1/6 Lecture : Introduction to Differential Equations

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- Textbook: Differential Equations 2nd Edition - Polking Boggess and Arnold

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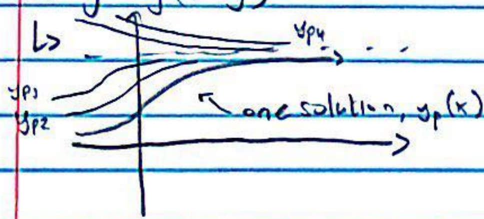
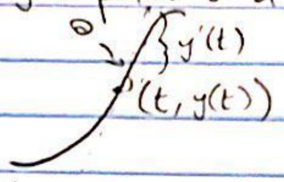
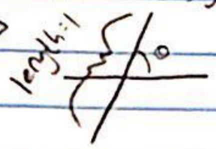
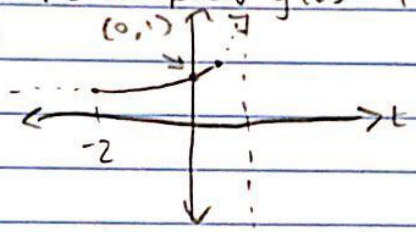
- Normal Form: form with the highest order on the left side
  - ↳ $f^{(n)} = \sigma(t, f, f', \dots, f^{(n-1)})$
  - ↳ $f \cdot f' - 1 = 0 \rightarrow f' = \frac{1}{f}$
  - ↳ $(f')^2 + f = 0 \rightarrow f' = \sqrt{-f}$ or $f' = -\sqrt{-f}$
- Solving Diff. Eq.s
  - ↳ $y' = y$
    - ↳ $\underbrace{y = e^x, \; y = 0}_{\text{particular solutions}}, \; \underbrace{y = ce^x}_{\text{general solution}} \rightarrow$ family of solutions
  - ↳ $y'' + 2y' + 3y = 0$
    - ↳ $y = c_1 f_1 + c_2 f_2$
- Initial Value Problem
  - ↳ diff. eq. + initial condition → usually gives a unique solution
  - ↳ Ex) $y' = y$, $y(0) = -1$
    - ↳ $y = -e^x$
- Picture of Solutions
  - ↳ $y' = y(1-y)$
  - ↳



1/8 Lecture: Seperable Variables

- Taking a point of a solution
  - ↳ 

  - ↳ slope of tangent line: $y'(t)$, $\theta = \tan^{-1}(y'(t))$
    - ↳ $y' = \varphi(t,y) = y(1-y)$
  - ↳ For each (t,y) we can construct a line segment
    - ↳ 

      $\tan\theta = \varphi(t,y)$
    - ↳ these are tangent to the solutions
- Ex) $y' = y^2$, $y(0) = 1$
  - ↳ Assume we know $y(t) = \frac{1}{1-t}$, $(-2, \frac{1}{2})$ is the solution
    - ↳ 

    - ↳ extend the interval from $(-\infty, 1)$
      - ↳ largest possible interval for a continuous solution - interval of existence
        - ↳ contains an initial condition
- Seperable Variables
  - ↳ $y' = g(t)f(y) = \frac{dy}{dt}$

    $\int \frac{dy}{f(y)} = \int g(t)\,dt$

    $F(y) = G(t) + C$
  - ↳ Ex) $y' = ty^2$

    $\int \frac{dy}{y^2} = \int t\,dt$ → loses solution $y = 0$

    $-\frac{1}{y} = \frac{1}{2}t^2 + C$ → implicit form

    $\boxed{y = -\frac{2}{t^2+C}}$ → explicit form
    - ↳ $\lim_{t\to\infty} \frac{-2}{t^2+C} = 0$
  - ↳ Ex) $y' = y(1-y)$

    $\frac{dy}{y(1-y)} = dt = \int \frac{1}{(1-y)}dy + \int \frac{1}{y}dy = \ln|y| - \ln|1-y| =$

    $t + C = \ln\left|\frac{y}{1-y}\right| \rightarrow e^{t+C} = \left|\frac{y}{1-y}\right| \rightarrow ke^t = \left|\frac{y}{1-y}\right| \rightarrow \boxed{y = \frac{Ce^t}{1+Ce^t}}$

    ↳ $k > 0$

1/10 Lecture: 1st Order Linear Diff Eq

· Definition: $y' = a(t) \cdot y + f(t)$ ⟵ forcing term

↳ C': contains all differentiable functions

↳ $\{f \mid f \cdot f' \text{ contains}\}$ $f, g \in C'$, then $c_1 f + c_2 g \in C'$

↳ $y' - a(t) \cdot y = f$

$\underbrace{\left(\frac{\partial}{\partial t} - a(t)\right)}_{\text{linear transformation from } C' \to C'} y = f \to A\vec{y} = \vec{f}$

↳ $A\vec{y} = \vec{0}$

↳ Solution: the kernel / null space → still a linear space

↳ $A\vec{y} = \vec{b}$

↳ ① Find one solution

$A\vec{y}_p = \vec{b}$

↳ ② $A\vec{y} = \vec{b}$

$A(\vec{y} - \vec{y}_p) = \vec{b} - \vec{b} = \vec{0} \to \vec{y} - \vec{y}_p \in Ker(A)$

$\vec{y} = \vec{y}_p + Ker(A)$

↳ $Ker(A) = \left(\frac{\partial}{\partial t} - a(t)\right)\vec{y} = \vec{0}$

$= y' - a(t)\vec{y} = \vec{0}$

$y' = a(t)\vec{y}$

$\int \frac{dy}{y} = \int a(t)\,dt$

$\ln|y| = A(t) + C$

$|y| = e^{A(t)} e^{C}$

$|y| = ke^{A(t)}, k > 0$

$y = ke^{A(t)}$ or $y = -ke^{A(t)}$

$y = Ce^{A(t)}, C \in \mathbb{R} \to$ 1-D linear space

↳ $y_h = e^{A(t)}$ → homogenous part to the diff eq. $y' = a(t)y + f$ (soln to $y' = a(t)y$

↳ $y' = a(t)y + f$

↳ Solution set is $Cy_h + y_p$, where y_p is a particular solution to $y' = a(t)y + f$

· How to find a particular solution $\vec{y}_p$

↳ ① Integrating factors

② Variation of parameter

↳ ① $\underbrace{y' - a(t)y}_{=(u(t)y)'} = f$

$= u'y + uy' = y' - a(t) \cdot y$

$((u' + a(t))y + (u+1)y' = 0$ X

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$\to u(t)(y' - a(t)y) = uf$

$uy' - ua(t)y = uy' + u'y$

$-ua(t)y = u'y$

$(ua(t) + u')y = 0$

always $ua(t) + u' = 0$

$\to u = e^{\int -a(t)\,dt}$

$\to uy = \int uf\,dt$

$y = \frac{1}{u}\int uf\,dt$

$= \frac{1}{u}(G(t) + C)$

$= \frac{G(t)}{u} + \frac{C}{u} \leftarrow C \cdot y_h$

1/13 Lecture: Solving for y_p

- Summary
  - $y' = a(x)y + f \rightarrow$ 1st order linear diff. eq.
    - solution of $Cy_h + y_p$
      - solve for y_p by:
        - integrating factors: $\rightarrow e^{\int -a(t)dt}$
  - Ex) $y' = y + e^{-t}$

$$IF = e^{\int -dt} = e^{-t}$$
$$y' - y = e^{-t}$$
$$e^{-t}y' - e^{-t}y = e^{-2t}$$
$$\int (e^{-t}y)' = \int e^{-2t}$$
$$e^{-t}y = -\tfrac{1}{2}e^{-2t} + C$$
$$\boxed{y = -\tfrac{1}{2}e^{-t} + Ce^{t}}$$

