Math 61 HW #1

1) $1+3+5+\ldots+(2n-1)=n^2$

$n=1: \; 1=1^2$ ✓

$n=2: \; (2-1)+(4-1)=2^2$ ✓

Assume: $1+3+5+\ldots+(2n-1)=n^2$

Prove: $1+3+5+\ldots+(2n-1)+(2n+1)=(n+1)^2$

$(1+3+5+\ldots+(2n-1))+(2n+1)=(n+1)^2$

$n^2+2n+1=(n+1)^2$

$\boxed{n^2+2n+1=n^2+2n+1 \checkmark}$

2) $1+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$

$n=1: \; 1^2=\frac{1(1+1)(2+1)}{6}=\frac{6}{6}=1$ ✓

$n=2: \; 1^2+2^2=\frac{2(2+1)(4+1)}{6}=5$ ✓

Assume: $1+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$

Prove: $1+2^2+3^2+\ldots+n^2+(n+1)^2=\frac{(n+1)(n+2)(2n+3)}{6}$

$(1+2^2+3^2+\ldots+n^2)+(n+1)^2=\frac{(n+1)(n+2)(2n+3)}{6}$

$\frac{n(n+1)(2n+1)}{6}+(n+1)^2=\frac{(n+1)(n+2)(2n+3)}{6}$

$\frac{n(2n+1)}{6}+(n+1)=\frac{(n+2)(2n+3)}{6}$

$\frac{n(2n+1)+(6n+6)}{6}=\frac{(2n^2+7n+6)}{6}$

$\frac{2n^2+n+6n+6}{6}=\frac{(2n^2+7n+6)}{6}$

$\boxed{\frac{2n^2+7n+6}{6}=\frac{2n^2+7n+6}{6} \checkmark}$

3) $\frac{1}{2^2-1}+\frac{1}{3^2-1}+\ldots+\frac{1}{(n+1)^2-1}=\frac{3}{4}-\frac{1}{2(n+1)}-\frac{1}{2(n+2)}$

$n=1: \; \frac{1}{2^2-1}=\frac{3}{4}-\frac{1}{2(2)}-\frac{1}{2(3)}=\frac{1}{3}$ ✓

$n=2: \; \frac{1}{2^2-1}+\frac{1}{3^2-1}=\frac{3}{4}-\frac{1}{2(3)}-\frac{1}{2(4)}=\frac{11}{24}$ ✓

Assume: $\frac{1}{2^2-1}+\frac{1}{3^2-1}+\ldots+\frac{1}{(n+1)^2-1}=\frac{3}{4}-\frac{1}{2(n+1)}-\frac{1}{2(n+2)}$

Prove: $\frac{1}{2^2-1}+\ldots+\frac{1}{(n+1)^2-1}+\frac{1}{(n+2)^2-1}=\frac{3}{4}-\frac{1}{2(n+2)}-\frac{1}{2(n+3)}$

$\left(\frac{1}{2^2-1}+\ldots+\frac{1}{(n+1)^2-1}\right)+\frac{1}{(n+2)^2-1}=\frac{3}{4}-\frac{1}{2(n+2)}-\frac{1}{2(n+3)}$

$\frac{3}{4}-\frac{1}{2(n+1)}-\frac{1}{2(n+2)}+\frac{1}{(n+2)^2-1}=\frac{3}{4}-\frac{1}{2(n+2)}-\frac{1}{2(n+3)}$

