Midterm 1 Review notes, Physics 1B, Winter 2020

These are review notes for some of the material covered in lecture 1-7. **Caveat emptor**: this is not all the material you are responsible for (see exam information) and there could be mistakes !

1 Oscillations

• Simple harmonic oscillator equation

$$\frac{d^2}{dt^2}x(t) + \omega^2 x(t) = 0 \tag{1}$$

• Most general solution

$$x(t) = A\cos(\omega t + \phi_0), \qquad \omega = \frac{2\pi}{T} \tag{2}$$

$A > 0$ is amplitude, ω angular frequency, T period of oscillation. velocity and acceleration

$$v(t) = \frac{d}{dt}x(t) = -A\omega\sin(\omega t + \phi_0) \tag{3}$$

$$a(t) = \frac{d}{dt}x(t) = -A\omega^2\cos(\omega t + \phi_0) \tag{4}$$

The phase constant ϕ_0 can be restricted to an interval of length 2π. By convention, we choose $-\pi < \phi_0 \le \pi$.

• The amplitude A can be determined by the value of $x(t)$ and $v(t)$ at any time (as long as both are taken to be at the same time, c.f. conservation of energy)

$$A = \sqrt{x(t)^2 + \frac{v(t)^2}{\omega^2}} \tag{5}$$

• Phase constant ϕ_0 is determined by the initial value of x and $v = \frac{dx}{dt}$ at $t = 0$, called x_0 and v_0. Setting $t = 0$ in the equations above gives

$$x_0 = A\cos\phi_0 \ , \qquad v_0 = -A\omega\sin\phi_0 \ . \tag{6}$$

Checking that these equations are satisfied always gives the right answer. They can be solved explicitly, though it is important to pay attention to the signs of x_0 and v_0 (recall that $\tan^{-1}(y) \in (-\pi/2, \pi/2)$),

$$\begin{aligned}
\phi_0 &= \tan^{-1}\left(\frac{-v_0}{\omega\ x_0}\right), && x_0 > 0 \\
\phi_0 &= \tan^{-1}\left(\frac{-v_0}{\omega\ x_0}\right) + \pi, && x_0 < 0, v_0 < 0 \\
\phi_0 &= \tan^{-1}\left(\frac{-v_0}{\omega\ x_0}\right) - \pi, && x_0 < 0, v_0 > 0 \ .
\end{aligned}$$

• Mass/spring system. Displacement: linear distance from equilibrium at $x = 0$.

Newtons equation:

$$m\frac{d^2}{dt^2}x + kx = 0 \tag{7}$$

Angular frequency:

$$\omega = \sqrt{\frac{k}{m}} \tag{8}$$

Kinetic and potential energy

$$K = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2, \qquad U = \frac{1}{2}kx^2 \tag{9}$$

• Torsion pendulum. Displacement angular displacement from equilibrium $\theta = 0$.

equation of motion

$$I\frac{d^2}{dt^2}\theta + \kappa\theta = 0 \tag{10}$$

Angular frequency:

$$\omega = \sqrt{\frac{\kappa}{I}} \tag{11}$$

Kinetic and potential energy

$$K = \frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2, \qquad U = \frac{1}{2}\kappa\theta^2 \tag{12}$$

• simple pendulum **small oscillations**.

Displacement angular displacement from equilibrium $\alpha = 0$. Small oscillations mean $\alpha < 25^0$.

equation of motion

$$\frac{d^2}{dt^2}\alpha + \frac{g}{l}\alpha = 0 \tag{13}$$

Angular frequency:

$$\omega = \sqrt{\frac{g}{l}} \tag{14}$$

Kinetic and potential energy

$$K = \frac{1}{2}ml^2\left(\frac{d\alpha}{dt}\right)^2, \qquad U = \frac{1}{2}mgl\alpha^2 \tag{15}$$

• The conservation of mechanical energy $E = K + U$ is often very useful in solving problems.

