Math 61 Lecture 1: Sets

- Roughly speaking, a set is a collection of objects
  - ↳ $A = \{1, 2, 3, 2020\}$]- elements (or members)
  - ↳ Order doesn't matter
  - ↳ Duplicates are not listed (assume members are unique)
- Can also describe a set by listing a property
  - ↳ $B = \{x \mid x$ is positive, bigger than $2020\}$
    - ↳ $B = \{2021, 2022, 2023 \ldots\}$
- Other examples:
  - ↳ $\mathbb{N} = \{1, 2, 3, \ldots\}$ natural #s, $\mathbb{Q} = \{\frac{p}{q} \mid p$ is integer, q is natural$\}$, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ integers
  - $\mathbb{N}_k = \{1, 2, 3, \ldots, k\}$
    - ↳ elements do <u>not</u> have to be related to each other
      - ↳ $W = \{a, 1,$ elmer, $\{c, d\}, \pi,$ Homero, $\Delta\}$
- Cardinality - # of elements in a finite set
  - ↳ If X has a finite # of elements, $|X|$ denotes its cardinality
    - ↳ $|X| =$ # of elements in X
  - ↳ Ex) $|A| = 4$, $|B| =$ Not finite
  - ↳ Cardinality is related to a concept that is perfectly illustrated with bidirectional arrows
    - ↳ $A = \{1, 3, 2020, 2\}$

      $\mathbb{N}_4 = \{1, 2, 3, 4\}$
      - ↳ 1-to-1 correspondence between 2 finite sets
- Given a set A.
  - ↳ $x \in A$ if x is an element of A
  - ↳ $x \notin A$ if x is not an element of A
- The empty set is the set w/ no elements, denoted ϕ, or $\{\}$
- The sets X and Y are equal iff X and Y have the same elements
  - ↳ $X = Y \iff \forall_x \in X \Rightarrow x \in Y$ and $\forall_y \in Y \Rightarrow y \in X$
- Ex) Prove $X = \{x \mid x^2 + x - 6 = 0\}$ and $Y = \{2, -3\}$ are equal

  $x^2 + x - 6 = 0 \rightarrow x = 2, x = -3$

  Case 1: If $x = 2$, then $x \in Y = \{2, -3\}$

  Case 2: If $x = -3$, then $x \in Y = \{2, -3\}$

- Subsets
  - ↳ X is a subset of Y ($X \subseteq Y$) iff $\forall_x \in X \Rightarrow x \in Y$
    - ↳ $X = Y \Leftrightarrow X \subseteq Y$ and $Y \subseteq X$
    - ↳ X is a proper subset of Y iff $X \subseteq Y$ and $X \neq Y$
      - ↳ $X \subset Y \Leftrightarrow \forall_x \in X \Rightarrow x \in Y$ and $\exists_y \in Y$ such that $y \in X$
      - ↳ Equivalently: $X \subset Y$ iff $X \subseteq Y$, but Y is not a subset of X
  - ↳ The empty set is a subset of any set
    - ↳ If X is a set, then $\emptyset \subseteq X$
- Power Sets
  - ↳ Given a set X, its power set, denoted $\mathcal{P}(x)$ is the set of all subsets of X
  - ↳ Ex) If $X = \{1, 2, 3\}$
    - ↳ 1 subset with 0 elements: $\emptyset$
    - ↳ 3 subsets with 1 element: $\{1\}, \{2\}, \{3\}$
    - ↳ 3 subsets with 2 elements: $\{1,2\}, \{2,3\}, \{1,3\}$
    - ↳ 1 subset with 3 elements: $\{1,2,3\}$
    - ↳ $|P(x)| = 8 = 2^{|x|}$

Math 61 Lecture 2: Functions

- DeMorgan's Laws -> $\cup = OR$, $\cap = AND$
  - ↳ $x \in \overline{X \cup Y} \rightarrow x \notin X \cup Y \rightarrow x \notin X$ AND $x \in Y$
- Defs:
  - ↳ $\overline{X}$ -> Complement
  - ↳ $X \cup Y$ -> Union
  - ↳ $X \cap Y$ -> Intersection
- Cartesian Product
  - ↳ Ordered pair -> $(a,b) \neq (b,a)$
  - ↳ Given sets A, B, define: $A \times B := \{(a,b) \mid a \in A, b \in B\}$
    - ↳ cartesian product
    - ↳ Ex) $A = \{1,2,3\}$, $B = \{a,b\}$

      $A \times B = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$
      - ↳ $(b,2)$ is <u>not</u> part of the cartesian product $(b \notin A)$
- A function f assigns to each member of a set X exactly one member of a set Y
  - ↳ Can be represented as a set of ordered pairs -> 1st entries from X, 2nd from Y
  - ↳ Let X and Y be sets. A function f from X to Y is a subset of the cartesian product $X \times Y$ having the property that:
    - ↳ $\forall_x \in X, \exists! y \in Y$ with $(x,y) \in f$
    - ↳ X - Domain, Y - Codomain, R - Range
  - ↳ Ex) $f = \{(1,a),(2,a),(3,b)\}$
    - a) $X = \{1,2,3\}$, $Y = \{a,b\}$ -> Yes
    - b) $X = \{1,2,3\}$, $Y = \{a,b,c\}$ -> Yes
    - c) $X = \{1,2,3,4\}$, $Y = \{a,b\}$ -> No
  - ↳ represent using $y = f(x)$ notation
  - ↳ a function can also be defined by a rule
    - ↳ rule, domain, and codomain must be specified
- One to One Functions
  - ↳ a function $f: X \rightarrow Y$ is said to be one-to-one (injection) if $\forall_{x_1,x_2} \in X$,