  - Ex) $(1+t^2)y' + 4ty = \frac{1}{1+t^2}, \; y(0) = 2$

$$(1+t^2)y' = \frac{1}{1+t^2} - 4ty$$
$$y' = \frac{1}{(1+t^2)^2} - \frac{4ty}{1+t^2}$$
$$y' = -\frac{4t}{1+t^2}y + \frac{1}{(1+t^2)^2}$$
$$a(t) = -\frac{4t}{1+t^2}$$
$$IF = e^{\int -a(t)dt}$$
$$= e^{\int 4t/(1+t^2)\,dt}$$
$$u = t^2, \; du = 2t\,dt$$
$$= e^{\int \frac{2du}{1+u}}$$
$$= e^{\ln(1+t^2)^2}$$
$$= (1+t^2)^2$$
$$(1+t^2)^2\left(y' + \frac{4t}{1+t^2}y\right) = (1+t^2)^2\left(\frac{1}{(1+t^2)^2}\right)$$
$$((1+t^2)^2 y)' = 1$$
$$(1+t^2)^2 y = t + C$$
$$y = \frac{t+C}{(1+t^2)^2}$$
$$C = 2$$
$$\boxed{y = \frac{t+2}{(1+t^2)^2}}$$

- Variation of Parameter $\rightarrow$ Duhamel's Principle
  - $y = \left(C + \frac{y_p}{y_h}\right)y_h$

$$= u(x) \cdot y_h(x)$$
$$(uy_h)' = a(x)(uy_h) + f$$

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$u'y_h + uy_h' = auy_h + f$

$u'y_h + u(y_h' - ay_h) = f \longleftarrow y_h' = ay_h$

$u'y_h = f$

$u' = \frac{f}{y_h}$

$u = \int \frac{f}{y_h} + C$

↳ must solve for y_h first: $y_h = e^{\int a(x)dx}$

↳ Ex) $(1+t^2)y' + 4ty = \frac{1}{1+t^2}$

$y' + \frac{4t}{(1+t^2)}y = \frac{1}{(1+t^2)^2} \leftarrow f(t)$

$y' + \frac{4t}{(1+t^2)}y = 0$ → get homogeneous part

$y' = -\frac{4t}{(1+t^2)}y$

$d\ln y = \frac{-4t}{(1+t^2)}dt$

$\ln y = -2\ln(1+t^2)$

$y_h = \frac{1}{(1+t^2)^2}$

$y = Cy_h$

$y' = ay + f$

$y = u(x)y_h$

$(uy_h)' = auy_h + f$

$u' = \frac{f}{y_h}$

$u' = \frac{\frac{1}{(1+t^2)^2}}{\frac{1}{(1+t^2)^2}}$

$u' = 1$

$u = t + C$

1/15 Lecture: Exact Equations

· Review.

↳ $x(t), y(t), w(x,y)$

$$\mathbb{R} \longrightarrow \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$$

$$t \longrightarrow (x(t),y(t)) \to w(x(t),y(t))$$

$$\frac{\partial}{\partial t} w(x(t),y(t)) = \frac{\partial w}{\partial t}$$

$$\frac{\partial w}{\partial x}\underbrace{\frac{\partial x}{\partial t}}_{x'} + \frac{\partial w}{\partial y}\underbrace{\frac{\partial y}{\partial t}}_{y'}$$

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy$$

· $y' = \varphi(x,y)$

$$\frac{dy}{dx} = \varphi(x,y)$$

$$dy - \varphi(x,y)\,dx = 0$$

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

· Ex) · $(x^2+y^2)\,dx + 2xy\,dy = 0$, let $w(x,y) = \frac{1}{3}x^3 + xy^2$

↳ The total derivative of $w(x,y)$ is the equation

$$dw = (x^2+y^2)\,dx + 2xy\,dy = 0$$

$$w = C$$

$$w(x,y) = w(x_0,y_0)$$

$$w(x_1,y_0) - w(x_0,y_0) = \frac{\partial w}{\partial x}\Big|(x-x_0)$$

· $P(x,y)\,dx + Q(x,y)\,dy$ is exact iff there exists $w(x,y)$ such that $\frac{\partial w}{\partial x} = P(x,y)$ and $\frac{\partial w}{\partial y} = Q(x,y)$

· Theorem: If $P, Q, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ are continuous and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then the differential equation is exact

↳ Ex) $(x^2+y^2)\,dx + 2xy\,dy = 0$

$$\frac{\partial P}{\partial y} = 2y, \quad \frac{\partial Q}{\partial x} = 2y$$

· How to find w when diff. eq. is exact

↳ $P(x,y) = \frac{\partial w}{\partial x}$

$$\int \frac{\partial w}{\partial x}dx = \int P(x,y)\,dx$$

↳ Let H be the antiderivative of $P(x,y)$

↳ Let $D = w(x,y) - H(x,y)$

$$\frac{\partial D}{\partial x} = 0 = \frac{\partial w}{\partial x} - \frac{\partial H}{\partial x} = P(x,y) - P(x,y)$$

$$D(x_1,y_0) - D(x_2,y_0) = \frac{\partial D}{\partial x}\Big|(x_1-x_2) = 0$$

↳ $D(x,y) = \sigma(y) = w - H$

$$w = \sigma(y) + H(x,y)$$

$$\frac{\partial w}{\partial y} = Q(x,y) = \frac{\partial}{\partial y}(\sigma(y) + H(x,y)) = \sigma'(y) + \frac{\partial H}{\partial y}(x,y) = Q(x,y)$$

$$\sigma'(y) = Q(x,y) - \frac{\partial H}{\partial y}(x,y)$$

$$\sigma(y) = \int \left(Q - \frac{\partial H}{\partial y}\right) dy + C$$

$$\frac{\partial w}{\partial x} = x^2 + y^2$$

$$w = \frac{1}{3}x^3 + xy^2 + \sigma(y)$$

$$\frac{\partial w}{\partial y} = 2xy + \sigma'(y) = 2xy = Q(x,y)$$

$$\sigma'(y) = 0$$

$$\sigma(y) = C$$

$$w(x,y) = \frac{1}{3}x^3 + xy^2 + C$$

$$\boxed{\frac{1}{3}x^3 + xy^2 = C}$$

1/17 Lecture: Integrating Factors of Exact Equations

- $P(x,y)\partial x + Q(x,y)\partial y = 0$
- Theorem: there exists a $\mu(x,y)$ such that $\mu(x,y)P(x,y) + \mu(x,y)Q(x,y) = 0$ is exact

