# 20W-MATH33B-1 Final Exam

CHARLES ZHANG

TOTAL POINTS

**91 / 100**

QUESTION 1

## 1 Question 1 10 / 10

✓ **- 0 pts** Correct

**- 1 pts** minus sign error

**- 2 pts** need to be in the form of "y = " form

**- 3 pts** algebraic mistake need to be ln(x)

**- 1 pts** miscelleneous algebraic mistake

**- 5 pts** only the homogeneous equation substitution was right

**- 2 pts** the position of "+c" needs to be in denominator

**- 8 pts** tried

QUESTION 2

## 2 Question 2 15 / 15

✓ **- 0 pts** Correct

**- 3 pts** some portion particular solution wrong/not found correctly

**- 1 pts** miscellaneous mistake

**- 6 pts** answer for part (b) wrong

**- 2 pts** general solution for 2(a) incomplete

QUESTION 3

## Question 3 15 pts

### 3.1 3 (a) 10 / 10

✓ **- 0 pts** Correct

**- 4 pts** Miscomputed one eigenvector.

**- 2 pts** You found fundamental solutions, but forgot C_1 and C_2.

**- 3 pts** Miscomputed both eigenvectors.

**- 1 pts** Write down the general solution, not just fundamental solutions.

### 3.2 3(b) 5 / 5

✓ **- 0 pts** Correct

**- 1 pts** Justification?

**- 2 pts** Eigenvectors graphed in incorrect quadrants.

**- 2 pts** Indicate direction travelled on solution curves.

**- 2 pts** The shape of your curves as t goes to infinity or - infinity is wrong

**- 3 pts** Draw solution curves in quadrants cut out by eigenvectors.

**- 3 pts** Draw the half-line solutions (the ones corresponding to the eigenvectors).

QUESTION 4

## 4 Question 4 10 / 10

✓ **- 0 pts** Correct

**- 2 pts** Your third fundamental solution is wrong/missing.

**- 2 pts** Your second fundamental solution is wrong/missing.

**- 1 pts** e^{2t} not e^t.

**- 2 pts** System of coefficients solved incorrectly/solution to IVP missing/incorrect.

**- 2 pts** Your first fundamental solution is wrong/missing.

QUESTION 5

## 5 Question 5 15 / 15

✓ **- 0 pts** Correct

**- 2 pts** Identify block matrices

**- 3 pts** Find eigenvalues for each block

**- 3 pts** Find (generalized) eigenvectors for each block

**- 4 pts** Construct solutions for each block

**- 3 pts** Combine solutions.

**- 1 pts** Minor calculation error

**- 2 pts** Moderate error in solution for one block

QUESTION 6

## Question 6 15 pts

### 6.1 6(a) **5 / 5**

✓ **- 0 pts** **Correct**

**- 2 pts** Knew to find zeros of RHS.

**- 2 pts** Correctly found at least one infinite family of solutions.

**- 1 pts** Found half of the solutions or made a computational mistake.

### 6.2 6(b) **5 / 5**

✓ **- 0 pts** **Correct**

**- 2 pts** Included equilibria

**- 3 pts** Solutions go in correct directions

**- 1 pts** Violates uniqueness

**- 1 pts** Small error

### 6.3 6(c) **5 / 5**

✓ **- 0 pts** **Correct**

**- 2 pts** Invoke hypotheses of existence and uniqueness.

**- 3 pts** Bound by equilibria

**- 1 pts** Minor error

**- 1 pts** Show it satisfies the hypotheses (show f and f' are continuous).

**- 1 pts** State that there are arbitrarily large or small equilibria.

**- 1 pts** Invoke existence and uniqueness.

QUESTION 7

## Question 7 10 pts

### 7.1 7(a) **3 / 5**

**- 0 pts** Correct

✓ **- 2 pts** **y_h /IF correct, didn't find y_p**

**- 2 pts** minor mistake / gap

**- 4 pts** Major mistake/gap

**- 5 pts** blank

**- 3 pts** y_h/IF minor mistake/not simplified, didn't find y_p

**- 1 pts** y_h not simplifed

**- 3 pts** y_h = ?

**- 1 pts** typo

### 7.2 7(b) **2 / 3**

**- 0 pts** Correct:

✓ **- 1 pts** **minor mistake (e.g. forget to say C can be anything), gap, logic flow not clear**

**- 2 pts** some meaningful writings. not much detail provided,  many gaps.

**- 3 pts** nothing meaningful

since C can be any number.