• Simple pendulum **large oscillations**. Note: this system is **not** a simple harmonic oscillator. Your best bet in solving problems is to use energy conservation.

equation of motion

$$ml^2\frac{d^2}{dt^2}\alpha + mgl\sin\alpha = 0 \tag{16}$$

Kinetic and potential energy

$$K = \frac{1}{2} m l^2 \left(\frac{d\alpha}{dt}\right)^2, \qquad U = mgl(1 - \cos\alpha) \tag{17}$$

- Damped oscillator. Simple harmonic oscillator with a viscous friction force $F = -bv$
  Newtons equation

$$\frac{d^2}{dt^2}x(t) + \frac{b}{m}\frac{d}{dt}x(t) + \frac{k}{m}x(t) = 0 \tag{18}$$

1. Under damped case $\frac{k}{m} > \frac{b^2}{4m}$, x oscillates but amplitude decays exponentially.

$$x(t) = Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi_0), \qquad \omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \qquad T = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}} \tag{19}$$

You can recognize an underdamped case by the fact that the displacement goes through $x = 0$ at least once (and in fact many times). During one oscillation the amplitude decreases by $e^{-\frac{b}{2m}T}$ and the energy decreases by $e^{-\frac{b}{m}T}$ (since it is quadratic in the amplitude, see above).

2. Critically damped case $\frac{k}{m} = \frac{b^2}{4m}$. The displacement decays exponentially

$$x(t) = A(1 + \frac{b}{2m}t)e^{-\frac{b}{2m}t} \tag{20}$$

This formula is only valid for releasing the system from rest (i.e. $v_0 = 0$), check this explicitly. The decay is faster than the overdamped case (most efficient dissipation of mechanical energy).

3. Overdamped $\frac{k}{m} < \frac{b^2}{4m}$ also decays exponentially but slower than the critically damped case with

$$x(t) = Ae^{-\left(\frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}\right)t} \tag{21}$$

The general solution for the overdamped case is more complicated than this one – it also contains another exponential with a + in front of the square root. However, this other exponential decays to zero more quickly, so the one written above is the generic way in which an overdamped system decays to zero.

- Forced oscillator. Damped oscillator with external driving force oscillating with frequency ω.

$$\frac{d^2}{dt^2}x(t) + \frac{b}{m}\frac{d}{dt}x(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\cos(\omega t) \tag{22}$$

Solution

$$x(t) = A\cos(\omega t + \phi_0), \qquad A = \frac{F_0}{\sqrt{m^2(\omega^2 - \omega_0^2)^2 + b^2\omega^2}} \tag{23}$$

Here $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural oscillation frequency of the undamped free oscillator. Amplitude A is maximal at the resonant frequency

$$\omega = \sqrt{\omega_0^2 - \frac{b^2}{2m^2}} \tag{24}$$

Phase shift ϕ_0 is approximately zero for ω much smaller than ω_0. ϕ_0 is approximately $\frac{\pi}{2}$ for ω close to ω_0 and ϕ_0 is approximately π for ω much larger than ω_0.

2 Waves

• Mechanical waves describe the propagation of a disturbance in a medium.

• The displacement u describe the departure from an equilibrium position in the medium

• We distinguish **transverse** waves, where the displacement is orthogonal to the direction the wave is traveling and **longitudinal** waves where the displacement is in the direction the wave is traveling

• For small displacement the propagation of any mechanical wave can be mathematically modelled by the wave equation.

$$\frac{\partial^2}{\partial x^2}y(x,t) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}y(x,t) = 0 \tag{25}$$

v is the wavespeed, the speed with which disturbances travel along the medium

• General solutions of the wave equation are rightmoving and leftmoving waves

$$y_{right}(x,t) = f(x - vt), \qquad y_{left}(x,t) = f(x + vt) \tag{26}$$

• A wavefunction $y(x,t)$ describes a snapshot at fixed time, if you fix t and view the wavefunction as a function of x.

• For a fixed x, $y(x,t)$ describes the time evolution of the displacement at the location x on the medium. The velocity and acceleration of the displacement at the x are the given by

$$v_y = \frac{\partial}{\partial t}y(x,t), \qquad a_y = \frac{\partial^2}{\partial t^2}y(x,t) \tag{27}$$

Note: Do not mix up this velocity with the wave speed v.