↳ P, Q defined on $\mathbb{R}^2$

↳ How to find $\mu(x,y)$? There's no general way

↳ μ is an integrating factor

↳ If we can find $\mu(x)$

↳ $\mu P\partial x + \mu Q\partial y = 0$

$\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q)$

$P\frac{\partial}{\partial y}\mu(x) + \frac{\partial}{\partial y}P\mu = \frac{\partial}{\partial x}\mu Q + \mu\frac{\partial Q}{\partial x}$

$0 + \mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \mu' Q$

$\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{\mu'(x)}{\mu(x)}$

↳ If $h(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ is a function of x only. Then $e^{\int h(x)\partial x}$ is an IF

↳ $g(y) = \frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ is a function of y only. Then $e^{-\int g(y)\partial y}$ is an IF

↳ $-xy\,\partial y + f(x)g(y)xy\,\partial x = 0$

↳ check if separable

↳ Ex) $(xy-2)\partial x + (x^2-xy)\partial y = 0$

↳ Step 1: check if separable → not

↳ Step 2: $\mu(x)$ or $\mu(y)$?

↳ $x - 2x + y = y - x \rightarrow \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$

$\frac{1}{x^2-xy}\cdot y - x = -\frac{1}{x} \rightarrow$ function of x

$\mu(x) = e^{\int -\frac{1}{x}} = \frac{1}{|x|}$

$\mu(x) = \frac{1}{x}$ or $-\frac{1}{x}$

$(y - \frac{2}{x})\partial x + (x-y)\partial y = 0$

$w = \int y - \frac{2}{x}\,\partial x = yx - 2\ln|x| + \varphi(y)$

$\frac{\partial w}{\partial y} = x + \varphi'(y) = x - y$

$\varphi'(y) = -y \rightarrow \varphi(y) = -\frac{1}{2}y^2 + C$

$yx - 2\ln|x| - \frac{1}{2}y^2 = C$

1/22 Lecture: Homogenous Equations

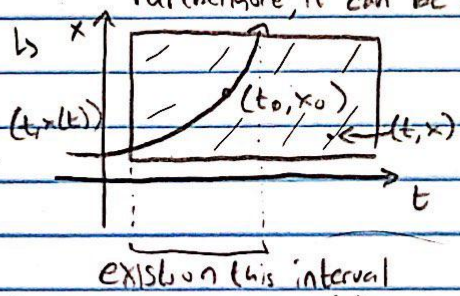
- $P(x,y) \to P(tx, ty) = t^n P(x,y)$
  - ↳ Homogenous function of degree n
  - ↳ $x^2+y^2 \to t^2x^2 + t^2y^2 \to t^2(xy)$, deg. 2
  - ↳ $e^{y/x} \to e^{ty/tx} \to e^{y/x} \to$ deg. 0
- $P(x,y)\,dx + Q(x,y)\,dy \to P$ and Q are homogenous of same degree
- How to solve:
  - ↳ Let $v(x) = \frac{y(x)}{x} \to y = vx$
  - ↳ Ex) $(x^2+y^2)\,dx + xy\,dy = 0$

$$(x^2+v^2x^2)\,dx + vx^2\,d(vx) = 0$$

↳ $d(vx) = dv(x) + v\,dx$

$$(x^2+v^2x^2)\,dx + vx^2(x\,dv + v\,dx) = 0$$
$$(x^2+v^2x^2+v^2x^2)\,dx + vx^3\,dv = 0$$
$$x^2(1+2v^2)\,dx + x^2(xv)\,dv = 0$$
$$(1+2v^2)\,dx + vx\,dv = 0 \to \text{guaranteed to be separable}$$
$$\frac{dx}{x} + \frac{v}{1+2v^2}\,dv = 0$$

- Existence/Uniqueness
  - ↳ Theorem: Consider IVP, $x' = f(t,x)$, $x(t_0) = x_0$
    - ↳ If $f(t,x)$ is continuous on R (a rectangle), then $(t_0, x_0) \in R$
      - ↳ Then, IVP has a solution defined on an interval contains t_0. Furthermore, it can be extended at least until $(t, x(t))$ leaves R
  - ↳

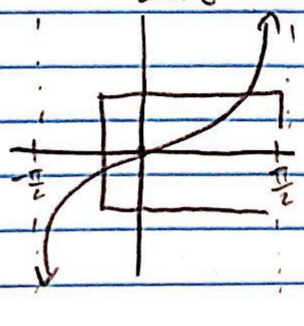


$f(t,x) = x'$

↳ $= \tan\theta$

- ↳ Ex) $x' = 1+x^2$, $x(0) = 0$

$$x = \tan t$$



1/24 Lecture: Uniqueness

- E/U

↳ $y' = y^{1/3}$, $y(0) = 0$

$y_1 = 0$ is a solution

$y_2 = \begin{cases} 0, & t \leq 0 \\ (\frac{2}{3}t)^{3/2}, & t > 0 \end{cases}$

↳ Theorem: $f(t,x)$ and $\frac{\partial f}{\partial x}(x,t)$ are continuous on a rectangle R (closed) containing (t_0, x_0) in the x-plane, then the IVP $x' = f(t,x)$, $x(t_0) = x_0$ has a unique solution inside R

↳ Ex) $x' = tx$ $f(t,x) = tx$, $\frac{\partial f}{\partial x} = t$ -> continuous on $\mathbb{R}^2$

↳ There is a unique solution

↳ Ex) $y' = \sin(y-x) + 1$, $y(0) = 1$

Prove that $y(x) > x$ for all $x \in \mathbb{R}$

$y = x$ is a solution

$\frac{\partial f}{\partial y}$ and f are continuous on $\mathbb{R}^2$

↳ The solution through whatever I.C. is unique -> solution curves do not intersect with each other

↳ let $y(x)$ be the solution through $(0,1)$, then $(x, y(x))$ does not intersect (x_0, x)

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Lecture 1/27: Autonomous Equations

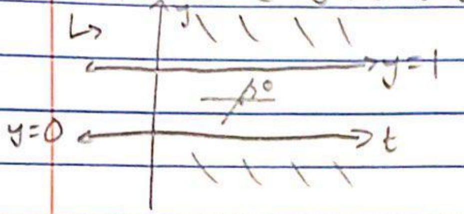
- $y' = f(y) \rightarrow$ one variable

↳ Ex) $y' = y(1-y) \rightarrow y(t) = 0, 1$

↳ If y_0 is a zero of $f(y)$, then $y(t) = y_0$ is a solution → equilibrium

- Sketch the solutions of the autonomous diff. eq.