### 7.3 7(c) **0 / 2**

**- 0 pts** Correct

**- 2 pts** wrong

✓ **- 1 pts** **didn't put in normal form y' = F(y,t) = 1/ (t^2 - a^2) y + ...**

✓ **- 1 pts** **didn't check/state that \partial F/ \partial y = 1/(t^2 - a^2) or calculation is wrong**

**- 1 pts** Gap

QUESTION 8

## 8 Question 8 **6 / 10**

**- 0 pts** Correct

**- 4 pts** gap: did not verify (A-aI)^2 = 0

**- 4 pts** minor mistake

**- 2 pts** lack essential detail / some typos

✓ **- 4 pts** **based on your flow, you didn't use math induction to give a proof for A^n**

**- 8 pts** Major mistake

1. (10 points) Solve the homogeneous equation (Your final answer should be in $y = f(x, C)$ form, e.g $y = \frac{1}{C+x}$):

$$(-xy + y^2)dx + x^2dy = 0.$$

$y = vx$ , $dy = vdx + xdv$

$(-x(vx) + (vx)^2)\,dx + x^2(vdx + xdv) = 0$

$(-vx^2 + v^2x^2)dx + vx^2dx + x^3dv = 0$

$-vx^2dx + v^2x^2dx + vx^2dx + x^3dv = 0$

$v^2x^2dx + x^3dv = 0$

$v^2dx + xdv = 0$

$v^2dx = -xdv$

$\frac{dx}{x} = -\frac{dv}{v^2}$

$C + \ln|x| = -\int v^{-2}dv$

$C + \ln|x| = \frac{1}{v}$

$v = \frac{y}{x}$

$C + \ln|x| = \frac{x}{y}$

$\boxed{y = \frac{x}{C + \ln|x|}}$

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## 1 Question 1 **10 / 10**

- ✓ **- 0 pts** **Correct**
- **- 1 pts** minus sign error
- **- 2 pts** need to be in the form of "y = " form
- **- 3 pts** algebraic mistake need to be ln(x)
- **- 1 pts** miscelleneous algebraic mistake
- **- 5 pts** only the homogeneous equation substitution was right
- **- 2 pts** the position of "+c" needs to be in denominator
- **- 8 pts** tried

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2. (a) (5 points) Find the general solution to the differential equation:

$$y'' - 2y' + y = 0$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0$$

$$\lambda = 1$$

$$y_1(t) = e^t, \; y_2(t) = te^t$$

$$\boxed{y(t) = c_1 e^t + c_2 t e^t}$$

(b) (10 points) Find a particular solution to the differential equation (Hint: split forcing term into two parts, check the table in P172 of your textbook):

$$y'' - 2y' + y = e^t(t+1) + e^t \sin t.$$

$y'' - 2y' + y = e^t(t+1)$

$e^{rt}(P(t))$

$y(t) = e^t(At+B)$ and $y(t) = te^t(At+B)$

result in exceptional cases

$y_1(t) = t^2 e^t (At+B) = e^t(At^3 + Bt^2)$

$y_1'(t) = e^t(At^3 + Bt^2) + e^t(3At^2 + 2Bt)$

$y_1''(t) = e^t(At^3 + Bt^2) + e^t(3At^2 + 2Bt) +$
$e^t(3At^2 + 2Bt) + e^t(6At + 2B)$

$\cancel{(At^3 + Bt^2)} + \cancel{2(3At^2 + 2Bt)} + (6At + 2B) -$
$\cancel{2(At^3 + Bt^2)} - \cancel{2(3At^2 + 2Bt)} + \cancel{(At^3 + Bt^2)} = t+1$

$6At + 2B = t + 1$

$6A = 1 \qquad 2B = 1$

$A = 1/6 \qquad B = 1/2$

$y_1(t) = t^2 e^t\left(\frac{1}{6}t + \frac{1}{2}\right)$

$y'' - 2y' + y = e^t \sin t$

$e^{rt}[a\cos\omega t + b\sin\omega t]$

$y_2(t) = e^t(a\cos t + b\sin t)$

$y_2'(t) = e^t(a\cos t + b\sin t) + e^t(-a\sin t + b\cos t)$

$y_2''(t) = e^t(a\cos t + b\sin t) + 2e^t(-a\sin t + b\cos t) +$
$e^t(-a\cos t - b\sin t)$

$\cancel{(a\cos t + b\sin t)} + \cancel{2(-a\sin t + b\cos t)} + (-a\cos t - b\sin t)$
$-\cancel{2(a\cos t + b\sin t)} - \cancel{2(-a\sin t + b\cos t)} +$
$\cancel{(a\cos t + b\sin t)} = \sin t$

