Math 33A MT2 Review

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Generalized Vector Spaces (Angelopoulos Spring 2017)

(a) *(10 points)* Let $\mathcal{V} = span\{\cos(x), \sin(x)\}$ be the space of functions spanned by the sine and cosine functions. Show that the transformation $T : \mathcal{V} \to \mathcal{V}$ given by

$$T(f) = f'' + af' + bf \text{ for constants } a, b \in \mathbb{R}$$

is linear and compute its matrix with respect to the basis

$$\mathcal{B} = \{\cos(x) - \sin(x), \cos(x) + \sin(x)\}$$

of $\mathcal{V}$.

(b) *(10 points)* State a condition for a, b so that T is an isomorphism.

Hint: Try to formulate the condition for T to be an isomorphism with respect to the matrix that you computed in part (a).

SOLUTION
(Angelopoulos Spring 2017)

a. Linearity conditions: $T(0) = 0$, $T(f+g) = T(f) + T(g)$, $T(cf) = cT(f)$

For this problem, $T(f) = f'' + af' + bf$, and every $f \in \mathcal{V}$

- $T(0) = 0'' + a0' + b0 = 0$, $c, d \in \mathbb{R}$, $f, g \in \mathcal{V}$
- $T(cf+dg) = (cf+dg)'' + a(cf+dg)' + b(cf+dg) \longrightarrow$ derivative is linear

$\longrightarrow cf'' + dg'' + acf' + adg' + bcf + bdg \longrightarrow$ split into f, g

$c(f'' + af' + bf) + d(g'' + ag' + bg) = cT(f) + dT(g)$

Because all conditions satisfied, T is linear.

SOLUTION
(Angelopoulos Spring 2017)

Now, basis $\beta = \{\underbrace{\cos(x) - \sin(x)}_{\vec{v}_1}, \underbrace{\cos(x) + \sin(x)}_{\vec{v}_2}\}$, apply transformation to each basis vector

$$T(\vec{v}_1) = -\cos(x) + \sin(x) - a\sin(x) - a\cos(x) + b\cos(x) - b\sin(x)$$
$$= -\vec{v}_1 - a\vec{v}_2 + b\vec{v}_1 = (b-1)\vec{v}_1 - a\vec{v}_2$$

In basis β, $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = (b-1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} - a\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b-1 \\ -a \end{bmatrix}$$

Can perform similar analysis for $\vec{v}_2$

$$T(\vec{v}_2) = -\cos(x) - \sin(x) - a\sin(x) + a\cos(x) + b\cos(x) + b\sin(x) = (b-1)\vec{v}_2 - a\vec{v}_1$$

$$\Rightarrow T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = (b-1)\begin{bmatrix} 0 \\ 1 \end{bmatrix} - a\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -a \\ b-1 \end{bmatrix}$$

Thus, our transformation matrix M in basis β is

$$[T]_\beta = M = \begin{pmatrix} b-1 & -a \\ -a & b-1 \end{pmatrix}$$

SOLUTION
(Angelopoulos Spring 2017)

b. For T to be an isomorphism, this means M is an isomorphism, which means M is invertible; for M invertible, $\det(M) \neq 0$, so

$$\det(M) = (b-1)^2 - a^2 \neq 0 \Rightarrow \underline{b-1 \neq \pm a}$$

Image and Rank (Gallauer Spring 2016)

3. (10 points) Consider the matrix

$$A = \begin{bmatrix} -2 & 2 & 1 \\ 2 & -2 & -1 \\ -6 & 8 & 6 \\ 8 & -12 & -10 \end{bmatrix}.$$

(a) Find a basis for $\text{image}(A)^{\perp}$.

(b) Compute $\text{rank}(A)$.

(c) Find all 2×2 matrices which are both orthogonal and skew-symmetric.

SOLUTION
(Gallauer Spring 2016)

a) $\text{image}(A)^{\perp} = \ker(A^T)$, so we can find a basis for $\ker(A^T)$

$$A^T = \begin{bmatrix} -2 & 2 & -6 & 8 \\ 2 & -2 & 8 & -12 \\ 1 & -1 & 6 & -10 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 - x_2 + 2x_4 = 0$ $\quad$ $x_4 = t$ $\quad$ $x_2 = r$

$x_3 = 2x_4$ $\quad$ $x_3 = 2t$ $\quad$ $x_1 = r - 2t$

$$\begin{pmatrix} r-2t \\ r \\ 2t \\ t \end{pmatrix} = r\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

