{e_1e_5} \\ &+ E_{e_1e_2} \wedge E_{e_1e_3} \wedge E_{e_1e_4} + E_{e_2e_3} \wedge E_{e_2e_4} \wedge E_{e_2e_5} = 0. \end{aligned} \tag{3.84}$$

We thus get

$$\mathrm{sgn}(e_ae_be_ce_de_f) * E_{e_be_c} \wedge E_{e_be_d} \wedge E_{e_be_f},\ \forall e_a \tag{3.85}$$

are either all outwards or all inwards, where we introduced the convention that e_b, e_c, e_d, e_f are the other 4 edges than e_a and $\mathrm{sgn}(e_ae_be_ce_de_f) = \epsilon_{e_ae_be_ce_de_f}$. Requiring them equal to $\pm\beta_{e_a}(v)N_{ve_a}$ implies that the resulting 4-simplex takes $\pm\beta_{e_a}(v)N_{ve_a}$ as its boundary volume vector. We thus get the equation for $E_{e_ae_b}$

$$\begin{aligned} E^{\pm}_{e_ae_b} + E^{\pm}_{e_be_a} &= 0 \\ E^{\pm}_{e_ae_b} + E^{\pm}_{e_be_c} + E^{\pm}_{e_ce_a} &= 0 \\ \mathrm{sgn}(e_ae_be_ce_de_f) * E^{\pm}_{e_be_c} \wedge E^{\pm}_{e_be_d} \wedge E^{\pm}_{e_be_f} &= \pm\beta_{e_a}(v)N_{ve_a}\sqrt{|V_4(v)|}. \end{aligned} \tag{3.86}$$

eq:equationsforE

where we introduced the 4-volume as

$$\begin{aligned} V_4(v) =& \det(\beta_{e_1}(v)N_{ve_1}, \beta_{e_1}(v)N_{ve_2}, \beta_{e_1}(v)N_{ve_3}, \beta_{e_1}(v)N_{ve_4}) \\ =& \det(-\beta_{e_1}(v)N_{ve_1}, -\beta_{e_1}(v)N_{ve_2}, -\beta_{e_1}(v)N_{ve_3}, -\beta_{e_1}(v)N_{ve_4}). \end{aligned} \tag{3.87}$$

In the last equation of (eq:equationsforE 3.86) , $\sqrt{|V_4(v)|}$ ensures that the dimensions of both sides match with each other, and we omit the factor 1/6 for convenience. Those equations in (eq:equationsforE 3.86) give us two sets of edges vectors $E^+_{e_ae_b}$ and $E^-_{e_ae_b}$, one of which generates a 4-simplex taking $\beta_{e_a}(v)N_{ve_a}$ as the outwards boundary volume vector, and the other of which generates a 4-simplex taking $\beta_{e_a}(v)N_{ve_a}$ as the inwards boundary volume vector.

According to the last equation in (eq:equationsforE 3.86), we get

$$\begin{aligned} &\begin{pmatrix} \pm\sqrt{|V_4(v)|}\beta_{e_1}(v)N^T_{ve_1} \\ \pm\sqrt{|V_4(v)|}\beta_{e_2}(v)N^T_{ve_2} \\ \pm\sqrt{|V_4(v)|}\beta_{e_3}(v)N^T_{ve_3} \\ \pm\sqrt{|V_4(v)|}\beta_{e_4}(v)N^T_{ve_4} \end{pmatrix} \cdot \eta \cdot \left(E^{\pm}_{e_5e_1},\ E^{\pm}_{e_5e_2},\ E^{\pm}_{e_5e_3},\ E^{\pm}_{e_5e_4}\right) \\ =&\mathrm{Diag}\begin{pmatrix} -\varepsilon_{IJKL}(E^{\pm}_{e_5e_1})^I(E^{\pm}_{e_5e_2})^J(E^{\pm}_{e_5e_3})^K(E^{\pm}_{e_5e_4})^L \\ \varepsilon_{IJKL}(E^{\pm}_{e_5e_2})^I(E^{\pm}_{e_5e_1})^J(E^{\pm}_{e_5e_3})^K(E^{\pm}_{e_5e_4})^L \\ -\varepsilon_{IJKL}(E^{\pm}_{e_5e_3})^I(E^{\pm}_{e_5e_1})^J(E^{\pm}_{e_5e_2})^K(E^{\pm}_{e_5e_4})^L \\ \varepsilon_{IJKL}(E^{\pm}_{e_5e_4})^I(E^{\pm}_{e_5e_1})^J(E^{\pm}_{e_5e_2})^K(E^{\pm}_{e_5e_3})^L \end{pmatrix}. \end{aligned} \tag{3.88}$$

eq:ENrelation

Determinant of both sides gives

$$|V_4(v)|^2\mathrm{sgn}(e_1e_2e_3e_4e_5)V_4(v)(-1)\det\left(E^{\pm}_{e_5e_1},\ E^{\pm}_{e_5e_2},\ E^{\pm}_{e_5e_3},\ E^{\pm}_{e_5e_4}\right) = \det\left(E^{\pm}_{e_5e_1},\ E^{\pm}_{e_5e_2},\ E^{\pm}_{e_5e_3},\ E^{\pm}_{e_5e_4}\right)^4 \tag{3.89}$$

leading to

$$\begin{aligned} V_4(v) =& -\mathrm{sgn}(e_1e_2e_3e_4e_5)\det\left(E^{\pm}_{e_5e_1}, E^{\pm}_{e_5e_2}, E^{\pm}_{e_5e_3}, E^{\pm}_{e_5e_4}\right) \\ =& -\mathrm{sgn}(e_ae_be_ce_de_f)\det\left(E^{\pm}_{e_ae_b}, E^{\pm}_{e_ae_c}, E^{\pm}_{e_ae_d}, E^{\pm}_{e_ae_f}\right). \end{aligned} \tag{3.90}$$

eq:volume4

Note that the signs $\pm$ will not affect the result of the LHS. Thus, $V_4(v)$ is the same for the 2 sets of solutions $E^{\pm}_{e_ae_b}$. Moreover, (3.88) also tells us

$$\left(E^{\pm}_{e_5e_1},\ E^{\pm}_{e_5e_2},\ E^{\pm}_{e_5e_3},\ E^{\pm}_{e_5e_4}\right)^{-1}=-\mathrm{sgn}(e_1e_2e_3e_4e_5)V_4(v)^{-1}\sqrt{|V_4(v)|}\begin{pmatrix}\pm\beta_{e_1}(v)N^T_{ve_1}\\ \pm\beta_{e_2}(v)N^T_{ve_2}\\ \pm\beta_{e_3}(v)N^T_{ve_3}\\ \pm\beta_{e_4}(v)N^T_{ve_4}\end{pmatrix}\cdot\eta, \tag{3.91}$$

where the minus sign in RHS comes from our definition $\varepsilon_{0123}=-1$.

With the relation between $E^{\pm}_{e_ae_b}$ and N_{ve_a}, we have

$$\begin{aligned}
&|V_4(v)| * (\pm)\beta_{e_a}(v)N_{ve_a}\wedge(\pm)\beta_{e_{a'}}N_{ve_{a'}}\\
=&\mathrm{sgn}(e_ae_be_ce_de_f)\mathrm{sgn}(e_{a'}e_{b'}e_{c'}e_{d'}e_{f'}) * (*E^{\pm}_{e_be_c}\wedge E^{\pm}_{e_be_d}\wedge E^{\pm}_{e_be_f})\wedge(*E^{\pm}_{e_{b'}e_{c'}}\wedge E^{\pm}_{e_{b'}e_{d'}}\wedge E^{\pm}_{e_{b'}e_{f'}})\\
=&\mathrm{sgn}(e_ae_be_ce_de_f)\mathrm{sgn}(e_{a'}e_{b'}e_{c'}e_{d'}e_{f'})\Big\{\det\Big(E^{\pm}_{e_be_c},E^{\pm}_{e_{b'}e_{c'}},E^{\pm}_{e_{b'}e_{d'}},E^{\pm}_{e_{b'}e_{f'}}\Big)E^{\pm}_{e_be_d}\wedge E^{\pm}_{e_be_f}+\\
&\det\Big(E^{\pm}_{e_be_d},E^{\pm}_{e_{b'}e_{c'}},E^{\pm}_{e_{b'}e_{d'}},E^{\pm}_{e_{b'}e_{f'}}\Big)E^{\pm}_{e_be_f}\wedge E^{\pm}_{e_be_c}+\det\Big(E^{\pm}_{e_be_f},E^{\pm}_{e_{b'}e_{c'}},E^{\pm}_{e_{b'}e_{d'}},E^{\pm}_{e_{b'}e_{f'}}\Big)E^{\pm}_{e_be_c}\wedge E^{\pm}_{e_be_d}\Big\}
\end{aligned} \tag{3.92}$$

where we used the definition $\varepsilon_{0123}=-1$ and the formula

$$\begin{aligned}
&(*(*A\wedge B\wedge C)\wedge(*E\wedge F\wedge S))_{MN}\\
=&\varepsilon^{I}{}_{JKL}A^JB^KC^L\varepsilon^{I'}{}_{J'K'L'}E^{J'}F^{K'}S^{L'}\varepsilon_{II'MN}\\
=&(-1)3!\delta^J_{[I'}\delta^K_M\delta^L_{N]}A_JB_KC_L\varepsilon^{I'}{}_{J'K'L'}E^{J'}F^{K'}S^{L'}\\
=&(-1)3!A_{[I'}B_MC_{N]}\varepsilon^{I'}{}_{J'K'L'}E^{J'}F^{K'}S^{L'}\\
=&-\Big(\varepsilon_{I'J'K'L'}A^{I'}E^{J'}F^{K'}S^{L'}B_M\wedge C_N+\varepsilon_{I'J'K'L'}B^{I'}E^{J'}F^{K'}S^{L'}C_M\wedge A_N+\varepsilon^{I'J'K'L'}C_{I'}E^{J'}F^{K'}S^{L'}A_M\wedge B_N\Big)
\end{aligned} \tag{3.93}$$

Now let us discuss the determinants. Take $\det\Big(E^{\pm}_{e_be_c},E^{\pm}_{e_{b'}e_{c'}},E^{\pm}_{e_{b'}e_{d'}},E^{\pm}_{e_{b'}e_{f'}}\Big)$ for instance. It is non-vanishing only if $b=a'$ or $c=a'$. When $b=a'$,c must be one of b',c',d',f', leading to $\det\Big(E^{\pm}_{e_{a'}e_c},E^{\pm}_{e_{b'}e_{c'}},E^{\pm}_{e_{b'}e_{d'}},E^{\pm}_{e_{b'}e_{f'}}\Big)=\det\Big(E^{\pm}_{e_{a'}e_{b'}},E^{\pm}_{e_{b'}e_{c'}},E^{\pm}_{e_{b'}e_{d'}},E^{\pm}_{e_{b'}e_{f'}}\Big)$. Similarly, when $c=a'$, b must be one of b',c',d',f', leading to $\det\Big(E^{\pm}_{e_be_{a'}},E^{\pm}_{e_{b'}e_{c'}},E^{\pm}_{e_{b'}e_{d'}},E^{\pm}_{e_{b'}e_{f'}}\Big)=\det\Big(E^{\pm}_{e_{b'}e_{a'}},E^{\pm}_{e_{b'}e_{c'}},E^{\pm}_{e_{b'}e_{d'}},E^{\pm}_{e_{b'}e_{f'}}\Big)$. Therefore, we have

$$\begin{aligned}
&\det\Big(E^{\pm}_{e_be_c},E^{\pm}_{e_{b'}e_{c'}},E^{\pm}_{e_{b'}e_{d'}},E^{\pm}_{e_{b'}e_{f'}}\Big)E^{\pm}_{e_be_d}\wedge E^{\pm}_{e_be_f}\\
=&\det\Big(E^{\pm}_{e_{b'}e_{a'}},E^{\pm}_{e_{b'}e_{c'}},E^{\pm}_{e_{b'}e_{d'}},E^{\pm}_{e_{b'}e_{f'}}\Big)\Big(-\delta_{ba'}E^{\pm}_{e_{a'}e_d}\wedge E^{\pm}_{e_{a'}e_f}+\delta_{ca'}E^{\pm}_{e_be_d}\wedge E^{\pm}_{e_be_f}\Big)
\end{aligned} \tag{3.94}$$

It is similar for the other two terms. We thus can simplify (eq:NwedgeNEE 3.92) as

$$
\begin{aligned}
&|V_4(v)| * (\pm)\beta_{e_a}(v) N_{ve_a} \wedge (\pm)\beta_{e_{a'}} N_{ve_{a'}} \\
=&\mathrm{sgn}(e_a e_b e_c e_d e_f)\mathrm{sgn}(e_{a'} e_{b'} e_{c'} e_{d'} e_{f'}) \det\Big(E^{\pm}_{e_{b'}e_{a'}}, E^{\pm}_{e_{b'}e_{c'}}, E^{\pm}_{e_{b'}e_{d'}}, E^{\pm}_{e_{b'}e_{f'}}\Big) \\
&\Bigg\{ -\delta_{ba'} E^{\pm}_{e_{a'}e_d} \wedge E^{\pm}_{e_{a'}e_f} + \delta_{ca'} E^{\pm}_{e_b e_d} \wedge E^{\pm}_{e_b e_f} - \delta_{ba'} E^{\pm}_{e_{a'}e_f} \wedge E^{\pm}_{e_{a'}e_c} + \delta_{da'} E^{\pm}_{e_b e_f} \wedge E^{\pm}_{e_b e_c} \\
&- \delta_{ba'} E^{\pm}_{e_{a'}e_c} \wedge E^{\pm}_{e_{a'}e_d} + \delta_{fa'} E^{\pm}_{e_b e_c} \wedge E^{\pm}_{e_b e_d} \Bigg\} \\
=&\mathrm{sgn}(e_a e_b e_c e_d e_f)\mathrm{sgn}(e_{a'} e_{b'} e_{c'} e_{d'} e_{f'}) \det\Big(E^{\pm}_{e_{b'}e_{a'}}, E^{\pm}_{e_{b'}e_{c'}}, E^{\pm}_{e_{b'}e_{d'}}, E^{\pm}_{e_{b'}e_{f'}}\Big) \\
&\Bigg\{ -\delta_{ba'} E^{\pm}_{e_c e_d} \wedge E^{\pm}_{e_c e_f} + \delta_{ca'} E^{\pm}_{e_b e_d} \wedge E^{\pm}_{e_b e_f} + \delta_{da'} E^{\pm}_{e_b e_f} \wedge E^{\pm}_{e_b e_c} + \delta_{fa'} E^{\pm}_{e_b e_c} \wedge E^{\pm}_{e_b e_d} \Bigg\} \\
=&\mathrm{sgn}(e_{a'} e_{b'} e_{c'} e_{d'} e_{f'}) \det\Big(E^{\pm}_{e_{b'}e_{a'}}, E^{\pm}_{e_{b'}e_{c'}}, E^{\pm}_{e_{b'}e_{d'}}, E^{\pm}_{e_{b'}e_{f'}}\Big) \\
&\Bigg\{ -\delta_{ba'} \mathrm{sgn}(e_a e_{a'} e_c e_d e_f) E^{\pm}_{e_c e_d} \wedge E^{\pm}_{e_c e_f} - \delta_{ca'} \mathrm{sgn}(e_a e_{a'} e_b e_d e_f) E^{\pm}_{e_b e_d} \wedge E^{\pm}_{e_b e_f} \\
&- \mathrm{sgn}(e_a e_{a'} e_b e_c e_f)\delta_{da'} E^{\pm}_{e_b e_c} \wedge E^{\pm}_{e_b e_f} - \mathrm{sgn}(e_a e_{a'} e_b e_c e_d)\delta_{fa'} E^{\pm}_{e_b e_c} \wedge E^{\pm}_{e_b e_d} \Bigg\} \\
=&-\mathrm{sgn}(e_{a'} e_{b'} e_{c'} e_{d'} e_{f'}) \det\Big(E^{\pm}_{e_{b'}e_{a'}}, E^{\pm}_{e_{b'}e_{c'}}, E^{\pm}_{e_{b'}e_{d'}}, E^{\pm}_{e_{b'}e_{f'}}\Big) \mathrm{sgn}(e_a e_{a'} e_c e_d e_f) E^{\pm}_{e_c e_d} \wedge E^{\pm}_{e_c e_f} \\
=&\mathrm{sgn}(e_{b'} e_{a'} e_{c'} e_{d'} e_{f'}) \det\Big(E^{\pm}_{e_{b'}e_{a'}}, E^{\pm}_{e_{b'}e_{c'}}, E^{\pm}_{e_{b'}e_{d'}}, E^{\pm}_{e_{b'}e_{f'}}\Big) \mathrm{sgn}(e_a e_{a'} e_c e_d e_f) E^{\pm}_{e_c e_d} \wedge E^{\pm}_{e_c e_f}
\end{aligned}
\tag{3.95}
$$

where in the last second step, it was observed that only one of those δ can be non-vanishing and that all the wedge products $E^{\pm}_{e_{\#1}e_{\#2}} \wedge E^{\pm}_{e_{\#1}e_{\#3}}$ include all of 3 edges $e_{\#1}, e_{\#2}, e_{\#2}$ other than e_a and $e_{a'}$, and the second step needs the calculation

$$
\begin{aligned}
&-\delta_{ba'} E^{\pm}_{e_{a'}e_d} \wedge E^{\pm}_{e_{a'}e_f} + \delta_{ca'} E^{\pm}_{e_b e_d} \wedge E^{\pm}_{e_b e_f} - \delta_{ba'} E^{\pm}_{e_{a'}e_f} \wedge E^{\pm}_{e_{a'}e_c} + \delta_{da'} E^{\pm}_{e_b e_f} \wedge E^{\pm}_{e_b e_c} \\
&-\delta_{ba'} E^{\pm}_{e_{a'}e_c} \wedge E^{\pm}_{e_{a'}e_d} + \delta_{fa'} E^{\pm}_{e_b e_c} \wedge E^{\pm}_{e_b e_d} \\
=&-\delta_{ba'} (E^{\pm}_{e_{a'}e_d} - E^{\pm}_{e_{a'}e_c}) \wedge (E^{\pm}_{e_{a'}e_c} + E^{\pm}_{e_c e_f}) + \delta_{ca'} E^{\pm}_{e_b e_d} \wedge E^{\pm}_{e_b e_f} + \delta_{da'} E^{\pm}_{e_b e_f} \wedge E^{\pm}_{e_b e_c} \\
&-\delta_{ba'} E^{\pm}_{e_{a'}e_c} \wedge E^{\pm}_{e_{a'}e_d} + \delta_{fa'} E^{\pm}_{e_b e_c} \wedge E^{\pm}_{e_b e_d} \\
=&-\delta_{ba'} E^{\pm}_{e_c e_d} \wedge E^{\pm}_{e_c e_f} + \delta_{ca'} E^{\pm}_{e_b e_d} \wedge E^{\pm}_{e_b e_f} + \delta_{da'} E^{\pm}_{e_b e_f} \wedge E^{\pm}_{e_b e_c} \\
&-\delta_{ba'} E^{\pm}_{e_{a'}e_c} \wedge (E^{\pm}_{e_{a'}e_d} - E^{\pm}_{e_c e_d}) + \delta_{fa'} E^{\pm}_{e_b e_c} \wedge E^{\pm}_{e_b e_d} \\
=&-\delta_{ba'} E^{\pm}_{e_c e_d} \wedge E^{\pm}_{e_c e_f} + \delta_{ca'} E^{\pm}_{e_b e_d} \wedge E^{\pm}_{e_b e_f} + \delta_{da'} E^{\pm}_{e_b e_f} \wedge E^{\pm}_{e_b e_c} + \delta_{fa'} E^{\pm}_{e_b e_c} \wedge E^{\pm}_{e_b e_d}
\end{aligned}
\tag{3.96}
$$

Substituting the definition of $V_4(v)$ given in (eq:volume4 3.90), we have

$$
*\beta_{e_a}(v) N_{ve_a} \wedge \beta_{e_{a'}} N_{ve_{a'}} = -\mathrm{sgn}(V_4(v))\mathrm{sgn}(e_a e_{a'} e_c e_d e_f) E^{\pm}_{e_c e_d} \wedge E^{\pm}_{e_c e_f} \tag{3.97}
$$

Combining with (eq:observables 3.75), we get

$$
\mathrm{sgn}(V_4(v))\tilde{\varepsilon}(v)\epsilon_{ef_{ee'}}(v) X_{vf_{ee'}} = \mathrm{sgn}(ee'e_c e_d e_f) * E^{\pm}_{e_c e_d}(v) \wedge E^{\pm}_{e_c e_f}(v) \tag{3.98}
$$

eq:XEWE

where we write $E^{\pm}_{e_a e_b}$ as $E^{\pm}_{e_a e_b}(v)$ to express that these edge vectors belongs to the 4-simplex dual to v.

Here, we have construct a 4-simplex dual to a vertex v. Now let us consider two adjacent 4-simplexes dual to vertices v and v' connected by the edge $e_o = (vv')$. We require the consistent orientation of the 2 4-simplex, i.e., for the 2 4-simplexes v and v' sharing the tetrahedron $e_o = [p_2(v)p_3(v)p_4(v)p_5(v)] = [p_2(v')p_3(v')p_4(v')p_5(v')]$ by the identity $p_a(v) \sim p_a(v')$ for all $a = 2, 3, 4, 5$, the 4-simplex v' should be oriented as $-[p_1(v')p_2(v')p_3(v')p_4(v')p_5(v')]$.

Applying the above procedure, we could construct two 4-simplexes dual to each of the 2 vertices. The first step in the procedure is to solve (eq:observables 3.75) to get the values of $\beta_e(v)$ for all edges e connected to v and the values of $\beta_{e'}(v')$ for all edges e' connected to v'. As we have discussed, for each vertex, we have 2 sets of solution for each of the 2 vertices, and this leads to a total of 4 sets of solutions for the pair $\{\beta_e(v), \beta_{e'}(v')\}$. They can be classified by the signs of $\beta_{e_o}(v)$ and $\beta_{e_o}(v')$. We will use, for instance, S_{+-}, to denote the solution with $\beta_{e_o}(v) > 0$ and $\beta_{e_o}(v) < 0$.

With the values of $\beta_e(v)$ and $\beta_{e'}(v')$, we could solve (3.86) for the edge vectors $E^{\pm}_{e_a e_b}(v) = p_b(v) - p_a(v)$ and $E^{\pm}_{e'_a e'_b}(v') = p_b(v') - p_a(v')$. It should be noted that, we added a sign ambiguity in (3.86). This ambiguity mirrors the sign choice made during the solution of $\{\beta_e(v), \beta_{e'}(v')\}$. Thus, in what follows, it is sufficient to consider only one of the two cases: either S_{++} or S_{--} [1]. Once this is fixed, we have totally 4 sets of edge vectors $\{E^+_{e_a e_b}(v), E^+_{e'_a e'_b}(v')\}$, $\{E^+_{e_a e_b}(v), E^-_{e'_a e'_b}(v')\}$, $\{E^-_{e_a e_b}(v), E^+_{e'_a e'_b}(v')\}$ and $\{E^+_{e_a e_b}(v), E^+_{e'_a e'_b}(v')\}$, solving (3.86).

Now let us come back to the values of $\beta_e(v)$ and $\beta_{e'}(v')$. We get the relation (3.75), i.e.,

$$\begin{aligned} \beta_{e_o} N_{ve_o} \wedge \beta_{e_a}(v) N_{ve'} &= \tilde{\varepsilon}(v) \epsilon_{e_o f_{e_o e_a}}(v) X_{v f_{e_o e_a}}, \\ \beta_{e_o} N_{v'e_o} \wedge \beta_{e'_a}(v') N_{v'e''} &= \tilde{\varepsilon}(v') \epsilon_{e_o f_{e_o e'_a}}(v') X_{v' f_{e_o e'_a}}. \end{aligned} \tag{3.99}$$

The edges are labelled such that e_a, e_o, e'_a span the face $f_{e_o e_a} = f_{e_o e'_a}$. The parallel transport relation gives us

$$\begin{aligned} (g_{v'e_o} g^{-1}_{ve_0}) \triangleright N_{ve_o} =& N_{v'e_o} \\ \tilde{\varepsilon}(v)(g_{v'e_o} g^{-1}_{ve_0}) \triangleright (\beta_{e_o}(v) N_{ve_o} \wedge \beta_{e_a}(v) N_{ve_a}) =& -\tilde{\varepsilon}(v') \beta_{e_o}(v') N_{v'e_o} \wedge \beta_{e'_a}(v') N_{v'e'_a} \end{aligned} \tag{3.100}$$

where we used the fact $\epsilon_{e_o f_{e_o e'}}(v) = -\epsilon_{e_o f_{e_o e''}}(v')$. These two equations implies

$$(g_{v'e_o} g^{-1}_{ve_0}) \triangleright N_{ve_a} = -\tilde{\varepsilon}(v)\tilde{\varepsilon}(v') \frac{\beta_{e_o}(v')\beta_{e'_a}(v')}{\beta_{e_o}(v)\beta_{e_a}(v)} N_{v'e'_a} + \alpha_{e_a} N_{v'e_o} \tag{3.101}$$

for some $\alpha_{e_a} \in \mathbb{R}$. Then, we have

$$\begin{aligned} V_4(v) =& \det\big(\beta_{e_o}(v)(g_{v'e_o} g^{-1}_{ve_0}) \triangleright N_{ve_o}, \beta_{e_1}(v)(g_{v'e_o} g^{-1}_{ve_0}) \triangleright N_{ve_1}, \beta_{e_2}(v)(g_{v'e_o} g^{-1}_{ve_0}) \triangleright N_{ve_2}, \beta_{e_3}(v)(g_{v'e_o} g^{-1}_{ve_0}) \triangleright N_{ve_3}\big) \\ =& \det\left(\beta_{e_o}(v) N_{v'e_o}, \frac{-\tilde{\varepsilon}(v)\tilde{\varepsilon}(v')\beta_{e_o}(v')}{\beta_{e_o}(v)} \beta_{e'_1}(v') N_{v'e'_1}, \frac{-\tilde{\varepsilon}(v)\tilde{\varepsilon}(v')\beta_{e_o}(v')}{\beta_{e_o}(v)} \beta_{e'_2}(v') N_{v'e'_2}, \frac{-\tilde{\varepsilon}(v)\tilde{\varepsilon}(v')\beta_{e_o}(v')}{\beta_{e_o}(v)} \beta_{e'_1}(v') N_{v'e'_1}\right) \\ =& (\tilde{\varepsilon}(v)\tilde{\varepsilon}(v'))^3 \left(\frac{\beta_{e_o}(v')}{\beta_{e_o}(v)}\right)^2 V_4(v') \\ =& \tilde{\varepsilon}(v)\tilde{\varepsilon}(v') \left(\frac{\beta_{e_o}(v')}{\beta_{e_o}(v)}\right)^2 V_4(v') \end{aligned} \tag{3.102}$$

where we used $V_4(v') = -\det\big(\beta_{e_o}(v') N_{v'e_o}, \beta_{e'_1}(v') N_{v'e_1}, \cdots, \beta_{e'_3}(v') N_{v'e'_3}\big)$ due to the consistent orientation. (3.102) implies

$$\operatorname{sgn}(V_4(v))\tilde{\varepsilon}(v) = \operatorname{sgn}(V_4(v'))\tilde{\varepsilon}(v') = \varepsilon, \tag{3.103}$$

i.e., ε is a global sign over the 2-complex. Moreover, we have

$$\frac{\beta_{e_o}(v')}{\beta_{e_o}(v)} = \left|\frac{\beta_{e_o}(v')}{\beta_{e_o}(v)}\right| = \sqrt{\frac{|V_4(v)|}{|V_4(v')|}} \tag{3.104}$$

where the first equality comes from our convention.

Let us use (3.91) to solve the edge vectors. Changing the notion to adapt the current convention, we get

$$\big(E^{s_v}_{e_4 e_o}(v),\ E^{s_v}_{e_4 e_1}(v),\ E^{s_v}_{e_4 e_2}(v),\ E^{s_v}_{e_4 e_3}(v)\big) = -V_4(v)\sqrt{|V_4(v)|}^{-1} \eta \cdot \begin{pmatrix} s_v \beta_{e_o}(v) N^T_{ve_o} \\ s_v \beta_{e_1}(v) N^T_{ve_1} \\ s_v \beta_{e_2}(v) N^T_{ve_2} \\ s_v \beta_{e_3}(v) N^T_{ve_3} \end{pmatrix}^{-1}, \tag{3.105}$$

[1] Here, we considered the further case where there are more than 2 vertices in our 2-complex. For instance, consider the case where we have 3 vertices connected by $e_o = (vv')$ and $e'_o = v'v''$. We can fix the sign ambiguity at e_o, e.g., choose $\beta_{e_o}(v) > 0$ and $\beta_{e_o}(v') > 0$. Then, we have a fixed sign of $\beta_{e'_o}(v')$ and we choose $\operatorname{sgn}(\beta_{e'_o}(v'')) = \beta_{e'_o}(v')$.

and

$$\left(E^{s_{v'}}_{e'_4 e_o}(v'),\ E^{s_{v'}}_{e'_4 e'_1}(v'),\ E^{s_{v'}}_{e'_4 e'_2}(v'),\ E^{s_{v'}}_{e'_4 e'_3}(v')\right) = V_4(v')\sqrt{|V_4(v')|}^{-1}\eta\cdot\begin{pmatrix} s_{v'}\beta_{e_o}(v')N^T_{v'e_o}\\ s_{v'}\beta_{e_1}(v')N^T_{v'e_1}\\ s_{v'}\beta_{e_2}(v')N^T_{v'e_2}\\ s_{v'}\beta_{e_3}(v')N^T_{v'e_3}\end{pmatrix}^{-1}, \qquad \text{eq:solveE2} \tag{3.106}$$

where in (eq:solveE2 3.106) there is no the minus sign in the right hand side because the consistent orientation leads to $\operatorname{sgn}(e_o e'_1 e'_2 e'_3 e'_4) = -1$.. Here we introduce $s_v, s_{v'} = \pm$, associated with a choice regarding which set of edge vectors is selected as the solution to equation (eq:equationsforE 3.86).

Applying the parallel transport to the (eq:solveE1 3.105) and using (eq:expandBByNp 3.101), we get, with $g_{v'v} := g_{v'e_o}g^{-1}_{ve_o}$,

$$\begin{aligned}
g_{v'v}\cdot\left(E^{s_v}_{e_4e_o}(v),\ E^{s_v}_{e_4e_1}(v),\ E^{s_v}_{e_4e_2}(v),\ E^{s_v}_{e_4e_3}(v)\right) &= -\frac{V_4(v)}{\sqrt{|V_4(v)|}}g_{v'v}\cdot\eta\cdot\begin{pmatrix} s_v\beta_{e_o}(v)N^T_{ve_o}\\ s_v\beta_{e_1}(v)N^T_{ve_1}\\ s_v\beta_{e_2}(v)N^T_{ve_2}\\ s_v\beta_{e_3}(v)N^T_{ve_3}\end{pmatrix}^{-1}\\
= -\frac{V_4(v)}{\sqrt{|V_4(v)|}}\eta\cdot(g^T_{v'v})^{-1}\cdot\begin{pmatrix} s_v\beta_{e_o}(v)N^T_{ve_o}\\ s_v\beta_{e_1}(v)N^T_{ve_1}\\ s_v\beta_{e_2}(v)N^T_{ve_2}\\ s_v\beta_{e_3}(v)N^T_{ve_3}\end{pmatrix}^{-1} &= -\frac{V_4(v)}{\sqrt{|V_4(v)|}}\eta\cdot\begin{pmatrix} s_v\beta_{e_o}(v)(g_{v'v}N_{ve_o})^T\\ s_v\beta_{e_1}(v)(g_{v'v}N_{ve_1})^T\\ s_v\beta_{e_2}(v)(g_{v'v}N_{ve_2})^T\\ s_v\beta_{e_3}(v)(g_{v'v}N_{ve_3})^T\end{pmatrix}^{-1}\\
= -\,\tilde\varepsilon(v)\tilde\varepsilon(v')\frac{V_4(v)}{\sqrt{|V_4(v)|}}\left(\frac{\beta_{e_o}(v')}{\beta_{e_o}(v)}\right)^{-1}\eta\cdot&\begin{pmatrix} -s_v\tilde\varepsilon(v)\tilde\varepsilon(v')\frac{\beta_{e_o}(v)^2}{\beta_{e_o}(v')}N^T_{v'e_o}\\ s_v(\beta_{e'_1}(v')N_{v'e'_1}+\tilde\alpha_{e_1}N_{v'e_o})^T\\ s_v(\beta_{e'_2}(v')N_{v'e'_2}+\tilde\alpha_{e_2}N_{v'e_o})^T\\ s_v(\beta_{e'_3}(v')N_{v'e'_3}+\tilde\alpha_{e_3}N_{v'e_o})^T\end{pmatrix}^{-1}\\
= -\,\tilde\varepsilon(v)\tilde\varepsilon(v')\frac{V_4(v)}{\sqrt{|V_4(v)|}}\sqrt{\frac{|V_4(v')|}{|V_4(v)|}}\eta\cdot&\begin{pmatrix} -s_v\tilde\varepsilon(v)\tilde\varepsilon(v')\frac{|V_4(v')|}{|V_4(v)|}\beta_{e_o}(v')N^T_{v'e_o}\\ s_v(\beta_{e'_1}(v')N_{v'e'_1}+\tilde\alpha_{e_1}N_{v'e_o})^T\\ s_v(\beta_{e'_2}(v')N_{v'e'_2}+\tilde\alpha_{e_2}N_{v'e_o})^T\\ s_v(\beta_{e'_3}(v')N_{v'e'_3}+\tilde\alpha_{e_3}N_{v'e_o})^T\end{pmatrix}^{-1}\\
= -\,\varepsilon\,\tilde\varepsilon(v')\sqrt{|V_4(v')|}\,\eta\cdot&\begin{pmatrix} -s_v\frac{V_4(v')}{V_4(v)}\beta_{e_o}(v')N^T_{v'e_o}\\ s_v(\beta_{e'_1}(v')N_{v'e'_1}+\tilde\alpha_{e_1}N_{v'e_o})^T\\ s_v(\beta_{e'_2}(v')N_{v'e'_2}+\tilde\alpha_{e_2}N_{v'e_o})^T\\ s_v(\beta_{e'_3}(v')N_{v'e'_3}+\tilde\alpha_{e_3}N_{v'e_o})^T\end{pmatrix}^{-1}\\
= -\,\frac{V_4(v')}{\sqrt{|V_4(v')|}}\,\eta\cdot&\begin{pmatrix} -s_v\frac{V_4(v')}{V_4(v)}\beta_{e_o}(v')N^T_{v'e_o}\\ s_v(\beta_{e'_1}(v')N_{v'e'_1}+\tilde\alpha_{e_1}N_{v'e_o})^T\\ s_v(\beta_{e'_2}(v')N_{v'e'_2}+\tilde\alpha_{e_2}N_{v'e_o})^T\\ s_v(\beta_{e'_3}(v')N_{v'e'_3}+\tilde\alpha_{e_3}N_{v'e_o})^T\end{pmatrix}^{-1}
\end{aligned}\tag{3.107}$$

where we substitute (eq:dividedbetabetap 3.104). We now have a matrix taking the form

$$A = \left(av_1, v_2+\alpha_2 v_1, v_3+\alpha_3 v_1, v_4+\alpha_4 v_1\right)^T \tag{3.108}$$

where v_i are column vectors. We have

$$\begin{aligned}
(A^{-1})_{im} &= \frac{1}{3!}\det(A)^{-1}\sum_{jklnpq}\epsilon_{ijkl}\epsilon_{mnpq}A_{jn}A_{kp}A_{lq}\\
\overset{\text{says } m=2}{\Rightarrow}(A^{-1})_{i2} &= \det(A)^{-1}\sum_{jkl}\epsilon_{ijkl}A_{j1}A_{k4}A_{l3}\\
&= a^{-1}\det(v_1,v_2,v_3,v_4)^{-1}\sum_{jkl}\epsilon_{ijkl}a\times v_1^j v_4^k v_3^l\\
&= ((v_1,v_2,v_3,v_4)^{-1})_{i2}.
\end{aligned}\tag{3.109}$$

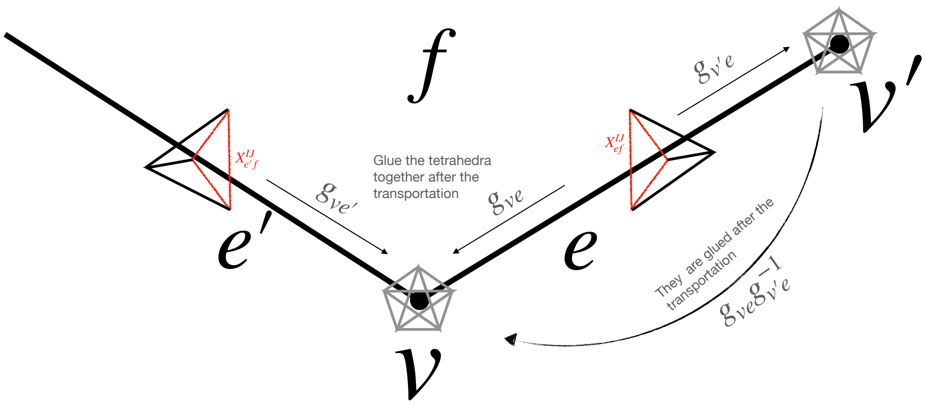


FIG. 6: A face in $\mathcal{K}$ with data

According to this formula, we have

$$
\begin{aligned}
& g_{v'v}\cdot\left(\text{a vector } \neq E^{s_v}_{e_oe_1}(v),\ E^{s_v}_{e_4e_1}(v),\ E^{s_v}_{e_4e_2}(v),\ E^{s_v}_{e_4e_3}(v)\right)\\
=& -\frac{V_4(v')}{\sqrt{|V_4(v')|}}\,\eta\cdot\begin{pmatrix} s_v\beta_{e_o}(v')N^T_{v'e_o}\\ s_v(\beta_{e'_1}(v')N^T_{v'e'_1}\\ s_v\beta_{e'_2}(v')N_{v'e'_2}\\ s_v\beta_{e'_3}(v')N_{v'e'_3}\end{pmatrix}^{-1} = -s_vs_{v'}\frac{V_4(v')}{\sqrt{|V_4(v')|}}\,\eta\cdot\begin{pmatrix} s_{v'}\beta_{e_o}(v')N^T_{v'e_o}\\ s_{v'}(\beta_{e'_1}(v')N^T_{v'e'_1}\\ s_{v'}\beta_{e'_2}(v')N_{v'e'_2}\\ s_{v'}\beta_{e'_3}(v')N_{v'e'_3}\end{pmatrix}^{-1}\\
=& -s_vs_{v'}\left(E^{s_{v'}}_{e'_4e_o}(v'),\ E^{s_{v'}}_{e'_4e'_1}(v'),\ E^{s_{v'}}_{e'_4e'_2}(v'),\ E^{s_{v'}}_{e'_4e'_3}(v')\right)
\end{aligned}
\tag{3.110}
$$

We thus get

$$
g_{v'e_o}g^{-1}_{ve_o}E^{s_v}_{e_de_f}(v) = -s_vs_{v'}E^{s_{v'}}_{e'_de'_f}(v') \equiv \mu_{e_o}E^{s_{v'}}_{e'_de'_f}(v'), \forall e_d, e_f, e'_d, e'_f \neq e_o. \tag{3.111}
$$

According to this equation, we conclude that the two 4-simplexes are glued by the geometrical parallel transport

$$
\Omega_{v'v} := \mu_{e_o}g_{v'e_o}g^{-1}_{ve_o}. \tag{3.112}
$$

With the edge vectors, we could get (3.98) for each of the vertex, i.e.,

$$
\begin{aligned}
\mathrm{sgn}(V_4(v))\tilde{\varepsilon}(v)\epsilon_{e_of_{e_oe_a}}(v)X_{vf_{e_oe_a}} =& \mathrm{sgn}(e_oe_ae_ce_de_f) * E^{\pm}_{e_ce_d}(v)\wedge E^{\pm}_{e_ce_f}(v)\\
\mathrm{sgn}(V_4(v'))\tilde{\varepsilon}(v')\epsilon_{e_of_{e_oe'_a}}(v')X_{vf_{e_oe'_a}} =& \mathrm{sgn}(e_oe'_ae'_ce'_de'_f) * E^{\pm}_{e'_ce'_d}(v')\wedge E^{\pm}_{e'_ce'_f}(v').
\end{aligned}
\tag{3.113}
$$

Note that this relation is true for all $e_a, e'_a \neq e_o$. According to the critical equation, we have

$$
(g_{v'e_o}g^{-1}_{ve_0})\triangleright(*E^{s_v}_{e_ce_d}(v)\wedge E^{s_v}_{e_ce_f}(v)) = *E^{s_{v'}}_{e'_ce'_d}(v')\wedge E^{s_{v'}}_{e'_ce'_f}(v'),\ \forall e_c, e_d, e_f, e'_c, e'_d, e'_f \neq e_o, \tag{3.114}
$$

where we used the fact $\epsilon_{e_of_{e_oe'}}(v) = -\epsilon_{e_of_{e_oe''}}(v')$, the equation (3.103) and $\mathrm{sgn}(e_oe_ae_ce_de_f) = -\mathrm{sgn}(e_oe'_ae'_ce'_de'_f)$. Given an $g\in \mathrm{SL}(2,\mathbb{C})$, we have

$$
L(g)^I{}_M L(g)^J{}_N L(g)^K{}_O L(g)^L{}_P \epsilon_{IJKL} = \det(L(g))\epsilon_{MNOP} = \epsilon_{MNOP} \Rightarrow g* = *g. \tag{3.115}
$$

We thus have from (3.114) that

$$
(g_{v'e_o}g^{-1}_{ve_0})\triangleright E^{s_v}_{e_ce_d}(v)\wedge E^{s_v}_{e_ce_f}(v) = E^{s_{v'}}_{e'_ce'_d}(v')\wedge E^{s_{v'}}_{e'_ce'_f}(v'). \tag{3.116}
$$

which can be guaranteed by (3.111).

In Fig. 6, we give a picture to illustrate the a face with spin foam data.

Finally, let us consider a vertices v connected to a boundary edge e_0. For such a vertex, the dual 4-simplex can still be constructed by applying the $\mathrm{SL}(2,\mathbb{C})$ data g_{ve} as above. However, now we have the boundary condition (3.54). The boundary spinors gives a boundary tetrahedron and we need to see the relation between the reconstructed 4-simplex with the boundary tetrahedron.

In the 2-complex, there are 4 faces taking e_0 as its boundary. For each face f of the 4 faces, ℓ_f takes n_0, the intersection of e_0 with the boundary, as either starting point or ending point. Then, the spinor corresponding to f at n_0 can be denoted as either $\xi_{s_{\ell_f}}$ for $n_0 = s_{\ell_f}$ or $\xi_{t_{\ell_f}}$ for $n_0 = t_{\ell_f}$ according to our previous convention. For convenience, we choose to uniformly use $\xi_{e_0 f}$ to denote the spinor at n_0 associated with face f, regardless of whether n_0 is the starting or ending point of the ℓ_f. This notation helps avoid the need to distinguish between the two cases.

To begin with, we construct the boundary tetrahedron. Each face f taking e_0 as its boundary edge is spanned by e_0 and e_a with $a = 1, 2, 3, 4$, giving the faces an order. We use f_a to denote the face spanned by e_0 and e_a.

The boundary spinors satisfy the closure condition

$$\sum_{f_a} 2\gamma j_f \epsilon_{e_0 f_a}(v) \vec{n}_{\xi_{e_0 f_a}} = 0, \;\; \vec{n}_{\xi_{e_0 f}} = \langle \xi_{e_0 f} | \vec{\sigma} | \xi_{e_0 f} \rangle. \tag{3.117}$$

The boundary tetrahedron is the tetrahedron taking $\epsilon_{e_0 f_a}(v)\vec{n}_{\xi_{e_0 f_a}}$ as its ingoing/outgoing boundary vectors. Let p_a be the vertex of the boundary tetrahedron opposite to the face f_a. As before, we consider the edge vectors $E_{f_a f_b}(e_0) := p_b - p_a$. There are 2 set of edge vectors satisfying such requirements, denoted by $E^{\pm}_{f_a f_b}(e_0$ respectively. They satisfy The edge vectors satisfy the equation

eq:fromntoE

$$\begin{aligned} \vec{E}^{\pm}_{f_a f_b}(e_0) + \vec{E}^{\pm}_{f_b f_a}(e_0) &= 0 \\ \vec{E}^{\pm}_{f_a f_b}(e_0) + \vec{E}^{\pm}_{f_b f_c}(e_0) + \vec{E}^{\pm}_{f_c f_a}(e_0) &= 0 \\ \mathrm{sgn}(f_a f_b f_c f_d) \vec{E}^{\pm}_{f_a f_b}(e_0) \times \vec{E}^{\pm}_{f_a f_c}(e_0) &= \pm 2\gamma j_f \epsilon_{e_0 f_d}(v) \vec{n}_{\xi_{e_0 f_d}} \end{aligned} \tag{3.118}$$

From (eq:XEWE 3.98), we know that the edge vectors of the 4-simplex satisfy

$$\varepsilon\, \epsilon_{e_0 f_{e_0 e_a}}(v) X_{v f_{e_0 e_a}} = \mathrm{sgn}(e_0 e_a e_c e_d e_f) * E^{\pm}_{e_c e_d}(v) \wedge E^{\pm}_{e_c e_f}(v). \tag{3.119}$$

Thus, we have

$$\begin{aligned} &\mathrm{sgn}(e_0 e_a e_c e_d e_f) \epsilon^{IJ}{}_{KL} [(g^{-1}_{v e_0} \triangleright E^{\pm}_{e_c e_d}(v))^K \wedge (g^{-1}_{v e_0} \triangleright E^{\pm}_{e_c e_f}(v))^L] = \varepsilon\, \epsilon_{e_0 f_{e_0 e_a}}(v) (g^{-1}_{v e_0} \triangleright X_{v f_{e_0 e_a}})^{IJ} \\ =&\varepsilon\, \epsilon_{e_0 f_{e_0 e_a}}(v) 2\gamma j_f u^I \wedge n^J_{\xi_{e_0 f_a}} \end{aligned} \tag{3.120}$$

where (eq:bivector4D 3.16) is applied. Then, we have

$$\mathrm{sgn}(f_a f_c f_d f_f) \epsilon^{0j}{}_{kl} [(g^{-1}_{v e_0} \triangleright E^{\pm}_{e_c e_d}(v))^k \wedge (g^{-1}_{v e_0} \triangleright E^{\pm}_{e_c e_f}(v))^l] = \varepsilon\, 2\gamma j_f \epsilon_{e_0 f_{e_0 e_a}}(v) n^j_{\xi_{e_0 f_a}} \tag{3.121}$$

where we used $\mathrm{sgn}(e_0 e_a e_c e_d e_f) = \mathrm{sgn}(f_a f_c f_d f_f)$. Noticing $\epsilon^{0j}{}_{kl} = \epsilon^j{}_{kl}$ and comparing this equation with (eq:fromntoE 3.118), we get, with $\mu_{e_0} = \pm$,

eq:parallelTransportE1

$$g^{-1}_{v e_0} \triangleright E^{\pm}_{e_c e_d}(v) = \begin{cases} \mu_{e_0} E^{+}_{f_c f_d}(e_0), & \varepsilon > 0, \\ \mu_{e_0} E^{-}_{f_c f_d}(e_0), & \varepsilon < 0. \end{cases} \tag{3.122}$$

According to the above discussion, the boundary tetrahedron, reconstructed from the boundary data, should be consistent with the value of ε, so that the boundary tetrahedron is equal to the corresponding tetrahedron in the 4-simplex up to a parallel transport. Once the consistent condition is satisfied, the edge vector in boundary tetrahedron is the same as the parallel transported edge vector from the 4-simplex up to a sign μ_{e_0}. In what follows, we will use $E^{\varepsilon}_{f_c f_d}(e_0)$ to denote the edge vectors of the consistent boundary tetrahedron.

4. the value of the action at the critical point: ***for internal face***

Considering the value of the action at the critical point, we substitute the first equation of (3.8) into the action to get

$$\begin{aligned}
S_f\Big|_{\text{critical}} =& \sum_{e\in\partial f} j_f \left(\ln\left(\frac{\left\langle \frac{\|Z_{s_eef}\|e^{i\alpha_{s_et_e}}}{\|Z_{t_eef}\|} Z_{t_eef}, Z_{t_eef}\right\rangle^2}{\left\langle \frac{\|Z_{s_eef}\|e^{i\alpha_{s_et_e}}}{\|Z_{t_eef}\|} Z_{t_eef}, \frac{\|Z_{s_eef}\|e^{i\alpha_{s_et_e}}}{\|Z_{t_eef}\|} Z_{t_eef}\right\rangle \langle Z_{t_eef}, Z_{t_eef}\rangle} \right) \right.\\
&\left. + i\gamma \ln\left(\frac{\left\langle \frac{\|Z_{s_eef}\|e^{i\alpha_{s_et_e}}}{\|Z_{t_eef}\|} Z_{t_eef}, \frac{\|Z_{s_eef}\|e^{i\alpha_{s_et_e}}}{\|Z_{t_eef}\|} Z_{t_eef}\right\rangle}{\langle Z_{t_eef}, Z_{t_eef}\rangle} \right) \right)\\
=& \sum_{e\in\partial f} j_f \left(\ln(e^{-2i\alpha_{s_et_e}}) + i\gamma \ln\left(\frac{\|Z_{s_eef}\|^2}{\|Z_{t_eef}\|^2}\right)\right)\\
=& \sum_{e\in\partial f} j_f \left(-2i\left(\phi^f_{et_e} - \phi^f_{es_e}\right) + 2i\gamma\Big[\ln(\|Z_{s_eef}\|) - \ln(\|Z_{t_eef}\|)\Big]\right)\\
=& -2ij_f \sum_{v\in\partial f} \left(\phi_{eve'} + \gamma\theta_{eve'}\right),
\end{aligned} \tag{3.123}$$

where for a face, $f \equiv f_{ee'}$ with $t_e = s_{e'} = v$, we define

$$\phi_{eve'} = \phi^f_{ev} - \phi^f_{e'v}, \quad \frac{\|Z_{vef}\|}{\|Z_{ve'f}\|} = e^{\theta_{eve'}}.$$

According to the second equation of (3.8), the difference between Z_{vef} and $Z_{ve'f}$, expressed through $\phi_{eve'}$ and $\theta_{eve'}$, encapsulate the information of the parallel transport. The connection between $\phi_{eve'}$ $\theta_{eve'}$ and the parallel transport is evident in the following equation, derived from (3.12) and (3.11),

$$\begin{aligned}
g_{e'v}g_{ve}\zeta_{ef} =& e^{i\phi_{eve'}+\theta_{eve'}}\zeta_{e'f}\\
g_{e'v}g_{ve}J\zeta_{ef} =& e^{-i\phi_{eve'}-\theta_{eve'}}J\zeta_{e'f}
\end{aligned} \tag{3.124}$$

where we define $g_{e'v} = g^{-1}_{ve'}$. This equation gives

$$g_{e'v}g_{ve} = e^{i\phi_{eve'}+\theta_{eve'}}\zeta_{e'f}\otimes\zeta^\dagger_{ef} + e^{-i\phi_{eve'}-\theta_{eve'}}J\zeta_{e'f}\otimes(J\zeta_{ef})^\dagger. \tag{3.125}$$

To understand this equation, we change the form of the RHS slightly to get

$$\begin{aligned}
g_{e'v}g_{ve} =& \left(\zeta_{e'f},\ J\zeta_{e'f}\right)\begin{pmatrix} e^{i\phi_{eve'}+\theta_{eve'}} & 0\\ 0 & e^{i\phi_{eve'}+\theta_{eve'}}\end{pmatrix}\begin{pmatrix}\zeta^\dagger_{ef}\\ (J\zeta_{ef})^\dagger\end{pmatrix}\\
=& \left(\zeta_{e'f},\ J\zeta_{e'f}\right) e^{(i\phi_{eve'}+\theta_{eve'})\sigma_3}\begin{pmatrix}\zeta^\dagger_{ef}\\ (J\zeta_{ef})^\dagger\end{pmatrix}.
\end{aligned} \tag{3.126}$$

For any normalized spinor ζ, one get

$$(\zeta,\ J\zeta)\begin{pmatrix}1\\0\end{pmatrix} = \zeta, \quad (\zeta,\ J\zeta)\begin{pmatrix}\zeta^\dagger\\ (J\zeta)^\dagger\end{pmatrix} = \zeta\otimes\zeta^\dagger + J\zeta\otimes(J\zeta)^\dagger = \mathbb{1}_2. \tag{3.127}$$

This implies that

$$(\zeta,\ J\zeta)\,\sigma_3\,(\zeta,\ J\zeta)^\dagger = \zeta\otimes\zeta^\dagger - J\zeta\otimes(J\zeta)^\dagger = \vec{n}_\zeta\cdot\vec{\sigma} \tag{3.128}$$

where $\vec{n}_\zeta$ is defined in (1.128), i.e., $\left(\zeta,\ J\zeta\right)$ is the SU(2) element transforming $\hat{z}$ into $\vec{n}_\zeta$. Thus, we could see that $g_{e'v}g_{ve}$ takes the form

$$g_{e'v}g_{ve} = n(\zeta_{e'f})e^{(i\phi_{eve'}+\theta_{eve'})\sigma_3}n(\zeta_{ef})^{-1}, \tag{3.129}$$

similar to the form of the $\mathrm{SL}(2,\mathbb{C})$ element labelling the coherent state.

