EXERCISES 2.1

GOAL *Use the concept of a linear transformation in terms of the formula $\vec{y} = A\vec{x}$, and interpret simple linear transformations geometrically. Find the inverse of a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ (if it exists). Find the matrix of a linear transformation column by column.*

Consider the transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined in Exercises 1 through 3. Which of these transformations are linear?

1. $\begin{aligned} y_1 &= 2x_2 \\ y_2 &= x_2 + 2 \\ y_3 &= 2x_2 \end{aligned}$

2. $\begin{aligned} y_1 &= 2x_2 \\ y_2 &= 3x_3 \\ y_3 &= x_1 \end{aligned}$

3. $\begin{aligned} y_1 &= x_2 - x_3 \\ y_2 &= x_1 x_3 \\ y_3 &= x_1 - x_2 \end{aligned}$

4. Find the matrix of the linear transformation

$$\begin{aligned} y_1 &= 9x_1 + 3x_2 - 3x_3 \\ y_2 &= 2x_1 - 9x_2 + x_3 \\ y_3 &= 4x_1 - 9x_2 - 2x_3 \\ y_4 &= 5x_1 + x_2 + 5x_3. \end{aligned}$$

5. Consider the linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^2$ with

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}7\\11\end{bmatrix}, \quad T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}6\\9\end{bmatrix},$$

$$\text{and} \quad T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}-13\\17\end{bmatrix}.$$

Find the matrix A of T.

6. Consider the transformation T from $\mathbb{R}^2$ to $\mathbb{R}^3$ given by

$$T\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\begin{bmatrix}1\\2\\3\end{bmatrix} + x_2\begin{bmatrix}4\\5\\6\end{bmatrix}.$$

Is this transformation linear? If so, find its matrix.

7. Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are arbitrary vectors in $\mathbb{R}^n$. Consider the transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ given by

$$T\begin{bmatrix}x_1\\x_2\\\vdots\\x_m\end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m.$$

Is this transformation linear? If so, find its matrix A in terms of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$.

EXERCISES 2.2

GOAL ***Use the matrices of orthogonal projections, reflections, and rotations. Apply the definitions of shears, orthogonal projections, and reflections.***

1. Sketch the image of the standard L under the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \vec{x}.$$

See Example 1.

2. Find the matrix of a rotation through an angle of 60° in the counterclockwise direction.

Find the matrices of the linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ given in Exercises 19 through 23. Some of these transformations have not been formally defined in the text. Use common sense. You may assume that all these transformations are linear.

19. The orthogonal projection onto the x–y-plane.

20. The reflection about the x–z-plane.

EXERCISES 2.3

GOAL *Compute matrix products column by column and entry by entry. Interpret matrix multiplication in terms of the underlying linear transformations. Use the rules of matrix algebra. Multiply block matrices.*

If possible, compute the matrix products in Exercises 1 through 13, using paper and pencil.

1. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

4. $\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 3 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

6. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

In the Exercises 17 through 26, find all matrices that commute with the given matrix A.

17. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

18. $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$

In Exercises 55 through 64, find all matrices X that satisfy the given matrix equation.

55. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

EXERCISES 2.4

GOAL *Apply the concept of an invertible function. Determine whether a matrix (or a linear transformation) is invertible, and find the inverse if it exists.*

Decide whether the matrices in Exercises 1 through 15 are invertible. If they are, find the inverse. Do the computations with paper and pencil. Show all your work.

1. $\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 2 & 5 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix}$

29. For which values of the constant k is the following matrix invertible?

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{bmatrix}$$