CS 161 Fundamentals of Artificial Intelligence
Lecture 6
Constraint Satisfaction Problem

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Outline

- CSP examples
- Constraint Propagation
- Backtracking search for CSPs
- Problem structure and problem decomposition

Constraint satisfaction problems (CSPs)

A constraint satisfaction problem (CSP) consists of three components: X, D, and C:

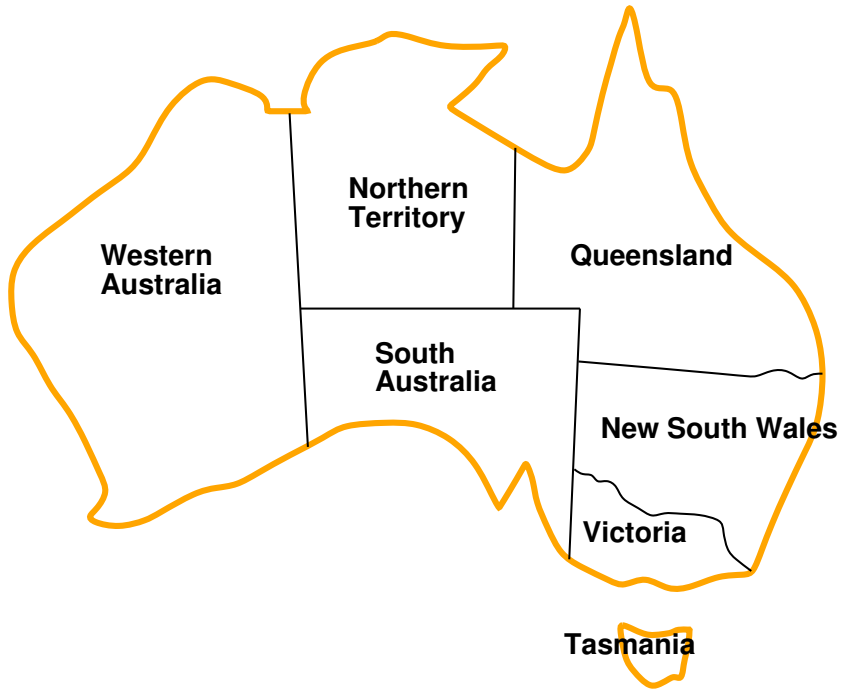
- X: a set of variables $\{X_1, ..., X_n\}$.
- D : a set of domains, $\{D_1, ..., D_n\}$, one for each variable.
- C : a set of constraints. $< scope, rel > \in C$: $scope$: tuple of variables; rel: values that those variables can take on

State of CSP: an assignment of values to some or all of the variables

- **Complete assignment**: every variable is assigned
- **Partial assignment**: assigns values to only some of the variables
- **Consistent**: not violate any constraints

Solution: a consistent, complete assignment

Example: Map-Coloring



Variables: WA, NT, Q, NSW, V, SA, T

Domains: $D_i = \{red, green, blue\}$

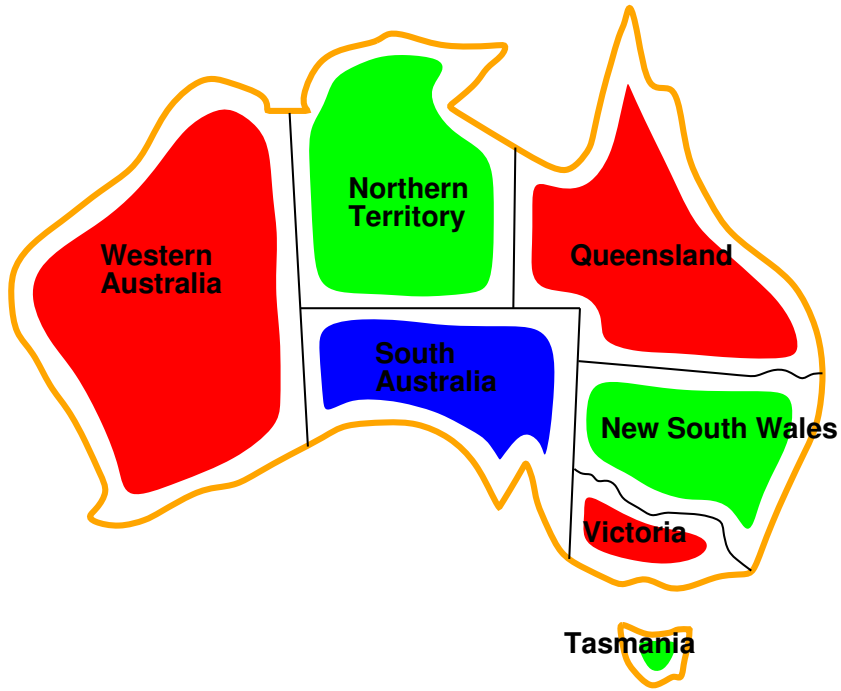
Constraints: adjacent regions must have different colors

e.g., $WA \neq NT$ (if the language allows this), or

$(WA, NT) \in$

$\{(red, green), (red, blue), (green, red), (green, blue), \ldots\}$

Example: Map-Coloring contd.



Solutions are assignments satisfying all constraints, e.g.,
$\{WA = red, NT = green, Q = red, NSW = green$
$V = red, SA = blue, T = green\}$

Varieties of CSPs

Discrete variables

finite domains; size d $\Rightarrow$ $O(d^n)$ complete assignments

- e.g., Boolean CSPs, incl. Boolean satisfiability (NP-complete)

infinite domains (integers, strings, etc.)

- e.g., job scheduling, variables are start/end days for each job
- need a **constraint language**, e.g., $StartJob_1 + 5 \leq StartJob_3$

Continuous variables

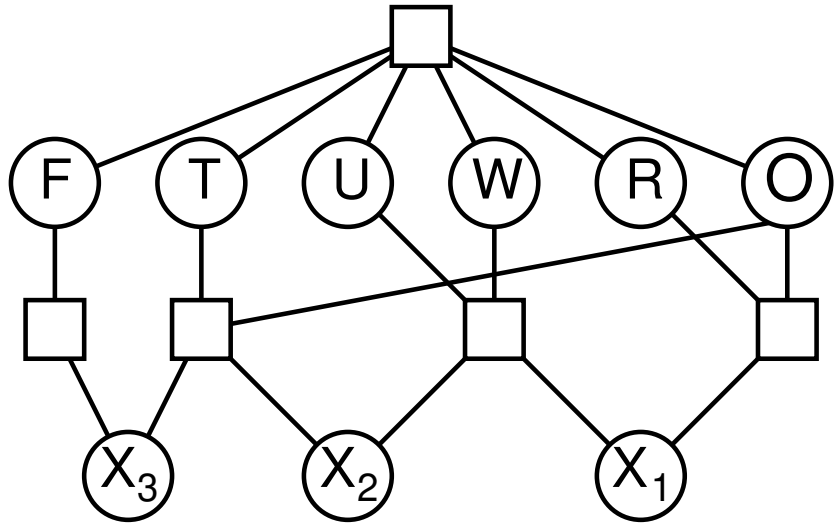
- e.g., start/end times for Hubble Telescope observations
- linear constraints solvable in poly time by linear programming

