Final Review Notes, Physics 1B, Winter 2020

These are review notes for some of the material covered in lecture 1-20. **Caveat emptor**: this is not all the material you are responsible for (see exam information) and there could be mistakes !

1 Oscillations

• Simple harmonic oscillator equation

$$\frac{d^2}{dt^2}x(t) + \omega^2 x(t) = 0 \tag{1}$$

• Most general solution

$$x(t) = A\cos(\omega t + \phi_0), \qquad \omega = \frac{2\pi}{T} \tag{2}$$

A is amplitude, ω angular frequency, T period of oscillation. velocity and acceleration

$$v(t) = \frac{d}{dt}x(t) = -A\omega\sin(\omega t + \phi_0) \tag{3}$$
$$a(t) = \frac{d}{dt}x(t) = -A\omega^2\cos(\omega t + \phi_0) \tag{4}$$

• The amplitude A can be determined by the value of $x(t)$ and $v(t)$ at any time (as long as they are the same)

$$A = \sqrt{x(t)^2 + \frac{v(t)^2}{\omega^2}} \tag{5}$$

• Phase constant ϕ_0 is determined by the initial value of x and $v = \frac{dx}{dt}$ at $t = 0$. The formula depends on the sign of x and v

$$\begin{aligned} \phi_0 &= tan^{-1}\left(\frac{-v(t=0)}{\omega\, x(t=0)}\right), && x > 0, v > 0 \text{ and } x > 0, v < 0 \\ \phi_0 &= tan^{-1}\left(\frac{-v(t=0)}{\omega\, x(t=0)}\right) + \pi, && x < 0, v > 0 \text{ and } x < 0, v < 0 \end{aligned} \tag{6}$$

• Mass/spring system. Displacement: linear distance from equilibrium at $x = 0$.

Newtons equation:

$$m\frac{d^2}{dt^2}x + kx = 0 \tag{7}$$

Angular frequency:

$$\omega = \sqrt{\frac{k}{m}} \tag{8}$$

Kinetic and potential energy

$$K = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2, \qquad U = \frac{1}{2}kx^2 \tag{9}$$

• Torsion pendulum. Displacement angular displacement from equilibrium $\theta = 0$.

equation of motion

$$I\frac{d^2}{dt^2}\theta + \kappa\theta = 0 \tag{10}$$

Angular frequency:

$$\omega = \sqrt{\frac{\kappa}{I}} \tag{11}$$

Kinetic and potential energy

$$K = \frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2, \qquad U = \frac{1}{2}\kappa\theta^2 \tag{12}$$

• simple pendulum **small oscillations**.

Displacement angular displacement from equilibrium $\alpha = 0$. Small oscillations mean $\alpha < 25^0$.

equation of motion

$$\frac{d^2}{dt^2}\alpha + \frac{g}{l}\alpha = 0 \tag{13}$$

Angular frequency:

$$\omega = \sqrt{\frac{g}{l}} \tag{14}$$

Kinetic and potential energy

$$K = \frac{1}{2}ml^2\left(\frac{d\alpha}{dt}\right)^2, \qquad U = \frac{1}{2}mgl\alpha^2 \tag{15}$$

• The conservation of mechanical energy $E = K + U$ is often very useful in solving problems.

• Simple pendulum **large oscillations**. Note: this system is **not** a simple harmonic oscillator. Your best bet in solving problems is to use energy conservation.

equation of motion

$$ml^2\frac{d^2}{dt^2}\alpha + mgl\sin\alpha = 0 \tag{16}$$

Kinetic and potential energy

$$K = \frac{1}{2}ml^2\left(\frac{d\alpha}{dt}\right)^2, \qquad U = mgl(1 - \cos\alpha) \tag{17}$$

• Damped oscillator. Simple harmonic oscillator with a viscous friction force $F = -bv$

Newtons equation

$$\frac{d^2}{dt^2}x(t) + \frac{b}{m}\frac{d}{dt}x(t) + \frac{k}{m}x(t) = 0 \tag{18}$$

1. Under damped case $\frac{k}{m} > \frac{b^2}{4m}$, x oscillates but amplitude decays exponentially.

$$x(t) = Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi_0), \qquad \omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \qquad T = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}} \tag{19}$$

You can recognize an underdamped case by the fact that the displacement goes through $x = 0$ at least once. During one oscillation the amplitude decreases by $e^{-\frac{b}{2m}T}$ and the energy decreases by $e^{-\frac{b}{m}T}$

2. Critically damped case $\frac{k}{m} = \frac{b^2}{4m}$. The displacement decays exponentially

$$x(t) = A(1 + \frac{b}{2m}t)e^{-\frac{b}{2m}t} \tag{20}$$

The decay is faster than the overdamped case (most efficient dissipation of mechanical energy).

3. Overdamped $\frac{k}{m} < \frac{b^2}{4m}$ also decays exponentially but slower than the critically damped case with

$$x(t) = Ae^{-\left(\frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{k}}\right)t} \tag{21}$$

• <u>Forced oscillator</u>. Damped oscillator with external driving force oscillating with frequency ω.

$$\frac{d^2}{dt^2}x(t) + \frac{b}{m}\frac{d}{dt}x(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\cos(\omega t) \tag{22}$$

Solution

$$x(t) = A\cos(\omega t + \phi_0), \qquad A = \frac{F_0}{\sqrt{m^2(\omega^2 - \omega_0^2)^2 + b^2\omega^2}} \tag{23}$$

Here $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural oscillation frequency of the undamped free oscillator. Amplitude A is maximal at the resonant frequency

$$\omega = \sqrt{\omega_0^2 - \frac{b^2}{2m^2}} \tag{24}$$

Phase shift ϕ_0 is approximately zero for ω much smaller than ω_0. ϕ_0 is approximately $\frac{\pi}{2}$ for ω close to ω_0 and ϕ_0 is approximately π for ω much larger than ω_0.

2 Waves

• Mechanical waves describe the propagation of a disturbance in a medium.

• The displacement u describe the departure from an equilibrium position in the medium

• We distinguish **transverse** waves, where the displacement is orthogonal to the direction the wave is traveling and **longitudinal** waves where the displacement is in the direction the wave is traveling

• For small displacement the propagation of any mechanical wave can be mathematically modelled by the wave equation.

$$\frac{\partial^2}{\partial x^2} y(x,t) - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} y(x,t) = 0 \tag{25}$$

v is the wavespeed, the speed with which disturbances travel along the medium

• General solutions of the wave equation are rightmoving and leftmoving waves

$$y_{right}(x,t) = f(x - vt), \qquad y_{left}(x,t) = f(x + vt) \tag{26}$$

• A wavefunction $y(x,t)$ describes a snapshot at fixed time, if you fix t and view the wavefunction as a function of x.

• For a fixed x, $y(x,t)$ describes the time evolution of the displacement at the location x on the medium. The velocity and acceleration of the displacement at the x are the given by

$$v_y = \frac{\partial}{\partial t} y(x,t), \qquad a_y = \frac{\partial^2}{\partial t^2} y(x,t) \tag{27}$$

Note: Do not mix up this velocity with the wave speed v.

• Superposition principle: For two solutions $u_1(x,t), u_2(x,t)$ of the wave equation (with the same wave speed v), the sum $u_{tot} = u_1(x,t) + u_2(x,t)$ is also a solution of the wave equation. Since the displacement can be positive or negative the total displacement can be zero.

• Reflection at boundaries. We distinguish two boundary conditions for ends of the medium.

Fixed boundary conditions enforce that the displacement at the boundary is vanishing. The reflected wave is inverted $u \to -u$. For sinusoidal wave this means that the reflected wave has a phase shift of π.

Free boundary conditions enforce that the spatial derivative of displacement at the boundary is vanishing. The reflected wave is not inverted inverted. For sinusoidal wave this means that the reflected wave has no phase shift.

• Sinusoidal waves are a special class of traveling waves where the wavefunction takes the form of a cosine or sine function. The most general form of left and rightmoving waves is

$$y_{right}(x,t) = A\cos(kx - \omega t + \phi_0), \qquad y_{left}(x,t) = A\cos(kx + \omega t + \phi_0), \tag{28}$$

• Sinusoidal waves are characterized by periodicity in both space and time

$$y(x+\lambda,t)=u(x), \qquad y(x,t+T)=u(t) \tag{29}$$

the period in space is given by wavelength λ and the period in time is given by the frequency $f=1/T$ which are related to the wave number k and angular frequency ω by

$$k=\frac{2\pi}{\lambda}, \qquad \omega=2\pi f=\frac{2\pi}{T} \tag{30}$$

A traveling sinusoidal wave the displacement at any point on the medium undergoes simple harmonic oscillator motion with angular frequency ω and amplitude $\bar{u}$.

• For sinusoidal waves the wavelength and the frequency are related to the wave speed.

$$\lambda f=v \tag{31}$$

Note: traveling sinusoidal waves can have any wavelength or frequency as long as relation (31) is satisfied.

