CS 161 Fundamentals of Artificial Intelligence
Lecture 4

Informed Search Algorithms

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Outline

- Best-first search
- A^* search
- Heuristics

Review: Tree search

```
function TREE-SEARCH( problem, fringe) returns a solution, or failure
    fringe ← INSERT(MAKE-NODE(INITIAL-STATE[problem]), fringe)
    loop do
        if fringe is empty then return failure
        node ← REMOVE-FRONT(fringe)
        if GOAL-TEST[problem] applied to STATE(node) succeeds return node
        fringe ← INSERTALL(EXPAND(node, problem), fringe)
```

A strategy is defined by picking the **order of node expansion**

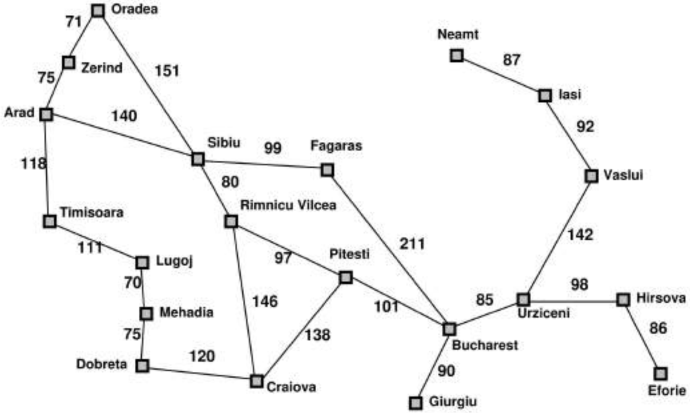
Best-first search

Idea: use an **evaluation function** for each node
– estimate of "desirability"
$\Rightarrow$ Expand most desirable unexpanded node

Implementation:
fringe is a queue sorted in decreasing order of desirability

Special cases:
greedy search
A^* search

Romania with step costs in km



Straight–line distance to Bucharest

Arad	366
Bucharest	0
Craiova	160
Dobreta	242
Eforie	161
Fagaras	176
Giurgiu	77
Hirsova	151
Iasi	226
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Urziceni	80
Vaslui	199
Zerind	374

Greedy search

Evaluation function $h(n)$ (**h**euristic)
$\quad = $ estimate of cost from n to the closest goal
E.g., $h_{\mathrm{SLD}}(n) =$ straight-line distance from node n to Bucharest
Greedy search expands the node that **appears** to be closest to goal

Greedy search example

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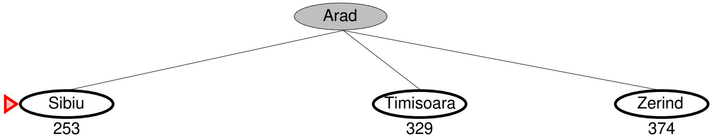
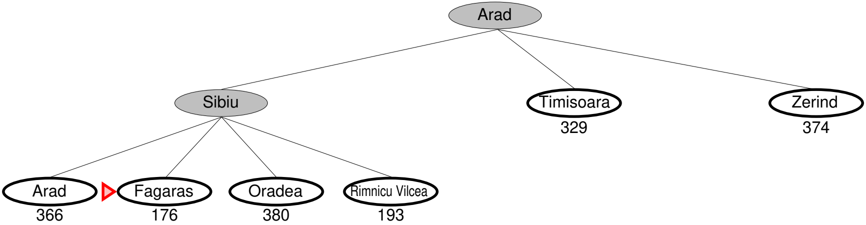


Greedy search example



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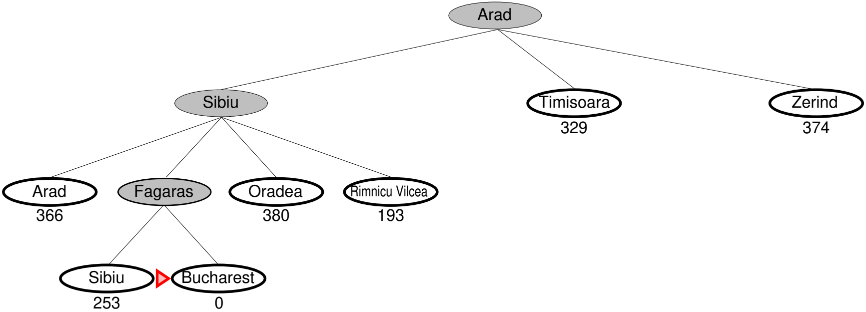

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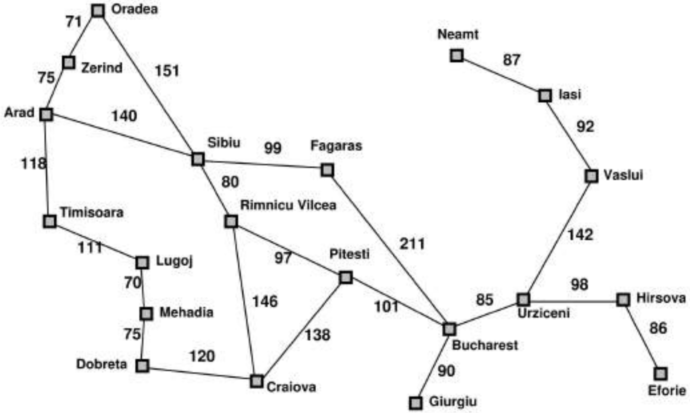
Optimal?? No