Varieties of constraints

Unary constraints involve a single variable,
e.g., $SA \neq green$
Binary constraints involve pairs of variables,
e.g., $SA \neq WA$
Higher-order(Global) constraints involve 3 or more variables,
e.g., cryptarithmetic column constraints, Alldif (all of the variables involved in the constraint must have different values)
Preferences (soft constraints), e.g., red is better than $green$
often representable by a cost for each variable assignment
$\rightarrow$ constrained optimization problems

Example: Cryptarithmetic

$$\begin{array}{cccc} & T & W & O \\ + & T & W & O \\ \hline F & O & U & R \end{array}$$



Squares represent constraints

Variables: $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$

Domains: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Constraints

$alldiff(F, T, U, W, R, O)$

$O + O = R + 10 \cdot X_1$,

$X_1 + W + W = U + 10 \cdot X_2$,

$X_2 + T + T = O + 10 \cdot X_3$,

$X_3 = F$

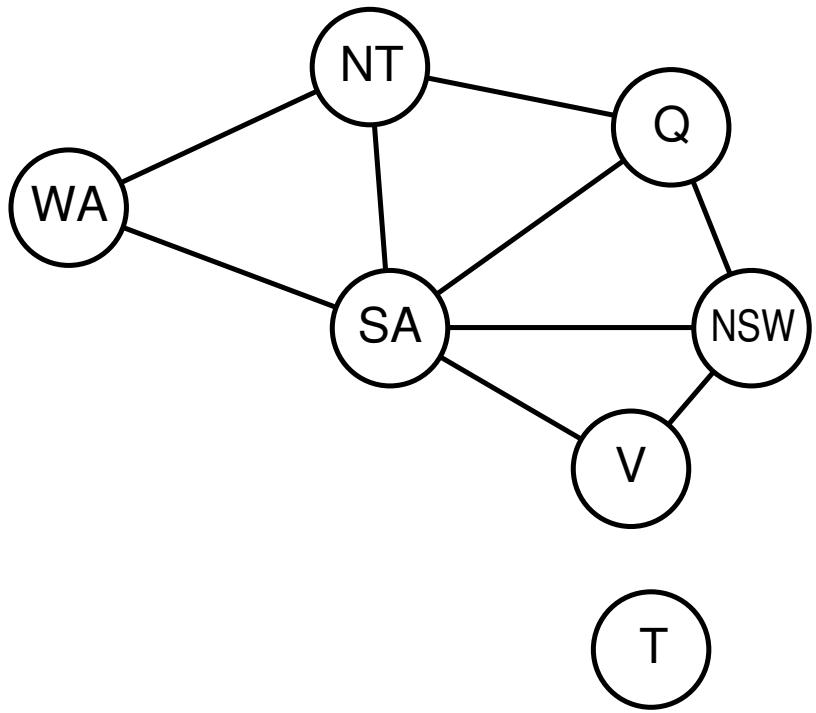
Real-world CSPs

Assignment problems
  e.g., who teaches what class
Timetabling problems
  e.g., which class is offered when and where?
Hardware configuration
Spreadsheets
Transportation scheduling
Factory scheduling
Floorplanning

Notice that many real-world problems involve real-valued variables

Constraint graph

Binary CSP: each constraint relates at most two variables
Constraint graph: nodes are variables, arcs show constraints



General-purpose CSP algorithms use the graph structure
to speed up search. E.g., Tasmania is an independent subproblem!

Constraint propagation

• **Constraint propagation**: using the constraints to reduce the number of legal values for a variable
• May be done as a preprocessing step, before search starts!
• **Node consistency**: all the values in the variable's domain satisfy the variable's unary constraints
• **Arc consistency**: every value in its domain satisfies the variable's binary constraints
For any variables X_i, X_j,

- X_i is arc-consistent with respect to X_j: for **every** value in D_i, there is some value in D_j that satisfies the binary constraint on the arc (X_i, X_j)

Arc consistency algorithm

- Each binary constraint becomes two arcs, one in each direction.
- AC-3: remove values from the domains of variables until no more arcs are in the queue

function AC-3(*csp*) **returns** the CSP, possibly with reduced domains
  inputs: *csp*, a binary CSP with variables $\{X_1, X_2, \ldots, X_n\}$
  local variables: *queue*, a queue of arcs, initially all the arcs in *csp*

  while *queue* is not empty **do**
    $(X_i, X_j) \leftarrow$ REMOVE-FIRST(*queue*)
    if REMOVE-INCONSISTENT-VALUES(X_i, X_j) **then**
      for each X_k **in** NEIGHBORS$[X_i]$ **do**
        add (X_k, X_i) to *queue*
  return *true*

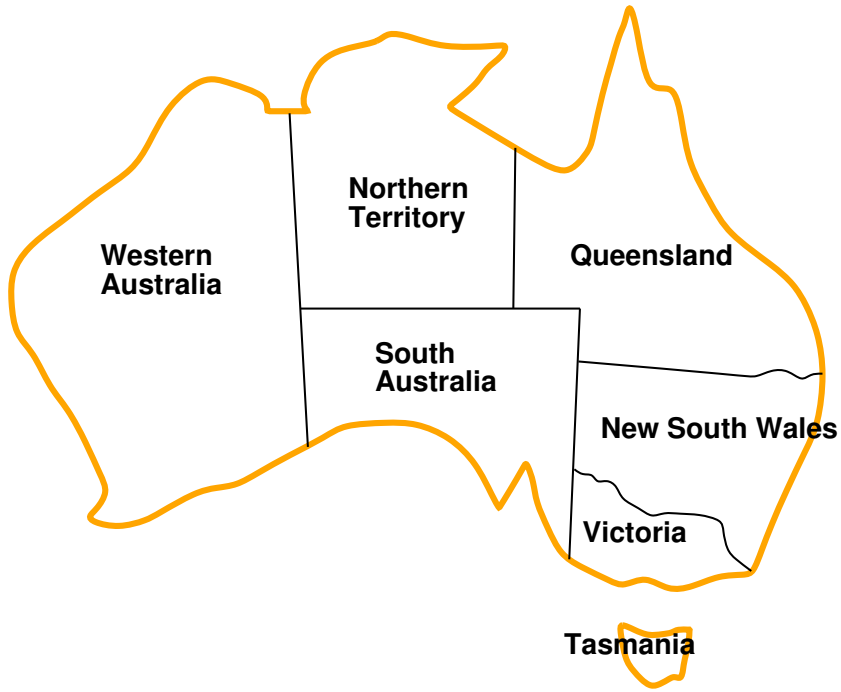
---

function REMOVE-INCONSISTENT-VALUES(X_i, X_j) **returns** true iff succeeds
  removed $\leftarrow$ *false*
  for each *x* **in** DOMAIN$[X_i]$ **do**
    if no value y in DOMAIN$[X_j]$ allows (*x*,*y*) to satisfy the constraint $X_i \leftrightarrow X_j$
      then delete *x* from DOMAIN$[X_i]$; *removed* $\leftarrow$ *true*
  return *removed*