    $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
    - ↳ $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
  - ↳ Ex) Show $f(n) = 2n-1$ is 1-to-1
    - ↳ $2n-1 = f(n) = f(m) = 2m-1$

      $2n = 2m$

      $n = m$

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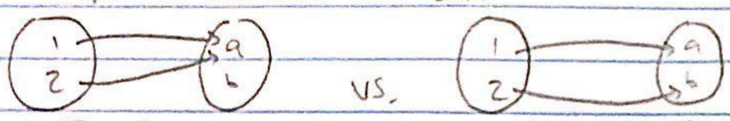
Math 61 Lecture 3: Functions

- Onto functions: A function $f: X \to Y$ such that $\forall y \in Y, \exists x \in X$ with $f(x) = y$
  - ↳ $R = \text{Range} = \{f(x) \in Y \mid x \in X\}$
  - ↳ f is onto if $Y = R$ → codomain = range
    - ↳ Y has no elements left out
  - ↳

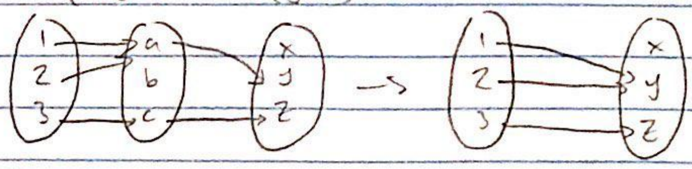


  - ↳ Ex) $\mathbb{E} = \{2, 4, 6, \ldots\}$ and $\mathbb{N} = \{1, 2, 3, \ldots\}$, show $f: \mathbb{E} \to \mathbb{N}$, $f(n) = \frac{n}{2}$ is 1-to-1 and onto

    Let n = any natural #

    $(2n) \to f(2n) = \frac{(2n)}{2} = n$

- Bijections (1-to-1 correspondence)
  - ↳ Definitions: Let $f: X \to Y$ be a function. f is said to be a bijection if it is both 1-to-1 and onto
  - ↳ invertible
- Inverse Functions: If $f: X \to Y$ is a bijection, then $\{(y, x) \mid (x, y) \in f\}$
  - ↳ Is a function → $\forall y \in Y, \exists! x \in X$ with $(y, x) \in f^{-1}$
    - ↳ order of entries reversed, prove uniqueness
  - ↳ Is onto
  - ↳ Is 1-to-1
- Composition of Functions
  - ↳ Let $g: X \to Y$ and $f: Y \to Z$, the composition is the new function $f \circ g: X \to Z$
    - ↳ $(f \circ g)(x) = f(g(x))$
  - ↳



- Misc.
  - ↳ Modulo: $x \bmod y$ = remainder when x is divided by y
  - ↳ Floor and ceiling:
    - ↳ Floor - $\lfloor x \rfloor$ - greatest integer $\leq x$
    - ↳ Ceiling $\lceil x \rceil$ - smallest integer $\geq x$
    - ↳ $\lfloor \pi \rfloor = 3$, $\lceil \pi \rceil = 4$, $\lfloor -\pi \rfloor = -4$, $\lceil -\pi \rceil = -3$
  - ↳ Binary operator is a function $f: X \circ X \to X$
  - ↳ Unary operator is a function $f: X \to X$
    - ↳ Ex) negative

Math 61 Lecture 4: Sequences and Strings

- Bird scooters charge \$1.00 to unlock and \$0.15 per min.

↳ \$1.15 \$1.30 \$1.45 \$1.60

1 2 3 4

↳ Ex) $D = \mathbb{N}$, $S_n = n$ gives $\{1, 2, 3, 4, \dots\}$

↳ Ex) $D = \mathbb{N} \cup \{0\}$, $S_n = q^n$, $q \in \mathbb{R}$ gives $\{1, q, q^2, \dots\}$

↳ Ex) $D = \{1, 4, 10\}$, $S_n = 2^n$ gives $\{2, 16, 1024\}$

↳ Ex) Define $S_n = 2^n + 4 \cdot 3^n$, $n \geq 0$

↳ $S_0 = 2^0 + 4 \cdot 3^0 = 1 + 4 = \boxed{5}$

↳ $S_1 = 2^1 + 4 \cdot 3^1 = 2 + 4 \cdot 3 = \boxed{14}$

↳ $S_k = \boxed{2^k + 4 \cdot 3^k}$

↳ $S_{n-1} = 2^{n-1} + 4 \cdot 3^{n-1} = \boxed{\frac{1}{2} 2^n + \frac{4}{3} 3^n}$