• The wave speed depends on the medium and its mechanical properties. A general conceptual formula is

$$v=\sqrt{\frac{\text{restoring force}}{\text{inertia}}} \tag{32}$$

1. Light is not a mechanical wave (it is an electromagnetic wave) the wave speed is

$$v=3\times 10^{8}\frac{m}{s} \tag{33}$$

2. For a string of **linear** mass density μ (mass per length) and tension T the wave speed is given by

$$v=\sqrt{\frac{T}{\mu}} \tag{34}$$

3. For sound waves there are two formulas

$$v=\sqrt{\frac{B}{\rho}} \tag{35}$$

Where B is the bulk modulus which relates the pressure to the change in volume of a gas or liquid $p=-BdV/V$.

$$v=\sqrt{\frac{\gamma RT}{M}} \tag{36}$$

where γ is the ratio of heat capacities ($\gamma=1.4$ for air), R is the gas constant $R=8.314J/molK$, M is the molar mass of the gas and T is the temperature (In Kelvin).

• Soundwaves are longitudinal waves where the displacement $u(x,t)$ is interpreted as the change of the (averaged) location of the air molecules from their equilibrium location. A second quantity is the change of the air pressure from its equilibrium value which is related to the displacement by

$$p(x,t) = -B\frac{\partial u(x,t)}{\partial x} \tag{37}$$

where B is the bulk modulus. This implies that the maximum value of the pressure (pressure amplitude) is given by

$$p_{max} = \bar{u}Bk \tag{38}$$

The relation (37) implies that extrema of the displacement are mapped to zeros of the pressure and vice versa.

• A traveling wave transports energy. For a transverse mechanical wave on a string with wavefunction $y(x,t) = A\cos(kx - \omega t)$ the <u>instantaneous power</u> is given by

$$P = \mu A^2\omega^2 v \sin^2(kx - \omega t) \tag{39}$$

The average power (averaged over one period T) is

$$\bar{P} = \frac{1}{2}\mu A^2\omega^2 v = \frac{1}{2}\sqrt{\mu T}A^2\omega^2 \tag{40}$$

where for the second formula we used (34) to replace v.

• A sound wave is extended in space (not limited to a one dimensional string). Instead of power the energy transport is measured by the averaged <u>intensity</u> $\bar{I}$ which is power per area (units W/m^2)

$$\bar{I} = \frac{1}{2}\sqrt{B\rho}\,\omega^2\bar{u}^2 = \frac{1}{2}\frac{p_{max}^2}{\sqrt{B\rho}} \tag{41}$$

Where p_{max} is the pressure amplitude (38).

• For point like sources (lighbulbs or speakers) the waves produced are spherical, for such waves the intensity decreases with distance r as

$$I_2 = I_1\frac{r_1^2}{r_2^2} \tag{42}$$

• The intensity of sound waves β is measured on a logarithmic scale in dB

$$\beta = 10dB\log_{10}\left(\frac{I}{I_0}\right) \tag{43}$$

where $I_0 = 10^{-12}W/m^2$ is a reference intensity (hearing threshold).

• Standing waves are superpositions of left and rightmoving waves of the same frequency and amplitude. Each point of the medium undergoes simple harmonic oscillator motion with angular frequency ω, where the amplitude depends on the location x on the medium. There are two special points:

1. Nodes are places where the amplitude is **zero**

2. Anti-nodes are places where the amplitude is **maximal**

• The distance between neighboring nodes (or neighboring anti-nodes) is $\lambda/2$. The distance between a node and an anti-node is $\lambda/4$.

• Standing waves on a medium of length L allow only for a discrete set of allowed wavelengths and frequencies (called resonances). The form of the wavefunction and values depend on the boundary conditions.

1. For a **fixed** end at $x = 0$ and a **fixed** end at $x = L$ the wavefunction is given by

$$u(x,t) = \bar{u}\sin\left(\frac{2\pi}{\lambda_n}x\right)\sin(2\pi f_n t + \phi_0) \tag{44}$$

where the allowed frequencies of the $n-th$ harmonic are

$$\lambda_n = \frac{2L}{n}, \qquad f_n = v\frac{n}{2L}, \qquad n = 1,2,3,\cdots \tag{45}$$

The n-th harmonic has n antinodes and $n-1$ nodes (not counting the nodes at the boundary)

2. For a **fixed** end at $x = 0$ and a **free** end at $x = L$ the wavefunction is given by

$$u(x,t) = \bar{u}\sin\left(\frac{2\pi}{\lambda_n}x\right)\sin(2\pi f_n t + \phi_0) \tag{46}$$

where the allowed frequencies of the $n-th$ harmonic are

$$\lambda_n = \frac{4L}{n}, \qquad f_n = v\frac{n}{4L}, \qquad n = 1,3,5,\cdots \tag{47}$$

Note that the even harmonic $n = 2,4,6,\cdots$ are missing (we follow the conventions of the textbook). The number of nodes and antinodes inside are $\frac{n-1}{2}$.

3. For a **free** end at $x = 0$ and a **fixed** end at $x = L$, the formulas for λ_n and f_n are also given by (47), the wavefunction is given by

$$u(x,t) = \bar{u}\cos\left(\frac{2\pi}{\lambda_n}x\right)\sin(2\pi f_n t + \phi_0) \tag{48}$$

4. For a **free** end at $x = 0$ and a **free** end at $x = L$ the wave function is given by

$$u(x,t) = \bar{u}\cos\left(\frac{2\pi}{\lambda_n}x\right)\sin(2\pi f_n t + \phi_0) \tag{49}$$

where the allowed frequencies of the $n-th$ harmonic are

$$\lambda_n = \frac{2L}{n}, \qquad f_n = v\frac{n}{2L}, \qquad n = 1, 2, 3, \cdots \tag{50}$$

• A general property of standing waves is that going from harmonic n to harmonic $n+1$ one adds a note and anti-node to the standing wave and that the fixed boundary corresponds to a node, whereas the free boundary corresponds to an anti-node.

• For standing sound waves **nodes of displacement** correspond to **anti-nodes of pressure** and vice versa.

• For standing sound wave in tubes or organ pipes the boundary conditions are determined as follows:

Open end: Node (fixed) for pressure, anti-node (free) for displacement

Closed end: Anti-Node (free) for pressure, node (fixed) for displacement

• Beats: If two traveling sinusoidal waves with two frequencies f_1, f_2 which are close (i.e. $|f_1 - f_2| << f_1$) in value the superposition can be interpreted as a traveling wave oscillating with the average frequency $f_{av} = \frac{1}{2}(f_1 + f_2)$ where the amplitude is modulated in time by a factor $\cos(2\pi f_{beat} t)$. The beat frequency is

$$f_{beat} = |f_1 - f_2| \tag{51}$$

• Doppler effect: The Doppler effect is the change of frequency of a sound wave due to the motion of the source and/or the listener. The following quantities are important

1. speed of sound v
2. speed of the source v_s
3. speed of the listener v_s
4. frequency of emitted wave from source (when it is at rest) f_s.

The frequency the listener L observes depends on wether she moves away/towards the source S and wether the source moves away/towards the listener, hence there are 4 cases.

Table 1: default

	L towards S	L away from S
S towards L	$f_L = \frac{v+v_L}{v-v_S} f_S$	$f_L = \frac{v-v_L}{v-v_S} f_S$
S away from L	$f_L = \frac{v+v_L}{v+v_S} f_S$	$f_L = \frac{v-v_L}{v+v_S} f_S$

One way to remember the signs is that v_L is in the numerator, v_S is denominator the signs are such that the frequency becomes larger in a towards case (+ sign for the numerator, - sign for the denominator).

3 Basic properties of charges

• There are two types of charges: positive and negative.

• Like charges (++ or - -) repel and unlike charges (+-) attract.

• The SI unit of charge is the Coulomb (C)

• The basic quantum of electric charge is that of an electron ($-1.602 \times 10^{-19} C$) and that of a proton ($+1.602 \times 10^{-19} C$).

• Charge is conserved, charges can be separated and transported, but the net amount of charge (positive - |negative|) remains the same in an isolated system.

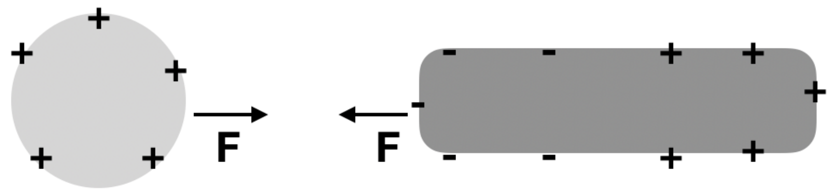
• A conductor has freely moving charges, if you put an excess charge on a conductor the repulsive force between them will make them spread out until an equilibrium is reached.

• Charge in an insulator or on its surface cannot move. Charge can be put/removed on an insulator by contact (e.g. rubbing a plastic rod with silk).