AC-3

- if AC-3 revises D_i, then we add (X_k, X_i), where X_k is a neighbor of X_i, since the change in D_i might enable further reductions in the domains of D_k
- Time complexity: $O(n^2 d^3)$
  - Each arc (X_k, X_i) can be inserted in the queue only d times because X_i has at most d values to delete
  - At most n^2 arcs
  - Checking consistency of an arc can be done in d^2 time

Arc consistency example: Map-Coloring



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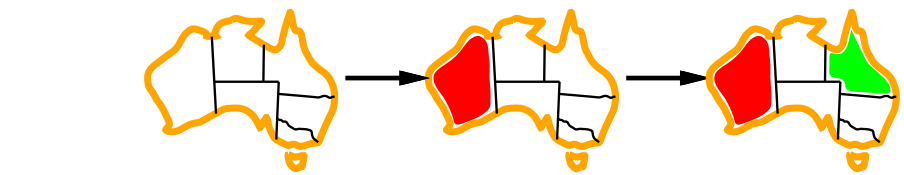
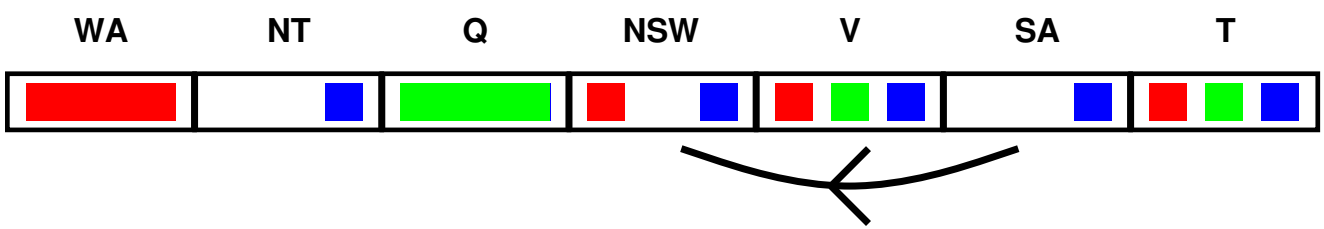
Arc consistency

Simplest form of propagation makes each arc **consistent**

$X \to Y$ is consistent iff

for **every** value x of X there is **some** allowed y





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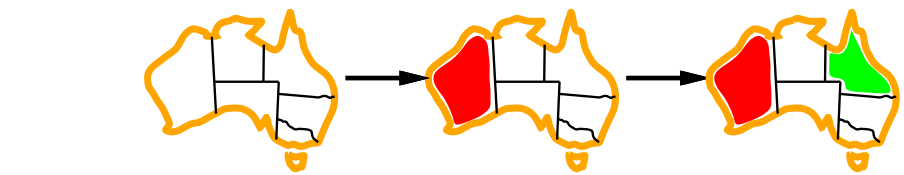

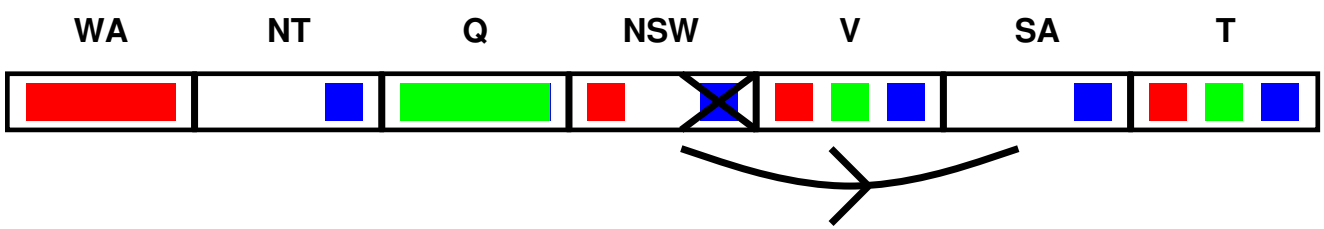

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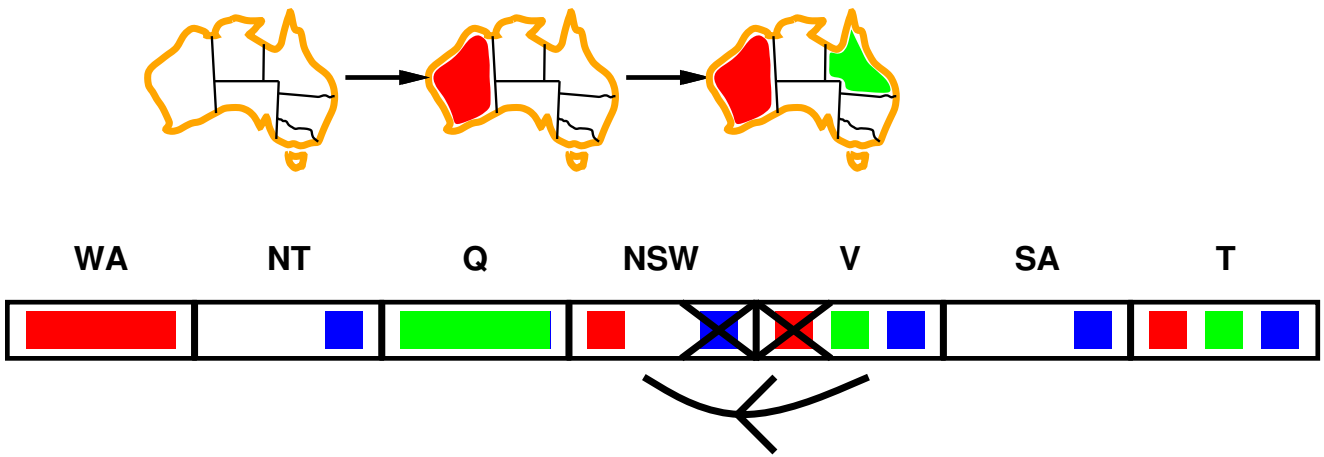


If X loses a value, neighbors of X need to be rechecked

Standard search formulation (incremental)

Let's start with the straightforward, dumb approach, then fix it
States are defined by the values assigned so far

- Initial state: the empty assignment, $\{\}$
- Successor function: assign a value to an unassigned variable that does not conflict with current assignment.
  $\Rightarrow$ fail if no legal assignments (not fixable!)
- Goal test: the current assignment is complete

1) This is the same for all CSPs! 
2) Every solution appears at depth n with n variables
   $\Rightarrow$ use depth-first search
3) Path is irrelevant, so can also use complete-state formulation
4) $b = (n - \ell)d$ at depth ℓ, hence $n!d^n$ leaves!!!! 