According to (3.126), we get

$$g_{e'v}g_{ve}g_{ve}^{\dagger}g_{e'v}^{\dagger}=(\zeta_{e'f},J\zeta_{e'f})e^{2\theta_{eve'}\sigma_3}(\zeta_{e'f},J\zeta_{e'f})^{\dagger}=\exp\Big(2\theta_{eve'}\vec{n}_{\zeta_{e'f}}\cdot\vec{\sigma}\Big). \tag{3.130}$$

By definition, we have

$$g_{ve}g_{ve}^{\dagger}=g_{ve}\sigma_0 g_{ve}^{\dagger}=N_{ve}^{I}\sigma_I. \tag{3.131}$$

Therefore, (3.130) can be rewritten as

$$g_{ve'}^{-1}N_{ve}^{I}\sigma_I g_{ve'}^{\dagger}{}^{-1}=\exp\Big(\theta_{eve'}\vec{n}_{\zeta_{e'f}}\cdot\vec{\sigma}\Big)\sigma_0\exp\Big(\theta_{eve'}\vec{n}_{\zeta_{e'f}}\cdot\vec{\sigma}\Big) \tag{3.132}$$

According to (3.15), we have

$$\vec{n}_{\zeta_{e'f}}\cdot\vec{\sigma}=\zeta_{e'f}\otimes\zeta_{e'f}^{\dagger}-J\zeta_{e'f}\otimes(J\zeta_{e'f})^{\dagger}=2\frac{X_{e'f}}{\|X_{e'f}\|} \tag{3.133}$$

where we define $\|X_{e'f}\|=\sqrt{\left|\frac{1}{2}(X_{e'f})^{IJ}(X_{e'f})_{IJ}\right|}$. Applying (3.75), we get

$$\begin{aligned}
&\frac{2}{\|X_{e'f}\|}X_{e'f}=\frac{2}{\|X_{vf}\|}g_{ve'}^{-1}X_{vf}g_{ve'}\\
=&\frac{2}{\|\beta_e(v)N_{ve}\wedge\beta_{e'}(v)N_{ve'}\|}\tilde{\varepsilon}(v)\epsilon_{ef}(v)g_{ve'}^{-1}\left(\beta_e(v)N_{ve}^{K}\wedge\beta_{e'}(v)N_{ve'}^{L}\frac{\sigma_K\tilde{\sigma}_L}{4}\right)g_{ve'}\\
=&2\tilde{\varepsilon}(v)\epsilon_{ef}(v)\mathrm{sgn}(\beta_e(v))\mathrm{sgn}(\beta_{e'}(v))\frac{\left[L(g_{ve'}^{-1})^{K}{}_{I}N_{ve}^{I}\right]\wedge u^{L}}{\left\|\left[L(g_{ve'}^{-1})\triangleright N_{ve}^{I}\right]\wedge u\right\|}\frac{\sigma_K\tilde{\sigma}_L}{4}.
\end{aligned} \tag{3.134}$$

Actually, the above calculation could tell us another result which is, even though, not directly useful for our calculation here,

$$\begin{aligned}
&2u^{I}\wedge n_{\zeta_{e'f}}^{J}\frac{\sigma_K\tilde{\sigma}_L}{4}=2\frac{X_{e'f}}{\|X_{e'f}\|}\\
=&2\tilde{\varepsilon}(v)\epsilon_{ef}(v)\mathrm{sgn}(\beta_e(v))\mathrm{sgn}(\beta_{e'}(v))\frac{\left[L(g_{ve'}^{-1})^{K}{}_{I}N_{ve}^{I}\right]\wedge u^{L}}{\left\|\left[L(g_{ve'}^{-1})\triangleright N_{ve}^{I}\right]\wedge u\right\|}\frac{\sigma_K\tilde{\sigma}_L}{4}
\end{aligned} \tag{3.135}$$

leading to

$$\frac{L(g_{ve'}^{-1})^{K}{}_{I}N_{ve}^{I}}{\left\|\left[L(g_{ve'}^{-1})\triangleright N_{ve}^{I}\right]\wedge u\right\|}=\tilde{\varepsilon}(v)\epsilon_{ef}(v)\mathrm{sgn}(\beta_e(v))\mathrm{sgn}(\beta_{e'}(v))n_{\zeta_{e'f}}^{K}+\alpha_0 u^{K},\ \text{for some }\alpha_0\in\mathbb{R}. \tag{3.136}$$

Substituting (3.134) into (3.132), we have

$$\begin{aligned}
\sigma_I L(g_{ve'}^{-1})^{I}{}_{K}N_{ve}^{K}=&\exp\left(2\theta_{eve'}\tilde{\varepsilon}(v)\epsilon_{ef}(v)\mathrm{sgn}(\beta_e(v))\mathrm{sgn}(\beta_{e'}(v))\frac{\left[L(g_{ve'}^{-1})^{K}{}_{I}N_{ve}^{I}\right]\wedge u^{L}}{\left\|\left[L(g_{ve'}^{-1})\triangleright N_{ve}^{I}\right]\wedge u\right\|}\frac{\sigma_K\tilde{\sigma}_L}{4}\right)u^{I}\sigma_I\\
&\exp\left(2\theta_{eve'}\tilde{\varepsilon}(v)\epsilon_{ef}(v)\mathrm{sgn}(\beta_e(v))\mathrm{sgn}(\beta_{e'}(v))\frac{\left[L(g_{ve'}^{-1})^{K}{}_{I}N_{ve}^{I}\right]\wedge u^{L}}{\left\|\left[L(g_{ve'}^{-1})\triangleright N_{ve}^{I}\right]\wedge u\right\|}\frac{\sigma_K\tilde{\sigma}_L}{4}\right)^{\dagger}.
\end{aligned} \tag{3.137}$$

Given a vector timelike $N^I=(\cosh(\theta),\sinh(\theta)\vec{n})$, we have

$$\frac{N^{I}\wedge u^{J}}{\|N\wedge u\|}\frac{\sigma_I\tilde{\sigma}_J}{4}=\frac{2\sinh(\theta)n^{i}}{|\sinh(\theta)|}\frac{\sigma_i\tilde{\sigma}_0}{4}=-\mathrm{sgn}(\theta)\frac{\vec{n}\cdot\vec{\sigma}}{2}. \tag{3.138}$$

Therefore, we have

$$\begin{aligned}&\exp\left(\phi\frac{N^I\wedge u^J}{\|N\wedge u\|}\frac{\sigma_I\tilde{\sigma}_J}{4}\right)u^I\sigma_I\exp\left(\phi\frac{N^I\wedge u^J}{\|N\wedge u\|}\frac{\sigma_I\tilde{\sigma}_J}{4}\right)^\dagger\\ =&\exp\left(-2\phi\mathrm{sgn}(\theta)\frac{\vec{n}\cdot\vec{\sigma}}{2}\right)=\cosh(-\mathrm{sgn}(\theta)\phi)\sigma_0+\sinh(-\mathrm{sgn}(\theta)\phi)\vec{n}\cdot\vec{\sigma}.\end{aligned}\tag{3.139}$$

According to this result, we get

$$N^I\sigma_I=\exp\left(\phi\frac{N^I\wedge u^J}{\|N\wedge u\|}\frac{\sigma_I\tilde{\sigma}_J}{4}\right)u^I\sigma_I\exp\left(\phi\frac{N^I\wedge u^J}{\|N\wedge u\|}\frac{\sigma_I\tilde{\sigma}_J}{4}\right)^\dagger\Rightarrow\phi=-|\theta|=-\mathrm{arccosh}(-u\cdot N)\tag{3.140}$$

Comparing this result with (3.137) where the vector analogous to N is $L(g_{ve'}^{-1})^I{}_K N_{ve}^K$, we conclude

$$2\theta_{eve'}\tilde{\varepsilon}(v)\epsilon_{ef}\mathrm{sgn}(\beta_e(v))\mathrm{sgn}(\beta_{e'}(v))=-\mathrm{arccosh}(-u_I L(g_{ve'}^{-1})^I{}_K N_{ve}^K)=-\mathrm{arccosh}(-N_{ve'}\cdot N_{ve}).\tag{3.141}$$

In other words, $2\theta_{eve'}\tilde{\varepsilon}(v)\epsilon_{ef}(v)\mathrm{sgn}(\beta_e(v))\mathrm{sgn}(\beta_{e'}(v))$ is the dihedral angle between N_{ve} and $N_{ve'}$. Here, we adopt the definition for dihedral angle as

$$\Theta=\begin{cases}-\mathrm{arccosh}(-\dfrac{M^IN_I}{\|M\|\|N\|}),\ \forall M \text{ and } N \text{ are both future pointing or both past pointing}\\ \quad\mathrm{arccosh}(\dfrac{M^IN_I}{\|M\|\|N\|}).\ \text{otherwise}\end{cases}\tag{3.142}$$

Let us define $\Theta_{eve'}$ as the dihedral angle between $\beta_e(v)N_{ve}$ and $\beta_{e'}(v)N_{ve'}$, i.e.,

$$\Theta_{eve'}=-\mathrm{sgn}(\beta_e(v))\mathrm{sgn}(\beta_{e'}(v))\mathrm{arccosh}(-N_{ve'}\cdot N_{ve}).\tag{3.143}$$

We get

$$2\theta_{eve'}=\tilde{\varepsilon}(v)\epsilon_{ef}(v)\Theta_{eve'}=\tilde{\varepsilon}(v)\Theta_{eve'}=\varepsilon\mathrm{sgn}(V_4(v))\Theta_{eve'},\tag{3.144}$$

where the second step is consequence of assuming the orientation of the face f, namely, the orientation $e\to v\to e'$.

To understand the physical meaning of the phase $\phi_{eve'}$, let us focus on the simplicial geometry constructed from the critical point. In the geometry, there is a triangle $t_{ee'}$ dual to the face f_{ee}. With the convention introduced in Sec. III A 3, the edge vectors of the triangle are $E_{e''e'''}(v)$ with $e'',e'''\neq e,e'$. They satisfy the conditions (see (3.86) and (3.98))

$$E_{e''e'''}(v)+E_{e'''e''''}(v)+E_{e''''e''}(v)=0,\quad \varepsilon\epsilon_{ef}(v)X_{vf}=\mathrm{sgn}(ee'e''e'''e'''')*E_{e''e'''}(v)\wedge E_{e''e''''}(v).\tag{3.145}$$

This triangle is located at the frame of the central vertex v. We could use the parallel transport g_{ve} and $g_{ve'}$ to move them to the frames of the edges e and e'. We get

$$E^{(v)}_{e''e'''}(e)=g_{ve}^{-1}\triangleright E_{e''e'''}(v),\quad E^{(v)}_{e''e'''}(e')=g_{ve'}^{-1}\triangleright E_{e''e'''}(v).\tag{3.146}$$

According to (3.145), we get, by applying (3.16)

$$*E^{(v)}_{e''e'''}(e)\wedge E^{(v)}_{e''e''''}(e)=\mathrm{sgn}(ee'e''e'''e'''')\varepsilon\epsilon_{ef}(v)X_{ef}=2\gamma j_f\mathrm{sgn}(ee'e''e'''e'''')\varepsilon\epsilon_{ef}(v)u\wedge n_{\zeta_{ef}}\tag{3.147}$$

We thus conclude that

- the vectors $E^{(v)}_{e''e'''}(e)$ for all $e'',e'''\neq e,e'$ are spatial vectors, and $\vec{n}_{\zeta_{ef}}$ is the spatial normal vector to the triangle spanned by these edge vectors; This is also true for $E^{(v)}_{e''e'''}(e')$ with $e'',e'''\neq e,e'$ and $\vec{n}_{\zeta_{e'f}}$.
- As a consequence of the last item, $E^{(v)}_{e''e'''}(e)$ and $E^{(v)}_{e''e'''}(e')$ are related by a SO(3) rotation.

According to (3.146), we have

$$E^{(v)}_{e''e'''}(e')=g_{ve'}^{-1}g_{ve}\triangleright E^{(v)}_{e''e'''}(e)=(\zeta_{e'f},\ J\zeta_{e'f})\,e^{i\phi_{eve'}+\theta_{eve'}\sigma_3}\begin{pmatrix}\zeta_{ef}^\dagger\\(J\zeta_{ef})^\dagger\end{pmatrix}\triangleright E^{(v)}_{e''e'''}(e)\tag{3.148}$$

where (3.126) is applied. As we have mentioned, $E^{(v)}_{e''e'''}(e)$ is orthogonal to $\vec{n}_{\zeta_{ef}}$. Therefore, $\begin{pmatrix}\zeta_{ef}^\dagger\\(J\zeta_{ef})^\dagger\end{pmatrix}\triangleright E^{(v)}_{e''e'''}(e)$ is orthogonal to $\begin{pmatrix}\zeta_{ef}^\dagger\\(J\zeta_{ef})^\dagger\end{pmatrix}\triangleright\vec{n}_{\zeta_{ef}}=\hat{z}$. In other words, $\begin{pmatrix}\zeta_{ef}^\dagger\\(J\zeta_{ef})^\dagger\end{pmatrix}\triangleright E^{(v)}_{e''e'''}(e)$ lies in the xy-plane. As a consequence, the boost part $\exp(\theta_{eve'}\sigma_3)$ preserves $\begin{pmatrix}\zeta_{ef}^\dagger\\(J\zeta_{ef})^\dagger\end{pmatrix}\triangleright E^{(v)}_{e''e'''}(e)$, i.e.,

$$\exp(\theta_{eve'}\sigma_3)\begin{pmatrix}\zeta_{ef}^\dagger\\(J\zeta_{ef})^\dagger\end{pmatrix}\triangleright E^{(v)}_{e''e'''}(e)=\begin{pmatrix}\zeta_{ef}^\dagger\\(J\zeta_{ef})^\dagger\end{pmatrix}\triangleright E^{(v)}_{e''e'''}(e). \tag{3.149}$$

Substituting this result into (3.148), we get

$$E^{(v)}_{e''e'''}(e')=g_{ve'}^{-1}g_{ve}\triangleright E^{(v)}_{e''e'''}(e)=(\zeta_{e'f},\ J\zeta_{e'f})\,e^{i\phi_{eve'}}\begin{pmatrix}\zeta_{ef}^\dagger\\(J\zeta_{ef})^\dagger\end{pmatrix}\triangleright E^{(v)}_{e''e'''}(e). \tag{3.150}$$

That is to say, the SU(2) parallel transport relating $E^{(v)}_{e''e'''}(e)$ and $E^{(v)}_{e''e'''}(e')$ is

$$G_{e've}=(\zeta_{e'f},\ J\zeta_{e'f})\,e^{i\phi_{eve'}\sigma_3}\begin{pmatrix}\zeta_{ef}^\dagger\\(J\zeta_{ef})^\dagger\end{pmatrix}. \tag{3.151}$$

It is easy to verify that

$$G_{e've}\vec{n}_{\zeta_{ef}}=\vec{n}_{\zeta_{e'f}}. \tag{3.152}$$

The vectors $E^{(v)}_{e''e'''}(e)$, $E^{(v)}_{e''e''''}(e)$ and $\vec{n}_{\zeta_{ef}}$ (also true for the correspondences at e') for a spatial "triad" at the reference frame of e, i.e., s_{ℓ_f}. Thus, $G_{e've}$ parallel transport the triad at s_{ℓ_f} to t_{ℓ_f}. In other words, $G_{e've}$ is the holonomy of the spin connection $\Gamma^i_{aj}=e^i_b\nabla_a e^b_j$, and, thus, $\phi_{eve'}$ can be regarded as the twist angle introduced in, e.g., *arXiv: 1005.2927*.

The above analysis explained the geometric meaning of $\theta_{eve'}$ and $\phi_{eve'}$. Now, let us consider an internal face f with boundary edges $e_0,e_1,\cdots,e_n$. The vertices surrounding the face is denoted by $v_n=e_{n-1}e_n$. Then, for each vertex, we have the dihedral angle

$$\theta_{e_{m-1}v_me_m}=\tilde{\varepsilon}(v_m)\Theta_{e_{m-1}v_me_m}, \tag{3.153}$$

and the twist angle $\phi_{e_{m-1}v_me_m}$.

To see the value of $\sum_{v_m\in\partial f}\phi_{e_{m-1}v_me_m}$, let us consider $G_{e_1v_0e_n}\cdots G_{e_2v_2e_1}G_{e_1v_1e_0}$. On the one hand, we have

$$G_{e_0v_0e_n}\cdots G_{e_2v_2e_1}G_{e_1v_1e_0}=(\zeta_{e_0f},\ J\zeta_{e_0f})\exp\left(i\sum_{v_m}\phi_{e_mv_me_{m-1}}\sigma_3\right)\begin{pmatrix}\zeta_{e_0f}^\dagger\\(J\zeta_{e_0f})^\dagger\end{pmatrix} \tag{3.154}$$

where $\sum_{v_m}\phi_{e_mv_me_{m-1}}$ is involved. On the other hand, according to (3.150) and (3.146), we have

$$\begin{aligned}&G_{e_0v_0e_n}\cdots G_{e_2v_2e_1}G_{e_1v_1e_0}\triangleright E^{(v_1)}_{e''e'''}(e_0)\\
=&g_{e_0v_0}g_{v_0e_n}\cdots g_{e_3v_2}g_{v_2e_1}g_{e_1v_1}g_{v_1e_0}\triangleright E^{(v_1)}_{e''e'''}(e_0)\\
=&g_{e_0v_0}g_{v_0e_n}\cdots g_{e_3v_2}g_{v_2e_1}g_{e_1v_1}g_{v_1e_0}\triangleright g_{v_0e_0}^{-1}\triangleright E_{e''e'''}(v_0)\\
=&g_{e_0v_0}g_{v_0e_n}\cdots g_{e_3v_2}g_{v_2e_1}g_{e_1v_1}g_{v_1e_0}g_{e_0v_0}\triangleright E_{e''e'''}(v_0)\\
=&\left(\prod_{e_m}\mu_{e_m}\right)g_{e_0v_0}\triangleright E_{e''e'''}(v_0)\\
=&\left(\prod_{e_m}\mu_{e_m}\right)E^{(v_1)}_{e''e'''}(e_0)\end{aligned} \tag{3.155}$$

where in the last second step we used (3.111). Substituting this result into (3.154), we have

$$\begin{aligned}\left(\prod_{e_m}\mu_{e_m}\right)E^{(v_1)}_{e''e'''}(e_0)&=(\zeta_{e_0f},\ J\zeta_{e_0f})\exp\left(i\sum_{v_m}\phi_{e_mv_me_{m-1}}\sigma_3\right)\begin{pmatrix}\zeta_{e_0f}^\dagger\\(J\zeta_{e_0f})^\dagger\end{pmatrix}\triangleright E^{(v_1)}_{e''e'''}(e_0)\\
&=\exp\left(i\sum_{v_m}\phi_{e_mv_me_{m-1}}\vec{n}_{\zeta_{e_0f}}\cdot\vec{\sigma}\right)\triangleright E^{(v_1)}_{e''e'''}(e_0).\end{aligned} \tag{3.156}$$

Since the rotation axis $\vec{n}_{\zeta_{e_0 f}}$ is orthogonal to $E^{(v_1)}_{e''e'''}(e_0)$ and $\exp\big(i\sum_{v_m}\phi_{e_m v_m e_{m-1}}\vec{n}_{\zeta_{e_0 f}}\cdot\vec{\sigma}\big)$ represents the rotation around the axis for $-2\sum_{v_m}\phi_{e_m v_m e_{m-1}}$ radians, we conclude from the last equation that

$$2\sum_{v_m}\phi_{e_m v_m e_{m-1}} = \left(1+\prod_{e_m}\mu_{e_m}\right)\pi. \tag{3.157}$$

Substituting all of the above result into (eq:actionCri 3.123), we finally have, for an internal face f,

$$S_{\text{internal } f}\Big|_{\text{critical}} = -ij_f\gamma\varepsilon\sum_{v\in\partial f}\operatorname{sgn}(V_4(v))\Theta_{eve'} - i\pi j_f\left(1+\prod_{e_m}\mu_{e_m}\right). \tag{3.158}$$

5. the value of the action at the critical point: ***for boundary face***

Now let us consider the boundary face starting from the half edge $(e_0 v_1)$ and ending at the half edge $(v_n e_n)$. We substitute the first 3 equations in (eq:djsvanishingFinal 3.43) into the boundary action (eq:actionBoundary 2.46) to get

$$\begin{aligned}
S_{f_b} =& j_f\Bigg[\sum_{\text{internal } e\in\partial f} i\gamma\ln\left(\frac{\langle Z_{s_e ef},Z_{s_e ef}\rangle}{\langle Z_{t_e ef},Z_{t_e ef}\rangle}\right) + \sum_{\text{internal edges } e}\ln\left(\frac{\langle Z_{s_e ef},Z_{t_e ef}\rangle^2}{\langle Z_{s_e ef},Z_{s_e ef}\rangle\langle Z_{t_e ef},Z_{t_e ef}\rangle}\right)+\\
& i\gamma\ln\left(\frac{\langle Z_{s_{e_n}e_n f},Z_{s_{e_n}e_n f}\rangle}{\langle Z_{t_{e_0}e_0 f},Z_{t_{e_0}e_0 f}\rangle}\right)+\ln\left(\frac{\langle Z_{s_{e_n}e_n f},g_\ell^{-1}Z_{t_{e_0}e_0 f}\rangle^2}{\langle Z_{s_{e_n}e_n f},Z_{s_{e_n}e_n f}\rangle\langle Z_{t_{e_0}e_0 f},Z_{t_{e_0}e_0 f}\rangle}\right)\Bigg] - \frac{t}{2}j_f(j_f+1)\\
=& j_f\Bigg[\sum_{\text{internal } e\in\partial f} i\gamma\ln\left(\frac{\|Z_{s_e ef}\|^2}{\|Z_{t_e ef}\|^2}\right) + \sum_{\text{internal edges } e}\ln\big(e^{-2i\alpha_{ef}}\big)+\\
& i\gamma\ln\left(\frac{\|Z_{s_{e_n}e_n f}\|^2}{\|Z_{t_{e_0}e_0 f}\|^2}\right)+\ln\Big(e^{i2(\alpha_{e_0 f}-\alpha_{e_n f})}\langle\xi_{t_{\ell_f}},g_\ell^{-1}\xi_{s_{\ell_f}}\rangle\Big)\Bigg] - \frac{t}{2}j_f(j_f+1)\\
=& j_f\Bigg[\sum_{\text{internal } e\in\partial f} i\gamma\ln\left(\frac{\|Z_{s_e ef}\|^2}{\|Z_{t_e ef}\|^2}\right) + \sum_{\text{internal edges } e}\ln\big(e^{-2i\alpha_{ef}}\big)+\\
& i\gamma\ln\left(\frac{\|Z_{s_{e_n}e_n f}\|^2}{\|Z_{t_{e_0}e_0 f}\|^2}\right)+\ln\Big(e^{i2(\alpha_{e_0 f}-\alpha_{e_n f})}\langle n_{t_{\ell_f}}\mathring{Z}(p),n_{t_\ell}e^{(ip-\phi)\tau_3}n^{-1}_{s_{\ell_f}}n_{s_{\ell_f}}\mathring{Z}(p)\rangle^2\Big)\Bigg] - \frac{t}{2}j_f(j_f+1)\\
=& j_f\Bigg[\sum_{\text{internal } e\in\partial f} i\gamma\ln\left(\frac{\|Z_{s_e ef}\|^2}{\|Z_{t_e ef}\|^2}\right) + \sum_{\text{internal edges } e}\ln\big(e^{-2i\alpha_{ef}}\big)+i\gamma\ln\left(\frac{\|Z_{s_{e_n}e_n f}\|^2}{\|Z_{t_{e_0}e_0 f}\|^2}\right)\\
& +\ln\Big(e^{i2(\alpha_{e_0 f}-\alpha_{e_n f})}\langle\mathring{Z}(p),e^{(ip-\phi)\frac{-i}{2}\sigma_3}\mathring{Z}(p)\rangle^2\Big)\Bigg] - \frac{t}{2}j_f(j_f+1)\\
=& j_f\Bigg[\sum_{\text{internal } e\in\partial f} i\gamma\ln\left(\frac{\|Z_{s_e ef}\|^2}{\|Z_{t_e ef}\|^2}\right) + \sum_{\text{internal edges } e}\ln\big(e^{-2i\alpha_{ef}}\big)+i\gamma\ln\left(\frac{\|Z_{s_{e_n}e_n f}\|^2}{\|Z_{t_{e_0}e_0 f}\|^2}\right)\\
& +\ln\Big(e^{i2(\alpha_{e_0 f}-\alpha_{e_n f})}e^{\operatorname{sgn}(p)(p+i\phi)}\Big)\Bigg] - \frac{t}{2}j_f(j_f+1)\\
=& j_f\Bigg[-2i\gamma\sum_{v\in\partial f}\theta_{eve'}+|p|+i\operatorname{sgn}(p)\phi-2i\sum_{\text{internal edges } e}\alpha_{ef}+2i(\alpha_{e_0 f}-\alpha_{e_n f})\Bigg] - \frac{t}{2}j_f(j_f+1)
\end{aligned} \tag{3.159}$$

eq:actionBoundaryCritical

According to (eq:phiev 3.10), we have

$$\alpha_{ef} = \phi^f_{ete} - \phi^f_{ese}. \tag{3.160}$$

Thus, we have

$$\begin{aligned}\sum_{\text{internal edges } e} \alpha_{ef} - (\alpha_{e_0 f} - \alpha_{e_n f}) &= \sum_{\text{internal edges } e} (\phi^f_{et_e} - \phi^f_{es_e}) - (\alpha_{e_0 f} - \alpha_{e_n f}) \\ &= (-\alpha_{e_0 f} - \phi^f_{e_1 v_1}) + \sum_{m\geq 2}^{n-2} \phi_{e_{m-1} v_m e_{m+1}} + (\phi^f_{e_{n-1} v_n} + \alpha_{e_n f}) \\ &= \sum_{v\in\partial f} \phi_{eve'}\end{aligned} \tag{3.161}$$

where $\phi_{eve'}$ is given by

$$\phi_{eve'} = \begin{cases} \phi^f_{ev} - \phi^f_{e'v}, & v \neq v_1,\ v\neq v_n \\ -\alpha_{e_0 f} - \phi^f_{e_1 v_1}, & v = v_1 \\ \phi^f_{e_{n-1} v_n} + \alpha_{e_n f}, & v = v_n. \end{cases} \tag{3.162}$$

Note that, $-\alpha_{e_0 f}$ plays a role of $\phi^f_{e_0 v}$, as in (3.49).

Thus, for a boundary face f the action at the critical point takes

$$S_f = -i j_f \left(\left(\sum_{v\in\partial f} 2\gamma \theta_{eve'} + 2\phi_{eve'} \right) - \operatorname{sgn}(p)\, \phi \right) + j_f |p| - \frac{t}{2} j_f (j_f + 1) \tag{3.163}$$

Substitute the area matching condition (3.40), the last term becomes

$$j_f |p| - \frac{t}{2} j_f (j_f+1) = \frac{(t-2p)^2}{8t} \tag{3.164}$$

which is not of order $O(p^0)$. However, it should be noticed that, the normalization factor of the Thiemann's coherent state has not been taken into account. The norm is

$$\|\psi^t_g\| = \frac{\sqrt{2\sqrt{\pi}} e^{t/8} \sqrt{|p|} e^{\frac{|p|^2}{2t}}}{t^{3/4}\sqrt{\sinh(|p|)}}, \tag{3.165}$$

This normalization factor, when we consider $\langle A_f | \psi^t_g\rangle / \|\psi^t_g\|$, will provide another exponential $\exp\left(-t/8 - \frac{|p|^2}{2t} + \frac{|p|}{2}\right)$, where the second terms comes from $\sinh(p)$. This exponential will provide a new term in the effective action. With the new term, we have

$$j_f |p| - \frac{t}{2} j_f (j_f+1) - \frac{|p|^2}{2t} + \frac{|p|}{2} - \frac{t}{8} = 0. \tag{3.166}$$

Thanks to the above discussion, we finally have the action at the critical point, taking

$$S_f = -i j_f \left(\left(\sum_{v\in\partial f} 2\gamma \theta_{eve'} + 2\phi_{eve'} \right) - \operatorname{sgn}(p)\, \phi \right) \tag{3.167}$$

In this expression, the particular variable we need to consider is $\theta_{e_0 v_1 e_1}$, $\theta_{e_{n-1} v_n e_n}$, $\phi_{e_0 v_1 e_1}$ and $\phi_{e_{n-1} v_n e_n}$. By definition, we have

$$e^{\theta_{e_0 v_1 e_1}} = \frac{\|Z_{v_1 e_0 f}\|}{\|Z_{v_1 e_1 f}\|}, \quad e^{\theta_{e_{n-1} v_n e_n}} = \frac{\|Z_{v_n e_{n-1} f}\|}{\|Z_{v_n e_n f}\|} \tag{3.168}$$

and the definition for ϕ is given in (3.162). Since we are considering a boundary face, the boundary condition has to be taken into account, which causes the difference in dealing with these variables. The boundary condition is given in (3.43), i.e.,

$$\frac{Z_{v_1 e_0 f}}{\|Z_{v_1 e_0 f}\|} = e^{i\alpha_{e_0 f}} \xi_{s_{\ell_f}}, \quad \frac{Z_{v_n e_n f}}{\|Z_{v_n e_n f}\|} = e^{i\alpha_{e_n f}} \xi_{t_{\ell_f}}, \tag{3.169}$$

With this boundary condition, we define

$$\zeta_{e_0f}=\xi_{s_{\ell_f}},\ \zeta_{e_nf}=\xi_{t_{\ell_f}},\ \phi^f_{e_0v_1}=-\alpha_{e_0f},\ \phi^f_{e_nv_n}=-\alpha_{e_nf}.$$

Then, according to (3.49) and its analogy for e_n, we get

$$g_{e_1v_1}g_{v_1e_0}\zeta_{e_0f}=e^{\theta_{e_0v_1e_1}+i\phi_{e_0v_1e_1}}\zeta_{e_1f},\quad g_{e_nv_n}g_{v_ne_{n-1}}\zeta_{e_{n-1}f}=e^{\theta_{e_{n-1}v_ne_n}+i\phi_{e_{n-1}v_ne_n}}\zeta_{e_nf} \tag{3.170}$$

similar form as in (3.124). We thus can use the results shown in Sec. III A 4 directly.

Now, let us discuss the value of $\sum_{v\in\partial f}\phi_{eve'}$. As in (3.154), we consider

$$G_{e_nv_ne_{n-1}}\cdots G_{e_2v_2e_1}G_{e_1v_1e_0}=(\zeta_{e_nf},\ J\zeta_{e_nf})\exp\left(i\sum_{v_m}\phi_{e_mv_me_{m-1}}\sigma_3\right)\begin{pmatrix}\zeta^\dagger_{e_0f}\\(J\zeta_{e_0f})^\dagger\end{pmatrix}. \tag{3.171}$$

This variable transform $\vec{n}_{\zeta_{e_0f}}$ to $\vec{n}_{\zeta_{e_nf}}$. As we know, in the boundary, we have 2 boundary tetrahedra at e_0 and e_n, denoted by τ_0 and τ_n respectively. $\vec{n}_{\zeta_{e_0f}}$ is the normal vector of the triangle dual to ℓ_f in τ_0, and $\vec{n}_{\zeta_{e_nf}}$ is the normal vector of the triangle dual to ℓ_f in τ_n. Now let us consider the triangle dual to ℓ_f in τ_0. As discussed in Sec. III A 3, the edge vectors of this triangle are denoted by $E^\varepsilon_{f_cf_d}(e_0)$ ($f_c,f_d\neq f$) satisfying (3.122). Since this triangle is orthogonal to $\vec{n}_{\zeta_{e_0f}}$, we that the boost part in $g_{e_1v_1}g_{v_1e_0}$, given by $\exp\Big(\theta_{e_0v_1e_1}n(\xi_{s_{\ell_f}})\sigma_3 n(\xi_{s_{\ell_f}})^{-1}\Big)$ preserves it. As a consequence, we have

$$\begin{aligned}&G_{e_nv_ne_{n-1}}\cdots G_{e_2v_2e_1}G_{e_1v_1e_0}\triangleright E^\varepsilon_{f_cf_d}(e_0)\\=&g_{e_nv_n}g_{v_ne_{n-1}}\cdots g_{e_2v_2}g_{v_2e_1}g_{e_1v_1}g_{v_1e_0}\triangleright E^\varepsilon_{f_cf_d}(e_0)\\=&\left(\prod_e\mu_e\right)E^\varepsilon_{f_c^{(n)}f_d^{(n)}}(e_n)\end{aligned} \tag{3.172}$$

where we applied (3.122) and (3.111), and $f_c^{(n)},\ f_d^{(n)}$ are the 2 faces at e_n corresponding to $f_c,\ f_d$ under a certain sense. The exact correspondence between $f_c^{(n)},\ f_d^{(n)}$ and $f_c,\ f_d$ does not matter for our further discussion and we will not define it properly.

At e_0, let us choose a triad $\vec{n}_{\zeta_{e_0f}}\equiv\vec{n}_{\xi_{s_{\ell_f}}}$, $E^\varepsilon_{f_cf_d}(e_0)$ and $E^\varepsilon_{f_af_b}(e_0)$ with $f_a,f_b,f_c,f_d\neq f$. Then, we have the corresponding triad $\vec{n}_{\zeta_{e_nf}}\equiv\vec{n}_{\xi_{t_{\ell_f}}}$, $E^\varepsilon_{f_c^{(n)}f_d^{(n)}}(e_0)$ and $E^\varepsilon_{f_a^{(n)}f_b^{(n)}}(e_0)$. The SU(2) element $G_{e_nv_ne_{n-1}}\cdots G_{e_2v_2e_1}G_{e_1v_1e_0}$ is nothing else but the parallel transport moving one triad to another. In other words, $G_{e_nv_ne_{n-1}}\cdots G_{e_2v_2e_1}G_{e_1v_1e_0}$ is the parallel transport with respect to the spin connection $\Gamma^i_{aj}=e^i_b\nabla_a e^b_j$ with e^i_b being the triad field whose values at e_0 and e_n are those aforementioned. According to (3.171),

$$2\sum_{v_m}\phi_{e_mv_me_{m-1}}:=\Phi_f$$

has the geometric interpretation of the twist angle for this holonomy. (see e.g. arXiv: 1005.2927 [2]).

Since the geometric meaning of $\theta_{eve'}$ has been given in the discussion in Sec. III A 4 and the result is still valid here. Finally ,we have

$$S_{\text{boundary }f}\Big|_{\text{critical}}=-ij_f\left(\gamma\varepsilon\sum_{v\in\partial f}\text{sgn}(V_4(v))\Theta_{eve'}+\Phi_f-\text{sgn}(p)\,\phi\right). \tag{3.173}$$

As we know, ϕ given by the holonomy of the boundary data encodes the information of twist angle and extrinsic curvature. In addition, $\sum_{v\in\partial f}\Theta_{eve'}$ has the geometric meaning of the extrinsic curvature. Thus, the terms in (3.173) coming from the pure spin foam amplitude coincide with the term coming from the boundary data.

---

[2] Indeed, according to the discussion in *arXiv: 1005.2927* (the paragraph above Eq. (16)), if the normal vectors $\vec{N}$ and $\tilde{N}$ therein are given by $\vec{N}=\langle\xi|\vec{\sigma}|\xi\rangle$ and $\tilde{N}=\langle\tilde{\xi}|\vec{\sigma}|\tilde{\xi}\rangle$, the SU(2) element V defined therein is just $V=g_\xi\exp(-i(\text{twist angle})\sigma_3/2)g_{\tilde{\xi}}^{-1}$

6. the parity symmetry of the critical points

The transformation preserving the action maps one critical to another. Besides these symmetries, there is another special symmetry, called the parity symmetry of the critical points.

Let (z_v, g_{ve}) be a critical point satisfying the critical equation (3.8). Then, consider another configuration

$$(\tilde{z}_v, \tilde{g}_{ve}) = \left(\frac{g_{ve} g_{ve}^\dagger z_v}{\|Z_{ve}\|^2}, J g_{ve} J^{-1}\right),$$

with $J g_{ve} J^{-1} = (g^\dagger)^{-1}$ by (1.118). According to the second equation of (3.8), $g_{ve} g_{ve}^\dagger z_v / \|Z_{ve}\|^2$ is independent of the choice of the edge e, and thus $\tilde{z}_v$ is well defined. By definition, we have

$$\tilde{g}_{ve}^\dagger \tilde{z}_v = \frac{Z_{ve}}{\|Z_{ve}\|^2} \tag{3.174}$$

According to the discussion in *arXiv: 1304.5626*, $(j_f, \tilde{z}_{vf}, \tilde{g}_{ve})$ gives another solution to the critical point equation. Consider a

It can be check that

$$\frac{1}{2}\operatorname{tr}\left(g\sigma_0 g^\dagger \tilde{\sigma}^0\right) - \frac{1}{2}\operatorname{tr}\left(g^{\dagger -1}\sigma_0 g^{-1}\tilde{\sigma}^0\right) = 0, \quad \frac{1}{2}\operatorname{tr}\left(g\sigma_0 g^\dagger \tilde{\sigma}^a\right) + \frac{1}{2}\operatorname{tr}\left(g^{\dagger -1}\sigma_0 g^{-1}\tilde{\sigma}^a\right) = 0, \forall a = 1,2,3, \tag{3.175}$$

we thus get that

$$N_{ve}^0 = \tilde{N}_{ve}^0, \quad N_{ve}^a = -\tilde{N}_{ve}^a, \ \forall a = 1,2,3 \tag{3.176}$$

where $\tilde{N}_{ve}$ is the normal vector obtained from the spinfoam data $(j_f, \tilde{g}_{ve}, \tilde{z}_f)$. Thus, the geometry from the data $(j_f, \tilde{g}_{ve}, \tilde{z}_f)$ is related to that from (j_f, g_{ve}, z_f) by a parity transformation $(t, x, y, z) \mapsto (t, -x, -y, -z)$.

B. some on timelike SF model

1. spacelike triangle

When the triangle is spacelike, we need to introduce the spinors

$$\xi^\alpha = \begin{pmatrix} \delta & \overline{\beta} \\ \beta & \overline{\delta} \end{pmatrix}^{-1\dagger} \xi_0^\alpha \text{ with } \xi_0^\alpha = \begin{cases} (1,0)^T, \ \alpha = + \\ (0,1)^T, \ \alpha = - \end{cases} \tag{3.177}$$

with $\begin{pmatrix} \delta & \overline{\beta} \\ \beta & \overline{\delta} \end{pmatrix}$ again belonging to SU(1,1). The parallel equations are

eq:criticaltimespacelike

$$\begin{aligned} g_{ve}\eta\xi_{ef}^\pm &= \frac{\overline{\zeta_{vef}}}{\zeta_{ve'f}} g_{ve'}\eta\xi_{e'f}^\pm \\ g_{ve}J\xi_{ef}^\pm &= \frac{\zeta_{ve'f}}{\zeta_{vef}} J\xi_{e'f}^\pm . \end{aligned} \tag{3.178}$$

We thus need to define the bivector

eq:spacelike

$$\begin{aligned} \Xi^\pm &= \pm\left(\eta\xi^\pm \otimes \xi^{\pm\dagger} - J\xi^\pm \otimes (J\eta\xi^\pm)^\dagger\right) \\ &= \pm\left(\eta\xi^\pm \otimes \xi^{\pm\dagger} + J\xi^\pm \otimes (J\xi^\pm)^\dagger\eta\right) \\ &= \pm\left(-\sigma_3\frac{1}{\sqrt{2}}\iota(J\xi^\pm)^I\tilde{\sigma}_I + \frac{1}{\sqrt{2}}\iota(J\xi^\pm)^I\sigma_I\tilde{\sigma}_3\right) \\ &= \pm\, 2\sqrt{2}\iota(J\xi^\pm)^I z^K J_{IK}, \end{aligned} \tag{3.179}$$

where the sign factor $\pm$ is introduced so that the closure condition can be written directly in terms of Ξ. The bivector $(\Xi^\pm)^{IJ}$ is

$$(\Xi^\pm)^{IJ} = \pm 2\sqrt{2}\iota(J\xi^\pm)^I \wedge z^J . \tag{3.180}$$

This bivector is glued after they are transported by g_{ve}.

2. *timelike triangle*

In timelike spinfoam model, once the triangle is timelike, we have the parallel equations

$$\begin{aligned}\frac{g_{ve}\eta l^+_{ef}}{\overline{\zeta_{vef}}} =& \frac{g_{ve'}\eta l^+_{e'f}}{\overline{\zeta_{ve'f}}},\\ g_{ve}^{-1\dagger}(l^-_{ef}+\alpha_{vef}l^+_{ef}) =& \frac{\zeta_{ve'f}}{\zeta_{vef}} g_{ve'}^{-1\dagger}(l^-_{e'f}+\alpha_{ve'f}l^+_{vef})\end{aligned} \tag{3.181}$$

eq:paralleltimelike

where, in contrast to the spacelike model, the spinors $l^{\pm}_{ef}$ in this model are additional and label the SU(1, 1) coherent state in the boundary Hilbert space of the vertex v, and $\eta=\sigma_3$ is the SU(1,1) invariant metric in $\mathbb{C}^2$. The spinors $l^{\pm}_{ef}$ are given by

$$l^{\pm} = \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha & \overline{\beta}\\ \beta & \overline{\alpha}\end{pmatrix}^{-1\dagger}\begin{pmatrix}1\\ \pm 1\end{pmatrix} \tag{3.182}$$

where the matrix $\begin{pmatrix}\alpha & \overline{\beta}\\ \beta & \overline{\alpha}\end{pmatrix}$ is an SU(1,1) element so that the metric $\eta=\sigma_3$ on $\mathbb{C}^2$ is preserved. One can check that

$${l^-_{ef}}^\dagger \eta l^+_{ef} = 1, \quad {l^+_{ef}}^\dagger \eta l^+_{ef} = 0 = {l^-_{ef}}^\dagger \eta l^-_{ef}. \tag{3.183}$$

With the spinors $l^{\pm}$, we define

$$\begin{aligned}\Xi =& i\left(\eta l^+\otimes(l^-+\alpha l^+)^\dagger - J(l^-+\alpha l^+)\otimes(J\eta l^+)^\dagger\right)\\ =& i\left(\eta l^+\otimes(l^-+\alpha l^+)^\dagger + J(l^-+\alpha l^+)\otimes(Jl^+)^\dagger\eta\right)\\ =& i\left(\eta l^+\otimes(l^-)^\dagger + \overline{\alpha}\eta l^+\otimes(l^+)^\dagger + Jl^-\otimes(Jl^+)^\dagger\eta + \overline{\alpha}Jl^+\otimes(Jl^+)^\dagger\eta\right)\\ =& i\left(\eta l^+\otimes(l^-)^\dagger + Jl^-\otimes(Jl^+)^\dagger\eta + \overline{\alpha}\eta l^+\otimes(l^+)^\dagger + \overline{\alpha}Jl^+\otimes(Jl^+)^\dagger\eta\right)\end{aligned} \tag{3.184}$$

eq:bivectortimelike

where we used in the second step

$$J\eta l^+ = \epsilon\overline{\sigma_3 l^+} = -\epsilon\overline{\sigma_3}\epsilon\epsilon\overline{l^+} = -\eta J l^+. \tag{3.185}$$

The representation of Ξ on the Minkowski space gives a bivector Ξ^{IJ}. To get the bivector, we have

$$\begin{aligned}\Xi =& \frac{1}{\sqrt{2}}\left(-\sigma_3\Big[\iota(iJl^{\pm})^I + i\overline{\alpha}\iota(Jl^+)^I\Big]\tilde{\sigma}_I + \Big[\iota(iJl^{\pm})^I + i\overline{\alpha}\iota(Jl^+)^I\Big]\sigma_I\tilde{\sigma}_3\right)\\ =& 2\sqrt{2}\Big[\iota(iJl^{\pm})^I + i\overline{\alpha}\iota(Jl^+)^I\Big]z^K\frac{\sigma_I\tilde{\sigma}_K-\sigma_K\tilde{\sigma}_I}{4}\end{aligned} \tag{3.186}$$

with $z^I=(0,0,0,1)^T$, where we utilized the equation $il^-\otimes(l^+)^\dagger = -\frac{1}{\sqrt{2}}\iota(iJl^{\pm})^I\tilde{\sigma}_I$ from (1.122) eq:spinorJspinor, with the definitions

$$\iota(iJl^{\pm})^I = \frac{1}{\sqrt{2}}\operatorname{tr}\left(iJl^-\otimes(Jl^+)^\dagger\tilde{\sigma}^I\right), \quad \iota(Jl^+)^I = \frac{1}{\sqrt{2}}\operatorname{tr}\left(Jl^+\otimes(Jl^+)^\dagger\tilde{\sigma}^I\right).$$

It can be checked that

$$\iota(iJl^{\pm})^I = \frac{1}{\sqrt{2}}(\vec{n}, i), \quad \text{with } \vec{n}.\eta.\vec{n}=1, \tag{3.187}$$

which implies that $z^{[I}\iota(iJl^{\pm})^{K]}$ are real. As demonstrated in [*ArXiv: 1810.09042*], when the tetrahedron contains both timelike and spacelike triangles, the closure condition will require that $\mathrm{Re}(\alpha)=0$, indicating that $i\alpha\iota(Jl^+)^I$ are real. We thus have the bivector Ξ^{IJ} as the representation of Ξ on the Minkowski space is

$$\Xi^{IK} = 2\sqrt{2}\Big[\iota(iJl^{\pm})^I + i\overline{\alpha}\iota(Jl^+)^I\Big]\wedge z^K. \tag{3.188}$$

This is the bivector which should be glued after they are transported by the SL(2, $\mathbb{C}$) elements g_{ve}, and the parallel equation (3.181) eq:paralleltimelike implies

$$g_{ve}\Xi_{ef}g_{ve}^{-1} = g_{ve'}\Xi_{e'f}g_{ve'}^{-1} \tag{3.189}$$

It is worth to deriving the closure condition for the action (eq:timelikeS 2.62) which is for the spacelike triangle in a timelike tetrahedron. To this end, we consider the variation of the action under the transformation

$$g_{ve}^{\dagger} \to e^{\lambda A_{ve}} g_{ve}^{\dagger} \cong (1+\lambda A_{ve}) g_{ve}^{\dagger}. \tag{3.190}$$

Then, we have

eq:closuretimelike

$$\begin{aligned}
\frac{\delta S_{fv}}{\delta\lambda} =& i\gamma j_f \left(\frac{\langle A_{ve'} Z_{ve'f}, Z_{ve'f}\rangle_\sigma + \langle Z_{ve'f}, A_{ve'} Z_{ve'f}\rangle_\sigma}{\langle Z_{ve'f}, Z_{ve'f}\rangle_\sigma} - \frac{\langle A_{ve} Z_{vef}, Z_{vef}\rangle_\sigma + \langle Z_{vef}, A_{ve} Z_{vef}\rangle_\sigma}{\langle Z_{vef}, Z_{vef}\rangle_\sigma} \right) \\
&- j_f \left(\frac{2\langle \xi_{ef}^{\alpha_f}, A_{ve} Z_{vef}\rangle_\sigma}{\langle \xi_{ef}^{\alpha_f}, Z_{vef}\rangle_\sigma} - \frac{\langle A_{ve} Z_{vef}, Z_{vef}\rangle_\sigma + \langle Z_{vef}, A_{ve} Z_{vef}\rangle_\sigma}{\langle Z_{vef}, Z_{vef}\rangle_\sigma} \right) \\
&- j_f \left(\frac{2\langle A_{ve'} Z_{ve'f}, \xi_{e'f}^{\alpha_f}\rangle_\sigma}{\langle Z_{ve'f}, \xi_{e'f}^{\alpha_f}\rangle_\sigma} - \frac{\langle A_{ve'} Z_{ve'f}, Z_{ve'f}\rangle_\sigma + \langle Z_{ve'f}, A_{ve'} Z_{ve'f}\rangle_\sigma}{\langle Z_{ve'f}, Z_{ve'f}\rangle_\sigma} \right) \\
=& i\gamma j_f \left(\frac{\langle A_{ve'} Z_{ve'f}, Z_{ve'f}\rangle_\sigma + \langle Z_{ve'f}, A_{ve'} Z_{ve'f}\rangle_\sigma}{\langle Z_{ve'f}, Z_{ve'f}\rangle_\sigma} - \frac{\langle A_{ve} Z_{vef}, Z_{vef}\rangle_\sigma + \langle Z_{vef}, A_{ve} Z_{vef}\rangle_\sigma}{\langle Z_{vef}, Z_{vef}\rangle_\sigma} \right) \\
&- j_f \left(\frac{2\langle \xi_{ef}^{\alpha_f}, A_{ve} Z_{vef}\rangle_\sigma}{\langle \xi_{ef}^{\alpha_f}, Z_{vef}\rangle_\sigma} - \frac{\langle A_{ve} Z_{vef}, Z_{vef}\rangle_\sigma + \langle Z_{vef}, A_{ve} Z_{vef}\rangle_\sigma}{\langle Z_{vef}, Z_{vef}\rangle_\sigma} \right) \\
&- j_f \left(\frac{2\langle A_{ve'} Z_{ve'f}, \xi_{e'f}^{\alpha_f}\rangle_\sigma}{\langle Z_{ve'f}, \xi_{e'f}^{\alpha_f}\rangle_\sigma} - \frac{\langle A_{ve'} Z_{ve'f}, Z_{ve'f}\rangle_\sigma + \langle Z_{ve'f}, A_{ve'} Z_{ve'f}\rangle_\sigma}{\langle Z_{ve'f}, Z_{ve'f}\rangle_\sigma} \right)
\end{aligned} \tag{3.191}$$

where we dropped the subscript v associated with α_{fv} and j_{fv} because they are actually independent v. At the critical point, we have

$$\mathrm{Re}(S_{fv}) = 0 \Rightarrow Z_{vef} = \zeta_{vef} \xi_{ef}^{\alpha_f} \tag{3.192}$$

for some complex number ζ_{vef}, i.e., Z_{vef} is parallel to $\xi_{ef}^{\alpha_f}$. Applying this relation, we can simplify (eq:closuretimelike 3.202) to be

eq:closuretimelike

$$\begin{aligned}
\frac{\delta S_{fv}}{\delta\lambda} =& i\gamma j_f \left(\frac{\langle A_{ve'} \xi_{e'f}^{\alpha_f}, \xi_{e'f}^{\alpha_f}\rangle_\sigma + \langle \xi_{e'f}^{\alpha_f}, A_{ve'} \xi_{e'f}^{\alpha_f}\rangle_\sigma}{\langle \xi_{e'f}^{\alpha_f}, \xi_{e'f}^{\alpha_f}\rangle_\sigma} - \frac{\langle A_{ve} \xi_{ef}^{\alpha_f}, \xi_{ef}^{\alpha_f}\rangle_\sigma + \langle \xi_{ef}^{\alpha_f}, A_{ve} \xi_{ef}^{\alpha_f}\rangle_\sigma}{\langle \xi_{ef}^{\alpha_f}, \xi_{ef}^{\alpha_f}\rangle_\sigma} \right) \\
&- j_f \left(\frac{2\langle \xi_{ef}^{\alpha_f}, A_{ve} \xi_{ef}^{\alpha_f}\rangle_\sigma}{\langle \xi_{ef}^{\alpha_f}, \xi_{ef}^{\alpha_f}\rangle_\sigma} - \frac{\langle A_{ve} \xi_{ef}^{\alpha_f}, \xi_{ef}^{\alpha_f}\rangle_\sigma + \langle \xi_{ef}^{\alpha_f}, A_{ve} \xi_{ef}^{\alpha_f}\rangle_\sigma}{\langle \xi_{ef}^{\alpha_f}, \xi_{ef}^{\alpha_f}\rangle_\sigma} \right) \\
&- j_f \left(\frac{2\langle A_{ve} \xi_{e'f}^{\alpha_f}, \xi_{e'f}^{\alpha_f}\rangle_\sigma}{\langle \xi_{e'f}^{\alpha_f}, \xi_{e'f}^{\alpha_f}\rangle_\sigma} - \frac{\langle A_{ve'} \xi_{e'f}^{\alpha_f}, \xi_{e'f}^{\alpha_f}\rangle_\sigma + \langle \xi_{e'f}^{\alpha_f}, A_{ve'} \xi_{e'f}^{\alpha_f}\rangle_\sigma}{\langle \xi_{e'f}^{\alpha_f}, \xi_{e'f}^{\alpha_f}\rangle_\sigma} \right) \\
=& i\gamma j_f \alpha_f \left(\left(\langle A_{ve'} \xi_{e'f}^{\alpha_f}, \xi_{e'f}^{\alpha_f}\rangle_\sigma + \langle \xi_{e'f}^{\alpha_f}, A_{ve'} \xi_{e'f}^{\alpha_f}\rangle_\sigma \right) - \left(\langle A_{ve} \xi_{ef}^{\alpha_f}, \xi_{ef}^{\alpha_f}\rangle_\sigma + \langle \xi_{ef}^{\alpha_f}, A_{ve} \xi_{ef}^{\alpha_f}\rangle_\sigma \right) \right) \\
&- j_f \alpha_f \left(\langle \xi_{ef}^{\alpha_f}, A_{ve} \xi_{ef}^{\alpha_f}\rangle_\sigma - \langle A_{ve} \xi_{ef}^{\alpha_f}, \xi_{ef}^{\alpha_f}\rangle_\sigma \right) - j_f \alpha_f \left(\langle A_{ve} \xi_{e'f}^{\alpha_f}, \xi_{e'f}^{\alpha_f}\rangle_\sigma - \langle \xi_{e'f}^{\alpha_f}, A_{ve'} \xi_{e'f}^{\alpha_f}\rangle_\sigma \right) \\
=& \alpha_f \left[i\gamma j_f \left(\langle A_{ve'} \xi_{e'f}^{\alpha_f}, \xi_{e'f}^{\alpha_f}\rangle_\sigma + \langle \xi_{e'f}^{\alpha_f}, A_{ve'} \xi_{e'f}^{\alpha_f}\rangle_\sigma \right) + j_f \left(\langle \xi_{e'f}^{\alpha_f}, A_{ve'} \xi_{e'f}^{\alpha_f}\rangle_\sigma - \langle A_{ve} \xi_{e'f}^{\alpha_f}, \xi_{e'f}^{\alpha_f}\rangle_\sigma \right) \right] \\
&- \alpha_f \left[i\gamma j_f \left(\langle A_{ve} \xi_{ef}^{\alpha_f}, \xi_{ef}^{\alpha_f}\rangle_\sigma + \langle \xi_{ef}^{\alpha_f}, A_{ve} \xi_{ef}^{\alpha_f}\rangle_\sigma \right) + j_f \left(\langle \xi_{ef}^{\alpha_f}, A_{ve} \xi_{ef}^{\alpha_f}\rangle_\sigma - \langle A_{ve} \xi_{ef}^{\alpha_f}, \xi_{ef}^{\alpha_f}\rangle_\sigma \right) \right].
\end{aligned} \tag{3.193}$$

It should be noted that the two terms associated with e and e' have the same form up to a sign because e starts at v, while e' ends at v. Since $A_{ve} = \sum_{j=1}^{3} \theta_{ve}^{j} \sigma_j$, the equation

$$\sum_{f:\partial f \ni e} \delta S_{fv} = 0 \tag{3.194}$$

leads to

eq:closure1

$$\sum_{f:\partial f \ni e} \alpha_f \epsilon_{ef}(v) \left[i\gamma j_f \left(\langle \vec{\sigma} \xi_{ef}^{\alpha_f}, \xi_{ef}^{\alpha_f}\rangle_\sigma + \langle \xi_{ef}^{\alpha_f}, \vec{\sigma} \xi_{ef}^{\alpha_f}\rangle_\sigma \right) + j_f \left(\langle \xi_{ef}^{\alpha_f}, \vec{\sigma} \xi_{ef}^{\alpha_f}\rangle_\sigma - \langle \vec{\sigma} \xi_{ef}^{\alpha_f}, \xi_{ef}^{\alpha_f}\rangle_\sigma \right) \right] = 0 \tag{3.195}$$

where we assumed that the triangles t_f for all f at e are spacelike for this discussion.

