Solutions for Midterm 1 (Math 33A, Fall 2019)

Problem 1 (10 points in total)

Consider the map $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

for any vector $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ in $\mathbb{R}^3$.

1. (2 points) Write down the definition of linear transformation.

   Solution:

   A map $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation if it satisfies the following:

   - $T(v + w) = T(v) + T(w)$ for all vectors v and w in $\mathbb{R}^m$
   - $T(cv) = cT(v)$ for all real numbers c and all vectors v in R^m.

2. (2 points) Show that the map T is a linear transformation.

   Solution: Let $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ be any vectors in $\mathbb{R}^m$, and c any real number.

   We have:

$$T(v + w) = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix},$$

and

$$T(v) + T(w) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

and therefore

$$T(v + w) = T(v) + T(w)$$

for all v, w in $\mathbb{R}^n$.

Similarly, we have on one hand $T(cv) = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$ and on the other $cT(v) = c \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and thus

$$T(cv) = cT(v)$$

for all c in $\mathbb{R}$ and v in $\mathbb{R}^n$. Therefore T is linear.

3. (3 points) Write down the matrix A such that $T(v) = Av$ for all vectors v in $\mathbb{R}^3$.

   Solution: We know that the entries in the ith column of the matrix A are given by the entries of $T(e_i)$, where e_i is the ith standard basis vector. Thus we have

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

4. (2 points) Compute the rank of A.

   Solution: The matrix A is in RREF, and thus we rank read off the rank directly. We have $\text{rank}(A) = 2$.

5. (1 point) Let B be any matrix of the same size as A. Can B have rank larger than A?

   Solution: No, because the rank of a matrix is bounded above by the number of columns and rows of the matrix.

Problem 2 (10 points in total)

1. (6 points) Give an example of three linear transformations $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $U\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that the matrices that represent T and S commute, while the matrices that represent S and U do not commute. (Recall: we say that an $n \times m$ matrix A represents a linear transformation $T\colon \mathbb{R}^m \to \mathbb{R}^n$ if $T(v) = Av$ for all $v \in \mathbb{R}^m$.)

   Solution: Let T be the counterclockwise rotation through $\pi/6$, let S be the counterclockwise rotation through $\pi/3$, and U be the reflection around the y axis. Then we have that T is represented by the matrix

   $$A = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix},$$

   S is represented by the matrix

   $$B = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix},$$

   and U is represented by the matrix

   $$C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

   We have $AB = BA$, while $BC \neq CB$.

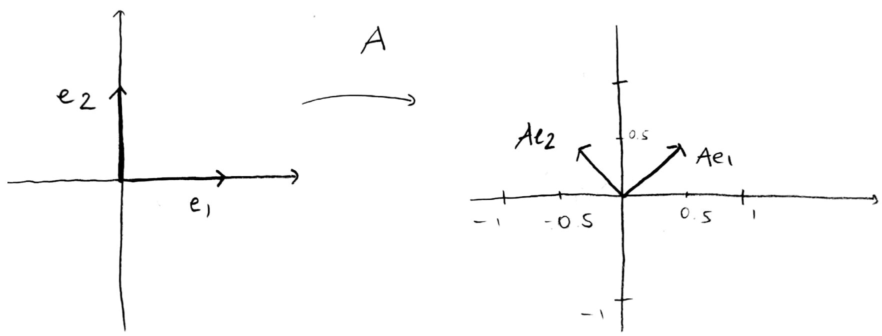
2. (4 points in total) Let $A = \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

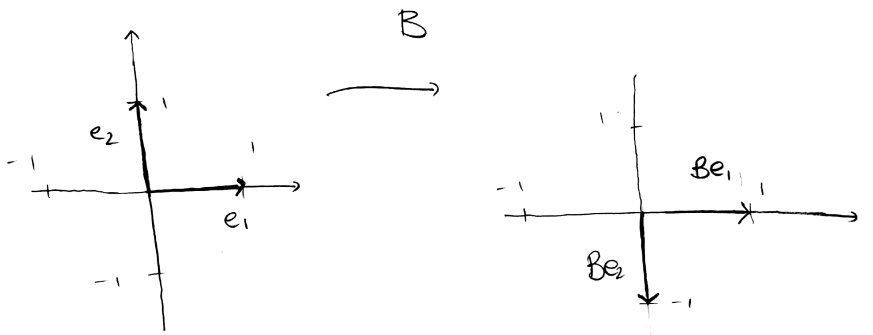
   - (2 points) Give a geometric interpretation of the transformations represented by A and B. (In words, or using a drawing.)