$-a\cos t - b\sin t = \sin t$

$a = 0, \; b = -1$

$y_2(t) = e^t(-\sin t)$

$$\boxed{y_p(t) = e^t\left(\frac{1}{6}t^3 + \frac{1}{2}t^2 - \sin t\right)}$$

## 2 Question 2 **15 / 15**

✓ **- 0 pts** Correct

**- 3 pts** some portion particular solution wrong/not found correctly

**- 1 pts** miscellaneous mistake

**- 6 pts** answer for part (b) wrong

**- 2 pts** general solution for 2(a) incomplete

3. (a) (10 points) Find the general solution ($y_{\text{general}} = C_1 y_1(t) + C_2 y_2(t)$) to the following $2 \times 2$ system $\mathbf{y}' = A\mathbf{y}$ , where

$$A = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}$$

$$\begin{bmatrix} -2-\lambda & -2 \\ 2 & 3-\lambda \end{bmatrix}$$

$$(-2-\lambda)(3-\lambda) - (-4) = 0$$

$$-6 - \lambda + \lambda^2 + 4 = 0$$

$$(\lambda^2 - \lambda - 2) = 0$$

$$(\lambda - 2)(\lambda + 1) = 0$$

$$\lambda_1 = 2, \lambda_2 = -1$$

$$\begin{bmatrix} -2-2 & -2 \\ 2 & 3-2 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 2 & 1 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

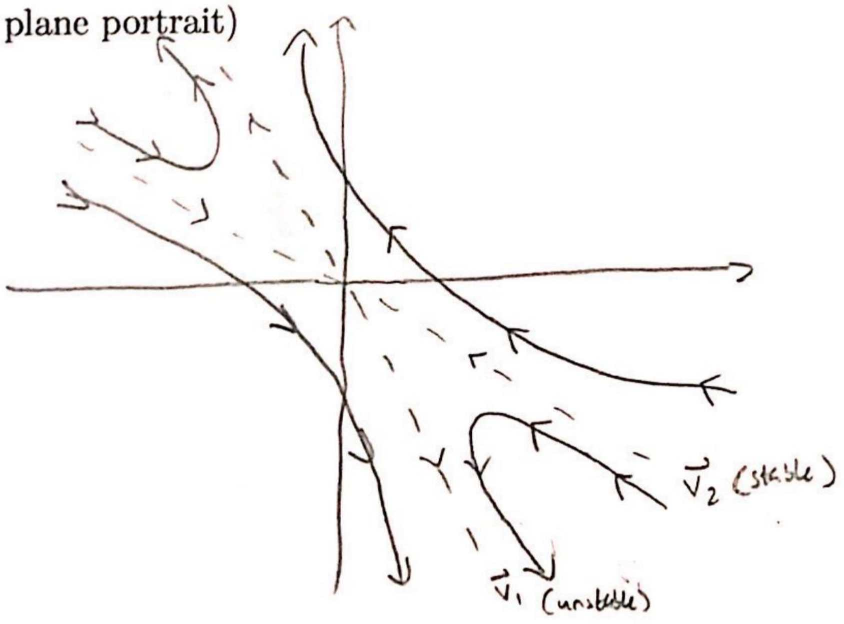
$$\begin{bmatrix} -2-(-1) & -2 \\ 2 & 3-(-1) \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & 4 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\boxed{y(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

saddle

(b) (5 points) Sketch the solutions on the phase plane. (i.e. Draw the phase plane portrait)

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3.1 3 (a) **10 / 10**

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- **- 4 pts** Miscomputed one eigenvector.
- **- 2 pts** You found fundamental solutions, but forgot C_1 and C_2.
- **- 3 pts** Miscomputed both eigenvectors.
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$$A = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}$$

$$\begin{bmatrix} -2-\lambda & -2 \\ 2 & 3-\lambda \end{bmatrix}$$

$$(-2-\lambda)(3-\lambda) - (-4) = 0$$
$$-6 - \lambda + \lambda^2 + 4 = 0$$
$$(\lambda^2 - \lambda - 2) = 0$$
$$(\lambda - 2)(\lambda + 1) = 0$$
$$\lambda_1 = 2, \lambda_2 = -1$$

$$\begin{bmatrix} -2-2 & -2 \\ 2 & 3-2 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 2 & 1 \end{bmatrix}$$
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$$\boxed{y(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