↳ $S_{n-2} = 2^{n-2} + 4 \cdot 3^{n-2} = \boxed{\frac{1}{4} 2^n + \frac{4}{9} \cdot 3^n}$

↳ Ex) $S_n = 2^n + 4 \cdot 3^n$, $n \geq 0$ → Show $S_n = 5 \cdot S_{n-1} - 6 S_{n-2}$

$$S_n = 5(2^{n-1} + 4 \cdot 3^{n-1}) - 6(2^{n-2} + 4 \cdot 3^{n-2})$$
$$= 5\left(\frac{1}{2} 2^n + \frac{4}{3} \cdot 3^n\right) - 6\left(\frac{1}{4} 2^n + \frac{4}{9} \cdot 3^n\right)$$
$$= \frac{5}{2} 2^n + \frac{20}{3} \cdot 3^n - \frac{3}{2} 2^n - \frac{8}{3} \cdot 3^n$$
$$= \boxed{2^n + 4 \cdot 3^n \checkmark}$$

- Definitions

↳ Increasing: if $i < j \Rightarrow S_i < S_j$

↳ Decreasing: if $i < j \Rightarrow S_i > S_j$

↳ Non-Decreasing: if $i < j \Rightarrow S_i \leq S_j$

↳ Non-Increasing: if $i < j \Rightarrow S_i \geq S_j$

↳ Ex) $S_n = \{1, 1, 1, 1, 1 \dots\}$ → Non-Dec., Non-Inc

↳ Ex) $S_n = \{1, 2, 2, 2, 3, 3, 3, 4, 4\}$ → Non-Dec.

- Notation - $\{u_1, u_2, u_3, \dots, u_k, \dots\} = \{u_k\}_{k=1}^{\infty}$
- If s is a sequence, a subsequence of s is obtained from s by choosing some elements of s in the same order in which they appear

↳ $\{S_{k_n}\} \subseteq \{S_n\}$ → where k_n is increasing

↳ Consider $\{1, 2, 3, 4, \dots, k, \dots\} = \{k\}_{k=1}^{\infty}$

↳ $\sigma_1 = 1$ $\qquad \pi_1 = 1$

$\sigma_2 = 1 + 2$ $\qquad \pi_2 = 1 \cdot 2$

$\sigma_3 = 1 + 2 + 3$ $\qquad \pi_3 = 1 \cdot 2 \cdot 3$

$\sigma_n = 1 + 2 + 3 + \dots + n$ $\qquad \pi_n = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$

$\sigma_n = \sum_{k=1}^{n} k$ $\qquad \pi_n = \prod_{k=1}^{n} k = n!$

Math 61 Lecture 5: Relations

· Strings - a finite sequence of characters

↳ A string over X (is a finite set) is a finite sequence of elements from $X \to \beta: \{1, 2, \dots, k\} \subseteq \mathbb{N} \to X$, $|X| < \infty$

↳ order matters: $baac \neq acab$

↳ power notation: $bbaaac = b^2a^3c$

↳ the null string (λ) contains no characters

↳ X^* = the set of all strings over X

↳ $X^+ := X^* - \{\lambda\}$ is the set of non-null strings over X

↳ The length of $\alpha \to |\alpha|$ is the # of elements in α

↳ α and β are strings $\to \alpha\beta$ is the concatenation of α and β

↳ a substring of α is some or all consecutive characters of α

↳ β is a substring of α if $\exists\, \gamma, \delta$ with $\alpha = \gamma\alpha\delta$

↳ the reverse string of α is $\alpha^R \to$ reversed characters of α

· Relations

↳ Recall: A function $f: X \to Y$ is a subset of the cartesian product with the 1-arrow per element property

↳ Definition: A binary relation R from X to Y is a subset of $X \times Y$

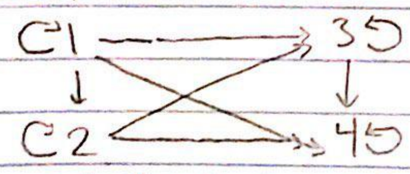
↳ $(x,y) \in R$ or $xRy \to x$ is related to y

↳ Digraphs → when $X = Y$, R is called a binary relation on X

↳ can be represented as a digraph

↳ Ex) Let $X = \{1, 2, 3, 4\}$ and $(x,y) \in R$ if $x \leq y$

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$



↳ Reflexive: if $\forall_x \in X$, $(x,x) \in R$

↳ Symmetric: if $\forall_x \in X$, $\forall_y \in X$, $(x,y) \in R \Rightarrow (y,x) \in R$

↳ Anti-Symmetric: $\forall_{x,y} \in X$, $(x,y) \in R$ and $(y,x) \in R \Rightarrow (x,y)$

↳ Transitive: if $\forall_{x,y,z} \in X$, $(x,y) \in R$ and $(y,z) \in R \Rightarrow (x,z) \in R$

Math 61 Lecture 6: Equivalance Relations

•Relation Properties:

↳ Reflexive: $\forall x \in X, (x,x) \in R$

↳ Symmetric: if $(x,y)\in R$, then $(y,x)\in R$

↳ Anti-Symmetric: $\forall x,y \in X$, $(x,y)\in R$ and $(y,x)\in R \Rightarrow x=y$

