HW2 .. Due 10/18 .

2.6 : 4. 6. 8, 10. 12, 20, 24, 28 30, 34.
36, 38. 40.

2.7 : 4. 6. 8. 14 26, 30, 32.

Lecture: Homogeneous Eqn (2.7).

Def: A function $G(x,y)$ is homogns of degree n.

if $G(tx, ty) = t^n G(x,y)$. $\left(G(x,y) = x^n \left(G(1, \frac{y}{x})\right) \right)$ for t

for all $t \neq 0$. $x, y \neq 0$.

Def: A DE $P\,dx + Q\,dy = 0$ is homogns. if P. Q. are homogns., of the same degree!

Q: How to Solve Homogns DE.?

Example: $\underbrace{(x^2+y^2)}_{\text{degree } 2}\,dx + xy\,dy = 0$.

Step1: set $v(x) = \frac{y}{x}$, change DE of $y(x)$ to a DE of $v(x)$.

$$y = vx \Rightarrow dy = (dv)\cdot x + v\cdot dx.$$

$$(x^2+y^2) = x^2\left(1+\left(\frac{y}{x}\right)^2\right) = x^2(1+v^2)$$

$$xy = x^2\cdot\frac{y}{x} = x^2 v$$

$$x^2(1+v^2)\,dx + x^2 v\,(dv\cdot x + v\cdot dx)$$

$$x^2(1+v^2+v^2)\,dx + x^3 v\,dv = 0.$$

$$\Leftrightarrow (1+2v^2)\,dx + xv\,dv = 0$$

Step2:

$\Rightarrow$ Separable Case.?

$$\frac{dx}{x} + \frac{v}{1+2v^2}\,dv = 0$$

$$\ln|x| + \underline{\int \frac{v}{1+2v^2}\,dv} = 0$$

$$\ln|x| + \frac{1}{2}\int \frac{dv^2}{1+2v^2} = 0$$

$$= \ln|x| + \frac{1}{4}\int \frac{d(2v^2)}{1+2v^2} = 0$$

$$= \ln|x| + \frac{1}{4}\ln\underbrace{|1+2v^2|}_{\text{positive}} = C$$

$$4\ln|x| + \ln|1+2v^2| = C$$

$$x^4 (1+2v^2) = e^C = C$$

$$\Rightarrow \quad x^4 + 2x^2y^2 = C$$

(It is always Seperable Variable Case !)

<u>Ex2</u>: Q37.

$$(3x+y)\,dx + x\,dy = 0 \quad \text{(Homogeneous Eq)}$$

Let $\frac{y}{x} = v \quad \Rightarrow y = xv$

$$dy = x\,dv + v\,dx$$

$$x(3+v)\,dx + x(x\,dv + v\,dx) = 0$$

$$(3+v+v)\,dx + x\,dv = 0$$

$$\frac{dx}{x} + \frac{dv}{3+2v} = 0 \quad \checkmark$$

$$v\left(\frac{1+v^2}{v^2-2} - 1\right)$$

$$v\left(\frac{1+v^2-v^2+2}{v^2-2}\right)$$

Q38: $$\frac{dy}{dx} = \frac{y(x^2+y^2)}{xy^2 - 2x^3}$$ Homo of degree 0 ✓

Fact. P. Q Homo of same degree.

$\Leftrightarrow$ $\frac{P}{Q}$ is homog of degree 0.

$$\frac{dy}{dx} = \frac{d\,xv}{dx} = \frac{\frac{y}{x}\left(1+\left(\frac{y}{x}\right)^2\right)}{\left(\frac{y}{x}\right)^2 - 2} = \frac{v(1+v^2)}{v^2-2}$$

$$\frac{v\,dx + x \cdot dv}{dx} = \frac{v(1+v^2)}{v^2-2}$$

$1+v^2 - v^2 + 2$

$$x \cdot \frac{dv}{dx} = \frac{v(1+v^2)}{v^2-2} - v$$

$$\frac{3v}{v^2-2}$$

$$\frac{(v^2-2)}{v} \cdot dv = dx$$

$$v - \frac{2}{v}$$

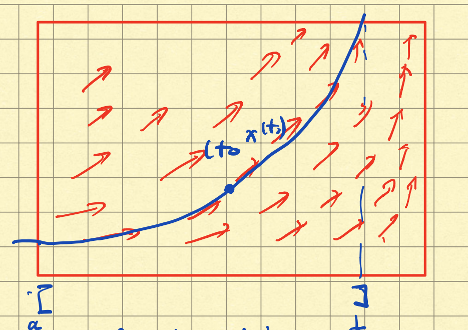
$$\frac{dv}{dx} = \cancel{\frac{v}{v^2-2}(2v^2-1)\frac{1}{x}}$$

