8. (a) Use implicit differentiation to show that $t^2 + y^2 = C^2$ implicitly defines solutions of the differential equation $t + yy' = 0$.

   (b) Solve $t^2 + y^2 = C^2$ for y in terms of t to provide explicit solutions. Show that these functions are also solutions of $t + yy' = 0$.

   (c) Discuss the interval of existence for each of the solutions in part (b).

   (d) Sketch the solutions in part (b) for $C = 1, 2, 3, 4$.

9. (a) Use implicit differentiation to show that $t^2 - 4y^2 = C^2$ implicitly defines solutions of the differential equation $t - 4yy' = 0$.

   (b) Solve $t^2 - 4y^2 = C^2$ for y in terms of t to provide explicit solutions. Show that these functions are also solutions of $t - 4yy' = 0$.

   (c) Discuss the interval of existence for each of the solutions in part (b).

   (d) Sketch the solutions in part (b) for $C = 1, 2, 3, 4$.

10. Show that $y(t) = 3/(6t - 11)$ is a solution of $y' = -2y^2$, $y(2) = 3$. Sketch this solution and discuss its interval of existence. Include the initial condition on your sketch.

11. Show that $y(t) = 4/(1 - 5e^{-4t})$ is a solution of the initial value problem $y' = y(4 - y)$, $y(0) = -1$. Sketch this solution and discuss its interval of existence. Include the initial condition on your sketch.

In Exercises 12–15, use the given general solution to find a solution of the differential equation having the given initial condition. Sketch the solution, the initial condition, and discuss the solution's interval of existence.

12. $y' + 4y = \cos t$, $y(t) = (4/17)\cos t + (1/17)\sin t + Ce^{-4t}$, $y(0) = -1$

13. $ty' + y = t^2$, $y(t) = (1/3)t^2 + C/t$, $y(1) = 2$

14. $ty' + (t + 1)y = 2te^{-t}$, $y(t) = e^{-t}(t + C/t)$, $y(1) = 1/e$

15. $y' = y(2 + y)$, $y(t) = 2/(-1 + Ce^{-2t})$, $y(0) = -3$

16. Maple, when asked for the solution of the initial value problem $y' = \sqrt{y}$, $y(0) = 1$, returns two solutions: $y(t) = (1/4)(t + 2)^2$ and $y(t) = (1/4)(t - 2)^2$. Present a thorough discussion of this response, including a check and a graph of each solution, interval of existence, and so on. *Hint:* Remember that $\sqrt{a^2} = |a|$.

In Exercises 17–20, plot the direction field for the differential equation by hand. Do this by drawing short lines of the appropriate slope centered at each of the integer valued coordinates (t, y), where $-2 \le t \le 2$ and $-1 \le y \le 1$.

17. $y' = y + t$

18. $y' = y^2 - t$

19. $y' = t\tan(y/2)$

20. $y' = (t^2 y)/(1 + y^2)$

EXERCISES

In Exercises 1–12, find the general solution of the indicated differential equation. If possible, find an explicit solution.

1. $y' = xy$
2. $xy' = 2y$
3. $y' = e^{x-y}$
4. $y' = (1 + y^2)e^x$
5. $y' = xy + y$
6. $y' = ye^x - 2e^x + y - 2$
7. $y' = x/(y + 2)$
8. $y' = xy/(x - 1)$
9. $x^2y' = y \ln y - y'$
10. $xy' - y = 2x^2y$
11. $y^3y' = x + 2y'$
12. $y' = (2xy + 2x)/(x^2 - 1)$

In Exercises 13–18, find the exact solution of the initial value problem. Indicate the interval of existence.

13. $y' = y/x$, $y(1) = -2$
14. $y' = -2t(1 + y^2)/y$, $y(0) = 1$
15. $y' = (\sin x)/y$, $y(\pi/2) = 1$
16. $y' = e^{x+y}$, $y(0) = 0$
17. $y' = (1 + y^2)$, $y(0) = 1$
18. $y' = x/(1 + 2y)$, $y(-1) = 0$

In Exercises 19–22, find exact solutions for each given initial condition. State the interval of existence in each case. Plot each exact solution on the interval of existence. Use a numerical solver to duplicate the solution curve for each initial value problem.

