

# Math 33A Sheet 9

## Chapter 7.1

Ex 1)  $A\vec{v} = \lambda\vec{v}$

$A^3\vec{v} = \lambda^3\vec{v}$

Yes, the eigenvalue is  $\lambda^3$

Ex 2)  $A\vec{v} = \lambda\vec{v}$

$A^{-1}A = I_n$

$A^{-1}A\vec{v} = A^{-1}\lambda\vec{v}$

$\vec{v} = A^{-1}\lambda\vec{v}$

$A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$

Yes, the eigenvalue is  $\frac{1}{\lambda}, \lambda \neq 0$

Ex 4)  $A\vec{v} = \lambda\vec{v}$

$7A\vec{v} = 7\lambda\vec{v}$

Yes, the eigenvalue is  $7\lambda$

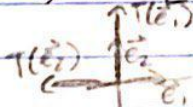
Ex 16)  $T(\vec{v}) = -\vec{v}$

$\vec{v} \in \mathbb{R}^2$  are eigenvectors

$\lambda = -1$  is the eigenvalue

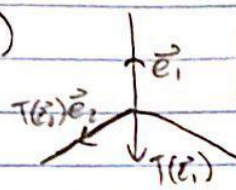
The transformation is diagonalisable

Ex 17)  $A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$



no eigenvalues/eigenvectors,  
T is not diagonalisable

Ex 18)



Eigenvalues are 1 and -1

Eigenvectors are  $\vec{v} \in V$  and  $\vec{v} \in V^\perp$

T is diagonalisable

Ex 37)

$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

a) rotation = orthogonal matrix =  
eigenvalues of  $\pm 1$

scaling  $\times 5 \rightarrow$  eigenvalues  $\pm 5$

b)  $\ker(A - (5)\lambda)$

$= \ker\left(\begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}\right)$

$= \text{span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$

$\ker(A - (-5)\lambda)$

$= \ker\left(\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\right)$

$= \text{span}\left(\begin{bmatrix} 1/2 \\ -1 \end{bmatrix}\right)$

eigenbasis:  $\begin{bmatrix} 2 & 1/2 \\ 1 & -1 \end{bmatrix}$

c)  $B = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$

## Chapter 7.2

Ex 1)  $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

$\begin{bmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix}$

$(1-\lambda)(3-\lambda) = 0$

$\lambda = 1, 3$

$\text{alm}(1) = 1$

$\text{alm}(3) = 1$



$$\text{Ex 13)} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\left| \begin{array}{ccc|c} -\lambda & 1 & 0 & -\lambda \\ 0 & -\lambda & 1 & 0 \\ 1 & 0 & -\lambda & 1 \end{array} \right|$$

$$-\lambda^3 + 1 = 0$$

$$-\lambda^3 = -1$$

$$\lambda^3 = 1$$

eigenvalue is 1

$$\text{almu}(1) = 3$$

$$\text{Ex 16)}^* A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

$$\begin{bmatrix} a-\lambda & b \\ b & c-\lambda \end{bmatrix}$$

$$(a-\lambda)(c-\lambda) - b^2 = 0$$

$$(a-\lambda)(c-\lambda) = b^2$$

$$ac - c\lambda - a\lambda + \lambda^2 = b^2$$

$$ac - (a+c)\lambda + \lambda^2 = b^2$$

$$\lambda^2 - (a+c)\lambda + (ac - b^2) = 0$$

$$\lambda = \frac{a+c \pm \sqrt{a^2 + 2ac + c^2 - 4ac + 4b^2}}{2}$$

$$\lambda = \frac{a+c \pm \sqrt{(a-c)^2 + 4b^2}}{2}$$

$$a \neq c \text{ or } b \neq 0$$

$$\text{Ex 4)} \begin{bmatrix} 0 & -1 \\ 1 & 2-\lambda \end{bmatrix}$$

$$-\lambda(2-\lambda) - (-1) = 0$$

$$-2\lambda + \lambda^2 + 1 = 0$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda-1)(\lambda-1) = 0$$

$$\lambda = 1$$

$$\ker \left( \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \right) = \text{span} \left( \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$$

$$\text{eigenbasis: } \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\text{gemu}(1) \neq n$$

not diagonalisable

$$\text{Ex 12)}^* \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left| \begin{array}{ccc|c} 1-\lambda & 1 & 0 & 1-\lambda \\ 0 & 1-\lambda & 1 & 0 \\ 0 & 0 & 1-\lambda & 0 \end{array} \right|$$

$$(1-\lambda)^2 = 0$$

$$\lambda = 1$$

$$\ker \left( \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right) = \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$\text{eigenbasis: } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{gemu} \neq n$$

