CS174A Lecture 3

Announcements & Reminders

- ***10/02/22: A1 due by Sunday midnight***
- ***10/16/22: A2 due; will be discussed during this week's TA session***
- ***10/27/22: Midterm Exam: 6:00 – 7:30 PM PST, in person, in class***
- ***Start forming your project teams (team size: 3-4)***
    - Project expectations scale with team size
    - 11/8/22: project proposals & teams due
    - 11/22/22: final proposals due

Last Lecture Recap

- ***A Basic Graphics System***
  - Input devices: keyboard, mouse, tablet, touchscreens
  - CPU/GPU
  - Frame Buffer: resolution, single vs. double buffering, color depth, interlaced vs. non-interlaced, refresh rate
  - Output devices: CRT (random-scan & raster), flat-panel (LED, LCD, Plasma), printers, plotters, head-mounted devices, stereo displays
- ***Linear Algebra***
  - Vectors: magnitude, unit vector, normalizing, addition, multiplication, properties
  - Linear combination of vectors: affine, convex, linear independence (today)

Next Up

- *Coordinate systems*
- *Finish up vectors: basis vectors, dot product, cross product*
- *Matrices: square, zero, identity, symmetric, matrix operations*
- *Homogeneous representations of points and vectors*
- *Representing shapes: lines, circles*
- *Transformations: translation, scaling, rotation, shear*

Summary of Scalar, Point & Vector Ops

Red font = makes sense for affine, does not make sense for linear operations

Operands	Operands	Add (+)	Subtract (-)	Multiply (*)
Scalar-Scalar	s_1, s_2	$s = s_1 + s_2$	$s = s_1 - s_2$	$s = s_1 * s_2$
Point-Point	P_1, P_2	$P = a_1*P_1 + a_2*P_2$	$v = P_2 - P_1$	X
Vector-Vector	v_1, v_2	$v = v_1 + v_2$	$v = v_1 - v_2$	X
Scalar-Point	s, P_1	X	X	$P = s * P_1$
Scalar-Vector	s, v_1	X	X	$v = s * v_1$
Point-Vector	P_1, v_1	$P_2 = P_1 + v_1$	$P_2 = P_1 - v_1$	X

Affine & Convex Combinations

- ***Parametric form of lines***
  - Line: infinite in both directions
  - Ray: infinite in one direction
  - Edge (or line segment): limited in both directions
  - Affine combination of points
  - Convex combination of points

- ***Parametric form of planes***
  - Affine combination of points
  - Convex combination of points

Generators and Base Vectors

How many vectors are needed to generate a vector space?

- Any set of vectors that generate a vector space is called a generator set
- Given a vector space $\mathbf{R}^n$ we can prove that we need minimum n vectors to generate all vectors $\mathbf{v}$ in $\mathbf{R}^n$
- A generator set with minimum size is called a basis for the given vector space

Standard Unit Vectors

$$\mathbf{v} = (x_1, \ldots, x_n), \quad x_i \in \Re$$

$$\begin{aligned}
(x_1, x_2, ..., x_n) = \; & x_1(1, 0, 0, \ldots, 0, 0) \\
& + x_2(0, 1, 0, \ldots, 0, 0) \\
& \ldots \\
& + x_n(0, 0, 0, \ldots, 0, 1)
\end{aligned}$$

Standard Unit Vectors

For any vector space R^n:

$$\mathbf{i}_1 = (1, 0, 0, \ldots, 0, 0)$$
$$\mathbf{i}_2 = (0, 1, 0, \ldots, 0, 0)$$
$$\ldots$$
$$\mathbf{i}_n = (0, 0, 0, \ldots, 0, 1)$$

The elements of a vector $\mathbf{v}$ in R^n are the scalar coefficients of the linear combination of the basis vectors

