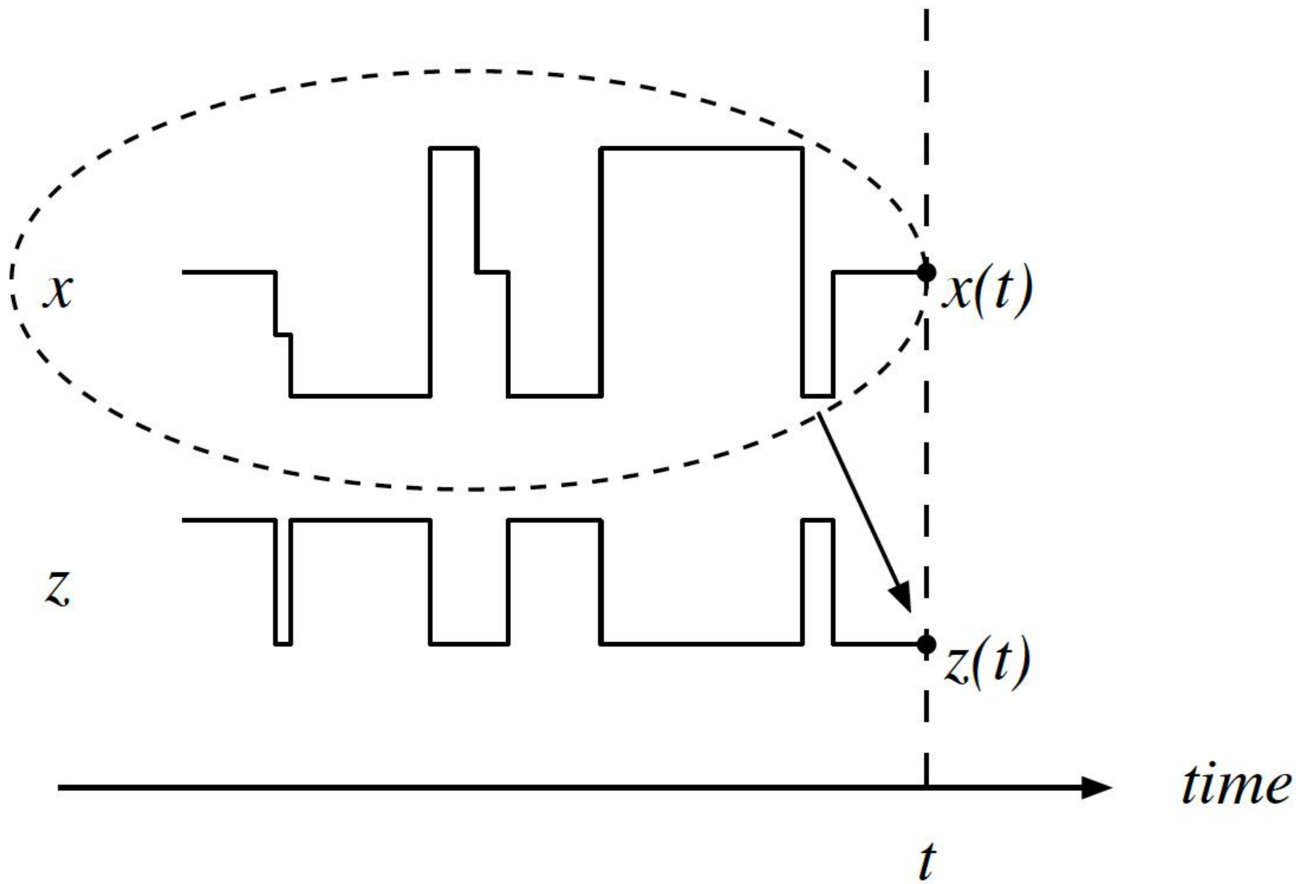
CS M51A
Logic Design of Digital Systems
Winter 2021

Some slides borrowed and modified from:
M.D. Ercegovac, T. Lang and J. Moreno, Introduction to Digital Systems.

