EXERCISES 5.3

GOAL *Use the various characterizations of orthogonal transformations and orthogonal matrices. Find the matrix of an orthogonal projection. Use the properties of the transpose.*

Which of the matrices in Exercises 1 through 4 are orthogonal?

1. $\begin{bmatrix} 0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix}$

2. $\begin{bmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{bmatrix}$

3. $\frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}$

4. $\frac{1}{7}\begin{bmatrix} 2 & 6 & -3 \\ 6 & -3 & 2 \\ 3 & 2 & 6 \end{bmatrix}$

If the $n \times n$ matrices A and B are orthogonal, which of the matrices in Exercises 5 through 11 must be orthogonal as well?

5. $3A$

6. $-B$

7. AB

8. $A + B$

9. B^{-1}

10. $B^{-1}AB$

11. A^T

If the $n \times n$ matrices A and B are symmetric and B is invertible, which of the matrices in Exercises 13 through 20 must be symmetric as well?

13. $3A$

14. $-B$

15. AB

16. $A + B$

17. B^{-1}

18. A^{10}

19. $2I_n + 3A - 4A^2$

20. AB^2A

If A and B are arbitrary $n \times n$ matrices, which of the matrices in Exercises 21 through 26 must be symmetric?

21. $A^T A$

22. BB^T

23. $A - A^T$

29. Show that an orthogonal transformation L from $\mathbb{R}^n$ to $\mathbb{R}^n$ preserves angles: The angle between two nonzero vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ equals the angle between $L(\vec{v})$ and $L(\vec{w})$. Conversely, is any linear transformation that preserves angles orthogonal?

30. Consider a linear transformation L from $\mathbb{R}^m$ to $\mathbb{R}^n$ that preserves length. What can you say about the kernel of L? What is the dimension of the image? What can you say about the relationship between n and m? If A is the matrix of L, what can you say about the columns of A? What is $A^T A$? What about AA^T? Illustrate your answers with an example where $m = 2$ and $n = 3$.

31. Are the *rows* of an orthogonal matrix A necessarily orthonormal?

32. **a.** Consider an $n \times m$ matrix A such that $A^T A = I_m$. Is it necessarily true that $AA^T = I_n$? Explain.

b. Consider an $n \times n$ matrix A such that $A^T A = I_n$. Is it necessarily true that $AA^T = I_n$? Explain.

EXERCISES 6.1

Find the determinants of the matrices A in Exercises 1 through 10, and find out which of these matrices are invertible.

1. $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 5 \\ 7 & 11 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 5 & 7 \\ 0 & 11 & 7 \\ 0 & 0 & 5 \end{bmatrix}$

6. $\begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

9. $\begin{bmatrix} 0 & 1 & 2 \\ 7 & 8 & 3 \\ 6 & 5 & 4 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$

EXERCISES 6.2

Use Gaussian elimination to find the determinant of the matrices A in Exercises 1 through 10.

1. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 2 & 2 & 5 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 6 & 8 \\ -2 & -4 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 3 & 2 & 4 \\ 1 & 6 & 4 & 8 \\ 1 & 3 & 0 & 0 \\ 2 & 6 & 4 & 12 \end{bmatrix}$

4. $\begin{bmatrix} 1 & -1 & 2 & -2 \\ -1 & 2 & 1 & 6 \\ 2 & 1 & 14 & 10 \\ -2 & 6 & 10 & 33 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 4 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 8 & -8 \end{bmatrix}$

7. $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}$

Consider a 4×4 *matrix* A *with rows* $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$. *If* $\det(A) = 8$, *find the determinants in Exercises 11 through 16.*

11. $\det \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ -9\vec{v}_3 \\ \vec{v}_4 \end{bmatrix}$

12. $\det \begin{bmatrix} \vec{v}_4 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_1 \end{bmatrix}$

13. $\det \begin{bmatrix} \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_1 \\ \vec{v}_4 \end{bmatrix}$

14. $\det \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 + 9\vec{v}_4 \\ \vec{v}_3 \\ \vec{v}_4 \end{bmatrix}$

15. $\det \begin{bmatrix} \vec{v}_1 \\ \vec{v}_1 + \vec{v}_2 \\ \vec{v}_1 + \vec{v}_2 + \vec{v}_3 \\ \vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4 \end{bmatrix}$

16. $\det \begin{bmatrix} 6\vec{v}_1 + 2\vec{v}_4 \\ \vec{v}_2 \\ \vec{v}_3 \\ 3\vec{v}_1 + \vec{v}_4 \end{bmatrix}$

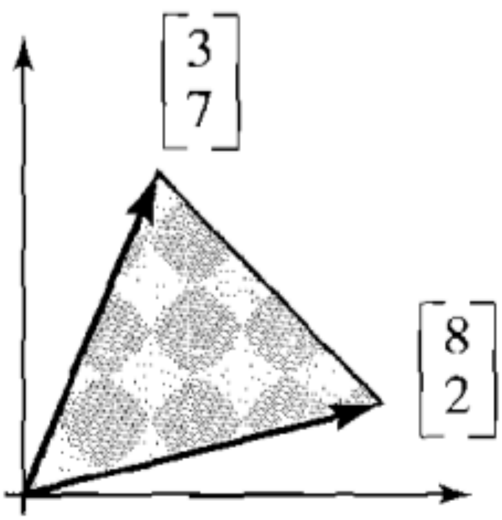
59. If the equation $\det A = \det B$ holds for two $n \times n$ matrices A and B, is A necessarily similar to B?

EXERCISES 6.3

GOAL *Interpret the determinant as an area or volume and as an expansion factor. Use Cramer's rule.*

1. Find the area of the parallelogram defined by $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$.

2. Find the area of the triangle defined by $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$.



6. What is the relationship between the volume of the tetrahedron defined by the vectors

$$\begin{bmatrix} a_1 \\ a_2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} b_1 \\ b_2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} c_1 \\ c_2 \\ 1 \end{bmatrix}$$

and the area of the triangle with vertices

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}?$$

See Exercises 4 and 5. Explain this relationship geometrically. *Hint:* Consider the top face of the tetrahedron.

19. A basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$ of $\mathbb{R}^3$ is called *positively oriented* if $\vec{v}_1$ encloses an acute angle with $\vec{v}_2 \times \vec{v}_3$. Illustrate this definition with a sketch. Show that the basis is positively oriented if (and only if) $\det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$ is positive.

20. We say that a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ *preserves orientation* if it transforms any positively oriented basis into another positively oriented basis. See Exercise 19. Explain why a linear transformation $T(\vec{x}) = A\vec{x}$ preserves orientation if (and only if) $\det A$ is positive.