Homework 2

Status: Final.

Please reach out to me via the CCLE forum for corrections and/or clarifications about statements found in this document.

Due date: Friday, April 17.

Regular exercises

Functions

Section 3.1 in course textbook.

1. Given
$$g = \{(1, b), (2, c), (3, a)\},$$
a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$, and
$$f = \{(a, x), (b, x), (c, z), (d, w)\},$$
a function from Y to $Z = \{w, x, y, z\}$, write $f \circ g$ as a set of ordered pairs and draw the arrow diagram of $f \circ g$.

2. Let f and g be functions from the positive integers to the positive integers defined by the equations
$$f(n) = 2n + 1, \qquad g(n) = 3n - 1.$$
Find the compositions $f \circ f$, $g \circ g$, $f \circ g$, and $g \circ f$.

3. Given
$$f = \{(a, b), (b, a), (c, b)\},$$
a function from $X = \{a, b, c\}$ to X:
   i. Write $f \circ f$ and $f \circ f \circ f$ as sets of ordered pairs.
   ii. Define
   $$f^n = f \circ f \circ \cdots \circ f$$
   to be the n-fold composition of f with itself. Write f^9, f^{623}, and f^{2020} as sets of ordered pairs.

4. Let f be the function from $X = \{0, 1, 2, 3, 4, 5\}$ to X defined by
$$f(x) = 4x \bmod 6.$$

Write f as a set of ordered pairs and draw the arrow diagram of f. Is f one-to-one? Is f onto?

5. Is the formula $\lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$ true for all real numbers? Prove it, or give a counter-example.

Sequences & strings

Section 3.2 in course textbook.

6. Determine whether the sequence s defined by $s_n = 2^n - n^2$ is *increasing*, *decreasing*, *nonincreasing*, or *nondecreasing*, for the given domain D.

   i. $D = \{1, 2, 3, 4\}$
   ii. $D = \{n \,|\, n \in \mathbf{Z},\ n \geq 2020\}$

7. For the sequence t defined by $t_n = 2n - 1$, $n \geq 1$.

   i. Find t_3, t_7, t_{100}, and t_{2020}.
   ii. Find $\prod_{i=3}^{6} t_i$, and $\sum_{i=3}^{7} t_i$.

8. For the sequence w defined by $w_n = \frac{1}{n} - \frac{1}{n+1}$, $n \geq 1$.

   i. Find $\sum_{i=1}^{3} w_i$, and $\sum_{i=1}^{10} w_i$.
   ii. Find $\sum_{i=1}^{2020} w_i$.

9. Let r be the sequence defined by $r_n = 3 \cdot 2^n - 4 \cdot 5^n$, $n \geq 0$.

   i. Find r_0, r_1, r_2, and r_3.
   ii. Find a formula for r_{n-1}, and r_{n-2}.
   iii. Prove that the sequence $\{r_n\}$ satisfies

$$r_n = 7r_{n-1} - 10r_{n-2}, \qquad n \geq 2.$$

10. Rewrite the sum

$$\sum_{i=1}^{n} i^2 r^{n-i}$$

replacing the index i by k, where $i = k + 1$.

11. Compute the given quantity using the strings

$$\alpha = baab, \qquad \beta = caaba, \qquad \gamma = bbab.$$

i. $\alpha\beta$
ii. $\beta\alpha$
iii. $\alpha\alpha$
iv. $\beta\beta$
v. $|\alpha\beta|$
vi. $|\beta\alpha|$

12. List all strings over $X = \{0, 1\}$ of ...

    i. lenght 2.
    ii. lenght 2 or less.
    iii. lenght 3.
    iv. lenght 3 or less.
    v. lenght 3 without two consecutive zeros.

13. Find all substrings of the string *babc*.

14. Let $X = \{a, b\}$. A *palindrome over* X is a string α for which $\alpha = \alpha^R$ (i.e., a string that reads the same forward and back-ward). An example of a palindrome over X is *bbaabb*. Define a function from X^* to the set of palindromes over X as $f(\alpha) = \alpha\alpha^R$. Is f one-to-one? Is f onto? Prove your answers.