SEQUENTIAL SYSTEMS

DEFINITION

$$z(t) = F(x(0,t))$$



Example ODD/EVEN

TIME-BEHAVIOR SPECIFICATION:

Input: $x(t) \in \{a, b\}$
Output: $z(t) \in \{0, 1\}$

Function: $z(t) = \begin{cases} 1 & \textbf{if} \quad x(0,t) \text{ contains an even number of } b'\text{s} \\ 0 & \textbf{otherwise} \end{cases}$

I/O SEQUENCE:

t	0	1	2	3	4	5	6	7
x, z	$a, 1$	$b,$	$b,$	$a,$	$b,$	$a,$	$b,$	$a,$

STATE DESCRIPTION OF ODD/EVEN

t	0	1	2	3	4	5	6	7
x, z	$a, 1$	$b, 0$	$b, 1$	$a, 1$	$b, 0$	$a, 0$	$b, 1$	$a, 1$

Input: $x(t) \in \{a, b\}$

Output: $z(t) \in \{0, 1\}$

State: $s(t) \in \{\text{EVEN, ODD}\}$

Initial state: s(0)$=$ EVEN

Functions: Transition and output functions

	$NS,\ z(t)$	
PS	$x(t) = a$	$x(t) = b$
EVEN	EVEN, 1	ODD, 0
ODD	ODD, 0	EVEN, 1

STATE DESCRIPTION OF ODD/EVEN

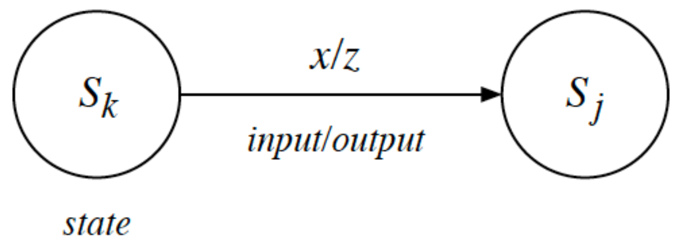
Input: $x(t) \in \{a, b\}$
Output: $z(t) \in \{0, 1\}$
State: $s(t) \in \{\text{EVEN, ODD}\}$
Initial state: s(0)= EVEN

Functions: Transition and output functions

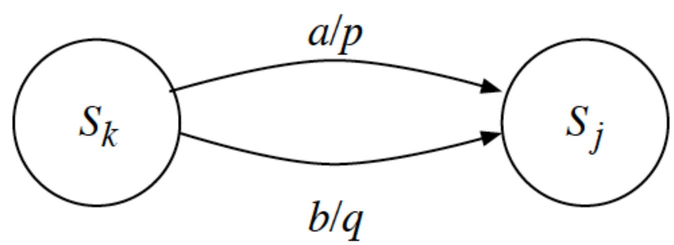
	$NS,\ z(t)$	
PS	$x(t) = a$	$x(t) = b$
EVEN	EVEN, 1	ODD, 0
ODD	ODD, 0	EVEN, 1

PS	$x(t) = a$	$x(t) = b$
EVEN	EVEN, 1	ODD, 0
ODD	ODD, 0	EVEN, 1
	$NS,\ z(t)$	

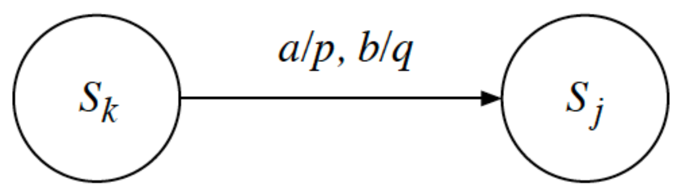
REPRESENTATION OF STATE-TRANSITION AND OUTPUT FUNCTIONS WITH STATE DIAGRAM



(a)



Complete state diagram



Simplified state diagram

(b)

What is the state diagram for this example?

Input: $x(t) \in \{a, b\}$
Output: $z(t) \in \{0, 1\}$
State: $s(t) \in \{\text{EVEN, ODD}\}$
Initial state: s(0)= EVEN

Functions: Transition and output functions

PS	$x(t) = a$	$x(t) = b$
EVEN	EVEN, 1	ODD, 0
ODD	ODD, 0	EVEN, 1
	$NS,\ z(t)$	

Example

Functions: The transition and output functions are

$s(t)$	$x(t)$	
	a	b
S_0	S_1, p	S_2, q
S_1	S_1, p	S_0, p
S_2	S_1, p	S_2, p
	$s(t+1), z(t)$	

What is the state diagram for this example?