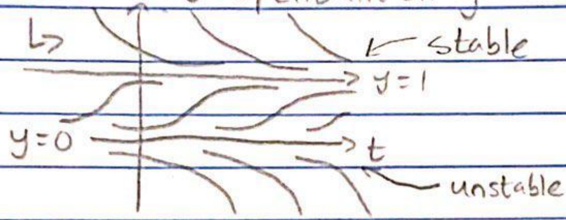
↳



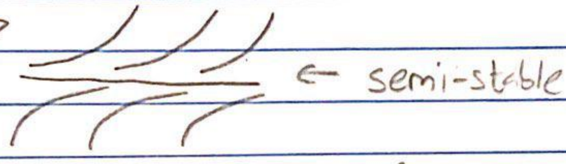
$\tan\theta = y' = f(y)$

↳ θ only dependant on y

↳



↳



↳ determine sign of $f(y)$ in each interval, then sketch the graph

↳ Given I.C. $y(0) = 1/2$

$y(t)$ is a solution, what is $\lim_{t\to\infty} y(t) = ?$, $\lim_{t\to-\infty} y(t) = ?$

↳ 1st, sketch the solutions, then follow the curve

$\lim_{t\to\infty} = 1$, $\lim_{t\to-\infty} = 0$

↳ Show that $0 < y(t) < 1$ for all $t \in (-\infty, \infty)$

↳ there are solutions at 0 and 1, therefore $y(t)$ cannot intersect with them

↳ proof: U/E → $f(y) = y(1-y)$ and $\frac{\partial f}{\partial y} = f'(y) = 1 - 2y$ is continuous,

so the solutions to the IVP are unique

because $y(t) = 0$ and $y(t) = 1$ are 2 equilibrium solutions, $(0, 1/2)$

is a point between the 2 solutions

Therefore, the solution going through $(0, 1/2)$ will not intersect

with $y(t) = 0$ or $y(t) = 1$

In other words $0 < y(t) < 1$ for all $t \in (-\infty, \infty)$

• Review → Integrating Factor formulas provided

↳ 1st Order Diff. Eq.

↳ First, put question in normal form

↳ Integrating Factor or Variation of Parameter

↳ Ex) $y' = \frac{1}{x}y + \sqrt{x}$, $y(1) = 0$

$y = 2x^{3/2} - 2x$

↳ Interval of existence: $(0, \infty)$

↳ includes both solution and original diff. eq.

↳ Homogeneous Equation:

↳ Ex) $(y^2 + 2xy)dx - x^2 dy = 0$

$y = xv$, $dy = xdv + vdx$

$(v^2x^2 + 2x^2v) + (-x^2)(xdv + vdx) = 0$

$(v^2 + 2v) + (-xdv - vdx) = 0$

$v^2 + 2v = xdv + vdx$

$\frac{dx}{x} = \frac{1}{v^2+v} dv$

$\frac{1}{v^2+v} = \frac{A}{v} + \frac{B}{v+1}$

$1 = Av + A + Bv$

$A = 1$, $B = -1$

$\frac{1}{v^2+v} = \frac{1}{v} - \frac{1}{v+1}$

$\frac{dx}{x} = \left(\frac{1}{v} - \frac{1}{v+1}\right)dv$

$\ln|x| = \ln|v| - \ln|v+1| + C$

$\ln|x| = \ln\left|\frac{v}{v+1}\right| + C$

$x = \frac{v}{v+1} + C$

$x = \frac{y/x}{y/x + 1} + C$

$\frac{y+x}{y/x} = C$

$\frac{x^2+xy}{y} = C$

$x^2 + xy = Cy$

$\boxed{x(x+y) = Cy}$

↳ Exact Equations:

↳ $Pdx + Qdy = 0$

↳ Suppose there is an I.F. of either $\mu(x)$ or $\mu(y)$ form, then solve the eq.

1/29 Lecture: Derivative of Solutions

- Ex) $y' = f_1(t,y)$, $y(t_0) = y_0$

$$y'(t_0) = f_1(t_0, y_0)$$
$$y''(t_0) = \frac{\partial}{\partial t} f_1(t,y)\,(t_0,y_0)$$
$$= \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial y}\frac{\partial y}{\partial t}$$
$$= \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f_1\right)(t_0, y_0)$$

↳ Ex) $y' = t + y$

$$y'' = 1 + 1(t+y)$$
$$y'' = 1 + t + y$$

- Higher Derivatives:

↳ $y^{(n-1)}(t) = f_{n-1}(t,y)$

$$\frac{\partial}{\partial t} y^{(n-1)}(t) = \frac{\partial f_{n-1}}{\partial t} + \frac{\partial f_{n-1}}{\partial y} \cdot \frac{\partial y}{\partial t}$$
$$y^{(n)}(t) = \frac{\partial f_{n-1}}{\partial t} + \frac{\partial f_{n-1}}{\partial y} f_1$$

↳ If we have t_0, y_0, then we know all $y^{(n)}(t_0)$

- Recall: for a function $f(t)$

↳ $y(t) = y(t_0) + \frac{y'(t_0)}{1!}(t-t_0) + \frac{y''(t_0)}{2!}(t-t_0)^2 + \dots + \frac{y^{(n)}(t_0)}{n!}(t-t_0)^n$

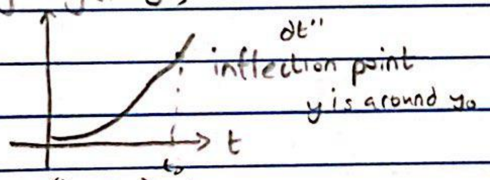
↳ Analytic Function: $f(t)$ = Taylor Series of $f(t)$

↳ If f_1 is analytic → $y(t)$ is analytic

↳ Consequently, if we know $y^{(n)}(t_0)$, then, $y(t) = \sum_{n=0}^{\infty} \frac{y^{(n)}(t_0)}{n!}(t-t_0)^n$

↳ y = # of daily infected patients

$y' = y(k-y)$



$y'' = (k-2y)y'$

$y'' = 0$ iff $y = \frac{1}{2}k$

- Second Order Differential Equations

↳ $y'' = f(t, y, y')$

↳ I.C. $y(t_0) = y_0$, $y'(t_0) = y_1$

↳ $y' = a(t)y + f$

↳ $y'' + p(t)y' + q(t)y = f$

↳ If $p(t), q(t), f(t)$ are continuous on $t \in (\alpha, \beta)$, then there exists a unique solution to the I.V.P.