(1) (3.195) is true for σ_3, which leads to

$$0=\sum_{f:\partial f\ni e} 2\alpha_f\epsilon_{ef}(v)i\gamma j_f\,\mathrm{tr}\Big(\sigma_3\sigma_3\xi_{ef}^{\alpha_f}\otimes\xi_{ef}^{\alpha_f\,\dagger}\Big) \tag{3.196}$$

(2) (3.195) is true for σ_a with $a=1,2$, which leads to

$$\begin{aligned}0&=\sum_{f:\partial f\ni e}\alpha_f\epsilon_{ef}(v)j_f\left(\mathrm{tr}\Big(\sigma_3\sigma_a\xi_{ef}^{\alpha_f}\otimes\xi_{ef}^{\alpha_f\,\dagger}\Big)-\mathrm{tr}\Big(\sigma_a\sigma_3\xi_{ef}^{\alpha_f}\otimes\xi_{ef}^{\alpha_f\,\dagger}\Big)\right)\\&=\sum_{f:\partial f\ni e}2i\alpha_f\epsilon_{ef}(v)j_f\epsilon_{3a}{}^{b}\,\mathrm{tr}\Big(\sigma_b\xi_{ef}^{\alpha_f}\otimes\xi_{ef}^{\alpha_f\,\dagger}\Big).\end{aligned} \tag{3.197}$$

We thus get

$$0=\sum_{f:\partial f\ni e}\alpha_f\epsilon_{ef}(v)j_f\,\mathrm{tr}\Big(\sigma_b\xi_{ef}^{\alpha_f}\otimes\xi_{ef}^{\alpha_f\,\dagger}\Big),\forall b=1,2. \tag{3.198}$$

Applying

$$\sigma_2=i\sigma_1\sigma_3,\quad \sigma_1=-i\sigma_2\sigma_3 \tag{3.199}$$

we get from (3.198)

$$0=\sum_{f:\partial f\ni e}\alpha_f\epsilon_{ef}(v)j_f\,\mathrm{tr}\Big(\sigma_b\sigma_3\xi_{ef}^{\alpha_f}\otimes\xi_{ef}^{\alpha_f\,\dagger}\Big),\forall b=1,2. \tag{3.200}$$

Combining (3.196) and (3.200) and applying (1.123), we finally get

$$\begin{aligned}0&=-\sum_{f:\partial f\ni e}\alpha_f\epsilon_{ef}(v)j_f\,\mathrm{tr}\Big(\sigma_3\xi_{ef}^{\alpha_f}\otimes\xi_{ef}^{\alpha_f\,\dagger}\sigma_i\Big)\tilde{\sigma}^i\\&=\sum_{f:\partial f\ni e}\epsilon_{ef}(v)j_f\alpha_f\left(\sigma_3\xi_{ef}^{\alpha_f}\otimes\xi_{ef}^{\alpha_f\,\dagger}-J\xi_{ef}^{\alpha_f}\otimes(J\sigma_3\xi_{ef}^{\alpha_f})^\dagger\right)\\&=\sum_{f:\partial f\ni e}\epsilon_{ef}(v)j_f\Xi_{ef}^{\alpha_f}\end{aligned} \tag{3.201}$$

We now finish the derivation of the closure for the case where the timelike tetrahedron contains only spacelike faces. However, in the timelike spinfoam model, we could consider an edge e containing either only spacelike faces or a mixture of spacelike and timelike faces. The closure condition reads

$$\sum_{f\text{ at }e}\epsilon_{ef}(v)2\gamma j_f\Xi_{ef}=0, \tag{3.202}$$

where the definition of Ξ depends on the type of face it corresponds to. For timelike faces, it is given by (3.184), while for spacelike faces with future-directed outward normals, it is given by Ξ^+ in (3.179). If the outward normals are past-directed, then Ξ is given by Ξ^- in (3.179).

The closure condition implies that when gluing the triangles dual to the faces f that share a common edge e, the area normal of the triangle t_f must be $\vec{A}_{ef}=\epsilon_{ef}(v)2\sqrt{2}\gamma j_f\Xi_{ef}$, where Ξ_{ef} is defined differently depending on whether the face is timelike or spacelike with future or past directed outward normal. The gluing gives rise to a tetrahedron with normals z^I, and each triangle t_f of the tetrahedron has a "spatial" normal $\Xi^{\pm IJ}z_J$.

IV. NUMERICAL CALCULATION OF SPINFOAM AMPLITUDE

A. Get spinfoam data for a single vertex

1. Normals of the tetrahedra

Consider the example where we have a 4-simplex with vertices

$$P_1=(0,0,0,0),\ P_2=(0,0,0,1),\ P_3=(0,0,1,1),\ P_4=(0,1,1,1),\ P_5=(\frac{1}{2},1,1,1). \tag{4.1}$$

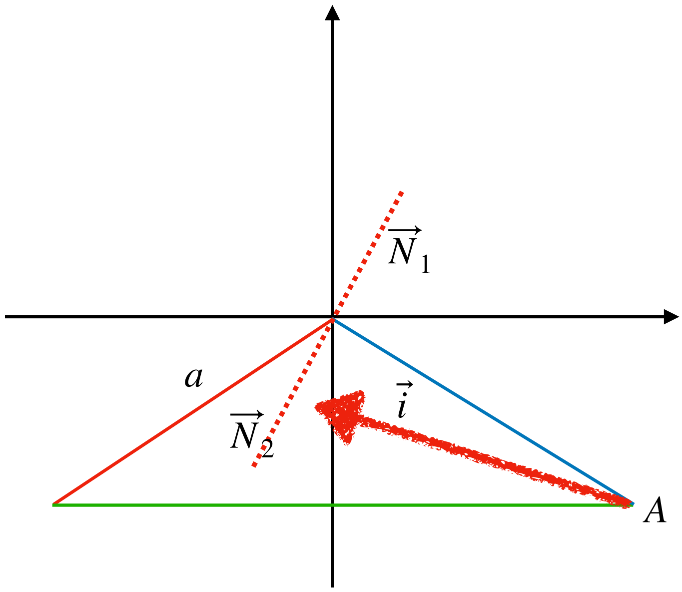


FIG. 7: How to decide the outwards direction.

We first calculate the normal of the tetrahedron, saying $[P_2P_3P_4P_5]$. Let $e_{ij} = P_j - P_i$ be the vector from P_i to P_j. We have the unnormalized normal $\tilde{N}_1^I$ of $[P_2P_3P_4P_5]$ as

$$\tilde{N}_1^I = \eta^{IJ}\epsilon_{JKLM}e_{23}^K e_{24}^L e_{25}^M, \tag{4.2}$$

so that the normalized normal denoted by M_1 reads

$$M_1^I = \tilde{N}_1^I / \sqrt{|\eta_{IJ}\tilde{N}_1^I\tilde{N}_1^J|}. \tag{4.3}$$

1) For M_1 being timelike, we choose the future-pointing normal, denoted by N_1 which is

$$N_1 = M_1 \mathrm{sgn}(M_1^0). \tag{4.4}$$

2) In the case where M_1 is spacelike, we select the outward normal N_1. To clarify this concept, we first define what is meant by an outward normal. As shown in Fig. (7) using the two-dimensional Minkowski case as an example, we need to determine whether $\vec{N}_1$ or $\vec{N}_2$ is the outward normal of edge a in the triangle. To do so, we consider an arbitrary vector $\vec{i}$ that starts at vertex A (opposite to edge a) and ends inside the triangle. We then determine which of $\vec{N}_1$ or $\vec{N}_2$ has a positive inner product with $\vec{i}$. The direction that satisfies this condition is defined as the outward normal of edge a, denoted by $\vec{N}_{\text{outwards}} := \mathrm{sgn}(\vec{i}\cdot\vec{N}_1)\vec{N}_1$.

   In our 4-dimensional case, we do the same to select the outwards direction. Thus we have

$$N_1 = M_1 \mathrm{sgn}(M_1^I \eta_{IJ} e_{12}^I). \tag{4.5}$$

The same for the other tetrahedra. This is the algorithm to calculate the normalized norm.

In our case, we have

$$\begin{pmatrix} N_1 & N_2 & N_3 & N_4 & N_5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \frac{2}{\sqrt{3}} & 1 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -1 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix} \tag{4.6}$$

2. *the* SL$(2,\mathbb{C})$ *elements*

sec:SL2C

Given a normal of a tetrahedra, saying N_a, we want to find an SL$(2,\mathbb{C})$ element g such that

$$g \triangleright N_{\rm ref} = N_a \tag{4.7}$$

eq:sl2c

where $N_{\rm ref} = (1,0,0,0)^T$ for timelike N_a and $N_{\rm ref} = (0,0,0,1)^T$ for spacelike N_a.

To this end, we first rewrite the spatial part of N_a in spherical coordinate as

$$N_a^I = (N_a^0, r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)^T. \tag{4.8}$$

Then, applying the results

$$g(\theta,\phi)(\sin\theta\cos\phi\tau_1 + \sin\theta\sin\phi\tau_2 + \cos\theta\tau_3)g(\theta,\phi)^{-1} = \tau_3 \tag{4.9}$$

where $\tau_j = -i\sigma_j/2$ and

$$g(\theta,\phi) = e^{-\theta(-\sin\phi\tau_1 + \cos\phi\tau_2)} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & e^{-i\phi}\sin\left(\frac{\theta}{2}\right) \\ -e^{i\phi}\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \in \mathrm{SU}(2) \tag{4.10}$$

we have

$$g(\theta,\phi) \triangleright N_a = (N_a^0, 0, 0, r)^T. \tag{4.11}$$

Then,

1) For timelike N_a, we have

$$-(N_a^0)^1 + r^2 = -1 \Rightarrow N_a^0 = \cosh(\theta),\ r = \sinh(\theta). \tag{4.12}$$

Applying

$$e^{-\theta\sigma_3}(\cosh\theta\sigma_0 + \sinh\theta\sigma_3) = \sigma_0 \tag{4.13}$$

we have

$$B(r) \triangleright (N_a^0, 0, 0, r)^T = (1,0,0,0)^T. \tag{4.14}$$

where

$$B(x) = e^{-\mathrm{arcsinh}(x)\sigma_3/2} = \begin{pmatrix} \frac{1}{\sqrt{\sqrt{x^2+1}+x}} & 0 \\ 0 & \sqrt{\sqrt{x^2+1}+x} \end{pmatrix} \tag{4.15}$$

2) For spacelike N_a, we have

$$-(N_a^0)^1 + r^2 = 1 \Rightarrow N_a^0 = \sinh(\theta),\ r = \cosh(\theta). \tag{4.16}$$

Applying

$$e^{-\theta\sigma_3}(\sinh\theta\sigma_0 + \cosh\theta\sigma_3) = \sigma_3 \tag{4.17}$$

we have

$$B(N_a^0) \triangleright (N_a^0, 0, 0, r)^T = (0,0,0,1)^T. \tag{4.18}$$

Finally, an SL$(2,\mathbb{C})$ element satisfying (4.7) is

$$g = [Bg(\theta,\phi)]^{-1} \tag{4.19}$$

with $B = B(r)$ for timelike N_a and $B = B(N_a^0)$ for spacelike N_a.

It should be noted that the SL$(2,\mathbb{C})$ elements can not be fixed by (4.7). Indeed, for timelike N_a, g can only be fixed up a rotation, i.e., all elements of the form

$$e^{\alpha\tau_1 + \beta\tau_2 + \gamma\tau_3} g \tag{4.20}$$

are solutions to (4.7). For spacelike N_a, g can only be fixed up to a rotation along the z-axis and boots along x- and y- axis, i.e., the solution to (4.7) could be

$$e^{\alpha\sigma_1/2 + \beta\sigma_2/2} e^{\gamma\tau_3} g. \tag{4.21}$$

Our objective is to determine the solution to the critical point equation using the 4-simplex dual to the central vertex. In Section IV A 2, we obtained the $\mathrm{SL}(2,\mathbb{C})$ element, and in this section, we focus on the spinors. The spinors appear in two equations: the parallel transport equation (such as Equation (3.14) for spacelike tetrahedra) and the closure condition. Both of these equations are expressed in terms of bivectors constructed from the spinors. The bivectors have a geometric interpretation as the binormals to the triangles of the tetrahedron. Therefore, we start by constructing the binormals to the triangles of the 4-simplex.

Consider a triangle, saying, $t = P_1P_2P_3$ of the 4-simplex. We first assign an orientation to t by sorting the points P_a. It is noted that the assignment of orientation is practically necessary. In practice, we do calculation in one 4-simplex at first and then go to another. This sort assigns a fixed orientation to each face, regardless of whether the face belongs to tetrahedra of two different 4-simplices. The binormal to the triangle is

$$X_t^{IJ} = \eta^{IK}\eta^{JL}\epsilon_{KLMN}E_{21}^M \wedge E_{31}^N, \tag{4.22}$$

where the edge vector E_{ij}^I is defined by $E_{ij}^I = P_j^I - P_i^I$. Then, we parallel transport the bivector from the central vertex to the edges by using the $\mathrm{SL}(2,\mathbb{C})$ element obtained in Section IV A 2. In this example where we consider $t = P_1P_2P_3$, the bivector can be parallel transported to the edge e_4 and e_5 dual to the tetrahedra $\tau_4 = P_1P_2P_3P_5$ and $\tau_5 = P_1P_2P_3P_4$. We get

$$X_{e_4f}^{IJ} = (g_{ve_4}^{-1})^I{}_K(g_{ve_4}^{-1})^J{}_L X_t^{KL}, \quad X_{e_5f}^{IJ} = (g_{ve_5}^{-1})^I{}_K(g_{ve_5}^{-1})^J{}_L X_t^{KL}. \tag{4.23}$$

Then, we get the bivector $X_{e_af}^{IJ}$ that satisfy the parallel transport equation, i.e,

$$(g_{ve_4})^I{}_K(g_{ve_4})^J{}_L X_{e_4f}^{KL} = (g_{ve_5})^I{}_K(g_{ve_5})^J{}_L X_{e_5f}^{KL}. \tag{4.24}$$

From the bivectors $X_f^{IJ} \equiv X_{e_af}^{IJ}$ [3], we get the $\mathfrak{sl}(2,\mathbb{C})$ element

$$X_f = \frac{1}{4}X_f^{IJ}\sigma_I\tilde{\sigma}_J. \tag{4.25}$$

X_f is the $\mathfrak{sl}(2,\mathbb{C})$ element satisfied the parallel transport equation. Then X_f is related to the spinors by (3.15) for spacelike tetrahedron and by (3.184) or (3.179) for timelike tetrahedron. The precise relation is discussed as follows case by case.

(1) When the tetrahedron is spacelike, according to (3.15), let us consider

$$\Xi_f = \zeta_f \otimes \zeta_f^\dagger - J\zeta_f \otimes (J\zeta_f)^\dagger. \tag{4.26}$$

Its norm, according to (1.106), is given by

$$\begin{aligned}
\|\Xi_f\|^2 &= -2\,\mathrm{tr}\Big(\Xi_f\Xi_f + \Xi_f^\dagger\Xi_f^\dagger\Big) = -4\,\mathrm{tr}(\Xi_f\Xi_f) \\
&= -4\,\mathrm{tr}\Big((\zeta_f \otimes \zeta_f^\dagger - J\zeta_f \otimes (J\zeta_f)^\dagger)(\zeta_f \otimes \zeta_f^\dagger - J\zeta_f \otimes (J\zeta_f)^\dagger)\Big) \\
&= -4\,\mathrm{tr}\Big(\zeta_f \otimes \zeta_f^\dagger\zeta_f \otimes \zeta_f^\dagger - J\zeta_f \otimes (J\zeta_f)^\dagger\zeta_f \otimes \zeta_f^\dagger - \zeta_f \otimes \zeta_f^\dagger J\zeta_f \otimes (J\zeta_f)^\dagger + J\zeta_f \otimes (J\zeta_f)^\dagger J\zeta_f \otimes (J\zeta_f)^\dagger\Big) \\
&= -4\,\mathrm{tr}\Big(\zeta_f \otimes \zeta_f^\dagger + J\zeta_f \otimes (J\zeta_f)^\dagger\Big) \\
&= -8.
\end{aligned} \tag{4.27}$$

where we used $\zeta^\dagger\zeta = 1$ and $\zeta^\dagger J\zeta = 0$. For X_f, we have

$$\|X_f\|^2 = 2\,\mathrm{tr}\Big(X_fX_f + X_f^\dagger X_f^\dagger\Big) = X_f^{IJ}X_f^{KL}\eta_{IK}\eta_{JL} \tag{4.28}$$

---

[3] In this context, we focus on a specific edge e_a while multiple faces share e_a. Therefore, we abbreviate X_{e_af} as X_f.

which can be obtained directly from (1.101) and (1.106)[4] Since the normal of the tetrahedron is $u^I = (1,0,0,0)$, i.e., u^I is one normal of the triangle, we have

$$X_f^{IJ} = u^I \wedge (X_f^{JK} u_K) \equiv u^I \wedge A_f^J \tag{4.30}$$

which leads to

$$X_f^{IJ}(X_f)_{IJ} = (u^I \wedge A_f^J)(u_I \wedge (A_f)_J) = 2(u^I A_f^J - u^J A_f^I) u_I (A_f)_J = -2\|A_f\|^2 \tag{4.31}$$

Eqs. (4.27) and (4.28) implies that

$$\Xi_f = 2\sqrt{\frac{-2}{X_f^{IJ}(X_f)_{IJ}}} X_f = \frac{2}{\|A_f\|} X_f. \tag{4.32}$$

Combining the above results, we have

$$\zeta_f \otimes \zeta_f^\dagger - J\zeta_f \otimes (J\zeta_f)^\dagger = \frac{1}{2} \frac{X_f^{IJ} \sigma_I \tilde{\sigma}_I}{\|A_f\|} = \frac{1}{2} \frac{A_f^J \sigma_0 \tilde{\sigma}_I - A_f^I \sigma_I \tilde{\sigma}_0}{\|A_f\|} = \frac{A_f^J \sigma_J}{\|A_f\|}. \tag{4.33}$$

where we used $A_f^0 = 0$.

To solve ζ_f from (4.33), we set

$$\zeta = U_\zeta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.34}$$

with $U_\zeta \in \mathrm{SU}(2)$. Then, (4.33) gives

$$\frac{A_f^I \sigma_I}{\|A_f\|} = U_\zeta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U_\zeta^\dagger - U_\zeta \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} U_\zeta^\dagger = U_\zeta \sigma_3 U_\zeta^\dagger. \tag{4.35}$$

Thus, U_ζ corresponds to the rotation which rotates the z axis to $\vec{A}_f$. To be compatible with the calculations for timelike tetrahedron, we will calculate U_ζ^{-1} by the following equation

$$U_\zeta^{-1} \frac{A_f^I \sigma_I}{\|A_f\|} U_\zeta^{-1\dagger} = \sigma_3. \tag{4.36}$$

(2) When the tetrahedron is timelike

---

[4] one can also do the following calculation to get the result

$$\begin{aligned}
\|X_f\|^2 &= 2\,\mathrm{tr}\left(X_f X_f + X_f^\dagger X_f^\dagger\right) = \frac{1}{8} X^{IJ} X^{KL}\, \mathrm{tr}(\sigma_I \tilde{\sigma}_J \sigma_K \tilde{\sigma}_L + \tilde{\sigma}_J \sigma_I \tilde{\sigma}_L \sigma_K) \\
&= \frac{1}{8} X^{IJ} X^{KL}\, \mathrm{tr}(2\eta_{JK} \sigma_I \tilde{\sigma}_L - \sigma_I \tilde{\sigma}_K \sigma_J \tilde{\sigma}_L + 2\eta_{IL} \tilde{\sigma}_J \sigma_K - \tilde{\sigma}_J \sigma_L \tilde{\sigma}_I \sigma_K) \\
&= \frac{1}{8} X^{IJ} X^{KL}\, \mathrm{tr}\left(2\eta_{JK} \sigma_I \tilde{\sigma}_L - \sigma_I \tilde{\sigma}_K \sigma_J \tilde{\sigma}_L + 2\eta_{IL} \tilde{\sigma}_J \sigma_K - \epsilon \sigma_J^T \epsilon \sigma_L \epsilon \sigma_I^T \epsilon \sigma_K\right) \\
&= \frac{1}{8} X^{IJ} X^{KL}\, \mathrm{tr}\left(2\eta_{JK} \sigma_I \tilde{\sigma}_L - \sigma_I \tilde{\sigma}_K \sigma_J \tilde{\sigma}_L + 2\eta_{IL} \tilde{\sigma}_J \sigma_K - \sigma_J^T \tilde{\sigma}_L^T \sigma_I^T \tilde{\sigma}_K^T\right) \\
&= \frac{1}{8} X^{IJ} X^{KL}\, \mathrm{tr}(2\eta_{JK} \sigma_I \tilde{\sigma}_L - \sigma_I \tilde{\sigma}_K \sigma_J \tilde{\sigma}_L + 2\eta_{IL} \tilde{\sigma}_J \sigma_K - \tilde{\sigma}_K \sigma_I \tilde{\sigma}_L \sigma_J) \\
&= \frac{1}{8} X^{IJ} X^{KL}\, \mathrm{tr}(2\eta_{JK} \sigma_I \tilde{\sigma}_L + 2\eta_{IL} \tilde{\sigma}_J \sigma_K - 2\sigma_I \tilde{\sigma}_{(L} \sigma_J \tilde{\sigma}_{K)}) \\
&= X^{IJ} X^{KL} \eta_{JK} \eta_{IL}.
\end{aligned} \tag{4.29}$$

where we used (1.96) and $\mathrm{tr}(A) = \mathrm{tr}(A^T)$

(2.1) for spacelike triangle, according to (eq:spacelike 3.179), we consider

$$\Xi_f = \alpha_f(\eta\xi^{\alpha_f}\otimes\xi^{\alpha_f\dagger} + J\xi^{\alpha_f}\otimes(J\xi^{\alpha_f})^\dagger\eta) \tag{4.37}$$

whose norm is

$$\begin{aligned}
\|\Xi_f\|^2 =& -2\,\mathrm{tr}\Big(\eta\xi^{\alpha_f}\otimes\xi^{\alpha_f\dagger}\eta\xi^{\alpha_f}\otimes\xi^{\alpha_f\dagger} + J\xi^{\alpha_f}\otimes(J\xi^{\alpha_f})^\dagger\eta\eta\xi^{\alpha_f}\otimes\xi^{\alpha_f\dagger}\\
&+\eta\xi^{\alpha_f}\otimes\xi^{\alpha_f\dagger}J\xi^{\alpha_f}\otimes(J\xi^{\alpha_f})^\dagger\eta + J\xi^{\alpha_f}\otimes(J\xi^{\alpha_f})^\dagger\eta J\xi^{\alpha_f}\otimes(J\xi^{\alpha_f})^\dagger\eta\\
&+\xi^{\alpha_f}\otimes\xi^{\alpha_f\dagger}\eta\xi^{\alpha_f}\otimes\xi^{\alpha_f\dagger}\eta + \eta J\xi^{\alpha_f}\otimes(J\xi^{\alpha_f})^\dagger\xi^{\alpha_f}\otimes\xi^{\alpha_f\dagger}\eta\\
&\xi^{\alpha_f}\otimes\xi^{\alpha_f\dagger}\eta\eta J\xi^{\alpha_f}\otimes(J\xi^{\alpha_f})^\dagger + \eta J\xi^{\alpha_f}\otimes(J\xi^{\alpha_f})^\dagger\eta J\xi^{\alpha_f}\otimes(J\xi^{\alpha_f})^\dagger\Big)\\
=& -2\alpha_f\,\mathrm{tr}\big(\eta\xi^{\alpha_f}\otimes\xi^{\alpha_f\dagger} + \xi^{\alpha_f}\otimes\xi^{\alpha_f\dagger}\eta - J\xi^{\alpha_f}\otimes(J\xi^{\alpha_f})^\dagger\eta - \eta J\xi^{\alpha_f}\otimes(J\xi^{\alpha_f})^\dagger\big)\\
=& -8,
\end{aligned} \tag{4.38}$$

where we used $\xi^{\alpha\dagger}\eta\xi^\alpha = \alpha = -(J\xi^\alpha)^\dagger\eta(J\xi^\alpha)$.
For the bivector X_f^{IJ} obtained from the 4-simplex, we have in this case

$$X_f^{IJ} = (X^{IK}z_K)\wedge z^J \equiv A_f^I\wedge z^J \tag{4.39}$$

whose norm is

$$X_f^{IJ}X_{fIJ} = 2A_f^I\wedge z^J A_{fI}z_J = 2\|A_f\|^2. \tag{4.40}$$

where it is worth noting that A_f is timelike and thus $\|A_f\|<0$. We then have

$$\alpha_f(\eta\xi^{\alpha_f}\otimes\xi^{\alpha_f\dagger} + J\xi^{\alpha_f}\otimes(J\xi^{\alpha_f})^\dagger\eta) = \frac{1}{2}\frac{X_f^{IJ}\sigma_I\tilde{\sigma}_J}{\sqrt{-\|A_f\|^2}} = \frac{A_f^J}{\sqrt{-\|A_f\|^2}}\frac{\sigma_J\tilde{\sigma}_3-\sigma_3\tilde{\sigma}_J}{2}. \tag{4.41}$$

eq:tmt1

To solve ξ^α from this equation, we set

$$\xi^\alpha = U^{-1\dagger}\xi_0^\alpha. \tag{4.42}$$

eq:xipmU

for $U\in$SU(1,1). Then, (eq:tmt1 4.41) gives

$$\begin{aligned}
\frac{\alpha_f A_f^J}{\sqrt{-\|A_f\|^2}}\frac{\sigma_J\tilde{\sigma}_3-\sigma_3\tilde{\sigma}_J}{2} =& \sigma_3U^{-1\dagger}\xi_0^{\alpha_f}\otimes(\xi_0^{\alpha_f})^\dagger U^{-1} - U\epsilon\xi_0^{\alpha_f}\otimes(\xi_0^{\alpha_f})^\dagger\epsilon U^\dagger\sigma_3\\
=& U\xi_0^{\alpha_f}\otimes(\xi_0^{\alpha_f})^\dagger U^\dagger\sigma_3 - U\epsilon\xi_0^{\alpha_f}\otimes(\xi_0^{\alpha_f})^\dagger\epsilon U^\dagger\sigma_3\\
=& U(\xi_0^{\alpha_f}\otimes(\xi_0^{\alpha_f})^\dagger - \epsilon\xi_0^{\alpha_f}\otimes(\xi_0^{\alpha_f})^\dagger\epsilon)U^\dagger\sigma_3\\
=& U\sigma_0U^\dagger\sigma_3,
\end{aligned} \tag{4.43}$$

where we used

$$U\sigma_3U^\dagger = \sigma_3,\ \ \sigma_3\xi_0^\pm\otimes(\xi_0^\pm)^\dagger = \xi_0^\pm\otimes(\xi_0^\pm)^\dagger\sigma_3,\quad \forall U\in\mathrm{SU}(1,1) \tag{4.44}$$

in the second step, and $A-\epsilon A\epsilon = \mathrm{tr}(A)\mathbb{1}$ in the fourth step. Multiplying σ_3 on the both side, we get

$$U\sigma_0U^\dagger = \sum_{a=0,1,2}\frac{\alpha_f A_f^a}{\sqrt{-\|A_f\|^2}}\sigma_a \tag{4.45}$$

eq:spaceliketriangleSpinor

due to $\sigma_3\tilde{\sigma}_a\sigma_3 = -\sigma_a$ for $a=0,1,2$ and $A_f^3=0$.
It should be emphasized that (eq:spaceliketriangleSpinor 4.45) is a equation not only on U but also on α_f. Since U cannot change the time direction, $\alpha_f A_f^a$ have to be future directed, i.e.,

$$\alpha_f A_f^0 > 0, \tag{4.46}$$

which solves the value of α_f. Once α_f is obtained, we can solve U^{-1} by

$$\sigma_0 = U^{-1}\left(\sum_{a=0,1,2}\frac{\alpha_f A_f^a}{\sqrt{-\|A_f\|^2}}\sigma_a\right)U^{\dagger-1}. \tag{4.47}$$

(2.2) for timelike triangle, according to (3.184), we consider

$$\Xi_f = i(\eta l_f^+ \otimes l_f^{-\dagger} + J l_f^- \otimes (J l_f^+)^\dagger \eta) \tag{4.48}$$

where l^- is the vector $l^- + \alpha l^+$ in (3.184). The norm of Ξ_f is

$$\begin{aligned}
\|\Xi_f\| =& -2(i)^2 \operatorname{tr} \Big[\eta l_f^+ \otimes l_f^{-\dagger} \eta l_f^+ \otimes l_f^{-\dagger} + J l_f^- \otimes (J l_f^+)^\dagger \eta \eta l_f^+ \otimes l_f^{-\dagger} \\
&+ \eta l_f^+ \otimes l_f^{-\dagger} J l_f^- \otimes (J l_f^+)^\dagger \eta + J l_f^- \otimes (J l_f^+)^\dagger \eta J l_f^- \otimes (J l_f^+)^\dagger \eta \\
&+ l_f^- \otimes l_f^{+\dagger} \eta l_f^- \otimes l_f^{+\dagger} \eta + \eta J l_f^+ \otimes (J l_f^-)^\dagger l_f^- \otimes l_f^{+\dagger} \eta \\
&+ l_f^- \otimes l_f^{+\dagger} \eta \eta J l_f^+ \otimes (J l_f^-)^\dagger + \eta J l_f^+ \otimes (J l_f^-)^\dagger \eta J l_f^+ \otimes (J l_f^-)^\dagger \Big] \\
=& 2 \operatorname{tr} \Big[\eta l_f^+ \otimes l_f^{-\dagger} - J l_f^- \otimes (J l_f^+)^\dagger \eta + l_f^- \otimes l_f^{+\dagger} \eta - \eta J l_f^+ \otimes (J l_f^-)^\dagger \Big] \\
=& 8.
\end{aligned} \tag{4.49}$$

For the bivector X_f^{IJ} obtained from the 4-simplex, it is of the same properties as these in Case (2.1), i.e.,

$$X_f^{IJ} = (X^{IK} z_K) \wedge z^J \equiv A_f^I \wedge z^J \tag{4.50}$$

whose norm is

$$X_f^{IJ} X_{f\,IJ} = 2 A_f^I \wedge z^J A_{f\,I} z_J = 2\|A_f\|^2. \tag{4.51}$$

We thus have

$$i(\eta l_f^+ \otimes l_f^{-\dagger} + J l_f^- \otimes (J l_f^+)^\dagger \eta) = \frac{A_f^J}{\sqrt{\|A_f\|^2}} \frac{\sigma_J \tilde{\sigma}_3 - \sigma_3 \tilde{\sigma}_J}{2}, \tag{4.52}$$

where it is worth noting that A_f^I is spacelike.
To get the spinor $l^\pm$, we set

$$l^\pm = U^{-1\dagger} (1, \pm 1)^T \tag{4.53}$$

for $U \in \mathrm{SU}(1,1)$. Then, (4.48) gives us

$$\begin{aligned}
\frac{A_f^I}{\sqrt{\|A_f\|^2}} \frac{\sigma_J \tilde{\sigma}_3 - \sigma_3 \tilde{\sigma}_J}{2} =& \frac{i}{2} \left(\sigma_3 U^{-1\dagger} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} U^{-1} - U \epsilon \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \epsilon U^\dagger \sigma_3 \right) \\
=& \frac{i}{2} \left(U \sigma_3 \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \sigma_3 U^\dagger \sigma_3 - U \epsilon \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \epsilon U^\dagger \sigma_3 \right) \\
=& -U \sigma_2 U^\dagger \sigma_3
\end{aligned} \tag{4.54}$$

We thus have

$$U \sigma_2 U^\dagger = -\sum_{a=0,1,2} \frac{A_f^a}{\sqrt{\|A_f\|^2}} \sigma_a. \tag{4.55}$$

By (4.53), it will be more convenient to solve U^{-1} by

$$\sigma_2 = U^{-1} \left(-\sum_{a=0,1,2} \frac{A_f^a}{\sqrt{\|A_f\|^2}} \sigma_a \right) U^{\dagger -1}. \tag{4.56}$$

4. the orientation $\epsilon_{ef}(v)$

We now already have the spinfoam data g_{ve} and the spinors (either ξ_{ef} or $\xi_{ef}^\pm$ or $l_{ef}^\pm$) such that the parallel equations hold. Finally, the orientation $\epsilon_{ef}(v)$ is solved from the closure condition

$$\sum_{f \text{ at } e} \epsilon_{ef}(v) \|A_{ef}\| \Xi_{ef} = 0, \tag{4.57}$$

where $\|A_{ef}\|$ denotes the area of the triangle dual to ef. The algorithm is presented as follows.

We first choose an edge, saying, e_1 connected with the faces denoted by f_2, f_3, f_4 and f_5, where f_a is the face expanded by the edges e_a and e_1. We then have the closure condition at e_1

eq:closure15
$$\sum_{a=2}^{5} \epsilon_{e_1 f_a}(v)\|A_{e_1 f_a}\|\Xi_{e_1 f_a} = 0. \tag{4.58}$$

Since either $\Xi^{IJ}_{e_1 f_a} u_I = 0$ for spacelike tetrahedron or $\Xi^{IJ}_{e_1 f_a} z_I = 0$ for timelike tetrahedron, (eq:closure15 4.58) can only determine $\epsilon_{e_1 f_a}$ for all f_a up to an overall sign. We can choose a convention, for instance, that sets $\epsilon_{e_1 f_2} = 1$ to a solution for $\epsilon_{e_1 f_a}(v)$. Then, we go to another edge, saying, e_2.

Associated with e_2, the closure condition reads

eq:closure25
$$\sum_{a\in\{1,2,3,4,5\}} \epsilon_{e_2 f_a}(v)\|A_{e_2 f_a}\|\Xi_{e_2 f_a} = 0. \tag{4.59}$$

It should be noted that

$$\epsilon_{e_1 f_2} = -\epsilon_{e_2 f_1}$$

which together with (eq:closure25 4.59) solves $\epsilon_{e_2 f_a}$ uniquely.

An issue arises when we consider the sign factor $\epsilon_{e_3 f_a}$. Besides the closure condition associated with e_3, we have the value of not only $\epsilon_{e_3 f_1} = -\epsilon_{e_1 f_3}$ but also $\epsilon_{e_3 f_2} = -\epsilon_{e_2 f_3}$. Then we encounter the issue that there are more equations than necessary to solve $\epsilon_{e_3 f_a}$. To address this, we utilize $\epsilon_{e_3 f_2} = -\epsilon_{e_2 f_3}$ as an extra equation, along with the closure condition, to determine the values of $\epsilon_{e_3 f_a}$. However, it is important to verify if the solution obtained satisfies $\epsilon_{e_3 f_1} = -\epsilon_{e_1 f_3}$, even though the expected result should be positive.

5. *the way to store the results*

In our mathematical codes, the results are restored as follows. We use

(1) `{Simplex[Label_],Tetrahedron[v1_,v2_,v3_,v4_]}` is used to denote a half edge where `Label_` is to label the 4-simplex and `v1_,v2_,v3_,v4_` are the labels of the four vertices of the tetrahedron. Note that the labels are sorted by the natural way, for applying the rules in future.

(2) `{Simplex[Label_],Tetrahedron[v1_,v2_,v3_,v4_],Triangle[v1_,v2_,v3_]}` is used to denote a segment of a face.

(3) the results are stored into a list `{SFDataRule,orientationRule}` where `SFDataRule` and `orientationRule` are both list of rules:

ma:sfdata1

Code 1 :

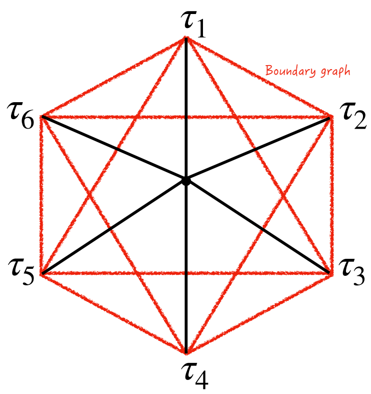
```
Out[1]=  SFDataRule={Simplex[Label_],Tetrahedron[v1_,v2_,v3_,v4_]}→ sl2c1,
             {Simplex[Label_],Tetrahedron[v5_,v6_,v7_,v8_]}→ sl2c2,
              ...,
             {Simplex[Label_],Tetrahedron[v1_,v2_,v3_,v4_],Triangle[v1_,v2_,v3_]}→spinor1
             {Simplex[Label_],Tetrahedron[v1_,v2_,v3_,v4_],Triangle[v2_,v3_,v4_]}→spinor2
             ...}
```

and

ma:sfdata2

Code 2 :

```
Out[2]= orientationRule=
        {Simplex[Label_],Tetrahedron[v1_,v2_,v3_,v4_],Triangle[v1_,v2_,v3_]}→εvef1
         {Simplex[Label_],Tetrahedron[v1_,v2_,v3_,v4_],Triangle[v2_,v3_,v4_]}→εvef2,...}
```



	p_1 •	p_2 •	p_3 •	p_4 •	p_5 •
τ_1	×	×	×	×	
τ_2		×	×	×	×
τ_3	×		×	×	×
τ_4		×		×	

FIG. 8: The left panel of the figure displays the 6-valence vertex introduced in the paper *arXiv:2105.06876*, with the red lines representing the boundary graph. The right panel demonstrates that it is impossible to find a tetrahedron τ_4 that shares two distinct faces with τ_2 and τ_3, while not sharing any faces with τ_1. In this illustration, the symbols $\times$ in the first line, for instance, indicates the tetrahedron $\tau_1 = p_1p_2p_3p_4$, which occupies the positions under p_1, p_2, p_3, and p_4. The requirement for τ_4 implies that it can occupy a maximum of two out of the four positions under p_1, p_2, p_3, and p_4. Otherwise, it would share faces with τ_1. For example, let us consider the scenario where these two positions are p_2 and p_4, as depicted in the figure. In this case, we would need to identify two additional positions from those under p_5, p_6, and so on, occupied by τ_4, such that τ_4 shares two distinct faces with τ_2 and τ_3. However, this cannot be achieved. fig:s

B. to do triangulation

1. *The issue in the work arXiv:2105.06876*

The work *arXiv:2105.06876* propose a dual graph comprising 6-valence vertices in the dual graph. The boundary graph of each 6-valence vertex is shown in Fig. fig:sixVGraph 8.

Now, let us assume this vertex dual to a 4-dimension cell. According to the boundary graph, the cell is enclosed by 6 tetrahedra. To construct the cell, it is necessary to properly arrange these 6 tetrahedra in a way that they share faces as indicated by the boundary graph. This can be done as follows.

(1) We find arbitrary 4 points p_1, p_2, p_3 and p_4 in $\mathbb{R}^4$ to construct the first tetrahedra $\tau_1 = p_1p_2p_3p_4$;

(2) According to the boundary graph, construct the second tetrahedron τ_2 in a way that it shares only one face with τ_1. To do this, introduce an additional point p_5 and construct τ_2 as $\tau_2 = p_2p_3p_4p_5$ or equivalently $\tau_2 = p_1p_2p_4p_5$, among other options. Without loss of generality, we choose $\tau_2 = p_2p_3p_4p_5$.

(3) According to the boundary graph, construct the third tetrahedron τ_3 in a way that it shares one face f_1 with τ_1 and another face f_2 with τ_2. It is important that $f_1 \neq f_2$. This leaves us with the options of $\tau_3 = p_1p_3p_4p_5$, $\tau_3 = p_1p_2p_3p_5$, or $\tau_3 = p_1p_2p_4p_5$. Without loss of generality, we choose $\tau_3 = p_1p_3p_4p_5$.

(4) Let us consider the fourth tetrahedron τ_4, which is required to share two faces $f_{42} \neq f_{43}$ with τ_2 and τ_3 respectively, but should not share any faces with τ_1 according to the boundary graph. However, we encounter a problem here: it is not possible to find a tetrahedron that satisfies this condition. One can refer to the right panel of Fig. fig:sixVGraph 8 for an illustration of the problem.

The discussion above suggests that the 6-valence vertex depicted in Figure fig:sixVGraph 8 cannot be dual to a 4-dimensional cell enclosed by tetrahedra. This raises the question of what this vertex is dual to. Upon careful examination of the construction described earlier, we discovered that the reason for the absence of a solution for τ_4 lies in the requirement that the adjacent boundary objects, i.e., the adjacent boundary tetrahedra, must share three points. Then, if we modify this requirement such that the adjacent boundary objects only share two points, it is possible to find solutions by following the aforementioned construction procedure. However, when the adjacent boundary objects share only two points, it implies that these boundary objects are no longer tetrahedra but rather quadrilaterals. Consequently, the cell dual to the vertex is enclosed by six quadrilaterals, making it a 3-dimensional cell. In fact, this cell corresponds to a hexahedron.

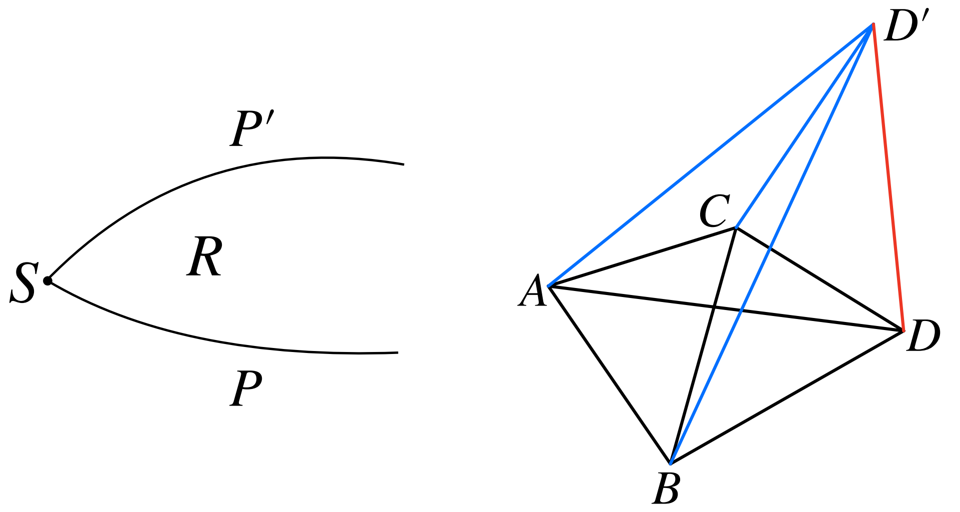


FIG. 9: triangulation of the region R. The red line DD' is the wordline of the vertex D. The triangle ABC lies in the plane S which does not evolute. The tetrahedron $ABCD'$ forms a triangulation of the future hypersurface P' and is called the future tetrahedron.

2. A systematic way to construct triangulation from the boundary triangulation

Consider a spacetime region R bounded by two spatial hypersurfaces P and P'. The intersection of these hypersurfaces forms a 2-plane denoted by S. In Figure 9, we depict a triangulation of the spatial surface P using the tetrahedra $ABCD$, where the triangle ABC lies in the 2-plane S. We call the vertex D as the dynamical vertex, which evolves from P to P' and eventually forms a vertex D' in P'. By connecting the vertex D' with the vertices A, B, and C, we obtain the triangulation $ABCDD'$ of the region R. It is worth noting that the tetrahedron $ABCD'$ forms a triangulation of the hypersurface P', which will be referred to as the future tetrahedron. Besides the future tetrahedron $ABCD'$, the 4-simplex contains other 4 boundary tetrahedra: the past tetrahedron $ABCD$, the tetrahedron $ACDD'$ as the world volume of the triangle ACD, the tetrahedron $ABDD'$ as the word volume of ABD and the tetrahedron $BCDD'$ as the world volume of the triangle BCD.