MEALY AND MOORE MACHINES

Mealy machine

$$z(t) = H(s(t), x(t))$$

$$s(t+1) = G(s(t), x(t))$$

Moore machine

$$z(t) = H(s(t))$$

$$s(t+1) = G(s(t), x(t))$$

Example: MOORE SEQUENTIAL SYSTEM

Input: $x(t) \in \{a, b, c\}$
Output: $z(t) \in \{0, 1\}$
State: $s(t) \in \{S_0, S_1, S_2, S_3\}$
Initial state: $s(0) = S_0$

Functions: Transition and output functions:

PS	Input			
	a	b	c	
S_0	S_0	S_1	S_1	0
S_1	S_2	S_0	S_1	1
S_2	S_2	S_3	S_0	1
S_3	S_0	S_1	S_2	0
	NS			Output

What is the state diagram for this example?

Input: $x(t) \in \{a, b, c\}$
Output: $z(t) \in \{0, 1\}$
State: $s(t) \in \{S_0, S_1, S_2, S_3\}$
Initial state: $s(0) = S_0$

Functions: Transition and output functions:

PS	Input			
	a	b	c	
S_0	S_0	S_1	S_1	0
S_1	S_2	S_0	S_1	1
S_2	S_2	S_3	S_0	1
S_3	S_0	S_1	S_2	0
	NS			Output

Example - state diagram

Input: $x(t) \in \{0, 1, 2, 3\}$
Output: $z(t) \in \{a, b\}$
State: $s(t) \in \{S_0, S_1\}$
Initial state: $s(0) = S_0$

Functions: The transition and output functions are

$$s(t+1) = \begin{cases} S_0 \ \textbf{if} \ (s(t) = S_0 \\ \quad \textbf{and} \ [x(t) = 0 \ \textbf{or} \ x(t) = 2]) \\ \quad \textbf{or} \ (s(t) = S_1 \ \textbf{and} \ x(t) = 3) \\ S_1 \ \textbf{otherwise} \end{cases}$$

$$z(t) = \begin{cases} a \ \textbf{if} \ s(t) = S_0 \\ b \ \textbf{if} \ s(t) = S_1 \end{cases}$$

- state diagram

Example

Input: $x(t) \in \{0, 1, 2, 3\}$

Output: $z(t) \in \{a, b\}$

State: $s(t) \in \{S_0, S_1\}$

Initial state: $s(0) = S_0$

Functions: The transition and output functions are

$$s(t+1) = \begin{cases} S_0 \ \textbf{if} \ \ (s(t) = S_0 \\ \qquad \textbf{and} \ [x(t) = 0 \ \textbf{or} \ x(t) = 2]) \\ \qquad \textbf{or} \ (s(t) = S_1 \ \textbf{and} \ x(t) = 3) \\ S_1 \ \textbf{otherwise} \end{cases}$$

$$z(t) = \begin{cases} a \ \textbf{if} \ \ s(t) = S_0 \\ b \ \textbf{if} \ \ s(t) = S_1 \end{cases}$$

PS	Input				
	0	1	2	3	
S_0					
S_1					
	NS				Output

Example - state diagram

Input:	$x(t) \in \{0, 1, 2, 3\}$
Output:	$z(t) \in \{a, b\}$
State:	$s(t) \in \{S_0, S_1\}$
Initial state:	$s(0) = S_0$

Functions: The transition and output functions are

$$s(t+1) = \begin{cases} S_0 \ \textbf{if} \ \ (s(t) = S_0 \\ \qquad \textbf{and} \ [x(t) = 0 \ \textbf{or} \ x(t) = 2]) \\ \qquad \textbf{or} \ (s(t) = S_1 \ \textbf{and} \ x(t) = 3) \\ S_1 \ \textbf{otherwise} \end{cases}$$

$$z(t) = \begin{cases} a \ \textbf{if} \ \ s(t) = S_0 \\ b \ \textbf{if} \ \ s(t) = S_1 \end{cases}$$

What is the state diagram for this example?

Clicker Question

FSM

If we start in state 10 and have the input stream $A = 0, 0, 1, 0, 0, 1, 1$, which state do we end up in?