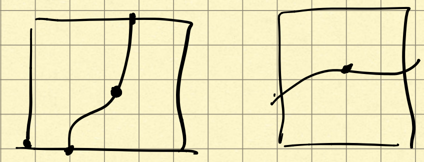
Lecture. Existence and Uniqueness.

Thm: (Existence). Suppose the function $f(t,x)$ is defined and contins on Rect R in the tx-plane. Then given a point $(t_0, x_0) \in R$, the initial value Problem

$$x' = f(t,x) \quad \text{and} \quad x(t_0) = x_0$$

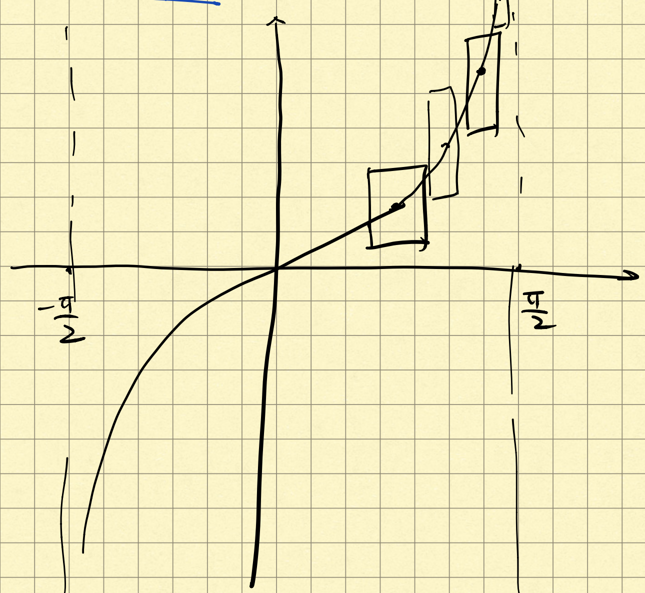
has a solution $x(t)$ defined in an interval containing t_0. Furthermore the solution will be defined at least until $t \to (t, x(t))$ leaves the Rectangle.





at least exists on $[a, b]$

Example: $x' = 1 + x^2$, $x(0) = 0$ $\Rightarrow$ $x = \tan t$

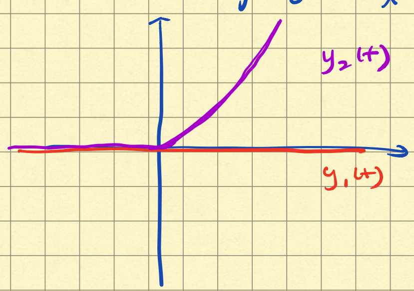


Lecture Uniqueness: 2.7 (cont.)

Example: $y' = y^{\frac{1}{3}}$ $y(0)=0$

① $y_1(t) \equiv 0$ is a solution.

② $y_2(t) = \begin{cases} (\frac{2}{3}t)^{\frac{3}{2}} & t>0 \\ 0 & t \le 0 \end{cases}$ is also a solution.



In this case. IVP DOES NOT have a unique Solution. We have to add more restrictions.

(Sufficient Condition)

Thm: Suppose. $f(t,x)$, $\frac{\partial f}{\partial x}(t,x)$ both continuous on a rectangle R in the tx-plane. ($[a,b] \times [c,d]$)

bounded. Closed Rectangle.!