↳ Transtive: if $\forall x,y,z \in X$, $(x,y)\in R$ and $(y,z)\in R \Rightarrow (x,z)\in R$

↳ Partial Order: reflexive, anti-symmetric, and transtive

↳ Ex) $X=\{\emptyset,\{1\},\{2\},\{1,2\}\}$, $A,B\in X$, $(A,B)\in R$ if $A\subseteq B$

$\emptyset\subseteq\{1\}\subseteq\{1,2\}\to(\emptyset,\{1\})\in R,(\{1\},\{1,2\})\in R,(\emptyset,\{1,2\})\in R$

$\emptyset\subseteq\{2\}\subseteq\{1,2\}\to(\emptyset,\{2\})\in R,(\{2\},\{1,2\})\in R,(\emptyset,\{1,2\})\in R$

↳ $x\preceq y$ if $(x,y)\in R$, $x\npreceq y$ if $(x,y)\notin R$

↳ $x\preceq y\to$ x and y are comparable, $x\npreceq y$ and $y\npreceq x\to$ x and y are incomparable

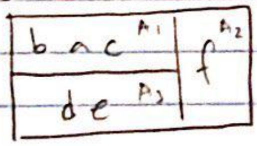
↳ Inverse Relation: $R^{-1}=\{(y,x)\mid(x,y)\in R\}\subseteq Y\times X$

↳ Composition: $R_2\circ R_1=\{(x,z)\mid(x,y)\in R_1$ and $(y,z)\in R_2$ for some $y\in Y\}$

↳ Ex) $X=\{1,2,3\}$, $R_1=\{(1,1),(1,2),(2,3)\}$, $R_2=\{(1,3),(2,3),(3,1)\}$

$R_2\circ R_1=\{(1,3),(2,1)\}$

•Equivalance Relations:

↳ 

$X=A_1\cup A_2\cup A_3$, $A_i\cap A_j=\emptyset$ if $i\neq j$

$R=\{(a,a),(a,b),(a,c),(b,b),(b,a),(b,c),(c,c),(c,b),(c,a),(d,d),(d,e),(e,e),(e,d),(f,f)\}$

↳ Define R: $(x,y)\in R$ if $x,y\in A_j\to$ x and y in same subset

↳ Reflexive ✓, symmetric ✓, transitive ✓

Math 61 Lecture 7: Equivalence Classes

- Partitions

↳ Let A_α be a collection of subsets of X satisfying $X = \cup_\alpha A_\alpha$ and $A_\alpha \cap A_\beta = \emptyset$ if $\alpha \neq \beta$, let $(x,y) \in R$ if $x, y \in A_\alpha$, then R is an equivalent relation

↳ R is reflexive: say $x \in X = \cup_\alpha A_\alpha$, $x \in A_\alpha \to x$ lives in same set as x

↳ R is symmetric: say $(x,y) \in R$, $x, y \in A_{\hat\alpha}$, $y, x \in A_{\hat\alpha} \to (y,x) \in R$

↳ R is transitive: say $(x,y) \in R$, $(y,z) \in R \to x \in A_\alpha$, $y \in A_\alpha$, $z \in A_\alpha \to$ $x, z \in A_\alpha \to (x,z) \in R$

- Equivalence Classes

↳ Recall $X = \{1, 2, 3 \ldots, 2020\}$, let $(x,y) \in R$ if 5 divides $x-y$ is an equivalence relation

↳ Let $[x] := \{y \in X \mid xRy\} \to [x]$ is the equivalence class of X

$[1] = \{1, 6, 11, 21, \ldots, 2011, 2016\}$

$[2] = \{2, 7, 12, 17, \ldots, 2012, 2017\}$

↳ 2 x's in the same eq. class have the same eq. class

↳ This collection of subsets is a partition of X

↳ every $x \in X$ belongs to its own equivalence class

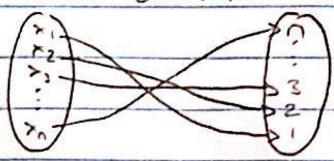
↳ $x \in X \to [x] \neq \emptyset$; by reflexivity $(x,x) \in R \Rightarrow x \in [x]$

↳ if $(x,y) \notin R$, then $[x] \cap [y] = \emptyset$

↳ say $[x] \cap [y] \neq$ then $z \in [x]$ and $z \in [y] \to (x,z) \in R$, $(y,z) \in R \to$ symmetry $(z,y) \in R \to (x,z)$ and $(z,y) \in R \to$ transitivity $(x,y) \in R$

- If X and Y are finite sets $\to |X \cup Y| = |X| + |Y|$
- Let X and Y be sets, we say $|X| = |Y|$ if $\exists f. X \to Y$, f-bijection

