Math 33A Sheet 6

Q1)* $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$v_1 = \dots = v_m = 0$

$c_1(0) + \dots + c_m(0) = 0$

↳ $c_1, \dots, c_m$ can be any real number

↳ there are infinitely many solutions to

$c_1v_1 + \dots + c_mv_m = 0$

↳ $\vec{0}$ is linearly dependent

Q2) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \text{rref}(A)$

$$\boxed{\begin{aligned} &\text{basis of } \operatorname{im}(A) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \neq \\ &\text{basis of } \operatorname{im}(\text{rref}(A)) : \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}}$$

Chapter 3.4

Ex22) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

a) $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] - I$

$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right] \times -\frac{1}{2}$

$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right] - II$

$\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$

$B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

$= \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}}$

b) $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 \Rightarrow T(\vec{x}) = A\vec{x} = 2c_1v_1$

$\downarrow \qquad \downarrow$

$[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \xrightarrow{B} [T(\vec{x})]_B = \begin{bmatrix} 2c_1 \\ 0 \end{bmatrix}$

$\boxed{B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}}$

c) $B = \begin{bmatrix} [T(\vec{v}_1)]_B & [T(\vec{v}_2)]_B \end{bmatrix}$

$B = \begin{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix}$

$\boxed{B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}}$

Ex27)* $A = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix} \begin{matrix} \times\frac{1}{4} \\ \\ +I \end{matrix}$

$= \begin{bmatrix} 1 & \frac{1}{2} & -1 \\ 2 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} - 2I$

$= \begin{bmatrix} 1 & \frac{1}{2} & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$T(\vec{x}) = A\vec{x} \rightarrow$

$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 + \frac{1}{2}c_2 + c_3 \\ 0 \\ 0 \end{bmatrix}$

$S = \left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ -2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \times\frac{1}{2} \\ \\ +I \end{matrix}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 \end{array}\right] - I$

$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 2 & -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 \end{array}\right] \times\frac{1}{2}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 \end{array}\right] - II$

$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{9}{4} & \frac{5}{4} & -\frac{1}{2} & 1 \end{array}\right] \times\frac{4}{9}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{5}{9} & -\frac{2}{9} & \frac{4}{9} \end{array}\right] \begin{matrix} -\frac{1}{2}III \\ +\frac{1}{4}III \\ \\ \end{matrix}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 1 & 0 & -\frac{1}{9} & \frac{4}{9} & \frac{1}{9} \\ 0 & 0 & 1 & \frac{5}{9} & -\frac{2}{9} & \frac{4}{9} \end{array}\right]$

$S^{-1} = \frac{1}{9}\begin{bmatrix} 2 & 1 & -2 \\ -1 & 4 & 1 \\ 5 & -2 & 4 \end{bmatrix}$

$B = \frac{1}{9}\begin{bmatrix} 2 & 1 & -2 \\ -1 & 4 & 1 \\ 5 & -2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ -2 & 1 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ -2 & 1 & 1 \end{bmatrix}$

$\boxed{B = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$

Ex 32) $T(\vec{x}) = \vec{x} \times \vec{v}_3$

$T(\vec{v}_1) = -\vec{v}_2$

$T(\vec{v}_2) = \vec{v}_1$

$T(\vec{v}_3) = 0$

$[T(\vec{v}_1)]_B = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

$[T(\vec{v}_2)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$[T(\vec{v}_3)]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\boxed{B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$

Rotation $\frac{\pi}{2}$ clockwise

Ex 37)* $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \vec{v}_1$

$\vec{v}_2 \perp \vec{v}_1$

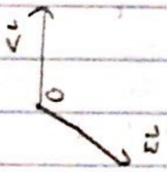
$\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

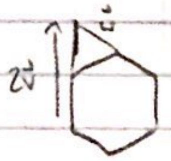
$\therefore T(\vec{x}) = c_1 \vec{v}_1, \ \vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2$

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \to \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$

$\boxed{B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}$

Ex 50) a)

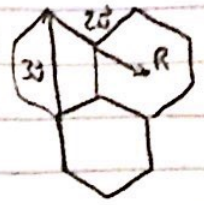




$\boxed{[\vec{OP}]_B = 2\vec{v} + \vec{w}}$

$\boxed{[\vec{OQ}]_B = \vec{v} + 2\vec{w}}$

b)