     Solution: The matrix A represents the composition of a rescaling by $1/\sqrt{2}$ and a rotation by $\pi/4$. (Note: here it would have been enough to say that it is a composition of a rescaling with a rotation, without giving exact values for the rescaling factor or the angle.)

     The matrix B represents the reflection around the x axis.

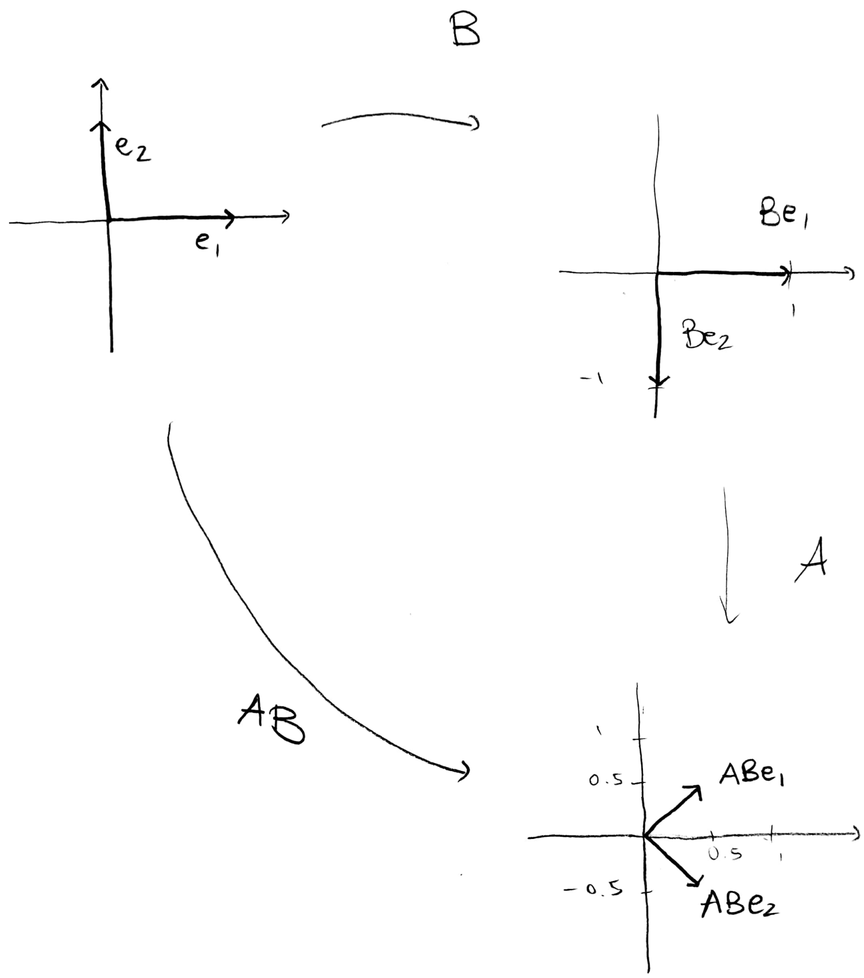
     Alternative solution: We draw the images of the standard basis vectors under the transformations represented by A, respectively B. We have:





- (2 points) Draw the images of the standard basis unit vectors of $\mathbb{R}^2$ under the linear transformation represented by AB.

Solution:



Problem 3 (10 points in total)

Consider the following system of three linear equations in the variables x_1, x_2, x_3, x_4:

$$\begin{aligned} 2x_2 + x_4 &= 1 \\ x_1 + x_3 &= 1 \\ x_4 &= 1 \end{aligned}$$

1. (4 points) Solve the system using the Gauss-Jordan elimination algorithm.

   Solution:

   We write the augmented matrix for the system, and put it into RREF using elementary row operations:

$$\left[\begin{array}{cccc|c} 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \overset{\text{interchange I and II}}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \overset{\frac{1}{2}\,\text{II}}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\overset{\text{II} - \frac{1}{2}\text{III}}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

   The solutions are:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t+1 \\ 0 \\ t \\ 1 \end{bmatrix}$$

   with $t \in \mathbb{R}$ arbitrary.