Then, let us go to a more complicated case. We triangulate the spatial surface P using the tetrahedron $ABCD$ where only the segment AB lies in the 2-plane S. Then, we will have two dynamical vertices C and D as shown in Fig. 10. The two dynamical vertices C and D evolves from P to P' and finally form the vertices C' and D'. To get the corresponding triangularization, we first consider the evolution of the tetrahedron $ABCD$ with D being the dynamical vertex, which will give us a 4-simplex $ABCDD'$ as did in the first case. In the 4-simplex, as we discussed before, there is the future tetrahedron $ABCD'$ as the world volume of the triangle ABC. Then, we consider the evolution of the tetrahedron $ABCD'$ with C being the dynamical vertex. This evolution gives a new 4-simplex $ABCD'C'$. The cell $ABCDC'D'$ is then triangulated by the 4-simplices $ABCDD'$ and $ABCD'C'$.

In another intriguing scenario, we perform a triangulation of the spatial surface P using two tetrahedra, specifically $ABCD$ and $BCDE$, which share the face BCD (refer to Figure 11). The resulting cell, denoted as $ABCDDC'D$, is obtained by evolving $ABCDE$ with C and D as the dynamic vertices, as shown in Figure 11. To perform the triangulation for the cell $ABCDEC'D'$, we follow a step-by-step process. First, we consider the tetrahedron $ABCD$ with D as the dynamic vertex and construct the 4-simplex $ABCDD'$. Next, we consider the evolution of the future tetrahedron $ABCD'$ with C as the dynamic vertex, resulting in the 4-simplex $ABCD'C'$. Then, we move on to the tetrahedron $BCDE$ with D as the dynamic vertex and construct the 4-simplex $BCDED'$. Finally, we consider the evolution of the future tetrahedron $BCED'$ with C as the dynamic vertex, resulting in the 4-simplex $BCED'C'$. As a result, the cell $ABCDEC'D'$ is obtained by gluing together the four 4-simplices, namely $ABCDD'$, $ABCD'C'$, $BCDED'$, and $BCED'C'$, as illustrated in Figure 11.

3. a modified triangulation

In the research paper *arXiv:2105.06876*, the focus is on examining the spacetime region bounded by Σ_1 and Σ_2, which intersect to form two spheres $S_\pm$ (refer to Figure 12). The triangulation of the past boundary Σ_2 is carried

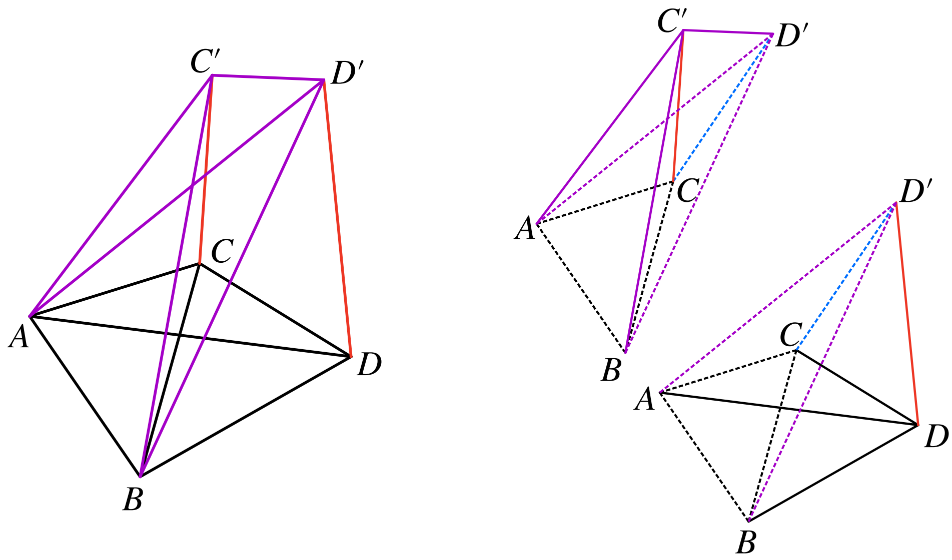


FIG. 10: The triangulation of the cell $ABCDD'C'$. In the right side, we have the 4-simplices which triangulate the cellular. The first 4-simplex $ABCDD'$ is given by evolution of the tetrahedron $ABCD$ with D being the dynamical vertex. The second 4-simplex $ABCC'D'$ is given by the evolution of the tetrahedron $ABCD'$ with C being the dynamical vertex. fig:t

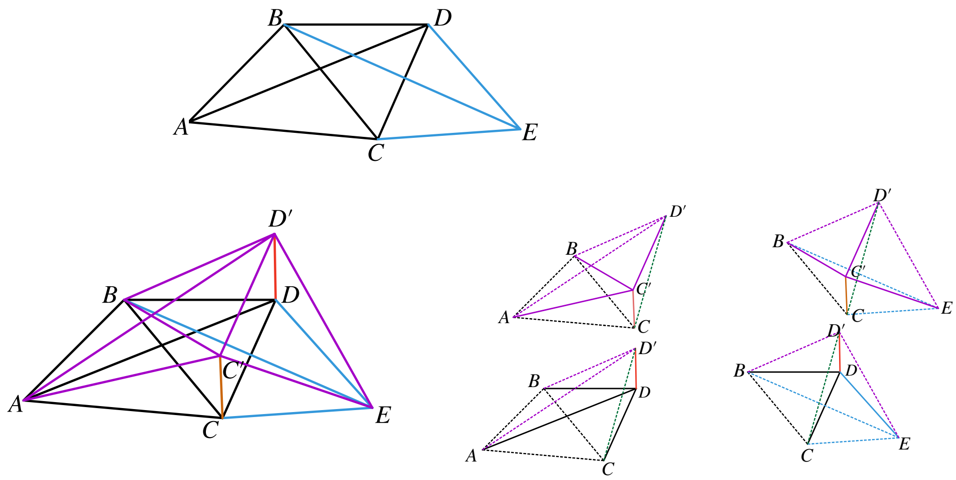


FIG. 11: The triangulation of the cell $ABCDED'C'$. fig:l

out in such a way that all the vertices lie on the spheres $S_{\pm}$. Thus there is no dynamical vertex in the boundary triangulation. This presents a challenge when attempting to triangulate the bulk using the previously mentioned systematic procedure. To address this issue, one possible approach is to let the vertices on the sphere S_+ be dynamical. This motivates the evolution of the sphere S_+ into a larger sphere denoted as S'_+. Consequently, we consider the spacetime region enclosed by Σ_2, Σ_0, and the world volume of S_+, as depicted in the right panel of Figure fig:new_solution 12. It is important to note that, accounting for the time reverse symmetry, we also include the upper region enclosed by Σ_0, Σ_1, and the world volume of S'_+, as illustrated in the right panel of Figure fig:new_solution 12.

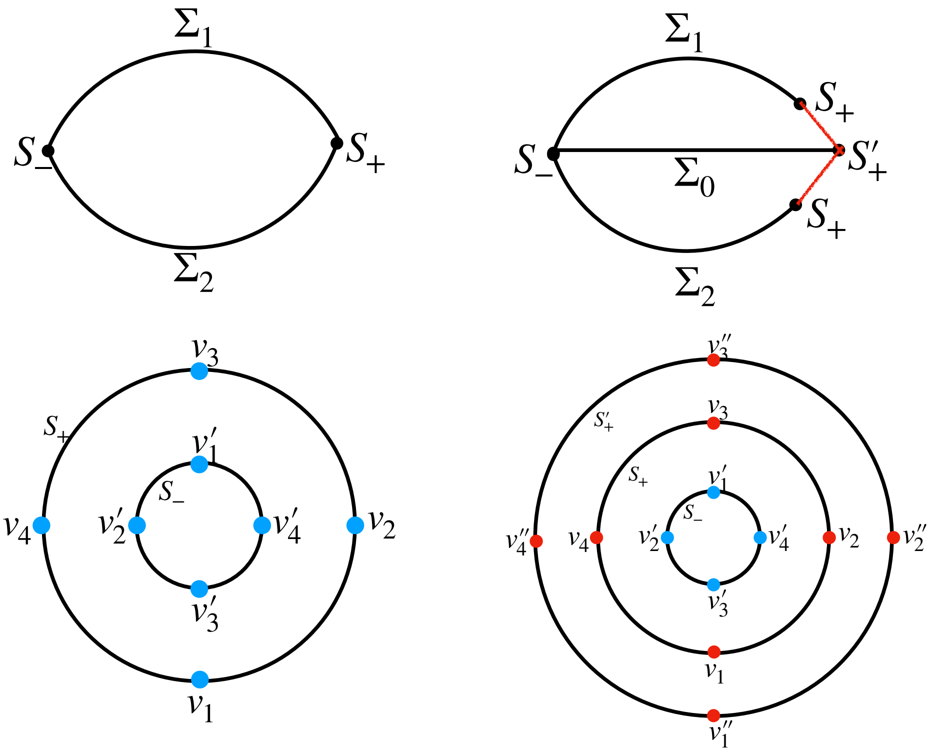


FIG. 12: The region discussed in *arXiv:2105.06876* is represented in the top-left panel. In this region, the triangulation of the past boundary Σ_2 is performed such that all the vertices lie on the spheres $S_\pm$, as shown in the bottom-left panel. As a result, there are no dynamical vertices in this triangulation. Consequently, the systematic procedure mentioned earlier cannot be applied to triangulate the bulk region. The top-right panel illustrates the region for the new solution. In this case, we consider the evolution of S_+, resulting in a spacetime region enclosed by Σ_2, Σ_0, and the world volume of S_+. To account for time reverse symmetry, we also include the upper region enclosed by Σ_0, Σ_1, and the world volume of S'_+.

C. numerics for multi-4-simplcies

1. *glue adjacent tetrahedra*

After obtaining the coordinates of the vertices of each 4-simplex, we can apply the algorithm mentioned earlier to obtain the spinfoam data. However, the resulting spinfoam data might not be appropriately connected. Indeed, a tetrahedron that connects two adjacent 4-simplices is associated with a set of spinors. In our algorithm, we calculate separately for each of the two adjacent 4-simplices, resulting in two sets of spinors for the tetrahedron connecting them. In the case where the spinfoam data are not connected, we need to do transformation for the data associated to one of the two adjacent 4-simplices. Let us discussion the issue case by case.

(1) **for the case with spacelike tetrahedron:** According to (3.11) and (3.12), we have the equation for the spinors ζ_{ef}

$$\begin{aligned}(g_{ve}^\dagger)^{-1}\zeta_{ef} =&\frac{\overline{\alpha_{ve'f}}}{\overline{\alpha_{vef}}}(g_{ve'}^\dagger)^{-1}\zeta_{e'f}\\ g_{ve}\zeta_{ef} =&\frac{\alpha_{vef}}{\alpha_{ve'f}}g_{ve'}\zeta_{e'f}\end{aligned} \tag{4.60}$$

with some complex numbers α_{vef} and $\alpha_{ve'f}$. Considering the data on a fixed edge e, we do the transformation

$$\zeta_{ef}\to\beta h\zeta_{ef}=\tilde{\zeta}_{ef},\quad g_{ve}\to g_{ve}h^{-1}=\tilde{g}_{ve},\quad \alpha_{vef}\to\beta\alpha_{vef}=\tilde{\alpha}_{vef}. \tag{4.61}$$

with a complex number β and an $\mathrm{SL}(2,\mathbb{C})$ element h. Obviously, $\tilde{\zeta}_{ef}$ and $\tilde{g}_{ve}$ are the solution to second equation of the above equations. If they are also the solution to to first equation, we have

$$\beta(g_{ve}^\dagger)^{-1}h^\dagger h\zeta_{ef} = \frac{\overline{\alpha_{ve'f}}}{\overline{\beta}\alpha_{vef}}(g_{ve'}^\dagger)^{-1}\zeta_{e'f} \tag{4.62}$$

which implies that

$$\left.\begin{aligned} h^\dagger h =&1\\ \beta\overline{\beta} =&1 \end{aligned}\right\} \Rightarrow h \in \mathrm{SU}(2), \quad \beta = e^{i\theta}. \tag{4.63}$$

Therefore, the transformation

$$\zeta_{ef} \to e^{i\theta_f}h\zeta_{ef}, \quad g_{ve} \to g_{ve}h^{-1}, \quad \alpha_{vef} \to e^{i\theta_f}\alpha_{vef}, \forall f \text{ at } e, \tag{4.64}$$

for $h \in \mathrm{SU}(2)$ and $\theta \in \mathbb{R}$ gives a new solution to the critical point equation. It is noted that this transformation is local, meaning that one can do this transformation only for a single edge.

Now, let us consider a tetrahedron t_e connecting two adjacent 4-simplices v and v'. We have two set of spinfoam data associated to vef and $v'ef$, denoted by (g_{ve}, ζ_{vef}) and $(g_{v'e}, \zeta_{v'ef})$ respectively. We want to do the transformation (4.64) for the data associated to vef such that

$$e^{i\theta_f}h\zeta_{vef} = \zeta_{v'ef}, \quad \forall f \text{ at } e. \tag{4.65}$$

We need to solve h and θ_f from this equation. To this end, we fixed a reference face f_o to get

$$U_{v'ef_o}\zeta_0 = e^{i\theta_f}hU_{vef_o}\zeta_0 = hU_{vef_o}e^{i\theta_f\sigma_3}\zeta_0 \tag{4.66}$$

where U_{vef} is the SU(2) element we set in (4.34) and we used $e^{i\theta_f\sigma_3}\zeta_0 = e^{i\theta_f}\zeta_0$. We thus get

$$h = U_{v'ef_o}e^{-i\theta_{f_o}\sigma_3}U_{vef_o}^{-1}. \tag{4.67}$$

Moreover, we find that this equation leads to

$$\begin{aligned} &h(\zeta_{vef}\otimes\zeta_{vef}^\dagger - (J\zeta_{vef})\otimes(J\zeta_{vef})^\dagger)h^\dagger = \zeta_{v'ef}\otimes\zeta_{v'ef}^\dagger - (J\zeta_{v'ef})\otimes(J\zeta_{v'ef})^\dagger\\ \Rightarrow &U_{v'ef_o}e^{-i\theta_{f_o}\sigma_3}U_{vef_o}^{-1}(\zeta_{vef}\otimes\zeta_{vef}^\dagger - (J\zeta_{vef})\otimes(J\zeta_{vef})^\dagger)U_{vef_o}e^{i\theta_f\sigma_3}U_{v'ef_o}^{-1} = \zeta_{v'ef}\otimes\zeta_{v'ef}^\dagger - (J\zeta_{v'ef})\otimes(J\zeta_{v'ef})^\dagger\\ \Rightarrow &U_{vef_o}^{-1}(\zeta_{vef}\otimes\zeta_{vef}^\dagger - (J\zeta_{vef})\otimes(J\zeta_{vef})^\dagger)U_{vef_o} = e^{i\theta_{f_o}\sigma_3}U_{v'ef_o}^{-1}(\zeta_{v'ef}\otimes\zeta_{v'ef}^\dagger - (J\zeta_{v'ef})\otimes(J\zeta_{v'ef})^\dagger U_{v'ef_o}e^{-i\theta_f\sigma_3} \end{aligned} \tag{4.68}$$

where we used $Jh\zeta = h^{\dagger -1}J\zeta = hJ\zeta$. This equation is automatically true for $f = f_0$. For $f \neq f_o$, the quantities $U_{vef_o}^{-1}(\zeta_{vef}\otimes\zeta_{vef}^\dagger - (J\zeta_{vef})\otimes(J\zeta_{vef})^\dagger)U_{vef_o}$ and $U_{v'ef_o}^{-1}(\zeta_{v'ef}\otimes\zeta_{v'ef}^\dagger - (J\zeta_{v'ef})\otimes(J\zeta_{v'ef})^\dagger U_{v'ef_o}$ defines two SU(2) Lie algebra elements $\sigma_{vef} = \vec{x}\cdot\vec{\sigma}$ and $\sigma_{v'ef} = \vec{x}'\cdot\vec{\sigma}$. Then, θ_{f_o} is solved by

$$\sigma_{vef} = e^{-i\theta_{f_o}\sigma_3}\sigma_{v'ef}e^{i\theta_f\sigma_3}, \tag{4.69}$$

meaning that $e^{-i\theta_{f_o}\sigma_3}$ rotates $\vec{x}'$ to $\vec{x}$. Note that to solve θ_{f_o}, we only need one surface $f_1 \neq f_o$. Once θ_{f_o} and thus h are obtained, one also need to check if (4.65) is true for the other two faces $f \neq f_o, f_1$.

(1) **for the case with timelike tetrahedron but spacelike triangle:** In this case, according to (3.178), the spinors satisfy the equation

$$\begin{aligned} g_{ve}\sigma_3\xi_{ef}^{\alpha_f} =& \frac{\overline{\zeta_{vef}}}{\overline{\zeta_{ve'f}}}g_{ve'}\sigma_3\xi_{e'f}^{\alpha_f}\\ g_{ve}J\xi_{ef}^{\alpha_f} =& \frac{\zeta_{ve'f}}{\zeta_{vef}}J\xi_{e'f}^{\alpha_f}. \end{aligned} \tag{4.70}$$

where we apply $\eta = \sigma_3$, it should be noted that ζ here denotes complex numbers and $\alpha_f = \pm$. Playing the same game as before, we find the following transformation

$$\zeta_{ef} \to e^{i\theta_f}h\zeta_{ef}, \quad g_{ve} \to g_{ve}h^{-1}, \quad \alpha_{vef} \to e^{i\theta_f}\alpha_{vef}, \forall f \text{ at } e, \tag{4.71}$$

with $h \in$SU(1,1) for a new solution.

Considering the two adjacent 4-simplices v v' connected by a tetrahedron t_e, we need to transform the data associated to vef such that

$$e^{i\theta_f} h\xi^{\alpha_f}_{vef} = \xi_{v'ef}. \tag{4.72}$$

Choosing a reference face f_o, we have

$$h\tilde{U}_{vef_o} e^{i\alpha_{f_o}\theta_{f_o}} \xi_0^{\alpha_f \sigma_3} = \tilde{U}_{v'ef_o} \xi_0^{\alpha_f} \tag{4.73}$$

where $\tilde{U} = U^{\dagger -1}$ with U introduced in (4.42). This leads to

$$h = \tilde{U}_{v'ef_o} e^{-i\alpha_{f_o}\theta_{f_o}\sigma_3} \tilde{U}^{-1}_{vef_o}. \tag{4.74}$$

For another face $f \neq f_o$, we have

$$\sigma_3 h\xi^{\alpha_f}_{vef} \otimes \xi^{\alpha_f}_{vef}{}^\dagger h^\dagger \sigma_3 = \sigma_3 \xi^{\alpha_f}_{v'ef} \otimes \xi^{\alpha_f}_{v'ef}{}^\dagger \sigma_3 \Rightarrow h^{\dagger -1} \sigma_3 \xi^{\alpha_f}_{vef} \otimes \xi^{\alpha_f}_{vef}{}^\dagger \sigma_3 h^{-1} = \sigma_3 \xi^{\alpha_f}_{v'ef} \otimes \xi^{\alpha_f}_{v'ef}{}^\dagger \sigma_3 \tag{4.75}$$

where we used $h^\dagger \sigma_3 h = \sigma_3$, and

$$J(h\xi^{\alpha_f}_{vef}) \otimes J(h\xi^{\alpha_f}_{vef})^\dagger = (J\xi^{\alpha_f}_{v'ef}) \otimes (J\xi^{\alpha_f}_{v'ef})^\dagger \Rightarrow h^{\dagger -1}(J\xi^{\alpha_f}_{vef}) \otimes J(\xi^{\alpha_f}_{vef})^\dagger h^{-1} = (J\xi^{\alpha_f}_{v'ef}) \otimes (J\xi^{\alpha_f}_{v'ef})^\dagger. \tag{4.76}$$

We thus have

$$h^{\dagger -1}\left(\sigma_3 \xi^{\alpha_f}_{vef} \otimes \xi^{\alpha_f}_{vef}{}^\dagger \sigma_3 + (J\xi^{\alpha_f}_{vef}) \otimes (J\xi^{\alpha_f}_{vef})^\dagger\right) h^{-1} = \sigma_3 \xi^{\alpha_f}_{v'ef} \otimes \xi^{\alpha_f}_{v'ef}{}^\dagger \sigma_3 + (J\xi^{\alpha_f}_{v'ef}) \otimes (J\xi^{\alpha_f}_{v'ef})^\dagger. \tag{4.77}$$

Since ξ is an SU(1,1) spinor, one can verify that

$$\sigma_3 \xi \otimes \xi^\dagger \sigma_3 + (J\xi) \otimes (J\xi)^\dagger = \sum_{a=0,1,2} x^a \sigma_a$$

for some vector $\vec{x}$ with $\vec{x} \cdot \eta \cdot \vec{x} = -1$. Thus, (4.77) leads to

$$\tilde{U}^\dagger_{vef_o} x^a \sigma_a \tilde{U}_{vef_o} = e^{i\alpha_{f_o}\theta_{f_o}\sigma_3} \tilde{U}^\dagger_{v'ef_o} x'^a \sigma_a \tilde{U}_{v'ef_o} e^{-i\alpha_{f_o}\theta_{f_o}\sigma_3}, \tag{4.78}$$

implying that $e^{i\alpha_{f_o}\theta_{f_o}\sigma_3}$ transform $\tilde{U}^\dagger_{v'ef_o} x'^a \sigma_a \tilde{U}_{vef_o}$ to $\tilde{U}^\dagger_{vef_o} x'^a \sigma_a \tilde{U}_{v'ef_o}$.

2. adjust the orientation to be consistent

After getting the coordinates of the vertices of each 4-simplex, we could use the algorithm introduced above to get the spinfoam data, including the orientation, i.e., the values of ϵ_{vef}. However, the orientations between two simplexes are not consistent with each other, where the consistent orientation means that $\epsilon_{vef} = -\epsilon_{v'ef}$ for two adjacent vertices v and v' connected by e. To get the consistent orientation, we do the followings.

For each 4-simplex v_a, we first find all other simplexes v_b adjacent to it and store the results as lists $\{v_a, v_b\}$. Then, we construct a graph

```
In[3]:= γ={UndirectedEdge[va, vb], UndirectedEdge[vb, vd], UndirectedEdge[va, ve],...}.
```

The next step is to find a path in the graph γ starting from the first simplex v_1 and ending at the last simplex v_n with length $n-1$, i.e., a path containing $n-1$ edges, ensuring that the path pass through all simplexes. Then, following the order of the simplexes shown by the path, we adjust the orientation of each simplex v by adding an overall -1 to all ϵ_{vef} associated to v if the orientation of v is not consistent with the one before it in the path.

3. get all faces

To get all faces of the triangulation, we first gather all of the objects `{Simplex[Label_],Tetrahedron[v1_,v2_,v3_,v4_],Triangle[v1_,v2_,v3_]}`, i.e., segments of face (vef), that have the same value `Triangle[___]`. Then, we get a list

```
Out[3]= {{Simplex[Label1],Tetrahedron[v1,v2,v3,v4],Triangle[v1,v2,v3]},
        {Simplex[Label2],Tetrahedron[v1,v5,v2,v3],Triangle[v1,v2,v3]},
         ...}
```

Then, we just delete all `Triangle[v1,v2,v3]` appearing in the resulting list to get

```
Out[4]= {{Simplex[Label1],Tetrahedron[v1,v2,v3,v4]},
        {Simplex[Label2],Tetrahedron[v1,v5,v2,v3]},...}
```

With this list, we construct a graph γ_f taking each sublist `{Simplex[_],Tetrahedron[___]}` as its edge. By checking if γ_f is close or not, we can determine whether the face is a boundary or an internal one. By examining the order in which they appear in γ_f, we can sort these `Simplex[_]` and `Tetrahedron[___]` to get a list whose order is either compatible with or opposite the orientation of the face. For a boundary face, we will get a list taking the pattern

```
Out[5]= F1={Tetrahedron[_ _ _], Simplex[_], Tetrahedron[_ _  _], Simplex[_],
        ...,Tetrahedron[_ _ _]}
```

because the boundary face start from a half edge and ends at another half edge connecting to the boundary. For a internal face, for the future convenience, we will choose one `Simplex[_]` object as its starting points, which indeed means that we choose the vertex `Simplex[_]` as the starting point of the face, to get a list of the pattern

```
Out[6]= F2={Simplex[x_], Tetrahedron[_ _  _], Simplex[_], ...,Tetrahedron[_ _ _], Simplex[x_]}
```

We then partition the lists $\mathtt{F_a}$ into sublists of length 2, with offset 1. Then, each of the resulting lists contains an object `Simplex[_]` and an object `Tetrahedron[___]`. We reorder each of them such that the object `Simplex[_]` is put at first, to get the list of half edges `Simplex[_],Tetrahedron[___]` whose order is either compatible with or opposite the orientation of the face. To make the order exactly compatible with the orientation of the face, we check the value of ϵ_{vef} associated to the fist half edge in the list. For boundary face, this value should be +1 while for an internal face, it should be -1, according to our convention (eq:orientationep 3.9). Finally, we store our results as a rule

ianglerule **Code 3 :**

```
Out[7]= Triangle[_ _ _]→{{Simplex[Label1],Tetrahedron[v1,v2,v3,v4]},
        {Simplex[Label2],Tetrahedron[v1,v5,v2,v3]},...  "Type"}
```

where the string `"Type"` takes either `"boundary"` or `"internal"` to tell the type of the face.

D. calculate the transition amplitude and its derivative

1. *calculate the transition amplitude associated to each face*

According to (eq:action 2.33), we will calculate the action associated to each face. It should be noted that the exact expression depends on whether the face is a boundary or not.

(1) For the case where the face is boundary, the associated configuration is shown in Fig. fig:configuration 13 (left panel). As shown in the figure, the face starts from one boundary edge and ends at the other boundary edge. The action is a function of the spinfoam spinors $z=(z_{v_1},z_{z_2},\cdots,z_{v_n})$, the SL$(2,\mathbb{C})$ elements $g=(g_{v_1e_1},g_{v_1e_2},\cdots,g_{v_ne_{n+1}})$ and the boundary spinors ξ_1 and ξ_2. Then the action reads,

$$S(j,g,z,\xi)=j\Bigg[\ln\left(\frac{\langle\xi_1,g^\dagger_{v_1e_1}z_{v_1}\rangle^2}{\langle\xi_1,\xi_1\rangle\langle g^\dagger_{v_1e_1}z_{v_1},g^\dagger_{v_1e_1}z_{v_1}\rangle}\right)+\sum_{k=1}^{n-1}\ln\left(\frac{\langle g^\dagger_{v_ke_{k+1}}z_{v_k},g^\dagger_{v_{k+1}e_{k+1}}z_{v_{k+1}}\rangle^2}{\langle g^\dagger_{v_ke_{k+1}}z_{v_k},g^\dagger_{v_ke_{k+1}}z_{v_k}\rangle\langle g^\dagger_{v_{k+1}e_{k+1}}z_{v_{k+1}},g^\dagger_{v_{k+1}e_{k+1}}z_{v_{k+1}}\rangle}\right)$$
$$+\ln\left(\frac{\langle g^\dagger_{v_ne_{n+1}}z_{v_n},\xi_2\rangle^2}{\langle g^\dagger_{v_ne_{n+1}}z_{v_n},g^\dagger_{v_ne_{n+1}}z_{v_n}\rangle\langle\xi_2,\xi_2\rangle}\right)+i\gamma(\text{imaginary part})\Bigg] \tag{4.79}$$

To do the calculation, we will first construct a sequence

$$\begin{aligned}Z=&\Big(\xi_1,g^\dagger_{v_1e_1}z_{v_1},g^\dagger_{v_1e_2}z_{v_1},g^\dagger_{v_2e_2}z_{v_2}\cdots,g^\dagger_{v_ne_{n+1}}z_{v_n},\xi_2\Big)\\=&\Big(\xi_1,Z_{v_1e_1},Z_{v_1e_2},Z_{v_2e_2},\cdots,Z_{v_ne_{n+1}},\xi_2\Big)\end{aligned} \tag{4.80}$$

eq:Zlist

Then, we partitions the sequence into nonoverlapping sublists of length 2, i.e.,

$$\tilde{Z} = \begin{pmatrix} \xi_1, & Z_{v_1 e_1} \\ Z_{v_1 e_2}, & Z_{v_2 e_2} \\ \cdots, & \cdots \\ Z_{v_n e_{n+1}}, & \xi_2. \end{pmatrix} \tag{4.81}$$

Then, we have

$$S(j,g,z,\xi) = j \sum_{k=1}^{n} \left(\ln\left(\frac{\langle Z_{k1}, Z_{k2}\rangle^2}{\langle Z_{k1}, Z_{k1}\rangle \langle Z_{k2}, Z_{k2}\rangle} \right) + i\gamma \ln\left(\frac{\langle Z_{k1}, Z_{k1}\rangle}{\langle Z_{k2}, Z_{k2}\rangle} \right) \right) \tag{4.82}$$

eq:numericalexpression

where each term of the summands is a function depending on each row of $\tilde{Z}$, i.e., a function of (Z_{k1}, Z_{k2}). Note that, in this case, our sequence starts from the boundary vertex, which implies that we start from the half edge (v_1, e_1) whose sign $\epsilon_{v_1 e_1} = 1$.

(2) For the case associated to the internal face, as shown in the right panel of Fig. fig:configuration 13, we need to start from a vertex and construct the sequence

$$Z = \left(g^\dagger_{v_1 e_1} z_{v_1}, g^\dagger_{v_2 e_1} z_{v_2}, g^\dagger_{v_2 e_2} z_{v_2} \cdots, g^\dagger_{v_n e_n} z_{v_n}, g^\dagger_{v_1 e_n} z_{v_1} \right). \tag{4.83}$$

Then we partition it into nonoverlapping subsequences with length 2, i.e.,

$$\tilde{Z} = \begin{pmatrix} g^\dagger_{v_1 e_1} z_{v_1}, & g^\dagger_{v_2 e_1} z_{v_2} \\ \cdots, & \cdots \\ g^\dagger_{v_n e_n} z_{v_n}, & g^\dagger_{v_1 e_n} z_{v_1}. \end{pmatrix} \tag{4.84}$$

so that (eq:numericalexpression 4.82) can still be applied.

It should be mentioned that, to get the sequence Z, we need to start from a vertex z_1 which implies that we start from a half edge (v_1, e_1) whose sign $\epsilon_{v_1 e_1} = -1$. This is a difference from the case of boundary edge.

To implement the calculation properly by our codes, we will construct a function taking the following objects as its inputs:

(1) `AreaRule` taking the pattern `Triangle[_]` $\to$ `j` and assigning to each face a spin.

(2) `TriangleRule` taking the pattern as shown in Code mma:trianglerule 3 and indicating face the action associated to.

(3) `assignSFDataRule`, taking the form as shown in Code mma:sfdata1 1 and assigning the SL$(2,\mathbb{C})$ elements and the spinors to the corresponding objects.

(4) `assignBdyDataRule`, taking the form

```
Out[8]=   assignBdyDataRule={Tetrahedron[_ _ _],  Triangle[_ _ _]}→ spinor₁,
              {Tetrahedron[_ _ _],  Triangle[_ _ _]}→ spinor₂,   ...}
```

and assigning the boundary spinors to the boundary objects.

(5) `orientationRule`, taking the form as shown in Code mma:sfdata2 2 and assigning value of ϵ_{vef} to the each (vef).

In our algorithm, we first apply the Input `TriangleRule` to create the list Z of boundary objects of the face:

(1) for boundary face:

```
Out[9]=  Z= {{Tetrahedron[bdy1_ _],Triangle[_ _]},
          {Simplex[_], Tetrahedron[bdy_ _ ],  Triangle[_ _ _]},
          {Simplex[_], Tetrahedron[_ _ _],  Triangle[_ _ _]},
          ...
          {Simplex[_], Tetrahedron[bdy2_ _ ],  Triangle[_ _ _]}
           {Tetrahedron[bdy2_ _ ],  Triangle[_ _ _]}
          }
```

(1) for boundary face:

```
Out[10]=  Z= {
           {Simplex[_], Tetrahedron[_ _ ],  Triangle[_ _ _]},
           {Simplex[_], Tetrahedron[_ _ _],  Triangle[_ _ _]},
           ...
           {Simplex[_], Tetrahedron[_ _ ],  Triangle[_ _ _]}
           }
```

Then, as what we mentioned in (4.80), we partitions the sequence Z into nonoverlapping sublist of length 2 to get $\tilde{Z}$. Then, we apply the rules `assignSFDataRule` and `assignBdyDataRule` to get the corresponding values of spinors which is related to the variables Z_{vef} by

$$\zeta_{vef} = \alpha_{vef}^{-1} Z_{vef} \tag{4.85}$$

whith α_{vef} being a complex number.

It should be noted that the two terms in the action dependent on the factor α_{vef}. We thus need to get the factor α_{vef} from the SL$(2,\mathbb{C})$ element and the spinors. Before doing that, let us see how the action depends on factor α_{vef}, where we will take the action associated to an internal face as an example. The imaginary part reads

$$\begin{aligned}
S_f =& j \sum_e \left(\ln\left(\frac{\langle Z_{s_e ef}, Z_{t_e ef}\rangle^2}{\|Z_{s_e ef}\|^2 \|Z_{t_e ef}\|^2} \right) + i\gamma \ln\left(\frac{\langle Z_{s_e ef}, Z_{s_e ef}\rangle}{\langle Z_{t_e ef}, Z_{t_e ef}\rangle} \right) \right) \\
=& j \sum_e \left(\ln\left(\frac{(\alpha^*_{s_e ef}\alpha_{t_e ef})^2 \langle \zeta_{s_e ef}, \zeta_{t_e ef}\rangle^2}{|\alpha_{s_e ef}|^2 |\alpha_{t_e ef}|^2 \|\zeta_{s_e ef}\|^2 \|\zeta_{t_e ef}\|^2} \right) + i\gamma \ln\left(\frac{|\alpha_{s_e ef}|^2 \langle \zeta_{s_e ef}, \zeta_{s_e ef}\rangle}{|\alpha_{t_e ef}|^2 \langle \zeta_{t_e ef}, \zeta_{t_e ef}\rangle} \right) \right) \\
=& j \sum_e \left(\ln\left(\langle \zeta_{s_e ef}, \zeta_{t_e ef}\rangle^2\right) + \ln\left(\frac{(\alpha^*_{s_e ef}\alpha_{t_e ef})^2}{|\alpha_{s_e ef}|^2 |\alpha_{t_e ef}|^2} \right) + i\gamma \ln\left(\frac{|\alpha_{s_e ef}|^2}{|\alpha_{t_e ef}|^2} \right) \right) \\
=& j \sum_e \ln\left(\langle \zeta_{s_e ef}, \zeta_{t_e ef}\rangle^2\right) + \sum_v \left(\ln\left(\frac{(\alpha^*_{ve_v f})^2 (\alpha_{ve_v f})^2}{|\alpha_{ve_v f}|^2 |\alpha_{vs_v f}|^2} \right) + i\gamma \ln\left(\frac{|\alpha_{ve_v f}|^2}{|\alpha_{vs_v f}|^2} \right) \right) \\
=& j \sum_e \ln\left(\langle \zeta_{s_e ef}, \zeta_{t_e ef}\rangle^2\right) + \sum_v \left(\ln\left(\frac{|\alpha_{ve_v f}|^2}{|\alpha_{vs_v f}|^2} \frac{(\alpha_{ve_v f})^2}{(\alpha_{ve_v f})^2} \right) + i\gamma \ln\left(\frac{|\alpha_{ve_v f}|^2}{|\alpha_{vs_v f}|^2} \right) \right)
\end{aligned} \tag{4.86}$$

where we used $z^*/|z| = |z|z^{-1}$ for any complex number and we regroup those Z_{vef} because the value of $\frac{|\alpha_{vs_v f}|^2}{|\alpha_{ve_v f}|^2}$ can be uniquely determined as follows. According to (4.85), we have

$$\begin{aligned}
\zeta_{vs_v f} &= \alpha^{-1}_{vs_v f} g^\dagger_{vs_v} z_{vf} \\
\zeta_{ve_v f} &= \alpha^{-1}_{ve_v f} g^\dagger_{ve_v} z_{vf}
\end{aligned} \tag{4.87}$$

which leads to

$$\begin{aligned}
&\alpha_{vs_v f} {g^\dagger_{vs_v}}^{-1} \zeta_{vs_v f} = \alpha_{ve_v f} {g^\dagger_{ve_v}}^{-1} \zeta_{ve_v f} \\
\Rightarrow & \frac{\alpha_{ve_v f}}{\alpha_{vs_v f}} = \zeta^\dagger_{ve_v f} g^\dagger_{ve_v} {g^\dagger_{vs_v}}^{-1} \zeta_{vs_v f} = \left(\zeta^\dagger_{vs_v f} g^{-1}_{vs_v} g_{ve_v} \zeta_{ve_v f} \right)^*
\end{aligned} \tag{4.88}$$

where we used $\zeta = U\zeta_0$ leading $\zeta^\dagger_{vef}\zeta_{vef} = 1$. We thus have

$$S_f = j \sum_e \ln\left(\langle \zeta_{s_e ef}, \zeta_{t_e ef}\rangle^2\right) + \sum_v \left(\ln\left(\frac{\left| \zeta^\dagger_{vs_v f} g^{-1}_{vs_v} g_{ve_v} \zeta_{ve_v f} \right|^2}{\left(\zeta^\dagger_{vs_v f} g^{-1}_{vs_v} g_{ve_v} \zeta_{ve_v f} \right)^{*\,2}} \right) + i\gamma \ln\left(\left| \zeta^\dagger_{vs_v f} g^{-1}_{vs_v} g_{ve_v} \zeta_{ve_v f} \right|^2 \right) \right) \tag{4.89}$$

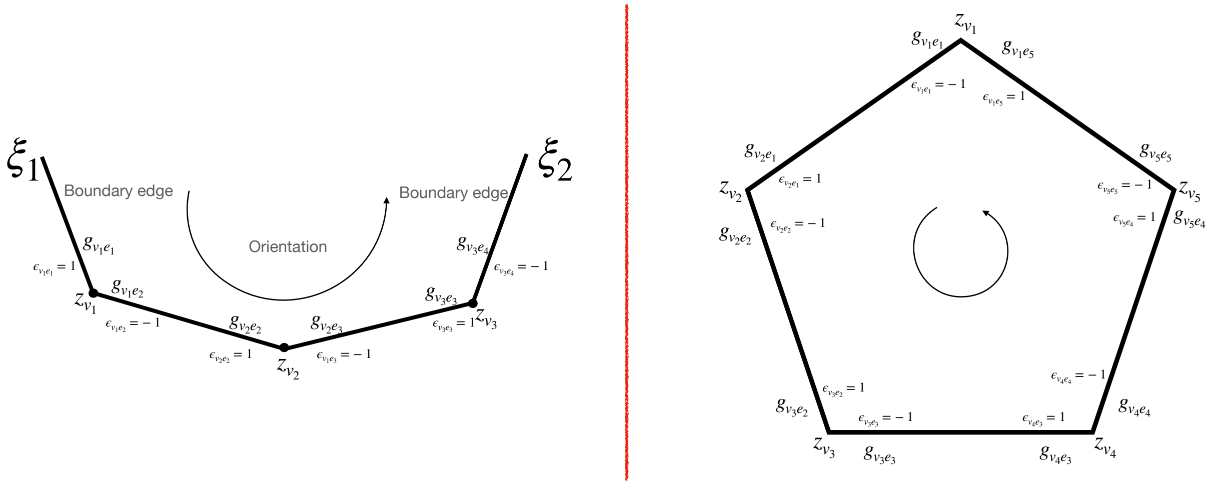


FIG. 13: the configuration associated to the boundary face (left panel) and the internal face (right panel). fig:c

2. calculate the derivative

According to the above discussion, the action is built from the block

$$
\begin{aligned}
\Theta(g_1,g_2,z_1,z_2) =& j\left(\ln\left(\frac{\langle g_1^\dagger z_1, g_2^\dagger z_2\rangle^2}{\langle g_1^\dagger z_1, g_1^\dagger z_1\rangle\langle g_2^\dagger z_2, g_2^\dagger z_2\rangle}\right)+i\gamma\ln\left(\frac{\langle g_1^\dagger z_1, g_1^\dagger z_1\rangle}{\langle g_2^\dagger z_2, g_2^\dagger z_2\rangle}\right)\right) \\
=& j\left(2\ln\left(\langle g_1^\dagger z_1, g_2^\dagger z_2\rangle\right)-(1-i\gamma)\ln\left(\langle g_1 z_1^\dagger, g_1 z_1^\dagger\rangle\right)-(1+i\gamma)\ln\left(\langle g_2^\dagger z_2, g_2^\dagger z_2\rangle\right)\right) \\
=& j\left(2\ln\left(z_1^\dagger g_1 g_2^\dagger z_2\right)-(1-i\gamma)\ln\left(z_1^\dagger g_1 g_1^\dagger z_1\right)-(1+i\gamma)\ln\left(z_2^\dagger g_2 g_2^\dagger z_2\right)\right)
\end{aligned}
\tag{4.90}
$$

We thus have

$$
\begin{aligned}
\partial_{z_1}\Theta &= -j(1-i\gamma)\frac{z_1^\dagger g_1 g_1^\dagger}{\langle g_1^\dagger z_1, g_1^\dagger z_1\rangle} \\
\partial_{z_1^\dagger}\Theta &= j\left(\frac{2 g_1 g_2^\dagger z_2}{\langle g_1^\dagger z_1, g_2^\dagger z_2\rangle}-(1-i\gamma)\frac{g_1 g_1^\dagger z_1}{\langle g_1^\dagger z_1, g_1^\dagger z_1\rangle}\right) \\
\partial_{z_2}\Theta &= j\left(\frac{2 z_1^\dagger g_1 g_2^\dagger}{\langle g_1^\dagger z_1, g_2^\dagger z_2\rangle}-(1+i\gamma)\frac{z_2^\dagger g_2 g_2^\dagger}{\langle g_2^\dagger z_2, g_2^\dagger z_2\rangle}\right) \\
\partial_{z_2^\dagger}\Theta &= -j(1+i\gamma)\frac{g_2 g_2^\dagger z_2}{\langle g_2^\dagger z_2, g_2^\dagger z_2\rangle}
\end{aligned}
\tag{4.91}
$$

We therefore have

$$
\begin{aligned}
\partial_{z_v} S_f =& j_f\left(\frac{2 z_{v_-}^\dagger g_{v_- s_v} g_{v s_v}^\dagger}{\langle g_{v_- s_v}^\dagger z_{v_-}, g_{v s_v}^\dagger z_v\rangle}-(1+i\gamma)\frac{z_v^\dagger g_{v s_v} g_{v s_v}^\dagger}{\langle g_{v s_v}^\dagger z_v, g_{v s_v}^\dagger z_v\rangle}-(1-i\gamma)\frac{z_v^\dagger g_{v e_v} g_{v e_v}^\dagger}{\langle g_{v e_v}^\dagger z_v, g_{v e_v}^\dagger z_v\rangle}\right) \\
\partial_{z_v^\dagger} S_f =& j_f\left(\frac{2 g_{v e_v} g_{v_+ e_v}^\dagger z_{v_+}}{\langle g_{v e_v}^\dagger z_v, g_{v_+ e_v}^\dagger z_{v_+}\rangle}-(1-i\gamma)\frac{g_{v e_v} g_{v e_v}^\dagger z_v}{\langle g_{v e_v}^\dagger z_v, g_{v e_v}^\dagger z_v\rangle}-(1+i\gamma)\frac{g_{v s_v} g_{v s_v}^\dagger z_v}{\langle g_{v s_v}^\dagger z_v, g_{v s_v}^\dagger z_v\rangle}\right)
\end{aligned}
\tag{4.92}
$$

eq:derivativeSz

where v_- is the vertex in front of v, v_+ is the one right after v, e_v denote the edge taking v as its starting point (the "ending" edge of v) and s_v denotes the one taking v as its ending point (the "starting" edge of v). In Fig. 14, we shows the configuration space that the derivative function of the action depends on. fig:derivative

It is convenient to rewrite (eq:derivativeSz 4.92) in terms of Z, we have

$$\begin{aligned}\partial_{z_v} S_f =& j_f\left(\frac{2Z^\dagger_{v-s_v} g^\dagger_{vs_v}}{\langle Z_{v-s_v}, Z_{vs_v}\rangle} - (1+i\gamma)\frac{Z^\dagger_{vs_v} g^\dagger_{vs_v}}{\langle Z_{vs_v}, Z_{vs_v}\rangle} - (1-i\gamma)\frac{Z^\dagger_{ve_v} g^\dagger_{ve_v}}{\langle Z_{ve_v}, Z_{ve_v}\rangle}\right)\\ \partial_{z^\dagger_v} S_f =& j_f\left(\frac{2g_{ve_v} Z_{v+e_v}}{\langle Z_{ve_v}, Z_{v+e_v}\rangle} - (1-i\gamma)\frac{g_{ve_v} Z_{ve_v}}{\langle Z_{ve_v}, Z_{ve_v}\rangle} - (1+i\gamma)\frac{g_{vs_v} Z_{vs_v}}{\langle Z_{vs_v}, Z_{vs_v}\rangle}\right)\end{aligned} \tag{4.93}$$

Substituting the relation $\zeta_{vef} = \alpha^{-1}_{vef} Z_{vef}$, we have

$$\begin{aligned}\partial_{z_v} S_f =& \alpha^{-1}_{ve_v} j_f\left(\alpha_{ve_v}\alpha^{-1}_{vs_v}\frac{2\zeta^\dagger_{v-s_v} g^\dagger_{vs_v}}{\langle \zeta_{v-s_v}, \zeta_{vs_v}\rangle} - (1+i\gamma)\alpha_{ve_v}\alpha^{-1}_{vs_v}\frac{\zeta^\dagger_{vs_v} g^\dagger_{vs_v}}{\langle \zeta_{vs_v}, \zeta_{vs_v}\rangle} - (1-i\gamma)\frac{\zeta^\dagger_{ve_v} g^\dagger_{ve_v}}{\langle \zeta_{ve_v}, \zeta_{ve_v}\rangle}\right)\\ \partial_{z^\dagger_v} S_f =& \alpha^{-1*}_{ve_v} j_f\left(\frac{2g_{ve_v}\zeta_{v+e_v}}{\langle \zeta_{ve_v}, \zeta_{v+e_v}\rangle} - (1-i\gamma)\frac{g_{ve_v}\zeta_{ve_v}}{\langle \zeta_{ve_v}, \zeta_{ve_v}\rangle} - (1+i\gamma)\alpha^*_{ve_v}\alpha^{-1*}_{vs_v}\frac{g_{vs_v}\zeta_{vs_v}}{\langle \zeta_{vs_v}, \zeta_{vs_v}\rangle}\right)\end{aligned} \tag{4.94}$$

where $\alpha_{ve_v}\alpha^{-1}_{vs_v}$ can be calculated as shown in (eq:alpharelation 4.88).

To calculate the derivative with respect to g, let us consider the variation $g \to g(1+\delta\vec{\xi}\vec{\sigma})$. Then, we will use ∂_g to denote $\delta s/\delta\xi$ and $\partial_{g^\dagger}$, $\delta s/\delta\bar{\xi}$. We have

$$\begin{aligned}\partial_{g_1}\Theta &= j\left(2\frac{\langle g^\dagger_1 z_1, \vec{\sigma} g^\dagger_2 z_2\rangle}{\langle g^\dagger_1 z_1, g^\dagger_2 z_2\rangle} - (1-i\gamma)\frac{\langle g^\dagger_1 z_1, \vec{\sigma} g^\dagger_1 z_1\rangle}{\langle g^\dagger_1 z_1, g^\dagger_1 z_1\rangle}\right)\\ \partial_{g^\dagger_1}\Theta &= -j(1-i\gamma)\frac{\langle g^\dagger_1 z_1, \vec{\sigma} g^\dagger_1 z_1\rangle}{\langle g^\dagger_1 z_1, g^\dagger_1 z_1\rangle}\\ \partial_{g_2}\Theta &= -j(1+i\gamma)\frac{\langle g^\dagger_2 z_2, \vec{\sigma} g^\dagger_2 z_2\rangle}{\langle g^\dagger_2 z_2, g^\dagger_2 z_2\rangle}\\ \partial_{g^\dagger_2}\Theta &= j\left(\frac{2\langle g^\dagger_1 z_1, \vec{\sigma} g^\dagger_2 z_2\rangle}{\langle g^\dagger_1 z_1, g^\dagger_2 z_2\rangle} - (1+i\gamma)\frac{\langle g^\dagger_2 z_2, \vec{\sigma} g^\dagger_2 z_2\rangle}{\langle g^\dagger_2 z_2, g^\dagger_2 z_2\rangle}\right)\end{aligned} \tag{4.95}$$

where we used $\sigma^\dagger = \sigma$. In Fig. fig:derivative 14, we shows the configuration space that the derivative function of the action depends on.