saddle

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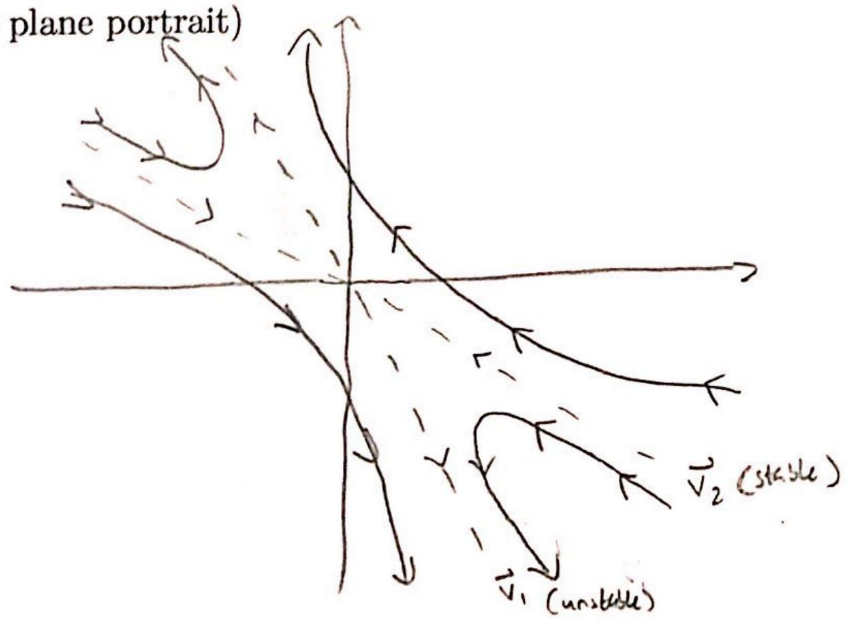

### 3.2 3(b) **5 / 5**

- ✓ **- 0 pts** **Correct**
- **- 1 pts** Justification?
- **- 2 pts** Eigenvectors graphed in incorrect quadrants.
- **- 2 pts** Indicate direction travelled on solution curves.
- **- 2 pts** The shape of your curves as t goes to infinity or - infinity is wrong
- **- 3 pts** Draw solution curves in quadrants cut out by eigenvectors.
- **- 3 pts** Draw the half-line solutions (the ones corresponding to the eigenvectors).

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4. (10 points) Find the solution $\mathbf{y}(t)$ to the following $3 \times 3$ system with given initial condition $\mathbf{y}(0) = (2, -2, 1)^T$:

$$\mathbf{y}' = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix} \mathbf{y}$$

(Hint: Use rational zero theorem to find integer roots of the characteristic polynomial)

$$\left| \begin{array}{ccc|cc} 3-\lambda & 1 & 1 & 3-\lambda & 1 \\ -1 & 1-\lambda & -1 & -1 & 1-\lambda \\ 1 & 1 & 2-\lambda & 1 & 1 \end{array} \right|$$

$(3-\lambda)(1-\lambda)(2-\lambda) - 2 - [-(2-\lambda) - (3-\lambda) + (1-\lambda)] = 0$

$(3-4\lambda+\lambda^2)(2-\lambda) - 2 - [-2+\lambda-3+\lambda+1-\lambda] = 0$

$(6-3\lambda-8\lambda+4\lambda^2+2\lambda^2-\lambda^3-2+2-\lambda+3-\lambda-1+\lambda) = 0$

$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$

$p: \pm1, \pm2, \pm4, \pm8, \; q = 1$

$\lambda$ cannot be a neg #

$p/q = 1, 2, 4, 8$

$\lambda = 2 \rightarrow (\lambda-2)^3 = 0$

check
$(\lambda-2)^3$
$(\lambda^2-4\lambda+4)(\lambda-2)$
$\lambda^3-2\lambda^2-4\lambda^2+8\lambda+4\lambda-8$ ✓

$A - \lambda I = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$

$\text{rref}(A-\lambda I) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$x + y = 0$
$z = 0$

$\vec{v}_1 = e^{2t}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$(A-\lambda I)^2 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\vec{v}_2 = e^{2t}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$

$\vec{v}_2 = e^{2t}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)$

$(A-\lambda I)^3 = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\vec{v}_3 = e^{2t}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = e^{2t}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)$

$y(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 e^{2t}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right) + c_3 e^{2t}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)$

$y(0) = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) + c_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$c_1 = 2, \; c_2 = 1, \; c_3 = 0$

$$\boxed{y(t) = 2e^{2t}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + e^{2t}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)}$$

## 4 Question 4 **10 / 10**

✓ **- 0 pts** Correct

**- 2 pts** Your third fundamental solution is wrong/missing.

**- 2 pts** Your second fundamental solution is wrong/missing.

**- 1 pts** e^{2t} not e^t.

**- 2 pts** System of coefficients solved incorrectly/solution to IVP missing/incorrect.

**- 2 pts** Your first fundamental solution is wrong/missing.

5. (15 points) Find the general solution(fundamental set) $\mathbf{y}(t)$ to the following $6 \times 6$ system:

$$\mathbf{y}' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} \mathbf{y}$$

(Hint : This is a block matrix. Try find a 1 by 1, 3 by 3, and 2 by 2 block.)

