EXERCISES 3.1

GOAL Use the concepts of the image and the kernel of a linear transformation (or a matrix). Express the image and the kernel of any matrix as the span of some vectors. Use kernel and image to determine whether a matrix is invertible.

For each matrix A in Exercises 1 through 13, find vectors that span the kernel of A. Use paper and pencil.

1.
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

2.
$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$\mathbf{3.} \ A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

4.
$$A = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$$

$$\mathbf{5.} \ \ A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$$

6.
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

7.
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\mathbf{8.} \ \ A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

For each matrix A in Exercises 14 through 16, find vectors that span the image of A. Give as few vectors as possible. Use paper and pencil.

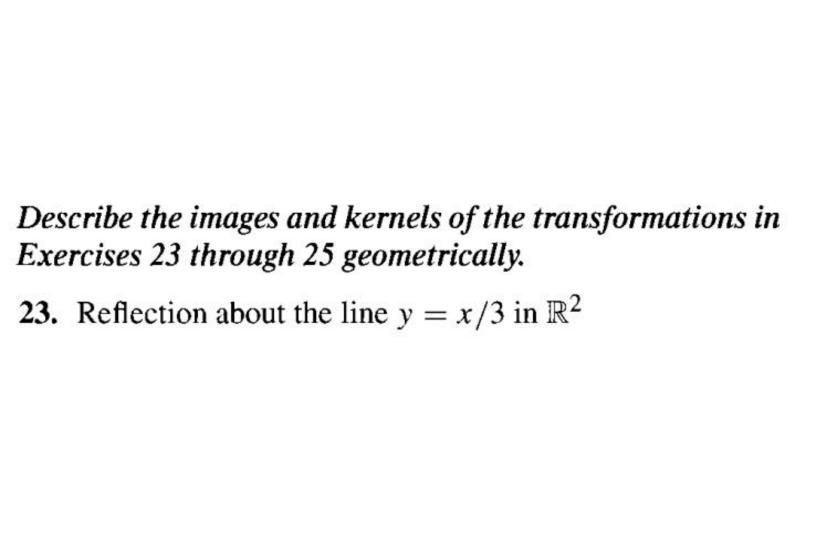
14.
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

15.
$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$
 16. $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$

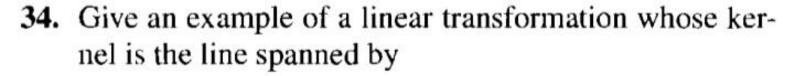
For each matrix A in Exercises 17 through 22, describe the image of the transformation $T(\vec{x}) = A\vec{x}$ geometrically (as a line, plane, etc. in \mathbb{R}^2 or \mathbb{R}^3).

17.
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
 18. $A = \begin{bmatrix} 1 & 4 \\ 3 & 12 \end{bmatrix}$

19.
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & -8 \end{bmatrix}$$



| 24. | Orthogonal | projection | onto the | plane | x + | 2y + | 3z | =0 | - |
|-----|-------------------|------------|----------|-------|-----|------|----|----|---|
| | in \mathbb{R}^3 | 277 8539 | | - T | | | | | |



$$\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

in \mathbb{R}^3 .

EXERCISES 3.2

GOAL Check whether or not a subset of \mathbb{R}^n is a subspace. Apply the concept of linear independence (in terms of Definition 3.2.3, Theorem 3.2.7, and Theorem 3.2.8). Apply the concept of a basis, both in terms of Definition 3.2.3 and in terms of Theorem 3.2.10.

Which of the sets W in Exercises 1 through 3 are subspaces of \mathbb{R}^3 ?

$$1. W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 1 \right\}$$

$$\mathbf{2.} \ \ W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \le y \le z \right\}$$

3.
$$W = \left\{ \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix} : x, y, z \text{ arbitrary constants} \right\}$$

- **4.** Consider the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ in \mathbb{R}^n . Is span $(\vec{v}_1, \ldots, \vec{v}_m)$ necessarily a subspace of \mathbb{R}^n ? Justify your answer.
- **5.** Give a geometrical description of all subspaces of \mathbb{R}^3 . Justify your answer.
- **6.** Consider two subspaces V and W of \mathbb{R}^n .
 - **a.** Is the intersection $V \cap W$ necessarily a subspace of \mathbb{R}^n ?
 - **b.** Is the union $V \cup W$ necessarily a subspace of \mathbb{R}^n ?
- 7. Consider a nonempty subset W of \mathbb{R}^n that is closed under addition and under scalar multiplication. Is W necessarily a subspace of \mathbb{R}^n ? Explain.
- **8.** Find a nontrivial relation among the following vectors:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

In Exercises 10 through 20, use paper and pencil to identify the redundant vectors. Thus determine whether the given vectors are linearly independent.

$$\begin{bmatrix} \mathbf{10.} & \begin{bmatrix} 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

11.
$$\begin{bmatrix} 7 \\ 11 \end{bmatrix}$$
, $\begin{bmatrix} 11 \\ 7 \end{bmatrix}$

12.
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
, $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$

13.
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

14.
$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$

15.
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
, $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

In Exercises 21 through 26, find a redundant column vector of the given matrix A, and write it as a linear combination of preceding columns. Use this representation to write a nontrivial relation among the columns, and thus find a nonzero vector in the kernel of A. (This procedure is illustrated in Example 8.)

21.
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
 22. $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ 23. $\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$
24. $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ 25. $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

Find a basis of the image of the matrices in Exercises 27 through 33.

27.
$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$
 28.
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
 29.
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

30.
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix}$$
 31.
$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 7 \\ 5 & 8 \end{bmatrix}$$