We thus have

$$\begin{aligned}\partial_{g_{s_e e}} S_f &= j_f\left(2\frac{\langle g^\dagger_{s_e e} z_{s_e}, \vec{\sigma} g^\dagger_{t_e e} z_{t_e}\rangle}{\langle g^\dagger_{s_e e} z_{s_e}, g^\dagger_{t_e e} z_{t_e}\rangle} - (1-i\gamma)\frac{\langle g^\dagger_{s_e e} z_{s_e}, \vec{\sigma} g^\dagger_{s_e e} z_{s_e}\rangle}{\langle g^\dagger_{s_e e} z_{s_e}, g^\dagger_{s_e e} z_{s_e}\rangle}\right)\\ \partial_{g^\dagger_{s_e e}} S_f &= -j_f(1-i\gamma)\frac{\langle g^\dagger_{s_e e} z_{s_e}, \vec{\sigma} g^\dagger_{s_e e} z_{s_e}\rangle}{\langle g^\dagger_{s_e e} z_{s_e}, g^\dagger_{s_e e} z_{s_e}\rangle}\\ \partial_{g_{t_e e}} S_f &= -j_f(1+i\gamma)\frac{\langle g^\dagger_{t_e e} z_{t_e} e, \vec{\sigma} g^\dagger_{t_e e} z_{t_e} e\rangle}{\langle g^\dagger_{t_e e} z_{t_e} e, g^\dagger_{t_e e} z_{t_e} e\rangle}\\ \partial_{g^\dagger_{t_e e}} S_f &= j_f\left(2\frac{\langle g^\dagger_{s_e e} z_{s_e}, \vec{\sigma} g^\dagger_{t_e e} z_{t_e}\rangle}{\langle g^\dagger_{s_e e} z_{s_e}, g^\dagger_{t_e e} z_{t_e}\rangle} - (1+i\gamma)\frac{\langle g^\dagger_{t_e e} z_{t_e}, \vec{\sigma} g^\dagger_{t_e e} z_{t_e}\rangle}{\langle g^\dagger_{t_e e} z_{t_e}, g^\dagger_{t_e e} z_{t_e}\rangle}\right)\end{aligned} \tag{4.96}$$

where s_e denotes the staring vertex of e, t_e denotes the targeting vertex of e

V. CALCULATE THE BOUNDARY DATA FROM THE BOUNDARY METRIC AND THE EXTRINSIC CURVATURE

A. the boundary we choose in the BH spacetime

Let us consider a spacetime region B that possesses the time reversal symmetry. The time reversal symmetry requires a hypersurface, referred to as the transition surface $\mathcal{T}$, in B which divides B into a union of the past region

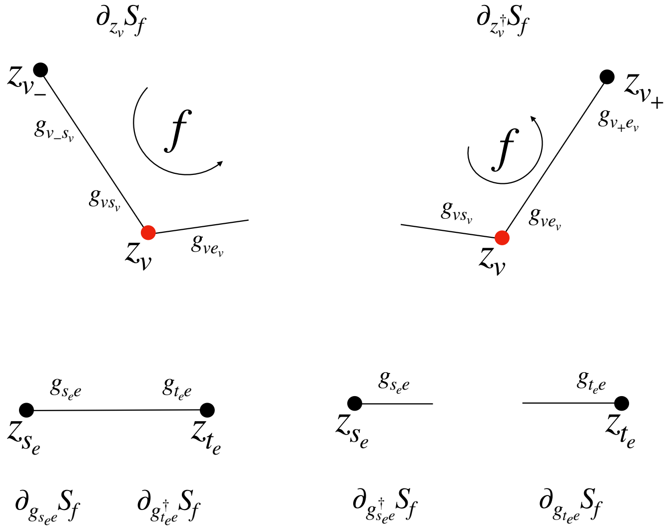


FIG. 14: It shows the configuration that the derivative functions of the action depends on. fig:d

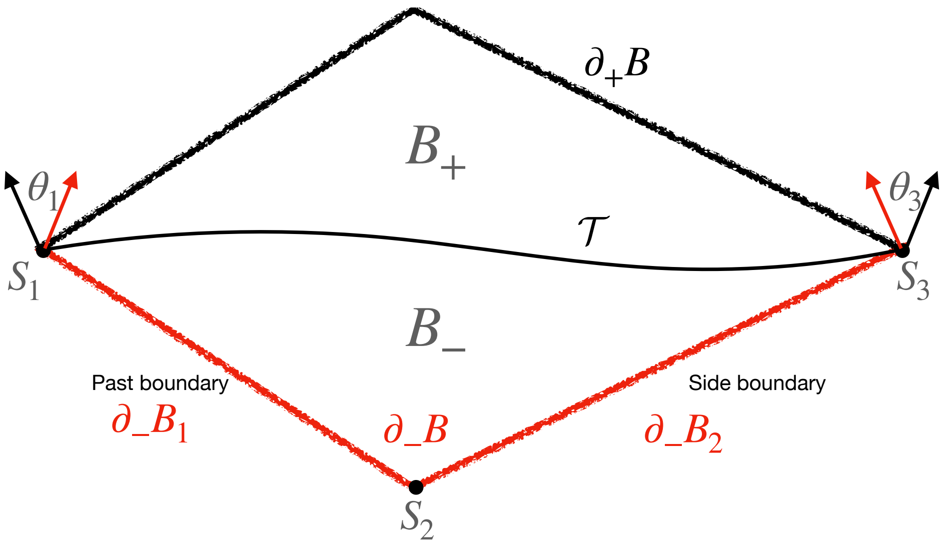


FIG. 15: The B region. fig:r

B_- and the future region B_+. Let $\partial_- B$ and $\partial_+ B$ be the remaining boundary of B_- and B_+ expect $\mathcal{T}$, so that $\partial B = \partial_- B \cup \partial_+ B$. Then, the time reversal symmetry implies that there exists a diffeomorphism, denoted by S, between $B_\pm$ preserving the transition surface $\mathcal{T}$ and the boundary conditions (see Fig. 15). In what follows, we only consider these triangulations of B adapted to the time reversal symmetry, i.e., these ones invariant under the diffeomorphism.

The way to construct the spinfoam model in the region is the following.

(1) Creat a triangulation in the boundary ∂B;

(2) Construct a triangulation in the B region adapted to the boundary triangulation in ∂B. More precisely, we need to construct a triangulation in B such that the boundary of the B-region triangulation take the same shape as the triangulation in ∂B, in the sense that the boundary of the B-region triangulation contains an identical count of vertices, edges, and faces as the triangulation in ∂B.

(3) Based on the dual graph of the B-region triangulation, write down the spinfoam amplitude, where the boundary state is choose to be a coherent state with some parameters $\phi_0(f)$ and $j_0(f)$ assigned to each boundary face.

(4) Consider the 3-metric and the extrinsic curvature on the boundary. Based on the boundary triangulation and the geometric fields on the boundary, calculate the parameters $\phi_0(f)$ and $j_0(f)$ associated to each phase.

(5) Substitute the parameters into the spinfoam amplitude and calculate the amplitude by applying the stationary phase approximation.

(6) In order to address the stationary phase equation, our initial step involves the creation of a flat Regge geometry. This geometry will adopt the identical triangulation as the one within region B and will approximate the boundary data given by the parameters $\phi_0(f)$ and $j_0(f)$ constructed before.

(7) Taking advantage of the flat geometry, we can construct an approximating solution to the stationary phase equations. Then, taking the approximating data as the initial data, we can search the true solution.

B. the boundary in BH and the boundary data

In the B region, we choose its boundary such that it takes the same reduced metric as the surface

$$t = \beta r + a \tag{5.1}$$

in the Minkowski spacetime. To do that, let us consider the Eddtington-Finkelsein coordinate. In the Eddington–Finkelstein coordinates we have the metric

$$\mathrm{d}s^2_{\mathrm{BH}} = -\, f(r)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega^2. \tag{5.2}$$

In the Minkowski spacetime, we have

$$\mathrm{d}s^2_{\mathrm{M}} = -\,\mathrm{d}\tilde{v}^2 + 2\mathrm{d}\tilde{v}\mathrm{d}r + r^2\mathrm{d}\Omega^2, \tag{5.3}$$

where $\tilde{v} = t + r$. In the surface $t = \beta r + a$, the reduced metric is given by

$$\mathrm{d}s^2_{\mathrm{M}} = -(1+\beta)^2\mathrm{d}r^2 + 2(1+\beta)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 = (1-\beta^2)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{5.4}$$

Let the surface in the BH spacetime which takes the same reduced metric as the surface $t = \beta r + a$ in the Minkowski spacetime given by

$$g(r,v) = 0 \Rightarrow \partial_r g\mathrm{d}r + \partial_v g\mathrm{d}v = 0 \Rightarrow \mathrm{d}v = -\frac{\partial_r g}{\partial_v g}\mathrm{d}r, \tag{5.5}$$

We thus have the reduced metric on the hypersurface

$$\mathrm{d}s^2_{\mathrm{BH}} = -\left(\frac{\partial_r g}{\partial_v g}\right)^2 f(r)\mathrm{d}r^2 - 2\frac{\partial_r g}{\partial_v g}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 = -\frac{\partial_r g}{\partial_v g}\left(\frac{\partial_r g}{\partial_v g}f(r) + 2\right)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{5.6}$$

We thus get

$$-\frac{\partial_r g}{\partial_v g}\left(\frac{\partial_r g}{\partial_v g}f(r) + 2\right) = 1 - \beta^2 \Rightarrow \frac{\partial_r g}{\partial_v g} = \frac{\pm\sqrt{\beta^2 f(r) - f(r) + 1} - 1}{f(r)}. \tag{5.7}$$

If $g(r,v)$ takes the form $g(r,v) = v + F(r)$, we have

$$\frac{\mathrm{d}F}{\mathrm{d}r} = -\frac{1 \pm \sqrt{1-(1-\beta^2)f(r)}}{f(r)}. \tag{5.8}$$

eq:eqF

1. properties of the hypersurface given by $v + F(r,v) = 0$

The conormal of the hypersurface is given by

$$N(r) = \mathrm{d}v + F'(r)\mathrm{d}r = \mathrm{d}v - \frac{1 \pm \sqrt{1-(1-\beta^2)f(r)}}{f(r)}\mathrm{d}r. \tag{5.9}$$

The following properties will be important for our further calculation.

(1) It is noticed that

$$\frac{1+\sqrt{1-(1-\beta^2)f(r)}}{f(r)} \tag{5.10}$$

is singular at the horizons $r = r_\pm$ since $f(r_\pm) = 0$.

(2) Moreover, we are concerned with the region $r \geq r_b$ where $r_b = (\alpha M/2)^{1/3}$ with $f(r_b) = 0$. We thus have

$$f(r) < 1 \quad \forall r \geq r_b. \tag{5.11}$$

Then, we have

$$1-(1-\beta^2)f(r) > 0,\ \forall r > r_b,\ \beta^2 < 1. \tag{5.12}$$

(3) We have

$$\frac{1-\sqrt{1-(1-\beta^2)f(r)}}{f(r)} > 0,\ \forall r > r_b\ \beta^2 < 1. \tag{5.13}$$

and with $\beta^2 < 1$,

$$\begin{aligned} \frac{1+\sqrt{1-(1-\beta^2)f(r)}}{f(r)} &> 0, \forall r \in [r_b, r_-) \cup (r_+, \infty), \\ \frac{1+\sqrt{1-(1-\beta^2)f(r)}}{f(r)} &< 0, \forall r \in (r_-, r_+). \end{aligned} \tag{5.14}$$

(4) The normal of N^a of the hypersurface

$$\begin{aligned} N^a =& g^{ab}N_b = (\partial_r)^a - \frac{1 \pm \sqrt{1-(1-\beta^2)f(r)}}{f(r)}(\partial_v)^a - f(r)\frac{1 \pm \sqrt{1-(1-\beta^2)f(r)}}{f(r)}(\partial_r)^a \\ =& -\frac{1 \pm \sqrt{1-(1-\beta^2)f(r)}}{f(r)}(\partial_v)^a - \Big(\pm\sqrt{1-(1-\beta^2)f(r)}\Big)(\partial_r)^a, \end{aligned} \tag{5.15}$$

since the inverse metric g^{ab} reads

$$g^{ab} = (\partial_v)^a(\partial_r)^b + (\partial_r)^a(\partial_v)^b + f(r)(\partial_r)^a(\partial_r)^b + \frac{1}{r^2}(\partial_\theta)^a(\partial_\theta)^b + \frac{1}{r^2\sin^2\theta}(\partial_\phi)^2. \tag{5.16}$$

(5) The norm of the normal is

$$\begin{aligned} \|N(r)\|^2 = F'(r)\left(f(r)F'(r)+2\right) = -\frac{1 \pm \sqrt{1-(1-\beta^2)f(r)}}{f(r)}\Big(1 \mp \sqrt{1-(1-\beta^2)f(r)}\Big) \\ = \frac{\sqrt{1-(1-\beta^2)f(r)}^2 - 1}{f(r)} = -(1-\beta^2) < 0. \end{aligned} \tag{5.17}$$

Thus, the hypersurface is spacelike.

(6) We have

$$\sqrt{h}g^{ab}K_{ab} = -(\pm)\frac{3\left(1-\beta^2\right)M + 2\beta^2 r}{\sqrt{-\alpha\left(1-\beta^2\right)M^2 + 2\left(1-\beta^2\right)Mr^3 + \beta^2 r^4}} \tag{5.18}$$

2. The boundary of the the B region

Consider another hypersurface $\tilde{H}$ which is the time reversal of the previous one, where the Killing time is chosen for the time reversal symmetry. This hypersurface is given by

$$0 = t - r_* - F(r) = v - 2r_* - F(r) \equiv v + \tilde{F}(r). \tag{5.19}$$

We have

$$\frac{\mathrm{d}\tilde{F}(r)}{\mathrm{d}r} = -\frac{2}{f(r)} - \frac{\mathrm{d}F(r)}{\mathrm{d}r} = \frac{-1 \mp \sqrt{1-(1-\beta^2)f(r)}}{f(r)} \tag{5.20}$$

taking the same form as (5.8). The norm of $\tilde{H}$ is

$$\tilde{N}(r) = \mathrm{d}v + \frac{-1 \mp \sqrt{1-(1-\beta^2)f(r)}}{f(r)}\mathrm{d}r. \tag{5.21}$$

We have

$$\|\tilde{N}(r)\| = -(1-\beta^2) < 0, \tag{5.22}$$

i.e., $\tilde{H}$ is spacelike.

Let r_o be the radial at which the two hypersurfaces intersect. The inner product between $N(r_o)$ and $\tilde{N}(r_o)$ is

$$g^{ab}(r_o)N_a(r_o)\tilde{N}_b(r_o) = -\frac{2-\left(1-\beta^2\right)f(r_o)}{f(r_o)}. \tag{5.23}$$

Considering the normalization of the normals, we have

$$\frac{g^{ab}(r_o)N_a(r_o)\tilde{N}_b(r_o)}{\|N(r_o)\|\|\tilde{N}(r_o)\|} = -\frac{2-(1-\beta^2)f(r_o)}{f(r_o)(1-\beta^2)} = 1 - \frac{2}{(1-\beta^2)f(r_o)} < 0, \tag{5.24}$$

where the right hand side is smaller than 0 because we assumed that r_o makes $0 < f(r_o) < 1$. Let θ be the dihedral angle of the two hypersurfaces H and $\tilde{H}$. We thus have

$$\beta^2 = 1 - \frac{2}{f(r_o)\left(\cosh(\theta)+1\right)}. \tag{5.25}$$

C. the model we considered

The triangulation of B is reduced to a triangulation of the boundaries $\partial_- B$ and $\partial_+ B$. The diffeomorphism then maps each triangle of the triangulation of $\partial_- B$ to a corresponding one in $\partial_+ B$. Let $f_- \subset \partial_- B$ and $f_+ \in \partial_+ B$ be two triangles related by the diffeomorphism transformation. In the spinfoam mode, the boundary condition is given by (j_f, ξ_f), a pair of a spin and a spinor, associated to each triangle in the boundary triangulation[5]. Then, the time reversal symmetry implies that the pairs satisfy

$$(j_{f_-}, \xi_{f_-}) = (j_{f_+}, \xi_{f_+}).$$

Given a set of boundary condition $\{(j_f, \xi_f)\}_{\forall f \in \partial B}$, one could write down the spinfoam transition amplitude associated to the boundary data as defined above. The path integral then is dominated by the critical points which are the solution to the equations of motion resulting from $\mathrm{Re}(S) = 0 = \delta S$. To find out the critical point associated to the boundary data, we usually take advantage of the geometric interpretation of the spinfoam critical point. The procedure involves constructing a flat geometry that is compatible with the boundary data and extracting the spin foam data from this geometry. However, a concern arises as to whether the geometry consistent with the boundary data is indeed flat. To have a flat geometry, it is necessary that the boundary data is provided in such a way that a

[5] Here let us first not consider the dihedral angle associated to the triangles.

flat geometry can be constructed in the neighborhoods of $\partial_\pm B$ respectively. Let us assume that this is actually the case. However, another issue arises: it is generally not possible to construct a globally flat geometry throughout the entire region of B.

While it may not be possible to construct a globally flat geometry throughout the entire region of B, it is still feasible to construct a flat geometry in the region B_- due to the assumption we made. Thus, let us do it and then extract the spin foam data (j_f, g_{ve}, z_{vf}) for all $v, e, f \subset B_-$. Given $v, e, f \subset B_-$, we get the corresponding objects $S(v), S(e), S(f)$ in B_+. Then, we can assign to $S(v), S(e), S(f)$ the spin foam data

$$(j_{S(f)}, g_{S(v)S(e)}, z_{S(v)S(f)}) = (j_f, g_{ve}, z_{vf})\,. \tag{5.26}$$

Indeed the data in the right hand side is the time parity transformation of j_f, g_{ve}, z_{vf} given in, e.g., *arXiv:1304.5626*. We will prove that the resulting spin foam data gives a set of critical point in the entire region B.

Let us consider only the spacelike spinfoam model.

In the process of proving, the difficulties typically arise in the equations that involve objects crossing the transition surface $\mathcal{T}$. Thus, let us first consider an edge e dual to a tetrahedron in $\mathcal{T}$. Let $v \in B_-$ be the starting point and $v' \in B_+$ be the ending point. Consider a face f taking e as one of its boundary edge. Since S preserves $\mathcal{T}$, we have $v' = S(v)$, $S(e) = e$ and $S(f) = f$. Thus, we have

$$z_{v'f} = z_{vf}\,,\;\; g_{v'e} = g_{ve}^\dagger, \tag{5.27}$$

which leads to

$$Z_{v'ef} = g_{ve}^\dagger z_{vf} = Z_{vef}. \tag{5.28}$$

Thus, the first equation of (3.8) is satisfied where $\alpha^f_{vv'} = 0$. Since the last two equations of (3.8) do not involve objects crossing $\mathcal{T}$, they are satisfied automatically. Finally, we only need to consider the equation $\partial_{j_f} S = 0$ for f crossing the transition surface. By definition (2.33), we have

$$\partial_{j_f} S = \sum_{e\in\partial f}\left[\ln\left(\frac{\langle Z_{s_e ef}, Z_{t_e ef}\rangle^2}{\langle Z_{s_e ef}, Z_{s_e ef}\rangle\langle Z_{t_e ef}, Z_{t_e ef}\rangle}\right) + i\gamma \ln\left(\frac{\langle Z_{s_e ef}, Z_{s_e ef}\rangle}{\langle Z_{t_e ef}, Z_{t_e ef}\rangle}\right)\right]. \tag{5.29}$$

For edges that are entirely contained in B_-, we have

$$\frac{Z_{s_e ef}}{\|Z_{s_e ef}\|} = e^{i\alpha^f_{s_e t_e}} \frac{Z_{t_e ef}}{\|Z_{t_e ef}\|}. \tag{5.30}$$

Therefore, one gets

$$\ln\left(\frac{\langle Z_{s_e ef}, Z_{t_e ef}\rangle^2}{\langle Z_{s_e ef}, Z_{s_e ef}\rangle\langle Z_{t_e ef}, Z_{t_e ef}\rangle}\right) = \ln\left(e^{i\alpha^f_{s_e t_e}}\right) \tag{5.31}$$

and

$$\ln\left(\frac{\langle Z_{s_e ef}, Z_{s_e ef}\rangle}{\langle Z_{t_e ef}, Z_{t_e ef}\rangle}\right) = \ln\left(\frac{\|Z_{s_e ef}\|^2}{\|Z_{t_e ef}\|^2}\right). \tag{5.32}$$

Now let us consider $e' = S(e)$. Then, we have $s_{e'} = S(t_e)$ and $t_{e'} = S(t_e)$. Thus, according to (5.30), one gets

$$\frac{Z_{s_{e'} e' f}}{\|Z_{s_{e'} e' f}\|} = e^{-i\alpha^f_{s_e t_e}} \frac{Z_{t_{e'}} e' f}{\|Z_{t_{e'}} e f\|} \tag{5.33}$$

which results in

$$\ln\left(\frac{\langle Z_{s_{e'} e' f}, Z_{t_{e'} e' f}\rangle^2}{\langle Z_{s_{e'} e' f}, Z_{s_{e'} e' f}\rangle\langle Z_{t_{e'} e' f}, Z_{t_{e'} e' f}\rangle}\right) = \ln\left(e^{-i\alpha^f_{s_e t_e}}\right) \tag{5.34}$$

and

$$\ln\left(\frac{\langle Z_{s_{e'} e' f}, Z_{s_{e'} e' f}\rangle}{\langle Z_{t_{e'} e' f}, Z_{t_{e'} e' f}\rangle}\right) = \ln\left(\frac{\|Z_{t_e ef}\|}{\|Z_{s_e ef}\|}\right). \tag{5.35}$$

We therefore have that for e_o satisfying $e_o \neq S(e_o)$,

$$\sum_{e\in\{e_o,S(e_o)\}}\left[\ln\left(\frac{\langle Z_{s_eef},Z_{t_eef}\rangle^2}{\langle Z_{s_eef},Z_{s_eef}\rangle\langle Z_{t_eef},Z_{t_eef}\rangle}\right)+i\gamma\ln\left(\frac{\langle Z_{s_eef},Z_{s_eef}\rangle}{\langle Z_{t_eef},Z_{t_eef}\rangle}\right)\right]=0. \tag{5.36}$$

For e with $e=S(e)$, the data on such edges satisfy (5.27). We thus have

$$\ln\left(\frac{\langle Z_{s_eef},Z_{t_eef}\rangle^2}{\langle Z_{s_eef},Z_{s_eef}\rangle\langle Z_{t_eef},Z_{t_eef}\rangle}\right)+i\gamma\ln\left(\frac{\langle Z_{s_eef},Z_{s_eef}\rangle}{\langle Z_{t_eef},Z_{t_eef}\rangle}\right)=0. \tag{5.37}$$

According (5.36) and (5.37), we finally get

$$\sum_{e\subset\partial f}\left[\ln\left(\frac{\langle Z_{s_eef},Z_{t_eef}\rangle^2}{\langle Z_{s_eef},Z_{s_eef}\rangle\langle Z_{t_eef},Z_{t_eef}\rangle}\right)+i\gamma\ln\left(\frac{\langle Z_{s_eef},Z_{s_eef}\rangle}{\langle Z_{t_eef},Z_{t_eef}\rangle}\right)\right]=0. \tag{5.38}$$

D. the map from the BH spacetime to the Minkowski spacetime

E. the flat geometry

The real critical point is obtained by considering the flat geometry. Let us consider only the region B_-. Its boundary is divided into 3 parts, referred to as the past boundary, the side boundary and the transition surface (see Fig. 15). Since the triangulation has been fixed, the geometry of the boundary is determined by the values of (t_1, r_1), (t_2, r_2) and (t_3, r_3), the time and radial coordinates of S_1, S_2 and S_3 respectively in the Minkowski spacetime. We will assume

$$r_3 > r_2 > 3r_1. \tag{5.39}$$

In our work, we use a regular tetrahedron to approximate the spheres. For convenience, let us denote the vertices on the sphere S_1 by V_1, V_2, V_3 and V_4, denote the vertices on the sphere S_2 by V_5, V_6, V_7 and V_8, and denote the vertices on S_3 by V_9, V_{10}, V_{11} and V_{12}. Given a 2-sphere with unit radial, the regular tetrahedron to approximate it is given by the vertices

$$\begin{aligned} v_1 &= \left(\sqrt{\frac{8}{9}},0,-\frac{1}{3}\right), v_2=\left(-\sqrt{\frac{2}{9}},\sqrt{\frac{2}{3}},-\frac{1}{3}\right),\\ v_3 &= \left(-\sqrt{\frac{2}{9}},-\sqrt{\frac{2}{3}},-\frac{1}{3}\right), v_4=(0,0,1)\,. \end{aligned} \tag{5.40}$$

Then, we set

$$\begin{aligned} V_a =&(t_1, r_1 v_a),\ \forall a=1,2,3,4,\\ V_b =&(t_2, -r_2 v_{b-4}),\ \forall b=5,6,7,8\\ V_c =&(t_3, -r_3 v_{c-8}),\ \forall c=9,10,11,12. \end{aligned} \tag{5.41}$$

In the triangulation, we triangulate the past boundary in the same way as what the work *arXiv:2105.06876* did. Then, we consider the "evolution" of the vertices on S_2. The vertex V_b on S_2 evolves to the vertex V_{b+4} on the sphere S_3. In the case where there are more than one vertices evolving in some tetrahedron, we let the vertex V_b with smaller value of b evolve at first.

In our analysis, we are interested in the dihedral angles that occur on the transition surface. These dihedral angles are concentrated on triangles that are shared by each pair of adjacent tetrahedra. Based on the values of these dihedral angles, the triangles can be classified into two types. The first type includes triangles for each of which two vertices lie on the sphere S_1, while the third vertex lies on the sphere S_3. An example of this type of triangle is the one formed by vertices V_1, V_2, and V_{10}. The second type comprises triangles for each of which only one vertex is on the sphere S_1, and the other two vertices are on the sphere S_3. An example of this type of triangles is the one formed by vertices V_1, V_{11}, and V_{12}.

Our purpose is to construct a spacelike transition surface with large dihedral angles. Our calculation shows that this could happen if

$$\frac{t_3 - t_1}{r_3 - 3r_1} = \frac{1}{3} - \epsilon. \tag{5.42}$$

Actually for $\epsilon = 0$, one of the adjacent tetrahedra connected by the second type triangle is null, leading to an infinite dihedral angle, but the adjacent tetrahedra connected by each first type triangles are all spacelike. The dihedral angle Δ_2 concentrated on the second type triangle is given by

$$\cosh(\Delta_2) = \frac{-3\left(\epsilon - \frac{1}{3}\right)^2 (r_3 - 3r_1) - r_1 + r_3}{\sqrt{3}\sqrt{(2-3\epsilon)\epsilon}\sqrt{\left|(r_1 - r_3)^2 - 3(r_3 - 3r_1)^2\left(\epsilon - \frac{1}{3}\right)^2\right|}} \sim \frac{\sqrt{\frac{2}{3}}r_3}{3\sqrt{\left|2r_1^2 - \frac{2r_3^2}{3}\right|}}\frac{1}{\sqrt{\epsilon}} + O(\sqrt{\epsilon}) \qquad \text{eq:dihedralep1} \tag{5.43}$$

The dihedral angle Δ_1 concentrated on the first type triangle is given by [6]

$$\begin{aligned} \cosh(\Delta_1) =& \frac{-r_1^2 + 3\left(\epsilon - \frac{1}{3}\right)^2 (r_3 - 3r_1)^2 + 4r_1 r_3 - 3r_3^2}{\sqrt{\left|\left((r_1 - 3r_3)^2 - (r_3 - 3r_1)^2(1-3\epsilon)^2\right)\left((r_1 - r_3)^2 - 3(r_3 - 3r_1)^2\left(\epsilon - \frac{1}{3}\right)^2\right)\right|}} \\ \sim& \frac{3r_1^2 + 3r_1 r_3 - 4r_3^2}{2\sqrt{3}\sqrt{|3r_1^4 - 4r_3^2 r_1^2 + r_3^4|}} + O(\epsilon) \end{aligned} \qquad \text{eq:dihedralep2} \tag{5.44}$$

Moreover, we want the side boundary to be spacelike too. This requirement needs

$$\frac{t_3 - t_2}{r_3 - r_2} < \frac{1}{3}. \tag{5.45}$$

The requirement that the past boundary should also be spacelike implies that

$$\frac{t_2 - t_1}{r_2 - 3r_1} < \frac{1}{3}. \tag{5.46}$$

Without loss of generality, let us set $t_1 = 0$. Then, we have

$$\begin{aligned} \frac{t_3}{r_3 - 3r_1} &= \frac{1}{3} - \epsilon \Rightarrow t_3 = \left(\frac{1}{3} - \epsilon\right)(r_3 - 3r_1), \\ \frac{t_3 - t_2}{r_3 - r_2} &< \frac{1}{3} \Rightarrow t_3 - t_2 < \frac{1}{3}(r_3 - r_2) \Rightarrow t_2 > \left(\frac{1}{3} - \epsilon\right)(r_3 - 3r_1) - \frac{1}{3}(r_3 - r_2) = \frac{1}{3}(r_2 - 3r_1) - \epsilon(r_3 - 3r_1), \\ \frac{t_2}{r_2 - 3r_1} &< \frac{1}{3} \Rightarrow t_2 < \frac{1}{3}(r_2 - 3r_1). \end{aligned} \tag{5.47}$$

In summary, we should choose the data such that

$$t_3 = \left(\frac{1}{3} - \epsilon\right)(r_3 - 3r_1), \quad \frac{1}{3}(r_2 - 3r_1) - \epsilon(r_3 - 3r_1) < t_2 < \frac{1}{3}(r_2 - 3r_1). \qquad \text{eq:t3} \tag{5.48}$$

We therefore choose t_2 to be

$$t_2 = \frac{1}{3}(r_2 - 3r_1) - \delta(r_3 - 3r_1), \text{ with some } \delta \text{ satisfying } 0 < \delta < \epsilon. \qquad \text{eq:t2} \tag{5.49}$$

Since S_3 is chosen to be in the future of S_2, we need $t_3 > t_2$, giving

$$t_3 - t_2 = \frac{1}{3}(r_3 - r_2) - (\epsilon - \delta)(r_3 - 3r_1) > 0 \Rightarrow \epsilon - \delta < \frac{r_3 - r_2}{3(r_3 - 3r_1)} \tag{5.50}$$

where we use the assumption $r_3 > 3r_1$.

[6] This is the formula for the dihedral angle where the simplexes are glued in such a way that any two adjacent simplexes meet only along their shared edges, faces, or surfaces, and not in their interior regions.

F. Construct the model

Let us assume that we can embed the past boundary and the side boundary of the B region into the Minkowski spacetime in such a way that

(1) for the past boundary, its embedding is given by

$$t = \frac{t_2}{r_2 - r_1}(r - r_1) = \frac{\frac{1}{3}(r_2 - 3r_1) - \delta\,(r_3 - 3r_1)}{r_2 - r_1}(r - r_1). \tag{5.51}$$

(2) for the side boundary, its embedding is given by

$$t - t_2 = \frac{t_3 - t_2}{r_3 - r_2}(r - r_2) = \frac{\frac{1}{3}(r_3 - r_2) - (\epsilon - \delta)(r_3 - 3r_1)}{r_3 - r_2}(r - r_2). \tag{5.52}$$

Note that here we used Eqs. (5.49) and (5.48). According to the discussions in Sec. V E, the geometry associated to the real critical point is determined once the extra two parameters ϵ and δ are given. However, these two parameters are not geometric. We would like to relate them with some geometrically meaningful variables. To this end, let us consider the dihedral angles θ_1 and θ_3 at S_1 and S_3 respectively (see Fig. 15 where the arrows represent the norms of the corresponding hypersurfaces). According the aforementioned discussion, these angles are related to the slopes of the past and side boundaries.

By (5.51) and (5.52), the slopes of the past boundary and the side boundary when they are embedded in the Minkowski spacetime are respectively

$$\begin{aligned}\beta_1 =& \frac{(r_2 - 3r_1) - 3\delta\,(r_3 - 3r_1)}{3(r_2 - r_1)} = \frac{1}{3} - \frac{2r_1/3 + \delta(r_3 - 3r_1)}{r_2 - r_1},\\ \beta_2 =& \frac{1}{3} - \frac{(\epsilon - \delta)(r_3 - 3r_1)}{r_3 - r_2}.\end{aligned} \tag{5.53}$$

Then, according to (5.24), the dihedral angles θ_1 and θ_3 are related to β_1 and β_3 by

$$\begin{aligned}\beta_1 &= \sqrt{1 - \frac{2}{f(r_1)(\cosh(\theta_1) + 1)}}\\ \beta_3 &= \sqrt{1 - \frac{2}{f(r_3)(\cosh(\theta_3) + 1)}}\end{aligned} \tag{5.54}$$

(5.53) and (5.54) relate δ and ϵ to the dihedral angles θ_1 and θ_3. More precisely, we have

$$\begin{aligned}\epsilon =& -\frac{1}{r_3 - 3r_1}\Bigg[(r_2 - r_1)\sqrt{1 - \frac{2}{(\cosh(\theta_1) + 1)f(r_1)}}\\ &+ (r_3 - r_1)\sqrt{1 - \frac{2}{(\cosh(\theta_3) + 1)f(r_3)}} - \frac{r_2 + r_3 - 4r_1}{3}\Bigg]\\ \delta =& -\frac{1}{r_3 - 3r_1}\left[(r_2 - r_1)\sqrt{1 - \frac{2}{(\cosh(\theta_1) + 1)f(r_1)}} - \frac{r_2 - 3r_1}{3}\right],\end{aligned} \tag{5.55}$$

which gives

$$\epsilon - \delta = -\frac{r_3 - r_1}{r_3 - 3r_1}\left[\sqrt{1 - \frac{2}{(\cosh(\theta_3) + 1)f(r_3)}} - \frac{1}{3}\right] \tag{5.56}$$

Then, we have

$$
\begin{aligned}
\delta > 0 &\Rightarrow (r_2 - r_1)\sqrt{1 - \frac{2}{(\cosh(\theta_1)+1)f(r_1)}} < \frac{r_2 - 3r_1}{3}, \\
\epsilon > 0 &\Rightarrow (r_2 - r_1)\sqrt{1 - \frac{2}{(\cosh(\theta_1)+1)f(r_1)}} + (r_3 - r_1)\sqrt{1 - \frac{2}{(\cosh(\theta_3)+1)f(r_3)}} < \frac{r_2 + r_3 - 4r_1}{3} \\
\epsilon < \frac{1}{3} &\Rightarrow (r_2 - r_1)\sqrt{1 - \frac{2}{(\cosh(\theta_1)+1)f(r_1)}} + (r_3 - r_1)\sqrt{1 - \frac{2}{(\cosh(\theta_3)+1)f(r_3)}} > \frac{r_2 - r_1}{3} \\
\epsilon - \delta > 0 &\Rightarrow (r_3 - r_1)\sqrt{1 - \frac{2}{(\cosh(\theta_3)+1)f(r_3)}} < \frac{r_3 - r_1}{3} \\
\epsilon - \delta < \frac{r_3 - r_2}{3(r_3 - 3r_1)} &\Rightarrow (r_3 - r_1)\sqrt{1 - \frac{2}{(\cosh(\theta_3)+1)f(r_3)}} > \frac{r_2 - r_1}{3}
\end{aligned}
\tag{5.57}
$$

To see the region where θ_1 and θ_2 satisfy the above inequalities, let us define

$$
x = (r_2 - r_1)\sqrt{1 - \frac{2}{(\cosh(\theta_1)+1)f(r_1)}} > 0, \quad y = (r_3 - r_1)\sqrt{1 - \frac{2}{(\cosh(\theta_3)+1)f(r_3)}} > 0. \tag{5.58}
$$

Then, in the (xy)-plane, the above inequalities leads to the region as shown in Fig. 16. According to the figure, we have

$$
\begin{aligned}
0 < \sqrt{1 - \frac{2}{(\cosh(\theta_1)+1)f(r_1)}} &< \frac{r_2 - 3r_1}{3(r_2 - r_1)} \\
\frac{r_2 - r_1}{3(r_3 - r_1)} < \sqrt{1 - \frac{2}{(\cosh(\theta_3)+1)f(r_3)}} &< \frac{1}{3}
\end{aligned}
\tag{5.59}
$$

which leads to the constraint on the values that θ_1 and θ_3 can take,

$$
\begin{aligned}
\frac{2}{f(r_1)} - 1 < \cosh(\theta_1) &< \frac{2}{f(r_1)}\left(1 - \left(\frac{r_2 - 3r_1}{3(r_2 - r_1)}\right)^2\right)^{-1} - 1 \\
\frac{2}{f(r_3)}\left(1 - \left(\frac{r_2 - r_1}{3(r_3 - r_1)}\right)^2\right)^{-1} - 1 < \cosh(\theta_3) &< \frac{9}{4f(r_3)} - 1
\end{aligned}
\tag{5.60}
$$

where we used $f(r_3), f(r_1) > 0$. Given θ_1 and θ_3, we can get ϵ by using (5.55). Then, with (5.43) and (5.44), we finally get the dihedral angle on the transition surface in terms of θ_1 and θ_3. Moreover, one can check that once θ_1 and θ_3 take all of the values satisfying the above condition (5.60), ϵ will span across all values within the interval $(0, 1/3)$, leading that the dihedral angle Δ_2 could encompass all the values from 0 to ∞.

To solve (5.8) to get the precise expression of $F(r)$, besides the value of β, we still need the initial data. Both of the past and the side boundaries pass through the sphere S_2. Thus the initial data can be given by assigning a value v_2 for S_2 so that $F(r_2) = -v_2$. Then, we solve the following equation with initial data

$$
\frac{\mathrm{d}F_a(r)}{\mathrm{d}r} = \frac{-1 + s_a\sqrt{1 - (1 - (\beta_a)^2)f(r)}}{f(r)}, \text{ with } s_1 = 1,\ s_3 = -1, \text{ with the initial condition } v_2 + F_a(r_2) = 0,\ \forall a = 1, 3 \tag{5.61}
$$

to get the equation $v + F_a(r) = 0$ of the past and the side boundaries.

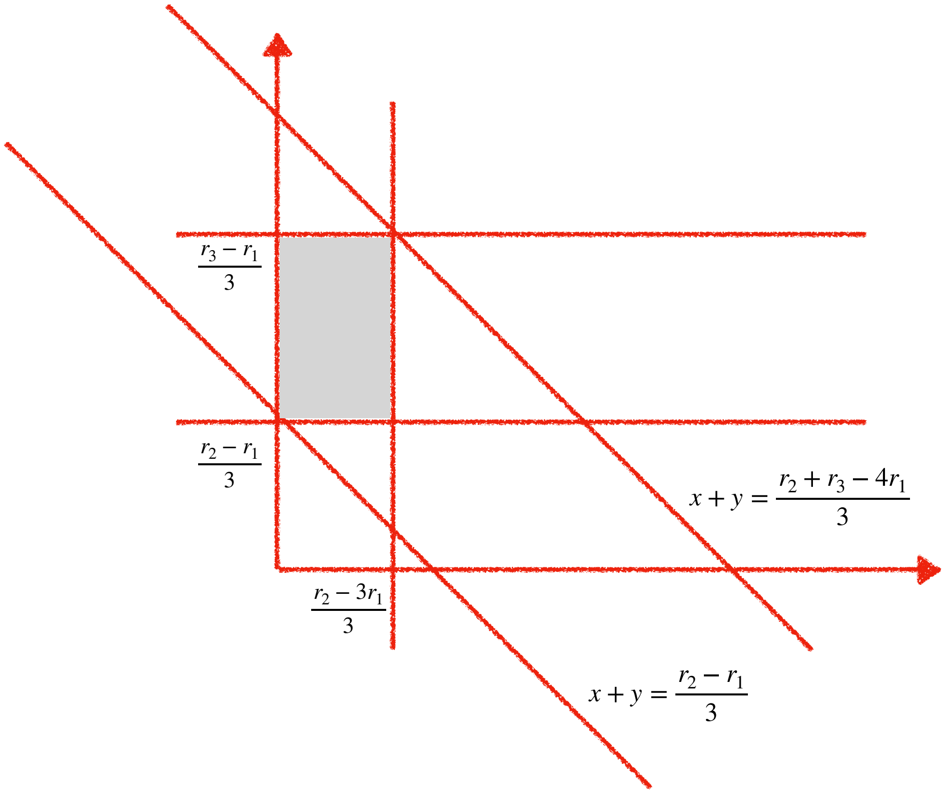


FIG. 16: The region given by the inequality.

fig:r

VI. THE BOUNDARY COHERENT STATE

A. The Hamiltonian analysis of Einstein-Hilbert acton in terms of co-frame field

Considering the following action

$$\begin{aligned} S_G =&\frac{1}{2\kappa\gamma}\int_{\mathcal{M}} e^\sigma_M e^\theta_N\left(\gamma\delta^{[M}_K\delta^{N]}_L+\frac{1}{2}\epsilon^{MN}{}_{KL}\right)\Omega^{KL}_{\sigma\theta}e\mathrm{d}t\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\\ =:&\frac{1}{2\kappa\gamma}\int_{\mathcal{M}} e^\sigma_M e^\theta_N P^{MN}{}_{KL}\Omega^{KL}_{\sigma\theta}e\mathrm{d}t\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\end{aligned} \tag{6.1}$$

where e^I_μ is the co-frame field. Note that in this equation, the volume element $e\mathrm{d}t\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3$ is a 4-form, rather than a pseudo 4-form. We thus should use the definition for the integral of a 4-form to do the integration. That is to say, in the integration, we should choose a right-hand coordinate in which the integral becomes $\int_{\Psi[\mathcal{M}]} e\, e^\sigma_M e^\theta_N P^{MN}{}_{KL}\Omega^{KL}_{\sigma\theta}e\mathrm{d}^4x$. Here, $\Psi(\mathcal{M})\subset\mathbb{R}^4$ denotes the domain of the coordinate chosen for the integration.

The metric is defined by

$$g_{\mu\nu}=\eta_{IJ}e^I_\mu e^J_\nu. \tag{6.2}$$

Let $\mathcal{M}$ be of the topology $\mathcal{M}=\mathbb{R}\times\Sigma$. The 3-manifold is given by $t=0$. Let ∂_t be a vector field satisfying $\partial_t t=1$. For each given point e^I_μ in the configuration space, we could define the unit conormal of Σ, denoted by n_μ. Specifically, n_μ is given by

$$n_\mu=\frac{(\mathrm{d}t)_\mu}{\|\mathrm{d}t\|},\quad \|\mathrm{d}t\|=\sqrt{|g^{\mu\nu}(\mathrm{d}t)_\mu(\mathrm{d}t)_\nu|}. \tag{6.3}$$

With n_μ, we have

$$n^\mu=\frac{g^{\mu\nu}(\mathrm{d}t)_\nu}{\|\mathrm{d}t\|}. \tag{6.4}$$

Then, we can introduced the lapse function N and the shift vector N^μ, given by

eq:lapseandshift

$$\begin{aligned} \left(\frac{\partial}{\partial t}\right)^\mu &= Nn^\mu + N^\mu \\ N^\mu n_\mu &= 0. \end{aligned} \tag{6.5}$$

The projection operator q^μ_ν is given by

$$q^\mu_\nu = \delta^\mu_\nu + n^\mu n_\nu. \tag{6.6}$$

With the projection, we could relate each tensor $T^{\mu\cdots}{}_{\nu\dots}$ to its spatial projection $T^{\mu\cdots}{}_{\nu\dots}q^{\mu'}_\mu\cdots q^\nu_{\nu'}$. Note that the projection $T^{\mu\cdots}{}_{\nu\dots}q^{\mu'}_\mu\cdots q^\nu_{\nu'}$ is still a 4-D tensor that maps n^ν and n_μ to 0. However, the property makes it possible to define a 1-to-1 and onto correspondence between the space of projection tensors and 3-D tensors. The correspondence between the tensor $T^{\mu\cdots}{}_{\nu\dots}q^{\mu'}_\mu\cdots q^\nu_{\nu'}$ and the 3-D tensor, denoted by $T^{a\cdots}{}_{b\dots}$, reads

$$T^{a\cdots}{}_{b\dots} = \text{the restriction of } T^{\mu\cdots}{}_{\nu\dots}q^{\mu'}_\mu\cdots q^\nu_{\nu'} \text{ on the space tangent to } \Sigma. \tag{6.7}$$

Now let us consider the co-frame field e^I_μ. We have

$$e^I_\mu = e^I_\nu q^\nu_\mu - n^\nu e^I_\nu n_\mu. \tag{6.8}$$

Now let us consider a coordinate system x^μ with $x^0 = t$. Then, the coordinate basis $\partial/\partial x_i$ is tangent to the spatial manifold. Then, under the coordinate we have

$$\{e^I_\mu\}^{I=0,3}_{\mu=0,3} = \begin{pmatrix} e^0_\mu(\partial_t)^\mu, & e^0_\nu(\partial_{x^k})^\nu \\ e^i_\mu(\partial_t)^\mu, & e^i_\nu(\partial_{x^k})^\nu \end{pmatrix} = \begin{pmatrix} e^0_\mu(\partial_t)^\mu, & e^0_\nu q^\nu_\mu(\partial_{x^k})^\mu \\ e^i_\mu(\partial_t)^\mu, & e^i_\nu q^\nu_\mu(\partial_{x^k})^\mu \end{pmatrix} \tag{6.9}$$

Substituting (eq:lapseandshift 6.5) into it, we get

$$e^I_\mu(\partial_t)^\mu = e^I_\mu(Nn^\mu + N^\mu) = Ne^I_\mu n^\mu + e^I_\mu q^\mu_\nu N^\nu. \tag{6.10}$$

Let the triad field, i.e., the restriction of $e^I_\mu(x)$ on the tangent space of $T_x\Sigma$, to be denoted by $\mathfrak{e}^I_a$. We have

$$e^I_\mu q^\mu_\nu N^\nu = \mathfrak{e}^I_a N^a,\ e^I_\mu q^\mu_\nu(\partial_{x^i})^\nu = \mathfrak{e}^I_a(\partial_{x^i})^a. \tag{6.11}$$

Thus, we get

$$\{e^I_\mu\}^{I=0,3}_{\mu=0,3} = \begin{pmatrix} Ne^0_\mu n^\mu + \mathfrak{e}^0_a N^a, & \mathfrak{e}^0_a(\partial_{x^k})^a \\ Ne^i_\mu n^\mu + \mathfrak{e}^i_a N^a, & \mathfrak{e}^i_a(\partial_{x^k})^a \end{pmatrix} \tag{6.12}$$

In terms of the co-frame field, there are the $\mathrm{SL}(2,\mathbb{C})$ gauge transformation. We partially fix this gauge by requiring

$$n^\mu e^I_\mu = (1,0,0,0)^T.$$

Then, we have

$$N^I n^J \eta_{IJ} = N^\mu e^I_\mu n^\nu e^J_\nu \eta_{IJ} = N^\mu n_\mu = 0 \tag{6.13}$$

leading to

$$N^I = (0,\vec{N})^T. \tag{6.14}$$

Similarly, we get

$$n_I\mathfrak{e}^I_a(\partial_{x^i})^a = n_I e^I_\mu(\partial_{x^i})^\mu = 0 \Rightarrow \mathfrak{e}^I_a(\partial_{x^i})^a = (0,\mathfrak{e}^l_a(\partial_{x^i})^a) \tag{6.15}$$

Thus, we have

eq:dete

$$\{e^I_\mu\}^{I=0,3}_{\mu=0,3} = \begin{pmatrix} N, & 0 \\ \mathfrak{e}^i_a N^a, & \mathfrak{e}^i_a(\partial_{x^k})^a \end{pmatrix} \Rightarrow \det(e) = N\det(\mathfrak{e}^i_a). \tag{6.16}$$

Due to

$$\begin{aligned} e^\mu_M e^\nu_N =& e^\sigma_M e^\theta_N (q^\mu_\sigma - n_\sigma n^\mu)(q^\nu_\theta - n_\theta n^\nu) \\ =& e^\sigma_M e^\theta_N (q^\mu_\sigma q^\nu_\theta - n_\sigma n^\mu q^\nu_\theta - q^\mu_\sigma n_\theta n^\nu + n_\sigma n^\mu n_\theta n^\nu) \end{aligned} \tag{6.17}$$

S_G becomes

$$\begin{aligned} S_G =& \frac{1}{2\kappa}\int_{\mathcal{M}} e\, e^\sigma_M e^\theta_N P^{MN}{}_{KL}\Omega^{KL}_{\sigma\theta} \\ =& \frac{1}{2\kappa}\int_{\mathcal{M}} e e^\sigma_M e^\theta_N \left(q^\mu_{[\sigma} q^\nu_{\theta]} - n_{[\sigma} n^\mu q^\nu_{\theta]} - q^\mu_{[\sigma} n_{\theta]} n^\nu\right) P^{MN}{}_{KL}\Omega^{KL}_{\mu\nu} \\ =& \frac{1}{2\kappa}\int_{\mathcal{M}} e e^\sigma_M e^\theta_N \left(q^\mu_{[\sigma} q^\nu_{\theta]} - 2n_{[\sigma} n^{[\mu} q^{\nu]}_{\theta]}\right) P^{MN}{}_{KL}\Omega^{KL}_{\mu\nu} \\ =& \frac{1}{2\kappa}\int_{\mathcal{M}} e(e^\sigma_M e^\theta_N q^\mu_\sigma q^\nu_\theta P^{MN}{}_{KL}\Omega^{KL}_{\mu\nu} - 2n_M \frac{t^\mu - N^\mu}{N} e^\theta_N q^\nu_\theta P^{MN}{}_{KL}\Omega^{KL}_{\mu\nu}). \end{aligned} \tag{6.18}$$

Because of (6.16), we have

$$S_G = \frac{1}{2\kappa}\int_{\mathcal{M}} \det(\mathfrak{e})(N e^\sigma_M e^\theta_N q^\mu_\sigma q^\nu_\theta P^{MN}{}_{KL}\Omega^{KL}_{\mu\nu} - 2n_M(t^\mu - N^\mu) e^\theta_N q^\nu_\theta P^{MN}{}_{KL}\Omega^{KL}_{\mu\nu}). \tag{6.19}$$

By the relation between Lie derivative and the exterior derivative, i.e.

$$\begin{aligned} \mathcal{L}_v\alpha = \mathrm{d}(i_v\alpha) + i_v(d\alpha),\ \text{more precisely,} \\ \mathcal{L}_v\alpha_{\mu_1,\cdots,\mu_n} = \mathrm{d}_{\mu_1}(v^\nu \alpha_{\nu\mu_2\cdots\mu_n}) + (\mathrm{d}\alpha)_{\nu\mu_1\cdots\mu_n} v^\nu \end{aligned} \tag{6.20}$$

one has

$$\begin{aligned} & t^\mu \Omega^{KL}_{\mu\nu} = t^\mu (\mathrm{d}\omega^{KL})_{\mu\nu} + t^\mu(\omega^K_{\mu I}\wedge\omega^{IL}_\nu) \\ =& \mathcal{L}_t\omega^{KL}_\nu - \partial_\nu(t^\mu\omega^{KL}_\mu) - \omega^L_{\nu I}\omega^{IK}_\mu t^\mu - \omega^K_{\nu I}\omega^{LI}_\mu t^\mu \\ =& \mathcal{L}_t\omega^{KL}_\nu - \nabla_\nu(t^\mu\omega^{KL}_\mu) \end{aligned} \tag{6.21}$$

Therefore, it is obtained

$$\begin{aligned} S_G =& \frac{1}{2\kappa}\int_{\mathcal{M}} \det(\mathfrak{e})(N e^\sigma_M e^\theta_N q^\mu_\sigma q^\nu_\theta P^{MN}{}_{KL}\Omega^{KL}_{\mu\nu} + 2n_M N^\mu e^\theta_N q^\nu_\theta P^{MN}{}_{KL}\Omega^{KL}_{\mu\nu}) \\ & - 2n_M e^\theta_N q^\nu_\theta P^{MN}{}_{KL}\left[\mathcal{L}_t\omega^{KL}_\nu - \nabla_\nu(t^\mu\omega^{KL}_\mu)\right] \\ =& \frac{1}{2\kappa}\int_{\mathcal{M}} \det(\mathfrak{e})(N\mathfrak{e}^a_M\mathfrak{e}^b_N P^{MN}{}_{KL}\Omega^{KL}_{ab} + 2n_M N^b \mathfrak{e}^a_N P^{MN}{}_{KL}\Omega^{KL}_{ba}) \\ & - 2n_M\mathfrak{e}^a_N P^{MN}{}_{KL}\left[\mathcal{L}_t\omega^{KL}_a - \nabla_a(t^\mu\omega^{KL}_\mu)\right]. \end{aligned} \tag{6.22}$$

It is easy to get

$$q^\mu_\sigma q^\nu_\theta \Omega^{IJ}_{\mu\nu} = q^\mu_\sigma q^\nu_\theta \partial_\mu(q^\alpha_\nu \omega^{IJ}_\alpha) - q^\mu_\sigma q^\nu_\theta \partial_\nu(q^\alpha_\mu\omega^{IJ}_\alpha) + [q^\mu_\sigma\omega_\mu, q^\nu_\theta\omega_\nu]^{IJ}. \tag{6.23}$$

meaning that $q^\mu_\sigma q^\nu_\theta\Omega^{IJ}_{\mu\nu}$ is the curvature of ω^{IJ}_a. Define $q^I_J = \delta^I_J + n_I n^J$. A^I_μ and B^{IJ}_μ are defined as

$$A^M_\nu := n_N P^{MN}{}_{KL}\omega^{KL}_\nu,\ B^{IJ}_\nu := q^I_M q^J_N P^{MN}{}_{KL}\omega^{KL}_\nu. \tag{6.24}$$

We thus have

$$\begin{aligned} & P^{MN}{}_{KL}\omega^{KL}_\mu = (q^M_{\tilde{M}} - n^M n_{\tilde{M}})(q^N_{\tilde{N}} - n^N n_{\tilde{N}})P^{\tilde{M}\tilde{N}}{}_{KL}\omega^{KL}_\mu \\ =& B^{MN}_\mu + n^M A^N_\mu - N^N A^M_\mu = 2n^{[M}A^{N]}_\mu + B^{MN}_\mu \end{aligned} \tag{6.25}$$

Moreover, we define

$$E^\mu_I := \det(\mathfrak{e})\mathfrak{e}^\mu_I. \tag{6.26}$$

Then

$$E_I^\mu n^I = E_I^\mu e_\nu^I n^\nu = \det(\mathfrak{e}) q_\sigma^\mu \delta_\nu^\sigma n^\nu = 0. \tag{6.27}$$

Then, we could see that the action S_G depends only on $(N, N^a, B_a^{ij}, t^\mu\omega_\mu^{IJ}, E_i^a, A_a^j)$.