• Superposition principle: For two solutions $u_1(x,t), u_2(x,t)$ of the wave equation (with the same wave speed v), the sum $u_{tot} = u_1(x,t) + u_2(x,t)$ is also a solution of the wave equation. Since the displacement can be positive or negative the total displacement can be zero.

• Reflection at boundaries. We distinguish two boundary conditions for ends of the medium.

Fixed boundary conditions enforce that the displacement at the boundary is vanishing. The reflected wave is inverted $u \to -u$. For sinusoidal wave this means that the reflected wave has a phase shift of π.

Free boundary conditions enforce that the spatial derivative of displacement at the boundary is vanishing. The reflected wave is not inverted. For sinusoidal wave this means that the reflected wave has no phase shift.

• Sinusoidal waves are a special class of traveling waves where the wavefunction takes the form of a cosine or sine function. The most general form of left and rightmoving waves is

$$y_{right}(x,t) = A\cos(kx - \omega t + \phi_0), \qquad y_{left}(x,t) = A\cos(kx + \omega t + \phi_0), \tag{28}$$

• Sinusoidal waves are characterized by periodicity in both space and time

$$y(x+\lambda,t) = u(x), \qquad y(x,t+T) = u(t) \tag{29}$$

the period in space is given by wavelength λ and the period in time is given by the frequency $f = 1/T$ which are related to the wave number k and angular frequency ω by

$$k = \frac{2\pi}{\lambda}, \qquad \omega = 2\pi f = \frac{2\pi}{T} \tag{30}$$

A traveling sinusoidal wave the displacement at any point on the medium undergoes simple harmonic oscillator motion with angular frequency ω and amplitude A.

• For sinusoidal waves the wavelength and the frequency are related to the wave speed.

$$\lambda f = v \tag{31}$$

Note: traveling sinusoidal waves can have any wavelength or frequency as long as relation (31) is satisfied.

• The wave speed depends on the medium and its mechanical properties. A general conceptual formula is

$$v = \sqrt{\frac{\text{restoring force}}{\text{inertia}}} \tag{32}$$

1. Light is not a mechanical wave (it is an electromagnetic wave) the wave speed is

$$v = 3\times 10^8 \frac{m}{s} \tag{33}$$

2. For a string of **linear** mass density μ (mass per length) and tension T the wave speed is given by

$$v = \sqrt{\frac{T}{\mu}} \tag{34}$$

3. For sound waves there are two formulas

$$v = \sqrt{\frac{B}{\rho}} \tag{35}$$

Where B is the bulk modulus which relates the pressure to the change in volume of a gas or liquid $p = -BdV/V$.

$$v = \sqrt{\frac{\gamma RT}{M}} \tag{36}$$

where γ is the ratio of heat capacities ($\gamma = 1.4$ for air), R is the gas constant $R = 8.314J/molK$, M is the molar mass of the gas and T is the temperature (In Kelvin).

• Soundwaves are longitudinal waves where the displacement $u(x,t)$ is interpreted as the change of the (averaged) location of the air molecules from their equilibrium location. A second quantity is the change of the air pressure from its equilibrium value which is related to the displacement by

$$p(x,t) = -B\frac{\partial u(x,t)}{\partial x} \tag{37}$$

where B is the bulk modulus. This implies that the maximum value of the pressure (pressure amplitude) is given by

$$p_{max} = ABk \tag{38}$$

The relation (37) implies that extrema of the displacement are mapped to zeros of the pressure and vice versa.

• A traveling wave transports energy. For a transverse mechanical wave on a string with wavefunction $y(x,t) = A\cos(kx - \omega t)$ the instantaneous power is given by

$$P = \mu A^2\omega^2 v \sin^2(kx - \omega t) \tag{39}$$

The average power (averaged over one period T) is

$$\bar{P} = \frac{1}{2}\mu A^2\omega^2 v = \frac{1}{2}\sqrt{\mu T}A^2\omega^2 \tag{40}$$

where for the second formula we used (34) to replace v, which means the symbol T appearing there is tension (not period) – always check the units!