Basis $\ker(A^T) = \text{image}(A)^{\perp}$ is $\boxed{\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}}$

SOLUTION
(Gallauer Spring 2016)

b) $\text{rank}(A) = \text{rank}(A^T) = \#$ of non-zero rows $= \boxed{2}$

c) <u>Skew-symmetric</u>

$A^T = -A$

<u>Orthogonal</u>

$A^T A = I_2$

Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

<u>Skew-symm</u>

$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$

$a = 0,$
$d = 0,$
$b = -c$

<u>orthogonal</u>

$\begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}\begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} c^2 & 0 \\ 0 & c^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad c = \pm 1$

$\boxed{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$

Image and Rank (Wilis Winter 2019)

Let $\mathcal{B} = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$ be a basis for $\mathbb{R}^2$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be two linear transformations that treat this basis in the following manner:

$$T(\vec{\mathbf{v}}_1) = \vec{\mathbf{v}}_2, \quad T(\vec{\mathbf{v}}_2) = \vec{\mathbf{v}}_1, \qquad S(\vec{\mathbf{v}}_1) = \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2, \quad S(\vec{\mathbf{v}}_2) = \vec{\mathbf{v}}_2$$

(a) [4 pts] Write down the matrix B for the composition $T \circ S$ in $\mathcal{B}$-coordinates.

[6 pts] Prove that, if we assume that S w an orthogonal transformation, then the basis vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ *cannot* be orthogonal to each other.

SOLUTION
(Wilis Winter 2019)

a) $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_B \qquad S = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}_B$

$$T \circ S = TS$$
$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}_B$$

SOLUTION
(Wilis Winter 2019)

b) Orthogonal transformations preserve length!

$$S(\vec{v_1}) = \vec{v_1} - \vec{v_2}$$

$$\|\vec{v_1}\| = \|\vec{v_1} - \vec{v_2}\|$$

$$\|\vec{v_1}\|^2 = \|\vec{v_1}\|^2 + \|\vec{v_2}\|^2 - 2\vec{v_1} \cdot \vec{v_2}$$

$$\Rightarrow \quad \|\vec{v_2}\|^2 = 2\vec{v_1} \cdot \vec{v_2}$$

$\because$ $\vec{v_2}$ is part of basis β, $\|\vec{v_2}\| \neq 0$

$\therefore$ $\vec{v_1} \cdot \vec{v_2} \neq 0$

Thus, $\vec{v_1}$ and $\vec{v_2}$ cannot be orthogonal. $\square$

Change of Basis (Gallauer Spring 2016)

1. (10 points) Let $\mathcal{B}$ be the basis of $\mathbb{R}^3$ given by the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -2 \\ -1 \\ 4 \end{bmatrix}.$$

(a) Suppose the $\mathcal{B}$-coordinate vector of $\vec{x}$ is $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. What is $\vec{x}$?

Change of Basis (Gallauer Spring 2016)

(b) Find the $\mathcal{B}$-coordinate vector of $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}$.

(c) Find the $\mathcal{B}$-matrix of $A = \begin{bmatrix} 5 & -5 & 1 \\ -1 & 2 & 0 \\ -31 & 37 & -5 \end{bmatrix}$.

SOLUTION
(Gallauer Spring 2016)

a. $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -2 \\ -1 \\ 4 \end{bmatrix}$ $\longrightarrow$ in our basis β, becomes $\longrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$[\vec{x}]_\beta = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\vec{x} = 0\vec{v}_1 + 1\vec{v}_2 + 2\vec{v}_3 = \underline{\begin{bmatrix} -3 \\ -1 \\ 10 \end{bmatrix}}$

b. Now, we do the inverse: if we notice, the above operation can be represented by

$$M[\vec{x}]_\beta = \vec{x}, \quad \text{for} \quad M = \begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & -1 \\ 1 & 2 & 4 \end{pmatrix}$$

We now want the opposite: to transform $y \longrightarrow [y]_\beta$, so we need the inverse of the matrix above, M, and then apply that to y

$$M^{-1} = \begin{pmatrix} -6 & 8 & -1 \\ 5 & -6 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \quad [y]_\beta = M^{-1}y = \underline{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}$$

SOLUTION
(Gallauer Spring 2016)

c. Finally, we use our change of basis formula, $[A]_\beta = M^{-1}AM$, to solve

$$[A]_\beta = \begin{pmatrix} -6 & 8 & -1 \\ 5 & -6 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 5 & -5 & 1 \\ -1 & 2 & 0 \\ -31 & 37 & -5 \end{pmatrix}\begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & -1 \\ 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