Standard Unit Vectors in 2D & 3D

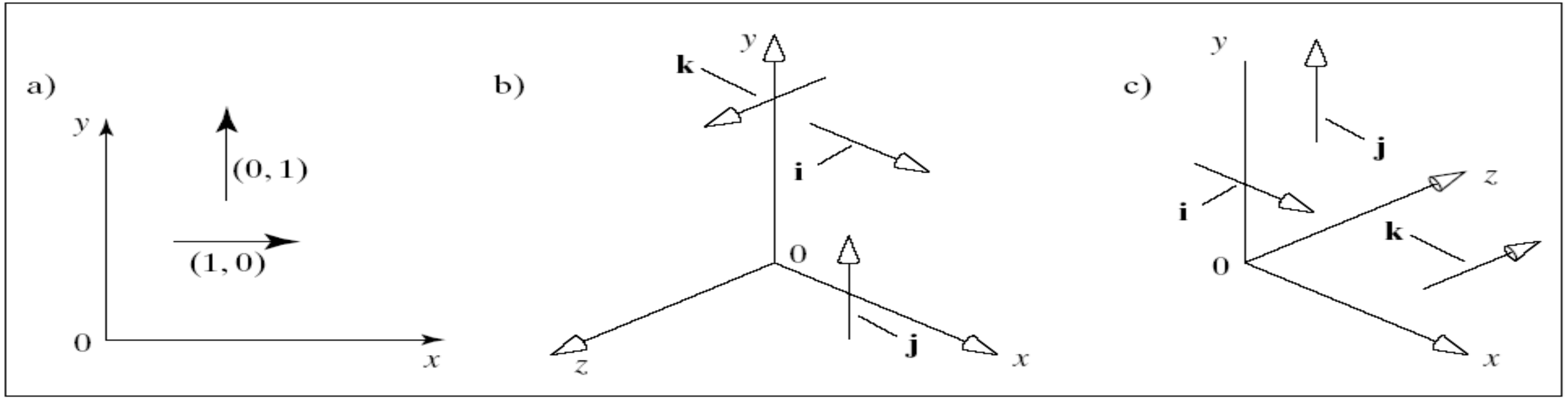
i = (1,0)

j = (0,1)

i = (1,0,0)

j = (0,1,0)

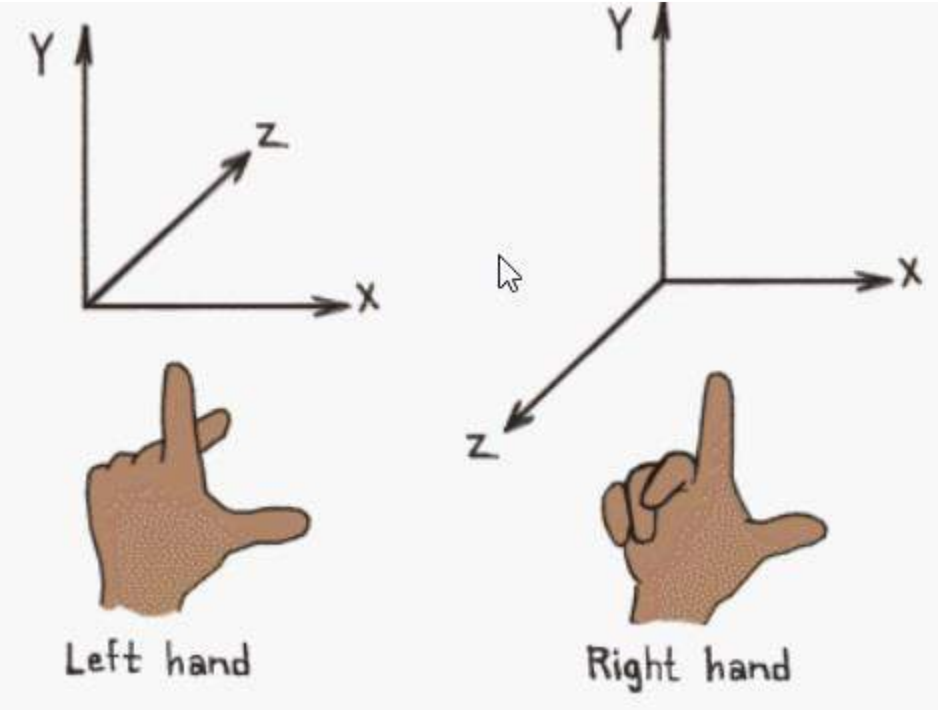
k = (0,0,1)