Suppose $(t_0, x_0) \in R$ and that the solution

$x' = f(t,x)$. $y' = f(t,y)$ are two solutions

Satisfies. $x(t_0) = y(t_0) = x_0$, Then as long as

$(t, x(t))$. and $(t, y(t))$ stay in R. we have

$$x(t) = y(t).$$

Back to the example

$$\boxed{f(t,y) = y^{\frac{1}{3}} \qquad \frac{\partial f}{\partial y} = \frac{1}{3}\frac{1}{y^{\frac{2}{3}}} \qquad \text{Not cont when } y=0}$$

Proof: (Not in the textbook)

$\left(\Leftarrow \left|\frac{\partial f}{\partial x}\right| \leq M. \text{ on } R\right)$

Fact: $\frac{\partial f}{\partial x}$ continuous on $R \Rightarrow \frac{\partial f}{\partial x}$ is bdd. on R.

optional.

Suppose $x(t)$. $y(t)$ are two solutions of the IVP.

let $h(t) = x(t) - y(t)$,

$$h'(t) = x'(t) - y'(t) = f(t,x) - f(t,y).$$

$$\overset{M.V.T}{=} \frac{\partial f}{\partial x}(t, \xi(t)) \cdot (x(t) - y(t))$$

$$= \frac{\partial f}{\partial x}(t, \xi(t)) \cdot h(t).$$

$$\left|\frac{h'}{h}\right| = \left|\frac{\partial f}{\partial x}(t, \xi(t))\right| \leq M.$$

$$\left|\int_{t_0}^{t} \frac{h'}{h}\,dt\right| \leq \int_{t_0}^{t} \left|\frac{h'}{h}\right| dt \leq M \cdot (t - t_0)$$

$$\Big|\, \ln|h| - \ln|h_0| \,\Big| \leq M \cdot (t - t_0)$$

$$\iff \quad \left| \ln \left| \frac{h}{h_0} \right| \right| \leq M\,(t - t_0)$$

$$\left| \frac{h}{h_0} \right| \leq e^{M(t-t_0)}$$

$$|x(t) - y(t)| \leq |x(t_0) - y(t_0)| \cdot e^{M(t-t_0)}$$

as $x_0 = y_0 \Rightarrow x(t) = y(t)$. on R.

<u>Example</u>: Check Existence and Uniqueness of the IVP:

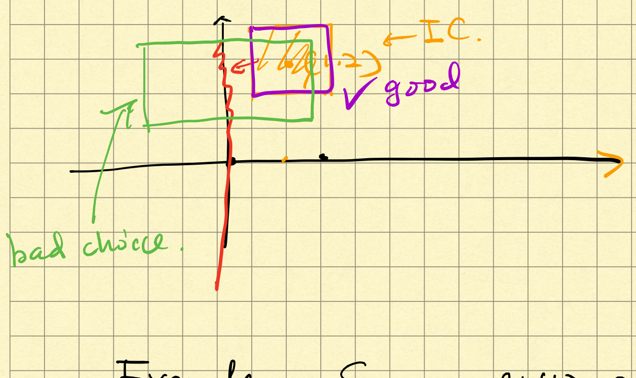
$$t\,y' = y + 3t^2. \qquad y(1) = 2.$$

Q: Is there a solution? If so, is the solution unique?

A: $y' = \frac{1}{t}(y + 3t^2) = f(t,y)$ cont when $t \neq 0$

$$\frac{\partial f}{\partial y} = \frac{1}{t} \quad \text{cont when } t \neq 0$$

$\left[\frac{1}{2}, 2\right] \times [a, 4]$



<u>Example</u>: Suppose $y(t)$ satisfies DE:

$$y' = (y-1) \cdot \cos(y \cdot t)$$

① If $y(0) = 1$, prove that $y(t) \equiv 1$

② Suppose $y(0)=2$, prove that $y(t)>1$ for all t

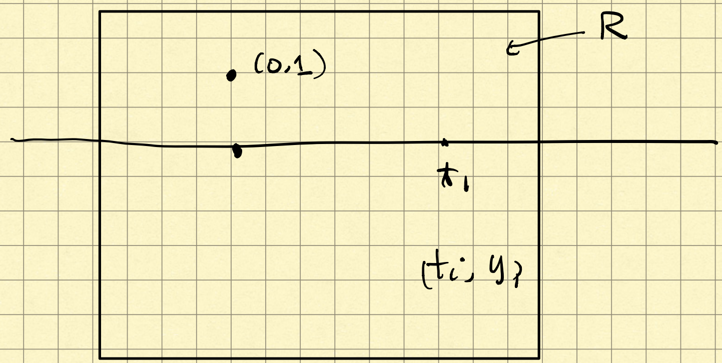
$$\frac{\partial}{\partial y} f(t,y) = -\cos(y\cdot t) + (1-y)\cdot y\cdot \sin(y\cdot t)$$