To conceptualize why this formula works, it is best to understand what every individual transformation does. We know A in our original basis (γ), so we can convert a given vector $[\vec{v}]_\beta$ to γ, then apply A, then convert it back to β. Since M transfers from β to γ, and M^{-1} transfers γ to β, we can put these together to create an overall transformation

$$[A]_\beta [v]_\beta = M^{-1}AM\,[v]_\beta$$

Geometric Interpretation (Keranen Spring 2017)

2. *a*) (4 pts) Let $\mathfrak{B}$ be the standard basis of $\mathbb{R}^2$. Describe geometrically the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose $\mathfrak{B}$-matrix is

$$A = \begin{pmatrix} -0.28 & 0.96 \\ 0.96 & 0.28 \end{pmatrix}.$$

(Hint: $(0.96)^2 + (0.28)^2 = 1$.)

b) (6 pts) Write down a basis $\mathfrak{B}'$ for $\mathbb{R}^2$ such that the $\mathfrak{B}'$-matrix of T is diagonal, and write down the $\mathfrak{B}'$-matrix of T.

SOLUTION

(Keranen Spring 2017)

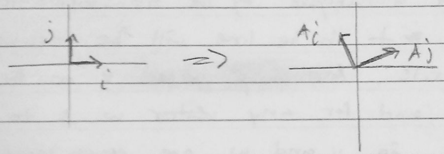
Keranen Spring 2017

Problem 2

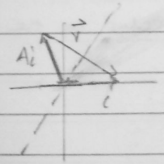
a) From the hint and the fact that the dot product of the columns of A is 0, we see that A is orthogonal.

Since A is orthogonal, there are two possible transformations – either A is a rotation or a reflection.

taking the transformation on the standard basis vectors, i and j, shows that A is a reflection about some line:



Now it remains to find the line of reflection.



The line of reflection is perpendicular to the vector $\vec{v}$, which begins at the head of vector Ai and ends at the head of vector i.

$\Rightarrow$ We can solve for $\vec{v}$: $\vec{v} = i - Ai = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -0.28 \\ 0.96 \end{bmatrix} = \begin{bmatrix} 1.28 \\ -0.96 \end{bmatrix}$

Then the vector $\perp$ to $\vec{v}$ is $\begin{bmatrix} 0.96 \\ 1.28 \end{bmatrix} \to \begin{bmatrix} 0.12 \\ 0.16 \end{bmatrix} \to \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix} \to \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

So the line of reflection is the line through the origin spanned by the vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

SOLUTION
(Keranen Spring 2017)

b) The basis β' of $\mathbb{R}^2$ s.t. $[T]_{\beta'}$ is diagonal is an eigenbasis for T.

One way to find this basis would be to directly find the eigenvectors of T via the characteristic polynomial: $\det(A - \lambda I) = 0$.

However, we can more easily find this basis from the geometric interpretation of the transformation.

$$\beta' = \left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \end{bmatrix} \right\}$$

$$[T]_{\beta'} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Since the transformation is the reflection about the line spanned by $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, we know that any vector on the line will be unchanged by the transformation, and any vector $\perp$ to the line will be rotated by 180°.

In other words, for any vector v on the line, $Av = 1 \cdot v$ and for any vector w $\perp$ to the line $Av = -1 \cdot v$. So v and w are eigenvectors and since they live in $\mathbb{R}^2$, they form an eigenbasis.

So our basis consists of any vector on the line and any vector $\perp$ to the line.

Dimension and Image (Nicole Spring 2014)

Problem 3: (10 pts)

Find an explicit 4-by-5 matrix B such that

$$3\dim(\mathrm{Ker}(B)) = 2\dim(\mathrm{Im}(B)) + 7,$$

or prove it does not exist.

SOLUTION
(Nicole Spring 2014)

Start with rank-nullity theorem for a 4x5 matrix B, which states

$\text{rank}(B) + \text{nullity}(B) = 5 \longrightarrow$ say $\text{nullity}(B) = a$, $\text{rank}(B) = b$

Assume that there exists B s.t.

$3\,\text{nullity}(B) = 2\,\text{rank}(B) + 7$, then $a, b \in \{\mathbb{N}, 0\}$ (must be whole numbers)

$\Rightarrow$ $3a = 2b + 7$, $a = 5 - b$ $\longrightarrow$ $3(5-b) = 2b + 7$, $15 - 3b = 2b + 7$ $\Rightarrow$ $b = \frac{8}{5}$ contradiction, so B DNE