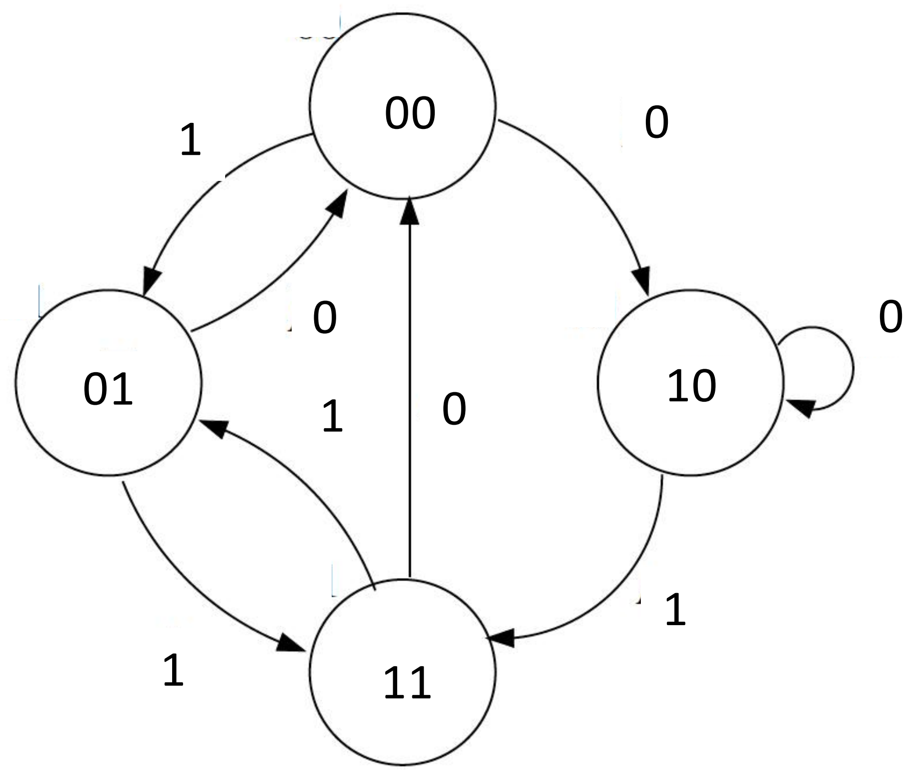
a 00

b 01

c 10

d 11

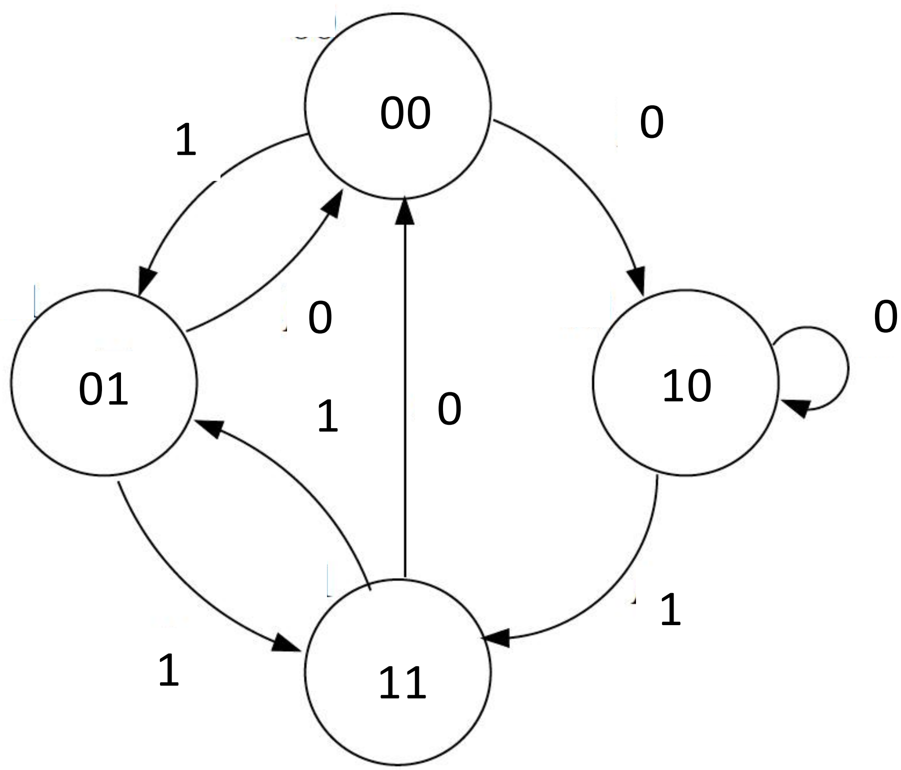
e none of the above



FSM

If we start in state 11 and have the input stream $A = 0, 0, 1, 0, 0, 1, 1$, which state do we end up in?