cont on $(-\infty,+\infty)\times(-\infty,+\infty)$

(cont on ANY Rectangle)

① Suppose $y(t_1) = y_1 \neq 1$

Find Rectangle R contains $(0,1)$, (t_1, y_1)



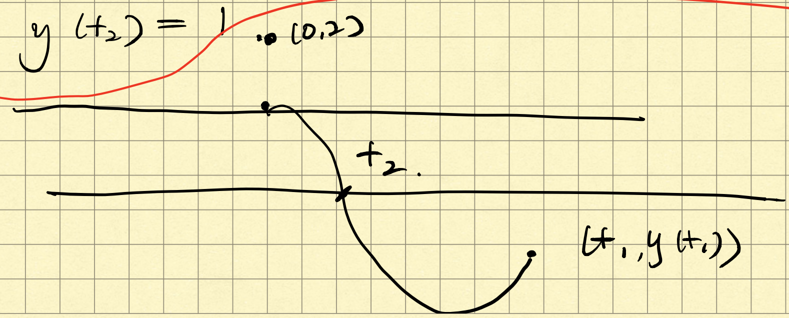
By Uniqueness. IVP on R has unique solution.

$\Rightarrow y(t) = 1$ on R. $\Rightarrow y(t_1) = 1$. ↯

② Suppose $y(t_1) = y_1 \leq 1$. , $y(t)$ cont

By IVT. $\exists\, t_2 \in [0, t_1]$ or $[t_1, 0]$

s.t. $y(t_2) = 1$



As before. if $y(t_2) = 1$. $y(t_2) \equiv 1$ is a solution and is unique. $\Rightarrow y(t) = 1$ on any Rectangle.

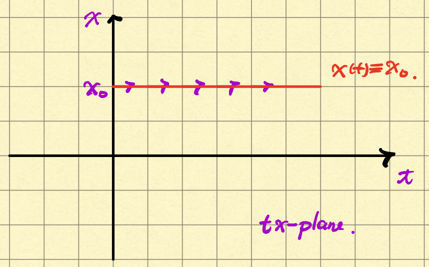
($\Rightarrow$ on $\mathbb{R}$)

Lecture

Autonomous Equation and Stability. (2.9)

Def: Autonomous DE $\iff$ $x' = f(x)$. $(x = x(t))$

Let x_0 be a root of $f(x)$. i.e. $f(x_0) = 0$. Then. $f(x) \equiv x_0$ is a solution.



x_0 is called an equilibrium point.

$x(t) \equiv x_0$ is called an equilibrium solution.

Analyzing the solutions of the autonomous equation.

Example: $y' = y(1-y)$.

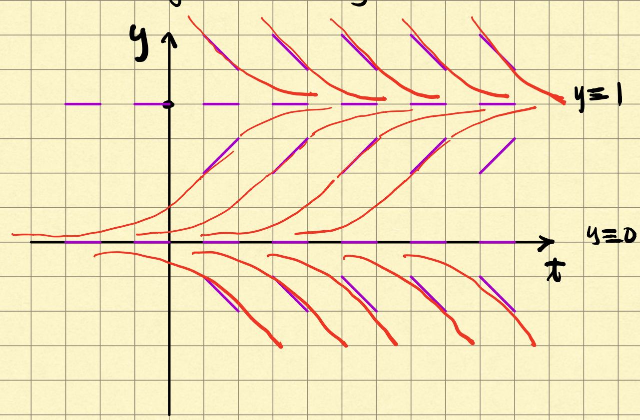
1) Sketch the directional fields
2) find Equilibrum.
3) Sketch the solutions.
4) Prove that if $y(0) = \frac{1}{2}$ then $0 < y(t) < 1$ for all $t \in (-\infty, \infty)$