2. (3 points) Let b_1, b_2, b_3 be arbitrary real numbers. How many solutions does the system

$$\begin{aligned} 2x_2 + x_4 &= b_1 \\ x_1 + x_3 &= b_2 \\ x_4 &= b_3 \end{aligned}$$

   have?

   Solution: Infinitely many. (The rank of the matrix of coefficients is 3, which is also the number of rows, thus the system can't be inconsistent. It can't have exactly one solution because the rank is smaller than the number of variables.)

3. (3 points) Let A be any $n \times n$ matrix. Is there always a sequence of elementary row operations that transforms the identity matrix I_n into A? You should motivate your answer.

   Solution: First note that if there is such a sequence then there is also a sequence of elementary row operations transforming A into the identity matrix. The answer is no, because when A is not invertible there can't be a sequence of elementary row operations that transforms it into the identity matrix (if there was such a sequence, then the identity matrix would be the RREF of A, and this can't be the case since A is not invertible).

Problem 4 (10 points in total)

1. (2 points) Write down the definition of invertible matrix.

   Solution: An $n \times n$ matrix A is invertible if it represents an invertible linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$.

2. (2 points) Give an example of a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that the matrix that represents T is not invertible.

   Solution: Let
   $$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$$
   for any vector $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in $\mathbb{R}^2$. Then
   $$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
   is not invertible.

3. (4 points) Consider the following matrix
   $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$
   Is A invertible? If yes, compute its inverse.

   Solution:
   $$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \overset{\substack{\text{II}-2\text{I} \\ \text{III}-3\text{I}}}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -5 & -2 & 1 & 0 \\ 0 & -5 & -7 & -3 & 0 & 1 \end{array}\right] \overset{\substack{(-1)\cdot\text{II} \\ \text{III}+5\text{II} \\ \text{I}-2\text{II}}}{\rightsquigarrow}$$
   $$\left[\begin{array}{ccc|ccc} 1 & 0 & -7 & -3 & 2 & 0 \\ 0 & 1 & 5 & 2 & -1 & 0 \\ 0 & 0 & 18 & 7 & -5 & 1 \end{array}\right] \overset{\substack{\frac{1}{18}\cdot\text{III} \\ \text{II}-5\cdot\text{III} \\ \text{I}+7\cdot\text{III}}}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{18} & \frac{1}{18} & \frac{7}{18} \\ 0 & 1 & 0 & \frac{1}{18} & \frac{7}{18} & -\frac{5}{18} \\ 0 & 0 & 1 & \frac{7}{18} & -\frac{5}{18} & \frac{1}{18} \end{array}\right].$$
   The matrix A is invertible because its RREF is the identity matrix. Its inverse is
   $$\begin{bmatrix} -\frac{5}{18} & \frac{1}{18} & \frac{7}{18} \\ \frac{1}{18} & \frac{7}{18} & -\frac{5}{18} \\ \frac{7}{18} & -\frac{5}{18} & \frac{1}{18} \end{bmatrix}.$$

4. (2 points) Let A be an $n \times n$ matrix. Assume that A is not invertible. How many solutions does the system $Ax = b$ have?

Solution: Either none or infinitely many.

Problem 5 (10 points total; 2 points each)

Answer the following questions with true or false.

1. Any 4×3 matrix with rank equal to 3 is invertible.

   Solution: FALSE

2. Let θ and η be any two angles with $\theta \neq \eta$. Let $T_\theta \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the counterclockwise rotation in $\mathbb{R}^2$ through θ, and $T_\eta \colon \mathbb{R}^2 \to \mathbb{R}^2$ the counterclockwise rotation in $\mathbb{R}^2$ through η. Then $T_\eta \circ T_\theta = T_\theta \circ T_\eta$.

   Solution: TRUE

3. There exists a real number a for which the following matrix is in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & a & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

   Solution: FALSE

4. The transformation $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1^2 + 2v_1 + 1 \\ v_1 + v_2 \end{bmatrix}$ is linear.

   Solution: FALSE

5. A linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if for every $v \in \mathbb{R}^n$ there exists a unique $w \in \mathbb{R}^n$ such that $T(v) = w$.

   Solution: FALSE