Backtracking search

Variable assignments are **commutative**, i.e.,
$[WA = red$ then $NT = green]$ same as $[NT = green$ then $WA = red]$
Only need to consider assignments to a single variable at each node
$\Rightarrow \quad b = d$ and there are d^n leaves

Depth-first search for CSPs with single-variable assignments
is called **backtracking** search
Backtracking search is the basic uninformed algorithm for CSPs

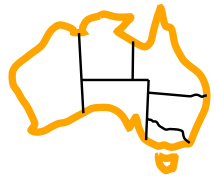
Backtracking search

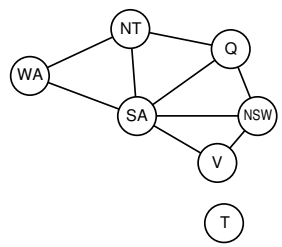
• **Backtracking search**: depth-first search that chooses values for one variable at a time and backtracks when a variable has no legal values left to assign

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
    return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment given CONSTRAINTS[csp] then
            add {var = value} to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failure then return result
            remove {var = value} from assignment
    return failure
```

Backtracking example





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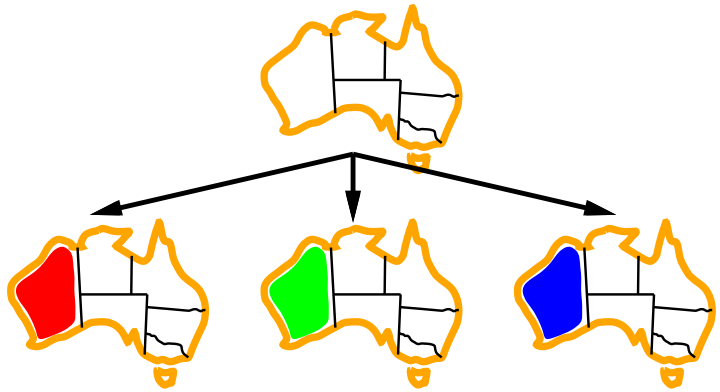

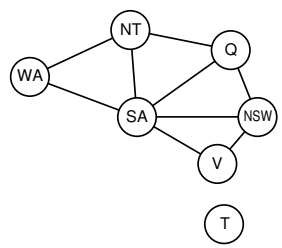

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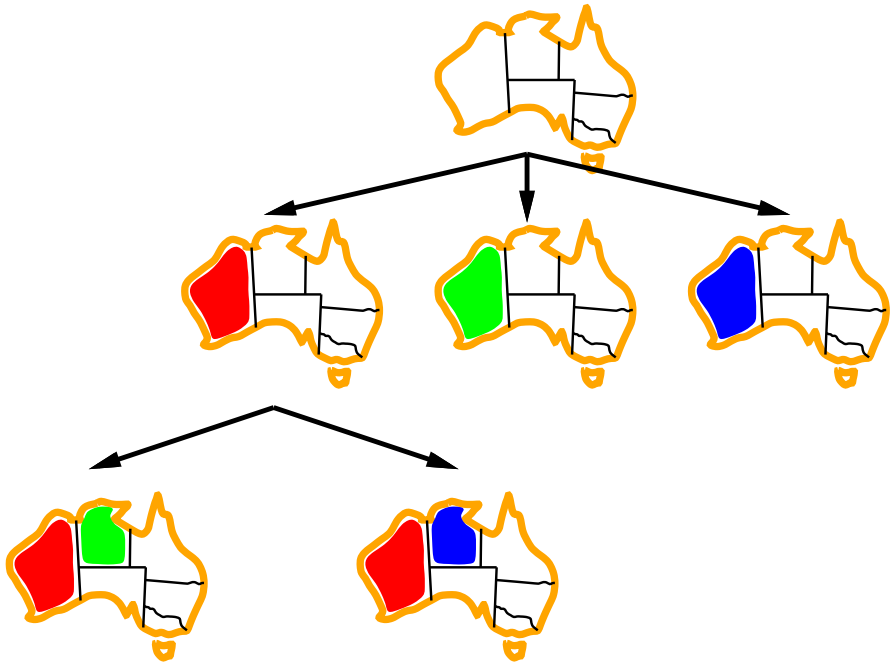

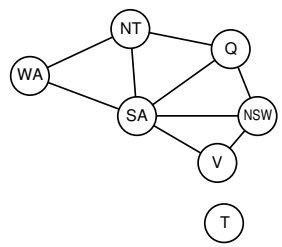