↳ $y'' = -py' - qy + f(t)$

↳ $\frac{\partial F}{\partial y} = -q$, $\frac{\partial F}{\partial y'} = -p$

2/3 Lecture: Solution Spaces

- $y'' + py' + qy = f$

↳ $\left(\frac{\partial}{\partial t}\frac{\partial}{\partial t} + p(t)\frac{\partial}{\partial t} + q(t)\right) y = f$

↳ 1st: solve homogeneous part

↳ $\left(\frac{\partial}{\partial t}\frac{\partial}{\partial t} + p(t)\frac{\partial}{\partial t} + q(t)\right) y = 0 \rightarrow$ find $\ker(A)$

↳ 2nd: find particular solution y_p

↳ Solution space $= \ker(A) + y_p$

↳ Goal: prove $\dim(\ker(A)) = 2$

↳ Def: $u(t)$ and $v(t)$ are finction on (α, β), they are linearly dependent iff $u(t) = cv(t)$ or $v(t) = cu(t)$ for some $c \in \mathbb{R}$

↳ Wronskian u, v

↳ $W(u,v) = \begin{vmatrix} u(t) & u'(t) \\ v(t) & v'(t) \end{vmatrix} = uv' - u'v$

↳ Proposition: u and v are linearly dependent $\iff W(u,v) = 0$ on (α, β)

↳ $uv' = u'v$

$\frac{v'}{v} = \frac{u'}{u} \rightarrow \ln|v| = \ln|u| + C \rightarrow \ln\left|\frac{v}{u}\right| = C$

$\frac{v}{u} = e^C \rightarrow \frac{v}{u} = k \rightarrow v = ku$

↳ Suppose u, v, are solutions to the homogeneous part ($y'' + py' + qy = 0$), then u, v, linearly independent / dependent $\iff W(u,v) \neq 0 / = 0$ on (α, β)

↳ If $W(u,v) = 0$ at t_0, then $W(u,v) = 0$ on (α, β)

$\begin{vmatrix} u(t_0) & v(t_0) \\ u'(t_0) & v'(t_0) \end{vmatrix} = 0 \iff \begin{pmatrix} u(t_0) \\ u'(t_0) \end{pmatrix} = c\begin{pmatrix} v(t_0) \\ v'(t_0) \end{pmatrix}$ or $\begin{pmatrix} v(t_0) \\ v'(t_0) \end{pmatrix} = c\begin{pmatrix} u(t_0) \\ u'(t_0) \end{pmatrix}$

↳ Let $y_1(t) = u(t)$ and $y_2(t) = cv(t) \rightarrow y_1$ and y_2 satisfy the same I.C

↳ By existence/uniqueness $\rightarrow y_1 = y_2$ on (α, β)

↳ Theorem: Suppose y_1 and y_2 are 2 linearly independent solutions to $y'' + py' + qy = 0$, then all solutions are linear combinations of y_1 and y_2

↳ $c_1y_1 + c_2y_2 = \text{span}\langle y_1, y_2 \rangle$

↳ Because y_1 and y_2 are linearly independent, we can find c_1, c_2 such that $c_1y_1 + c_2y_2$ satisfies some initial condition (y_0, y_1)

↳ $c_1y_1(t_0) + c_2y_2(t_0) = y_0$

$c_1y_1'(t_0) + c_2y_2'(t_0) = y_1$

215 Lecture: Constant Coefficient 2nd Order Diff. Eq.

- $y'' + p(x)y' + q(x)y = f \to y'' + py' + qy = f$, where p/q are constants
  - ↳ Step 1: Solve Homogeneous part
    - $y_1, y_2 \to c_1 y_1 + c_2 y_2$
  - ↳ Step 2: Get a particular solution
    - ↳ Method 1: Undetermined Coefficient
      - ↳ Fast/easy to calculate, requirement on f
    - ↳ Method 2: Variation of Parameter
      - ↳ No requirements, relatively hard to compute
  - ↳ Ex) $\lambda^2 + \lambda p + q = f$
    - ↳ $\lambda^2 + \lambda p + q = 0 \to \lambda_1, \lambda_2$

      $\left(\frac{\partial}{\partial t}\frac{\partial}{\partial t} + p\frac{\partial}{\partial t} + q\right) = 0$

      $\left(\frac{\partial}{\partial t} - \lambda_1\right)\left(\frac{\partial}{\partial t} - \lambda_2\right) = 0$

      $\ker(BA) = \ker\left(\frac{\partial}{\partial t} - \lambda_1\right) + \ker\left(\frac{\partial}{\partial t} - \lambda_2\right)$

      $\frac{\partial}{\partial t}y - \lambda_1 y = 0,\ Ce^{\lambda_1 t} = y_1$

      $\frac{\partial}{\partial t}y - \lambda_2 y = 0,\ Ce^{\lambda_2 t} = y_2$

      $y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 y_2}$
      - ↳ $\lambda_1, \lambda_2 \in \mathbb{R} \to e^{\lambda_1 t}, e^{\lambda_2 t}$
      - ↳ $\lambda_1 = \alpha + \beta i,\ \lambda_2 = \alpha - \beta i \to e^{\alpha t}\cos\beta t,\ e^{\alpha t}\sin\beta t$
      - ↳ $\lambda_1 = \lambda_2 = \lambda \to e^{\lambda t}, te^{\lambda t}$
  - ↳ Ex) $8y'' + 2y' - y = 0,\ y(-1) = 1,\ y'(-1) = 0$

    $\lambda_1 = \frac{1}{4},\ \lambda_2 = -\frac{1}{2}$

    $e^{1/4t} = y_1,\ y_2 = e^{-1/2t}$

    $y = C_1 e^{1/4t} + C_2 e^{-1/2t} \to 1 = c_1 e^{-1/4} + c_2 e^{1/2}$

    $y' = \frac{1}{4}C_1 e^{1/4t} - \frac{1}{2}C_2 e^{-1/2t} \to 0 = \frac{1}{4}C_1 e^{-1/4} - \frac{1}{2}C_2 e^{1/2}$

    $c_1 = \frac{2}{3}e^{1/4},\ c_2 = \frac{1}{3}e^{-1/2}$

2/7 Lecture: Undetermined Coefficient

- $y'' + py' + qy = f$

↳ Restrictions on f:

↳ 1) $f: Ce^{at}$

2) $f: A\cos\omega t + B\sin\omega t$

3) $f =$ polynomial

- Ex) $y'' - y' - 2y = 2e^{-2t}$

↳ Step 1: $\lambda^2 - \lambda - 2 = 0$

$\lambda = -1, 2$

$y_1 = e^{-t}, y_2 = e^{2t}$

↳ Step 2: find y_p

↳ in form $y = Ce^{-2t}$ → C is the undetermined coefficient

$y_p' = -2Ce^{-2t}, y_p'' = 4Ce^{-2t}$

$4Ce^{-2t} - (-2Ce^{-2t}) - 2Ce^{-2t} = 2e^{-2t}$

$4Ce^{-2t} = 2e^{-2t}$

$C = \frac{1}{2}$

$y_p = \frac{1}{2}e^{-2t}$

↳ $\boxed{y = C_1 e^{-t} + C_2 e^{2t} + \frac{1}{2}e^{-2t}}$

- Ex) $y'' - y' - 2y = e^{2t}$

↳ $y_1 = e^{-t}, y_2 = e^{2t}$

↳ Solution: let $y_p = Cte^{2t}$, solve same way as above

- Ex) $y'' - 2y' + y = e^t$

$\lambda^2 - 2\lambda + 1 = 0$

$\lambda = 1$, repeated → $y_1 = e^t, y_2 = te^t$

↳ $y_p = t^2 e^t$

- Ex) $y'' + 2y' - 3y = 5\sin 3t + 0\cos 3t$

$\lambda^2 + 2\lambda - 3 = 0$

$(\lambda + 3)(\lambda - 1) = 0$

$\lambda = -3, 1$

$y_p = A\cos 3t + B\sin 3t$

$y_p' = -3A\sin 3t + 3B\cos 3t$

$y_p'' = -9A\cos 3t - 9B\sin 3t$

$(-9A\cos 3t - 9B\sin 3t) + 2(-3A\sin 3t + 3B\cos 3t) - 3(A\cos 3t + B\sin 3t)$

$(-12A + 6B)\cos 3t + (-6A - 12B)\sin 3t = 5\sin 3t + 0\cos 3t$

$A = -1/6, B = -1/3$

$\boxed{y_p = -\frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t}$

$\hookrightarrow iy'' + 2iy' - 3iy = 5i\sin 3t$

$+ \quad x'' + 2x' - 3x = 5\cos 3t$

$= (x+iy)'' + 2(x+iy)' - 3(x+iy) = 5\underbrace{(\cos 3t + i\sin 3t)}_{\hookrightarrow = e^{3ti}}$