A* search

Idea: avoid expanding paths that are already expensive
Evaluation function $f(n) = g(n) + h(n)$
$g(n) =$ cost so far to reach node n
$h(n) =$ estimated cost to goal from n
$f(n) =$ estimated total cost of path through n to goal
A* search uses an **admissible** heuristic
i.e., $h(n) \leq h^*(n)$ where $h^*(n)$ is the **true** cost from n.
(Also require $h(n) \geq 0$, so $h(G) = 0$ for any goal G.)
E.g., $h_{\mathrm{SLD}}(n)$ never overestimates the actual road distance
Theorem: A* search is optimal

Romania with step costs in km



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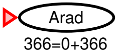

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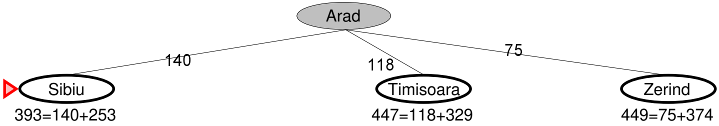

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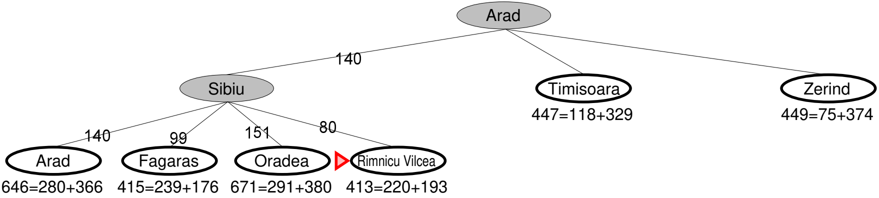

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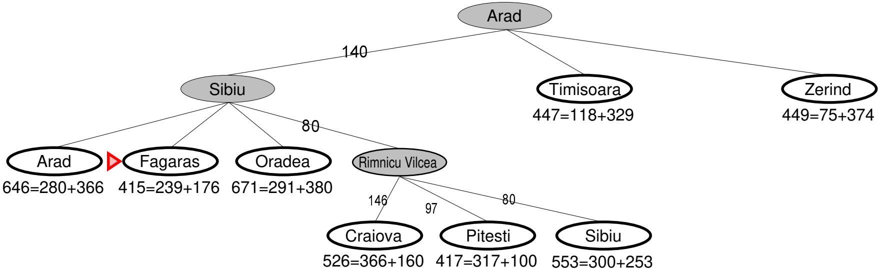

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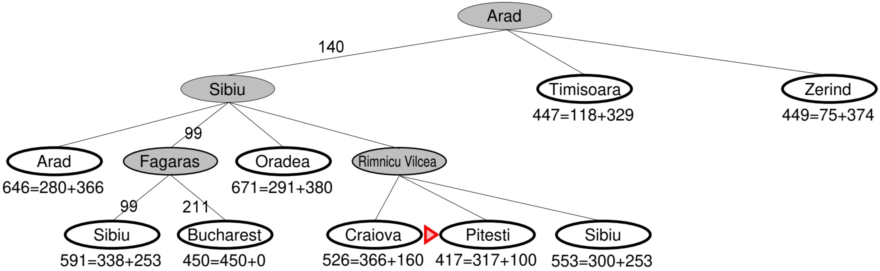

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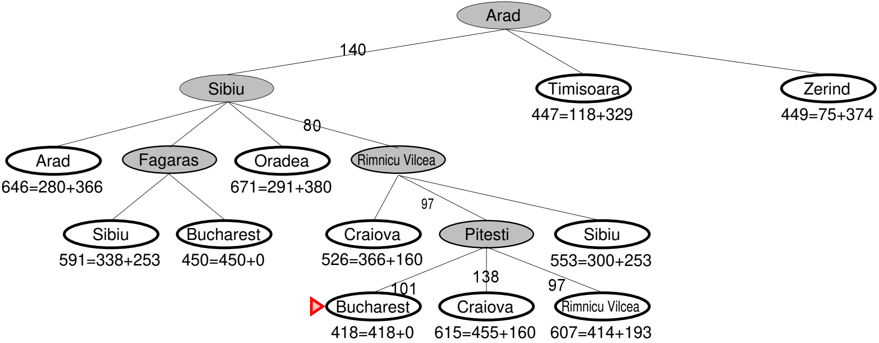
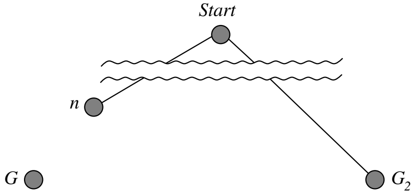


Optimality of A* (standard proof)

Suppose some suboptimal goal G_2 has been generated and is in the queue. Let n be an unexpanded node on a shortest path to an optimal goal G.