15. Let L be the set of all strings, including the null string, that can be constructed by repeated application of the following rules:

    - If $\alpha \in L$, then $a\alpha b \in L$, and $b\alpha a \in L$.
    - If $\alpha \in L$ and $\beta \in L$, then $\alpha\beta \in L$.

    For example, ab is in L, for if we take $\alpha = \lambda$, then $\alpha \in L$ and the first rule states that $ab = a\alpha b \in L$. Similarly, $ba \in L$. As a final example, $abba$ is in L, for if we take $\alpha = ab$ and $\beta = ba$, then $\alpha \in L$ and $\beta \in L$; by the second rule, $abba = \alpha\beta \in L$.

    i. Show that $aaabbb$ is in L.
    ii. Show that $baabab$ is in L.
    iii. Show that aab is not in L.

Relations

Section 3.3 in course textbook.

16. Draw the *digraph* of the following relations:

    i. $R = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$ on $\{1, 2, 3, 4\}$.
    ii. $R = \{(1, 2), (2, 1), (3, 3), (1, 1), (2, 2)\}$ on $\{1, 2, 3\}$.

17. Let R be the relation on the set $\{1,2,3,4,5\}$ defined by the rule $(x,y) \in R$ if 3 divides $x - y$.

    i. List the elements of R.
    ii. List the elements of R^{-1}.
    iii. Repeat the previous two parts when R is defined by the rule $(x,y) \in R$ if $x + y \leq 6$.

18. For the given relation R on the set of positive integers, determine if it is *reflexive*, *symmetric*, *antisymmetric*, *transitive*, and/or a *partial order*.

    i. $(x,y) \in R$ if 3 divides $x - y$.
    ii. $(x,y) \in R$ if $x \geq y$.

19. Let X be a nonempty set. Define a relation on $\mathcal{P}(X)$, the power set of X, as $(A,B) \in R$ if $A \subseteq B$. Is this relation *reflexive*, *symmetric*, *antisymmetric*, *transitive*, and/or a *partial order*?

20. Let R_1 and R_2 be the relations on $\{1,2,3,4\}$ given by

$$R_1 = \{(1,1),(1,2),(3,4),(4,2)\} \tag{1}$$
$$R_2 = \{(1,1),(2,1),(3,1),(4,4),(2,2)\}. \tag{2}$$

List the elements of $R_1 \circ R_2$ and $R_2 \circ R_1$.

Miscellaneous exercises

- A store offers a (fixed, nonzero) percentage off the price of certain items. A coupon is also available that offers a (fixed, nonzero) amount off the price of the same items. The store will honor both discounts. Show that regardless of the price of an item, the percentage off the price, and amount off the price, it is always cheapest to use the coupon first.

- Let $f : \mathbf{N} \to \mathbf{Z}$ be the function defined by the formula

$$f(n) = (-1)^{n+1} \left\lfloor \frac{n}{2} \right\rfloor.$$

  Is f *one-to-one*? Is it *onto*? Is it a *bijection*?

- Given

$$f = \{(x, x^2) \mid x \in X\},$$

  a function from $X = \{-5, -4, ..., 4, 5\}$ to the set of integers, write f as a set of ordered pairs and draw the arrow diagram of f. Is f one-to-one or onto?

- Let $a_n = n^2 - 3n + 3$, for $n = 1, 2, \ldots$ Find

  i. $\sum\limits_{n=1}^{k} a_{2020}$

  ii. $\sum\limits_{n=1}^{2020} a_n$

  iii. $\sum\limits_{n=1}^{k} a_n$

- Use induction to prove that

  $$\sum \frac{1}{n_1 \cdot n_2 \cdots n_k} = n,$$

  for all $n \geq 1$, where the sum is taken over all nonempty subsets $\{n_1, n_2, \ldots, n_k\}$ of $\{1, 2, \ldots, n\}$.

- Find all substrings of the string *aabaabb*.

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