The DE has two equilibrium.. $y \equiv 0$ and $y \equiv 1$

Let's sketch the directional fields of $y' = y(1-y) = f(y)$

$y > 1$, $f(y) < 0$ \
$y = 1$, $f(y) = 0$ —
$0 < y < 1$, $f(y) > 0$ /
$y = 0$, $f(y) = 0$ —
$y < 0$, $f(y) < 0$ \

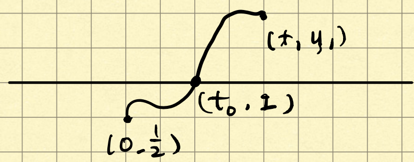
The dir $\tan\theta = f(y)$



two Equilibrium solution.

Step1 WLOG (Without loss of gen.)

4) **Proof** = Suppose $y(t_1) = y_1 > 1$. $t_1 > 0$

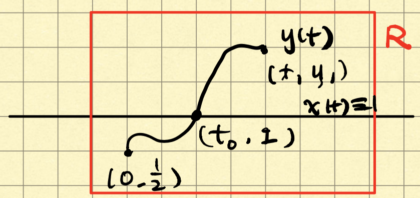


Note that, the solution $y(x)$ of DE is a continous function. (Actually Dif $\Rightarrow$ cont)

As $y(0) = \frac{1}{2} < 1$

and $y(t_1) = y_1 > 1$.

By I.V.T, there exist $t_0 \in (0, t_1)$ s.t $y(t_0) = 1$.



Step2 = $\exists$ Rectangle R contains $(0, \frac{1}{2})$. $(t_0, 1)$. (t_1, y_1)

We know that $\frac{\partial f}{\partial y} = 1 - 2y$ is continous on R.

By Uniqueness Thm, the solution go through $(t_0, 1)$ is unique!

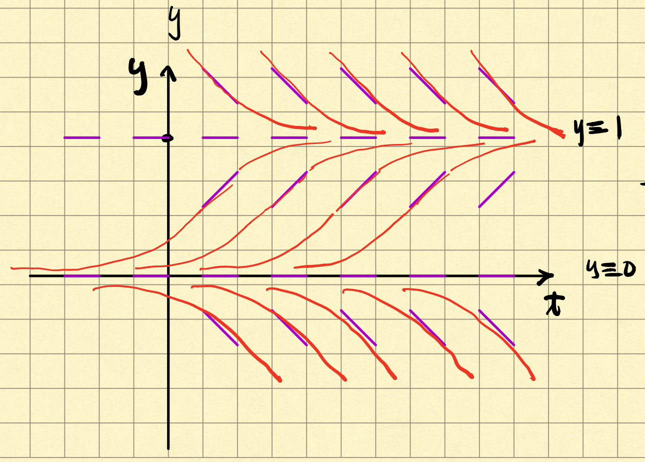
However, we have two (different) solutions $y(t)$ and $x(t) \equiv 1$.

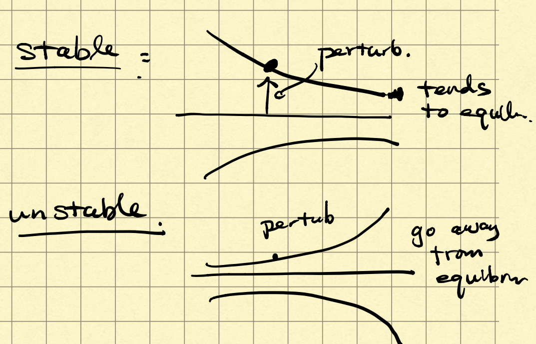
They are different, b/c. $x(0) = 1$. $y(0) = \frac{1}{2}$.