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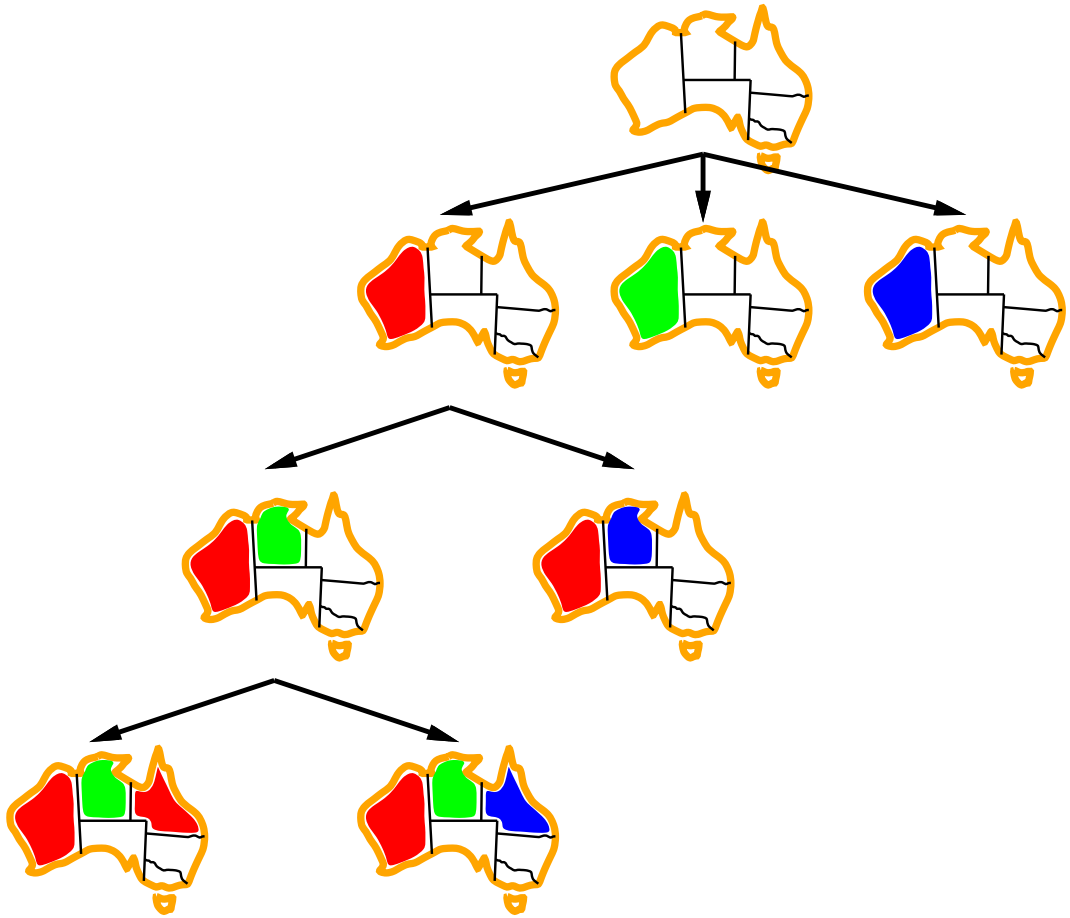
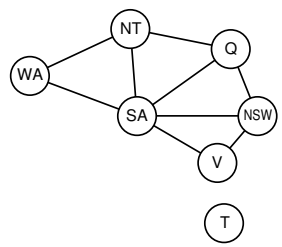




Backtracking example: 4-Queens as a CSP

Assume one queen in each column. Which row does each one go in?

Variables Q_1, Q_2, Q_3, Q_4

Domains $D_i = \{1, 2, 3, 4\}$

Constraints

$Q_i \neq Q_j$ (cannot be in same row)

$|Q_i - Q_j| \neq |i - j|$ (or same diagonal)

Translate each constraint into set of allowable values for its variables

E.g., values for (Q_1, Q_2) are $(1, 3)$ $(1, 4)$ $(2, 4)$ $(3, 1)$ $(4, 1)$ $(4, 2)$

Backtracking example: 4-Queens as a CSP

Backtracking search:
(Q_1, Q_2, Q_3, Q_4):
$(1, X, X, X) \rightarrow (1, 3, X, X) \rightarrow$ No legal assign for Q_3, backtracking
$(1, 4, X, X) \rightarrow (1, 4, 2, X) \rightarrow$ No legal assign for Q_4, backtracking
$(2, X, X, X) \rightarrow (2, 4, X, X) \rightarrow (2, 4, 1, X) \rightarrow (2, 4, 1, 3)$, Bingo!

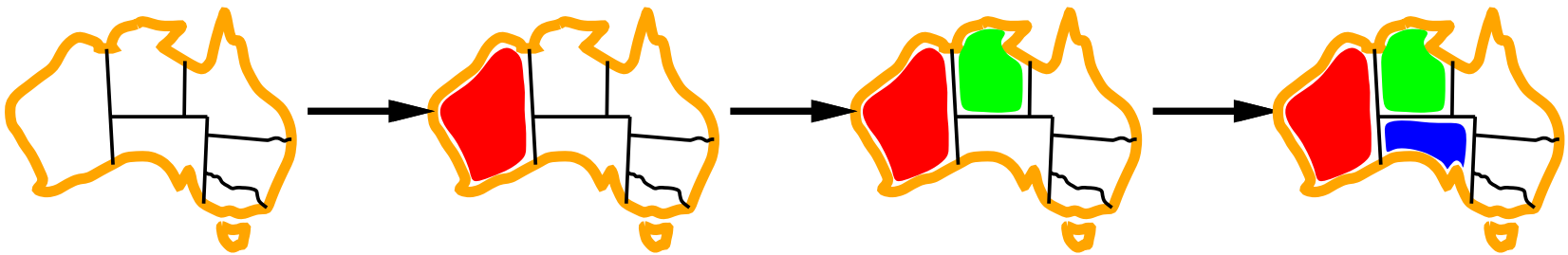
Improving backtracking efficiency

General-purpose methods can give huge gains in speed:

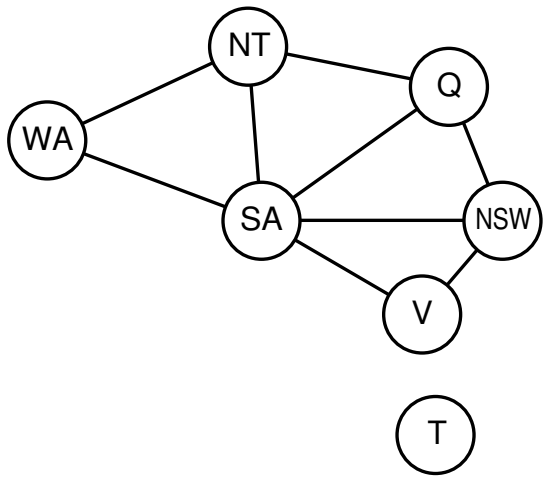
1. Which variable should be assigned next?
2. In what order should its values be tried?
3. Can we detect inevitable failure early?
4. Can we take advantage of problem structure?

Minimum remaining values

- Minimum remaining values (MRV): choose the variable with the fewest legal values



