20W-MATH33B-1 Midterm 2

CHARLES ZHANG

TOTAL POINTS

43 / 50

QUESTION 1

Q1 10 pts

1.1 Q1a 2 / 2

✓ **- 0 pts** Correct

- 1 pts Arithmatic mistake

- 1 pts didn't verify solutions.

- 1 pts didn't verify linear independence

1.2 Q1b 6 / 8

- 0 pts Correct

- 1 pts Minor computational mistake

✓ **- 2 pts** Forgot to put in normal form

- 2 pts Incorrect formula for variation of parameters

- 1 pts Did not correctly evaluate integrals

- 2 pts Did not correctly find y after finding v's.

- 2 pts Did not find v's.

- 4 pts Bad guess for method of undetermined coefficients.

QUESTION 2

Q2 15 pts

2.1 Q2a 5 / 5

✓ **- 0 pts** Correct

- 2 pts Minor mistake

- 2 pts Solution?

- 1 pts one solution is not correct

2.2 Q2b 6 / 10

- 0 pts Correct

- 1 pts minor mistake 1

- 1 pts minor mistake 2

✓ **- 1 pts** minor mistake 3

- 2 pts major mistake 1: exp

- 2 pts major mistake 2: trig

- 2 pts major mistake 3: polynomial

- 6 pts VoP calculation not finished

- 4 pts VoP Calc wrong

✓ **- 3 pts** y_p_1 = ?

- 7 pts VoP No Calculation.

- 9 pts nothing meaningful

- 8 pts Click here to replace this description.

QUESTION 3

3 Q3 10 / 10

✓ **- 0 pts** Correct

- 2 pts not correct eigenvalues

- 3 pts not correct eigenvectors

- 8 pts tried

- 1 pts miscellineous mistake

- 1 pts one of eigenvalue incorrect

- 1.5 pts one of eigenvector incorrect

- 2 pts didn't know the definition of fundamental set

QUESTION 4

Q4 15 pts

4.1 a 11 / 12

- 0 pts Correct

- 2 pts k is incorrect

- 5 pts Didn't find v2 and y2

- 2 pts Your v2 is incorrect.

- 3 pts Found v1 and v2, but form of general solution is wrong

- 2 pts Your v1 is incorrect

- 2 pts Your v2 should not be an eigenvector too.

- 2 pts Your characteristic polynomial is incorrect

- 1 Point adjustment

Should be + tv1 where I'm pointing

4.2 b **3 / 3**

- **- 0 pts** Correct
- ✓ **- 0 pts Incorrect, but correct given answer in (a)**
- **- 2 pts** System solved/written incorrectly
- **- 1 pts** Write down the actual solution, not just the coefficients.
- **- 1 pts** System solved correctly, solution written down incorrectly

Midterm 2

Last Name: Zhang

First Name: Charles

Student ID: 305413659

Signature:

Section:

	Tuesday:	Thursday:	
	1A	1B	TA: YIH, SAMUEL
	1C	1D	TA: KIM, BOHYUN
	1E	1F	TA: BOSCHERT, NICHOLAS

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. You must **show your work** to receive credit. Please **circle or box your final answers.**

Please do not write below this line.

Question	Points	Score
1	10	
2	15	
3	10	
4	15	
Total:	50	

$t^2 \to 2t \to 2$

1. (a) (2 points) Verify that $y_1(t) = t$ and $y_2(t) = t^2$ are two linearly independent solutions to the differential equation:

$$t^2y'' - 3ty' + 3y = 0.$$

$y_1'(t) = 1$ $\quad y_2'(t) = 3t^2$

$y_1''(t) = 0$ $\quad y_2''(t) = 6t$

y_1: $t^2y_1'' - 3ty_1' + 3y_1 = 0$

$t^2(0) - 3t(1) + 3t = 0$

$-3t + 3t = 0$

$0 = 0$ ✓

y_2. $t^2(6t) - 3t(3t^2) + 3t^3 = 0$

$6t^3 - 9t^3 + 3t^3 = 0$

$0 = 0$ ✓

(b) (8 points) Find a particular solution y_p to the following differential equation:

$$t^2y'' - 3ty' + 3y = t^2 + 1$$

$y(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$

check

$\cancel{y'(t) = \frac{4}{3}t^3 - \frac{3}{2}t^2 - t}$

$\cancel{y''(t) = 4t^2 - 3t - 1}$

$\cancel{4t^4 - 3t^3 - t^2 - 4t^4 - \frac{9}{2}t^3 - 3t^2} +$

t^4

$y'(t) = \frac{4}{3}t^3 \cancel{- 2t}$

$\cancel{y''(t) = 4t^2 - 2}$

$\cancel{4t^4 - 2t^2 - 4t^3 + 6t + t^4 - 3t^2}$

$\cancel{y(t) = at^2 + bt + c}$

$\cancel{y'(t) = 2at + b}$

$\cancel{y''(t) = 2a}$

$\cancel{2at^2 - 3(2at + b)t + 3(at^2 + bt + c) = t^2 + 1}$

$\cancel{2at^2 - 6at^2 - 3bt + 3at^2 + 3bt + 3c = t^2 + 1}$

$y_1(t) = t$, $y_2(t) = t^3$

$W(t) = y_1y_2' - y_1'y_2 = t(3t^2) - 1(t^3)$

$W(t) = 2t^3$

$v_1(t) = \int \frac{-y_2 f}{W} = \int \frac{-t^3(t^2+1)}{2t^3} = -\int \frac{t^5 + t^3}{2t^3} = -\frac{1}{2}\int (t^2 + 1)\,dt$

$= -\frac{1}{2}\left[\frac{1}{3}t^3 + t\right]$

$v_1 = -\frac{1}{6}t^3 - \frac{1}{2}t$

$v_2(t) = \int \frac{y_1 f}{W} = \int \frac{t(t^2+1)}{2t^3} = \frac{1}{2}\int \frac{t^3 + t}{t^3}$

$= \frac{1}{2}\int (1 + t^{-2})\,dt$

$= \frac{1}{2}\left[t - \frac{1}{t}\right]$

$v_2 = \frac{1}{2}t - \frac{1}{2t}$

$y(t) = (-\frac{1}{6}t^3 - \frac{1}{2}t)t + (\frac{1}{2}t - \frac{1}{2t})t^3$

$y(t) = -\frac{1}{6}t^4 - \frac{1}{2}t^2 + \frac{1}{2}t^4 - \frac{1}{2}t^2$

$\boxed{y(t) = \frac{1}{3}t^4 - t^2}$

check

$\cancel{y = -\frac{1}{6}t^3 - \frac{1}{2}t}$

$\cancel{y' = -\frac{1}{2}t^2 - \frac{1}{2}}$

$\cancel{y'' = -t}$

$\cancel{t^2(-t) - 3t(-\frac{1}{2}t^2 - \frac{1}{2}) + 3(-\frac{1}{6}t^3 - \frac{1}{2}t) = t^2 + 1}$

$\cancel{-t^3 + \frac{3}{2}t^3 + \frac{3}{2}t - \frac{1}{2}t^3 - \frac{3}{2}t = t^2 + 1}$

Last six digits of UID: 413659

2. (a) (5 points) Find the general solution to the differential equation:

$$y'' + y' - 2y = 0$$

$(\lambda^2 + \lambda - 2) = 0$ $\quad y(t) = e^{\lambda t}$

$(\lambda + 2)(\lambda - 1) = 0$

$\lambda = -2, 1$

$y_1(t) = e^{-2t}$, $y_2(t) = e^t$

$\boxed{y(t) = C_1 e^{-2t} + C_2 e^t}$

(b) (10 points) Find a particular solution to the differential equation (Hint: split forcing term into three parts): $Ae^{-2t} \to -2Ae^{-2t} \to 4Ae^{-2t}$

$$y'' + y' - 2y = 3e^{-2t} + 10\cos t + t + 1.$$

$y'' + y' - 2y = 3e^{-2t}$

$y(t) = Ae^{-2t}$

$y'(t) = -2Ae^{-2t}$

$y''(t) = 4Ae^{-2t}$

$4Ae^{-2t} + (-2Ae^{-2t}) - 2Ae^{-2t} = 3e^{-2t}$

$4Ae^{-2t} - 2Ae^{-2t} - 2Ae^{-2t} = 3e^{-2t}$

$4A - 2A - 2A = 3$ ✗

$y_1(t) = Ae^{-2t}$, $y_1'(t) = -2Ae^{-2t}$, ?