Right handed

Left handed

Right & Left Hand Coordinate Systems



Representation of Vectors Through Basis Vectors

Given a vector space R^n,
a set of basis vectors B $\{\mathbf{b}_i$ in R^n, $i=1,\ldots n\}$ and
a vector $\mathbf{v}$ in R^n
we can always find scalar coefficients such that:

$$\mathbf{v} = a_1\mathbf{b}_1+\ldots+a_n\mathbf{b}_n$$

So, vector $\mathbf{v}$ expressed with respect to B is:

$$\mathbf{v}_B = (a_1, \ldots, a_n)$$

Dot Products in Graphics

- Another problem dot products solve: Comparing Vectors
    - Trig measurements!

Dot (Scalar) Product

Definition:

$$\mathbf{w}, \mathbf{v} \in \Re^n$$

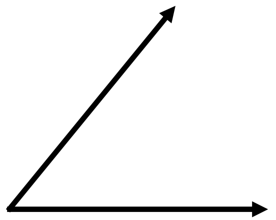
$$\mathbf{w} \cdot \mathbf{v} = \sum_{i=1}^{n} w_i v_i$$

Properties

1. Symmetry: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
2. Linearity: $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$
3. Homogeneity: $(s\mathbf{a}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b})$
4. $|\mathbf{b}|^2 = \mathbf{b} \cdot \mathbf{b}$
5. $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$

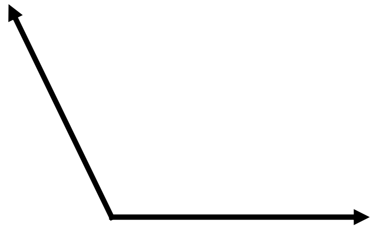
Dot Product and Perpendicularity

From Property 5:



$\mathbf{a} \cdot \mathbf{b} > 0$

$\mathbf{a} \cdot \mathbf{b} = 0$



$\mathbf{a} \cdot \mathbf{b} < 0$

Perpendicular Vectors

Definition

Vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular iff $\mathbf{a} \cdot \mathbf{b} = 0$

Also called "normal" or "orthogonal" vectors

It is easy to see that the standard unit vectors form an orthogonal basis:

$$\mathbf{i} \cdot \mathbf{j} = 0, \qquad \mathbf{j} \cdot \mathbf{k} = 0, \qquad \mathbf{i} \cdot \mathbf{k} = 0$$

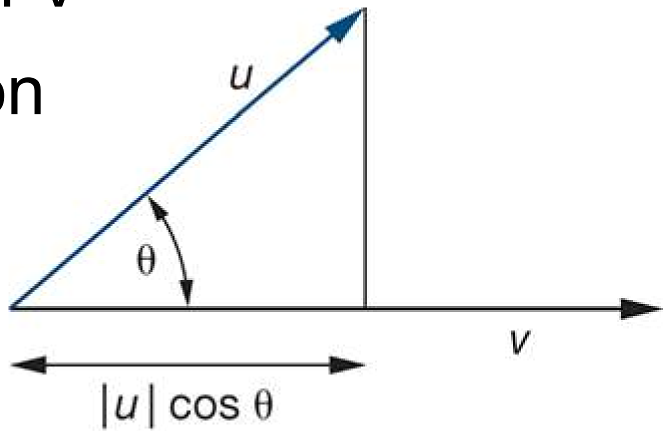
Dot Product: Projection

$u \cdot v = |u||v|\cos(\theta)$

$|u|\cos(\theta) = \frac{u \cdot v}{|v|}$

= projection of vector u on unit vector v

= projection of vector u in v's direction



Dot Products in Graphics

- The problem dot products solve in graphics:
  - Dot with a vector of coefficients. Now you have a linear function that maps a point onto a scalar

    ```
    3x + 4y + 5z = ?
    ```

  - Predictable effect as you adjust a coordinate

Dot Products and Matrices

- What if we want a function that produces not a scalar, but a new point?
  - *This would become a tool for moving points somewhere new!*
- How do we generate three scalar outputs instead of one?