not diagonalisable

Chapter 7.3

$$\text{Ex 1)} \begin{bmatrix} 7 & 9 \\ 0 & 9 \end{bmatrix}$$

$$(7-\lambda)(9-\lambda) = 0$$

$$\lambda = 7, 9$$

$$\ker \left( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) = \text{span} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$\ker \left( \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \right) = \text{span} \left( \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right)$$

$$\text{eigenbasis: } \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Ex 47)} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

$$\lambda = 1$$

$$\ker \left( \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \right)$$

$$ax_2 - bx_3 = 0$$

$$cx_3 = 0$$

$$x_2 \text{ and } x_3 \neq 0$$

$$a = b = c = 0$$



## Chapter 8.1

Ex 1)  $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

$$(1-\lambda)(2-\lambda) = 0$$

$$\lambda = 1, 2$$

$$\ker\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$\ker\left(\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

Ex 8)  $A = \begin{bmatrix} 3 & 3 \\ 3 & -5 \end{bmatrix}$

$$\begin{bmatrix} 3-\lambda & 3 \\ 3 & -5-\lambda \end{bmatrix}$$

$$(3-\lambda)(-5-\lambda) - 9 = 0$$

$$-15 + 2\lambda + \lambda^2 - 9 = 0$$

$$\lambda^2 + 2\lambda - 24 = 0$$

$$(\lambda + 6)(\lambda - 4) = 0$$

$$\lambda = -6, 4$$

$$\ker\left(\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right)$$

$$\ker\left(\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$$

$$\vec{v}_1 \cdot \vec{v}_2 = 3 - 3 = 0 \rightarrow \text{they're orthogonal}$$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$S = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}$$

$$\left[ \begin{array}{cc|cc} 1/\sqrt{10} & 3/\sqrt{10} & 1 & 0 \\ -3/\sqrt{10} & 1/\sqrt{10} & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{cc|cc} 1/\sqrt{10} & 3/\sqrt{10} & 1 & 0 \\ 0 & 10/\sqrt{10} & +3 & 1 \end{array} \right]$$

$$\left[ \begin{array}{cc|cc} 1 & 3 & \sqrt{10} & 0 \\ 0 & 10 & +3\sqrt{10} & \sqrt{10} \end{array} \right]$$

$$\left[ \begin{array}{cc|cc} 1 & 3 & \sqrt{10} & 0 \\ 0 & 1 & +3/\sqrt{10} & 1/\sqrt{10} \end{array} \right]$$

$$\left[ \begin{array}{cc|cc} 1 & 0 & 1/\sqrt{10} & -3/\sqrt{10} \\ 0 & 1 & 3/\sqrt{10} & 1/\sqrt{10} \end{array} \right]$$

$$S^{-1} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \frac{1}{\sqrt{10}}$$

$$D = S^{-1}AS$$

$$D = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \frac{1}{10}$$

$$= \begin{bmatrix} -6 & 18 \\ 12 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \frac{1}{10}$$

$$= \frac{1}{10} \begin{bmatrix} -60 & 0 \\ 0 & 40 \end{bmatrix}$$

$$D = \begin{bmatrix} -6 & 0 \\ 0 & 4 \end{bmatrix}$$

Ex 12)  $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

a) need orthogonal vectors to  $\vec{v}$ ,

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_3 = 0$$

$$x_1 = -2x_3$$

$$x_3 = 1, x_1 = -2$$

$$\vec{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{u}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_3 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$b) \begin{bmatrix} 1/\sqrt{5} & 0 & -2/\sqrt{5} \\ 0 & 1 & 0 \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \end{bmatrix}$$

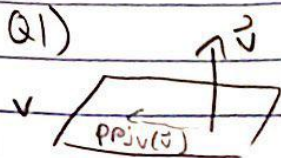
$$B = [T(\vec{e}_1) \ T(\vec{e}_2) \ T(\vec{e}_3)]$$

$$B = \begin{bmatrix} 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 \\ 2/\sqrt{5} & 0 & -1/\sqrt{5} \end{bmatrix}$$

$$c) A = [T(\vec{e}_1) \ T(\vec{e}_2) \ T(\vec{e}_3)]$$

$$A = \begin{bmatrix} 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 \\ -2/\sqrt{5} & 0 & -1/\sqrt{5} \end{bmatrix}$$





all projections are not parallel to  $\vec{v}$  unless  $\vec{v} \in V$

$\vec{v}$  must be in the subspace  $V$

$\therefore \vec{v} \in V$  are eigenvectors  $\leftarrow$  length doesn't change

eigenvalue = 0, 1  $\leftarrow$

diagonalisable?

0 is the kernel of the transformation

1 is the eigenvalue for the remaining eigenvectors