$z''(t) + 2z'(t) - 3z(t) = 5e^{3it}$ → not a root

$z_p = Ce^{3it}$

$z_p' = 3iCe^{3it}$

$z_p'' = -9Ce^{3it}$

$(-9C + 6iC - 3C)e^{3it} = 5e^{3it}$

$(-12C + 6iC) = 5$

$C = \frac{-5}{12-6i} = -\frac{1}{6}(2+i)$

$z_p = -\frac{1}{6}(2+i)e^{3it} = x_p + y_p$

$y_p = \mathrm{Im}(z_p)$

$= -\frac{1}{6}(2+i)(\cos 3t + \sin 3t\, i)$

$= -\frac{1}{6}((2\cos 3t - \sin 3t) + (\cos 3t + 2\sin 3t)i)$

2/10 Lecture: Variation of Parameter

- $y'' + py' + qy = f_1 \rightsquigarrow y_{p_1}$
  $y'' + py' + qy = f_2 \rightsquigarrow y_{p_2}$
  $y'' + py' + qy = f_1 + f_2 \rightsquigarrow y_{p_1} + y_{p_2}$

↳ Ex) $y'' - y' - 2y = e^{-2t} - 3e^{-t}$

$f_1: y_{p_1} = \frac{1}{4}e^{-2t}$ $\boxed{y_p = y_{p_1} + y_{p_2} = \frac{1}{4}e^{-2t} + te^{-t}}$

$f_2: y_{p_2} = te^{-t}$

- $y'' + p(t)y' + q(t)y = f(t)$

↳ Suppose we have 2 linearly independent solutions to $y'' + p(t)y' + q(t) = 0$,

$y_1(t), y_2(t)$

↳ $y_h = c_1 y_1(t) + c_2 y_2(t)$

↳ $y_p = v_1(t)y_1(t) + v_2(t)y_2(t)$

↳ $v_1(t) = \int \frac{-y_2(t)f(t)}{y_1y_2' - y_1'y_2} = \int \frac{-y_2 f}{W(t)}$

$v_2(t) = \int \frac{y_1(t)f(t)}{y_1y_2' - y_1'y_2} = \int \frac{y_1 f}{W(t)}$

↳ Ex) $y'' + y = \tan t$ → 1st consider undetermined coefficient

↳ Step 1. y_h

$\lambda^2 + 1 = 0 \to \lambda = \pm i$

$y_1 = \cos t$, $y_2 = \sin t$

↳ Step 2: $W(t)$

$\cos t(\cos t) + \sin t(\sin t)$

$= 1$

↳ Step 3: $V(t)$

$v_1 = \int -\sin t(\tan t)$

$= \int \frac{-\sin^2 t}{\cos t} = \int \frac{\cos^2 t - 1}{\cos t}$

$= \cancel{\int \cos t\, dt - \int \frac{1}{\cos t} dt}$

$= \sin t + \ln|\sec t - \tan t|$

↳ Ex) $t^2y'' + 3ty' - 3y = \frac{1}{t}$

$y_1 = t$, $y_2 = t^{-3}$

$W(t) = -3t^{-3} - t^{-3} = -4t^{-3}$

$v_1 = \int \frac{-t^{-3}(\frac{1}{t^3})}{-4t^{-3}} = \frac{1}{4}\int \frac{1}{t^3} dt$

$= -\frac{1}{8}t^{-2}$

$v_2 = \int \frac{t(\frac{1}{t^3})}{-4t^{-3}} = -\frac{1}{4}\int t\, dt$

$= -\frac{1}{8}t^2$

$y_p = -\frac{1}{8}t^{-1} - \frac{1}{8}t^{-1} = \boxed{-\frac{1}{4t}}$

2/12 Lecture: Linear Systems

- $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, A(t) = \begin{bmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{bmatrix}, f(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$
  - ↳ $x'(t) = A(t) \cdot \vec{x}(t) + f(t)$
  - ↳ Theorem: Solution space for $\vec{x}' = A\vec{x}$ is $c_1\vec{x}^1(t) + \dots + c_n\vec{x}^n(t)$
    - ↳ $\vec{x}^1(t) \dots \vec{x}^n(t)$ are linearly independent as $n \times 1$ vectors
- 2×2 systems
  - ↳ $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} a x_1(t) \\ b x_2(t) \end{bmatrix}$$

$x_1(t) = e^{at}, x_2(t) = e^{bt}$

$$\boxed{\vec{x}(t) = \begin{bmatrix} e^{at} \\ e^{bt} \end{bmatrix}}$$

- Eigenvalue/Eigenvector
  - ↳ $A\vec{v} = \lambda\vec{v}, \vec{v} \neq 0$
  - ↳ Suppose $e^{\lambda t}\vec{v} = \vec{x}(t)$

$$\frac{\partial}{\partial t}(e^{\lambda t}\vec{v}) \stackrel{?}{=} A(e^{\lambda t}\vec{v}) = e^{\lambda t}A\vec{v} = \lambda e^{\lambda t}\vec{v}$$

$$\frac{\partial}{\partial t}e^{\lambda t} \cdot \vec{v} + e^{\lambda t}\left(\frac{\partial}{\partial t}\vec{v}\right) \leftarrow 0$$

$$\lambda e^{\lambda t}\vec{v} = \lambda e^{\lambda t}\vec{v} \checkmark$$

- Goal: find 2 linearly independent solutions to $\vec{x}' = A\vec{x}$
  - ↳ Case 1: A has 2 real eigenvalues, $\lambda_1 \neq \lambda_2$

$\vec{x}^1 = e^{\lambda_1 t}\vec{v}_1, \quad \vec{x}^2 = e^{\lambda_2 t}\vec{v}_2$

$$\vec{x}' = \begin{bmatrix} -5 & 1 \\ -2 & -2 \end{bmatrix}\vec{x}$$

$A\vec{v} = \lambda\vec{v} \rightarrow (A - \lambda I)\vec{v} \rightarrow \ker(A - \lambda I) \neq 0 \rightarrow \det(A - \lambda I) = 0$