- a 00
- b 01
- c 10
- d 11
- e none of the above



Example

Input: $x(t) \in \{0, 1\}$

Output: $z(t) \in \{0, 1\}$

Function: $z(t) = \begin{cases} 1 & \textbf{if} \;\; x(t-3, t) = 1101 \\ 0 & \textbf{otherwise} \end{cases}$

- PATTERN DETECTOR $\Rightarrow$ DETECT SUBPATTERNS

State	indicates that
S_{init}	Initial state; also no subpattern
S_1	First symbol (1) of pattern has been detected
S_{11}	Subpattern 11 has been detected
S_{110}	Subpattern 110 has been detected

What is the state diagram for this example?

STATE DESCRIPTION $\Rightarrow$ I/O SEQUENCE

Initial state: $s(0) = S_2$

Functions: Transition and output functions are

PS	$x(t)$			
	a	b	c	
S_0	S_0	S_1	S_1	p
S_1	S_2	S_0	S_1	q
S_2	S_2	S_3	S_0	q
S_3	S_0	S_1	S_2	p
		NS		$z(t)$

t	0	1	2	3	4
x	a	b	c	a	
s	S_2				
z	q				

Clicker Question

INPUT: $x(t) \in \{0,1\}$

OUTPUT: $z(t) \in \{0,1\}$

FUNCTION: $z(t) = \begin{cases} 1 \textbf{ if } \ x(t-2,t) = 101 \\ 0 \textbf{ otherwise} \end{cases}$

t	0	1	2	3	4	5	6	7	8
x	0	0	1	0	1	0	1	0	0
z	0	0	0	A	B	C	D	0	0

a) A=1, B= 1, C= 0, D=0
b) A=0, B= 1, C= 0, D=0
c) A=0, B= 1, C= 0, D=1
d) A=1, B= 0, C= 0, D=1
e) none

INPUT: $x(t) \in \{0, 1\}$

OUTPUT: $z(t) \in \{0, 1\}$

$$\text{FUNCTION: } z(t) = \begin{cases} 1 \textbf{ if } & x(t-2, t) = 101 \\ 0 \textbf{ otherwise} \end{cases}$$

t	0	1	2	3	4	5	6	7	8
x	0	0	1	0	1	0	1	0	0
z	0	0	0	A	B	C	D	0	0

What is the state diagram for this system?

Reduction of the State Set

- k-DISTINGUISHABLE STATES: DIFF. OUTPUT SEQUENCES

$$z(x(t, t+k-1), S_v) \neq z(x(t, t+k-1), S_w)$$

- k-EQUIVALENT STATES: NOT DISTINGUISHABLE FOR SEQUENCES OF LENGTH k
- P_k: PARTITION OF STATES INTO k-EQUIVALENT CLASSES

Reduction of the State Set

Input: $x(t) \in \{a, b, c\}$
Output: $z(t) \in \{0, 1\}$
State: $s(t) \in \{A, B, C, D, E, F\}$
Initial state: $s(0) = A$

Functions: TRANSITION AND OUTPUT

PS	$x = a$	$x = b$	$x = c$
A	$E, 0$	$D, 1$	$B, 0$
B	$F, 0$	$D, 0$	$A, 1$
C	$E, 0$	$B, 1$	$D, 0$
D	$F, 0$	$B, 0$	$C, 1$
E	$C, 0$	$F, 1$	$F, 0$
F	$B, 0$	$C, 0$	$F, 1$
	NS, z		

Reduction of the State Set

Input: $x(t) \in \{a, b, c\}$
Output: $z(t) \in \{0, 1\}$
State: $s(t) \in \{A, B, C, D, E, F\}$
Initial state: $s(0) = A$

Functions: TRANSITION AND OUTPUT

PS	$x=a$	$x=b$	$x=c$
A	$E,0$	$D,1$	$B,0$
B	$F,0$	$D,0$	$A,1$
C	$E,0$	$B,1$	$D,0$
D	$F,0$	$B,0$	$C,1$
E	$C,0$	$F,1$	$F,0$
F	$B,0$	$C,0$	$F,1$
		$NS,\ z$	

A and B ARE 1-DISTINGUISHABLE BECAUSE

$$z(b, A) \neq z(b, B)$$

A and C ARE 1-EQUIVALENT BECAUSE

$$z(x(t), A) = z(x(t), C), \quad for\ all\ x(t) \in I$$

Reduction of the State Set

Input: $x(t) \in \{a, b, c\}$
Output: $z(t) \in \{0, 1\}$
State: $s(t) \in \{A, B, C, D, E, F\}$
Initial state: $s(0) = A$