Cross (Vector) Product

Defined only for 3D vectors and with respect to the standard unit vectors

Definition

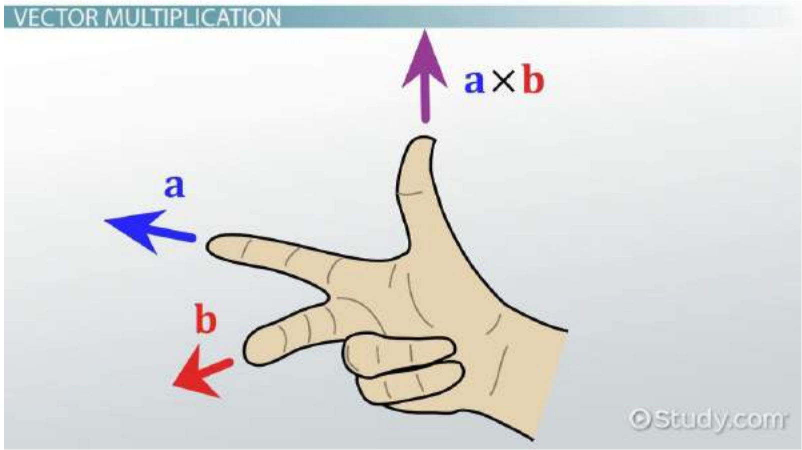
$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$$

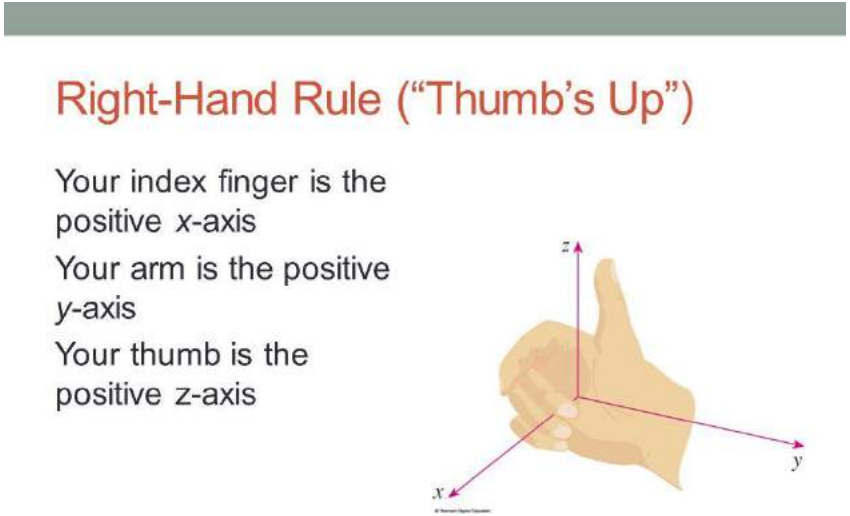
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Properties of the Cross Product