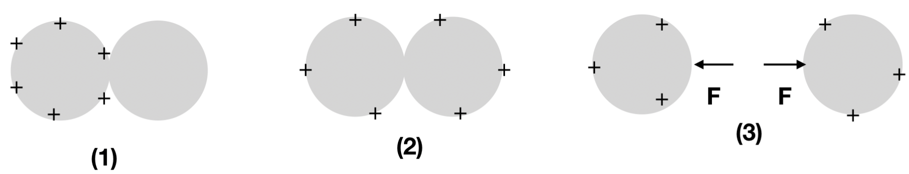
• "Ground" is an infinite reservoir of positive or negative charges which can be accessed at no energy cost.

4 Basic electrostatic phenomena

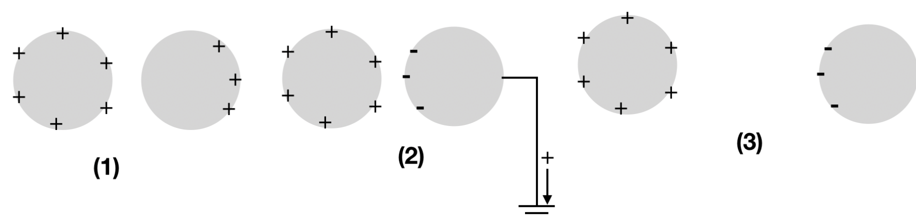
• Separation of charge in a conductor due to presence of an external charge, leads to an attractive force.



• Transfer of charge to a neutral conductor by contact, leads to a repulsive force



• Charging by induction, only works for conductors: 1) Bringing an external charge near a neutral conductor leads to a separation of charge. 2. Grounding removes one type of charge and leaves the conductor charged (3).



• The presence of an external charge nearby an insulator leads to polarization, where the "center of charge " for positive and negative charge in each individual atom is moved in opposite directions, leading to a net attractive force.

5 Coulomb's law

• Force that two point charges exert on each other.

• Magnitude F of force depends on charges q_1 and q_2 and distance r between the charges.

$$F = \frac{1}{4\pi\epsilon_0}\frac{|q_1 q_2|}{r^2} \tag{52}$$

where ϵ_0 is the dielectric constant

$$\epsilon_0 = 8.854 \times 10^{-12}\frac{C^2}{Nm^2} \tag{53}$$

Direction: pointing towards the other charge for unlike charges (attractive), pointing away from other charge for like charges (repulsive).

• Vector form: $F_{1\leftarrow 2}$ force of charged object 2 acting on charged object 1 (i.e. the force experienced by object 1). $\vec{r}_1$ location of charged object 1, $\vec{r}_2$ location of charged object 2

$$\vec{F}_{1\leftarrow 2} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2}\hat{r}_{12}, \qquad \hat{r}_{12} = \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|} \tag{54}$$

Here $\hat{r}_{12}$ is the unit vector pointing from object 2 to object 1. A simple special case is when $\vec{r}_2 = 0$, then $\hat{r}_{12} = \frac{\vec{r}_1}{|\vec{r}_1|}$ is the radial unit vector pointing from the origin to $\vec{r}_1$.
• Note that the force of 1 acting on 2 and 2 acting on one are force pairs obeying Newton's third law

$$\vec{F}_{1\leftarrow 2} = -\vec{F}_{2\leftarrow 1} \tag{55}$$

• For more than one point charge acting on a point charge q_1 the superposition principle holds: the total force acting on q_1 is the sum of all the forces

$$\vec{F}_{1,tot} = \sum_{i=2}^{n} \vec{F}_{1\leftarrow i} = \sum_{i=2}^{n} \frac{1}{4\pi\epsilon_0}\frac{q_1 q_i}{|\vec{r}_1 - \vec{r}_i|^2}\hat{r}_{1i} \tag{56}$$

• For a charge distribution with a 3dim charge density (charge per volume) $\rho(\vec{x})$ the sum over point charges turns into a integral over the charge density

$$\vec{F}_{1,tot} = \frac{1}{4\pi\epsilon_0}\int d^3x \frac{q_1\rho(\vec{x})}{|\vec{r}_1 - \vec{x}|^2}\hat{r}_{1x}, \qquad \hat{r}_{1x} = \frac{\vec{r}_1 - \vec{x}}{|\vec{r}_1 - \vec{x}|} \tag{57}$$

Note that this formula needs to be modified when the volume charge density $\rho(\vec{x})$ is replaced with a linear charge density $\mu(\vec{x})$ (charge/length, sometimes also denoted $\lambda(\vec{x})$), or a surface charge density $\sigma(\vec{x})$ (charge/area). In these cases the volume integral over d^3x is replaced by a suitable line or surface integral.

• The direction of the total force or the vanishing of the total force can often be determined by a symmetry argument. If you can perform a transformation (for example a reflection about a surface or a point, a rotation or a translation) which leaves configuration of all the charges unchanged, then the resulting force must be unchanged by the transformation too.

6 Electric field

• Introduce a (very small) test charge q_t at a location $\vec{r}$, it feels a force produced by the electric field $\vec{E}(\vec{r})$ defined by

$$\vec{F}_{q_t,tot} = q_t\vec{E}(\vec{r}), \qquad \vec{E}(\vec{r}) = \frac{\vec{F}_{q_t,tot}}{q_t} \tag{58}$$

• $\vec{E}$ is a vector field, each point in space ($\vec{r}$) is assigned a vector $\vec{E}(\vec{r})$

• Electric field produced by a point charge q at $\vec{r} = 0$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \tag{59}$$

Where $\hat{r}$ is the unit vector pointing radially outward from 0. The electric field produced by a point charge points away from a positive point charge and points towards a negative point charge.

• If you know the electric field, then the force on a test charge is given by (58).

• For several point charges the electric field is the superposition (vector sum) of the electric field produced by the individual point charges

$$\vec{E}(\vec{r}) = \sum_{i=1}^{n} \frac{1}{4\pi\epsilon_0}\frac{q_i}{|\vec{r}-\vec{r}_i|^2}\hat{r}_{ri}, \qquad \hat{r}_{ri} = \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|} \tag{60}$$

Here $\hat{r}_{ri}$ is the unit vector pointing from the location $\vec{r}$ where you want to evaluate the electric field and the location $\vec{r}_i$ of the i-th point charge.

• Superposition for a three dimensional charge density $\rho(\vec{x})$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3x \frac{\rho(\vec{x})}{|\vec{r}-\vec{x}|^2}\hat{r}_{rx}, \qquad \hat{r}_{rx} = \frac{\vec{r}-\vec{x}}{|\vec{r}_1-\vec{x}|} \tag{61}$$

As above this is replaced by an appropriate surface or line integral if ρ is replaced by a surface charge density σ or a linear charge density μ (or λ).

• Electric field from an infinite thin sheet (extended in x,y direction) with uniform charge density (charge per area) σ, at $z = 0$

$$\vec{E}(\vec{r}) = \begin{cases} \frac{\sigma}{2\epsilon_0}\hat{e}_z & z > 0 \\ -\frac{\sigma}{2\epsilon_0}\hat{e}_z & z < 0 \end{cases} \tag{62}$$

• Electric field from inside two infinite thin sheets (extended in x,y direction) with uniform charge density $+\sigma$, at $z = 0$ and $-\sigma$, at $z = d, d > 0$

$$\vec{E}(\vec{r}) = \begin{cases} 0 & z > d \\ \frac{\sigma}{\epsilon_0}\hat{e}_z & 0 < z < d \\ 0 & z < 0 \end{cases} \tag{63}$$

• Infinite line charge with linear charge density (charge per length) μ, wire is extending in the z-direction and is at $r = 0$ in cylindrical coordinates (z, r, θ)

$$\vec{E}(z, r) = \frac{1}{2\pi\epsilon_0}\frac{\mu}{r}\hat{e}_r \tag{64}$$

where $\hat{e}_r$ is the unit vector pointing radially outward in cylindrical coordinates.

• A spherically symmetric charge distribution centered around the origin with total charge q, has an electrical field (<u>outside</u> the charge distribution)

$$\vec{E}(z,r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{e}_r \tag{65}$$

Where $\hat{e}_r$ is the unit vector pointing radially outward in spherical coordinates, and r is the distance from the origin

• A efficient way to visualize electric fields is using electric field lines, they follows these rules

1. Electric field $\vec{E}$ at a point is tangent to the field line at that point, pointing in the direction the field line points.
2. The higher the density of field lines the larger the magnitude of the electric field.
3. Electric field lines originate at positive charges and terminate at negative charges.
4. In conductors carrying charge electric field lines are orthogonal to the surface. (There is no electric field, and hence no field lines, inside conductors.)
5. field lines do not cross
6. field lines do not form closed loops
7. charges test particle do not necessarily follow field lines, the direction of the field line gives the direction of the force (and hence the acceleration).