19. $y' = x/y$, $y(0) = 1$, $y(0) = -1$
20. $y' = -x/y$, $y(0) = 2$, $y(0) = -2$
21. $y' = 2 - y$, $y(0) = 3$, $y(0) = 1$
22. $y' = (y^2 + 1)/y$, $y(1) = 2$
23. Suppose that a radioactive substance decays according to the model $N' = N_0e^{-\lambda t}$. Show that the half-life of the radioactive substance is given by the equation

$$T_{1/2} = \frac{\ln 2}{\lambda}. \tag{2.38}$$

24. The half-life of ^{238}U is 4.47×10^7 yr.
    (a) Use equation (2.38) to compute the ***decay constant*** λ for ^{238}U.
    (b) Suppose that 1000 mg of ^{238}U are present initially. Use the equation $N = N_0e^{-\lambda t}$ and the decay constant determined in part (a) to determine the time for this sample to decay to 100 mg.
25. Tritium, ^3H, is an isotope of hydrogen that is sometimes used as a biochemical tracer. Suppose that 100 mg of ^3H decays to 80 mg in 4 hours. Determine the half-life of ^3H.
26. The isotope Technetium 99m is used in medical imaging. It has a half-life of about 6 hours, a useful feature for radioisotopes that are injected into humans. The Technetium, having such a short half-life, is created artificially on scene by harvesting from a more stable isotope, ^{99}Mb. If 10 g of ^{99m}Tc are "harvested" from the Molybdenum, how much of this sample remains after 9 hours?
27. The isotope Iodine 131 is used to destroy tissue in an overactive thyroid gland. It has a half-life of 8.04 days. If a hospital receives a shipment of 500 mg of ^{131}I, how much of the isotope will be left after 20 days?

36. Consider the equation

$$y' = f(at + by + c),$$

where a, b, and c are constants. Show that the substitution $x = at + by + c$ changes the equation to the separable equation $x' = a + bf(x)$. Use this method to find the general solution of the equation $y' = (y + t)^2$.

EXERCISES

In Exercises 1–12, find the general solution of the first-order, linear equation.

1. $y' + y = 2$
2. $y' - 3y = 5$
3. $y' + (2/x)y = (\cos x)/x^2$
4. $y' + 2ty = 5t$
5. $x' - 2x/(t+1) = (t+1)^2$
6. $tx' = 4x + t^4$
7. $(1+x)y' + y = \cos x$
8. $(1+x^3)y' = 3x^2y + x^2 + x^5$
9. $L(di/dt) + Ri = E, \quad L, R, E$ real constants
10. $y' = my + c_1e^{mx}, \quad m, c_1$ real constants
11. $y' = \cos x - y \sec x$
12. $x' - (n/t)x = e^t t^n, \quad n$ an positive integer
13. (a) The differential equation $y' + y\cos x = \cos x$ is linear. Use the integrating factor technique of this section to find the general solution.

    (b) The equation $y' + y\cos x = \cos x$ is also separable. Use the separation of variables technique to solve the equation and discuss any discrepancies (if any) between this solution and the solution found in part (a).

In Exercises 14–17, find the solution of the initial value problem.

14. $y' = y + 2xe^{2x}, \quad y(0) = 3$
15. $(x^2+1)y' + 3xy = 6x, \quad y(0) = -1$
16. $(1+t^2)y' + 4ty = (1+t^2)^{-2}, \quad y(1) = 0$
17. $x' + x\cos t = \frac{1}{2}\sin 2t, \quad x(0) = 1$

In Exercises 18–21, find the solution of the initial value problem. Discuss the interval of existence and provide a sketch of your solution.

18. $xy' + 2y = \sin x, \quad y(\pi/2) = 0$
19. $(2x+3)y' = y + (2x+3)^{1/2}, \quad y(-1) = 0$
20. $y' = \cos x - y\sec x, \quad y(0) = 1$
21. $(1+t)x' + x = \cos t, \quad x(-\pi/2) = 0$
22. The presence of nonlinear terms prevents us from using the technique of this section. In special cases, a change of variable will transform the nonlinear equation into one that is linear. The equation known as ***Bernoulli's equation***,

    $$x' = a(t)x + f(t)x^n, \quad n \neq 0, 1,$$

    was proposed for solution by James Bernoulli in December 1695. In 1696, Leibniz pointed out that the equation can be reduced to a linear equation by taking x^{1-n} as the dependent variable. Show that the change of variable, $z = x^{1-n}$, will transform the nonlinear Bernoulli equation into the linear equation

    $$z' = (1-n)a(t)z + (1-n)f(t).$$

    Hint: If $z = x^{1-n}$, then $dz/dt = (dz/dx)(dx/dt) = (1-n)x^{-n}(dx/dt)$.