• A sound wave is extended in space (not limited to a one dimensional string). Instead of power the energy transport is measured by the averaged intensity $\bar{I}$ which is power per area (units W/m^2)

$$\bar{I} = \frac{1}{2}\sqrt{B\rho}\omega^2 A^2 = \frac{1}{2}\frac{p_{max}^2}{\sqrt{B\rho}} \tag{41}$$

Where p_{max} is the pressure amplitude (38).

• For point like sources (lighbulbs or speakers) the waves produced are spherical, for such waves the intensity decreases with distance r as

$$I_2 = I_1\frac{r_1^2}{r_2^2} \tag{42}$$

• The intensity of sound waves β is measured on a logarithmic scale in dB

$$\beta = 10dB\log_{10}\left(\frac{I}{I_0}\right) \tag{43}$$

where $I_0 = 10^{-12}W/m^2$ is a reference intensity (hearing threshold).

• Standing waves can be thought of as superpositions of left and rightmoving waves of the same frequency and amplitude. Each point of the medium undergoes simple harmonic oscillator motion with angular frequency ω, where the amplitude depends on the location x on the medium. There are two special points:

1. Nodes are places where the amplitude is always **zero**

2. Anti-nodes are places where the amplitude is **maximal**

• The distance between neighboring nodes (or neighboring anti-nodes) is $\lambda/2$. The distance between a node and an anti-node is $\lambda/4$.

• Standing waves on a medium of length L allow only for a discrete set of allowed wavelengths and frequencies (called resonances). The form of the wavefunction and values depend on the boundary conditions.

1. For a **fixed** end at $x = 0$ and a **fixed** end at $x = L$ the wavefunction is given by

$$u(x,t) = A \sin\left(\frac{2\pi}{\lambda_n} x\right) \sin(2\pi f_n t + \phi_0) \tag{44}$$

where the allowed frequencies of the $n-th$ harmonic are

$$\lambda_n = \frac{2L}{n}, \qquad f_n = v\frac{n}{2L}, \qquad n = 1, 2, 3, \cdots \tag{45}$$

The n-th harmonic has n antinodes and $n-1$ nodes (not counting the nodes at the boundary)

2. For a **fixed** end at $x = 0$ and a **free** end at $x = L$ the wavefunction is given by

$$u(x,t) = A \sin\left(\frac{2\pi}{\lambda_n} x\right) \sin(2\pi f_n t + \phi_0) \tag{46}$$

where the allowed frequencies of the $n-th$ harmonic are

$$\lambda_n = \frac{4L}{n}, \qquad f_n = v\frac{n}{4L}, \qquad n = 1, 3, 5, \cdots \tag{47}$$

Note that the even harmonic $n = 2, 4, 6, \cdots$ are missing (we follow the conventions of the textbook). The number of nodes and antinodes inside are $\frac{n-1}{2}$.

3. For a **free** end at $x = 0$ and a **fixed** end at $x = L$, the formulas for λ_n and f_n are also given by (47), the wavefunction is given by

$$u(x,t) = A \cos\left(\frac{2\pi}{\lambda_n} x\right) \sin(2\pi f_n t + \phi_0) \tag{48}$$

4. For a **free** end at $x = 0$ and a **free** end at $x = L$ the wave function is given by

$$u(x,t) = A \cos\left(\frac{2\pi}{\lambda_n} x\right) \sin(2\pi f_n t + \phi_0) \tag{49}$$

where the allowed frequencies of the $n-th$ harmonic are

$$\lambda_n = \frac{2L}{n}, \qquad f_n = v\frac{n}{2L}, \qquad n = 1, 2, 3, \cdots \tag{50}$$

• A general property of standing waves is that going from harmonic n to harmonic $n+1$ one adds a note and anti-node to the standing wave and that the fixed boundary corresponds to a node, whereas the free boundary corresponds to an anti-node.

• For standing sound waves **nodes of displacement** correspond to **anti-nodes of pressure** and vice versa.

• For standing sound wave in tubes or organ pipes the boundary conditions are determined as follows:

Open end: Node (fixed) for pressure, anti-node (free) for displacement

Closed end: Anti-Node (free) for pressure, node (fixed) for displacement