With these variables, we have

$$\det(\mathfrak{e}) n_M e_N^a P^{MN}{}_{KL} \mathcal{L}_t \omega_a^{KL} = -E_i^a \mathcal{L}_t A_a^i \tag{6.28}$$

leading to

eq:actionSGsymp

$$S_G = \frac{1}{\kappa\gamma} \int \mathrm{d}^4x \left(E_i^a \mathcal{L}_t A_a^i + \text{constraint terms} \right). \tag{6.29}$$

Here, we also have the Lagrangian multiplier B_a^{ij} which does not appears in the standard connection dynamics. This implies that we are actually now far away from the standard connection dynamics. Indeed, a direct calculation shows that we will arrive at a second class constraint system on the phase space consisting of $(N, N^a, B_a^{ij}, t^\mu\omega_\mu^{IJ}, E_i^a, A_a^j)$. Then, we need to solve the second class constraints $\delta S/\delta(t\cdot\omega)^{0i} \cong 0$ and $\delta S/\delta B_a^{ij} \cong 0$ to finally get the standard connection constraint. We are not doing the constraint analysis here, since it is not very necessary for our following analysis.

In what follows, we will use e_i^a and e_a^i instead of $\mathfrak{e}_i^a$ and $\mathfrak{e}_a^i$ to represent the triad and cotriad field for convenience. We have the flux field E_i^a given by

eq:densityfieldp

$$E_i^a = \det(e) e_i^a. \tag{6.30}$$

By this definition, E_i^a is a coordinate dependent vector fields. In other words, we could think of E_i^a as some abstract object which is represented by a vector field under a certain coordinate. The representations under different coordinate systems are different. For two given coordinate systems x^k and y^k, we use $\tilde{E}_i^a$ and $\underline{E}_i^a$ to denote the representations. We have

$$\tilde{E}_i^a = \det\left(\{(\partial_{x^k})^b e_b^i\}_{k=1,3}^{i=1,3} \right) e_i^a = \det\left(\frac{\partial y}{\partial x}\right) \det\left(\{(\partial_{y^k})^b e_b^i\}_{k=1,3}^{i=1,3} \right) e_i^a = \det\left(\frac{\partial y}{\partial x}\right) \overline{E}_i^a \tag{6.31}$$

which gives the transformation law between the representations under different coordinate systems. We will see in the following that the abstract object for E_i^a is the densitized frame field.

Let us summarize the structures we have to define the Hamiltonian theory:

(1) The 4-manifold is orientable so that we could define the integration in action S_G. The orientation is denoted by $O_{abcd}^{(4)}$

(2) A diffeomorphism $\Psi : \mathbb{R} \times \Sigma \to \mathcal{M}$;

(3) With the diffeomorphism Ψ, an orientation o on Σ is given by requiring $\Psi^*(O) = \mathrm{d}t \otimes o$ with $\mathrm{d}t$ a given orientation on $\mathbb{R}$.

(4) The we could write the action S_G as

$$S_G = \int_{\mathbb{R}} \mathrm{d}t \int_\Sigma \left(E_i^a \mathcal{L}_t A_a^i + \text{constraint term} \right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \tag{6.32}$$

where $\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ is right-hand with respect to the orientation o. Here this orientation is necessary because $(E_i^a \mathcal{L}_t A_a^i + \cdots)\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ is a 3-form and defining its integral needs us to choose a right-hand coordinate. In other words, if we naively represent $(E_i^a \mathcal{L}_t A_a^i + \cdots)$ as a scalar field under a coordinate and do the integration with respect to the lebesgue measure d^3x in the coordinate domain, the results depend on the orientation of the coordinate. Precisely, the result in a right hand coordinate is the negative of the result in a left hand coordinate.

According to the above items, we have the Poisson bracket

eq:poisson0

$$\{A_a^i(x), E_j^b(y)\} = \kappa\gamma \delta_a^b \delta_j^i \delta^3(x-y) \tag{6.33}$$

where the δ-function is the delta function with respect to the orientation o, i.e., $\int \delta^3(x)\mathrm{d}^3x = 1$ for a right hand coordinate system.

B. the classical phase space in terms of principal bundle

Let Σ be a 3-dimensional Riemann manifold. $\pi : P \to \Sigma$ is an SU(2) principal bundle on it. By the definition of a principal bundle[7], there is an SU(2) right-action on P and a local trivialization $\phi_\alpha : \pi^{-1}[U_\alpha] \mapsto U_\alpha \times U$ identifying $U_\alpha \times G$ with $\pi^{-1}[U_\alpha]$. The Ashtekar connection is a connection on P.

1. *Frame bundle*

To construct the densitized triad, we consider the frame bundle defined as follows. Given $x \in \Sigma$, let $T_x\Sigma$ be the tangent space at x. A frame $e^a_i(x)$ at x is a linear map

$$e^a_i : \mathbb{R}^3 \mapsto T_x\Sigma. \tag{6.35}$$

The set of frame at x is denoted by F_x. The GL($\mathbb{R}^3$)-action on F_x is defined by

$$(e \cdot g)^a_i = e^a_j g^j{}_i. \tag{6.36}$$

The frame bundle is defined as the disjoint union

$$F = \bigsqcup_{x\in\Sigma} F_x. \tag{6.37}$$

We now equip F with a topology and smooth structure as follows. Let (U_α, ϕ_α) be a chart with U_α an open subset in Σ. Then, we could construct the following map $\tilde{\phi} : F_\alpha \to \mathbb{R}^{n+n^2}$, with $F_\alpha = \bigsqcup_{x\in U_\alpha} F_x$,

$$\tilde{\phi}_\alpha : e^a_i(x) \mapsto (\phi_\alpha(x), e^a_i(\mathrm{d}\phi_\alpha)_a) \in \mathbb{R}^{n+n^2}. \tag{6.38}$$

We use these maps to define the topology and smooth structure on F.

Let (U_α, ϕ_α) be the atlas covering Σ with $\phi_\alpha : U_\alpha \to \mathbb{R}^3$. Given $e^a_i(x)$ for $x \in U_\alpha$, we define

$$T_\alpha(E) = (x, \mathrm{d}\phi_\alpha \circ e) \in U_\alpha \times \mathrm{GL}(\mathbb{R}^3). \tag{6.39}$$

For $\tilde{e}^a_i = e^a_j g^j{}_i$, we have

$$\mathrm{d}\phi_\alpha \circ \tilde{e} = (\mathrm{d}\phi_\alpha \circ e) \circ g, \tag{6.40}$$

implying that T_α is local trivialization.

---

[7] Let P be a manifold which is acted freely by a Lie group G as:

$$P \times G \to P; (p, g) \mapsto pg,$$

where we mean by a free action that the stabilizer of every point is trivial. A **principle fibre bundle** is the pair (P, G) with the following structures

- $M = P/G$ is a manifold (called the base);
- The nature projection $\pi : P \to \Sigma$ taking a point to its orbit is a smooth surjection.
- Local triviality: For any open cover U_α there exists a diffeomorphism

  $$\psi_\alpha : \pi^{-1}[U_\alpha] \to U_\alpha \times G; p \mapsto (\pi(p), g_\alpha(p))$$

  where $g_\alpha : \pi^{-1}[U_\alpha] \to G$ is some G-valued function on $\pi^{-1}[U_\alpha]$ satisfying

  $$g_\alpha(pg) = g(p)g. \tag{6.34}$$

  eq:eqG

  Because the action of G is free, $g_\alpha(\cdot)$ is determined by its value at some base point p_0.

Now let us consider the space $\mathbb{R}$ with the GL($\mathbb{R}^3$)-action on it as

$$g \cdot \rho = \det(g)^{-1}\rho. \tag{6.41}$$

The density bundle is defined as the associated bundle with respect to the action (6.41). Specifically, we consider $P = F \times R$ with the right GL($\mathbb{R}^3$)-action

$$(e, \rho) \cdot g = (e \cdot g, \det\left(g^{-1}\right)^{-1}\rho) = (e \cdot g, \det(g)\rho). \tag{6.42}$$

Then, we identity elements in $F \times R$ related by this action to define the associated bundle as $D = F \times R/G$. Denote the equivalence class of (e, ρ) by $[e, \rho]$. $\pi_D([e, \rho]) \mapsto \pi(e)$ is well-defined and gives the projection from D to Σ. Consider the trivialization (U_α, T_α) in P. Note that

$$(e, \rho) \sim (\tilde{e}, \tilde{\rho}) \Rightarrow e \cdot g_\alpha(e)^{-1} = \tilde{e} \cdot g_\alpha(\tilde{e})^{-1}, \text{ and } \quad \det(g_\alpha(e))^{-1}\rho = \det(g_\alpha(\tilde{e}))^{-1}\tilde{\rho} \tag{6.43}$$

where $g_\alpha(e) \in$GL($\mathbb{R}^3$) is given by $(x, g_\alpha(e)) := T_\alpha(e)$ for $e \in F_x$, leading to $T_\alpha(e \cdot g_\alpha(e)^{-1}) = (x, \mathbb{1}_g)$. We have the generated local trivialization $(U_\alpha, \tilde{T}_\alpha)$ on D given by

$$\tilde{T}_\alpha([e, \rho]) = (x, \det(g_\alpha(e))^{-1}\rho). \tag{6.44}$$

The above equation is explained as follows. Thanks to the trivialization T_α, we could identify each $e \in F_x$ with $(x, g_\alpha) \in \{x\}\times$GL($\mathbb{R}^3$). Then, we could find the special element $e_o = e \cdot g_\alpha(e)^{-1} \in F_x$ such that e_o is identified with $(x, \mathbb{1}_g)$. Next, in the equivalence class $[e, \rho]$ we find the element $(e_o, \rho_o) \sim (e, \rho)$ and define $\tilde{T}_\alpha([e, \rho]) = (x, \rho_o)$. Now let us consider two local trivial localization T_α and T_β. For $x \in U_\alpha \cap U_\beta$ and $e \in \pi^{-1}[\{x\}]$, we have

$$\tilde{T}_\alpha([e, \rho]) = (x, \det(g_\alpha(e))^{-1}\rho), \quad \tilde{T}_\beta([e, \rho]) = (x, \det(g_\beta(e))^{-1}\rho) \tag{6.45}$$

leading to

$$\mathrm{Pr}_2(\tilde{T}_\alpha([e, \rho])) = (\det(g_\alpha(e)g_\beta(e))^{-1})^{-1}\mathrm{Pr}_2(\tilde{T}_\beta([e, \rho])) \tag{6.46}$$

where $\mathrm{Pr}_2((x, y)) = y$. By definition, we have

$$g_\alpha(e)g_\beta(e)^{-1} = (\mathrm{d}\phi_\alpha \circ e)\,(\mathrm{d}\phi_\beta \circ e)^{-1} = \mathrm{d}\phi_\alpha \circ \mathrm{d}\phi_\beta^{-1}. \tag{6.47}$$

We thus get

$$\mathrm{Pr}_2(\tilde{T}_\alpha([e, \rho])) = \det\left(\mathrm{d}\phi_\alpha \circ \mathrm{d}\phi_\beta^{-1}\right)^{-1}\mathrm{Pr}_2(\tilde{T}_\beta([e, \rho])), \tag{6.48}$$

where $\mathrm{d}\phi_\alpha \circ \mathrm{d}\phi_\beta^{-1}$ is just the Jacobin matrix of the two coordinate system. Now let us consider a section $\sigma : \Sigma \to D; x \mapsto \sigma(x) = [e(x), \rho(x)]$. A trivialization T_α make σ be a scalar field σ_α with

$$\sigma_\alpha(x) := \mathrm{Pr}_2 T_\alpha(\sigma(x)). \tag{6.49}$$

The scalar field is the representation of the σ under the trivialization T_α. (6.48) gives the transformation law between the representations.

Given a symmetric $(0, 2)$-type tensor λ_{ab}. We could define a coordinate dependent function

$$\lambda_\alpha = \mathrm{sgn}(\phi_\alpha)\sqrt{|\det(\lambda_{ij})|} \tag{6.50}$$

where λ_{ij} are the coefficients of λ_{ab} under the coordinate system $\phi_\alpha : U_\alpha \mapsto \mathbb{R}^3$, $\mathrm{sgn}(\phi_\alpha) = 1$ for right hand side coordinate ϕ and -1 otherwise. Considering another coordinate system ϕ_β, we get

$$\lambda_\beta = \det\left(\mathrm{d}\phi_\alpha \circ \mathrm{d}\phi_\beta^{-1}\right)\lambda_\alpha \tag{6.51}$$

which coincides with (6.48). Thus the coordinate dependent functions $\{\lambda_\alpha, \forall\alpha\}$ defines a section in the density bundle.

In addition, we have that

$$
\begin{aligned}
\int_{U_\alpha\cap U_\beta}\sigma(x) &:= \int_{\phi_\alpha(U_\alpha\cup U_\beta)} \operatorname{sgn}(\phi_\alpha)\mathrm{Pr}_2(\tilde{T}_\alpha(\sigma(x))\mathrm{d}^3\phi_\alpha(x) \\
&= \int_{\phi_\alpha(U_\alpha\cup U_\beta)} \operatorname{sgn}(\phi_\alpha)|\mathrm{d}\phi_\alpha\circ\mathrm{d}\phi_\beta^{-1}|^{-1}\mathrm{Pr}_2(\tilde{T}_\beta(\sigma(x))\mathrm{d}^3\phi_\alpha(x) \\
&= \int_{\phi_\beta(U_\alpha\cup U_\beta)} \operatorname{sgn}(\phi_\beta)\mathrm{Pr}_2(\sigma(x)))\mathrm{d}^3\phi_\beta(x),
\end{aligned}
\tag{6.52}
$$

is well-defined, i.e., independent of the choice of coordinate system.

3. densitized frame bundle

Consider $V = \mathrm{GL}(\mathbb{R}^3)$ with the left $\mathrm{GL}(\mathbb{R}^3)$-action

$$g\cdot h = (\det(g))^{-1}gh \tag{6.53}$$

The associated bundle is defined as

$$\tilde{F} = F\times V/\mathrm{GL}(\mathbb{R}^3), \tag{6.54}$$

where the $\mathrm{GL}(\mathbb{R}^3)$ action on $F\times V$ is

$$g\cdot(e,h) = (e\cdot g, \det(g)g^{-1}h). \tag{6.55}$$

The equivalence class of (e,h) is denoted as $[e,h]$. We have the local trivialization $(U_\alpha, \overline{T}_\alpha)$ associated with the local trivialization (U_α, T_α),

$$\overline{T}_\alpha([e,h]) = (x, \det\big(g_\alpha(e)^{-1}\big)g_\alpha(e)h), \quad \text{for } T_\alpha(e) = (x, g_\alpha(e)). \tag{6.56}$$

Given 2 local trivialization (U_α, T_α) and (U_β, T_β), for $x\in U_\alpha\cap U_\beta$, we have

$$\overline{T}_\alpha([e,h]) = (x, \det\big(g_\alpha(e)^{-1}\big)g_\alpha(e)h), \quad \overline{T}_\beta([e,h]) = (x, \det\big(g_\beta(e)^{-1}\big)g_\beta(e)h). \tag{6.57}$$

This implies

$$\mathrm{Pr}_2(\overline{T}_\alpha([e,h])) = \det\big(g_\alpha(e)g_\beta(e)^{-1}\big)^{-1}g_\alpha(e)g_\beta(e)^{-1}\mathrm{Pr}_2(\overline{T}_\beta([e,h])) \tag{6.58}$$

Substituting (eq:transition 6.47), we get

$$\mathrm{Pr}_2(\overline{T}_\alpha([e,h])) = \det\Big(\mathrm{d}\phi_\alpha\circ\mathrm{d}\phi_\beta^{-1}\Big)^{-1}\mathrm{d}\phi_\alpha\circ\mathrm{d}\phi_\beta^{-1}\mathrm{Pr}_2(\overline{T}_\beta([e,h])). \tag{6.59}$$

eq:transformationDensityFrame

To compare (eq:transformationDensityFrame 6.59) with the usual definition of the density triad, we consider

$$E^a_i = \det(e)e^a_i$$

as given in (eq:densityfieldp 6.30) and calculate its transformation from coordinate ϕ_α to coordinate ϕ_β. We have

$$
\begin{aligned}
(\mathrm{d}\phi_\alpha)\circ E_i &= \det\big(e^k_a(\partial_{\phi_\alpha})^a\big)e^a_i(\mathrm{d}\phi_\alpha)_a \\
&= \det\big(e^k_a(\partial_{\phi_\beta})^a\big)\det\left(\frac{\partial\phi_\beta}{\partial\phi_\alpha}\right)e^a_i(\mathrm{d}\phi_\beta)_a\frac{\partial\phi_\alpha}{\partial\phi_\beta} \\
&= \det\left(\frac{\partial\phi_\beta}{\partial\phi_\alpha}\right)(\mathrm{d}\phi_\alpha\circ\mathrm{d}\phi_\beta^{-1})[(\mathrm{d}\phi_\beta)\circ E_i]
\end{aligned}
\tag{6.60}
$$

which is compatible with (eq:transformationDensityFrame 6.59). Thus, the densitized triad E^a_i is a section $\sigma:\Sigma\to\tilde{F}$. Conversely, given a section $\sigma:\Sigma\to\tilde{F}$, the coordinate dependent vector field corresponding to σ is

$$\mathrm{d}\phi_\alpha^{-1}\mathrm{Pr}_2(\overline{T}_\alpha[\sigma]) = \mathrm{Pr}_2(\overline{T}_\alpha[\sigma])^i{}_j\frac{\partial}{\partial\phi^i_\alpha} \tag{6.61}$$

For V being the space of all isomorphism from $\mathfrak{su}(2)$ to $\mathbb{R}^3$, i.e.

$$V = \{a : \mathfrak{su}(2) \to \mathbb{R}^3, a \text{ is invertible}\}. \tag{6.62}$$

We have the action left $\mathrm{GL}(\mathbb{R}^3)\times\mathrm{SU}(2)$ action on it as

$$(g, h) \cdot a = \det(g)^{-1} g \circ a \circ \mathrm{Ad}_{h^{-1}}. \tag{6.63}$$

Then, we consider $Q = P \times F \times V$ where P is the principal bundle considered before with the right action

$$(g, h) \cdot (p, e, a) = (p \cdot h, e \cdot g, \det(g) g^{-1} \circ a \circ \mathrm{Ad}_h). \tag{6.64}$$

Then, we can identify elements in Q related by this action to define the quotient space $\tilde{Q} = Q/\sim$. Let the equivalence class of (p, e, a) be denoted by $[p, e, a]$. The projection

$$\tilde{\pi} : \tilde{Q} \to \Sigma; [p, e, a] \mapsto \pi_F(e) = \pi_P(h) \tag{6.65}$$

is well defined. Given a local trivialization (U_α, T_α) for F and (U_α, ϕ_α) for P, we defined

$$\Psi_\alpha : [p, e, a] \mapsto (x, \det(g_\alpha(e))^{-1} g_\alpha(e) \circ a \circ \mathrm{Ad}_{h_\alpha(p)^{-1}}) \tag{6.66}$$

where we used the $g_\alpha(e)$ is given by $(x, g_\alpha(e)) = T_\alpha(e)$, $h_\alpha(p)$ is given by $(p, h_\alpha(p)^{-1}) = \phi_\alpha(p)$, and we used the fact

$$(p, e, a) \sim (p', e', a') \Rightarrow \det(g_\alpha(e))^{-1} g_\alpha(e) \circ a \circ \mathrm{Ad}_{h_\alpha(p)^{-1}} = \det(g_\alpha(e'))^{-1} g_\alpha(e') \circ a' \circ \mathrm{Ad}_{h_\alpha(p')^{-1}}. \tag{6.67}$$

Indeed, $(p, e, a) \sim (p', e', a')$ implies that there exist (g_o, h_o) such that

$$(p', e', a') = (p \cdot h_o, e \cdot g_o, \det(g_o) g_o^{-1} \circ a \circ \mathrm{Ad}_{h_o}). \tag{6.68}$$

Then, the intertwing property of g_α and h_α leads to

$$\begin{aligned} &\det(g_\alpha(e'))^{-1} g_\alpha(e') \circ a' \circ \mathrm{Ad}_{h_\alpha(p')^{-1}} = \det(g_\alpha(e \cdot g_o))^{-1} g_\alpha(e \cdot g_o) \circ (\det(g_o) g_o^{-1} \circ a \circ \mathrm{Ad}_{h_o}) \circ \mathrm{Ad}_{h_\alpha(p \cdot h_o)^{-1}} \\ =\,&\det(g_\alpha(e))^{-1} g_\alpha(e) \circ a \circ \mathrm{Ad}_{h_\alpha(p)^{-1}}. \end{aligned} \tag{6.69}$$

It is easy to say that

$$\Psi_\alpha : \tilde{\pi}[U_\alpha] \to V \tag{6.70}$$

is a surjection, indicating that $\tilde{\pi} : Q \to \Sigma$ is a bundle with fibers $\tilde{\pi}^{-1}[\{x\}] \cong V$. Now, let us consider another local trivialization Ψ_β. For $x \in U_\alpha \cap U_\beta$, we have

$$\Psi_\beta([p, e, a]) = (x, \det(g_\beta(e))^{-1} g_\beta(e) \circ a \circ \mathrm{Ad}_{h_\beta(p)^{-1}}) \tag{6.71}$$

leading to

$$\begin{aligned} &\mathrm{Pr}_2\Psi_\alpha([p, e, a]) = \det\big(g_\alpha(e) g_\beta(e)^{-1}\big)^{-1} g_\alpha(e) g_\beta(e)^{-1} \circ \mathrm{Pr}_2\Psi_\beta([p, e, a]) \circ \mathrm{Ad}_{h_\beta(p) h_\alpha(p)^{-1}}) \\ =\,&\det\Big(\mathrm{d}\phi_\alpha \circ \mathrm{d}\phi_\beta^{-1}\Big)^{-1} \mathrm{d}\phi_\alpha \circ \mathrm{d}\phi_\beta^{-1} \circ \mathrm{Pr}_2\Psi_\beta([p, e, a]) \circ \mathrm{Ad}_{h_{\alpha\beta}(x)^{-1}}) \end{aligned} \tag{6.72}$$

eq:sectionAshtekarBundle

where we used (6.59) and $h_{\alpha\beta}(x) = h_\alpha(p) h_\beta(p)^{-1}$ is the transition function. (6.72) contains both the transformation of densitized triad under coordinate change and the gauge transformation of it. This implies that the Ashtekar densitized triad is a section $\sigma : \Sigma \to \tilde{Q}$.

For our following calculation, it will be useful to consider the densitized 3-form bundle.

At first, let us consider the 3-form bundle on Σ. Since the space of 3-forms at each point $x \in \Sigma$ is a one-dimensional vector space, the 3-form bundle is thus a line bundle. It is constructed as follows. Let $\Lambda^3 T^*_x\Sigma$ be the space of 3-forms at x. The 3-form bundle is defined as

$$\Lambda^3 T\Sigma = \bigsqcup_x \Lambda^3 T^*_x \Sigma. \tag{6.73}$$

As in the frame bundle, the topology and smooth structure on $\Lambda^3 T\Sigma$ are defined by the map

$$\tilde{\phi}_\alpha : \lambda_{abc}(x) \mapsto (\phi_\alpha(x), \lambda_{abc}(x)(\partial^1_{\phi_\alpha})^a(\partial^2_{\phi_\alpha})^b(\partial^3_{\phi_\alpha})^c) \in \mathbb{R}^{3+1}. \tag{6.74}$$

To define $\Lambda^3 T\Sigma$ as a vector bundle, we need to define the local trivialization $T_\alpha : \pi^{-1}[U_\alpha] \to U_\alpha \times \mathbb{R}$ such that $T_\alpha \restriction \pi^{-1}[\{x\}] : \pi^{-1}[\{x\}] \to \mathbb{R}$ is a linear isomorphism. We define T_α as

$$T_\alpha : \lambda_{abc} = (x, \lambda_{abc}(\partial^1_{\phi_\alpha})^a(\partial^2_{\phi_\alpha})^b(\partial^3_{\phi_\alpha})^c), \quad \text{for } \lambda_{abc} \in \pi^{-1}[\{x\}] \tag{6.75}$$

for which one can verify that its restriction at $\pi^{-1}[\{x\}]$ is a linear isomorphism to $\mathbb{R}$. Given another trivialization (U_β, T_β), we have for $x \in U_\alpha \cap U_\beta$

$$T_\beta : \lambda_{abc} = (x, \lambda_{abc}(\partial^1_{\phi_\beta})^a(\partial^2_{\phi_\beta})^b(\partial^3_{\phi_\beta})^c), \tag{6.76}$$

leading to that

$$\mathrm{Pr}_2 T_\alpha(\lambda) = \frac{\lambda_{abc}(\partial^1_{\phi_\alpha})^a(\partial^2_{\phi_\alpha})^b(\partial^3_{\phi_\alpha})^c}{\lambda_{abc}(\partial^1_{\phi_\beta})^a(\partial^2_{\phi_\beta})^b(\partial^3_{\phi_\beta})^c} \mathrm{Pr}_2 T_\beta(\lambda) = \det\big(\mathrm{d}\phi_\beta \circ \mathrm{d}\phi_\alpha^{-1}\big) \mathrm{Pr}_2 T_\beta(\lambda). \tag{6.77}$$

From this equation, we can see that the transition function $\det\big(\mathrm{d}\phi_\beta \circ \mathrm{d}\phi_\alpha^{-1}\big)$ is a function on Σ, i.e., it is independent on the choice of element in the fiber.

We are going to define a densitized 3-form bundle, which is the tensor product bundle of $\Lambda^3 T\Sigma$ and the density bundle of weight -1. Here the density bundle of weight -1 is the bundle constructed similarly as the density bundle but with replacing (6.41) by

$$g \cdot \rho = \det(g)\rho. \tag{6.78}$$

We will use $\underline{D}$ to denote the density bundle of weight -1. Then, we have the densitized 3-form bundle

$$\underline{\Lambda^3 T\Sigma} = \Lambda^3 T\Sigma \otimes \underline{D}. \tag{6.79}$$

The tensor product of the two bundles are defined as follows. Let U_α and U_β be the common local trivialization of $\Lambda^3 T\Sigma$ and $\underline{D}$. In $U_\alpha \cap U_\beta$, we have two transition functions

$$g^{(1)}_{\alpha\beta} = \det\big(\mathrm{d}\phi_\beta \circ \mathrm{d}\phi_\alpha^{-1}\big), \quad g^{(2)}_{\alpha\beta} = \mathrm{d}\phi_\alpha \circ \mathrm{d}\phi_\beta^{-1} \tag{6.80}$$

where we used (6.48) but it is noted that $\underline{D}$ is the density bundle with weight -1. With $g^{(1)}_{\alpha\beta}$ and $g^{(2)}_{\alpha\beta}$, we could define a new transition function

$$g_{\alpha\beta}(x) = g^{(1)}_{\alpha\beta}(x) \otimes g^{(2)}_{\alpha\beta}(x) \in \mathrm{GL}(\mathbb{R} \otimes \mathbb{R}). \tag{6.81}$$

The action of $g_{\alpha\beta}(x)$ on $\mathbb{R} \otimes \mathbb{R} \cong \mathbb{R}$ is

$$g_{\alpha\beta}(x) \cdot (\lambda \otimes \rho) = (g^{(1)}_{\alpha\beta}(x)\lambda) \otimes (g^{(2)}_{\alpha\beta}(x)\rho) = \lambda \otimes \rho. \tag{6.82}$$

Now let us consider $(U_\alpha \sqcup U_\beta) \times (\mathbb{R} \otimes \mathbb{R})$, where we could define the equivalence relation

$$(\{U_\alpha, x\}, \lambda \otimes \rho) \sim (\{U_\beta, x\}, \lambda \otimes \rho) \tag{6.83}$$

where $\{U_\alpha, x\} \in U_\alpha$ denotes the element x treated as an element in U_α and $\{U_\beta, x\} \in U_\beta$ denotes the same element x but treated as an element in U_β. Then tensor product bundle is

$$\underline{\Lambda^3 T\Sigma} = \Lambda^3 T\Sigma \otimes \underline{D} = \left(\bigsqcup_\alpha U_\alpha\right) \times (\mathbb{R} \otimes \mathbb{R})/\sim . \tag{6.84}$$

In $\underline{\Lambda^3 T\Sigma}$, an element can be denoted by $[(U_\alpha, x), \lambda\otimes\rho]$ which is the equivalence class of $((U_\alpha, x), \lambda\otimes\rho)$. The projection is

$$\pi : [(U_\alpha, x), \lambda \otimes \rho] \mapsto x. \tag{6.85}$$

The local trivialization is

$$T_\alpha : [(U_\alpha, x), \lambda \otimes \rho] \mapsto (x, \lambda\rho) \in U_\alpha \times \mathbb{R}. \tag{6.86}$$

Consider another trivialization T_β, for $x \in U_\alpha \cap U_\beta$, we have

$$T_\beta : [(U_\alpha, x), \lambda \otimes \rho] = [(U_\beta, x), \mathrm{sgn}(\det\big(\mathrm{d}\phi_\beta \circ \mathrm{d}\phi_\alpha^{-1}\big))\lambda \otimes \rho] = (x, \lambda\rho), \tag{6.87}$$

leading to

eq:transformationepsilon

$$\mathrm{Pr}_2(T_\alpha(\sigma(x))) = \mathrm{Pr}_2(T_\beta(\sigma(x))) \tag{6.88}$$

This equation tells us that a section $\sigma : \Sigma \to \underline{\Lambda^3 T\Sigma}$ can be represented by a set of coordinate independent functions. A specific such function is defined as

$$\varepsilon_\alpha = 1 = \varepsilon_\beta. \tag{6.89}$$

Given a section $\sigma : \Sigma \mapsto \underline{\Lambda^3 T\Sigma}$, for $x \in U_\alpha$, we introduce

$$\sigma_\alpha = \mathrm{Pr}_2(T_\alpha(\sigma(x)))\mathrm{d}\phi^1_\alpha \wedge \mathrm{d}\phi^2_\alpha \wedge \mathrm{d}\phi^3_\alpha, \tag{6.90}$$

which is a coordinate-dependent 3-form on Σ. For the section ε_α, we have

eq:densityepsilon

$$\varepsilon_{abc} = (\mathrm{d}\phi^1_\alpha)_a \wedge (\mathrm{d}\phi^2_\alpha)_b \wedge (\mathrm{d}\phi^3_\alpha)_c. \tag{6.91}$$

6. *Volume form*

pseudoform

Consider an orientable 3-manifold Σ over which we defines the density bundle of weight 1 and the densitized 3-form bundle of weight -1. Consider a section ρ and the section ε. Then, on U_α, we have the following 3 form

$$\mathrm{vol}^\alpha_\rho = \rho_\alpha \varepsilon_{abc} = \rho_\alpha (\mathrm{d}\phi^1_\alpha)_a \wedge (\mathrm{d}\phi^2_\alpha)_b \wedge (\mathrm{d}\phi^3_\alpha)_c. \tag{6.92}$$

Now, let us consider another trivialization U_β. In $U_\alpha \cap U_\beta$, under the trivialization U_β, we have

$$\begin{aligned}\mathrm{vol}^\beta_\rho =&\rho_\beta (\mathrm{d}\phi^1_\beta)_a \wedge (\mathrm{d}\phi^2_\beta)_b \wedge (\mathrm{d}\phi^3_\beta)_c\\ =&\rho_\alpha \det\Big(\mathrm{d}\phi_\alpha \circ \mathrm{d}\phi_\beta^{-1}\Big)(\mathrm{d}\phi^1_\beta)_a \wedge (\mathrm{d}\phi^2_\beta)_b \wedge (\mathrm{d}\phi^3_\beta)_c\\ =&\rho_\alpha (\mathrm{d}\phi^1_\alpha)_a \wedge (\mathrm{d}\phi^2_\alpha)_b \wedge (\mathrm{d}\phi^3_\alpha)_c\\ =&\mathrm{vol}^\alpha_\rho.\end{aligned} \tag{6.93}$$

We can see that vol_ρ defines a 3-form.

C. From the continuous phase space to discrete phase space

In the classical theory, we need an oriented 3-Riemann manifold Σ. The phase space Γ comprise pairs of fields (A^i_a, E^b_j) where A^i_a is the SU(2) connection and E^b_j is the densitized triad field of weight 1. The Poisson bracket on Γ is

$$\{A^i_a(x), E^b_j(y)\} = \kappa\gamma\delta^b_a\delta^i_j\delta^3(x, y) \tag{6.94}$$

with $\kappa = 8\pi G$. Note that, according to our discussion for (6.33), the δ-function is the delta function with respect to the orientation o. Therefore, when calculating the Poisson bracket, we need to choose a right hand coordinate for our calculation so that we can use $\int \delta^3(x)\mathrm{d}^3x = 1$.

Given an oriented edge $e : t \mapsto e(t)$, the holonomy along it is defined by the equation

$$\partial_t h_{e(t)}(A) = h_{e(t)}(A)A_a(e(t))\dot{e}^a(t),\ \ h_{e(0)} = 1. \tag{6.95}$$

Its solution takes

$$h_e(A) = \mathcal{P}\exp\int_e A = 1 + \sum_{n=1}^{\infty}\int_0^1 \mathrm{d}t_n \int_0^{t_n}\mathrm{d}t_{n-1}\cdots\int_0^{t_2}\mathrm{d}t_1 A(t_1)\cdots A(t_n). \tag{6.96}$$

Let f^i be a $\mathfrak{su}(2)$ value field on Σ. Then, we have the following functional on Γ

$$E[f] = \frac{1}{\kappa\beta}\int E^a_i(x)f^i_a(x)\mathrm{d}^3x.$$

Note that we have assumed that x in the above integral is a right hand coordinate. It can be calculated that

$$\begin{aligned}
&\{h_e(A), E[f]\} = \int \frac{\delta h_e(A)}{\delta A^i_a(x)}f^i_a(x)\mathrm{d}^3x\\
=&\int_0^1 \mathrm{d}t_1 f^i_a(t_1)\dot{e}^a(t_1)\tau_i + \int_0^1\mathrm{d}t_2\int_0^{t_1}\mathrm{d}t_1 A(t_1)f^i_a(t_2)\dot{e}^a(t_2)\tau_i + \int_0^1\mathrm{d}t_2\int_0^{t_1}\mathrm{d}t_1 f^i_a(t_1)\dot{e}^a(t_1)\tau_i A(t_2) + \cdots\\
=&\int_0^1 \mathrm{d}t f^i_a(t)\dot{e}^a(t)\tau_i + \int_0^1 \mathrm{d}t f^i_a(t)\dot{e}^a(t)\int_0^t \mathrm{d}T_1 A(T_1)\tau_i + \int_0^1\mathrm{d}t f^i_a(t)\dot{e}^a(t)\int_t^1\mathrm{d}T_1\tau_i A(T_1) + \cdots\\
=&\sum_{n=1}^{\infty}\sum_{k=1}^{n}\int_0^1\mathrm{d}t f^i_a(t)\dot{e}^a(t)\int_t^1\mathrm{d}T_{n-1}\int_t^{T_{n-1}}\mathrm{d}T_{n-2}\cdots\int_t^{T_{k+1}}\mathrm{d}T_k\int_0^t\mathrm{d}T_{k-1}\cdots\int_0^{T_2}\mathrm{d}T_1 A(T_1)\cdots\tau_i\cdots A(T_{n-1})\\
=&\sum_{n=1}^{\infty}\sum_{k=1}^{n}\int_0^1\mathrm{d}t f^i_a(t)\dot{e}^a(t)\left(\int_0^t\mathrm{d}T_{k-1}\cdots\int_0^{T_2}\mathrm{d}T_1 A(T_1)\cdots A(T_{k-1})\right)\tau_i\times\\
&\left(\int_t^1\mathrm{d}T_{n-1}\int_t^{T_{n-1}}\mathrm{d}T_{n-2}\cdots\int_t^{T_{k+1}}\mathrm{d}T_k A(T_k)\cdots A(T_{n-1})\right)\\
=&\int_0^1\mathrm{d}t f^i_a(t)\dot{e}^a(t)h_{e[0,t]}\tau_i h_{e[t,1]}.
\end{aligned} \tag{6.97}$$

where $f^i(t) \equiv f^i(e(t))$ is the value of the field at $e(t) \in e$.

Since $E^a_i(x)$ is a density field of weight 1, it is dual to a 2-form $E^a_i(x)\varepsilon_{abc}$. Actually, to see $E^a_i(x)\varepsilon_{abc}$ is a 2-form, let us consider the precise definition of $E^a_i(x)$ by using fiber bundle. From the point of view of fiber bundle, $E^a_i(x)$ is actually a section in the densitized frame bundle. Given a section denoted by σ, the field $E^a_i(x)$ as a representation of the section under a trivialization is

$$E^a_i(x) = \mathrm{Pr}_2(\overline{T}_\alpha[\sigma(x)])^k{}_i\left(\frac{\partial}{\partial\phi^k_\alpha}\right)^a$$

where $\overline{T}_\alpha$ is the trivialization. Acording to (6.59), we have the transformation law between different trivialization

$$\mathrm{Pr}_2(\overline{T}_\alpha\sigma(x))^k{}_i\frac{\partial}{\partial\phi^k_\alpha} = \det\Big(\mathrm{d}\phi_\alpha\otimes\mathrm{d}\phi^{-1}_\beta\Big)^{-1}\mathrm{Pr}_2(\overline{T}_\beta\sigma(x))^k{}_i\frac{\partial}{\partial\phi^k_\beta} \tag{6.98}$$

Thus, we get the transformation law for $E^a_i\varepsilon_{abc}$ under different coordinate system, i.e.,

$$\begin{aligned}
&\mathrm{Pr}_2(\overline{T}_\alpha\sigma(x))^k{}_i\frac{\partial}{\partial\phi^k_\alpha}\lrcorner(\mathrm{d}\phi^1_\alpha\wedge\mathrm{d}\phi^2_\alpha\wedge\mathrm{d}\phi^3_\alpha) = \det\Big(\mathrm{d}\phi_\alpha\otimes\mathrm{d}\phi^{-1}_\beta\Big)^{-1}\left[\mathrm{Pr}_2(\overline{T}_\beta\sigma(x))^k{}_i\frac{\partial}{\partial\phi^k_\beta}\right]\lrcorner(\mathrm{d}\phi^1_\alpha\wedge\mathrm{d}\phi^2_\alpha\wedge\mathrm{d}\phi^3_\alpha)\\
=&\mathrm{Pr}_2(\overline{T}_\beta\sigma(x))^k{}_i\frac{\partial}{\partial\phi^k_\beta}\lrcorner(\det\Big(\mathrm{d}\phi_\alpha\otimes\mathrm{d}\phi^{-1}_\beta\Big)^{-1}\mathrm{d}\phi^1_\alpha\wedge\mathrm{d}\phi^2_\alpha\wedge\mathrm{d}\phi^3_\alpha) = \mathrm{Pr}_2(\overline{T}_\beta\sigma(x))^k{}_i\frac{\partial}{\partial\phi^k_\beta}\lrcorner(\mathrm{d}\phi^1_\beta\wedge\mathrm{d}\phi^2_\beta\wedge\mathrm{d}\phi^3_\beta).
\end{aligned} \tag{6.99}$$

This implies that $E^a_i \varepsilon_{abc}$ is coordinate independent. In other words, $E^a_i \varepsilon_{abc}$ is a 2-form. The integration of a 2-form over a surface is background independent. We thus introduce the flux of E as

$$E_S(f) := \frac{1}{\kappa\beta} \int_S f^i E^a_i \varepsilon_{abc} \tag{6.100}$$

with $f : S \to \mathfrak{su}(2)$, and S being a 2-surface. Actually, the integration (6.101) is not very clear. We need a more precise definition on the integration which is describe as follows.

We know how to define the integration (6.101) if we treat the surface S as a 2-D manifold independently. In such treatment, we need to choose a right hand coordinate $\Psi : x \mapsto (\sigma^1(x), \sigma^2(x)) \in \mathbb{R}^2$ on S and then

$$E_S(f) = \frac{1}{\kappa\beta} \int_{\Psi[S]} f^i E^a_i \varepsilon_{abc} \left(\frac{\partial}{\partial \sigma^1}\right)^b \left(\frac{\partial}{\partial \sigma^2}\right)^c \mathrm{d}\sigma^1 \mathrm{d}\sigma^2. \tag{6.101}$$

where we treat $(\partial/\partial\sigma^k)^a$ as the 3-D vector tangent to the surface S to do the contraction. Moreover, one needs to choose an extra 3-D coordinate covering the surface S to get the 2-form $f^i E^a_i \varepsilon_{abc}$. As we showed before, the resulting 2-form is independent of the choice of the 3-D coordinate. In other words, the choice of the 3-D coordinate will not affect our result.

From this definition, we can see that $E_S(f)$ depends on the orientation of the 2-surface S. To define the orientation of S, we need to choose an 'outwards' direction t^a of S, i.e., a continuously varying vector field over S that is not tangent to S anywhere. The orientation of S then is given by $t^a o_{abc} \restriction S$ with o_{abc} being the orientation of the 3-manifold Σ. A right-hand coordinate system $\{\sigma^1, \sigma^2\}$ on S is the coordinate system such that

$$t^a o_{abc} \left(\partial_{\sigma^1}\right)^b \left(\partial_{\sigma^2}\right)^c > 0,\ \forall \text{ points} \in S. \tag{6.102}$$

Now let us consider a 3-D coordinate system $\{x^1, x^2, x^3\}$ such that

$$\text{the surface } S \text{ is given by } x^1 = 0 \tag{6.103}$$

It can be verify

$$\operatorname{sgn}(o_{abc} t^a (\partial_{x^2})^b (\partial_{x^3})^c) = \operatorname{sgn}(t^a (\mathrm{d}x^1)_a) \operatorname{sgn}(o_{abc} (\partial_{x^1})^a (\partial_{x^2})^b (\partial_{x^3})^c) \tag{6.104}$$

Therefore, $\{x^2, x^3\}$, as a coordinate on S, is right hand if $\operatorname{sgn}(t^a(\mathrm{d}x^1)_a) o_{abc} (\partial_{x^1})^a (\partial_{x^2})^b (\partial_{x^3})^c > 0$ and left hand if $\operatorname{sgn}(t^a(\mathrm{d}x^1)_a) o_{abc} (\partial_{x^1})^a (\partial_{x^2})^b (\partial_{x^3})^c < 0$. As a consequence, using the coordinate x^i, we get

$$\begin{aligned} E_S(f) =& \frac{\operatorname{sgn}(t^a(\mathrm{d}x^1)_a)\operatorname{sgn}(o_{abc}(\partial_{x^1})^a(\partial_{x^2})^b(\partial_{x^3})^c)}{\kappa\beta} \int_{\Psi[S]} f^i E^a_i \varepsilon_{abc} \left(\frac{\partial}{\partial x^2}\right)^b \left(\frac{\partial}{\partial x^3}\right)^c \mathrm{d}x^2 \mathrm{d}x^3 \\ =& \frac{\operatorname{sgn}(t^a(\mathrm{d}x^1)_a)\operatorname{sgn}(o_{abc}(\partial_{x^1})^a(\partial_{x^2})^b(\partial_{x^3})^c)}{\kappa\beta} \int_{\Psi[S]} f^i E^a_i (\mathrm{d}x^1)_a \mathrm{d}x^2 \mathrm{d}x^3, \end{aligned} \tag{6.105}$$

where we still use $\Psi(S)$ to denote the domain of S under the coordinate $S \ni x \mapsto \{x^2, x^3\} \in \mathbb{R}^2$, and in the last line E^a_i should be treated as the representing vector field of the density E^a_i under the coordinate $\{x^1, x^2, x^3\}$, because in the last equality we have applied $\varepsilon_{abc} = (\mathrm{d}x^1)_a \wedge (\mathrm{d}x^2)_b \wedge (\mathrm{d}x^3)_c$, i.e. represented ε_{abc} under the coordinate $\{x^1, x^2, x^3\}$.

Let us consider a general case where we have a 3-D coordinate which does not satisfy the condition (6.103). In this coordinate, we assume that the surface S can be given by the equation

$$x^1 - g(x^2, x^3) = 0.$$

Then, we could consider a new coordinate system $\{y^1, y^2, y^3\}$ with

$$\begin{aligned} y^1 &= x^1 - g(x^2, x^3) \\ y^2 &= x^2 \\ y^3 &= x^3. \end{aligned} \tag{6.106}$$

The coordinate $\{y^1, y^2, y^3\}$ is such a coordinate used in (6.105). We thus have

$$
\begin{aligned}
E_S(f) =& \frac{\mathrm{sgn}(t^a(\mathrm{d}y^1)_a)\mathrm{sgn}(o_{abc}(\partial_{y^1})^a(\partial_{y^2})^b(\partial_{y^3})^c)}{\kappa\beta} \int_{\Psi[S]} f^i E_i^a (\mathrm{d}y^1)_a \mathrm{d}y^2\mathrm{d}y^3 \\
=& \frac{\mathrm{sgn}(t^a(\mathrm{d}y^1)_a)\mathrm{sgn}(o_{abc}(\partial_{x^1})^a(\partial_{x^2})^b(\partial_{x^3})^c)}{\kappa\beta} \times \\
& \int_{\Psi[S]} f^i \det\left(\frac{\partial(y^1,y^2,y^3)}{\partial(x^1,x^2,x^3)}\right) E_i^a \left((\mathrm{d}x^1)_a - (\partial_{x^2} g)(\mathrm{d}x^2)_a - (\partial_{x^3} g)(\mathrm{d}x^3)_a\right) \mathrm{d}x^2\mathrm{d}x^3 \\
=& \frac{\mathrm{sgn}(t^a(\mathrm{d}y^1)_a)\mathrm{sgn}(o_{abc}(\partial_{x^1})^a(\partial_{x^2})^b(\partial_{x^3})^c)}{\kappa\beta} \times \\
& \int_{\Psi[S]} f^i E_i^a \left((\mathrm{d}x^1)_a - (\partial_{x^2} g)(\mathrm{d}x^2)_a - (\partial_{x^3} g)(\mathrm{d}x^3)_a\right) \mathrm{d}x^2\mathrm{d}x^3
\end{aligned}
\tag{6.107}
$$

Let us define

$$
\begin{aligned}
n_a =& \mathrm{sgn}\left(t^b(\mathrm{d}y^1)_b\right)\left((\mathrm{d}x^1)_a - (\partial_{x^2} g)(\mathrm{d}x^2)_a - (\partial_{x^3} g)(\mathrm{d}x^3)_a\right) \\
=& \mathrm{sgn}\left[t^b\left((\mathrm{d}x^1)_b - (\partial_{x^2} g)(\mathrm{d}x^2)_b - (\partial_{x^3} g)(\mathrm{d}x^3)_b\right)\right]\left((\mathrm{d}x^1)_a - (\partial_{x^2} g)(\mathrm{d}x^2)_a - (\partial_{x^3} g)(\mathrm{d}x^3)_a\right)
\end{aligned}
\tag{6.108}
$$

Then, n_a has the geometric interpretation of the conormal of S compatible with its orientation. Then, we have

$$
E_S(f) = \frac{\mathrm{sgn}(o_{abc}(\partial_{x^1})^a(\partial_{x^2})^b(\partial_{x^3})^c)}{\kappa\beta} \int_{\Psi[S]} f^i E_i^a n_a \mathrm{d}x^2\mathrm{d}x^3, \tag{6.109}
$$

where we assumed that the restriction of x^2 and x^3 gives a well-defined coordinate system in S.

Proposition 1. *(6.109) is independent of the choice of the 3-D coordinate.*

Proof. To begin with, we consider the right hand coordinate $\{x^1, x^2, x^3\}$ such that S is given by $x^1 = 0$ and $t^a(\mathrm{d}x_1)_a > 0$. Then, the integral is (6.109)

$$
E_S(f) = \frac{1}{\kappa\beta}\int_{\Psi[S]} f^i(0,x^2,x^3)E_i^1(0,x^2,x^3)\mathrm{d}x^2\mathrm{d}x^3. \tag{6.110}
$$

Then, we go to a general coordinate $\{y^1, y^2, y^3\}$ where the restriction of y^2 and y^3 on S gives a coordinate of S. We have

$$
\begin{aligned}
E_S(f) =& \frac{\mathrm{sgn}(o_{abc}(\partial_{y^1})^a(\partial_{y^2})^b(\partial_{y^3})^c)\mathrm{sgn}(t^a(\mathrm{d}y^1)_a)}{\kappa\beta} \\
& \int_{\Psi'[S]} f^i(y^1,y^2,y^3)\left(\tilde{E}_i^1(y^1,y^2,y^3) - \frac{\partial y^1}{\partial y^2}\tilde{E}_i^2(y^1,y^2,y^3) - \frac{\partial y^1}{\partial y^3}\tilde{E}_i^3(y^1,y^2,y^3)\right)\mathrm{d}y^2\mathrm{d}y^3,
\end{aligned}
\tag{6.111}
$$

where y^1 in the second line is treated as a function $y^1(y^2,y^3)$, we use $\tilde{E}$ to denote the representing vector field of the density E under the y-coordinate, and we used that

$$
\mathrm{sgn}\left[t^b\left((\mathrm{d}y^1)_b - (\partial_{y^2}y^1)(\mathrm{d}y^2)_b - (\partial_{y^3}y^1)(\mathrm{d}y^3)_b\right)\right] = \mathrm{sgn}(t^b(\mathrm{d}y^1)_b). \tag{6.112}
$$

Now let us prove that it is the same as the integral given in (6.110).

