In Exercises 19 through 24, find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$. For practice, solve each problem in three ways: (a) Use the formula $B = S^{-1}AS$, (b) use a commutative diagram (as in Examples 3 and 4), and (c) construct B "column by column."

19. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

20. $A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$; $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

21. $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$; $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

22. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$; $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

23. $A = \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}$; $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

24. $A = \begin{bmatrix} 13 & -20 \\ 6 & -9 \end{bmatrix}$; $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

In Exercises 25 through 30, find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_m)$.

25. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

26. $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}; \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

27. $A = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix};$

$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Let $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ be any basis of $\mathbb{R}^3$ consisting of perpendicular unit vectors, such that $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2$. In Exercises 31 through 36, find the $\mathfrak{B}$-matrix B of the given linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$. Interpret T geometrically.

31. $T(\vec{x}) = \vec{v}_2 \times \vec{x}$

32. $T(\vec{x}) = \vec{x} \times \vec{v}_3$

33. $T(\vec{x}) = (\vec{v}_2 \cdot \vec{x})\vec{v}_2$

34. $T(\vec{x}) = \vec{x} - 2(\vec{v}_3 \cdot \vec{x})\vec{v}_3$

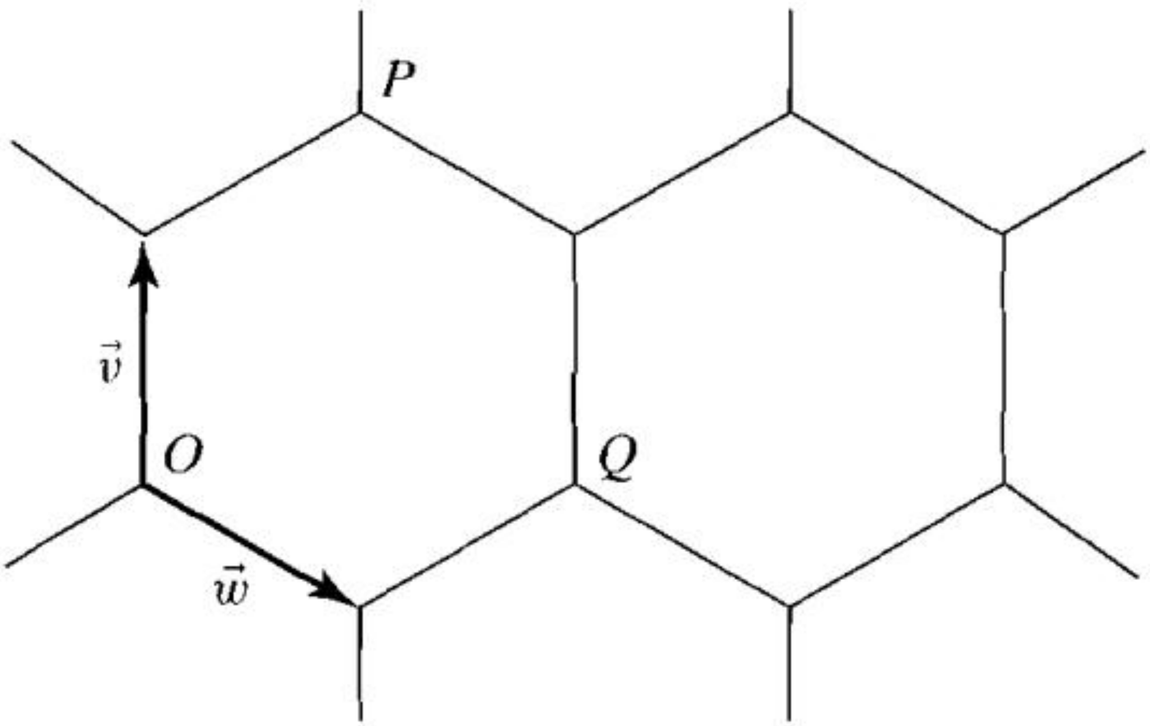
35. $T(\vec{x}) = \vec{x} - 2(\vec{v}_1 \cdot \vec{x})\vec{v}_2$

36. $T(\vec{x}) = \vec{v}_1 \times \vec{x} + (\vec{v}_1 \cdot \vec{x})\vec{v}_1$

In Exercises 37 through 42, find a basis $\mathfrak{B}$ of $\mathbb{R}^n$ such that the $\mathfrak{B}$-matrix B of the given linear transformation T is diagonal.

37. Orthogonal projection T onto the line in $\mathbb{R}^2$ spanned by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

50. Given a hexagonal tiling of the plane, such as you might find on a kitchen floor, consider the basis $\mathfrak{B}$ of $\mathbb{R}^2$ consisting of the vectors $\vec{v}, \vec{w}$ in the following sketch:



a. Find the coordinate vectors $\left[\overrightarrow{OP}\right]_{\mathfrak{B}}$ and $\left[\overrightarrow{OQ}\right]_{\mathfrak{B}}$.
Hint: Sketch the coordinate grid defined by the basis $\mathfrak{B} = (\vec{v}, \vec{w})$.

b. We are told that $\left[\overrightarrow{OR}\right]_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Sketch the point R. Is R a vertex or a center of a tile?

c. We are told that $\left[\overrightarrow{OS}\right]_{\mathfrak{B}} = \begin{bmatrix} 17 \\ 13 \end{bmatrix}$. Is S a center or a vertex of a tile?

61. Find a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that the $\mathfrak{B}$-matrix of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} -5 & -9 \\ 4 & 7 \end{bmatrix} \vec{x} \quad \text{is} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

EXERCISES 5.1

GOAL *Apply the basic concepts of geometry in $\mathbb{R}^n$: length, angles, orthogonality. Use the idea of an orthogonal projection onto a subspace. Find this projection if an orthonormal basis of the subspace is given.*

Find the length of each of the vectors $\vec{v}$ in Exercises 1 through 3.

1. $\vec{v} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$

2. $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

3. $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$

'ind the angle $\boldsymbol{\theta}$ between each of the pairs of vectors $\vec{\mathbf{u}}$ and in Exercises 4 through 6.

4. $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$

5. $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

6. $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$

0. For which value(s) of the constant k are the vectors

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 1 \\ k \\ 1 \end{bmatrix}$$

perpendicular?

16. Consider the vectors

$$\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

in $\mathbb{R}^4$. Can you find a vector $\vec{u}_4$ in $\mathbb{R}^4$ such that the vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$ are orthonormal? If so, how many such vectors are there?

17. Find a basis for $W^{\perp}$, where

$$W = \text{span}\left(\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}5\\6\\7\\8\end{bmatrix}\right).$$

28. Find the orthogonal projection of

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

onto the subspace of $\mathbb{R}^4$ spanned by

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$