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Disc. 1D
Fri. 10am - 12pm
Song

1.

- Algorithm: // Assuming a root node is given and adj. list representation
    - Initialize an empty tree *T*
    - Initialize a list *L* and iterator variable *i* as 0
    - Take the root node, mark it as discovered, and add it to L[*i*]
    - Add the root node to *T*
    - While L[*i*] is not empty:
        - Initialize L[*i* + 1]
        - For all nodes in L[*i*]:
            - Search each edge (*x*, *y*)
            - If *y* is undiscovered:
                - Set *y* as discovered
                - Add *y* to L[*i* + 1]
                - Add *y* to *T*
                - Add the edge (*x*, *y*) to *T*
            - Else:
                - Ignore *y*
        - Increment *i* by 1
- Proof: // Assuming proof that levels = distance in BFS tree
    - Assuming our BFS tree contains a node who's level doesn't reflect its shortest distance, there must be some node *x* on level *i* who's shortest distance is *j*, such that $j < i$
    - For this to occur there must be some path in the original graph that reaches *x* in *j* edges
    - Due to a how BFS searches all neighbors of each node, we know that for each step *k* in the "shortest path", the node at that step will be found at level *k* or earlier in the BFS tree
    - As a result, we know that the step *j* where *x* implies that *x* will be found at level *j* or earlier in the BFS tree
    - This contradicts the assumption that $j < i$, as this result shows that $j >= i$
- Time Complexity:
    - For each node in the given graph, the algorithm will analyze each of its neighbors
        - Since we are analyzing each node, we automatically have an O(n) runtime at minimum
        - In the worst case, each node has to search n - 1 nodes, but that's overcounting, so instead we can say that, throughout the course of the algorithm's execution, we explore every edge once, adding O(e) to our runtime
    - Our overall runtime for BFS is O(n + e), where n is the number of nodes and e is the number of edges

2.

- Algorithm:
    - Initialize a set of lists *s* with 1 list, which contains the first interval
    - Set a counter variable *i* at 1
    - Delete the first interval
    - While there are intervals to check:
        - Take the start time of the next interval
        - For all lists in *s*:
            - If the end time of the last interval of the current list is less than or equal to the start time of the current interval:
                - Add the current interval to the list
                - Delete the current interval
        - If the current interval is not yet deleted:
            - Initialize a new element of *s*, *s*[*i* + 1]
            - Add the current interval to *s*[*i* + 1]
            - Increment *i* by 1
    - Return *i*
- Time Complexity:
    - This algorithm will take each interval and perform operations on it, resulting in an O(n) runtime at least
    - The for loop in the algorithm will run through all existing lists to check if the interval can fit into any of them
        - In the worst case, this will take O(n) runtime
    - The creation of a new list will take constant time
    - As a result, this algorithm has a worst case runtime complexity of $O(n^2)$, but it is likely to be less, unless the number of processors needed is similar to the number of total intervals
- Proof:
    - In order for this algorithm to break, one of 2 things must occur
    - We undercount:
        - For this to happen, an interval must have been assigned to a list where it cannot actually exist
        - Since we check the end times and start times each time we add an interval to the list, this error is simply impossible
    - We overcount:
        - For this to happen, the lists must have created a new list when a new list wasn't needed
        - This can only occur if the algorithm decided that an interval that could be added to a list wasn't
        - However, we iterate through each list and check start times and end times, therefore each relevant end time is checked
        - It is impossible for our algorithm to skip a valid insertion for our interval
    - Proof by contradiction

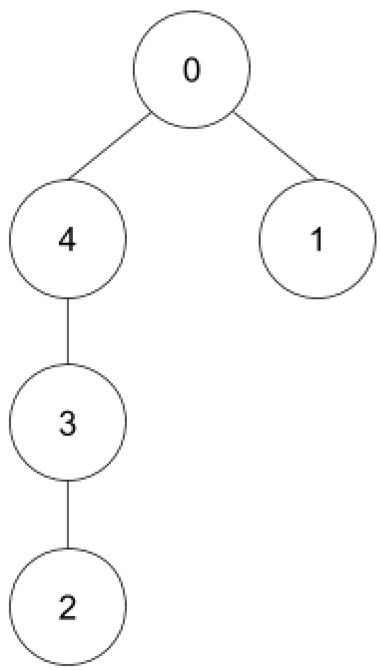
3.