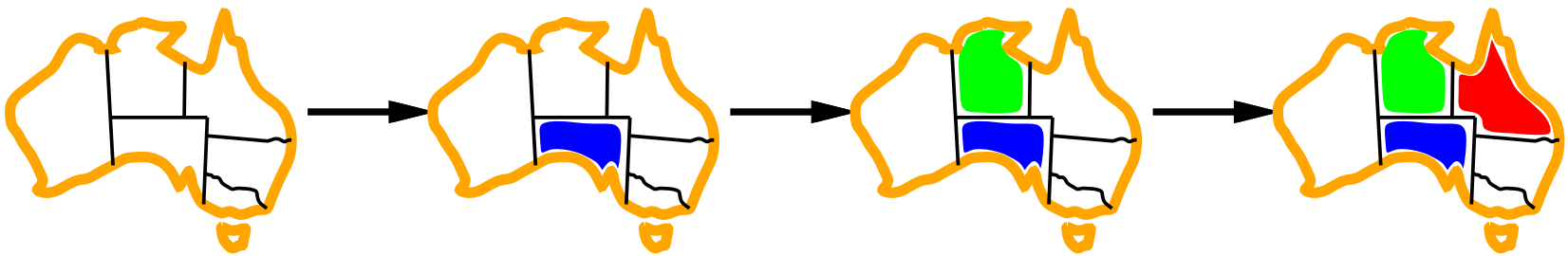
- WA:3 $\rightarrow$ NT: 2 $\rightarrow$ SA:1



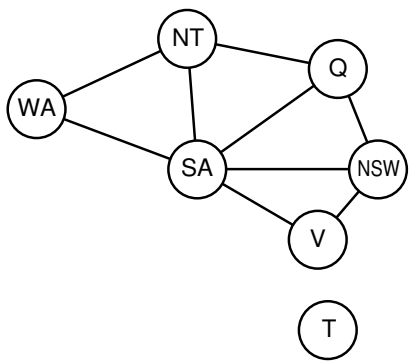
Degree heuristic

Tie-breaker among MRV variables

Degree heuristic: choose the variable with the most constraints on remaining variables

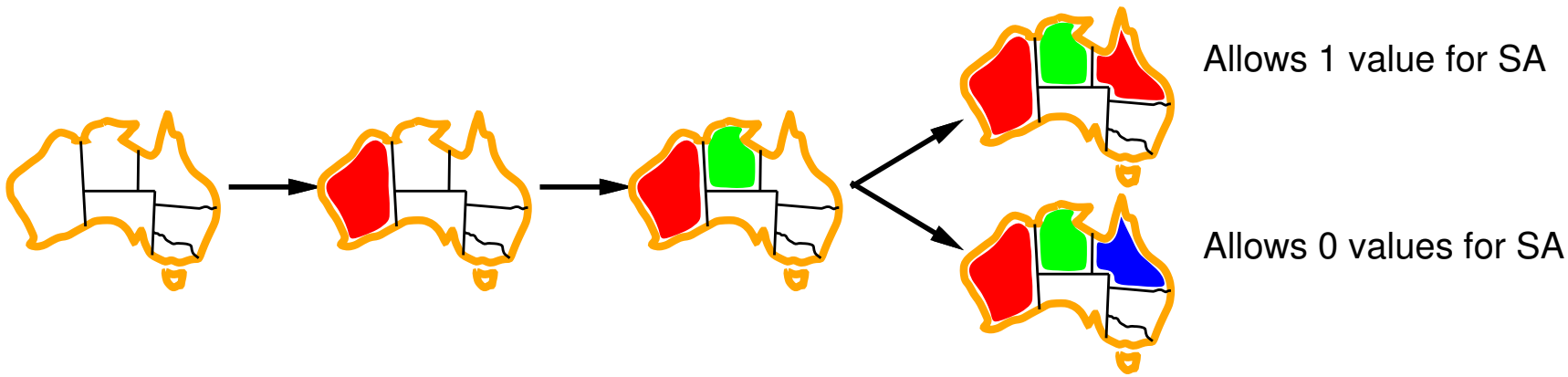


- SA:5 $\rightarrow$ NT: 3 $\rightarrow$ Q:3



Least constraining value

Given a variable, choose the **least constraining value**: the one that rules out the fewest values in the remaining variables

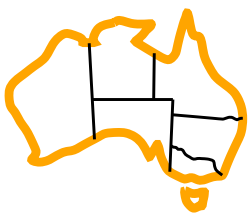


• Leave the maximum flexibility for subsequent variable assignments

Forward checking

- How do we detect failure early (if there is no solution?)
- If we do AC-3 before the search, already know if there is failure!
- What if we do not do AC-3 before?

Idea: Keep track of remaining legal values for unassigned variables
Terminate search when any variable has no legal values



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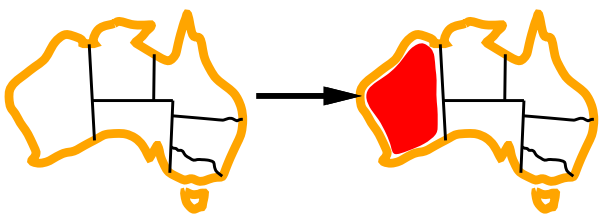

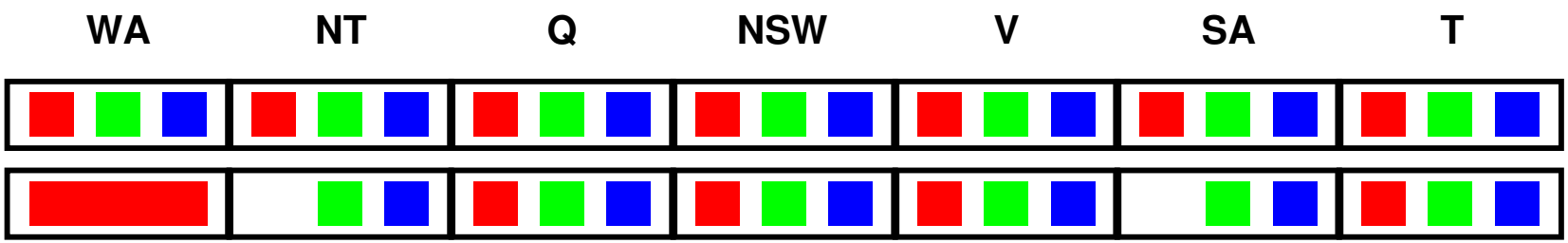

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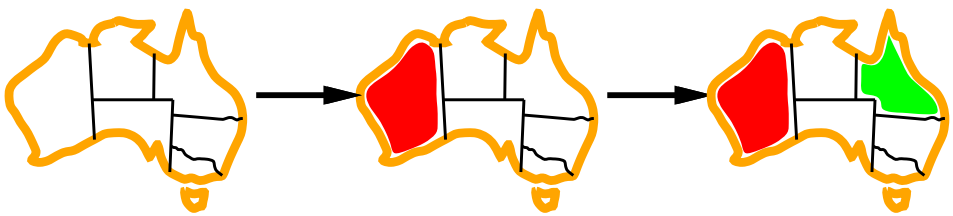

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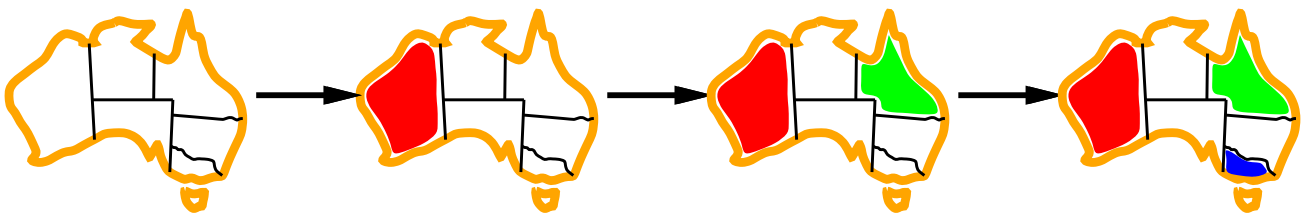
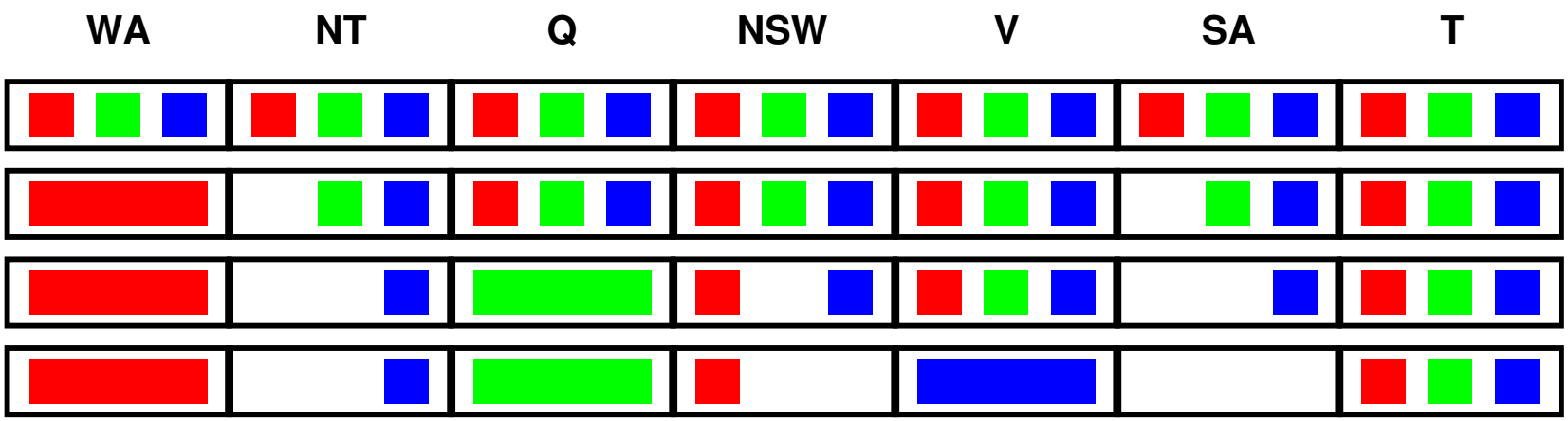
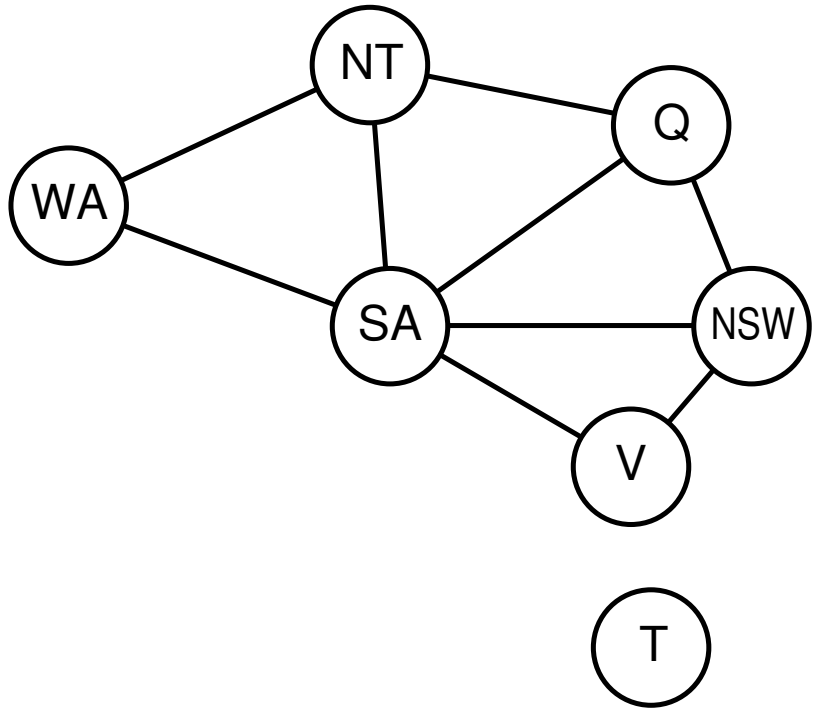