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$y' = Ay$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & -1 & -\lambda \end{vmatrix} \begin{matrix} -\lambda & 0 \\ 0 & 2-\lambda \\ 0 & -1 \end{matrix}$$

$$= \lambda^2(2-\lambda) - \lambda$$

$$= 2\lambda^2 - \lambda^3 - \lambda$$

$$\lambda^3 - 2\lambda^2 + \lambda = 0$$

$$\lambda(\lambda^2 - 2\lambda + 1) = 0$$

$$\lambda(\lambda - 1)^2 = 0$$

$$\lambda_1 = 0, \lambda_2 = 1$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix} \to \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \to \vec{v}_2 = e^t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$(A - \lambda_2 I)^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}^2 =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{v}_3 = e^t\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$\vec{v}_3 = e^t\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right)$$

$$= e^t \begin{bmatrix} 0 \\ t \\ 1-t \end{bmatrix}$$

$$B = [1]$$

$$\det(B - \lambda I) = 1 - \lambda$$

$$\lambda = 1$$

$$y' = [1]y$$

$$y = [e^t]$$

$$\vec{v} = [e^t]$$

$$C = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

$$y' = Cy$$

$$y' = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} y$$

$$\vec{y}_1 = \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix}, \vec{y}_2 = \begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix} = \begin{bmatrix} -2\sin 2t \\ 2\cos 2t \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix} = \begin{bmatrix} -2\cos 2t \\ -2\sin 2t \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ e^t \\ -e^t \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ te^t \\ e^t(1-t) \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ e^t \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cos 2t \\ \sin 2t \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2\cos 2t \\ -2\sin 2t \end{bmatrix}$$

## 5 Question 5 **15 / 15**

- ✓ **- 0 pts** Correct
- **- 2 pts** Identify block matrices
- **- 3 pts** Find eigenvalues for each block
- **- 3 pts** Find (generalized) eigenvectors for each block
- **- 4 pts** Construct solutions for each block
- **- 3 pts** Combine solutions.
- **- 1 pts** Minor calculation error
- **- 2 pts** Moderate error in solution for one block

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6. Consider the autonomous equation:

$$y' = \sin y + \cos y$$

(a) (5 points) Find all equilibria of the differential equation.

$0 = \sin y + \cos y$

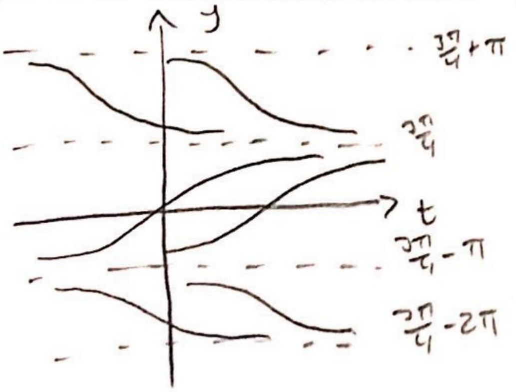
$-\sin y = \cos y$

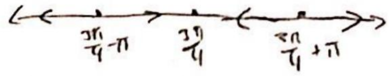
$-\tan y = 1$

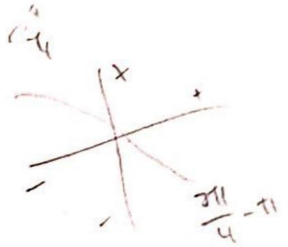
$\tan y = -1$

$\boxed{y = \frac{3\pi}{4} + n\pi, \text{ where } n \text{ is an integer}}$

(b) (5 points) Sketch the solutions on the $t-y$ plane.




(c) (5 points) Prove that if $y(t)$ is a solution, then $y(t)$ is a bounded function. (In other words, given a solution $y(t)$, there exists $m, M$ such that, $m < y(t) < M$ for all $t \in (-\infty, +\infty)$)

$f(t,y)$ is continuous for $(-\infty, \infty)$, and so is its derivative, $\frac{\partial f}{\partial y} = \cos y - \sin y$. Therefore, this function must satisfy uniqueness. This means that if $y(t)$ is a solution, then it cannot intersect with another solution. Therefore, since equilibrium solutions are defined by $y = \frac{3\pi}{4} + n\pi$ (there are infinitely many) $y(t)$ can always be bounded by equilibrium solutions if it is a non-equilibrium solution, or by trivial values if it is an equilibrium solution (ie. $\pi/2$ and $\pi$ bound $y = \frac{3\pi}{4}$).

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### 6.1 6(a) 5 / 5

- ✓ **- 0 pts** Correct
- **- 2 pts** Knew to find zeros of RHS.
- **- 2 pts** Correctly found at least one infinite family of solutions.
- **- 1 pts** Found half of the solutions or made a computational mistake.