↳ Particular case: $|X| = n$



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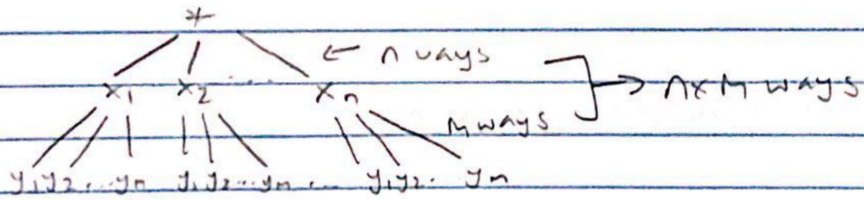
Math 61 Lecture 8: Counting Principles

- Multiplication Principle
  - ↳ If an activity can be done in t successive steps, where
    - step 1 can be done in n_1 ways
    - step 2 can be done in n_2 ways
    - step t can be done in n_t ways
    - ↳ the # of different possible activities is $n_1 \cdot n_2 \cdot \ldots \cdot n_t$
  - ↳ Cardinality of Cartesian Product
    - ↳ Given X and Y (finite sets), what is the cardinality of $X \times Y$
      - ↳ Activity: form an ordered pair (x, y)
      - ↳ Step 1: pick $x \to |X|$ ways
      - ↳ Step 2: pick $y \to |Y|$ ways
      - ↳ $|X \times Y| = |X| \times |Y|$
- Using Decision Trees
  - ↳

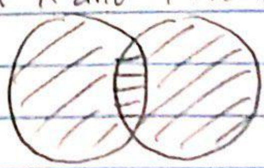


- Counting Strings
  - ↳ How many strings of length 4 can be formed using ABCDE
    - ↳ Step 1: 1st char → 5 ways
    - ↳ Step 2: 2nd char → 5 ways
    - ↳ Step 3: 3rd char → 5 ways
    - ↳ Step 4: 4th char → 5 ways
    - = 625 ways
  - ↳ How many strings of part A begin with the letter B
    - ↳ $\underline{1}\ \underline{5}\ \underline{5}\ \underline{5} \to 125$ ways
  - ↳ How many do not?
    - ↳ $\underline{4}\ \underline{5}\ \underline{5}\ \underline{5} \to 500$ ways
  - ↳ How many strings with no repetitions?
    - ↳ $\underline{5}\ \underline{4}\ \underline{3}\ \underline{2} \to 120$ ways
- Addition Principle
  - ↳ If $\{x_1, \ldots\}$ do not intersect, then the number in the union is $n_1 + n_2 + \ldots n_t$

Math 61 Lecture 9: Inclusion - Exclusion

- Ex) A 6 person committee composed of A, B, C, D, E, F must select a chairperson, secretary, and treasurer
  - ↳ How many ways can this be done
    - ↳ 6 5 4 1 1 1 → Mult. Principle
      C S T 4 5 6
      - ↳ $\boxed{120 \text{ ways}}$ → $6 \times 5 \times 4$
  - ↳ C or F must be chairperson
    - ↳ C 5 4
      F 5 4
      - ↳ $5 \times 4 + 5 \times 4 = \boxed{40 \text{ ways}}$
  - ↳ How many ways if E must hold an office?
    - ↳ E 5 4
      5 E 4
      5 4 E
      $3(5 \times 4) = \boxed{60 \text{ ways}}$
  - ↳ How many ways if A and B must hold offices
    - ↳ A B 4
      4 A B
      A 4 B
      B A 4
      4 B A
      B 4 A
      - ↳ $6 \times 4 = \boxed{24 \text{ ways}}$

- Inclusion- Exclusion
  - ↳ If X and Y are finite sets. $|X \cup Y| = |X| + |Y| - |X \cap Y|$



Math 61 Lecture 10: Permutations and Combinations

- Ex) 4 candidates are running for the same office, how many distinct ballots can there be?

  1st: $\boxed{4}$
  2nd: $\boxed{3}$
  3rd: $\boxed{2}$
  4th: $\boxed{1}$

  $\rightarrow 4 \times 3 \times 2 \times 1 = \boxed{24 \text{ ways}} = 4!$

- Permutations: a permutation of n distinct elements $x_1, \dots, x_n$ is an ordering of the n elements $x_1, \dots, x_n$
- Theorem: Given a set with n distinct elements, there are $n!$ permutations of the members of the set

↳ Ex) How many permutations of ABCDEF contain DEF?

DEF = 1 element → 4 elements

$4! = \boxed{24 \text{ ways}}$

↳ DEF together in any order?

$4! \times 3! = \boxed{144 \text{ ways}}$

↳ Ex) How many ways can 6 people wait in a line?

$6! = \boxed{720 \text{ ways}}$

↳ in a circle?