Due to transformation law between the representing vector fields of E under different coordinate, we have

$$
\begin{aligned}
& \tilde{E}_i^1 - \frac{\partial y^1}{\partial y^2}\tilde{E}_i^2 - \frac{\partial y^1}{\partial y^3}\tilde{E}_i^3 \\
=& \det\left(\frac{\partial(x^1,x^2,x^3)}{\partial(y^1,y^2,y^3)}\right)\left(E_i^a(\mathrm{d}y^1)_a - \frac{\partial y^1}{\partial y^2}E_i^a(\mathrm{d}y^2)_a - \frac{\partial y^1}{\partial y^3}E_i^a(\mathrm{d}y^3)_a\right) \\
=& \det\left(\frac{\partial(x^1,x^2,x^3)}{\partial(y^1,y^2,y^3)}\right)\left(E_i^\mu\frac{\partial y^1}{\partial x^\mu} - \frac{\partial y^1}{\partial y^2}E_i^\mu\frac{\partial y^2}{\partial x^\mu} - \frac{\partial y^1}{\partial y^3}E_i^\mu\frac{\mathrm{d}y^3}{\partial x^\mu}\right)
\end{aligned}
\tag{6.113}
$$

In the y-coordinate, the conormal of S is given by

$$n_a = (\mathrm{d}y^1)_a - \frac{\partial y^1}{\partial y^2}(\mathrm{d}y^2)_a - \frac{\partial y^1}{\partial y^3}(\mathrm{d}y^3)_a \tag{6.114}$$

which should proportional to $(\mathrm{d}x^1)_a$, i.e.,

$$n_a = \beta(\mathrm{d}x^1)_a. \tag{6.115}$$

Therefore, one get

$$\frac{\partial y^1}{\partial x^\mu} - \frac{\partial y^1}{\partial y^2}\frac{\partial y^2}{\partial x^\mu} - \frac{\partial y^1}{\partial y^3}\frac{\mathrm{d}y^3}{\partial x^\mu} = \left(\frac{\partial}{\partial x^\mu}\right)^a n_a = \delta_{\mu,1}\left(\frac{\partial}{\partial x^1}\right)^a n_a \equiv \beta\delta_{1,\mu}. \tag{6.116}$$

We thus have

$$\left(E_i^\mu \frac{\partial y^1}{\partial x^\mu} - \frac{\partial y^1}{\partial y^2}E_i^\mu\frac{\partial y^2}{\partial x^\mu} - \frac{\partial y^1}{\partial y^3}E_i^\mu\frac{\mathrm{d}y^3}{\partial x^\mu}\right) = \beta E_i^1. \tag{6.117}$$

For the determinant of the Jacobian matrix, we divided it into blocks such that

$$\frac{\partial(y^1,y^2,y^3)}{\partial(x^1,x^2,x^3)} = \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^1}{\partial x^2} & \frac{\partial y^1}{\partial x^3} \\ \frac{\partial y^2}{\partial x^1} & \frac{\partial y^2}{\partial x^2} & \frac{\partial y^2}{\partial x^3} \\ \frac{\partial y^3}{\partial x^1} & \frac{\partial y^3}{\partial x^2} & \frac{\partial y^3}{\partial x^3} \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad D := \begin{pmatrix} \frac{\partial y^2}{\partial x^2} & \frac{\partial y^2}{\partial x^3} \\ \frac{\partial y^3}{\partial x^2} & \frac{\partial y^3}{\partial x^3} \end{pmatrix}. \tag{6.118}$$

Here we divide the Jacobian matrix into blocks such that D is the Jacobian matrix for change of variables on the surface. Indeed, since the surface is given by $x^1 = 0$, the Jacobian matrix from the y-coordinate on the surface to the x-coordinate on the surface is just D, while the inverse one, i.e., the Jacobian matrix from x-coordinate to the y-coordinate on the surface is more complicated. Applying the formula for determinant of block matrix, we get

$$\det\left(\frac{\partial(y^1,y^2,y^3)}{\partial(x^1,x^2,x^3)}\right) = \det(D)(A - BD^{-1}C) \tag{6.119}$$

Therefore, we have

$$\tilde{E}_i^1 - \frac{\partial y^1}{\partial y^2}\tilde{E}_i^2 - \frac{\partial y^1}{\partial y^3}\tilde{E}_i^3 = \det(D)^{-1}(A - BD^{-1}C)^{-1}\beta E_i^1. \tag{6.120}$$

On the other hand, applying the formula for the inverse of a block matrix, we have

$$(A - BD^{-1}C)^{-1} = \left[\left(\frac{\partial(y^1,y^2,y^3)}{\partial(x^1,x^2,x^3)}\right)^{-1}\right]_{11} = \frac{\partial x^1}{\partial y^1} = \left(\frac{\partial}{\partial y^1}\right)^a (\mathrm{d}x^1)_a \tag{6.121}$$

Since $(\mathrm{d}x^1)_a = \beta^{-1}n_a$, we have

$$(A - BD^{-1}C)^{-1} = \beta^{-1}\left(\frac{\partial}{\partial y^1}\right)^a n_a = \beta^{-1}, \tag{6.122}$$

where (6.114) is applied. We therefore get

$$\tilde{E}_i^1 - \frac{\partial y^1}{\partial y^2}\tilde{E}_i^2 - \frac{\partial y^1}{\partial y^3}\tilde{E}_i^3 = \det(D)^{-1}E_i^1. \tag{6.123}$$

Therefore, we have

$$\begin{aligned} &\mathrm{sgn}(o_{abc}(\partial_{y^1})^a(\partial_{y^2})^b(\partial_{y^3})^c)\mathrm{sgn}(t^a(\mathrm{d}y^1)_a)\times \\ &\int_{\Psi'[S]} f^i(y^1,y^2,y^3)\left(\tilde{E}_i^1(y^1,y^2,y^3) - \frac{\partial y^1}{\partial y^2}\tilde{E}_i^2(y^1,y^2,y^3) - \frac{\partial y^1}{\partial y^3}\tilde{E}_i^3(y^1,y^2,y^3)\right)\mathrm{d}y^2\mathrm{d}y^3 \\ =&\int_{\Psi'[S]} f^i(y^1,y^2,y^3)|\det(D)|^{-1}E_i^1\mathrm{d}y^2\mathrm{d}y^3 \\ =&\int_{\Psi(S)} f^i(0,x^2,x^3)E_i^1(0,x^2,x^3)\mathrm{d}x^2\mathrm{d}x^3, \end{aligned} \tag{6.124}$$

implying that $E_S(f)$ does not depend on the choice of coordinate. Here we used that $\mathrm{sgn}(o_{abc}(\partial_{y^1})^a(\partial_{y^2})^b(\partial_{y^3})^c)\mathrm{sgn}(t^a(\mathrm{d}y^1)_a) = 1$ for $\{y^2,y^3\}$ being right hand on S and -1 for $\{y^2,y^3\}$ being left hand. $\square$

We are going to calculate the Poisson bracket between the flux and the holonomy, we thus choose the right hand coordinate $\{x^1, x^2, x^3\}$. Consider the region $\mathcal{R} = \{(x^1, x^2, x^3), -1 \leq x^1 \leq 1\}$ containing S. We define the regularized version of E_S^f as

$$E_S^\epsilon(f) := \frac{1}{\kappa\beta}\int_{\mathcal{R}} \delta^\epsilon(x^1) E_i^1(x^1, x^2, x^3)\tilde{f}^i(x^1, x^2, x^3)\mathrm{d}^3x \tag{6.125}$$

where $\tilde{f}$ is an extension of f to $\mathcal{R}$ and $\delta^\epsilon(t)$ is a family of functions tending to δ distribution on $\mathbb{R}$. Define

$$g_c^i(x^1, x^2, x^3) = \begin{cases} \delta^\epsilon(x^1)(\mathrm{d}x^1)_c\tilde{f}^i(x^1, x^2, x^3), & x \in \mathcal{R} \\ 0\ , & x \notin \mathcal{R} \end{cases} \tag{6.126}$$

By using (eq:poisson 6.97), we get

$$\{h_e(A), E_S^\epsilon[f]\} = \{h_e(A), E[g]\} = \int_0^1 \mathrm{d}\xi\, g_a^i(e(\xi))\dot{c}^a(\xi) h_{e[0,\xi]}\tau_i h_{e[\xi,1]} \tag{6.127}$$

If e is interact S, the result is non-vanishing and reads

$$\{h_e(A), E_S^\epsilon[f]\} = \int_0^1 \mathrm{d}\xi\, \delta^\epsilon\Big(x^1(e(\xi))\Big)(\mathrm{d}x^1)_a\tilde{f}^i(e(\xi))\dot{e}^a(\xi) h_{e[0,\xi]}\tau_i h_{e[\xi,1]} \tag{6.128}$$

where $x^1(e(\xi))$ is the x^1 coordinate of the point $e(\xi) \in \mathcal{R}$. By definition, $(\mathrm{d}x^1)_a\dot{e}^a(\xi) = \mathrm{d}x^1/\mathrm{d}\xi$. Without loss of generality, we can assume the function $x(\xi)$ along the edge e is monotonous. Then the curve $e(t)$ can be re-parameterized by using x^1, leading to

$$\{h_e(A), E_S(f)\} := \lim_{\epsilon\to 0}\{h_e(A), E_S^\epsilon[f]\} = \lim_{\epsilon\to 0}\int_{x^1[0,1]} \mathrm{sgn}(\mathrm{d}x^1/\mathrm{d}\xi)\delta^\epsilon(x^1)\, \tilde{f}^i(e(x^1))h_{c[0,x^1]}\tau_i h_{c[x^1,1]}\mathrm{d}x^1, \tag{6.129}$$

where $x^1[0,1]$ denotes the interval $[\min\{x^1(e(0)), x^1(e(1)\}, \max\{x^1(e(0)), x^1(e(1)\}]$. Therefore, we have

(1) If e lies on S, then $x(e(0)) = x^1(e(1))$. Hence

$$\{h_e(A), E_S(f)\} = 0. \tag{6.130}$$

(2) If e intersects S at $p = c(t_0)$ which is not an end point of e. Then

$$\{h_e(A), E_S(f)\} = \begin{cases} f^i(p)h_{c[0,t_0]}\tau_i h_{c[t_0,1]}, \text{ if } \mathrm{d}x^1/\mathrm{d}\xi > 0, \text{ i.e. the orientations of } S \text{ and } e \text{ are coincide.} \\ -f^i(p)h_{c[0,t_0]}\tau_i h_{c[t_0,1]}, \text{ if } \mathrm{d}x^1/\mathrm{d}\xi < 0, \text{ i.e. the orientations of } S \text{ and } e \text{ are opposite} \end{cases} \tag{6.131}$$

eq:case2

(3) If e intersects S at $p = c(t_0)$ which is an end point of e. In this case, either $t(0) = 0$ or $t(1) = 0$. Then

$$\{h_e(A), E_S(f)\} = \begin{cases} \frac{1}{2}f^i(p)\tau_i h_e, \text{ if } \mathrm{d}x^1/\mathrm{d}\xi > 0\ \&\ x^1(0) = 0, \text{ i.e. the orientations of } S \text{ and } e \text{ are coincide.} \\ \frac{1}{2}f^i(p)h_e\tau_i, \text{ if } \mathrm{d}x^1/\mathrm{d}\xi > 0\ \&\ x^1(1) = 0, \text{ i.e. the orientations of } S \text{ and } e \text{ are coincide.} \\ -\frac{1}{2}f^i(p)\tau_i h_e, \text{ if } \mathrm{d}x^1/\mathrm{d}\xi < 0\ \&\ x^1(0) = 0, \text{ i.e. the orientations of } S \text{ and } e \text{ are opposite} \\ -\frac{1}{2}f^i(p)h_e\tau_i, \text{ if } \mathrm{d}x^1/\mathrm{d}\xi < 0\ \&\ x^1(1) = 0, \text{ i.e. the orientations of } S \text{ and } e \text{ are opposite} \end{cases} \tag{6.132}$$

eq:case3

It is noticed that the factor 1/2 makes (eq:case3 6.132) and (eq:case2 6.131) compatible.

1. *covariant flux*

Given an edge e, we consider a surface S_e dual to the edge and intersecting e at $p = e(1/2)$. Let t_a be the outwards direction of S_e and t_a is defined to be any covector such that $t_a t^a = 1$. Let $\dot{e}_p^a$ denote the tangent vector of e at p.

The covariant flux operator is defined to be

$$\begin{aligned} p^i_{s,e}(A,E) &:= \frac{-1}{\kappa\beta}\,\mathrm{tr}\left[-2\tau^i \int_{S_e} h(\rho^s_e(\sigma))E^c(\sigma)h(\rho^s_e(\sigma)^{-1})\varepsilon_{cab}\right] \\ p^i_{t,e}(A,E) &:= \frac{1}{\kappa\beta}\,\mathrm{tr}\left[-2\tau^i \int_{S_e} h(\rho^t_e(\sigma))E^c(\sigma)h(\rho^t_e(\sigma)^{-1})\varepsilon_{cab}\right] \end{aligned} \tag{6.133}$$

eq:flux

where $\rho^s_e(x)$ ($\rho^t_e(x)$ resp.) is the path starting at the source (target resp.) of e and traveling along e till $e \cap S_e$, then running in S_e till $x \in S_e$.

Now let us compute the commutation relation between the covariant fluxes and the holonomy. According to the following calculation, we will have

$$\begin{aligned} \{h_e, p^i_{s,e}\} &= -\mathrm{sgn}(t_a\dot{e}^a_p)\tau^i h_e, \\ \{h_e, p^i_{t,e}\} &= \mathrm{sgn}(t_a\dot{e}^a_p)h_e\tau^i, \\ \{p^i_{s,e}, p^j_{s,e}\} &= -\mathrm{sgn}(t_a\dot{e}^a_p)\epsilon_{ijl}p^i_{s,e}, \\ \{p^i_{t,e}, p^j_{t,e}\} &= -\mathrm{sgn}(t_a\dot{e}^a_p)\epsilon_{ijl}p^i_{t,e}, \\ \{p^i_{s,e}, p^j_{t,e}\} &= 0. \end{aligned} \tag{6.134}$$

(1) $p^i_{s,e}$ can be written as the form $E_{S_e}(f^i)$ of (6.101) with eq:surface

$$(f^i)^k(\sigma) \equiv f^{ik}(\sigma) = -\,\mathrm{tr}\left[-2\tau^i h(\rho^s(\sigma))\tau^k h(\rho^s(\sigma))^{-1}\right] \tag{6.135}$$

Hence, the Poisson bracket between h_e and $p^i_{s,e}$ is

$$\begin{aligned} \{h_e, p^i_{s,e}\} =& \mathrm{sgn}(t_a\dot{e}^a_p)f^{ik}(p)h_{e[0,\frac{1}{2}]}\tau_k h_{e[\frac{1}{2},1]} \\ =& -\mathrm{sgn}(t_a\dot{e}^a_p)\,\mathrm{tr}\left[-2\tau^i h(\rho^s(p))\tau^k h(\rho^s(p))^{-1}\right] h_{e[0,\frac{1}{2}]}\tau_k h_{e[\frac{1}{2},1]}. \end{aligned} \tag{6.136}$$

By definition, one has $\rho^s(p) = e[0,\frac{1}{2}]$. Therefore, we have

$$\begin{aligned} \{h_e, p^i_{s,e}\} =& -\mathrm{sgn}(t_a\dot{e}^a_p)\,\mathrm{tr}\left[-2\tau^i h_{e[0,\frac{1}{2}]}\tau^k h^{-1}_{e[0,\frac{1}{2}]}\right] h_{e[0,\frac{1}{2}]}\tau_k h_{e[\frac{1}{2},1]} \\ =& -\mathrm{sgn}(t_a\dot{e}^a_p)\,\mathrm{tr}\left[-2h^{-1}_{e[0,\frac{1}{2}]}\tau^i h_{e[0,\frac{1}{2}]}\tau^k\right] h_{e[0,\frac{1}{2}]}\tau_k h_{e[\frac{1}{2},1]} \\ =& -\mathrm{sgn}(t_a\dot{e}^a_p)h_{e[0,\frac{1}{2}]}h^{-1}_{e[0,\frac{1}{2}]}\tau^i h_{e[0,\frac{1}{2}]}h_{e[\frac{1}{2},1]} \\ =& -\mathrm{sgn}(t_a\dot{e}^a_p)\tau^i h_e, \end{aligned} \tag{6.137}$$

where we used

$$-2\,\mathrm{tr}\big(v_k\tau^k\sigma_i\big)\tau^i = v_i\tau^i \Rightarrow -2\,\mathrm{tr}(\#\tau_i)\tau^i = \#. \tag{6.138}$$

(2) Similarly

$$\begin{aligned} \{h_e(A), p^i_{t,e}\} &= \mathrm{sgn}(t_a\dot{e}^a_p)\,\mathrm{tr}\left[-2\tau^i h_{e[1,\frac{1}{2}]}\tau^k h^{-1}_{e[1,\frac{1}{2}]}\right] h_{e[0,\frac{1}{2}]}\tau_k h_{e[\frac{1}{2},1]} \\ =&\mathrm{sgn}(t_a\dot{e}^a_p)h_{e[0,\frac{1}{2}]}h^{-1}_{e[1,\frac{1}{2}]}\tau^i h_{e[1,\frac{1}{2}]}h_{e[\frac{1}{2},1]} \\ =&\mathrm{sgn}(t_a\dot{e}^a_p)h_e\tau^i. \end{aligned} \tag{6.139}$$

(3) Now let us consider the Poisson bracket between $p^i_{s,e}$ and $p^j_{s,e}$. We have

$$
\begin{aligned}
&\{p^i_{s,e},p^j_{s,e}\}=\{E_{S_e}(f^i),E_{S_e}(f^j)\}\\
=&\int_{S_e}\varepsilon_{cab}E^c_k(\sigma)\{f^{ik}(\sigma),E_{S_e}(f^j)\}+\int_{S_e}\varepsilon_{cab}f^{ik}(\sigma)\{E^c_k(\sigma),E_{S_e}(f^j)\}\\
=&\int_{S_e}\varepsilon_{cab}E^c_k(\sigma)\{f^{ik}(\sigma),E_{S_e}(f^j)\}+\int_{S_e}\varepsilon_{cab}\int_{S_e}\varepsilon_{da'b'}f^{ik}(\sigma)\{E^c_k(\sigma),f^{jl}(\sigma')\}E^d_l(\sigma')\\
=&\int_{S_e}\varepsilon_{cab}E^c_k(\sigma)\{f^{ik}(\sigma),E_{S_e}(f^j)\}+\int_{S_e}\varepsilon_{da'b'}\{E_{S_e}(f^i),f^{jl}(\sigma')\}E^d_l(\sigma')\\
=&\int_{S_e}\varepsilon_{cab}E^c_k(\sigma)\{f^{ik}(\sigma),E_{S_e}(f^j)\}-\int_{S_e}\varepsilon_{da'b'}E^d_l(\sigma')\{f^{jl}(\sigma'),E_{S_e}(f^i)\}\\
=&\int_{S_e}\varepsilon_{cab}E^c_k(\sigma)\{f^{ik}(\sigma),E_{S_e}(f^j)\}-(i\leftrightarrow j)
\end{aligned}
\tag{6.140}
$$

By definition of f^{ik}, we have

$$
\begin{aligned}
&\{f^{ik}(\sigma),E_{S_e}(f^j)\}\\
=&\{-\operatorname{tr}\left[-2\tau^i h(\rho^s(\sigma))\tau^k h(\rho^s(\sigma))^{-1}\right],E_{S_e}(f^j)\}\\
=&-\operatorname{tr}\left[-2\tau^i\{h(\rho^s(\sigma)),E_{S_e}(f^j)\}\tau^k h(\rho^s(\sigma))^{-1}\right]-\operatorname{tr}\left[-2\tau^i h(\rho^s(\sigma))\tau^k\{h(\rho^s(\sigma))^{-1},E_{S_e}(f^j)\}\right]
\end{aligned}
\tag{6.141}
$$

For each $\sigma\in S_e$, the path $\rho^s(\sigma)$ satisfy

$$
\rho^s(\sigma)=e[0,\frac{1}{2}]\circ\mathfrak{p}
\tag{6.142}
$$

with $\mathfrak{p}$ being the part lying in S_e. Therefore, we have

$$
\begin{aligned}
&\{h(\rho^s(\sigma),E_{S_e}(f^j)\}=\{h_{e[0,\frac{1}{2}]},E_{S_e}(f^j)\}h_{\mathfrak{p}}\\
=&\operatorname{sgn}(t_a\dot{e}^a_p)\frac{1}{2}f^{jk}(p)h_{e[0,\frac{1}{2}]}\tau_k=-\operatorname{sgn}(t_a\dot{e}^a_p)\frac{1}{2}\operatorname{tr}\left[-2\tau^j h_{e[0,\frac{1}{2}]}\tau^k h^{-1}_{e[0,\frac{1}{2}]}\right]h_{e[0,\frac{1}{2}]}\tau_k h_{\mathfrak{p}}\\
=&-\operatorname{sgn}(t_a\dot{e}^a_p)\frac{1}{2}h_{e[0,\frac{1}{2}]}h^{-1}_{e[0,\frac{1}{2}]}\tau^j h_{e[0,\frac{1}{2}]}h_{\mathfrak{p}}=-\operatorname{sgn}(t_a\dot{e}^a_p)\frac{1}{2}\tau^j h_{e[0,\frac{1}{2}]}h_{\mathfrak{p}}\\
=&-\operatorname{sgn}(t_a\dot{e}^a_p)\frac{1}{2}\tau^j h(\rho^s(\sigma)).
\end{aligned}
\tag{6.143}
$$

On the other hand, we have

$$
\{h(\rho^s(\sigma))^{-1},E_{S_e}(f^j)\}=-h(\rho^s(\sigma))^{-1}\{h(\rho^s(\sigma)),E_{S_e}(f^j)\}h(\rho^s(\sigma))^{-1}=\operatorname{sgn}(t_a\dot{e}^a_p)\frac{1}{2}h(\rho^s(\sigma))^{-1}\tau^j.
\tag{6.144}
$$

Therefore, one has

$$
\begin{aligned}
\{f^{ik}(\sigma),E_{S_e}(f^j)\}=&\operatorname{tr}\left[\operatorname{sgn}(t_a\dot{e}^a_p)\tau^i\tau^j h(\rho^s(\sigma))\tau^k h(\rho^s(\sigma))^{-1}\right]-\operatorname{tr}\left[\operatorname{sgn}(t_a\dot{e}^a_p)\tau^i h(\rho^s(\sigma))\tau^k h(\rho^s(\sigma))^{-1}\tau^j\right]\\
=&\operatorname{sgn}(t_a\dot{e}^a_p)\operatorname{tr}\left[(\tau^i\tau^j-\tau^j\tau^i)h(\rho^s(\sigma))\tau^k h(\rho^s(\sigma))^{-1}\right]\\
=&\operatorname{sgn}(t_a\dot{e}^a_p)\epsilon_{ijl}\operatorname{tr}\left[\tau^l h(\rho^s(\sigma))\tau^k h(\rho^s(\sigma))^{-1}\right]
\end{aligned}
\tag{6.145}
$$

Then

$$
\begin{aligned}
&\{p^i_{s,e},p^j_{s,e}\}=\int_{S_e}\varepsilon_{abc}E^c_k(\sigma)\{f^{ik}(\sigma),E_{S_e}(f^j)\}-i\leftrightarrow j\\
=&\operatorname{sgn}(t_a\dot{e}^a_p)\epsilon_{ijl}\int_{S_e}E^c_k(\sigma)\operatorname{tr}\left[\tau^l h(\rho^s(\sigma))\tau^k h(\rho^s(\sigma))^{-1}\right]\\
=&-\operatorname{sgn}(t_a\dot{e}^a_p)\epsilon_{ijl}\int_{S_e}E^c_k(\sigma)\operatorname{tr}\left[-2\tau^l h(\rho^s(\sigma))\tau^k h(\rho^s(\sigma))^{-1}\right]\\
=&-\operatorname{sgn}(t_a\dot{e}^a_p)\epsilon_{ijl}p^l_{s,e}.
\end{aligned}
\tag{6.146}
$$

(4) For the Poisson bracket between $p^i_{t,e}$ and $p^j_{t,e}$, let us use the trick

$$\rho^t(\sigma) = e[1,0] \circ \rho^s(\sigma) \tag{6.147}$$

which results in

$$p^i_{t,e} = -2\,\mathrm{tr}\left[\tau^i h_e^{-1} p^j_s \tau_j h_e\right] = -2 p^j_{s,e}\,\mathrm{tr}\left[\tau^i h_e^{-1}\tau_j h_e\right]. \tag{6.148}$$

Therefore, we have

$$\begin{aligned}
&\{p^i_{t,e}, p^j_{t,e}\} = \{2p^k_{s,e}\,\mathrm{tr}\left[\tau^i h_e^{-1}\tau_k h_e\right], 2p^l_{s,e}\,\mathrm{tr}\left[\tau^j h_e^{-1}\tau_l h_e\right]\}\\
=&4p^l_{s,e}\{p^k_{s,e}, \mathrm{tr}\left[\tau^j h_e^{-1}\tau_l h_e\right]\}\,\mathrm{tr}\left[\tau^i h_e^{-1}\tau_k h_e\right] + 4p^k_{s,e}\{\mathrm{tr}\left[\tau^i h_e^{-1}\tau_k h_e\right], p^l_{s,e}\}\,\mathrm{tr}\left[\tau^j h_e^{-1}\tau_l h_e\right]\\
&+ 4\{p^k_{s,e}, p^l_{s,e}\}\,\mathrm{tr}\left[\tau^j h_e^{-1}\tau_l h_e\right]\mathrm{tr}\left[\tau^i h_e^{-1}\tau_k h_e\right]\\
=&4\mathrm{sgn}(t_a\dot{e}^a_p)p^l_{s,e}\,\mathrm{tr}\left[\tau^j h_e^{-1}[\tau_l,\tau_k]h_e\right]\mathrm{tr}\left[\tau^i h_e^{-1}\tau_k h_e\right] - 4\mathrm{sgn}(t_a\dot{e}^a_p)p^k_{s,e}\,\mathrm{tr}\left[\tau^i h_e^{-1}[\tau_k,\tau_l]h_e\right]\mathrm{tr}\left[\tau^j h_e^{-1}\tau_l h_e\right]\\
&+ 4\{p^k_s, p^l_s\}\,\mathrm{tr}\left[\tau^i h_e^{-1}\tau_k(e) h_e\right]\mathrm{tr}\left[\tau^j h_e^{-1}\tau_l(e) h_e\right]\\
=&4\mathrm{sgn}(t_a\dot{e}^a_p)p^l_{s,e}\epsilon_{lkm}\,\mathrm{tr}\left[\tau^j h_e^{-1}\tau_m h_e\right]\mathrm{tr}\left[\tau^i h_e^{-1}\tau_k h_e\right] - 4\mathrm{sgn}(t_a\dot{e}^a_p)p^k_{s,e}\epsilon_{klm}\,\mathrm{tr}\left[\tau^i h_e^{-1}\tau_m h_e\right]\mathrm{tr}\left[\tau^j h_e^{-1}\tau_l h_e\right]\\
&- 4\mathrm{sgn}(t_a\dot{e}^a_p)\epsilon_{klm}p^m_s(e)\,\mathrm{tr}\left[\tau^i h_e^{-1}\tau_k h_e\right]\mathrm{tr}\left[\tau^j h_e^{-1}\tau_l h_e\right]\\
=&4\mathrm{sgn}(t_a\dot{e}^a_p)\epsilon_{lkm}p^l_{s,e}\,\mathrm{tr}\left[\tau^i h_e^{-1}\tau_k h_e\right]\mathrm{tr}\left[\tau^j h_e^{-1}\tau_m h_e\right] - 4\mathrm{sgn}(t_a\dot{e}^a_p)\epsilon_{kml}p^l_{s,e}\,\mathrm{tr}\left[\tau^i h_e^{-1}\tau_k h_e\right]\mathrm{tr}\left[\tau^j h_e^{-1}\tau_m h_e\right]\\
&- 4\mathrm{sgn}(t_a\dot{e}^a_p)\epsilon_{kml}p^l_{s,e}\,\mathrm{tr}\left[\tau^i h_e^{-1}\tau_k h_e\right]\mathrm{tr}\left[\tau^j h_e^{-1}\tau_m h_e\right]\\
=&- 4\mathrm{sgn}(t_a\dot{e}^a_p)p^l_{s,e}\epsilon_{lmk}\,\mathrm{tr}\left[\tau^i h_e^{-1}\tau_m h_e\right]\mathrm{tr}\left[\tau^j h_e^{-1}\tau_k h_e\right]\\
=&- 4\mathrm{sgn}(t_a\dot{e}^a_p)(-2)\,\mathrm{tr}([\tau_m,\tau_k]\tau_l)p^l_{s,e}\,\mathrm{tr}\left[\tau^i h_e^{-1}\tau_m h_e\right]\mathrm{tr}\left[\tau^j h_e^{-1}\tau_k h_e\right]\\
=&2\mathrm{sgn}(t_a\dot{e}^a_p)p^l_s\,\mathrm{tr}\left(\left[\mathrm{tr}\left[-2\tau^i h_e^{-1}\tau_m h_e\right]\tau^m, \mathrm{tr}\left[-2\tau^j h_e^{-1}\tau_k h_e\right]\tau^k\right]\tau_l\right)\\
=&2\mathrm{sgn}(t_a\dot{e}^a_p)p^l_s\,\mathrm{tr}\left(\left[h_e\tau^i h_e^{-1}, h_e\tau^j h_e^{-1}\right]\tau_l\right)\\
=&2\mathrm{sgn}(t_a\dot{e}^a_p)p^l_s\,\mathrm{tr}\left(h_e\tau^k h_e^{-1}\tau_l\right)\epsilon_{ijk}\\
=&2\mathrm{sgn}(t_a\dot{e}^a_p)p^l_s\,\mathrm{tr}\left(\tau^k h_e^{-1}\tau_l h_e\right)\epsilon_{ijk}\\
=&- \mathrm{sgn}(t_a\dot{e}^a_p)\epsilon_{ijk}p^k_t.
\end{aligned} \tag{6.149}$$

(5) Finally, for $p^i_{t,e}$ and $p^j_{s,e}$ we have

$$\begin{aligned}
&\{p^i_{t,e}, p^j_{s,e}\}\\
=&\{-2p^k_{s,e}\,\mathrm{tr}\left[\tau^i h_e^{-1}\tau_k h_e\right], p^j_{s,e}\}\\
=&2\mathrm{sgn}(t_a\dot{e}^a_p)\epsilon^{kjl}p^l_{s,e}\,\mathrm{tr}\left[\tau^i h_e^{-1}\tau_k h_e\right] - 2p^k_{s,e}\{\mathrm{tr}\left[\tau^i h_e^{-1}\tau_k h_e, p^j_{s,e}\right]\}\\
=&2\mathrm{sgn}(t_a\dot{e}^a_p)\epsilon^{kjl}p^l_{s,e}\,\mathrm{tr}\left[\tau^i h_e^{-1}\tau_k h_e\right] + 2\mathrm{sgn}(t_a\dot{e}^a_p)p^k_{s,e}\,\mathrm{tr}\left[\tau^i h_e^{-1}[\tau_k,\tau_j]h_e\right]\\
=&2\mathrm{sgn}(t_a\dot{e}^a_p)\epsilon^{kjl}p^l_{s,e}\,\mathrm{tr}\left[\tau^i h_e^{-1}\tau_k h_e\right] + 2\mathrm{sgn}(t_a\dot{e}^a_p)\epsilon_{kjl}p^k_{s,e}\,\mathrm{tr}\left[\tau^i h_e^{-1}\tau_l h_e\right]\\
=&0
\end{aligned} \tag{6.150}$$

2. *the quantization operator*

Quantization of the flux operator O will be

$$\hat{O} = -i\hbar\{\cdot, O\}, \tag{6.151}$$

so that we have

$$\begin{aligned}
&\hat{O}_1\hat{O}_2\psi = -\hbar^2\{\{\psi, O_2\}, O_1\} = \hbar^2\{\{O_2, O_1\},\psi\} + \hbar^2\{\{O_1,\psi\}, O_2\}\\
=&i\hbar(-i\hbar\{\psi,\{O_1,O_2\}\}) - \hbar^2\{\{\psi, O_1\}, O_2\} = -i\hbar(-i\hbar\{\psi,\{O_1,O_2\}\}) + \hat{O}_2\hat{O}_1\psi,
\end{aligned} \tag{6.152}$$

implying

$$[\hat{O}_1, \hat{O}_2] = i\hbar\widehat{\{O_1, O_2\}}. \tag{6.153}$$

Therefore, we have

eq:operatoraction

$$\begin{aligned}\hat{p}^i_{s,e}\psi(h_e) &= i\mathrm{sgn}(t_a\dot{e}^a_p)\hbar\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\psi(e^{t\tau_i}h_e),\\ \hat{p}^i_{t,e}\psi(h_e) &= i\mathrm{sgn}(t_a\dot{e}^a_p)\hbar\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\psi(h_e e^{-t\tau_i}).\end{aligned} \tag{6.154}$$

The commutation relation is

$$\begin{aligned}[h_e,\hat{p}^j_{s,e}] &= -i\hbar\,\mathrm{sgn}(t_a\dot{e}^a_p)\tau^j h_e\\ [h_e,\hat{p}^j_{t,e}] &= i\hbar\,\mathrm{sgn}(t_a\dot{e}^a_p)h_e\tau^j\\ [\hat{p}^j_{s,e},\hat{p}^k_{s,e}] &= -i\hbar\,\mathrm{sgn}(t_a\dot{e}^a_p)\epsilon^{jk}{}_l\hat{p}^l_{s,e}\\ [\hat{p}^j_{t,e},\hat{p}^k_{t,e}] &= -i\hbar\,\mathrm{sgn}(t_a\dot{e}^a_p)\epsilon^{jk}{}_l\hat{p}^l_{t,e}\end{aligned} \tag{6.155}$$

D. The coherent state

In this section, we do our calculation on the edge e.
Consider the coherent state

$$\psi^t_g(h) = \sum_j d_j e^{-\frac{t}{2}j(j+1)}\sum_m D^j_{mm}(hg^{-1}),\ \ g = e^{-i\vec{p}\cdot\vec{\tau}}u. \tag{6.156}$$

We have

$$\psi^t_g(h) = \sum_j d_j e^{-\frac{t}{2}j(j+1)}\sum_m D^j_{mm}(hu^{-1}e^{i\vec{p}\cdot\vec{\tau}}) \tag{6.157}$$

Let us parametrize $\vec{p}$ as

$$\vec{p} = p(\cos\phi\sin\theta,\sin\phi\sin\theta,\cos\theta) \tag{6.158}$$

so that we define $n(\vec{p}/p)$ such that

$$n(\vec{p}/p)\sigma_3 n(\vec{p}/p)^{-1} = \vec{p}\cdot\vec{\sigma}/p, \tag{6.159}$$

leading to

$$g = e^{-i\vec{p}\cdot\vec{\tau}}u = n(\vec{p}/p)e^{-ip\tau_3}n(\vec{p}/p)^{-1}u. \tag{6.160}$$

Due to $n(\vec{p}/p)^{-1}u \in \mathrm{SU}(2)$, we could parametrize it as

eq:npu

$$n(\vec{p}/p)^{-1}u = e^{\alpha\tau_3}e^{-\gamma\tau_3}e^{-\beta\tau_2}e^{\gamma\tau_3} = e^{\alpha\tau_3}n(\vec{p'}/p)^{-1}, \tag{6.161}$$

where $\vec{p'}$ is given by

$$\vec{p'} := p(\sin\beta\cos\gamma,\sin\beta\sin\gamma,\cos\beta). \tag{6.162}$$

According to (eq:npu 6.161), we get

$$u = n(\vec{p}/p)e^{\alpha\tau_3}n(\vec{p'}/p)^{-1}, \tag{6.163}$$

resulting in

$$u^{-1}(\vec{p}\cdot\vec{\sigma})u = pu^{-1}n(\vec{p}/p)\sigma_3 n(\vec{p}/p)^{-1}u = pn(\vec{p'}/p)\sigma_3 n(\vec{p'}/p)^{-1} = \vec{p'}\cdot\vec{\sigma}. \tag{6.164}$$

With the above notations, we have

$$g = n(\vec{p}/p)e^{(-ip+\alpha)\tau_3}n(\vec{p'}/p)^{-1}. \tag{6.165}$$

Substituting this expression into the coherent state, we get

$$
\begin{aligned}
\psi_g^t(h) =& \sum_j d_j e^{-\frac{t}{2}j(j+1)} \sum_m D^j_{mm}(hg^{-1}) \\
=& \sum_j d_j e^{-\frac{t}{2}j(j+1)} \sum_m D^j_{mm}(hn(\vec{p'})e^{(ip-\alpha)\tau_3}n(\vec{p})^{-1}) \\
=& \sum_j d_j e^{-\frac{t}{2}j(j+1)} \sum_m D^j_{mm}(n(\vec{p})^{-1}hn(\vec{p'})e^{-i(ip-\alpha)\sigma_3/2}) \\
=& \sum_j d_j e^{-\frac{t}{2}j(j+1)} \sum_m D^j_{mm}(n(\vec{p})^{-1}hn(\vec{p'}))e^{-i(ip-\alpha)m} \\
=& \sum_j d_j e^{-\frac{t}{2}j(j+1)} \sum_m D^j_{mm}(n(\vec{p})^{-1}hn(\vec{p'}))e^{(p+i\alpha)m} \\
\cong& \sum_j d_j e^{-\frac{t}{2}j(j+1)+j(|p|+i\mathrm{sgn}(p)\alpha)} D^j_{\mathrm{sgn}(p)j,\mathrm{sgn}(p)j}(n(\vec{p})^{-1}hn(\vec{p'})) \\
=& \exp\left(\frac{(t-2|p|)^2}{8t}\right)\sum_j d_j \exp\left(-\frac{1}{2}t\left(j-\frac{|p|}{t}+\frac{1}{2}\right)^2 + i\alpha\mathrm{sgn}(p)j\right) D^j_{\mathrm{sgn}(p)j,\mathrm{sgn}(p)j}(n(\vec{p})^{-1}hn(\vec{p'}))
\end{aligned}
\tag{6.166}
$$

Now let us consider the expectation values of the operators $p^i_{s,e}$, $p^i_{t,e}$ and h_e. As usual, we do calculation at first for $n(\vec{p}) = n(\vec{p'}) = 1$, i.e., for ψ_{g_o} given by

$$
\psi^t_{g_o} = \sum_j d_j e^{-\frac{t}{2}j(j+1)} \sum_m D^j_{mm}(he^{(ip-\alpha)\tau_3}). \tag{6.167}
$$

As we have known that the expectation values of $\hat{p}^1_{s,e}, \hat{p}^2_{s,e}, \hat{p}^1_{t,e}, \hat{p}^2_{t,e}$ with respect to $\psi^t_{g_o}$ are vanishing. We only need to focus on $\hat{p}^3_{s,e}$ and $\hat{p}^3_{s,e}$.

(1) For $\hat{p}^3_{s,e}$, (eq:operatoraction 6.154) leads to

$$
\begin{aligned}
\hat{p}^3_{s,e}\psi^t_{g_o}(h) =& i\mathrm{sgn}(t_a\dot{e}^a_p)\,\hbar\sum_j d_j e^{-\frac{t}{2}j(j+1)} \sum_m D^j_{mm}(\tau_3 he^{(ip-\alpha)\tau_3}) \\
=& i\mathrm{sgn}(t_a\dot{e}^a_p)\,\hbar\sum_j d_j e^{-\frac{t}{2}j(j+1)} \sum_m D^j_{mm}\left(he^{-i(ip-\alpha)\sigma_3/2}(-i)\frac{\sigma_3}{2}\right) \\
=& \mathrm{sgn}(t_a\dot{e}^a_p)\,\hbar\sum_j d_j e^{-\frac{t}{2}j(j+1)} \sum_m D^j_{mm}(h)e^{-i(ip-\alpha)m}m \\
=& \mathrm{sgn}(t_a\dot{e}^a_p)\,\hbar\sum_j d_j e^{-\frac{t}{2}j(j+1)} \sum_m D^j_{mm}(h)e^{(p+i\alpha)m}m.
\end{aligned}
\tag{6.168}
$$

Therefore, for the expectation value, we have

$$
\begin{aligned}
\langle\psi^t_{g_o}, \hat{p}^3_{s,e}\psi^t_{g_o}\rangle =& \mathrm{sgn}(t_a\dot{e}^a_p)\,\hbar\sum_j d_j^2 e^{-tj(j+1)}\frac{1}{d_j}\sum_m e^{2pm}m \\
=& \mathrm{sgn}(t_a\dot{e}^a_p)\,\hbar\sum_j d_j e^{-tj(j+1)}\sum_m e^{2pm}m \\
=& \mathrm{sgn}(t_a\dot{e}^a_p)\,\hbar\frac{\partial_p}{2}\sum_j d_j e^{-tj(j+1)}\sum_m e^{2pm} \\
\cong& \|\psi^t_{g_o}\|^2\mathrm{sgn}(t_a\dot{e}^a_p)\frac{\hbar}{t}p.
\end{aligned}
\tag{6.169}
$$

(2) For the operator $\hat{p}^3_{t,e}$, we get

$$\begin{aligned}\hat{p}^3_{t,e}\psi^t_{g_o}(h) =& i\,\mathrm{sgn}(t_a\dot{e}^a_p)\hbar\sum_j d_j e^{-\frac{t}{2}j(j+1)}\sum_m D^j_{mm}(-h\tau_3 e^{(ip-\alpha)\tau_3})\\ =& i\,\mathrm{sgn}(t_a\dot{e}^a_p)\hbar\sum_j d_j e^{-\frac{t}{2}j(j+1)}\sum_m D^j_{mm}(hi\frac{\sigma_3}{2}e^{-i(ip-\alpha)\sigma_3/2})\\ =& -\mathrm{sgn}(t_a\dot{e}^a_p)\hbar\sum_j d_j e^{-\frac{t}{2}j(j+1)}\sum_m D^j_{mm}(h)e^{-i(ip-\alpha)m}m\\ =& -\mathrm{sgn}(t_a\dot{e}^a_p)\hbar\sum_j d_j e^{-\frac{t}{2}j(j+1)}\sum_m D^j_{mm}(h)e^{(p+i\alpha)m}m.\end{aligned} \tag{6.170}$$

Thus for the expectation value, we have

$$\begin{aligned}\langle\psi^t_{g_o},\hat{p}^3_{t,e}\psi^t_{g_o}\rangle =& -\mathrm{sgn}(t_a\dot{e}^a_p)\hbar\sum_j d_j^2 e^{-tj(j+1)}\frac{1}{d_j}\sum_m e^{2pm}m\\ =& -\mathrm{sgn}(t_a\dot{e}^a_p)\hbar\sum_j d_j e^{-tj(j+1)}\sum_m e^{2pm}m\\ =& -\mathrm{sgn}(t_a\dot{e}^a_p)\hbar\frac{\partial_p}{2}\sum_j d_j e^{-tj(j+1)}\sum_m e^{2pm}\\ \cong& -\mathrm{sgn}(t_a\dot{e}^a_p)\|\psi^t_{g_o}\|^2\frac{\hbar}{t}p.\end{aligned} \tag{6.171}$$

(3) For the holonomy operator, we have

$$\begin{aligned}D^{\frac12}_{mn}(h)\psi^t_{g_o} =& \sum_j d_j e^{-\frac{t}{2}j(j+1)}\sum_k D^{\frac12}_{mn}(h)D^j_{kk}(he^{(ip-\alpha)\tau_3})\\ =& \sum_j d_j e^{-\frac{t}{2}j(j+1)}\sum_k D^{\frac12}_{mn}(h)D^j_{kk}(h)e^{-i(ip-\alpha)k}\\ =& \sum_j d_j e^{-\frac{t}{2}j(j+1)}\sum_k e^{-i(ip-\alpha)k}\sum_{J=j\pm\frac12} d_J(-1)^{M-N}\begin{pmatrix}\frac12 & j & J\\ m & k & -M\end{pmatrix}\begin{pmatrix}\frac12 & j & J\\ n & k & -N\end{pmatrix}D^J_{MN}(h)\\ =& \sum_j d_j e^{-\frac{t}{2}j(j+1)}\sum_k e^{-i(ip-\alpha)k}\sum_{J=j\pm\frac12} d_J(-1)^{m-n}\begin{pmatrix}\frac12 & j & J\\ m & k & -(m+k)\end{pmatrix}\begin{pmatrix}\frac12 & j & J\\ n & k & -(n+k)\end{pmatrix}D^{j+1/2}_{m+k,n+k}(h)\\ =& \sum_j d_j e^{-\frac{t}{2}j(j+1)}\sum_k e^{-i(ip-\alpha)k}(2j+2)(-1)^{m-n}(-1)^{\frac12+m-j+k}\sqrt{\frac{j+1+2mk}{2(j+1)(2j+1)}}\\ & (-1)^{\frac12+n-j+k}\sqrt{\frac{j+1+2nk}{2(j+1)(2j+1)}}D^{j+1/2}_{m+k,n+k}(h)\\ & +\sum_j d_j e^{-\frac{t}{2}j(j+1)}\sum_k e^{-i(ip-\alpha)k}2j(-1)^{m-n}(-1)^{j-k+1}\sqrt{\frac{j-2mk}{2j(2j+1)}}\\ & (-1)^{j-k+1}\sqrt{\frac{j-2nk}{2j(2j+1)}}D^{j-1/2}_{m+k,n+k}(h)\\ =& \sum_j e^{-\frac{t}{2}j(j+1)}\sum_k e^{(p+i\alpha)k}\sqrt{j+1+2mk}\sqrt{j+1+2nk}D^{j+1/2}_{m+k,n+k}(h)\\ & +\sum_j e^{-\frac{t}{2}j(j+1)}\sum_k e^{(p+i\alpha)k}(-1)^{m-n}\sqrt{j-2mk}\sqrt{j-2nk}D^{j-1/2}_{m+k,n+k}(h)\end{aligned} \tag{6.172}$$

where we used

$$\begin{pmatrix}\frac12 & j & j+\frac12\\ m & k & -k-m\end{pmatrix} = (-1)^{\frac12+m-j+k}\sqrt{\frac{j+1+2mk}{2(j+1)(2j+1)}} \tag{6.173}$$

and

$$\begin{pmatrix} \frac{1}{2} & j & j-\frac{1}{2} \\ m & k & -k-m \end{pmatrix} = (-1)^{j-k+1}\sqrt{\frac{j-2mk}{2j(2j+1)}} \tag{6.174}$$

Then, we have for the expectation value

$$\begin{aligned}
\langle\psi^t_{g_o}, D^{\frac{1}{2}}_{m,n}\psi^t_{g_o}\rangle =&\delta_{mn}\sum_j e^{-\frac{t}{2}(j(j+1)+(j+1/2)(j+3/2))}\sum_{k=-j}^{j} e^{(p+i\alpha)k+(p-i\alpha)(m+k)}(j+1+2mk) \\
&+\delta_{m,n}\sum_j e^{-\frac{t}{2}(j(j+1)+(j-1/2)(j+1/2))}\sum_{k=-j}^{j} e^{(p+i\alpha)k+(m+k)(p-i\alpha)}(j-2mk) \\
=&\delta_{mn}e^{m(p-i\alpha)}\sum_j e^{-\frac{t}{2}(j(j+1)+(j+1/2)(j+3/2))}\sum_{k=-j}^{j} e^{2pk}(j+1+2mk) \\
&+\delta_{m,n}e^{m(p-i\alpha)}\sum_j e^{-\frac{t}{2}(j(j+1)+(j-1/2)(j+1/2))}\sum_{k=-j}^{j} e^{2p}(j-2mk) \\
=&\delta_{mn}e^{m(p-i\alpha)}\sum_{n\geq 1} e^{-\frac{t}{8}\left(2n^2+2n-1\right)}(\frac{n+1}{2}+m\partial_p)\frac{\sinh(np)}{\sinh(p)} \\
&+\delta_{m,n}e^{m(p-i\alpha)}\sum_{n\geq 1} e^{-\frac{t}{8}\left(2n^2-2n-1\right)}(\frac{n-1}{2}-m\partial_p)\frac{\sinh(np)}{\sinh(p)} \\
=&\delta_{m,n}e^{m(p-i\alpha)}\sum_{n\in\mathbb{Z}} e^{-\frac{t}{8}\left(2n^2-2n-1\right)}(\frac{n-1}{2}-m\partial_p)\frac{\sinh(np)}{\sinh(p)} \\
\cong&\delta_{m,n}e^{m(p-i\alpha)}\int \mathrm{d}x e^{-\frac{t}{8}\left(2x^2-2x-1\right)}(\frac{x-1}{2}-mx+m\coth(p))\frac{e^{xp}}{\sinh(p)} \\
&-\delta_{m,n}e^{m(p-i\alpha)}\int \mathrm{d}x e^{-\frac{t}{8}\left(2x^2-2x-1\right)}(\frac{x-1}{2}+mx+m\coth(p))\frac{e^{-xp}}{\sinh(p)} \\
\cong&\|\psi^t_{g_o}\|^2 e^{-im\alpha}\delta_{m,n}
\end{aligned} \tag{6.175}$$

Now we can consider the expectation values with respect to arbitrary state ψ^t_g. We have

$$\begin{aligned}
&\frac{p_k}{p}\hat{p}^k_{s,e}\psi^t_g(h) = i\,\mathrm{sgn}(t_a\dot{e}^a_p)\hbar\sum_j d_j e^{-\frac{t}{2}j(j+1)}\sum_m D^j_{mm}\left(\frac{p^k}{p}\tau_k h n(\vec{p'})e^{(ip-\alpha)\tau_3}n(\vec{p})^{-1}\right) \\
=&i\,\mathrm{sgn}(t_a\dot{e}^a_p)\hbar\sum_j d_j e^{-\frac{t}{2}j(j+1)}\sum_m D^j_{mm}\left(n(\vec{p})^{-1}\frac{p^k}{p}\tau_k n(\vec{p})n(\vec{p})^{-1}hn(\vec{p'})e^{(ip-\alpha)\tau_3}\right) \\
=&i\,\mathrm{sgn}(t_a\dot{e}^a_p)\hbar\sum_j d_j e^{-\frac{t}{2}j(j+1)}\sum_m D^j_{mm}(\tau_3 n(\vec{p})^{-1}hn(\vec{p'})e^{(ip-\alpha)\tau_3}) \\
=&i\,\mathrm{sgn}(t_a\dot{e}^a_p)\hbar\sum_j d_j e^{-\frac{t}{2}j(j+1)}\sum_m D^j_{mm}(n(\vec{p})^{-1}hn(\vec{p'})e^{-i(ip-\alpha)\sigma_3/2}(-i)\frac{\sigma_3}{2}) \\
=&\mathrm{sgn}(t_a\dot{e}^a_p)\hbar\sum_j d_j e^{-\frac{t}{2}j(j+1)}\sum_m D^j_{mm}(n(\vec{p})^{-1}hn(\vec{p'}))e^{-i(ip-\alpha)m}m \\
=&\mathrm{sgn}(t_a\dot{e}^a_p)\hbar\sum_j d_j e^{-\frac{t}{2}j(j+1)}\sum_m D^j_{mm}(n(\vec{p})^{-1}hn(\vec{p'}))e^{(p+i\alpha)m}m.
\end{aligned} \tag{6.176}$$

Due to the invariance properties of the Haar measure in the inner produce, we have

$$\langle\psi^t_g, \frac{p_k}{p}\hat{p}^k_{s,e}\psi^t_g\rangle = \|\psi^t_g\|^2\,\mathrm{sgn}(t_a\dot{e}^a_p)\frac{\hbar}{t}p \tag{6.177}$$

leading to

$$\frac{\langle\psi_g^t, \hat{p}_{s,e}^k \psi_g^t\rangle}{\|\psi_g^t\|^2} = \operatorname{sgn}(t_a \dot{e}_p^a)\frac{\hbar}{t} p^k. \tag{6.178}$$

eq:psexpect

Similarly for $\hat{p}_{t,e}^k$, we have

$$\begin{aligned}
\operatorname{sgn}(t_a \dot{e}_p^a)\frac{p'_k}{\|\vec{p'}\|} p_{t,e}^k \psi_g^t(h) =& \, i\hbar \sum_j d_j e^{-\frac{t}{2}j(j+1)} \sum_m D_{mm}^j(-h\frac{p'^k}{\|\vec{p'}\|}\tau_k n(\vec{p'}) e^{(ip-\alpha)\tau_3} n(\vec{p})^{-1}) \\
=& i\hbar \sum_j d_j e^{-\frac{t}{2}j(j+1)} \sum_m D_{mm}^j(n(\vec{p})^{-1} h n(\vec{p'}) i \frac{\sigma_3}{2} e^{-i(ip-\alpha)\sigma_3/2}) \\
=& -\hbar \sum_j d_j e^{-\frac{t}{2}j(j+1)} \sum_m D_{mm}^j(n(\vec{p})^{-1} h n(\vec{p'})) e^{-i(ip-\alpha)m} m \\
=& -\hbar \sum_j d_j e^{-\frac{t}{2}j(j+1)} \sum_m D_{mm}^j(n(\vec{p})^{-1} h n(\vec{p'})) e^{(p+i\alpha)m} m.
\end{aligned} \tag{6.179}$$

leading to

$$\frac{\langle\psi_g^t, \hat{p}_{t,e}^k \psi_g^t\rangle}{\|\psi_g^t\|^2} \cong -\operatorname{sgn}(t_a \dot{e}_p^a)\frac{\hbar}{t} p'^k. \tag{6.180}$$

For the holonomy operator, we have

$$\begin{aligned}
D_{mn}^{\frac{1}{2}}(n(\vec{p})^{-1} h n(\vec{p'}))\psi_g^t =& \sum_j d_j e^{-\frac{t}{2}j(j+1)} \sum_k D_{mn}^{\frac{1}{2}}(n(\vec{p})^{-1} h n(\vec{p'})) D_{kk}^j(h n(\vec{p'}) e^{(ip-\alpha)\tau_3} n(\vec{p})^{-1}) \\
=& \sum_j e^{-\frac{t}{2}j(j+1)} \sum_k e^{(p+i\alpha)k} \sqrt{j+1+2mk}\sqrt{j+1+2nk} D_{m+k,n+k}^{j+1/2}(n(\vec{p})^{-1} h n(\vec{p'})) \\
&+ \sum_j e^{-\frac{t}{2}j(j+1)} \sum_k e^{(p+i\alpha)k} (-1)^{m-n} \sqrt{j-2mk}\sqrt{j-2nk} D_{m+k,n+k}^{j-1/2}(n(\vec{p})^{-1} h n(\vec{p'})).
\end{aligned} \tag{6.181}$$

Applying the invariance properties of the Haar measure in the inner produce, we have

$$\langle\psi_g^t, D_{mn}^{\frac{1}{2}}(n(\vec{p})^{-1} h n(\vec{p'}))\psi_g^t\rangle = \|\psi_{g_o}^t\|^2 \delta_{mn} e^{-i\alpha m} \tag{6.182}$$

leading to

$$\frac{\langle\psi_g^t | D^{\frac{1}{2}}(h) | \psi_g^t\rangle}{\|\psi_g^t\|^2} = n(\vec{p}) e^{\alpha\tau_3} n(\vec{p'})^{-1}. \tag{6.183}$$

In terms of the expectation values, the $\mathrm{SL}(2,\mathbb{C})$ variable g can be written as

$$g = e^{-i\vec{p}\cdot\vec{\tau}} u = \exp\bigl(-it \operatorname{sgn}(t_a \dot{e}^a)\langle\hat{p}_{s,e}^k\rangle\tau_i\bigr)\langle D^{\frac{1}{2}}(h)\rangle \tag{6.184}$$

By this relation, as illustrated in Fig. fig:geometryVecP 17, $\vec{p}$ has the geometric interpretation as the norm of the surface S taking the orientation opposite to the direction of the edge, independent the value of $\operatorname{sgn}(\dot{e}^a t_a)$.