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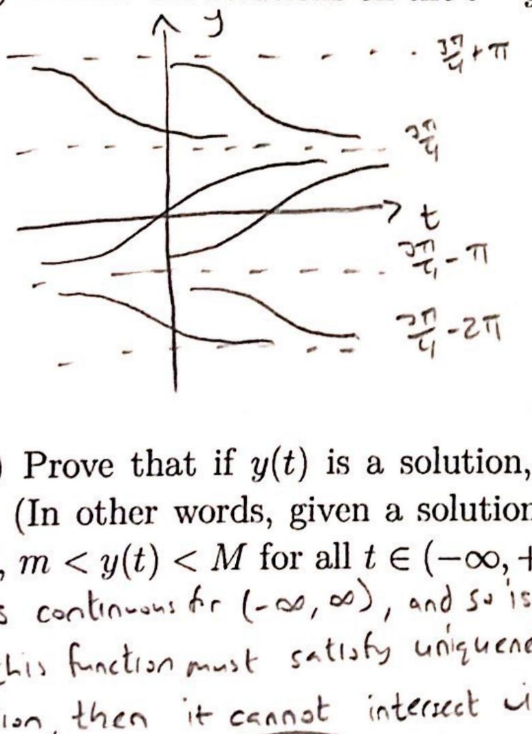
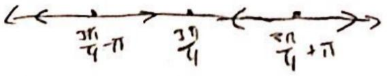
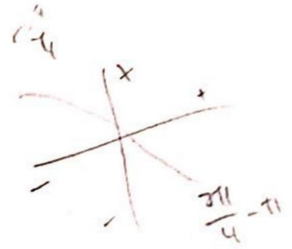
(b) (5 points) Sketch the solutions on the $t - y$ plane.



(c) (5 points) Prove that if $y(t)$ is a solution, then $y(t)$ is a bounded function. (In other words, given a solution $y(t)$, there exists $m, M$ such that, $m < y(t) < M$ for all $t \in (-\infty, +\infty)$)

$f(t,y)$ is continuous for $(-\infty, \infty)$, and so is its derivative, $\frac{\partial f}{\partial y} = \cos y - \sin y$. Therefore, this function must satisfy uniqueness. This means that if $y(t)$ is a solution, then it cannot intersect with another solution. Therefore, since equilibrium solutions are defined by $y = \frac{3\pi}{4} + n\pi$ (there are infinitely many) $y(t)$ can always be bounded by equilibrium solutions if it is a non-equilibrium solution, or by trivial values if it is an equilibrium solution (ie. $\pi/2$ and $\pi$ bound $y = \frac{3\pi}{4}$).

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### 6.2 6(b) **5 / 5**

✓ **- 0 pts** Correct

**- 2 pts** Included equilibria

**- 3 pts** Solutions go in correct directions

**- 1 pts** Violates uniqueness

**- 1 pts** Small error

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6. Consider the autonomous equation:

$$y' = \sin y + \cos y$$

(a) (5 points) Find all equilibria of the differential equation.

$0 = \sin y + \cos y$

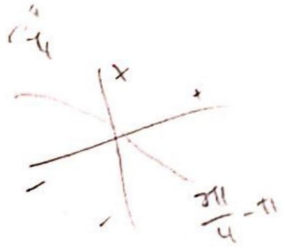
$-\sin y = \cos y$

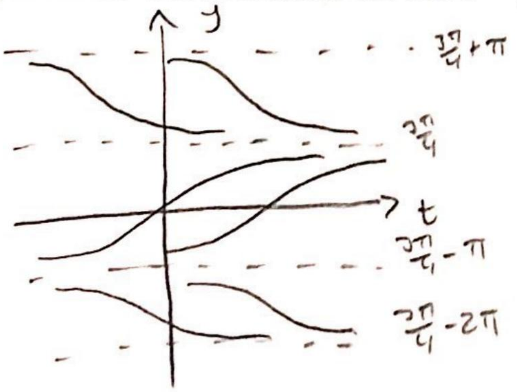
$-\tan y = 1$

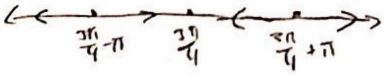
$\tan y = -1$

$\boxed{y = \frac{3\pi}{4} + n\pi, \text{ where } n \text{ is an integer}}$

(b) (5 points) Sketch the solutions on the $t - y$ plane.