Problem structure

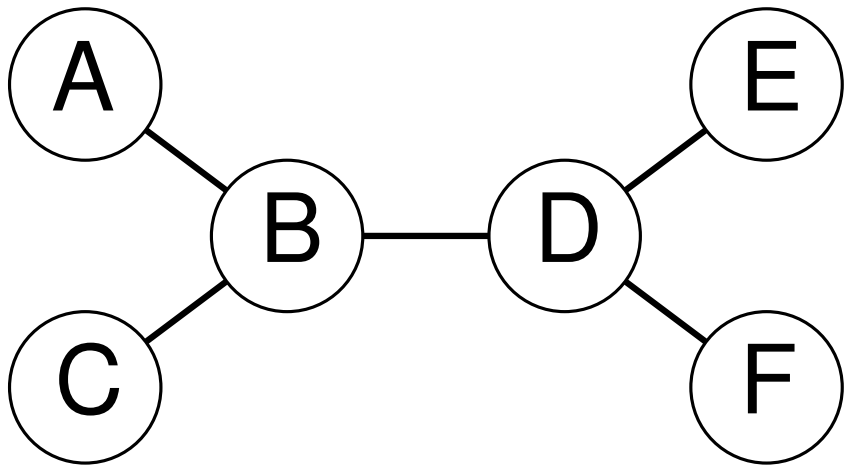


Tasmania and mainland are **independent subproblems**
Identifiable as **connected components** of constraint graph

Problem structure contd.

Suppose each subproblem has c variables out of n total
Worst-case solution cost is $n/c \cdot d^c$, **linear** in n
E.g., $n = 80$, $d = 2$, $c = 20$
2^{80} = 4 billion years at 10 million nodes/sec
$4 \cdot 2^{20}$ = 0.4 seconds at 10 million nodes/sec

Tree-structured CSPs



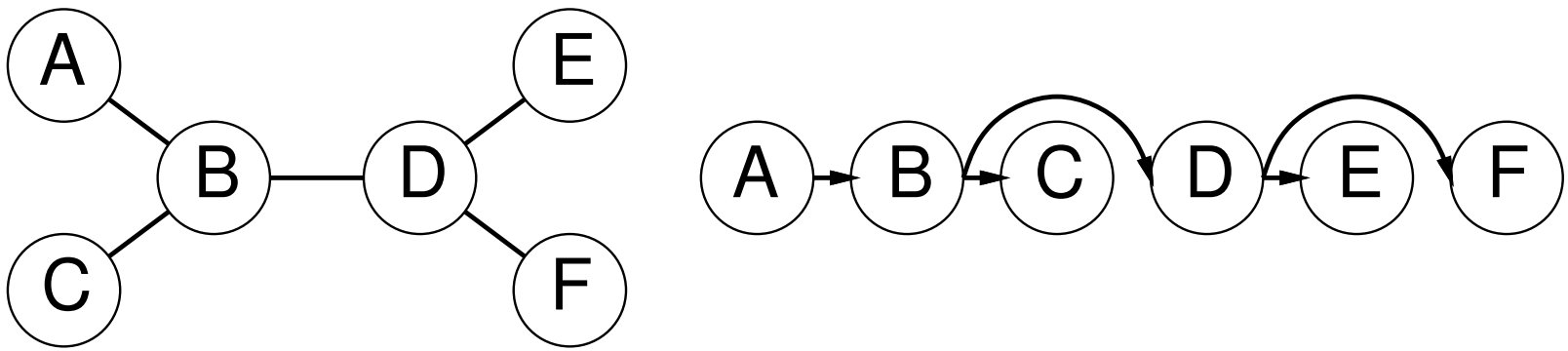
Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n\,d^2)$ time

Compare to general CSPs, where worst-case time is $O(d^n)$

This property also applies to logical and probabilistic reasoning: an important example of the relation between syntactic restrictions and the complexity of reasoning.

Algorithm for tree-structured CSPs

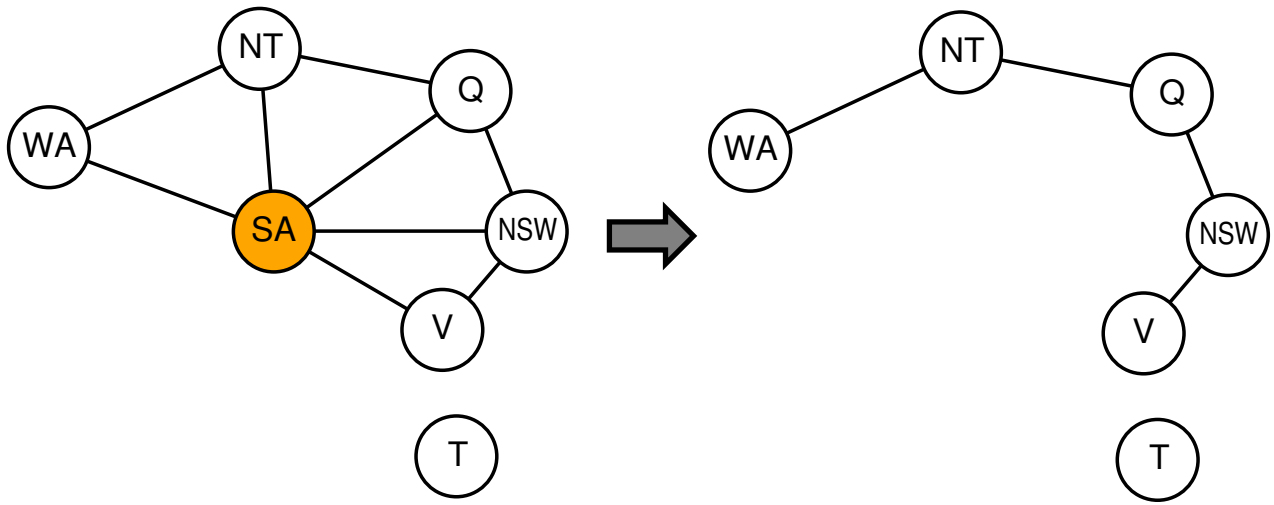
1. Choose a variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering



2. For j from n down to 2, apply $\textsc{RemoveInconsistent}(Parent(X_j), X_j)$
3. For j from 1 to n, assign X_j consistently with $Parent(X_j)$

Nearly tree-structured CSPs

Conditioning: instantiate a variable, prune its neighbors' domains



Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree

Cutset size c $\Rightarrow$ runtime $O(d^c \cdot (n-c)d^2)$, very fast for small c