$-\frac{1}{2(n+1)}+\frac{1}{(n+2)^2-1}=-\frac{1}{2(n+3)}$

$\frac{-(n+2)^2-1-2(n+1)}{[2(n+1)][(n+2)^2-1]}=\frac{-1}{2(n+3)}$

$2(n+3)\left[(n+2)^2-1-2n-2\right]=2n+2((n+2)^2-1))$

$(2n+6)(n^2+2n+1)=(2n+2)(n^2+4n+3)$

$\boxed{2n^3+10n^2+14n+6=2n^3+10n^2+14n+6 \checkmark}$

4) 7^n-1 is divisible by 6 for all $n\geq1$

$n=1: \; 7^1-1=6 \to$ divisible $\to 6\times1$

$n=2: \; 7^2-1=48 \to 6\times8$

Assume: 7^n-1 is divisible

Prove: $7^{n+1}-1$ is divisible

$\frac{7^n-1}{6}=c$

$7^n-1=6c$

$7^{n+1}-1=7(7^n)-1$

$7^{n+1}-1=7(7^n-1)+6$

$7^{n+1}-1=7(6c)+6$

$7^{n+1}-1=42c+6=$

$\boxed{7^{n+1}-1=6(7c+1) \checkmark}$

5) 11^n-6 is divisible by 5 for all $n\geq1$

$n=1: \; 11^1-6=5 \to \frac{5}{5}=1$

$n=2: \; 11^2-6=115 \to \frac{115}{5}=23$

Assume: 11^n-6 is divisible by 5

Prove: $11^{n+1}-6$ is divisible by 5

$\frac{11^n-6}{5}=c$

$11^n-6=5c$

$11^{n+1}-6=11(11^n-6)+60$

$11^{n+1}-6=11(5c)+60$

$\boxed{11^{n+1}-6=5(11c+12) \checkmark}$

6) $\frac{1}{2n}\leq\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots(2n)}$, for $n\geq1$

$n=1: \; \frac{1}{2}\leq\frac{(2-1)}{2}=\frac{1}{2}$ ✓

$n=2: \; \frac{1}{4}\leq\frac{1\cdot3}{2\cdot4}=\frac{3}{8}$ ✓

Assume: $\frac{1}{2n}\leq\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots(2n)}$

Prove: $\frac{1}{2n+2}\leq\frac{1\cdot3\cdot\ldots\cdot(2n-1)\cdot(2n+1)}{2\cdot4\cdot\ldots\cdot(2n+2)}$

$\frac{1}{2n+2}\leq\left(\frac{1\cdot3\cdot\ldots\cdot(2n-1)}{2\cdot4\cdot2n}\right)\left(\frac{2n+1}{2n+2}\right)$