Functions: TRANSITION AND OUTPUT

PS	$x=a$	$x=b$	$x=c$
A	$E,0$	$D,1$	$B,0$
B	$F,0$	$D,0$	$A,1$
C	$E,0$	$B,1$	$D,0$
D	$F,0$	$B,0$	$C,1$
E	$C,0$	$F,1$	$F,0$
F	$B,0$	$C,0$	$F,1$
		NS, z	

A and C ARE ALSO 2-EQUIVALENT BECAUSE

$$
\begin{aligned}
z(aa, A) &= z(aa, C) &= 00 \\
z(ab, A) &= z(ab, C) &= 01 \\
z(ac, A) &= z(ac, C) &= 00 \\
z(ba, A) &= z(ba, C) &= 10 \\
z(bb, A) &= z(bb, \mathrm{C}) &= 10 \\
z(bc, A) &= z(bc, C) &= 11 \\
z(ca, A) &= z(ca, C) &= 00 \\
z(cb, A) &= z(cb, C) &= 00 \\
z(cc, A) &= z(cc, C) &= 01
\end{aligned}
$$

Reduction of the State Set

PS	$x(t) = a$	$x(t) = b$	$x(t) = c$
A	0	1	0
B	0	0	1
C	0	1	0
D	0	0	1
E	0	1	0
F	0	0	1
		NS, z	

- 1-EQUIVALENT IF SAME "row pattern"

$$P_1 = (A, C, E) \ \ (B, D, F)$$

Reduction of the State Set

- NUMBER THE CLASSES IN P_1
- TWO STATES ARE IN THE SAME CLASS OF P_2
  IF THEIR SUCCESSOR COLUMNS HAVE THE SAME NUMBERS

PS	$x = a$	$x = b$	$x = c$
A	$E, 0$	$D, 1$	$B, 0$
B	$F, 0$	$D, 0$	$A, 1$
C	$E, 0$	$B, 1$	$D, 0$
D	$F, 0$	$B, 0$	$C, 1$
E	$C, 0$	$F, 1$	$F, 0$
F	$B, 0$	$C, 0$	$F, 1$
	NS, z		

	1			2		
P_1	$(A,$	$C,$	$E)$	$(B,$	$D,$	$F)$
a	1	1	1	2	2	2
b	2	2	2	2	2	1
c	2	2	2	1	1	2

BY IDENTIFYING IDENTICAL COLUMNS OF SUCCESSORS, WE GET

$$P_2 = (A, C, E)\ \ (B, D)\ \ (F)$$

Reduction of the State Set

- SAME PROCESS TO OBTAIN THE NEXT PARTITION:

PS	$x=a$	$x=b$	$x=c$
A	$E,0$	$D,1$	$B,0$
B	$F,0$	$D,0$	$A,1$
C	$E,0$	$B,1$	$D,0$
D	$F,0$	$B,0$	$C,1$
E	$C,0$	$F,1$	$F,0$
F	$B,0$	$C,0$	$F,1$
	NS, z		

	1			2		3
P_2	$(A,$	$C,$	$E)$	$(B,$	$D),$	(F)
a	1	1	1	3	3	
b	2	2	3	2	2	
c	2	2	3	1	1	

$$P_3 = (A, C)\ \ (E)\ \ (B, D)\ \ (F)$$

- SIMILARLY, WE DETERMINE $P_4 = (A, C)\ (E)\ (B, D)\ (F)$

BECAUSE $P_4 = P_3$ THIS IS ALSO THE EQUIVALENCE PARTITION P

Reduction of the State Set

THE MINIMAL SYSTEM:

PS	$x = a$	$x = b$	$x = c$
A	$E, 0$	$D, 1$	$B, 0$
B	$F, 0$	$D, 0$	$A, 1$
C	$E, 0$	$B, 1$	$D, 0$
D	$F, 0$	$B, 0$	$C, 1$
E	$C, 0$	$F, 1$	$F, 0$
F	$B, 0$	$C, 0$	$F, 1$
	NS, z		

PS	$x = a$	$x = b$	$x = c$
A	$E, 0$	$B, 1$	$B, 0$
B	$F, 0$	$B, 0$	$A, 1$
E	$A, 0$	$F, 1$	$F, 0$
F	$B, 0$	$A, 0$	$F, 1$
	NS, z		