$720/6 = \boxed{120 \text{ ways}}$

- r-Permutations → an ordering of an r-element subset

↳ $P(n,r) = \frac{n!}{(n-r)!}$

↳ Ex) 7 distinct dwarves and 5 distinct Elves, no 2 Elves can stand together

_ D_1 _ D_2 _ D_3 _ D_4 _ D_5 _ D_6 _ D_7 _

$7!(P(8,5)) = \boxed{\frac{7!\,8!}{3!}}$

- r-Combination → an unordered selection of r-elements of set X

Math 61 Lecture 11: Generalizations

- $C(n,r) = \frac{n!}{r!(n-r)!}$

↳ How many 8 bit strings contain exactly 4 1s
  ↳ Order of 1s does not matter → $\boxed{\frac{8!}{4!4!}}$

↳ How many ways can we select a committee of 2 women and 3 men from 5 women and 6 men
  ↳ Step 1: Select 2/5 women → $\frac{5!}{3!2!}$
  ↳ Step 2: Select 3/6 men → $\frac{6!}{3!3!}$
    ↳ $\boxed{\frac{5!6!}{3!3!3!2!}}$

- How many strings can be formed with MISSISSIPPI
  ↳ $\ell = 12$ letters, 4 S's, 5 I's, 2 P's
    ↳ Step 1: $\binom{12}{4}$ for S's
    ↳ Step 2: $\binom{8}{5}$ for I's
    ↳ Step 3: $\binom{3}{2}$ for P's
    ↳ Step 4: $\boxed{\frac{12!}{4!5!2!1!}}$

- A sequence S of n items and n_1 identical objects of type 1, n_2 identical objects of type 2, etc. has orderings equal to $\frac{n!}{n_1!n_2!\ldots}$
- Ex) A library has 6+ copies of 3 different books, how many ways can we select 6 books?

↳ _ B _ B _ B _ B _ B _ B _ → ${}_8C_2 = \boxed{\frac{8!}{6!2!}}$

Math 61 Lecture 12: Pigeonhole Principle

- By counting subsets of a set with n elements:
  $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$

↳ Proof:

$X = \{x_1, x_2, \ldots, x_{n-1}, x_n\}$

If $A \subseteq X$: $A = \{x_2, \ldots, x_k, \ldots\} \rightarrow n$ spots

↳ # of subsets = $|P(x)| = 2 \cdot 2 \cdot 2 \cdot 2 \ldots \cdot 2 = 2^n$

↳ # of subsets w/ 0 elements: $1 \rightarrow \emptyset \rightarrow \binom{n}{0}$

# of subsets w/ 1 element: $n \rightarrow \{x_1\}, \{x_2\}, \ldots, \{x_n\} \rightarrow \binom{n}{1}$

# of subsets w/ k elements: $\binom{n}{k}$

# of subsets w/ n elements: $1 = \binom{n}{n}$

↳ Combinations → order irrelevant in sets

- Pigeonhole Principle

↳ Is there an item having a given property? → Doesn't explain how to find

↳ If n pigeons fly into k pigeonholes and $k < n$, some pigeonhole contains at least 2 pigeons

↳ Ex) 10 people have 1st names Alice, Angie, and Alma and last names Galvez, Hess, and Alvarez, show that 2 people have the same name

↳ n = # of people = 10

k = # of possible names = $3 \times 3 = 9$

$k < n$, by pigeonhole principle, $\exists$ at least 1 repeated name

↳ Assume all people have different names:

Alice Galvez	Angie Galvez	Alma Galvez
Alice Hess	Angie Hess	Alma Hess
Alice Alvarez	Angie Alvarez	Alma Alvarez

↳ Second form: If f is a function from $X \rightarrow Y$ and $|X| > |Y|$, then f cannot be one-to-one

Math 61 Lecture 13: Recurrance Relations

- An equation that relates a_n to its predecessors $a_0, a_1, \ldots, a_{n-1}$
  - Ex) Start w/ 5, add 3 to each term
    - $a_n = a_{n-1} + 3$
    - 5 → initial condition
      - need as many initial conditions as predecessor terms in the equation
  - Ex) Fibonacci Sequence → $f_n = f_{n-1} + f_{n-2}$, $n \geq 3$, $f_1 = 1$, $f_2 = 1$
    - $f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8, f_7 = 13, f_8 = 21 \ldots$
  - Ex) \$1000 invested at 12% interest annually, A_n = amount after n yrs
    - $A_0 = \$1000$

      $A_1 = \$1000(1+0.12) = (1.12)(1000)$

      $A_2 = (1.12)(\$1000)(1+0.12) = (1.12)^2(\$1000)$

      $A_n = A_0(1.12)^n = 1.12(A_{n-1})$
  - Ex) Let S_n denote the number of n-bit strings that do not contain the pattern 111
    - $\underset{1}{\_\_}\ \underset{2}{\_\_} \ldots \underset{n-1}{\_\_}\ \underset{n}{\_\_}$
      - Case 1: Least bit
        - SC1: $n = 0$
        - SC2: $n = 1$

          } assume string does not contain 111
          - If previous string ends w/ 11 → issue found
      - Case 2:
        - SC1: Ends in 01 → $S_n = \underbrace{S_{n-1} + S_{n+2}}_{\text{disjoint}} + \ldots$
        - SC2: End in 11
          - If previous string ends w/ 1 → issue found
      - Case 3: 3rd to last bit, ends in 11
        - SC1: 0 → $S_n = S_{n-1} + S_{n-2} + S_{n-3}$