$\frac{1}{2n+2}\leq\frac{1}{2n}\left(\frac{2n+1}{2n+2}\right)$

$\frac{1}{2n+2}\leq\frac{2n+1}{2n}\left(\frac{1}{2n+2}\right)$

$\boxed{1\leq1+\frac{1}{2n} \checkmark}$

Scanned with CamScanner

7) $2n+1 \le 2^n$ for $n \ge 3$

$n=3$: $6+1 \le 2^3 = 8$ ✓

$n=4$: $8+9 \le 2^4 = 16$ ✓

Assume: $2n+1 \le 2^n$

Prove: $2n+3 \le 2^{n+1}$

$2n+3 \le 2^n(2)$

$2n+3 \le (2n+1)(2)$

$2n+3 \le 4n+2$

$1 \le 2n$

$\boxed{n \ge \frac{1}{2} \checkmark}$

8) $2^n \ge n^2$ for $n \ge 4$

$n=4$: $2^4 = 16 \ge 16$ ✓

$n=5$: $2^5 = 32 \ge 5^2 = 25$ ✓

Assume: $2^n \ge n^2$

Prove: $2^{n+1} \ge (n+1)^2$

$2(2^n) \ge (n+1)^2$

$2n^2 \ge n^2 + 2n + 1$

$n^2 + n^2 \ge n^2 + 2n + 1$

for $n \ge 4$, $n^2 > 2n+1$

$\boxed{n^2 + n^2 \ge n^2 + 2n + 1 \checkmark}$

9) $r^0 + r^1 + \dots + r^n < \frac{1}{1-r}$, $n \ge 0$, $0 < r < 1$

$S_n = \frac{a(r^n - 1)}{r-1}$

$n=0$: $r^0 = 1 < \frac{1}{1-r}$ ✓

$n=1$: $r^0 + r^1 = 1 + r < \frac{1}{1-r}$ ✓

Assume: $r^0 + r^1 + \dots + r^n < \frac{1}{1-r}$

Prove: $r^0 + r^1 + \dots + r^{n+1} < \frac{1}{1-r}$

$\frac{a(r^n - 1)}{r-1} < \frac{1}{1-r}$

$(r^0 + r^1 + \dots + r^n) + r^{n+1} < \frac{1}{1-r}$

$\frac{1}{1-r} + r^{n+1} < \frac{1}{1-r}$

$\frac{r^{n+1} - r^{n+2}}{1-r} < \frac{1}{1-r}$

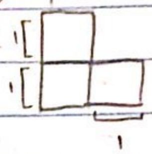
$r^{n+1} - r^{n+2} < 1$

$r^{n+1} < 1 + r^{n+2}$

$\boxed{1 < 1 + r^n \checkmark}$

10) Any $2^n \times 2^n$ L-shape can be tiled w/ trominoes

$n=1$:



✓

Assume: $2^n \times 2^n$ L-shape works

Prove: $2^{n+1} \times 2^{n+1}$ L-shape works

?