In Exercises 23–26, use the technique of Exercise 22 to transform the Bernoulli equation into a linear equation. Find the general solution of the resulting linear equation.

23. $y' + x^{-1}y = xy^2$
24. $y' + y = y^2$
25. $xy' + y = x^4y^3$
26. $P' = aP - bP^2$
27. The equation

    $$\frac{dy}{dt} + \psi y^2 + \phi y + \chi = 0,$$

    where ψ, ϕ, and χ are functions of t, is called the ***generalized Riccati equation***. In general, the equation is not integrable by quadratures. However, suppose that one solution, say $y = y_1$, is known.

    (a) Show that the substitution $y = y_1 + z$ reduces the generalized Riccati equation to

    $$\frac{dz}{dt} + (2y_1\psi + \phi)\,z + \psi z^2 = 0,$$

    which is an instance of Bernoulli's equation (see Exercise 22).

    (b) Use the fact that $y_1 = 1/t$ is a particular solution of

    $$\frac{dy}{dt} = -\frac{1}{t^2} - \frac{y}{t} + y^2$$

    to find the equation's general solution.
28. Suppose that you have a closed system containing 1000 individuals. A flu epidemic starts. Let $N(t)$ represent the number of infected individuals in the closed system at time t. Assume that the rate at which the number of infected individuals is changing is jointly proportional to the number of infected individuals and to the number of uninfected individuals. Furthermore, suppose that when 100 individuals are infected, the rate at which individuals are becoming infected is 90 individuals per day. If 20 individuals are infected at time $t = 0$, when will 90% of the population be infected? *Hint:* The assumption here is that there are only healthy individuals and sick individuals. Furthermore, the resulting model can be solved using the technique introduced in Exercise 22.
29. In Exercise 33 of Section 2.2, the time of death of a murder victim is determined using Newton's law of cooling. In particular it was discovered that the proportionality constant in Newton's law was $k = \ln(5/4) \approx 0.223$. Suppose we discover another murder victim at midnight with a body temperature of 31°C. However, this time the air temperature at midnight is 0°C, and is falling at a constant rate of 1°C per hour. At what time did the victim die? (Remember that the normal body temperature is 37°C.)

In Exercises 30–35, use the variation of parameters technique to find the general solution of the given differential equation.

In Exercises 36–41, use the variation of parameters technique to find the general solution of the given differential equation. Then find the particular solution satisfying the given initial condition.

36. $y' - 3y = 4, \quad y(0) = 2$

37. $y' + (1/2)y = t, \quad y(0) = 1$

38. $y' + y = e^t, \quad y(0) = 1$

39. $y' + 2xy = 2x^3, \quad y(0) = -1$

40. $x' - (2/t^2)x = 1/t^2, \quad x(1) = 0$

41. $(t^2 + 1)x' + 4tx = t, \quad x(0) = 1$

42. Consider anew Newton's law of cooling, where the temperature of a body is modeled by the equation

$$T' = -k(T - A), \tag{4.43}$$

where T is the temperature of the body and A is the temperature of the surrounding medium (*ambient temperature*). Although this equation is linear and its solution can be found by using an integrating factor, Theorem 4.41 provides a far simpler approach.

(a) Find a solution T_h of the homogenous equation $T' + kT = 0$.

(b) Find a particular solution T_p of the inhomogeneous equation (4.43). *Note:* This equation is autonomous. See Section 2.9.

(c) Form the general solution $T = T_h + T_p$.

(d) Add a source of constant heat (like a heater in a room) to the model (4.43), as in

$$T' = -k(T - A) + H. \tag{4.44}$$

Use the technique outlined in parts (a)–(c) to find the general solution of equation (4.44).