• A dipole is positive charge $+q$ separated by a distance d from a negative charge $-q$. In a constant electric field $\vec{E}$. There is no net force on the dipole but a torque on the dipole

$$\tau = q|\vec{E}|d\sin\theta \tag{66}$$

where θ is the angle between the dipole axis and the electric field vector. Defining the dipole moment as

$$\vec{p} = q\vec{d} \tag{67}$$

where $\vec{d}$ is the distance vector pointing from the negative charge to the positive charge. The torque is then given by

$$\vec{\tau} = \vec{p} \times \vec{E} \tag{68}$$

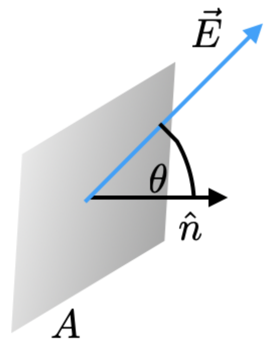
7 Electric flux and Gauss's law

• For a rectangular surface with area A, one defines the area vector $\vec{A} = A\hat{n}$ where $\hat{n}$ is the unit normal vector (a vector with unit length and orthogonal to the surface).

• For a constant electric field $\vec{E}$ and a rectangular surface A, the flux of the electric field through the surface A is given by

$$\Phi = \vec{A}\cdot\vec{E} = |\vec{A}||\vec{E}|\cos\theta \tag{69}$$

where θ is the angle between the normal vector $\hat{n}$ and the electric field $\vec{E}$.



• For an infinitesimal area dA the infinitesimal flux $d\Phi$ through the surface is defined the same way

$$d\Phi = d\vec{A} \cdot \vec{E} = |dA||\vec{E}| \cos\theta \tag{70}$$

• For a curved surface or non-constant electric field one sums up infinitesmal flux by a surface integral

$$\Phi = \oint_S d\vec{A} \cdot \vec{E} \tag{71}$$

• For Gauss's law we are particularly interested in closed surfaces, which are surfaces which don't have any holes or boundaries. For such a surface there is an obvious notion of inside and outside. The normal vector $\hat{n}$ to the surface is the one which points outward.

• Using flux lines we can get an intuitive understanding of electric flux: Count a fluxlines which pierces the closed surface and goes out of the surface as $+1$ and count a flux line which pierces the closed surface and goes into the surface as -1.

• Fluxes are additive, say a closed surface S is made by combining two surfaces S_1 and S_2 (for example make a sphere by putting together the northern and southern hemisphere). Then the we have

$$\Phi_s = \Phi_1 + \Phi_2 \tag{72}$$

• Gauss's law relates the electric flux through a surface S to the enclosed charge q_{encl} inside S

$$\Phi_S = \frac{q_{encl}}{\epsilon_0} \tag{73}$$

• It does not matter how the charge is distributed inside S, only the total enclosed charge matters

• Gauss's law is very often a good way to determine the electric field following the steps, especially is symmetric configurations

1. Use the symmetry to determine the electric field pattern (which way is $\vec{E}$ pointing)

2. Choose a closed surface (called Gaussian surface), a good choice is usually informed by symmetry, especially if $d\vec{A}$ and $\vec{E}$ are either parallel or orthogonal.
3. Work out the flux integral $\Phi = \oint_S d\vec{A} \cdot \vec{E}$
4. work out the enclosed charge q_{encl}
5. determine E using Gauss's law

• Good choices for Gaussian surfaces:

1. spherical symmetric charge distribution: concentric spherical surfaces
2. infinite line charge, or charge distribution with cylindrical symmetry: concentric cylinder
3. infinite sheet with constant charge density: rectangular prism parallel to surface.

• In electrostatics we define electrostatic equilibrium as a situation where no charge moves (in the absence of batteries or generators etc. a charge configuration will settle into this state)
• Consequences of Gauss's law for conductors (in electrostatic equilibrium)

1. Inside a conductor the electric field vanishes everywhere.
2. All the excess charge (charge in excess of neutral) resides on the surface of a conductor
3. The electric field on the surface of a conductor is orthogonal to the surface

8 Electrostatic potential

• The electrostatic potential V of a charge distribution is defined in analogy with the electric field: For a very small test charge q_t define the potential energy $U(\vec{x})$ as the work it takes to move the test charge from infinity to $\vec{x}$. (Recall $U > 0$ is you have to do positive work on the system, and $U < 0$ if the work is negative). The the electrostatic potential $V(\vec{x})$ is then defined by

$$V(\vec{x}) = \frac{U(\vec{x})}{q_t} \tag{74}$$

• The unit of electrostatic potential is $Volt = Joule/Coulomb$.

• Given V the potential energy of a test charge is given by $U(\vec{x}) = q_t V(\vec{x})$. In particular this potential energy can be used to solve motion problem of test charges using energy conservation

$$U_i + K_i + U_{i,other} = U_f + K_f + U_{f,other} \tag{75}$$

Here $U_{i,f}$ is the electrostatic potential energy at the initial and final location and $U_{i,f,other}$ are other potential energies (such as spring or gravitational potential energies).

• The electrostatic potential of a point charge q located at the origin is

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \tag{76}$$

where $r = |\vec{r}|$ is the distance to the origin

• The electrostatic potential for several point charges q_i located at $\vec{r}_i$ is given by the superposition principle

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\vec{r} - \vec{r}_i|} \tag{77}$$

• For a charge distribution with charge density $\rho(x)$ the potential is given

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3x \frac{\rho(\vec{x})}{|\vec{r} - \vec{x}|} \tag{78}$$

As for electric fields, this needs to be changed to a surface or line integral over any surface or linear charge distributions that may be present (i.e. $\rho(\vec{x})d^3x \to \sigma(\vec{x})dA$ or $\mu(\vec{x})d\ell$.)

• If the charge distribution goes all the way to infinity (infinite line charge, infinite sheet) the integral (78) can be divergent, but the divergence will only be in an overall constant (which is immaterial in the definition of V).

• Potential in between infinitely large charged sheets at $z = 0$ with charge density $-\sigma$ and $z = d$ with charge density $+\sigma$.

$$V(z) = \begin{cases} \frac{\sigma}{\epsilon_0}d & z > d \\ \frac{\sigma}{\epsilon_0}z & 0 < z < d \\ 0 & z < 0 \end{cases} \tag{79}$$

• Potential of uniform spherical shell of radius R and total charge q

$$V(r) = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{q}{r} & r > R \\ \frac{1}{4\pi\epsilon_0}\frac{q}{R} & r \leq R \end{cases} \tag{80}$$

• Potential difference of a infinite line charge along the z direction with charge density μ

$$V(r_b) - V(r_a) = \frac{\mu}{2\pi\epsilon_0}\ln\left(\frac{r_a}{r_b}\right) \tag{81}$$

• Note that the superposition principle means that if the charge distribution is build up from several simple ones the total potential is the sum of the potentials from the individual charge distributions calculated as if the other charge distributions were not there.

• In order to check signs it is useful to use a rule of thumb: roughly speaking, the electrostatic potential decreases in the direction of the field lines (i.e. in the direction the electric field is pointing). Note that "decrease" includes signs, i.e. -100 is smaller than +10 !

• A second way to obtain the electrostatic potential is by a line integral.

$$V(\vec{r}_b) - V(\vec{r}_a) = -\int_{C_{a\to b}} \vec{E} \cdot d\vec{x} \tag{82}$$

Here $C_{a\to b}$ is a path in space going from $\vec{r}_a$ to $\vec{r}_b$.

• The potential difference (82) does not depend on the choice of path. Often it's possible to choose a path which makes the calculation easy (For example the electric field can be calculated using Gauss's law or path goes through a conductor and henc $\vec{E}$ is zero everywhere along the path).

• The line integral of the electric field along a closed loop C vanishes (the electrostatic force is conservative).

$$\oint_C \vec{E} \cdot d\vec{x} = 0 \tag{83}$$

• In a conductor in electrostatic equilibrium all points have the same value of the potential.

• A way to visualize the potential is to use equipotential surfaces (or equipotential lines in two dimensional diagrams).

1. All points on an equipotential surfaces have the same potential
2. The denser equipotential surface lie together the larger the change of the potential over a distance.
3. equipotential surfaces are orthogonal to electric field lines
4. the surfaces of conductors in electrostatic equilibrium are equipotential surfaces
5. the electric field points towards equipotentials which decrease.

• One can obtain the electric field $\vec{E}$ from the potential V by taking the gradient

$$\vec{E} = -\vec{\nabla} V \tag{84}$$

or in components

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z} \tag{85}$$

Useful formulas when evaluating gradients. For $r = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}, \tag{86}$$

• The potential energy of a charge configuration is defined as follows. Bring the first charge to its location from infinity: that does not produce any potential energy. Then bring the second charge in from infinity in the field of the first charge and calculate the potential energy, the bring in the third charge in from infinity in the field of the two charges and calculate the potential energy etc, at the end add all the potential energies. The final result for n charges q_i is

$$U_{tot} = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} \tag{87}$$

9 Capacitance

• Two conductors separated by an insulator form a capacitor. The charge stored in the capacitor is related to the potential difference by

$$Q = CV_{ab} \tag{88}$$

The unit of capacitance is Farad or Coulomb/Volt.