Math 61 Lecture 14: Linear Homogeneous Recurrence Relations

- Cardinality of power sets:
  - ↳ Find a recurrence relation for the cardinality of the power set of a non-empty set with n elements
    - ↳ $X_n = \{x_1, x_2, x_3, \ldots, x_n\}$, let $a_n = |P(x_n)|$
    - ↳ Given a subset $S \subseteq \underbrace{\{x_1, x_2, \ldots, x_{n-1}\}}_{x_{n-1}}$
      - ↳ $a_n = 2a_{n-1}$

        $a_0 = |P(\emptyset)| = 1$

        $a_1 = |P(\{x_1\})| = |\{\emptyset, \{x_1\}\}| = 2$
        - ↳ $a_{n-1} = 2a_{n-2}$

          $a_n = 2(a_{n-1}) = 2(2a_{n-2})$

          $a_n = 2^2 a_{n-2}$
          - ↳ $a_n = 2^k a_{n-k}$
- Linear homogeneous relation
  - ↳ $a_n = C_1 a_{n-1} + C_2 a_{n-2} + \ldots + C_k a_{n-k}$
    - ↳ $a_0 = C_0, a_1 = C_1, \ldots, a_{k-1} = C_{k-1}$
    - ↳ $a_n = r^n$
  - ↳ $a_n = 3a_{n-1}a_{n-2}$ -> not linear (no products)
  - ↳ $S_n = 2S_{n-1}$ -> linear homogenous -> order 1
  - ↳ $f_n = f_{n-1} + f_{n-2}$ -> linear homogeneous
  - ↳ $a_n = a_{n-1} + 2n$ -> linear, constant coefficient, not homogeneous
  - ↳ Cardinality of power sets
    - ↳ Try $a_n = \alpha r^n$ -> Find α and r

      $a_n = 2a_{n-1}, a_0 = 1$

      $a_n = 2a_{n-1}$

      $\alpha r^n = 2(\alpha r^{n-1})$

      $r^n = 2r^{n-1}$

      $r^n - 2r^{n-1} = 0$

      $r^{n-1}(r-2) = 0$
      - ↳ $r = 0$ or $r = 2$

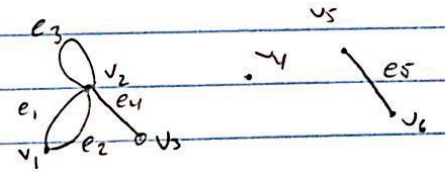
        If $r=0$: $a_0 = 1 = \alpha(0)^0$ -> indeterminate

        If $r=2$: $a_0 = 1 = \alpha(2)^0$ -> $\alpha = 1$

        $a_n = 2^n$

Math 61 Lecture 15: Graphs

- The Traveling Salesperson Problem
- A graph G is a set of V verticies and a set of E edges such that each edge $e \in E$ is associated with an unordered pair of verticies
  - ↳ if only one $e \in E$ is associated with $v, w \in V$, we write $e = (v,w)$ or $e = (w,v)$
- A digraph G consists of a set V of verticies and a set E of edges such that each edge $e \in E$ is associated with an ordered pair of verticies
  - ↳ $e = (v,w) \neq e = (w,v)$
- An edge e is associated w/ v and w
  - ↳ e is incident on v and w
  - ↳ v and w are incident on e
  - ↳ v and w are adjacent verticies
- If G is a graph w/ V verticies and E edges -> $G = (V,E)$
- Ex)

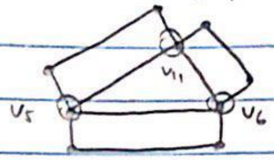


↳ $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$ -> $G = (V,E)$ is one graph

- More definitions:
  - ↳ Distinct edges associated w/ the same pair of verticies are parallel edges -> e_1 and e_2
  - ↳ An edge incident on a single vertex is called a loop -> e_3
  - ↳ A vertex that is not incident on any edge is called an isolated vertex -> v_4
  - ↳ A graph w/ no loops or parallel edges is a simple graph
  - ↳ A graph w/ numbers on the edges is a weighted graph
    - ↳ The length of a path is the sum of weights along that path
    - ↳ The path of minimum length is the optimal path
  - ↳ A complete graph is a simple graph with an edge between every pair of distinct verticies
  - ↳ A bipartite graph is a graph that has subsets V_1 and V_2 of V such that $V_1 \cap V_2 = \phi$, $V_1 \cup V_2 = V$ and each edge in E is incident on one vertex in V_1 and one in V_2

Math 61 Lecture 16: Paths

- Ex) Prove the graph is not bipartite



↳ Say v_4 is on 1 side and v_5 is on the other

↳ v_6 cannot be in either team if the graph was bipartite due to (v_4, v_6) and (v_5, v_6)

- Complete bipartite graph is both complete and bipartite (edge set contains all (v_1, v_2))
- Ex) For what values of n is K_n bipartite

↳ $K_1 \to \bullet v_1 \to V_1 = \{v_1\}, V_2 = \emptyset \to$ Bipartite

↳ $K_2 \to v_1 \bullet\!\!-\!\!\bullet v_2 \to V_1 = \{v_1\}, V_2 = \{v_2\} \to$ Bipartite

↳ $K_3+ \to$ Not bipartite

- A path from v_0 to v_n of length n is an alternating sequence of n edges and $n+1$ vertices beginning w/ v_0 and ending w/ v_n