11) $U = \{1, 2, 3, \dots, 10\}$

$A = \{1, 4, 7, 10\}$

$B = \{1, 2, 3, 4, 5\}$

$C = \{2, 4, 6, 8\}$

a) $A \cup B - C$

$A \cup B = \{1, 2, 3, 4, 5, 7, 10\}$

$\boxed{A \cup B - C = \{1, 3, 5, 7, 10\}}$

b) $A \cap B \cup C$

$A \cap B = \{1, 4\}$

$\boxed{A \cap B \cup C = \{1, 2, 4, 6, 8\}}$

c) $(A \cup B) - (C - B)$

$A \cup B = \{1, 2, 3, 4, 5, 7, 10\}$

$C - B = \{6, 8\}$

$\boxed{(A \cup B) - (C - B) = \{1, 2, 3, 4, 5, 7, 10\}}$

12) a) Cardinality of $\emptyset$

$\boxed{|\emptyset| = 0}$

b) Cardinality of $\{\emptyset\}$

$\boxed{|\{\emptyset\}| = 1}$

c) Cardinality of $\{a, b, a, c\}$

$\boxed{|\{a, b, a, c\}| = 3}$

d) Cardinality of $\{\{a\}, \{a, b\}, \{a, c\}, a, b\}$

$\boxed{= 5}$

13) Show $A \neq B$

a) $A=\{1,2\}$, $B\{x \mid x^3-2x^2-x+2=0\}$

If $A=B$, then $A \subseteq B$ and $B \subseteq A$

$x=1 \to 1^3-2(1)^2-1+2=0$ ✓

$x=2 \to 2^3-2(2)^2-2+2=0$ ✓

$\therefore A \subseteq B$

$x=-1 \to (-1)^3-2(-1)^2-(-1)+2=0$ ✓

$\therefore -1 \in B$, but $-1 \notin A$

$\boxed{A \neq B}$

b) $A=\{1,3,5\}$, $B=\{n \in \mathbb{Z} \mid n>0 \text{ and } n^2-1 \leq n\}$

If $A=B$, then $A \subseteq B$ and $B \subseteq A$

$x=1 \to 1^2-1=0 \leq 1$ ✓

$x=3 \to 3^2-1=8 \not\leq 3$

$\therefore 3 \in A$, but $3 \notin B$

$\therefore \boxed{A \neq B}$

14) Prove A is not a subset of B

a) $A=\{1,2,3\}$, $B=\{1,2\}$

$A \subseteq B$ iff $\forall_a \in A \Rightarrow a \in B$

$3 \in X$ but $3 \notin Y$

$\therefore$ $\boxed{\text{A is not a subset of B}}$

b) $A=\{1,2,3\}$, $B=\emptyset$

$A \subseteq B$ iff $\forall_a \in A \Rightarrow a \in B$

$1 \in A$, but $1 \notin B$

$\therefore$ $\boxed{\text{A is not a subset of B}}$

15) $U=\{\text{ppl. surveyed}\}$, $|U|=151$

$A=\{\text{ppl. watched L/D}\}$, $|A|=68$

$B=\{\text{ppl. watched 25}\}$, $|B|=61$

$C=\{\text{ppl. watched Ten.}\}$, $|C|=52$

$|A \cap B|=16$, $|A \cap C|=25$

$|B \cap C|=19$, $N=\{\text{none}\}$, $|N|=26$

$|A \cap B \cap C|=?$

$|A \cup B|=113$, $|B \cup C|=94$, $|C \cup A|=95$

$\overline{A \cup B}=12$

12 people watch only Ten.

$|C|-12=40$ ppl. watched Ten + other

x ppl. watched all 3

44 ppl. watched Ten + other

$44-40-x=0$

$x=4$

$\boxed{\text{4 people watched all 3}}$

16) $X=\{1,2\}$, $Y=\{a\}$, $Z=\{\alpha,\beta\}$

a) $X \times Y \times Z$

$X \times Y=\{(1,a),(2,a)\}$

$\boxed{X \times Y \times Z=\{((1,a),\alpha),((1,a),\beta),((2,a),\alpha),((2,a),\beta)\}}$

b) $X \times X \times X$

$X \times X=\{(1,1),(1,2),(2,1),(2,2)\}$

$\boxed{X \times X \times X=\{((1,1),1),((1,1),2),((1,2),1),((1,2),2),((2,1),1),((2,1),2),((2,2),1),((2,2),2)\}}$

c) $Z \times Y \times X$

$Z \times Y=\{(\alpha,a),(\beta,a)\}$

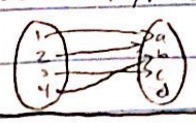
$\boxed{Z \times Y \times X=\{((\alpha,a),1),((\alpha,a),2),((\beta,a),1),((\beta,a),2)\}}$

17) $X=\{1,2,3,4\}$, $Y=\{a,b,c,d\}$

a) $\{(1,a),(2,a),(3,c),(4,b)\}$

Yes, neither 1-to-1 nor onto

Domain: X, Range: $\{a,c,b\}$

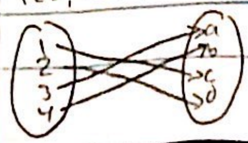


b) $\{(1,c),(2,a),(3,b),(4,c),(2,d)\}$

$\boxed{\text{No}}$

c) $\{(1,c),(2,d),(3,a),(4,b)\}$

Yes, both



D: X

R: Y

18) a) $f(n) = n+1$

1-to-1: $f(m) = f(n)$

$m+1 = n+1$

$m = n$ ✓

onto: $y = n+1$

$n = y-1$ ✓

$\boxed{\text{Both}}$

b) $f(n) = n^2 - 1$

1-to-1: $f(m) = f(n)$

$m^2 - 1 = n^2 - 1$

$m^2 = n^2$

$\pm m = \pm n$ X

onto: $y = n^2 - 1$

$n = \sqrt{y-1}$

$y \geq 1$ X

$\boxed{\text{Neither}}$

c) $f(n) = n^3$

1-to-1: $f(m) = f(n)$

$m^3 = n^3$

$m = n$ ✓

onto: $y = n^3$

$n = \sqrt[3]{y}$ X

↳ Ex) $y = 2 \rightarrow n \notin \mathbb{Z}$

$\boxed{\text{1-to-1}}$

19) A function $f(x)$ s.t. $\forall x_1, x_2 \in X$, if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

20) A function s.t. $\forall y \in Y$, $\exists x \in X$ s.t. $f(x) = y$