2/14 Lecture: 2nd Order Linear Systems

- Case 2: $\lambda = \alpha + \beta i$, $\bar{\lambda} = \alpha - \beta i$

↳ $(A - \lambda I)\vec{v} = 0$

$e^{\lambda t}\vec{v} = e^{\alpha t + \beta t i}\vec{v} = \vec{z}^1(t)$

$\vec{z}^2(t) = e^{\bar{\lambda} t}(\bar{\vec{v}}) = (\overline{e^{\lambda t}\vec{v}})$

$\vec{z} = c_1\vec{z}^1 + c_2\vec{z}^2$

$\vec{z}^1(t) = \vec{u}(t) + \vec{w}(t)i$ – real/imaginary parts are linearly independent

$\therefore \vec{y} = c_1\vec{u}(t) + c_2\vec{w}(t)$

↳ Ex) $\vec{y}' = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} y$

↳ $(\lambda - 0)(\lambda - 2) + 2 = 0$

$\lambda^2 - 2\lambda + 2 = 0$

$\lambda = 1 \pm i$

$\lambda = 1 + i$, $\bar{\lambda} = 1 - i$

$(\lambda I - A)\vec{v} = 0$

$\begin{bmatrix} 1+i & -1 \\ -2 & -1+i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $a, b \in \mathbb{C}$

$(1+i)a - b = 0$

let $a = 1$, $b = 1 + i$

$\vec{v} = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$

$e^{(1+i)t}\begin{bmatrix} 1 \\ 1+i \end{bmatrix} = \vec{y}^1(t)$

$e^t e^{it}\begin{bmatrix} 1 \\ 1+i \end{bmatrix}$

$e^t(\cos t + i\sin t)\begin{bmatrix} 1 \\ 1+i \end{bmatrix}$

$\begin{bmatrix} e^t\cos t + (e^t\sin t)i \\ e^t(\cos t - \sin t) + e^t(\cos t + \sin t)i \end{bmatrix}$

$$\boxed{\vec{y} = c_1\begin{bmatrix} e^t\cos t \\ e^t(\cos t - \sin t) \end{bmatrix} + c_2\begin{bmatrix} e^t\sin t \\ e^t(\cos t + \sin t) \end{bmatrix}}$$

- Case 3: $\lambda_1 = \lambda_2$

$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, $\begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}$, $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, $\begin{bmatrix} \lambda & a \\ 0 & \lambda \end{bmatrix}$

$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, $y' = Ay$

↳ $\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} \lambda x_1'(t) \\ \lambda x_2'(t) \end{bmatrix}$

$x_1 = c_1 e^{\lambda t}, \quad x_2 = c_2 e^{\lambda t}$

$c_1 e^{\lambda t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c_2 e^{\lambda t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

↳ $A = \begin{bmatrix} \lambda & a \\ 0 & \lambda \end{bmatrix}$

↳ Theorem: suppose $\vec{v}_1$ is the only eigenvector for A, $\vec{w}(\neq 0)$ is a vector linearly independent to $\vec{v}_1$

$(A-\lambda I)^2 = 0$

$(A-\lambda I)(A-\lambda I)\vec{w} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$(A-\lambda I)\vec{w} \in \ker(A-I\lambda), \quad \dim(\ker(A-I\lambda)=1)$

$(A-\lambda I)\vec{w} = k(\neq 0)\vec{v}_1$

$v_2 = \frac{1}{k}\cdot\vec{w}, \quad (A-\lambda I)\vec{v}_2 = \vec{v}_1$

Fundamental set $\begin{cases} e^{\lambda t}\vec{v}_1 \\ e^{\lambda t}(\vec{v}_2 + t\vec{v}_1) \end{cases}$

↳ Ex) $A = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$

$\lambda = -2$ repeated

$A - \lambda I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0$

eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{v}_1$

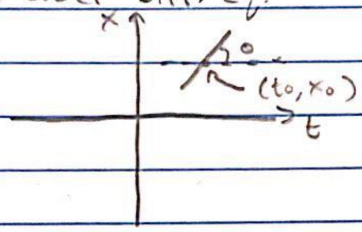
let $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{x}_1(t)$

$e^{-2t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \vec{x}_2(t)$

2/21 Lecture: U/E of Linear Systems

- $\vec{x}'(t) = A(t)\vec{x}(t) + \vec{f}(t)$
  - ↳ When $\vec{f}(t) = \vec{0}$, this is homogeneous
- Theorem: Suppose $A(t)\vec{f}(t)$ is continuous on (α, β)
  - ↳ Then for an IVP: $\vec{x}' = A\vec{x}(t) + f(t)$, $\vec{x}(t_0) = \vec{v}$, it has a "unique" solution on (α, β)
- Directional Field:
  - ↳ 1st order diff. eq.



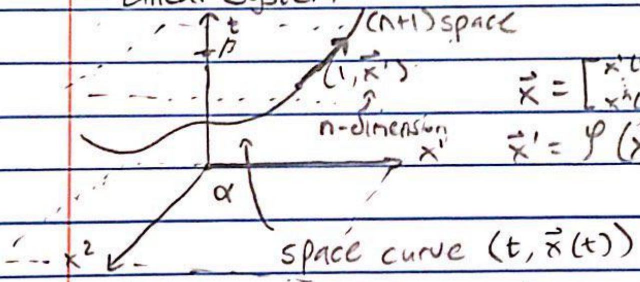
$\tan\theta = x' = f(t_0, x_0)$

$t \to (t, x(t))$

$\downarrow$

$(1, x'(t))$

  - ↳ Linear System



$\vec{x} = \begin{bmatrix} x^1(t) \\ \vdots \\ x^n(t) \end{bmatrix}$

$\vec{x}' = \varphi(x^1, \dots, x^n, t)$

- If $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are solutions to $\vec{x}' = A(t)\vec{x} + f(t)$, then $\vec{x}_1 - \vec{x}_2$ is also a solution
  - ↳ If $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are solutions to $\vec{x}' = A(t)\vec{x}(t)$, then $c_1\vec{x}_1 + c_2\vec{x}_2$ is still a solution
  - ↳ Solution space for $\vec{x}' = A(t)\vec{x} + \vec{f}(t)$: $\underbrace{\ker\left(\frac{\partial}{\partial t} - A(t)\right)}_{\text{solution space for } \vec{x}' = A\vec{x}} + \vec{y}_p$
    - ↳ $\frac{\partial}{\partial t}(\vec{x}(t)) = A\vec{x} \to \left(\frac{\partial}{\partial t} - A(t)\right)\vec{x} = 0$
  - ↳ If $\vec{x}_1(t), \dots, \vec{x}_n(t)$ are solutions to $\vec{x}' = A(t)\vec{x}$, then $\vec{x}_1, \dots, \vec{x}_n$ are linearly independent on $(\alpha, \beta) \iff \vec{x}_1(t_0), \dots, \vec{x}_n(t_0)$ are linearly independent
  - ↳ Study space $\ker\left(\frac{\partial}{\partial t} - A(t)\right)$:

    $IVP_1: \vec{x}_1 = A\vec{x}, \ \vec{x}_1(t_0) = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}$

    $IVP_k = \vec{x}_k = A\vec{x}, \ \vec{x}_k(t_0) = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ ← kth coord = 1