(c) (5 points) Prove that if $y(t)$ is a solution, then $y(t)$ is a bounded function. (In other words, given a solution $y(t)$, there exists $m, M$ such that, $m < y(t) < M$ for all $t \in (-\infty, +\infty)$)

$f(t,y)$ is continuous for $(-\infty, \infty)$, and so is its derivative, $\frac{\partial f}{\partial y} = \cos y - \sin y$. Therefore, this function must satisfy uniqueness. This means that if $y(t)$ is a solution, then it <u>cannot</u> intersect with another solution. Therefore, since equilibrium solutions are defined by $y = \frac{3\pi}{4} + n\pi$ (there are infinitely many) $y(t)$ can always be bounded by equilibrium solutions if it is a non-equilibrium solution, or by trivial values if it is an equilibrium solution (ie. $\pi/2$ and $\pi$ bound $y = \frac{3\pi}{4}$).

6.3 6(c) **5 / 5**

✓ **- 0 pts** Correct

**- 2 pts** Invoke hypotheses of existence and uniqueness.

**- 3 pts** Bound by equilibria

**- 1 pts** Minor error

**- 1 pts** Show it satisfies the hypotheses (show f and f' are continuous).

**- 1 pts** State that there are arbitrarily large or small equilibria.

**- 1 pts** Invoke existence and uniqueness.

7. Let $a$ be a positive integer(it is a fixed unknown number). Consider the following differential equation:

$$(t^2 - a^2)y' = y + t^2 - t - a^2.$$

(a) (5 points) Find the general solution to the above differential equation.

$(t^2-a^2)y' = y+t^2-t-a^2$

$y' = \frac{y+t^2-t-a^2}{(t^2-a^2)}$

$y' = \frac{1}{t^2-a^2}y - \frac{t}{(t^2-a^2)} + 1$

$a(t) = \frac{1}{t^2-a^2}$

$IF = e^{\int a(t)dt}$

$IF = e^{-\int \frac{1}{t^2-a^2}}$

$= e^{\ln\left|\frac{t-a}{t+a}\right|^{+2a}}$

$= \left|\frac{t-a}{t+a}\right|^{2a}$

$\left|\frac{t-a}{t+a}\right|^{2a}y' - \left|\frac{t-a}{t+a}\right|^{2a}\left(\frac{y}{t^2-a^2}\right) = -\frac{t}{(t^2-a^2)}\left(\frac{t-a}{t+a}\right)^{2a} + \left(\frac{t-a}{t+a}\right)^{2a}$

$\int\left(\left(\frac{t-a}{t+a}\right)^{2a}y\right)' = \int \frac{-t}{(t^2-a^2)}\left(\frac{t-a}{t+a}\right)^{2a} + \left(\frac{t-a}{t+a}\right)^{2a} dt$

$\left(\frac{t-a}{t+a}\right)^{2a}y = \int \frac{t^2-t-a^2}{(t^2-a^2)}\left(\frac{t-a}{t+a}\right)^{2a} dt$

I've tried every method that we've been taught, and I've gotten an impossible integral every time :(

$\boxed{\text{Assume: } y = C\left(\frac{t-a}{t+a}\right)^{\frac{1}{2a}} + a}$

$\frac{1}{t+a}C_1$

$b = \frac{1}{2a}C_1$

$C_1 = 2ab$

$\frac{1}{t+a}2ab$

(b) (3 points) Consider the above differential equation together with the initial condition $y(a) = b$ (initial value problem), where $b$ is a real number. Prove that,

- if $b = a$, there are infinite many solution to the initial value problem. (i.e. go through the initial condition.)
- if $b \neq a$, there is no solutions to the initial value problem.

Assuming $y = C\left(\frac{t-a}{t+a}\right)^{\frac{1}{2a}} + a$

$y(a) = C\left(\frac{0}{2a}\right)^{\frac{1}{2a}} + a$

$y(a) = a \rightarrow$ if $b=a$, then $a=a \rightarrow$ infinite solutions

$y(a) = a \rightarrow$ if $b \neq a$ then $b=a$ is false $\rightarrow$ no solution

(c) (2 points) Does the above (weird) result contradict with the existence and uniqueness theorem? Why?

This doesn't contradict U/E because the function/its derivative is not continuous over the necessary interval.

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$(-y - t^2 + t + a^2)\,dt + (t^2 - a^2)\,dy = 0$

$P = -y - t^2 + t + a^2 \qquad Q = t^2 - a^2$

$\dfrac{\partial P}{\partial y} = -1 \qquad \dfrac{\partial Q}{\partial t} = 2t$

$\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial t} = -1 - 2t = -(2t+1)$