• Formulas for capacitance

1. Plate capacitor: with Area A, distance d (making the simplifying assumption that the field between the capacitor is given by an infinite place capacitor)

$$C = \epsilon_0 \frac{A}{d} \tag{89}$$

2. spherical capacitor: inner radius r_a and outer radius r_b.

$$C = 4\pi\epsilon_0 \frac{r_a r_b}{r_b - r_a} \tag{90}$$

3. cylindrical capacitor: length L, inner radius r_a and outer radius r_b

$$C = 2\pi\epsilon_0 \frac{L}{\ln\left(\frac{r_b}{r_a}\right)} \tag{91}$$

Capacitance depends on the geometry of the capacitor, it increases with the size and if the plates of the conductors are closer together.

• Series and parallel capacitors

1. for capacitors in series the stored charge is the same for each capacitor. The equivalent capacity for two capacitors in series

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \tag{92}$$

2. for capacitors in parallel the potential difference is the same for each capacitor. The equivalent capacity for two capacitors in series

$$C_{eq} = C_1 + C_2 \tag{93}$$

3. More complicated networks of capacitors can be reduced to combinations of capacitors step by step.

• Energy stored in a capacitor have 3 equivalent forms

$$U = \frac{Q^2}{2C} = \frac{CV^2}{2} = \frac{QV}{2} \tag{94}$$

Which formula to use is determined by which quantity is known.

• Dielectrics: If you insert a dielectric with dielectric constant K in a capacitor the electric field inside the dielectric gets smaller

$$\vec{E}_{diel} = \frac{1}{K}\vec{E}_{vacuum} \tag{95}$$

The capacitance of the capacitor

$$C_{diel} = KC_{vacuum} \tag{96}$$

This effect is due to the creation of a surface charge in the dielectric due to polarization. The energy stored in a capacitor changes, but one has to distinguish the case of fixed potential difference (i.e. C is connected to a battery) or fixed charge (C is charged and then disconnected)

$$\begin{aligned} \text{fixed charge}: \quad & U_{diel} = \frac{1}{K}U_{vacuum} \\ \text{fixed voltage}: \quad & U_{diel} = KU_{vacuum} \end{aligned} \tag{97}$$

10 Resistance, emf

• Current is the amount of charge flowing through a wire per unit time

$$I = \frac{dQ}{dt} \qquad \text{unit Ampere} = \left[\frac{C}{s}\right] \tag{98}$$

• Whereas the current I is the net amount of charge, the current density $\vec{J}$ has the magnitude

$$|\vec{J}| = \frac{I}{A} \tag{99}$$

where A is the cross section through which the current is flowing and the direction is the direction in which the conventional (i.e. positive) charge is flowing. The current through an area A with area vector $\vec{A}$ (defined the same way as for the electric flux) is

$$I = \vec{J} \cdot \vec{A} \tag{100}$$

In most circumstances (wires) all vectors are aligned and (99) suffices.

• Relation of J to drift velocity $\vec{v}_d$

$$\vec{J} = nq\vec{v}_d \tag{101}$$

where q is the charge of the charge carrier and n is the number density of charge carrier.

• "microscopic" Ohm's law

$$\vec{J} = \sigma \vec{E} \tag{102}$$

σ is the conductivity, the resistance $\rho = 1/\sigma$. Both are intrinsic qualities of the material of the conductor and the temperature but not the geometry of the conductor.

• Resistance R for resistor length l, cross sectional area A

$$R = \frac{\rho l}{A} \tag{103}$$

• Ohm's law, related resistance to current I lowing through resistor and voltage drop across resistor V_{ab}.

$$V_{ab} = IR \tag{104}$$

11 DC circuits

• equivalent resistance for two resistors in series

$$R_{eq} = R_1 + R_2 \tag{105}$$

• equivalent resistance for two resistors in parallel

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \tag{106}$$

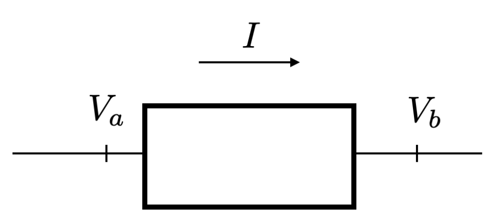
• An ideal battery supplies the voltage difference (or emf) ϵ no matter how much current goes through it.

• A real battery has an internal resistance and the voltage difference (terminal voltage) V_{ab} depends on the current

$$V_{ab} = \epsilon - Ir \tag{107}$$

where ϵ is the voltage difference at zero load ($I = 0$).

• The power (energy/time) a circuit element takes out/puts into the circuit depends on the voltage difference and the current going through it.



$$P = (V_a - V_b)I \tag{108}$$

The sign conventions are as follows: If $V_a > V_b$ the power is positive and this means that energy is taken out of the circuit to heat up the resistor, light a lightbulb, drive a motor etc. If $V_a < V_b$ the power is negative and this means energy is put into the circuit. This is the case for a battery, power supply or generator. (The power on a resistor is always positive, a resistor never puts energy into the circuit.)

• Using Ohm's law the formula for the power can be rewritten in other forms, which are useful to determine how the power depends on resistance (when current or voltage drop is held constant)

$$P = I^2 R, \qquad P = \frac{V_{ab}^2}{R} \tag{109}$$

• Kirchhoff's rules are used to determine currents and voltage drops in DC circuits.

Junction rule: At a junction (three or more wires connected at a point) the sum of all currents is zero.

$$\sum_i I_i = 0 \tag{110}$$

Loop rule: Along a closed loop in the circuit, the sum of all the voltage drops is zero

$$\sum_i \Delta V_i = 0 \tag{111}$$

• Strategy in analyzing curcuits using Kirchhoff's rules

1. label all the circuit elements, conductors, batteries, resistors, capacitors and junctions.
2. if there are combinations of resistors and capacitors which can be replaced by equivalent resistors replace them to simplify the circuit.
3. along any conductor draw a current I_i with an arrow denoting the direction (this does not need to be the real direction, if the current flows the other way the math will give you a negative I).
4. along a conductor and through resistors and batteries, the current is everywhere the same if there are no junctions.
5. ideal batteries have voltage drops ϵ, real batteries can be represented by an ideal battery with emf ϵ in series with a resitor with resistance r.
6. Use Kirchhoff's junction law to reduce the number of independent currents, remember I's going into the junction count postive, I's going out count negative
7. label the voltage differences across resistors and batteries, use Ohm's law to relate voltage differences to currents (sign depends on orientation of current).

8. apply Krichhoff's loop law for a minimal set of loops (as many loops as the curcuit has faces, the loops cover each part of the circuit at least once). This gives you the equations which will determine the unknowns.

9. If we consider the circuit after capacitors are fully charged no current flows through a capacitor and the charge in the capacitor is determined by the voltage difference at the ends of the capacitor, this is determined by analyzing the curcuit with the capacitor removed.

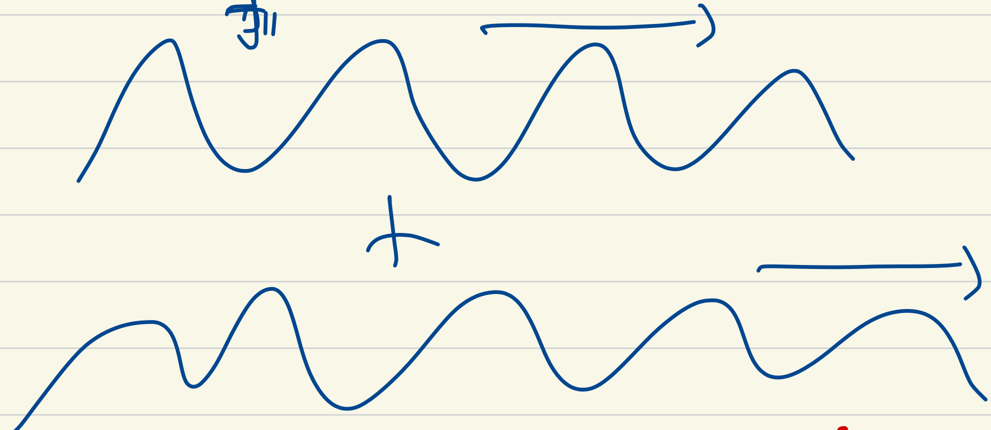
1B Review Session

- doppler, beats
- capacitance
- resistance / ohms
- circuits

Beats:

$f_1 \sim f_2$

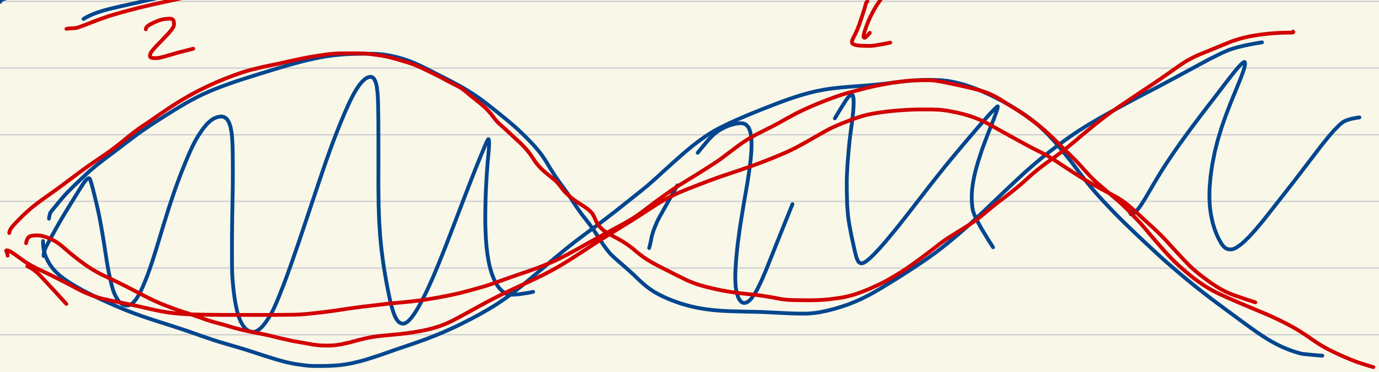
f_1



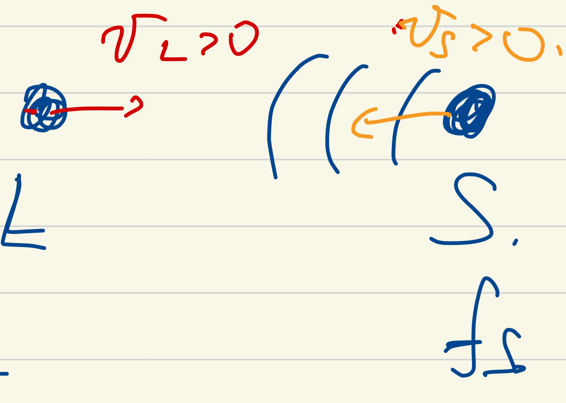
f_2

$f_{beat} = |f_1 - f_2|$

$f_{av} = \frac{f_1 + f_2}{2}$



Doppler Effect:



$$f_L = \left(\frac{v + v_L}{v - v_S}\right) f_S.$$

$$f_L = \left(\frac{1 + v_L/v}{1 - v_S/v}\right) f_S$$

V = Speed of sound in air

$1 + v_L/v > 1.$

$1 - v_S/v < 1.$

---

ex: Bat $\longrightarrow$ v_B.

emitted by bat $\rightarrow$ $f = 1.7$ kHz

f_L

f_L.

heard by bat $\rightarrow$ f'

L. $\quad v_L = 0.$

$$f_L = \underbrace{\frac{v}{v - v_B}}_{\frac{1}{1 - V_B/V}} \cdot f.$$

$$f_{beat} = |f' - f| = 8 \text{ Hz}.$$

Bat $\quad V_B$ = (Listener)'

$f_L = f_S$ wall source

$$f' = \left(\frac{V + V_B}{V}\right) \cdot f_L = \left[(V + V_B)/_{V - V_B}\right] f$$

(3.)

$$f' = \underbrace{\frac{v+v_B}{v-v_B}}_{\frac{1+v_B/v}{1-v_B/v}} \underbrace{f}_{1.7\text{ kHz}} > f.$$

$$\underset{f+8\,Hz}{\overset{\shortparallel}{}}$$

$$\underbrace{\frac{1+v_B/v}{1-v_B/v}}_{\substack{v \\ 7.}}$$ ← Taylor expand

$$0 < v_B < v$$

$\underline{v_B \ll v}$

$$\approx \left(1+v_B/v\right)\left(1+v_B/v\right)$$

$$\approx 1+\frac{2v_B}{v} + \cdots$$

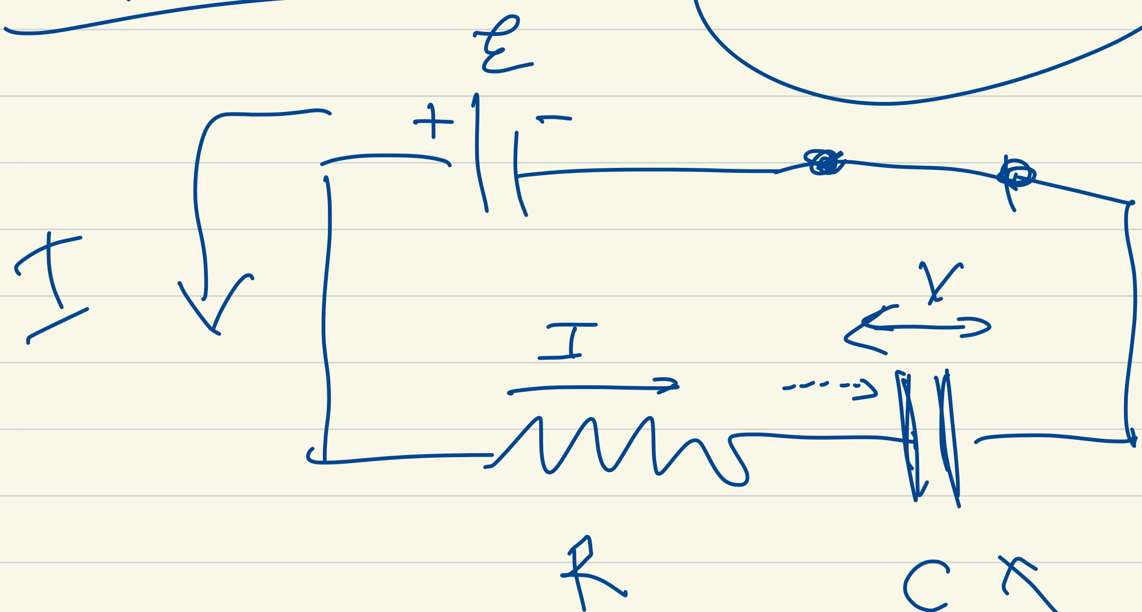
$$\underbrace{f'}_{\left(1+\frac{2v_B}{v}\right)\cdot f} = f + 8\,Hz$$

$$\Rightarrow \quad \frac{\cancel{2}v_B}{v} = \frac{8\,Hz}{f} = \frac{\cancel{8}^{\,4}}{1700}$$

$$v_B = \underbrace{v}_{340\frac{m}{s}} \cdot \underbrace{\frac{4}{1700}}_{\approx\frac{1}{400}} \sim 0.8\ m/s.$$

Capacitance:

$Q = CV$



$-IR$.

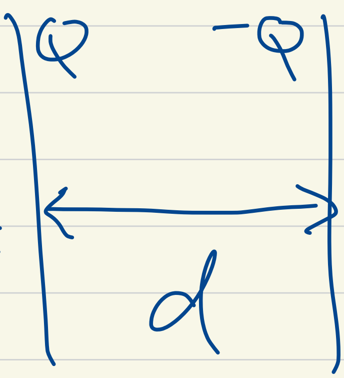
$Q(0) = 0$.

$t = 0$

$t \gg RC$, $V = \mathcal{E}$. $Q = C \cdot \mathcal{E}$.