$y_1''(t) = 4Ae^{-2t}$

$y_3(t) = at + b$

$y_3'(t) = a$

$y_3''(t) = 0$

$a - 2(at + b) = t + 1$

$-2at + a - b = t + 1$

$-2a = 1$

$a = -1/2$

$a - b = 1$

$-1/2 - b = 1$

$b = -3/2$

$y_3(t) = -\frac{1}{2}t - \frac{3}{2}$

$y'' + y' - 2y = 10\cos t$

$y(t) = a\cos t + b\sin t$

$y'(t) = -a\sin t + b\cos t$

$y''(t) = -a\cos t - b\sin t$

$(-a\cos t - b\sin t) + (-a\sin t + b\cos t) - 2(a\cos t + b\sin t) = 10\cos t$

$(-a + b - 2a)\cos t = 10\cos t$

$(-b - a - 2b)\sin t = 0\sin t$

$-3b - a = 0$

$a = -3b$

$(3b + b + 6b) = 10$

$b = 1$, $a = -3$

$y_2(t) = -3\cos t + \sin t$

$y(t) = y_1(t) + y_2(t) + y_3(t)$

$\boxed{y(t) = y_1(t) - 3\cos t + \sin t - \frac{1}{2}t - \frac{3}{2}}$

3. (10 points) Find the fundamental set (in other words, you have to find two linearly indepedent solutions) of the following 2×2 system $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$$

$$\begin{bmatrix} 2-\lambda & 4 \\ 3 & 1-\lambda \end{bmatrix}$$

$$(2-\lambda)(1-\lambda)-12=0$$

$$2-3\lambda+\lambda^2-12=0$$

$$\lambda^2-3\lambda-10=0$$

$$(\lambda-5)(\lambda+2)=0$$

$$\lambda=5,-2$$

$$\lambda_1=5, \quad \lambda_2=-2$$

$$\begin{bmatrix} 2-5 & 4 \\ 3 & 1-5 \end{bmatrix} \qquad \begin{bmatrix} 2+2 & 4 \\ 3 & 1+2 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix}=0 \qquad \begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix}=0$$

$$v_1=\begin{bmatrix} 4 \\ 3 \end{bmatrix} \qquad v_2=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\boxed{y(t)=c_1 e^{5t}\begin{bmatrix} 4 \\ 3 \end{bmatrix}+c_2 e^{-2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

Last six digits of UID: 413659

4. (a) (12 points) Find the general solution ($y_{\text{general}} = C_1 y_1(t) + C_2 y_2(t)$) to the following 2×2 system $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} 4 & 4 \\ -1 & 8 \end{pmatrix}.$$

$$\begin{bmatrix} 4-\lambda & 4 \\ -1 & 8-\lambda \end{bmatrix}$$

$$(4-\lambda)(8-\lambda)+4=0$$

$$32-12\lambda+\lambda^2+4=0$$

$$(\lambda^2-12\lambda+36)=0$$

$$(\lambda-6)^2=0$$

$$\lambda=6$$

$$\begin{bmatrix} 4-6 & 4 \\ -1 & 8-6 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}=0$$

$$v_1=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$(A-\lambda I)v_2=v_1$$

$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$v_2=\begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$y(t)=c_1e^{\lambda t}v_1+c_2e^{\lambda t}(v_2-tv_1)$$

$$\boxed{y(t)=c_1e^{6t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}+c_2e^{6t}\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}-t\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)}$$

(b) (3 points) Find a solution to the above differential equation with initial condition $y(0) = (1,1)^T$. (In other words, determine coefficient C_1 and C_2. Try simplify your answers.)

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}=c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}+c_2\begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$c_1=1,\ c_2=1$$

$$y(t)=e^{6t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}+e^{6t}\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}-t\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

$$=e^{6t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}+e^{6t}\begin{bmatrix} -1 \\ 0 \end{bmatrix}-te^{6t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$=\boxed{e^{6t}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}-t\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)}$$

Scratch Paper and Some useful formulas, etc:

Variation of Parameters, (2nd Order Differential Equations)

$$v_1(x) = -\int \frac{1}{W} y_2(x) f(x) \mathrm{d}x$$

$$v_2(x) = \int \frac{1}{W} y_1(x) f(x) \mathrm{d}x$$