- Algorithm:
  - Initialize an empty tree *T*
  - Initialize a list *L* and iterator variable *i* as 0
  - Take an arbitrary node, mark it as discovered, add it to L[*i*], and color the vertex color 1
  - Set the color *c* to color 2
  - While L[i] is not empty:
    - Initialize L[i +1]
    - For all nodes in level i:
      - Search each edge (x, y)
      - If y is undiscovered:
        - Mark it as discovered
        - Add y to L[i + 1]
      - Else:
        - If y is not the same color as c:
          - Return false
      - Mark the edge (x, y) as discovered
    - Change the color c to the other color
    - Increment i by 1
  - Return the graph
- Proof:
  - The algorithm is essentially constructing a BFS tree, with some added operations
  - We know by the definition of a BFS tree that if we guarantee there are no odd cycles in the graph, we can 2 color the graph
    - This is because odd cycles cannot be bipartite, and are therefore impossible to 2 color
    - We also know this because in a BFS tree, we can simply alternate coloring each level of the tree
  - The algorithm attempts to construct a BFS tree, alternating color assignments per level
  - Base Case:
    - There is only one node
    - The algorithm colors it appropriately
    - Base case is correct behavior
  - Induction: when it discovers a new node, there are 2 possibilities:
    - The node hasn't been discovered yet, which means it will be on the next level
      - By the definitions we laid out above, this node can then be colored appropriately
    - The node has been discovered and colored
      - From here there are 2 possibilities:
        - The node is the right color

            - This simply means we've discovered an even cycle, and since it's the right color, we're fine
        - The node is the wrong color
            - This means we've discovered an odd cycle, and since you can't 2 color an odd cycle, we return false from the algorithm
    - All of these possibilities exhibit the correct behavior, and therefore the algorithm will work for all general cases
- Time Complexity:
    - For each node in the given graph, the algorithm will analyze each of its neighbors
        - Since we are analyzing each node, we automatically have an O(n) runtime at minimum
        - In the worst case, each node has to search n - 1 nodes, but that's overcounting, so instead we can say that, throughout the course of the algorithm's execution, we explore every edge once, adding O(e) to our runtime
        - For each of these nodes, color assignments and checks are performed, which are simply constant time operations
    - Our overall runtime for 2 coloring is O(n + e) like BFS, where n is the number of nodes and e is the number of edges

4.

- Steps:
  - Stack *S* is initialized
  - 0 pushed onto stack, *S* = {0}
  - 0 discovers 4 and pushes it onto stack, *S* = {4, 0}
  - 4 discovers 3 and pushes it onto stack, *S* = {3, 4, 0}
  - 3 discovers 2 and pushes it onto stack, *S* = {2, 3, 4, 0}
  - 2 has no more children to discover, 2 is popped from stack, *S* = {3, 4, 0}
  - 3 has no more children to discover, 3 is popped from stack, *S* = {4, 0}
  - 4 has no more children to discover, 4 is popped from stack, *S* = {0}
  - 0 attempts to discover 3, 3 has already been discovered, *S* = {0}
  - 0 discovers 1 and pushes it onto stack, *S* = {1, 0}
  - 1 attempts to discover 2, 2 has already been discovered, *S* = {1, 0}
  - 1 attempts to discover 3, 3 has already been discovered, *S* = {1, 0}
  - 1 has no more children to discover, 1 is popped from stack, *S* = {0}
  - 0 has no more children to discover, 0 is popped from stack, *S* = {}
  - Algorithm terminates
- Final tree:



5.

- Algorithm:
    - Initialize a max and min variable to the first element of L
    - Initialize an iterator to *i*
    - While *i* + 1 < size of L:
        - If L[i] > L[i + 1]:
            - Set variable local max to L[i]
            - Set variable local min to L[i + 1]
        - Else:
            - Set variable local min to L[i]
            - Set variable local max to L[i + 1]
        - Set max to maximum of local max and current value
        - Set min to minimum of local min and current value
        - Increment i by 2
    - If i doesn't equal the size of L:
        - Set max to maximum of L[i] and current value
        - Set min to minimum of L[i] and current value
    - Return max and min
- Proof:
    - Base Case:
        - An array of size 1:
            - The 1 element is both the maximum and minimum
            - The first element is assigned as both max and min
            - The correct answer is returned
        - Base case correct
    - Inductive Step:
        - Assume the algorithm works for an array of size n
        - Prove it works for an array of size n + 1
        - 2 cases:
            - n + 1 is even:
                - This means that n was an odd value
                - Since the first value is handled separately, the loop handles an odd number of values
                - Assuming the first n values are handled correctly, the n + 1st value is handled by the if statement outside the loop, which is a brute force comparison to the global max and min
                - If the n + 1st value should be a max or min, it will be assigned as such
            - n + 1 is odd:
                - This means that n was an even value
                - Since the first value is handled separately, the loop handles an even number of values

        - For each pair of values, we perform a comparison to determine a local maximum and minimum
        - It is impossible for a local maximum to be a global minimum and vice versa
        - As a result, only the local maximum needs to be compared to the global maximum, and the same applies for the minimums
        - At this point, we effectively have a brute force comparison to the global max and min
        - If the n + 1st value should be a max or min, it will be assigned as such
    - Proof by induction
- Time Complexity:
    - For each 2 elements, a simple approach would make 4 comparisons: both elements would be compared to the global maximum and minimum
    - By first comparing the 2 elements once, we know one is greater than the other, invalidating one for maximum consideration and one from minimum consideration
        - This allows us to gain the information of 2 comparisons in 1 comparison
    - We then follow by performing a single comparison to the global max and a single comparison to the global min
    - As a result, for every 2 elements, our method uses 3 comparisons, resulting in approximately 3n/2 total comparisons