$I = 0$

$u \sim 1/d \, V^2$

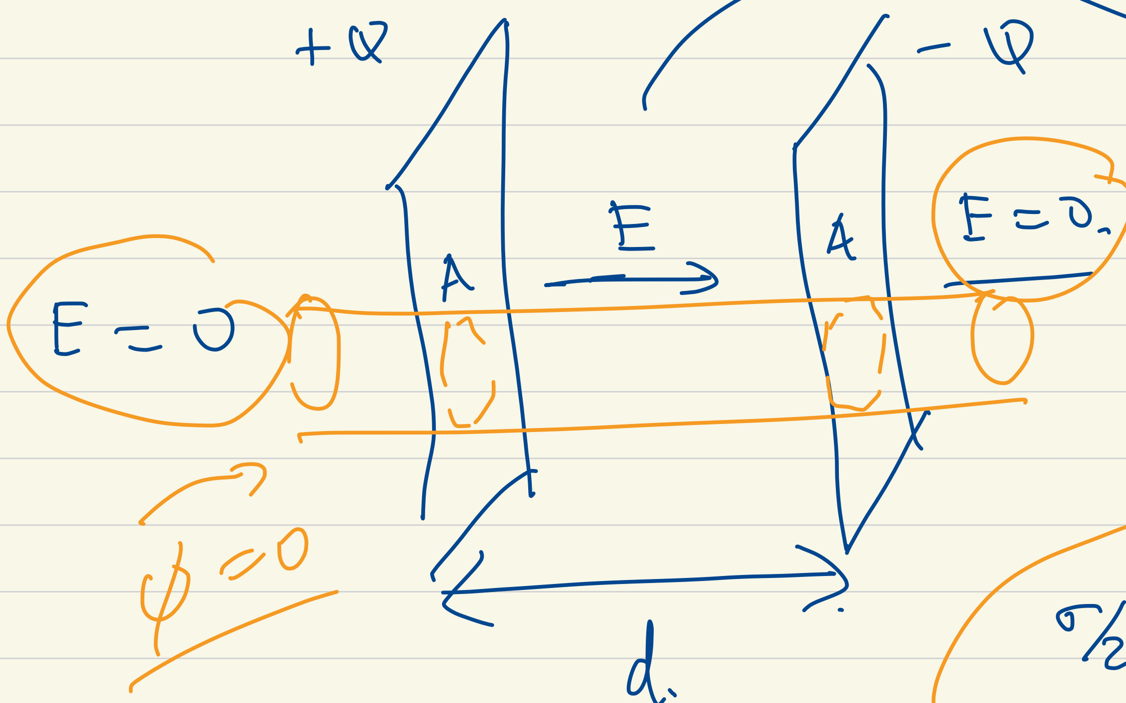


$C = \epsilon_0 \frac{A}{d}$

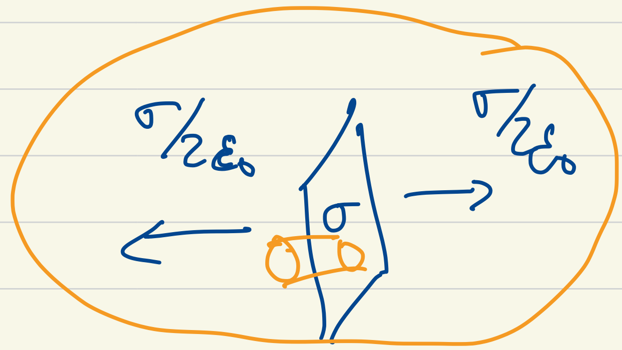
$U = \frac{1}{2} C \cdot V^2 = \frac{Q^2}{2C}$

$C \sim 1/d$.

$u \sim d \cdot Q^2$



$$E = \sigma/\varepsilon_0.$$



$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 \cdot A} = \text{const.}$$

$$\sigma = \frac{Q}{A}$$

$$V = \underline{E \cdot d} = \frac{Qd}{\varepsilon_0 \cdot A}.$$

$$\int \vec{E}' \cdot d\vec{x}'$$

$$C = Q/V = \frac{\cancel{Q}}{\cancel{Q}d/\varepsilon_0 A} = \varepsilon_0 \frac{A}{d}$$

6.

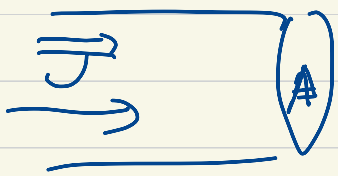
ohm

$$\vec{J} = \sigma \vec{E}$$

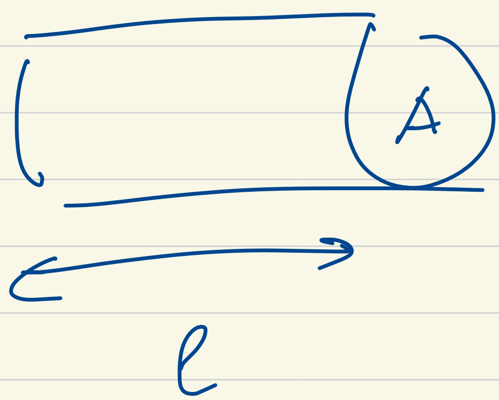
$$\sigma = \frac{1}{\rho}$$

conductivity

resistivity.



$I = J \cdot A$.



$$R = \rho \cdot \frac{l}{A}$$

Ohm:

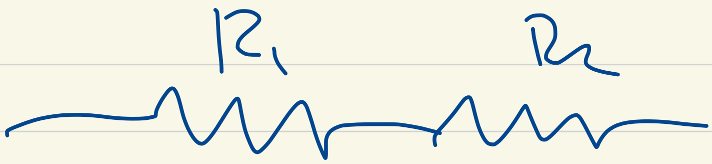
$$V = IR$$

$$I = \frac{V}{R}$$

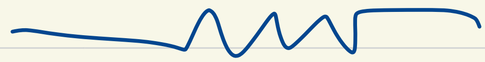
$$J = \underbrace{\frac{1}{\rho}}_{\sigma} \underbrace{\left(\frac{V}{l}\right)}_{E}$$

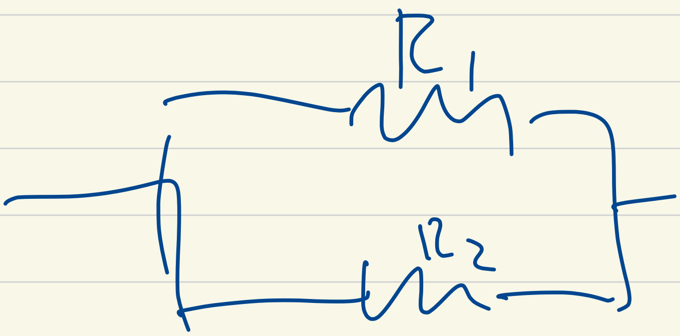
$$\underbrace{\frac{I}{A}}_{J} = \frac{V}{\underbrace{A \cdot R}_{\rho l}}$$

7.



$\longrightarrow$ $R_{eq} = R_1 + R_2$



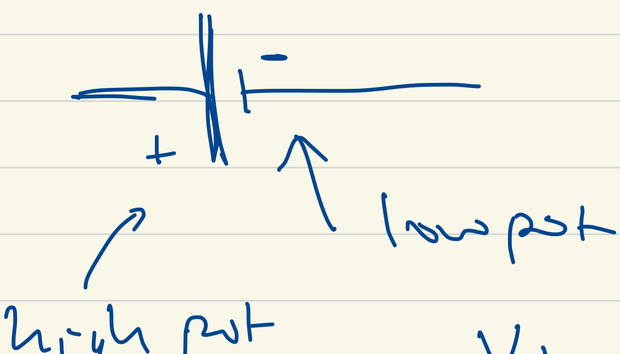


$\longrightarrow$

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

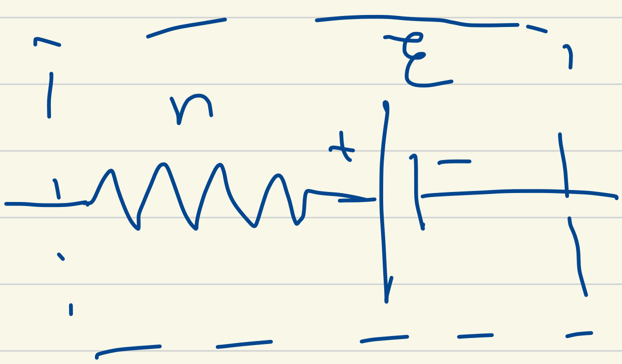
$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

Batteries:



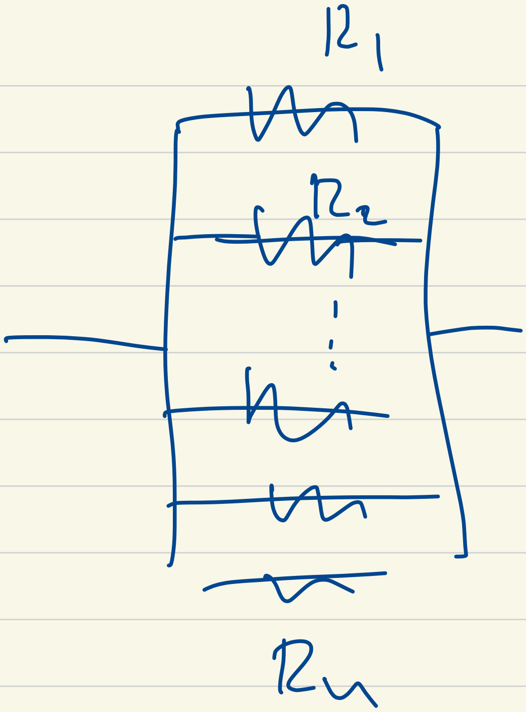
$V_{high} - V_{low} = \underbrace{e.m.f}_{\mathcal{E}}$

"ideal battery"

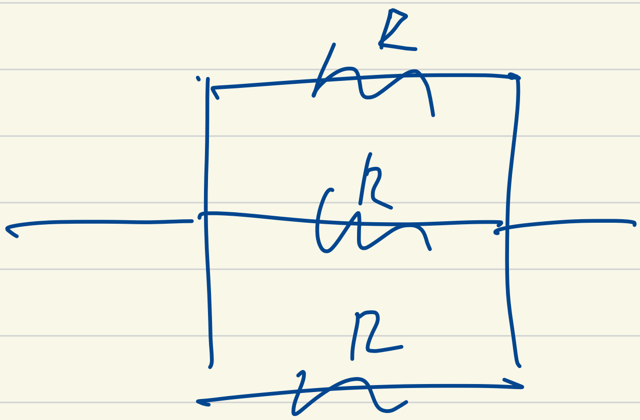


real battery w/ internal resistance.