↳ $(v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n)$

- Connected graph - you can reach any vertex from a given vertex
- Let $G = (V, E)$ be a graph → (V', E') is a subgraph of G if $V' \subseteq V$, $E' \subseteq E$, and for every edge $e' \in E'$, if e' is incident on v' and w', then $v', w' \in V'$

↳ Ex) Find all subgraphs of K_2 w/ at least 1 vertex

↳ G_1: $V' = \{v_1\}$, $E' = \emptyset$

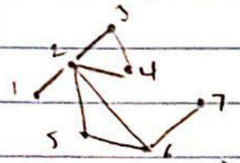
G_2: $V' = \{v_2\}$, $E' = \emptyset$

G_3: $V' = \{v_1, v_2\}$, $E' = \emptyset$

G_4: $V' = \{v_1, v_2\}$, $E' = \{(v_1, v_2)\}$

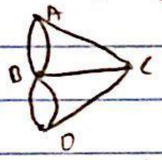
Math 61 Lecture 17: Cycles

- Simple path - a path from v to w with no repeated vertices
- Cycle - a path of nonzero length from v to v with no repeated edges
- Simple Cycle - a cycle where there are no repeated vertices other than the beginning and ending
- (Ex)

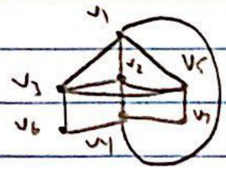


	Simple Path	Cycle	Simple Cycle
(6,5,2,4,3,2,1)	No	No	No
(6,5,2,4)	Yes	No	No
(2,6,5,2,4,3,2)	No	Yes	No
(5,6,2,5)	No	Yes	Yes
(7)	Yes	No	No

- The Königsberg Bridge Problem

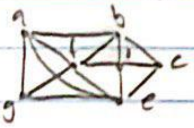


↳ impossible because C has an odd # of edges
- Euler Cycles - a cycle in a graph G that includes all edges and vertices
- The Degree of a Vertex - $\delta(v)$ is the # of edges incident on v - each loop is 2
- If a graph G has an Euler cycle, then G is connected and every vertex has an even degree
-



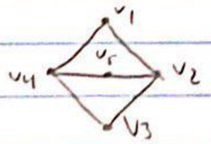
Math 61 Lecture 18: Hamilton Cycles

- A graph G has a path with no repeated edges from v to w $(v \neq w)$ containing all the edges and verticles iff
  - ↳ G is connected
  - ↳ v and w are the only vertices having odd degree
  - ↳ Proof: Take G is connected, v/w have odd degree
    - ↳ We are 1 edge away from an Euler cycle -> add it and find the E.C.
    - ↳ Travel the cycle starting anywhere, then remove it
- Hamilton's puzzle:
  - ↳ start at any city, travel along the edges, visit each city exactly one time, and return to the initial city
  - ↳ a Hamiltonian cycle is a cycle that visits each vertex exactly once
  - ↳ Ex)



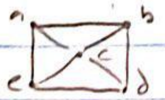
$\boxed{\{a, b, c, d, e, f, g, a\}}$

    - ↳ There are exactly n edges travelled, where $n = |V|$
  - ↳ Ex)



→ The cycle visits v_1

    - ↳ The cycle also visits v_3
    - ↳ v_4, v_2 have been visited, v_5 cannot be
  - ↳ Ex)

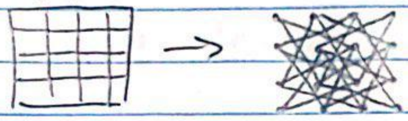


    - ↳ a, b, e, d contribute 2 edges to HC → remove 4 edges → remain 4 < 5
      - ↳ incorrect EEAC method due to repetitions

Math 61 Lecture 19: Applications of Hamiltonian Cycles

- The Traveling Salesperson → find the shortest Hamiltonian Cycle
- The Knight's Tour - visit each square exactly once, return to original spot

↳ GK_4



↳ bipartite, connected

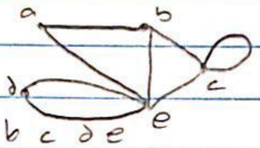
↳ Necessary: If GK_n has a Hamiltonian cycle, n is even

↳ Not Sufficient: GK_2 and GK_4 do not have Hamiltonian cycles

Math 61 Lecture 20: Representation of Graphs

- Adjacency Matrix - the entry in row i, column j is 2x the # of loops incident on i (i=j)

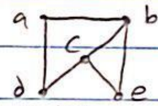
↳ Ex)



$$\begin{array}{c|c} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

→ symmetric

↳ Ex)



$$\begin{array}{c|c} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array} = A$$

↳ $$A^2 = \begin{bmatrix} 2 & 0 & 2 & 0 & 0 \\ 0 & 3 & 1 & 2 & 1 \\ 2 & 1 & 3 & 0 & 0 \\ 0 & 2 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

↳ entries indicate # of paths from i→j of length n

- Incidence Matrix → the entry for row v and column e is 1 if e is incident on v and 0 otherwise

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