$x(t_1) = 1$ $y(t_1) = y_1 > 1$ ↯

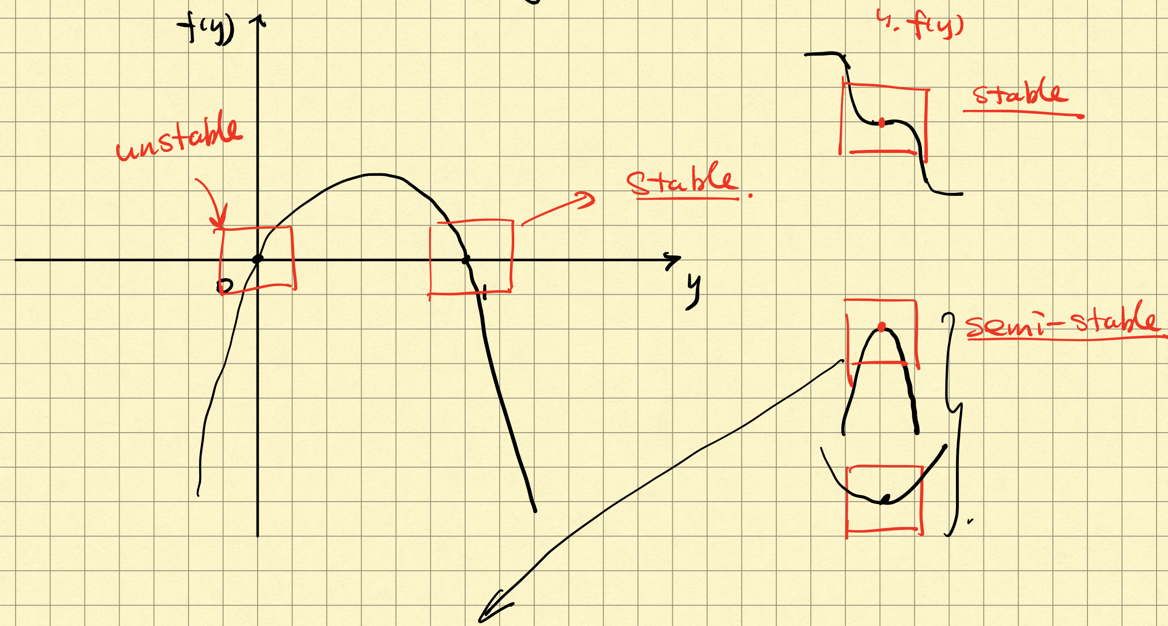
Classification of Equilibrium

$y' = f(y)$

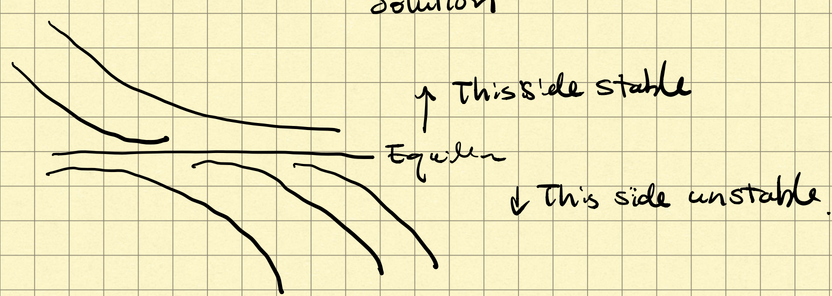




Also, we can check the graph of f(y)



Solution



Example: Analyze the solutions of

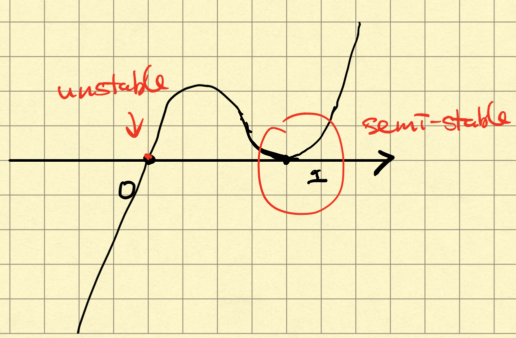
$$x' = x^3 - 2x^2 + x = x(x-1)^2 = f(x)$$

Step 1: Draw graph of f(x)

$x \in (1, +\infty)$ $f > 0$

$x \in (0, 1)$ $f > 0$

$x \in (-\infty, 0)$ $f < 0$



Step2 :