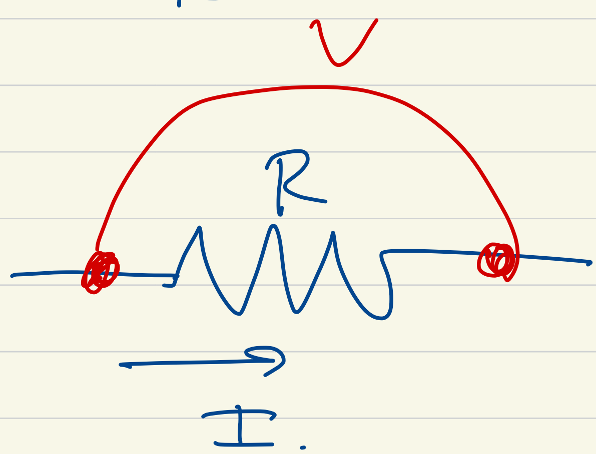
8



$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \cdots + \frac{1}{R_n}$$



$$\Rightarrow \frac{1}{R_{eq}} = \frac{3}{R} \Rightarrow R_{eq} = \frac{R}{3}$$

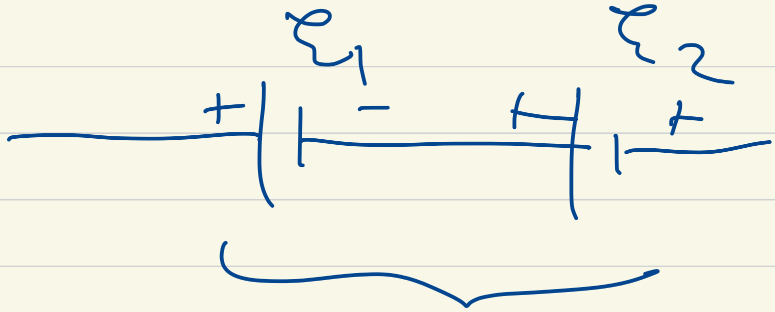


Power dissipated

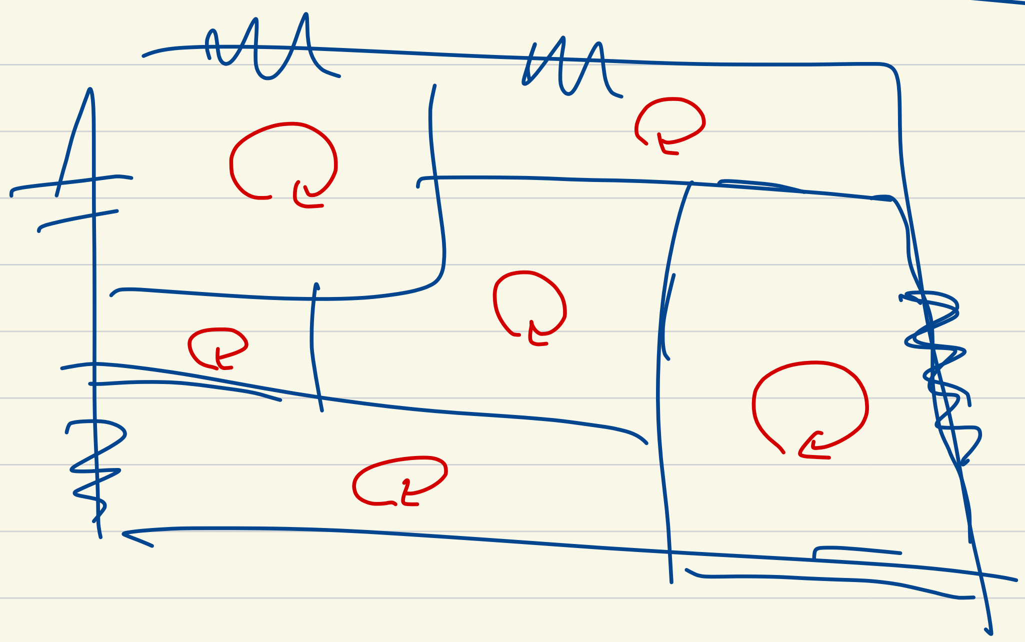
$$P_{dissipated} = I \cdot \underbrace{V}_{I \cdot R}$$

$$= I^2 R = V^2/R.$$

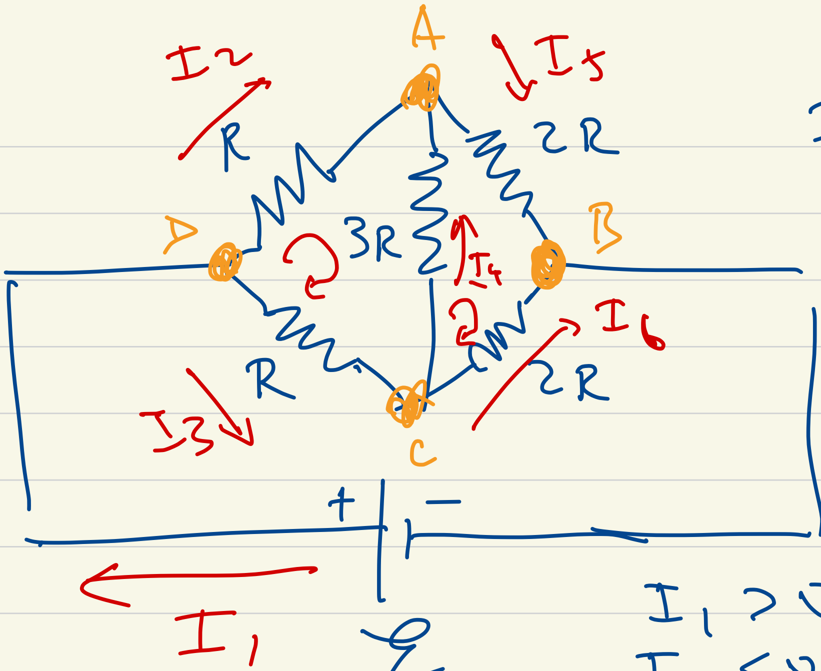
9.



$\mathcal{E}_1 + \mathcal{E}_2$



10.



$I_2 = I_3$

$I_5 = I_6$.

$I_4 = 0$

$I_1 > 0$ current ←

$I_1 < 0$ current →

What is the current in resistor 3R?

A: zero Prove it!

(A) $$\sum_{\text{incoming}} I_i = \sum_{\text{outgoing}} I_j$$

$$\sum_{\text{inc.}} I_i - \sum_{\text{out}} I_j = 0.$$

in: I_2, I_4 out I_5

(A) $\boxed{I_2 + I_4 = I_5}$

(B) $\boxed{I_5 + I_6 = I_1}$

(C) $I_3 = I_4 + I_6$

(D) $I_1 = I_2 + I_3$

$I_1, I_2, I_3, I_4, I_5, I_6 \rightarrow$ 6 unk.

11.

A $\boxed{I_2 + I_4 = I_5}$

B $\boxed{I_5 + I_6 = I_1}$

C $I_3 = I_4 + I_6$

D $I_1 = I_2 + I_3$

known

unknown.

$I_5 = I_2 + I_4$

$I_3 = I_4 + I_6$

known

$I_1 = I_2 + I_3$

$= I_2 + I_4 + I_6$

B $\underbrace{I_5}_{I_2 + I_4} + I_6 = \underbrace{I_1}_{I_2 + I_4 + I_6} \Longrightarrow 0 = 0$

Redundant

I_2, I_4, I_6 still unknown

$-\cancel{R} I_2 + 3\cancel{R} I_4 + I_3 \cancel{R} = 0$

$\boxed{-I_2 + 4 \cdot I_4 + I_6 = 0}$

$$-3\cancel{R}\cdot I_4 - 2\cancel{R}\, I_5 + 2\cancel{R}\cdot I_6 = 0$$

$$I_5 = I_2 + I_4$$

$$-3I_4 - 2\cdot I_4 - 2\cdot I_2 + 2\cdot I_6 = 0$$

$$\boxed{-5\cdot I_4 - 2\cdot I_2 + 2I_6 = 0}$$

$$+\frac{\mathcal{E}}{R} - I_3\cancel{R} - 2\cancel{R}I_6 = 0$$

$$I_3 = I_4 + I_6$$

$$\mathcal{E}/R = I_4 + \underbrace{I_6 + 2I_6}_{3I_6.}$$

$$\boxed{3I_6 + I_4 = \mathcal{E}/R}$$

$$I_2 = 4 \cdot I_4 + I_6$$

$$-5 \cdot I_4 - 2 \cdot I_2 + 2 I_6 = 0$$

$$\Rightarrow -5 I_4 - 8 I_4 - \cancel{2 I_6} + \cancel{2 I_6} = 0$$

$$-13 \cdot I_4 = 0$$

$$\Rightarrow I_4 = 0$$

$$\Rightarrow \boxed{\begin{array}{l} I_2 = I_6 = \dfrac{\varepsilon}{3R} \\ \parallel \\ I_3 = I_5 \end{array}}$$