    $IVP_n = \vec{x}_n = A\vec{x}, \ \vec{x}_n(t_0) = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$

    - ↳ for $\vec{x}(t_0) = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix}$: $\vec{y} = k_1\vec{x}_1 + k_n\vec{x}_n = k_1\begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} + \dots, k_n\begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$
- Theorem: Suppose $A(t), \vec{f}(t)$ is continuous on (α, β), then the solution space for $\vec{x}' = A(t)\vec{x} + \vec{f}(t) = \text{span}(\vec{x}_1(t), \dots, \vec{x}_n(t)) + \vec{x}_p(t)$, where $\vec{x}_1, \dots, \vec{x}_n$ are solutions to $\vec{x}' = A\vec{x}$

2/26 Lecture: Phase Plane Portraits

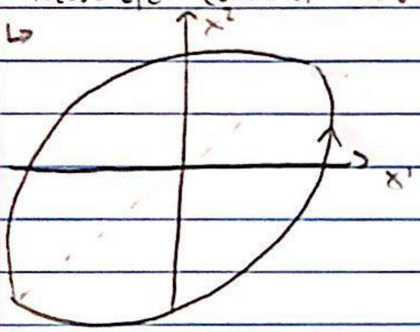
- $A_{2\times 2}$
- Case 2: Complex Root

$\lambda = \alpha + \beta i$, $\vec{w} = \vec{v}_1 + \vec{v}_2 i$

$\bar{\lambda} = \alpha - \beta i$, $\bar{w} = \vec{v}_1 - \vec{v}_2 i$

↳ $\vec{x}(t) = C_1 e^{\alpha t}(\cos\beta t\,\vec{v}_1 - \sin\beta t\,\vec{v}_2) + C_2 e^{\alpha t}(\sin\beta t\,\vec{v}_1 + \cos\beta t\,\vec{v}_2)$

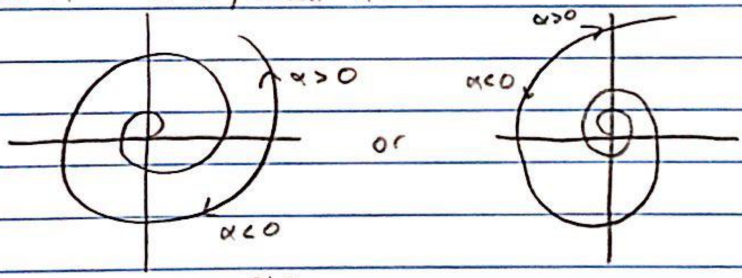
↳



↳ Norm of $\vec{x}(t)$

$$\frac{|\vec{x}(t)|}{\lim_{t\to\infty}|\vec{x}(t)|} = \begin{cases} +\infty, \text{ when } \alpha > 0 \\ 0, \text{ when } \alpha < 0 \end{cases}$$

↳



↳ Direction: pick $\begin{bmatrix}1\\0\end{bmatrix}$

$t \to (t, x^1(t), x^2(t))$

direction: $(1, x^{1\prime}(t), x^{2\prime}(t))$

↳ Project to Phase Plane $\frac{\partial}{\partial t}\begin{bmatrix}x^1(t)\\x^2(t)\end{bmatrix} = A\begin{bmatrix}x^1(t)\\x^2(t)\end{bmatrix}$

$A = \begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix} \to$ direction based on a_{21}

↳ Consider $A = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

$\vec{x}^1(t) = \begin{bmatrix}1\\0\end{bmatrix}$, $\vec{x}^2(t) = \begin{bmatrix}0\\1\end{bmatrix}$

$C_1\begin{bmatrix}1\\0\end{bmatrix} + C_2\begin{bmatrix}0\\1\end{bmatrix}$

$A = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$

$= C_1 e^t\begin{bmatrix}1\\0\end{bmatrix} + C_2\begin{bmatrix}0\\1\end{bmatrix}$

$= \begin{bmatrix}C_1 e^t\\C_2\end{bmatrix}$

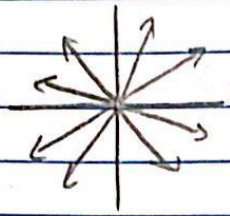
2/28 Lecture: Exp. of Matlix

- Repealed Root ($\lambda \neq 0$)

$\begin{bmatrix} \lambda & a \\ a & \lambda \end{bmatrix}$ or $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

↳ $c_1 e^{\lambda t} \vec{v}_1 + c_2 e^{\lambda t} \vec{v}_2$

Defective Node



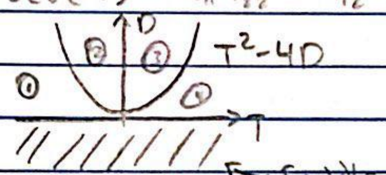
Star Node

- Trace-Determinant Plane:

↳ $\lambda^2 + p\lambda + q = 0$ → characteristic polynomial

↳ $p = -tr(A) = a_{11} + a_{22} = -T$

↳ $q = det(A) = a_{11}a_{22} - a_{12}a_{21} = D$

↳ 

↳ $T = 0, D > 0$ → Non-generic

↳ ①: Nodal Sink, ②: Spiral Sink, ③: Spiral Source, ④: Nodal Source

↳ Generic

↳ Repeated Root when $T^2 - 4D = 0$

- Exp of Matlix

↳ 1st: $x' = a \cdot x$, $x(t) = e^{at}$

n-dim $\vec{x}' = A\vec{x}$, $\vec{x} = e^{At}\vec{v}$

↳ $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots + \frac{t^n}{n!}$

↳ Def: $e^A = I^n + A + \frac{A \cdot A}{2!} + \dots + \frac{A^n}{n!}$

↳ Well-defined: $(e^{At})_{ij}$ should be a series, and the sum of the series must converge

↳ Def: $\|A\|$ (Norm of matrix) $= \max\{A_{ij}\}$ → $\left\| \begin{matrix} 2 & 1 \\ 3 & 4 \end{matrix} \right\| = 4 = M$

↳ $|A_{ij}| \leq M$ ($|I_{ij}| \leq 1$), $|(A \cdot A)_{ij}| = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{n1} & \ddots & \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{n1} & \ddots & \end{bmatrix} = |a_{i1}a_{1j} + \dots + a_{in}a_{nj}| \leq$

$|a_{i1}a_{1j}| + \dots + |a_{in}a_{nj}| \leq n \cdot M^2$

$|(A^k)_{ij}| \leq (n \cdot M)^{k-1} \cdot M = \frac{n^k M^k}{n}$

$|(e^A)_{ij}| \leq 1 + \frac{1}{1!}M + \dots + \frac{1}{k!}\frac{(M \cdot n)^k}{n}$ → converges by Ratio Test

↳ $e^{tA} = 1 + tA + \frac{(tA)^2}{2!} + \dots + \frac{(tA)^n}{n!}$, $\frac{d}{dt}(e^{tA}) = \frac{d}{dt}\left(\sum_{n=0}^{\infty} \frac{t^n A^n}{n!}\right) = Ae^{tA}$