$\boxed{\text{Center of a tile}}$

c) $\begin{bmatrix} 17 \\ 13 \end{bmatrix}$

odd # $\vec{v}$ = from upper vertex

odd # $\vec{w}$ = at a vertex

$\boxed{\text{vertex}}$

Ex 61) $T(\vec{x}) = \begin{bmatrix} -5 & -9 \\ 4 & 7 \end{bmatrix} \vec{x}, \ B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2$

$[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \xrightarrow{B} [T(\vec{x})]_B = \begin{bmatrix} c_1 + c_2 \\ c_2 \end{bmatrix}$

$\begin{bmatrix} -5 & -9 \\ 4 & 7 \end{bmatrix} \xrightarrow{S} \begin{bmatrix} c_1 + c_2 \\ c_2 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} c \\ d \end{bmatrix}$

$S = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = [T(\vec{v}_1)]_B \to A\vec{v}_1.$

$\begin{bmatrix} -5 & -9 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -5a - 9b \\ 4a + 7b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$-5a - 9b = 1$

$4a + 7b = 0$

$-a - \frac{9}{5} b = \frac{1}{5}$

$-4a - \frac{36}{5} b = \frac{4}{5}$

$-\frac{1}{5} b = \frac{4}{5}$

$b = -4, \ a = 7$

$\begin{bmatrix} -5 & -9 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$-5c - 9d = 1$

$4c + 7d = 1$

$-4c - \frac{36}{5} d = \frac{4}{5}$

$-\frac{1}{5} d = \frac{9}{5}$

$d = -9, \ c = 16$

$\boxed{\begin{bmatrix} 7 \\ -4 \end{bmatrix}, \begin{bmatrix} 16 \\ -9 \end{bmatrix}}$

Chapter 5.1

Ex 1)* $\vec{v} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$

$\|\vec{v}\| = \sqrt{7^2 + 11^2} = \boxed{\sqrt{170}}$

Ex 4) $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$

$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$

$\theta = \arccos\left(\frac{18}{\sqrt{2}(\sqrt{170})}\right)$

$\boxed{\theta = \arccos\left(\frac{18}{\sqrt{340}}\right)}$

Ex 10) $\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ k \\ 1 \end{bmatrix}$

$2(1) + 3k + 4(1) = 0$

$6 + 3k = 0$

$\boxed{k = -2}$

Ex 16) $\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$

$\vec{u}_4 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$x_1 + x_2 + x_3 + x_4 = 0$

$x_1 + x_2 - x_3 - x_4 = 0$

$x_1 - x_2 + x_3 - x_4 = 0$

$-2x_3 - 2x_4 = 0$

$-2x_2 - 2x_4 = 0$

$x_3 + x_4 = 0$

$x_2 + x_4 = 0$

$x_1 - x_4 = 0$

$x_4 = t$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ -t \\ t \end{bmatrix}$

$|t| = 1/2 \leftarrow$ unit vector

$\boxed{\vec{u}_4 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \text{ or } \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}}$

Ex 17)* $W = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}\right)$

$W^\perp \to$ all vectors perpendicular to W

$\therefore$ find perpendicular to basis of W

$0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = 0$

$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$

$5x_1 + 6x_2 + 7x_3 + 8x_4 = 0$

$-4x_2 - 8x_3 - 12x_4 = 0$

$x_2 + 2x_3 + 3x_4 = 0$

$x_1 - x_3 - 2x_4 = 0$

$x_3 = t, x_4 = s$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t + 2s \\ -2t - 3s \\ t \\ s \end{bmatrix}$

$= t\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

$\boxed{\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}}$

Ex 28)* $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\text{proj}_V(\vec{x}) = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \ldots + (\vec{u}_3 \cdot \vec{x})\vec{u}_3$

$\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$

$= \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\left(\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}\right) \ldots$

$= \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 1/4 \\ -1/4 \\ -1/4 \end{bmatrix} + \begin{bmatrix} 1/4 \\ -1/4 \\ -1/4 \\ 1/4 \end{bmatrix}$

$= \begin{bmatrix} 3/4 \\ 1/4 \\ -1/4 \\ 1/4 \end{bmatrix}$

$\boxed{= \begin{bmatrix} 3/4 \\ 1/4 \\ -1/4 \\ 1/4 \end{bmatrix}}$