$\dfrac{1}{t^2-a^2}(-2t+1) \leftarrow$ only in terms of t

$IF: e^{\int f(t)} = e^{\int \frac{-2t+1}{t^2-a^2}\,dt}$

$= e^{\int \frac{-2t}{t^2-a^2} + \int \frac{1}{t^2-a^2}}$

$= e^{-\ln|t^2-a^2| + \ln\left|\frac{t-a}{t+a}\right|^{1/2a}}$

$= e^{\ln|t^2-a^2|^{-1}} \, e^{\ln\left|\frac{t-a}{t+a}\right|^{1/2a}}$

$\dfrac{1}{t^2-a^2}\left(\dfrac{t-a}{t+a}\right)^{1/2a} = IF$

$P = \dfrac{1}{t^2-a^2}\left(\dfrac{t-a}{t+a}\right)^{1/2a}(-y - t^2 + t + a^2)$

$Q = \cancel{\dfrac{1}{t^2-a^2}(t^2-a^2)}\left(\dfrac{t-a}{t+a}\right)^{1/2a}$

$\int Q\,dy = \int \left(\dfrac{t-a}{t+a}\right)^{1/2a} dy =$

$\left(\dfrac{(t-a)^{1/2a}}{(t+a)^{1/2a}}\right) y + \phi(t) = F(t,y)$

$f(x) = y(t-a)^{1/2a}, \quad f'(x) = \dfrac{y}{2a}(t-a)^{1/2a-1}$

$g(x) = (t+a)^{1/2a}, \quad g'(x) = \dfrac{1}{2a}(t+a)^{\frac{1}{2a}-1}$

$\dfrac{\partial F}{\partial t} = \dfrac{\frac{y}{2a}(t+a)^{1/2a}(t-a)^{\frac{1}{2a}-1} - \frac{y}{2a}(t-a)^{1/2a}(t+a)^{\frac{1}{2a}-1}}{(t+a)^{1/a}} + \phi'(t)$

AWESOME

Pretending that $y = a\left(\dfrac{t-a}{t+a}\right)^{\frac{1}{2a}} + a$, from now on

**7.1** 7(a) **3 / 5**

- **- 0 pts** Correct
- ✓ **- 2 pts** **y_h /IF correct, didn't find y_p**
- **- 2 pts** minor mistake / gap
- **- 4 pts** Major mistake/gap
- **- 5 pts** blank
- **- 3 pts** y_h/IF minor mistake/not simplified, didn't find y_p
- **- 1 pts** y_h not simplifed
- **- 3 pts** y_h = ?
- **- 1 pts** typo

7. Let $a$ be a positive integer(it is a fixed unknown number). Consider the following differential equation:

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(a) (5 points) Find the general solution to the above differential equation.

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$y(a) = C\left(\frac{0}{2a}\right)^{\frac{1}{2a}} + a$

$y(a) = a \to$ if $b = a$, then $a = a \to$ infinite solutions

$y(a) = a \to$ if $b \neq a$ then $b = a$ is false $\to$ no solution

(c) (2 points) Does the above (weird) result contradict with the existence and uniqueness theorem? Why?

This doesn't contradict U/E because the function/its derivative is not continuous over the necessary interval.

7.2 7(b) **2 / 3**

- **- 0 pts** Correct:
- ✓ **- 1 pts minor mistake (e.g. forget to say C can be anything), gap, logic flow not clear**
- **- 2 pts** some meaningful writings. not much detail provided,  many gaps.
- **- 3 pts** nothing meaningful

since C can be any number.

7. Let $a$ be a positive integer(it is a fixed unknown number). Consider the following differential equation:

$$(t^2 - a^2)y' = y + t^2 - t - a^2.$$

(a) (5 points) Find the general solution to the above differential equation.

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(c) (2 points) Does the above (weird) result contradict with the existence and uniqueness theorem? Why?

This doesn't contradict U/E because the function/its derivative is not continuous over the necessary interval.

7.3 7(c) **0 / 2**

- **- 0 pts** Correct
- **- 2 pts** wrong
- ✓ **- 1 pts** didn't put in normal form y' = F(y,t) = 1/ (t^2 - a^2) y + ...
- ✓ **- 1 pts** didn't check/state that \partial F/ \partial y = 1/(t^2 - a^2) or calculation is wrong
- **- 1 pts** Gap

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8. (10 points) Calculate $e^{tA}$, where $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ ← never truncates

(Hint: Use truncation formula) ?

$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \cdots$

$A^2 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}$

$A^3 = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}$

$A^4 = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^4 & 4a^3b \\ 0 & a^4 \end{bmatrix}$

$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^n}{n!}A^n$

$$\boxed{I + t\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix} + \cdots + \frac{t^n}{n!}\begin{bmatrix} a^n & n(a^{n-1}b) \\ 0 & a^n \end{bmatrix}}$$

## 8 Question 8 **6 / 10**

- **- 0 pts** Correct
- **- 4 pts** gap: did not verify (A-aI)^2 = 0
- **- 4 pts** minor mistake
- **- 2 pts** lack essential detail / some typos
- ✓ **- 4 pts** **based on your flow, you didn't use math induction to give a proof for A^n**
- **- 8 pts** Major mistake