Iterative algorithms for CSPs

Hill-climbing, simulated annealing typically work with "complete" states, i.e., all variables assigned

To apply to CSPs:

- allow states with unsatisfied constraints
- operators **reassign** variable values

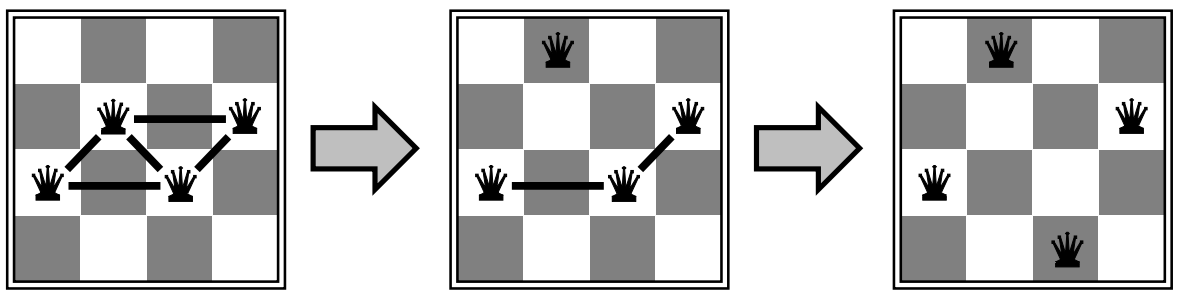
Variable selection: randomly select any conflicted variable

Value selection by **min-conflicts** heuristic:

- choose value that violates the fewest constraints
- i.e., hillclimb with $h(n) =$ total number of violated constraints

Example: 4-Queens

States: 4 queens in 4 columns ($4^4 = 256$ states)
Operators: move queen in column
Goal test: no attacks
Evaluation: $h(n)$ = number of attacks



Summary

CSPs are a special kind of problem:
  states defined by values of a fixed set of variables
  goal test defined by **constraints** on variable values
Backtracking = depth-first search with one variable assigned per node
Variable ordering and value selection heuristics help significantly
Forward checking prevents assignments that guarantee later failure
Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
The CSP representation allows analysis of problem structure
Tree-structured CSPs can be solved in linear time
Iterative min-conflicts is usually effective in practice

Acknowledgment

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