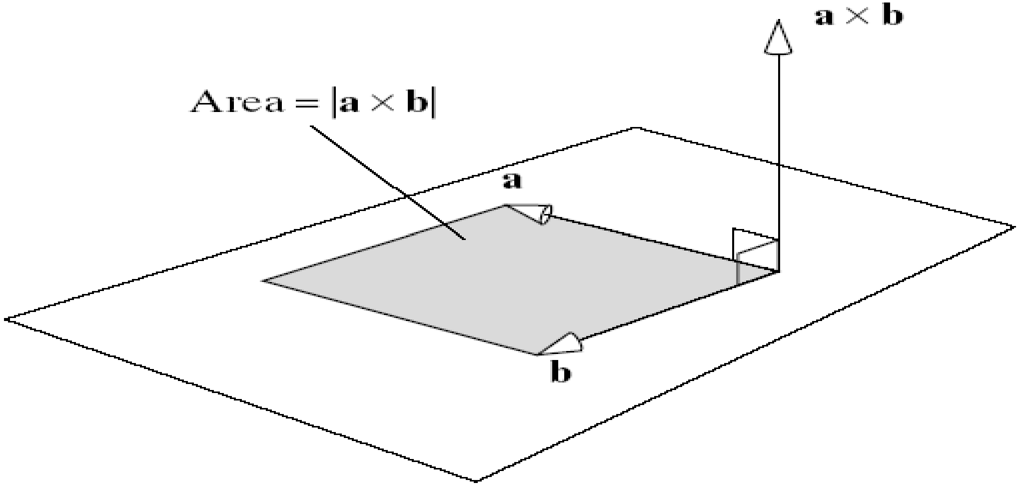
1. $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}$
2. Antisymmetry: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
3. Linearity: $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. Homogeneity: $(s\mathbf{a}) \times \mathbf{b} = s(\mathbf{a} \times \mathbf{b})$
5. The cross product is normal to both vectors: $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$
6. $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\theta)$

Direction of Cross Product





Geometric Interpretation of the Cross Product



Matrices

Rectangular arrangement of scalar elements

Matrix:
Bold upper-case

$$\mathbf{A}_{3\times 3} = \begin{pmatrix} -1 & 2.0 & 0.5 \\ 0.2 & -4.0 & 2.1 \\ 3 & 0.4 & 8.2 \end{pmatrix}$$

$$\mathbf{A} = (\mathbf{A}_{ij})$$

Special Square (n x n) Matrices

Zero matrix: $\mathbf{A}_{ij} = 0$ *for all i,j*

Identity matrix: $\mathbf{I}_n = \begin{cases} \mathbf{I}_{ii} = 1 & \text{for all } i \\ \mathbf{I}_{ij} = 0 & \text{for } i \neq j \end{cases}$

Symmetric matrix: $(\mathbf{A}_{ij}) = (\mathbf{A}_{ji})$

Operations with Matrices

Addition:

$$\mathbf{A}_{m\times n} + \mathbf{B}_{m\times n} = (a_{ij} + b_{ij})$$

Properties:

1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
2. $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$
3. $f(\mathbf{A} + \mathbf{B}) = f\mathbf{A} + f\mathbf{B}$
4. Transpose: $\mathbf{A}^T = (a_{ij})^T = (a_{ji})$

Multiplication

Definition:

$$\mathbf{C}_{m\times r} = \mathbf{A}_{m\times n}\mathbf{B}_{n\times r}$$

$$(\mathbf{C}_{ij}) = (\sum_{k=1}^{n} a_{ik}b_{kj})$$

Properties:

1. $\mathbf{AB} \neq \mathbf{BA}$
2. $\mathbf{A(BC)} = \mathbf{(AB)C}$
3. $f(\mathbf{AB}) = (f\mathbf{A})\mathbf{B}$
4. $\mathbf{A(B+C)} = \mathbf{AB} + \mathbf{AC}$,
   $\mathbf{(B+C)A} = \mathbf{BA} + \mathbf{CA}$
5. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

Inverse of a Square Matrix

Definition

$\mathbf{MM}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$

Important property

$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\,\mathbf{A}^{-1}$

Dot Product as a Matrix Multiplication

Representing vectors as column matrices:

$$
\begin{aligned}
\mathbf{a} \cdot \mathbf{b} &= \mathbf{a}^T \mathbf{b} \\
&= \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\
&= a_1 b_1 + a_2 b_2 + a_3 b_3
\end{aligned}
$$