1. *the holonomy and flux from classical variables*

Let us consider a state (A_a^i, E_i^a) whose corresponding geometry can be approximated by a Regge geometry. As shown in Fig. fig:reggeapproximation 18, we assume that in the geometry there exist a geodesic $\gamma(t)$ along which the extrinsic curvature is $K_{ab} \propto \dot{\gamma}_a \dot{\gamma}_b$.

Let us first consider the the holonomy of the spin connection along the geodesic γ

$$\partial_t G_{\gamma(t)} = G_{\gamma(t)} \Gamma_a(t) \dot{\gamma}^a(t), \; G_{\gamma(0)} = 0. \tag{6.185}$$

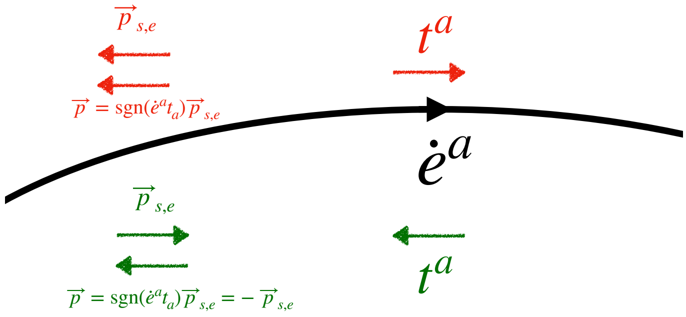


FIG. 17: The geometric interpretation of $\vec{p}$ in the coherent label. We could see that $\vec{p}$ is the always the norm of the surface S taking the orientation opposite to the direction of the edge, independent the value of $\mathrm{sgn}(\dot{e}^a t_a)$. fig:g

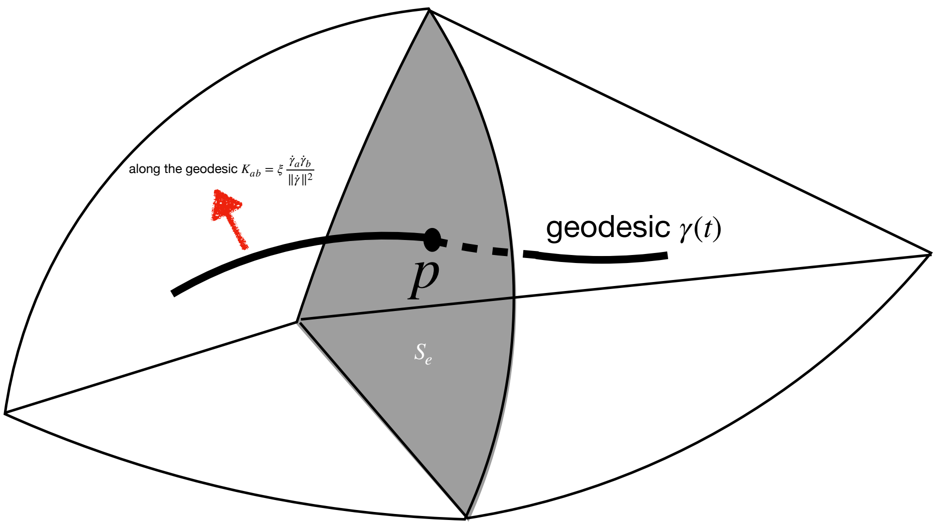


FIG. 18: a region where the geometry can be approximated by a Regge geometry. fig:r

By definition of the spin connection, we have

$$\Gamma^i_{aj} = e^i_b \nabla_a e^b_j. \tag{6.186}$$

Then, we get

$$\begin{aligned}
&\dot{\gamma}^c(t) e^j_c(t) \partial_t (G_{\gamma(t)})^i{}_j = \dot{\gamma}^c(t) e^j_c(t) (G_{\gamma(t)})^i{}_k \Gamma^k_{aj}(t) \dot{\gamma}^a(t) \\
=&(G_{\gamma(t)})^i{}_k e^k_b(t) \dot{\gamma}^a(t) \dot{\gamma}^c(t) e^j_c(t) \nabla_a e^b_j(t) \\
=&-(G_{\gamma(t)})^i{}_k e^k_b(t) \dot{\gamma}^a(t) e^b_j(t) \nabla_a [e^j_c(t) \dot{\gamma}^c(t)] \\
=&-(G_{\gamma(t)})^i{}_j \dot{\gamma}^a(t) \nabla_a [e^j_c(t) \dot{\gamma}^c(t)] \\
=&-(G_{\gamma(t)})^i{}_j \partial_t [e^j_c(t) \dot{\gamma}^c(t)]
\end{aligned} \tag{6.187}$$

where we used $e^j_c \nabla_a e^b_j = -e^b_j \nabla_a e^j_c$, $\dot{\gamma}^a \nabla_a \dot{\gamma}^b = 0$ and should note that ∇_a in the above calculation has no action on the interior index i. We thus get

$$0 = \dot{\gamma}^c(t) e^j_c(t) \partial_t (G_{\gamma(t)})^i{}_j + (G_{\gamma(t)})^i{}_j \partial_t [e^j_c(t) \dot{\gamma}^c(t)] = \partial_t \left[(G_{\gamma(t)})^i{}_j e^j_c(t) \dot{\gamma}^c(t) \right], \tag{6.188}$$

i.e.,

eq:invariantgammadot

$$(G_{\gamma(t)})^i{}_j e^j_c(t) \dot{\gamma}^c(t) = e^j_c(0) \dot{\gamma}^c(0) \equiv \dot{\gamma}^j(0) \Rightarrow G^i{}_j \dot{\gamma}^j(1) = \dot{\gamma}^j(0). \tag{6.189}$$

Let n_s and n_t be given by

eq:nsntgamma

$$n_s \sigma_3 n_s^{-1} = \frac{\dot{\gamma}^i(0)\sigma_i}{\|\dot{\gamma}(0)\|}, \quad n_t \sigma_3 n_t^{-1} = \frac{\dot{\gamma}^i(1)\sigma_i}{\|\dot{\gamma}(1)\|}. \tag{6.190}$$

The curve γ is a geodesic leading that $\|\dot{\gamma}(t)\|$ is preserved. Therefore, in the spin-$\frac{1}{2}$ representation, we get

$$G\frac{(\dot{\gamma}^i(1)\sigma_i)}{\|\dot{\gamma}(1)\|}G^{-1} = \frac{\dot{\gamma}^i(0)\sigma_i}{\|\dot{\gamma}(0)\|} \Rightarrow G n_t \sigma_3 n_t^{-1} G^{-1} = n_s \sigma_3 n_s^{-1} \Rightarrow G = n_s e^{\alpha\tau_3} n_t^{-1} \tag{6.191}$$

for some $\alpha \in \mathbb{R}$.

The holonomy of the Ashtekar variable is given by the equation

$$\partial_t h_{e(t)} = h_{e(t)}(\Gamma + \beta K), \tag{6.192}$$

where we used β to denote the Immirzi parameter. Assuming $h_{e(t)} = V_{e(t)} G_{e(t)}$, we have

eq:equationV

$$\partial_t V_{e(t)} = \beta V_{e(t)} G_{e(t)} K(e(t)) G_{e(t)}^{-1}. \tag{6.193}$$

Let us assume that along the curve γ

eq:assumeK

$$K_{ab}(t) = \frac{\xi(t)}{\|\dot{\gamma}(t)\|^2} n_a(t) n_b(t) \Rightarrow K_a{}^b(t) = \frac{\xi(t)}{\|\dot{\gamma}(t)\|^2} n_a(t)\dot{\gamma}^b(t) \tag{6.194}$$

where we define $n_a(t) = g_{ab}(t)\dot{\gamma}^b(t)$ and the denominator $\|\dot{\gamma}\|^2$ ensures the independence of the parametrization of the curve γ. Indeed, with this assumption, we have

$$\xi(t) = K_{ab}(t)\frac{\dot{\gamma}^a\dot{\gamma}^b}{\|\dot{\gamma}\|^2} = \mathrm{tr}(K(t)). \tag{6.195}$$

We have

$$\dot{\gamma}^a(t) K_a^i(t) = \dot{\gamma}^a(t) K_a{}^b(t) e_b^i(t) = \dot{\gamma}^a(t)\frac{\xi(t)}{\|\dot{\gamma}\|^2} n_a(t)\dot{\gamma}^b(t) e_b^i(t) = \xi(t)\dot{\gamma}^i(t). \tag{6.196}$$

Thus, we get

$$R(G_{e(t)})^i{}_j \left[\dot{\gamma}^a(t) K_a^j(t)\right] = \|\dot{\gamma}(t)\|^2 \frac{\xi(t)}{\|\dot{\gamma}(t)\|^2} R(G_{e(t)})^i{}_j \dot{\gamma}^j(t) = \xi(t)\dot{\gamma}^i(0), \tag{6.197}$$

where in the last step we used (eq:invariantgammadot 6.189). In the spin-$\frac{1}{2}$ representation, this equation becomes

eq:GKGconstant

$$G_{e(t)}\left(\dot{\gamma}^a(t) K_a(t)\right)(G_{e(t)})^{-1} = \xi(t)\dot{\gamma}^i(0)\tau_i. \tag{6.198}$$

Thus, the solution to (eq:equationV 6.193) is

$$V = \exp\left(\beta\left(\int_\gamma \xi(t)\mathrm{d}t\right)\dot{\gamma}^i(0)\tau_i\right). \tag{6.199}$$

Note that the integral $\left(\int_\gamma \xi(t)\mathrm{d}t\right)\dot{\gamma}^i(0)\tau_i$ is independent of the parametrization of γ. Consider a new parametrization $t' = at + b$ where a and b are constant because t and t' are affine parameter. We have $\mathrm{d}t' = a\mathrm{d}t$ while the integration interval for t' becomes $[t'_i, t'_f] = [at_i + b, at_f + b]$. Therefore we have

$$\left(\int_{t_i}^{t_f} \xi(t)\mathrm{d}t\right)\dot{\gamma}^i(0)\tau_i = \left(\int_{t'_i}^{t'_f} \xi(t')\frac{\mathrm{d}t}{\mathrm{d}t'}\mathrm{d}t'\right)\dot{\gamma}^i(0)\tau_i = \left(\int_{t'_i}^{t'_f} \xi(t')\mathrm{d}t'\right)\frac{\mathrm{d}t}{\mathrm{d}t'}\dot{\gamma}^i(0)\tau_i = \left(\int_{t'_i}^{t'_f} \xi(t')\mathrm{d}t'\right)\dot{\gamma}'^i(0)\tau_i \tag{6.200}$$

where $\dot{\gamma}'^i$ is the tangent vector of γ with respect to the new parameter t'. Then, we have

eq:holonomybetaalpha

$$\begin{aligned} h_e = VG &= \exp\left(\beta\left(\int_\gamma \xi(t)\mathrm{d}t\right)\dot{\gamma}^i(0)\tau_i\right) n_s e^{\alpha\tau_3} n_t^{-1} = n_s \exp\left(\beta\left(\int_\gamma \xi(t)\mathrm{d}t\right) n_s^{-1}\dot{\gamma}^i(0)\tau_i n_s\right) e^{\alpha\tau_3} n_t^{-1} \\ &= n_s \exp\left(\left[\beta\left(\int_\gamma \xi(t)\|\dot{\gamma}(0)\|\mathrm{d}t\right) + \alpha\right]\tau_3\right) n_t^{-1} = n_s \exp\left(\left[\beta\left(\int_\gamma \xi(l)\mathrm{d}l\right) + \alpha\right]\tau_3\right) n_t^{-1} \end{aligned} \tag{6.201}$$

where we used (6.190), and $\mathrm{d}l = \|\dot\gamma(t)\|\mathrm{d}t = \|\dot\gamma(0)\|\mathrm{d}t$ due to $\|\dot\gamma(t)\| = \|\dot\gamma(0)\|$ for the affine parameter, with l being the arc length associated to γ.

Now, let us come to the flux $p^i_{t,e}$ and $p^i_{s,e}$ associated with a surface S_e, which intersect γ at x_o. Let $h_{e[0,x_o]}$ denote the holonomy along the segment $[\gamma(0), x_o] \subset \gamma$, and $h_{e[x_o,1]}$, the holonomy along $[x_o, \gamma(1)] \subset \gamma$. Then, by definition of the flux (6.133), we have

$$h^{-1}_{e[0,x_o]} \triangleright p^i_{s,e} = -h_{e[x_o,1]} \triangleright p^i_{t,e} = \frac{-1}{\kappa\beta}\,\mathrm{tr}\left[-2\tau^i \int_{S_e} h(\rho_{x_o}(\sigma)) E^c(\sigma) h(\rho_{x_o}(\sigma)^{-1})\varepsilon_{cab}\right] \tag{6.202}$$

where $\rho_{x_o}(x)$ is the path starting at x_o and running in S_e till $x \in S_e$. For a given S_e, let $p_e \in \mathbb{R}$ be the real number such that

$$(p_e)^2 = \|\vec p_{s,e}\|^2. \tag{6.203}$$

Let t_c denote the unit conormal of S_e at x_o that coincides with it orientation. We could choose S_e properly such that

$$q^{ab}(x_o)t_b \text{ is parallel to } \frac{\dot\gamma^a(x_o)}{\|\dot\gamma(x_o)\|} \tag{6.204}$$

and

$$\frac{-1}{\kappa\beta}\,\mathrm{tr}\left[-2\tau^i \int_{S_e} h(\rho_{x_o}(\sigma)) E^c(\sigma) h(\rho_{x_o}(\sigma)^{-1})\varepsilon_{cab}\right] = -\mathrm{sgn}(e)|p_e|e^a_i t_a = -\mathrm{sgn}(e)\mathrm{sgn}(t_a\dot\gamma^a)|p_e|\frac{\dot\gamma^a(x_o)}{\|\dot\gamma(x_o)\|}. \tag{6.205}$$

We have

$$p^i_{s,e} = -h_{e[0,x_o]} \triangleright \left(\mathrm{sgn}(e)\mathrm{sgn}(t_a\dot\gamma^a)|p_e|\frac{\dot\gamma^i(x_o)}{\|\dot\gamma(x_o)\|}\right) = -\mathrm{sgn}(e)\mathrm{sgn}(t_a\dot\gamma^a)|p_e|\frac{\dot\gamma^i(0)}{\|\dot\gamma(0)\|}, \tag{6.206}$$

and

$$p^i_{t,e} = h^{-1}_{e[x_o,1]} \triangleright \left(\mathrm{sgn}(t_a\dot\gamma^a)\mathrm{sgn}(e)|p_e|\frac{\dot\gamma^i(x_o)}{\|\dot\gamma(x_o)\|}\right) = \mathrm{sgn}(e)|p_e|\mathrm{sgn}(t_a\dot\gamma^a)\frac{\dot\gamma^i(1)}{\|\dot\gamma(1)\|}. \tag{6.207}$$

Let us define $n\Big(\mathrm{sgn}(t_a\dot\gamma^a)\vec p_{s,e}/p_e\Big)$ and $n\Big(-\mathrm{sgn}(t_a\dot\gamma^a)\vec p_{t,e}/p_e\Big) \in \mathrm{SU}(2)$ by

$$\begin{aligned} n\Big(\mathrm{sgn}(t_a\dot\gamma^a)\vec p_{s,e}/p_e\Big)\tau_3 n\Big(\mathrm{sgn}(t_a\dot\gamma^a)\vec p_{s,e}/p_e\Big)^{-1} &= \mathrm{sgn}(t_a\dot\gamma^a)\frac{p^i_{s,e}\tau_i}{p_e},\\ n\Big(-\mathrm{sgn}(t_a\dot\gamma^a)\vec p_{t,e}/p_e\Big)\tau_3 n\Big(-\mathrm{sgn}(t_a\dot\gamma^a)\vec p_{t,e}/p_e\Big)^{-1} &= \mathrm{sgn}(t_a\dot\gamma^a)\frac{-p^i_{t,e}\tau_i}{p_e} \end{aligned} \tag{6.208}$$

where the sign factor $\mathrm{sgn}(t_a\dot\gamma^a)$ is involved because $\mathrm{sgn}(t_a\dot\gamma^a)p^i_{s,e}$ coincides with the parameter in the coherent state label (see (6.178)). Then, we get

$$\begin{aligned} n\Big(\mathrm{sgn}(t_a\dot\gamma^a)\vec p_{s,e}/p_e\Big)\tau_3 n\Big(\mathrm{sgn}(t_a\dot\gamma^a)\vec p_{s,e}/p_e\Big)^{-1} &= -\mathrm{sgn}(e)\mathrm{sgn}(p_e)\frac{\dot\gamma^i(0)\tau_i}{\|\dot\gamma(0)\|},\\ n\Big(-\mathrm{sgn}(t_a\dot\gamma^a)\vec p_{t,e}/p_e\Big)\tau_3 n\Big(-\mathrm{sgn}(t_a\dot\gamma^a)\vec p_{t,e}/p_e\Big)^{-1} &= -\mathrm{sgn}(e)\mathrm{sgn}(p_e)\frac{\dot\gamma(1)^i\tau_i}{\|\dot\gamma(1)\|}. \end{aligned} \tag{6.209}$$

leading to

$$h_e = n\Big(\mathrm{sgn}(t_a\dot\gamma^a)\vec p_{s,e}/p_e\Big)\exp\left(\left[-\mathrm{sgn}(e)\mathrm{sgn}(p_e)\beta\left(\int_\gamma \xi(l)\mathrm{d}l\right) + \tilde\alpha\right]\tau_3\right) n\Big(-\mathrm{sgn}(t_a\dot\gamma^a)\vec p_{t,e}/p_e\Big)^{-1}, \tag{6.210}$$

where $\tilde\alpha$ is given by

$$n_s e^{\alpha\tau_3} n_t^{-1} = n\Big(\mathrm{sgn}(t_a\dot\gamma^a)\vec p_{s,e}/p_e\Big)\exp(\tilde\alpha\tau_3) n\Big(-\mathrm{sgn}(t_a\dot\gamma^a)\vec p_{t,e}/p_e\Big)^{-1}. \tag{6.211}$$

Moreover, we have

$$\int_\gamma \xi(l)\mathrm{d}l = \int_\gamma \mathrm{tr}(K)\mathrm{d}l = \int_\gamma \frac{K_{ab}\dot{\gamma}^a\dot{\gamma}^b}{\|\dot{\gamma}\|^2}\mathrm{d}l. \tag{6.212}$$

Now let us consider a region $\mathcal{R}$ containing the curve γ. Let A be the surface which is transversely passed by γ. Moreover, we assume that the surface A intersects γ at its middle point and take $\dot{\gamma}^a$ as its normal. Then, we have

$$\int_\gamma \xi(l)\mathrm{d}l \cong \frac{\int_{\mathcal{R}} \sqrt{q}\frac{K_{ab}\dot{\gamma}^a\dot{\gamma}^b}{\|\dot{\gamma}\|^2}d^3x}{\int_A \sqrt{q}\dot{\gamma}^a\epsilon_{abc}} \tag{6.213}$$

Let us choose a right hand coordinate $\{x^1, x^2, x^3\}$ such that the curve γ is given by $x^2 = 0 = x^3$ and $x^1 = l$ along the curve γ. Then, we have

$$q = \begin{pmatrix} N^2 + \vec{N}^T h\vec{N} & h\vec{N} \\ \vec{N}^T h & h \end{pmatrix} \tag{6.214}$$

Then, we have

$$\det(q) = \det(h)(N^2 + \vec{N}^T h\vec{N} - \vec{N}^T h h^{-1} h\vec{N}) = N^2 \det(h). \tag{6.215}$$

$$\int_{\Psi[\mathcal{R}]} \sqrt{q}\frac{K_{ab}\dot{\gamma}^a\dot{\gamma}^b}{\|\dot{\gamma}\|^2}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 = \int_\Sigma \mathrm{d}x^2\mathrm{d}x^3 \int_\gamma \sqrt{q}\frac{K_{ab}\dot{\gamma}^a\dot{\gamma}^b}{\|\dot{\gamma}\|^2}\mathrm{d}x^1 \tag{6.216}$$

2. *The integral of the extrinsic curvature along an edge*

Consider the holonomy $h_e(A)$, satisfying the equation

eq:holonomy

$$\partial_t h_{e(t)} = h_{e(t)} A_a(e(t))\dot{e}^a(t),\ \ h_{e(0)} = 1. \tag{6.217}$$

In this convention, $h_{e(t)}$ is the parallel transportation of covector field, i.e., $v_I(t) = v_I(0)(h_{e(t)})^I{}_J$ is the parallel transported field on the curve e, and given a parallel transported field $v^I(t)$, $v^I(0) = (h_{e(t)})^I{}_J v^J(t)$.

By definition, we have

$$A_a = \Gamma_a + \beta K_a \Rightarrow \partial_t h_{e(t)} = h_{e(t)}(\Gamma + \beta K). \tag{6.218}$$

Assuming

$$h_{e(t)} = V_{e(t)} G_{e(t)}$$

where $G_{e(t)}$ is the holonomy of Γ, i.e.

eq:spinholo

$$\partial_t G_{e(t)} = G_{e(t)}\Gamma(e(t)) \tag{6.219}$$

we get

eq:holonomyDivision

$$\partial_t h_{e(t)} = V_{e(t)}\partial_t G_{e(t)} + \partial_t V_{e(t)} G_{e(t)} = V_{e(t)} G_{e(t)}\Gamma(e(t)) + \partial_t V_{e(t)} G_{e(t)}. \tag{6.220}$$

Combining this equation with (eq:holonomy 6.217), we finally get

eq:equV

$$\beta V_{e(t)} G_{e(t)} K(e(t)) G^{-1}_{e(t)} = \partial_t V_{e(t)}. \tag{6.221}$$

Thus we have

$$h_e(A) = \mathcal{P}\exp\Big(\int_e \beta G_{e(t)} K(e(t)) G^{-1}_{e(t)}\Big)\mathcal{P}\exp\Big(\int_e \Gamma(e(t))\Big). \tag{6.222}$$

Now let us investigate the geometric interpretation of $V_{e(t)}$.

Embed our spatial manifold Σ into a spacetime $(M, g_{\mu\nu})$. Let n_μ be the unit conormal field of Σ, i.e., $n_\mu n^\mu = -1$. We have

$$K_{\mu\nu} = q^\sigma_\mu \nabla_\sigma n_\nu \tag{6.223}$$

where $q^\nu_\mu = \delta^\nu_\mu + n_\mu n^\nu$ is the projection projecting tensors to be the spatial ones.

Let $\gamma : t \mapsto \gamma_t \in \Sigma$ be a curve lying in Σ. Considering the parallel transport $g^\mu{}_\nu$ of the Levi-Civita connection $\Gamma^\mu_{\sigma\nu}$, it is given by the equation

$$\partial_t g^\mu{}_\nu(\gamma_t) = g^\mu{}_\alpha(\gamma_t)\Gamma^\alpha_{\sigma\nu}(\gamma_t)\dot{\gamma}^\sigma_t \equiv g^\mu{}_\alpha(\gamma_t)\Gamma^\alpha_{t\nu}(\gamma_t) \tag{6.224}$$

where we employ the convention $\Gamma_t \equiv \dot{\gamma}^\sigma_t \Gamma_\sigma$ for now on Then, for a co-vector field $v_\mu(t) = v_\mu(0) g^\mu{}_\nu(t)$ on γ with $v_\mu(0) \in T^*_{\gamma_0}M$, we have

$$\nabla_t v_\mu = \partial_t v_\mu(t) - \Gamma^\nu_{t\mu} v_\nu(t) = v_\nu(0)\partial_t g^\nu_\mu(t) - \Gamma^\nu_{t\mu} v_\sigma(0) g^\sigma_\nu(t) = 0. \tag{6.225}$$

This equation implies that $v_\mu(t)$ is a parallel transported vector field along the curve γ.

For $(\partial_t)^\mu K_{\mu\nu}(t) \equiv K_{t\nu}(t)$, we have

$$K_{t\nu} = \partial_t n_\nu - \Gamma^\mu_{t\nu} n_\nu = \partial_t n_\nu - n_\mu (g^{-1})^\mu{}_\alpha \partial_t g^\alpha{}_\mu = \partial_t \left(n_\mu (g^{-1})^\mu{}_\alpha \right) g^\alpha{}_\nu \tag{6.226}$$

where (6.224) is applied. Let e^μ_I be a vielbein field in M such that on the hypersurface Σ, $e^\mu_0 = -n^\mu$ and e^μ_i for $i = 1, 2, 3$ are tangent to Σ (time gauge). With e^μ_I, we have the spin connection $\omega^I_\mu{}_J$ defined by

$$\nabla_\mu e^\nu_I =: \omega^J_\mu{}_I e^\nu_J. \tag{6.227}$$

Then, we can define the holonomy $p^I{}_J$ of the spin connection $\omega^I_\mu{}_J$, given by

$$\partial_t p^I{}_J = p^I{}_K \omega^K_t{}_J. \tag{6.228}$$

Proposition 2. *Along the curve γ, we have*

$$e^\nu_J(\gamma_t) = (g^{-1})^\nu{}_\mu(t) p^I{}_J(t) e^\mu_I(\gamma_0). \tag{6.229}$$

Proof. We have

$$\begin{aligned}
&\partial_t \left((g^{-1})^\nu{}_\mu p^I{}_J e^\mu_I(\gamma_0) \right) = \partial_t (g^{-1})^\nu{}_\mu) p^I{}_J e^\mu_I(\gamma_0) + g^{-1})^\nu{}_\mu \partial_t p^I{}_J e^\mu_I(\gamma_0) \\
&= -(g^{-1})^\nu{}_\sigma \partial g^\sigma{}_\alpha (g^{-1})^\alpha{}_\mu p^I{}_J e^\mu_I(\gamma_0) + (g^{-1})^\nu{}_\mu \partial_t p^I{}_J e^\mu_I(\gamma_0) \\
&= -\Gamma^\nu_{t\,\alpha} (g^{-1})^\alpha{}_\mu p^I{}_J e^\mu_I(\gamma_0) + (g^{-1})^\nu{}_\mu p^I{}_K \omega^K_t{}_J e^\mu_I(\gamma_0).
\end{aligned} \tag{6.230}$$

This equation implies that, with $\tilde{e}^\nu_J \equiv (g^{-1})^\nu{}_\mu p^I{}_J e^\mu_I(\gamma_0)$

$$\partial_t \tilde{e}^\nu_J + \Gamma^\nu_{t\sigma} \tilde{e}^\sigma_J - \tilde{e}^\nu_K \omega^K_t{}_J = 0, \quad \text{with the initial data } \tilde{e}^\nu_J(0) = e^\nu_J(\gamma_0). \tag{6.231}$$

We thus complete the proof. □

Contracting the vielbein field and the holonomy p^{-1} with the both sides of (6.226), we have

$$K_{t\nu} e^\nu_I (p^{-1})^I{}_J = \partial_t (n_\nu (g^{-1})^\nu{}_\alpha) g^\alpha{}_\nu e^\nu_I (p^{-1})^I{}_J = \partial_t (n_\nu (g^{-1})^\nu{}_\alpha) e^\alpha_J(\gamma_0) = \partial_t (\tilde{n}_\alpha e^\alpha_J(\gamma_0)). \tag{6.232}$$

with $\tilde{n}_\alpha = n_\nu (g^{-1})^\nu{}_\alpha$. It is useful to recognize the geometric meaning of the right hand side. By definition, $n_\nu (g^{-1})^\nu{}_\alpha = \tilde{n}_\alpha$ is the vector obtained by parallel transporting $n_\mu(\gamma_t)$ to the starting point γ_0 of γ. Then, $\tilde{n}_\alpha e^\alpha_J(\gamma_0)$ is the inner product of $n_\mu(\gamma_t)$ with the veilbein at γ_0, giving the information of the direction of $\tilde{n}$ with respect to $e^\alpha_J(\gamma_0)$. Due to $e^\alpha_0(\gamma_0) = n^\alpha(\gamma_0)$, $|\tilde{n}_\alpha e^\alpha_0(\gamma_0)| = \cosh(\text{the dihedral angle})$.

In the formula (6.232), the holonomy p^{-1} is the one of the 4-D spin connection. However, our objective is to incorporate the 3-D connection and extrinsic curvature. Therefore, we need to reduce the 4-D objects into those 3-D ones. To this end, we introduce the following notation.

We choose the time gauge, i.e., we choose the veilbein field e^ν_I such that

$$n_I := e^\nu_I n_\nu = (1, 0, 0, 0). \tag{6.233}$$

Taking advantage of the projection operator q^I_J

$$q^I_J = \delta^I_J + n^I n_J, \tag{6.234}$$

we get

$$\begin{aligned}
&\omega_\mu{}^I{}_J = \omega_\mu{}^K{}_L \left(q^L_J - n^L n_J\right)\left(q^I_K - n^I n_K\right) = \tilde{\omega}_t{}^I{}_J - \tilde{K}_\mu{}^I{}_J,\\
&\tilde{\omega}_\mu{}^I{}_J := \omega_\mu{}^K{}_L q^L_J q^I_K,\\
&\tilde{K}_\mu{}^I{}_J = \omega_\mu{}^K{}_L \left(q^L_J n^I n_K + n^L n_J q^I_K\right).
\end{aligned} \tag{6.235}$$

One thus has the holonomy p satisfying the equation

$$\partial_t p^I{}_J = p^I{}_K(\tilde{\omega}_t{}^K{}_J - \tilde{K}^K{}_J). \tag{6.236}$$

Playing the same game as in (eq:holonomyDivision 6.220), we get

$$p^I{}_J = p_1{}^I{}_K p_2{}^K{}_J,\ \partial_t p_2{}^I{}_J = p_2{}^I{}_K \tilde{\omega}_t{}^K{}_J, \quad \partial_t p_1{}^I{}_J = -p_1{}^I{}_K (p_2 \tilde{K}_t p_2^{-1})^K{}_J. \tag{6.237}$$

The following propositions will be helpful for our further computation.

Proposition 3. *Given a* $\mathrm{SL}(2,\mathbb{C})$ *holonomy* g *with respect to some* $\mathfrak{sl}(2,\mathbb{C})$ *connection* A*, i.e.,*

$$\partial_t g = g A_a \dot{\gamma}^a, \tag{6.238}$$

the $SO(1,3)$ *correspondence of* g*, denoted by* $g^I{}_J$ *will be the holonomy of the corresponding* $\mathfrak{so}(1,3)$ *connection* $A^I{}_J$*, i.e.*

$$\partial_t g^I{}_J = g^I{}_K A_a{}^K{}_J \dot{\gamma}^a. \tag{6.239}$$

Proof. According to (eq:sl2cso13 1.105) and (eq:sl2ctoso13group 1.107), we have

$$\begin{aligned}
g^I{}_K A_t{}^K{}_J =& \frac{1}{4}\operatorname{tr}\left(\tilde{\sigma}^I g \sigma_K g^\dagger\right)\operatorname{tr}\Big(\tilde{\sigma}^K (A_t\sigma_J + \sigma_J A_t^\dagger)\Big)\\
=& \frac{1}{4}\operatorname{tr}\Big(\operatorname{tr}\left(g^\dagger \tilde{\sigma}^I g \sigma_K\right)\tilde{\sigma}^K (A_t\sigma_J + \sigma_J A_t^\dagger)\Big)\\
=& \frac{1}{2}\operatorname{tr}\Big(g^\dagger \tilde{\sigma}^I g A_t \sigma_J + A_t^\dagger g^\dagger \tilde{\sigma}^I g \sigma_J\Big)\\
=& \frac{1}{2}\operatorname{tr}\left(g^\dagger \tilde{\sigma}^I \partial_t g \sigma_J + \partial_t g^\dagger \tilde{\sigma}^I g \sigma_J\right)\\
=& \frac{1}{2}\operatorname{tr}\left(\partial_t (g^\dagger \tilde{\sigma}^I g \sigma_J)\right)\\
=& \partial_t g^I{}_J.
\end{aligned} \tag{6.240}$$

where we used

$$\frac{1}{2}\operatorname{tr}\left(v_K \tilde{\sigma}^K \sigma_I\right)\tilde{\sigma}^I = v_I \tilde{\sigma}^I \Rightarrow \frac{1}{2}\operatorname{tr}(\#\sigma_I)\tilde{\sigma}^I = \#. \tag{6.241}$$

eq:twosigmatrick

□

This proposition motivate us to introduce the holonomy p_1 and p_2 as follows.
According to (eq:xiIJxi 1.101), the $\mathfrak{sl}(2,\mathbb{C})$ connection $\tilde{\omega}_\mu$ corresponding to $\tilde{\omega}_\mu{}^I{}_J$ is given by

$$\tilde{\omega}_\mu = \frac{1}{4}\tilde{\omega}_\mu^{IJ}\sigma_I \tilde{\sigma}_J. \tag{6.242}$$

By definition of $\tilde{\omega}_\mu{}^I{}_J$, we have

$$\tilde{\omega}_\mu = \frac{1}{4}\tilde{\omega}_\mu^{ij}\sigma_i\tilde{\sigma}_j = \frac{1}{8}\tilde{\omega}_\mu^{ij}[\sigma_i,\sigma_j] = \frac{1}{4}\tilde{\omega}_\mu^{mn}\epsilon_{mn}{}^k(i\sigma_k) \equiv -\frac{1}{2}\tilde{\omega}_\mu^k \tau_k \in \mathfrak{su}(2), \tag{6.243}$$

resulting in that the holonomy p_2 is purely SU(2). Indeed, p_2 is just holonomy $G_{e(t)}$ given in (eq:spinholo 6.219), as a consequence of the following fact

eq:identifyp2G
$$\dot{\gamma}^\mu(t)\tilde{\omega}_\mu = -\frac{1}{2}\dot{\gamma}^\mu(t)\omega_\mu^{mn}\epsilon_{mn}{}^k\tau_k = -\frac{1}{2}\dot{\gamma}^a(t)\Gamma_a^{mn}\epsilon_{mn}{}^k\tau_k = \dot{\gamma}^a(t)\Gamma_a^k\tau_k. \tag{6.244}$$

For $\tilde{K}^I_{\mu\,J}$, we have its correspondence

eq:tilldeK
$$\begin{aligned}\tilde{K}_\mu =& \frac{1}{4}\tilde{K}_\mu^{IJ}\sigma_I\tilde{\sigma}_J = \frac{1}{4}\omega_\mu^K{}_L(q^{LJ}n^In_K + n^Ln^Jq_K^I)\sigma_I\tilde{\sigma}_J\\ =&\frac{1}{4}\omega_\mu^{KL}q_L^Jn^In_K\sigma_I\tilde{\sigma}_J + \omega_\mu^{KL}n_Ln^Jq_K^I\sigma_I\tilde{\sigma}_J)\\ =&\frac{1}{4}\left(-\omega_\mu^{0l}\sigma_0\tilde{\sigma}_l - \omega_\mu^{k0}\sigma_k\tilde{\sigma}_0\right)\\ =&\frac{1}{2}\omega_\mu^{l0}\sigma_l.\end{aligned} \tag{6.245}$$

where we used $n^I = \eta^{IJ}n_J = (-1,0,0,0)^T$, $\omega_\mu^{0l} = -\omega_\mu^{l0}$ and $\sigma_0 = \mathbb{1}_2$. Thus, the SL$(2,\mathbb{C})$ holonomy p_1 is given by the equation

eq:p1
$$\partial_t p_1 = -p_1(p_2\tilde{K}_tp_2^{-1}) = -p_1\left[p_2\left(\frac{1}{2}\omega_t^{l0}\sigma_l\right)p_2^{-1}\right] \tag{6.246}$$

Now let us come to the left hand side of (eq:KI4D 6.232), we have

$$K_{t\nu}e_I^\nu = -(\partial_t)^\mu n_\nu\nabla_\mu e_I^\nu = -n_\nu\omega_t^J{}_Ie_J^\nu = -\omega_t^0{}_I \tag{6.247}$$

Then, we get

$$\begin{aligned}K_{t\nu}e_I^\nu(p^{-1})^I{}_J =& -\omega_t^0{}_I(p_2^{-1})^I{}_K(p_1^{-1})^K{}_J = -\frac{1}{2}\operatorname{tr}\left(\omega_t^0{}_I\tilde{\sigma}^Ip_2^{-1}\sigma_Kp_2(p_1^{-1})^K{}_J\right)\\ =&-\frac{1}{2}\operatorname{tr}\left(p_2\omega_t^0{}_I\tilde{\sigma}^Ip_2^{-1}p_1^{-1}\sigma_Jp_1^{-1\dagger}\right)\\ =&-\frac{1}{2}\operatorname{tr}\left(p_1^{-1\dagger}p_2\omega_t^{0I}\tilde{\sigma}_Ip_2^{-1}p_1^{-1}\sigma_J\right)\\ =&\frac{1}{2}\operatorname{tr}\left(p_1^{-1\dagger}p_2\omega_t^{i0}\sigma_ip_2^{-1}p_1^{-1}\sigma_J\right)\end{aligned} \tag{6.248}$$

where we used (eq:sl2ctoso13group 1.107), $p_2\in$ SU(2), (eq:gsigma 1.108) and $\omega_t^{00} = 0$. Applying (eq:twosigmatrick 6.241), we get

eq:kesigmafinal
$$K_{t\nu}e_I^\nu(p^{-1})^I{}_J\tilde{\sigma}^J = p_1^{-1\dagger}p_2\omega_t^{i0}\sigma_ip_2^{-1}p_1^{-1} = \partial_t(p_1^{-1\dagger}p_1^{-1}), \tag{6.249}$$

where we used the following consequence of (eq:p1 6.246)

$$\begin{aligned}\partial_t(p_1^{-1\dagger}p_1^{-1}) =& -\partial_tp_1^{-1\dagger}p_1^{-1} + p_1^{-1\dagger}\partial_tp_1^{-1}\\ =&p_1^{-1\dagger}\partial_tp_1^\dagger p_1^{-1\dagger}p_1^{-1} - p_1^{-1\dagger}p_1^{-1}\partial_tp_1p_1^{-1}\\ =&p_1^{-1\dagger}\left[p_2\left(\frac{1}{2}\omega_t^{l0}\sigma_l\right)p_2^{-1}\right]p_1^{-1} - p_1^{-1\dagger}\left[p_2\left(\frac{1}{2}\omega_t^{l0}\sigma_l\right)p_2^{-1}\right]p_1^{-1}\\ =&p_1^{-1\dagger}\left[p_2(\omega_t^{l0}\sigma_l)p_2^{-1}\right]p_1^{-1}.\end{aligned} \tag{6.250}$$

Combining the result (eq:kesigmafinal 6.249) with (eq:KI4D 6.232), we get

$$\partial_t(p_1^{-1\dagger}p_1^{-1}) = \partial_t(\tilde{n}_\alpha e_J^\alpha(\gamma_0)\tilde{\sigma}^J) \Rightarrow p_1^{-1\dagger}p_1^{-1} = \tilde{n}_\alpha e_J^\alpha(\gamma_0)\tilde{\sigma}^J + \text{constant} \tag{6.251}$$

where the constant is determined by the initial condition. Since $(p_1^{-1\dagger}p_1^{-1})\restriction\gamma_0 = \mathbb{1}_2$ and $(\tilde{n}_\alpha e_J^\alpha(\gamma_0)\tilde{\sigma}^J)\restriction\gamma_0 = n_\alpha(\gamma_0)e_J^\alpha(\gamma_0)\tilde{\sigma}^J = n_\alpha(\gamma_0)(-n^\alpha(\gamma_0))\tilde{\sigma}^0 = \mathbb{1}_2$, we have

$$p_1^{-1\dagger}p_1^{-1} = \tilde{n}_\alpha e_J^\alpha(\gamma_0)\tilde{\sigma}^J \equiv \tilde{n}_J\tilde{\sigma}^J. \tag{6.252}$$

leading to

$$p_1 p_1^\dagger = -\tilde{n}_J \sigma^J = -\tilde{n}^J \sigma_J \Leftrightarrow p_1 n^I \sigma_I p_1^\dagger = \tilde{n}^J \sigma_J. \tag{6.253}$$

where we used

$$-\tilde{n}_J \sigma^J \tilde{n}_I \tilde{\sigma}^I = -\tilde{n}_{(I} \tilde{n}_{J)} \sigma^J \tilde{\sigma}^I = -\tilde{n}_I \tilde{n}_J \eta^{IJ} \mathbb{1} = \mathbb{1} \tag{6.254}$$

Since $p_1 p_1^\dagger$ for $p_1 \in \mathrm{SL}(2,\mathbb{C})$ is purely boost, it takes the form

$$\exp\left(\theta \frac{x^i \sigma_i}{\sqrt{x^i x_i}}\right) = \cosh(\theta)\sigma_0 + \sinh(\theta) \frac{x^k \sigma_k}{\sqrt{x^i x_i}}, \tag{6.255}$$

for some x^i. Its being equal to $-\tilde{n}^J \sigma_J$ implies

$$\tilde{n}^0 < 0. \tag{6.256}$$

In order to make the right hand side of (6.255) equal to $-n^I \sigma_I$, let us consider the setting with $x^i = \tilde{n}^i$ and $\theta = \pm\mathrm{arccosh}(|\tilde{n}^0|)$ in (6.255), to get

$$\exp\left(\pm\mathrm{arccosh}(|\tilde{n}^0|) \frac{\tilde{n}^i \sigma_i}{\sqrt{\tilde{n}^i \tilde{n}_i}}\right) = |\tilde{n}^0|\sigma_0 \pm \sqrt{\tilde{n}_0^2 - 1} \frac{\tilde{n}^k \sigma_k}{\sqrt{\tilde{n}^i \tilde{n}_i}} = |\tilde{n}^0|\sigma_0 \pm \tilde{n}^k \sigma_k. \tag{6.257}$$

Considering $\tilde{n}^0 < 0$, we get

$$\exp\left(-\mathrm{arccosh}(-\tilde{n}^0) \frac{\tilde{n}^i \sigma_i}{\sqrt{\tilde{n}^i \tilde{n}_i}}\right) = -\tilde{n}^0 \sigma_0 - \tilde{n}^k \sigma_k = -\tilde{n}^I \sigma_I. \tag{6.258}$$

We thus get

$$p_1 p_1^\dagger = \exp\left(-\mathrm{arccosh}(-\tilde{n}^0) \frac{\tilde{n}^i \sigma_i}{\sqrt{\tilde{n}^i \tilde{n}_i}}\right) = \exp\left(-\mathrm{arccosh}(-n_I \tilde{n}^J) \frac{\tilde{n}^i \sigma_i}{\sqrt{\tilde{n}^i \tilde{n}_i}}\right) \tag{6.259}$$

This implies

$$p_1 = \exp\left(-\frac{1}{2}\mathrm{arccosh}(-n_I \tilde{n}^I) \frac{\tilde{n}^i \sigma_i}{\sqrt{\tilde{n}^i \tilde{n}_i}}\right) u = \exp\left(-\frac{1}{2}\mathrm{arccosh}(-n_\alpha \tilde{n}^\alpha) \frac{\tilde{n}^i \sigma_i}{\sqrt{\tilde{n}^i \tilde{n}_i}}\right) u \tag{6.260}$$

with u being an element in SU(2).

To relate p_1 to $V_{e(t)}$ considered in (6.221), we need the relation between $\tilde{K}_\mu \dot{\gamma}^\mu(t)$ in (6.245) and the extrinsic curvature $K_a^i \dot{\gamma}^a(t)$

$$\dot{\gamma}^\mu(t) \tilde{K}_\mu = \frac{1}{2} \dot{\gamma}^\mu(t) \omega_\mu^{l0} \sigma_l = i \dot{\ell}^a(t) K_a^l \tau_l. \tag{6.261}$$

Since we have identified p_2 with the holonomy $G_{e(t)}$ in (6.244), the equation for p_1 now becomes

$$\partial_t p_1(\gamma(t)) = -i p_1(\gamma(t)) G_{\gamma(t)} K_a(\gamma(t)) \dot{\gamma}(t)^a G_{\gamma(t)}^{-1}. \tag{6.262}$$

Focusing on the special case considered in (6.194) and applying (6.198), we could get

$$p_1(\gamma(t)) = \exp\left(-\left(\int_{\gamma[0,t]} \xi(s) \mathrm{d}s\right) \frac{\dot{\gamma}^a(0) e_a^i(\gamma(0))}{\|\dot{\gamma}(0)\|} \frac{\sigma_i}{2}\right), \tag{6.263}$$

where $\gamma[0,t] \subset \gamma$ denotes the segment of γ with the parameter taking values in $[0,t]$. Comparing this result with (6.260), we get $u = 1$ and

$$\int_{\gamma[0,t]} \xi(s) \mathrm{d}s = \mathrm{arccosh}(-n_\alpha(\gamma(0)) \tilde{n}^\alpha(t)), \quad \frac{\dot{\gamma}^a(0) e_a^i(\gamma_0)}{\|\dot{\gamma}(0)\|} = \frac{\tilde{n}^\alpha(t) e_\alpha^i(\gamma(0))}{\|\tilde{n}^\alpha(t) e_\alpha^i(\gamma(0))\|},$$

or

$$\int_{\gamma[0,t]} \xi(s) \mathrm{d}s = -\mathrm{arccosh}(-n_\alpha(\gamma(0)) \tilde{n}^\alpha(t)), \quad \frac{\dot{\gamma}^a(0) e_a^i(\gamma(0))}{\|\dot{\gamma}(0)\|} = -\frac{\tilde{n}^\alpha(t) e_\alpha^i(\gamma(0))}{\|\tilde{n}^\alpha(t) e_\alpha^i(\gamma(0))\|}. \tag{6.264}$$

Here we considered the fact that the vectors involved are unit vectors. By definition of the dihedral angle (3.142), (6.264) means that $\int_\gamma \xi(s)\mathrm{d}s$ is the signed dihedral angle between $n_\alpha(\gamma(0))$ and $\tilde{n}^\alpha(1)$, i.e.,

$$\int_\gamma \xi(s)\mathrm{d}s = \mp\left(-\operatorname{arccosh}(-n_\alpha(\gamma(0))\tilde{n}^\alpha(1))\right)$$

where the sign depends on whether the hypersurface Σ is 'convex', i.e., $\dot{\gamma}^a(0)e^i_a(\gamma(0)) \propto \tilde{n}^\alpha(t)e^i_\alpha(\gamma(0))$, or not convex, i.e., $\dot{\gamma}^a(0)e^i_a(\gamma(0)) \propto -\tilde{n}^\alpha(t)e^i_\alpha(\gamma(0))$. In Regge calculus, the angle $-\operatorname{arccosh}(-n_\alpha(\gamma(0))\tilde{n}^\alpha(1))$ becomes the dihedral angle between the 4-D normals of the two adjacent tetrahedra dual to the points $\gamma(0)$ and $\gamma(1)$. Using Θ_ℓ to denote this angle, we have

$$\int_\gamma \xi(s)\mathrm{d}s = \begin{cases} -\Theta_\gamma, & \Sigma \text{ is convex,} \\ \Theta_\gamma, & \Sigma \text{ is not convex.} \end{cases} \tag{6.265}$$