Math 33A Sheet 4

Chapter 3.1

Ex.2) $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

$\text{rref}(\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \end{array}\right]) =$

$x_1 = -2x_2 - 3x_3$

$x_2 = t \quad x_3 = s$

$\begin{bmatrix} -2t-3s \\ t \\ s \end{bmatrix}$

$= t\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

$\boxed{\ker(A) = \text{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right)}$

Ex 3) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$0 = 0$

$\boxed{\ker(A) = \mathbb{R}^2}$

Ex 8)* $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$\text{rref}(A) =$

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 + x_2 + x_3 = 0$

$x_1 = -x_2 - x_3$

$x_2 = t, \; x_3 = s$

$\begin{bmatrix} -t-s \\ t \\ s \end{bmatrix}$

$= t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\boxed{\ker(A) = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)}$

Ex 14) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

$\text{rref}(A) = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 + 2x_2 + 3x_3 = 0$

$x_2 = \frac{1}{2}x_1, \quad x_3 = \frac{1}{3}x_1$

all variables depend on each other

$\boxed{\ker(A) = \text{span}\left(\begin{bmatrix} 1 \\ \cdot \\ \cdot \end{bmatrix}\right)}$

Ex 17) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$T(\vec{v}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$v_1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + v_2\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$\boxed{\text{img}(T) \text{ is the plane spanned by } \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 4 \end{bmatrix} \text{ in } \mathbb{R}^2}$

Ex 19)* $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & -8 \end{bmatrix}$

$T(\vec{v}) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & -8 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$

$= v_1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + v_2\begin{bmatrix} 2 \\ -4 \end{bmatrix} + v_3\begin{bmatrix} 3 \\ -6 \end{bmatrix} + v_4\begin{bmatrix} 4 \\ -8 \end{bmatrix}$

$\boxed{\text{img}(T) \text{ is the line spanned by } \begin{bmatrix} 1 \\ -2 \end{bmatrix} \text{ in } \mathbb{R}^2}$

Ex 24) Orthogonal projection onto the plane $x+2y+3z=0$ in $\mathbb{R}^3$

The image consists of all points in the function's target space, therefore

$\boxed{img(T) = x+2y+3z=0 \text{ in } \mathbb{R}^3}$

A projection of a vector perpendicular to the plane makes $A\vec{v}=0$, therefore

$\boxed{\ker(T) = span\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right)}$

Ex 34) $\ker(T) = span\left(\begin{bmatrix}-1\\1\\2\end{bmatrix}\right)$ in $\mathbb{R}^3$

$t\begin{bmatrix}-1\\1\\2\end{bmatrix} \to \begin{bmatrix}-t\\t\\2t\end{bmatrix}$

$x_1=-x_2,\ x_1=2x_3$

$x_1+x_2=0 \quad x_1-2x_3=0$

$A = \begin{bmatrix}1&1&0\\1&0&-2\\0&0&0\end{bmatrix}$

$\boxed{T(\vec{v}) = \begin{bmatrix}1&1&0\\1&0&-2\\0&0&0\end{bmatrix}\vec{v}}$

Chapter 3.2

Ex 1)* $W = \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} : x+y+z=1\right\}$

$0+0+0 \neq 1$

$\boxed{\text{Not a subspace of } \mathbb{R}^3\text{, doesn't contain the zero vector of } \mathbb{R}^3}$

Ex 4) $span(\vec{v}_1, \ldots \vec{v}_m) = img(\vec{v}_1, \ldots, \vec{v}_m)$

the image is a subspace, therefore the $span(\vec{v}_1, \ldots, \vec{v}_m)$ must be a subspace

Ex 8) $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}$

$\begin{bmatrix}1&2&3\\2&3&4\end{bmatrix} -2(I)$

$\begin{bmatrix}1&2&3\\0&-1&-2\end{bmatrix} \begin{matrix}+2(II)\\ \div -1\end{matrix}$

$\begin{bmatrix}1&0&-1\\0&1&2\end{bmatrix}$

$\boxed{-1\begin{bmatrix}1\\2\end{bmatrix} + 2\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}}$

$\begin{bmatrix}3\\4\end{bmatrix}$ is a lin. combo. of $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\3\end{bmatrix}$

Ex 13) $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}$

$\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$

$\boxed{\begin{bmatrix}1\\2\end{bmatrix} \text{ is redundant, they are linearly dependant}}$

Ex 15) $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}$

$\begin{bmatrix}1&2&3\\2&3&4\end{bmatrix} -2(I)$

$\begin{bmatrix}1&2&3\\0&-1&-2\end{bmatrix} \begin{matrix}+2(II)\\ \div -1\end{matrix}$

$\begin{bmatrix}1&0&-1\\0&1&2\end{bmatrix}$

$-\begin{bmatrix}1\\2\end{bmatrix} + 2\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}$

$\boxed{\begin{bmatrix}3\\4\end{bmatrix} \text{ is redundant, linearly dependant}}$

Scanned with CamScanner

Ex22) $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{matrix} \\ -2(I) \end{matrix}$

$\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

$3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$3\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \vec{0}$

$\boxed{\ker(A) = \text{span}\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right)}$

Ex24) $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ is redundant

$2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \vec{0}$

$2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{0}$

$\boxed{\ker(A) = \text{span}\left(\begin{bmatrix} 2 \\ 3 \\ -1 \\ 0 \end{bmatrix}\right)}$

Ex27) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{matrix} \\ -4(I) \end{matrix}$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} \begin{matrix} \\ \times -\frac{1}{3} \end{matrix}$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \begin{matrix} -2(II) \\ \\ \end{matrix}$

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

$-1\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$\boxed{\text{basis of } A \text{ is } \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}}$

Ex30)* $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix} \begin{matrix} \\ -I \\ -I \end{matrix}$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 2 & 6 \end{bmatrix} \begin{matrix} -II \\ \\ -2(II) \end{matrix}$

$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & -2 \end{bmatrix} \begin{matrix} \\ +III \\ \times -\frac{1}{2} \end{matrix}$

$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} +3(III) \\ \\ \\ \end{matrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

$\boxed{\text{basis of } A \text{ is } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}}$

Q1) Image:

$\vec{v}_1 = T(\vec{w}_1), \vec{v}_2 = T(\vec{w}_2)$

$\vec{v}_1 + \vec{v}_2 = T(\vec{w}_1) + T(\vec{w}_2) = T(\vec{w}_1 + \vec{w}_2)$

↳ closed under addition ✓

$\vec{v}_1 = T(\vec{w})$

$k\vec{v}_1 = kT(\vec{w}) = T(k\vec{w})$

↳ closed under multiplication ✓

$\vec{0} = A(\vec{0}) = T(\vec{0})$

↳ contains the zero vector ✓

Kernel:

$T(\vec{v}) + T(\vec{w}) = T(\vec{v} + \vec{w})$

$T(\vec{0}) + T(\vec{0}) = T(\vec{0} + \vec{0}) = \vec{0}$

↳ closed under addition

$kT(\vec{v}) = T(k\vec{v})$

$kT(\vec{0}) = T(k\vec{0}) = \vec{0}$

↳ closed under multiplication

The kernel is all zeros of the linear transformation

↳ contains the origin