$$
\begin{aligned}
f(G_2) &= g(G_2) && \text{since } h(G_2) = 0 \\
&> g(G) && \text{since } G_2 \text{ is suboptimal} \\
&= g(n) + h^*(n) && \text{By the definition of } h^*(n) \\
&\geq g(n) + h(n) && \text{since } h \text{ is admissible} \\
&= f(n)
\end{aligned}
$$

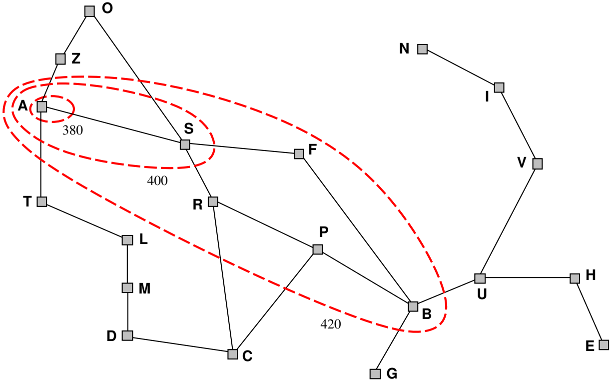
Since $f(G_2) > f(n)$, A* will never select G_2 for expansion

Optimality of A* (more useful)

Lemma: A* expands nodes in order of increasing f value*

Gradually adds "f-contours" of nodes (cf. breadth-first adds layers)

Contour i has all nodes with $f \leq f_i$, where $f_i < f_{i+1}$



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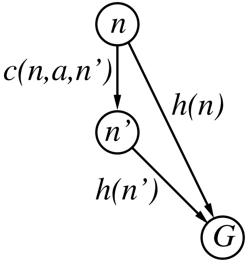
Optimal?? Yes—cannot expand f_{i+1} until f_i is finished
A* expands all nodes with $f(n) < C^*$, C^* is the cost of optimal solution path
A* expands some nodes with $f(n) = C^*$
A* expands no nodes with $f(n) > C^*$

Proof of lemma: Consistency

A heuristic is **consistent** if

$$h(n) \leq c(n, a, n') + h(n')$$



where $c(n, a, n')$ is the cost of path from n to n' by choosing action a

If h is consistent, we have

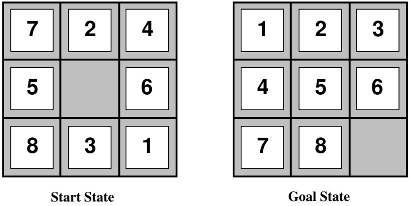
$$\begin{aligned} f(n') &= g(n') + h(n') \\ &= g(n) + c(n, a, n') + h(n') \\ &\geq g(n) + h(n) \\ &= f(n) \end{aligned}$$

I.e., $f(n)$ is nondecreasing along any path. **Thus, the goal state with the lowest f-cost will be found first**

Every consistent heuristic is admissible!

Admissible heuristics

E.g., for the 8-puzzle:
$h_1(n)$ = number of misplaced tiles
$h_2(n)$ = total **Manhattan** distance
(i.e., no. of squares from desired location of each tile)



$\underline{h_1(S) =}$??
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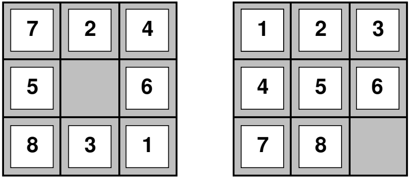


Start State Goal State

$h_1(S) =$?? 6
$h_2(S) =$?? 4+0+3+3+1+0+2+1 = 14

Relaxed problems

Admissible heuristics can be derived from the **exact** solution cost of a **relaxed** version of the problem

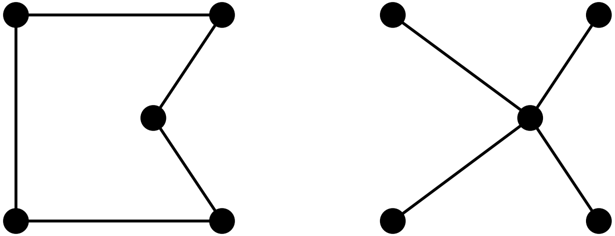
If the rules of the 8-puzzle are relaxed so that a tile can move **anywhere**, then $h_1(n)$ gives the shortest solution

If the rules are relaxed so that a tile can move to **any adjacent square**, then $h_2(n)$ gives the shortest solution

Key point: the optimal solution cost of a relaxed problem is no greater than the optimal solution cost of the real problem

Relaxed problems contd.

Well-known example: **travelling salesperson problem** (TSP)
Find the shortest tour visiting all cities exactly once



Minimum spanning tree can be computed in $O(n^2)$
and is a lower bound on the shortest (open) tour

Summary

Heuristic functions estimate costs of shortest paths

Good heuristics can dramatically reduce search cost

Greedy best-first search expands lowest h

– incomplete and not always optimal

A* search expands lowest $g + h$

– complete and optimal

– also optimally efficient (up to tie-breaks, for forward search)

Admissible heuristics can be derived from exact solution of relaxed problems

Acknowledgment

The slides are adapted from Stuart Russell, Guy Van den Broeck et al.