EXERCISES

In Exercises 1–8, calculate the differential dF for the given function F.

1. $F(x, y) = 2xy + y^2$
2. $F(x, y) = x^2 - xy + y^2$
3. $F(x, y) = \sqrt{x^2 + y^2}$
4. $F(x, y) = 1/\sqrt{x^2 + y^2}$
5. $F(x, y) = xy + \tan^{-1}(y/x)$
6. $F(x, y) = \ln(xy) + x^2 y^3$
7. $F(x, y) = \ln(x^2 + y^2) + x/y$
8. $F(x, y) = \tan^{-1}(x/y) + y^4$

In Exercises 9–21, determine which of the equations are exact and solve the ones that are.

9. $(2x + y)\,dx + (x - 6y)\,dy = 0$
10. $(1 - y\sin x)\,dx + (\cos x)\,dy = 0$
11. $\left(1 + \dfrac{y}{x}\right) dx - \dfrac{1}{x}\,dy = 0$
12. $\dfrac{x}{\sqrt{x^2 + y^2}}\,dx + \dfrac{y}{\sqrt{x^2 + y^2}}\,dy = 0$
13. $\dfrac{dy}{dx} = \dfrac{3x^2 + y}{3y^2 - x}$
14. $\dfrac{dy}{dx} = \dfrac{x}{x - y}$
15. $(u + v)\,du + (u - v)\,dv = 0$
16. $\dfrac{2u}{u^2 + v^2}\,du + \dfrac{2v}{u^2 + v^2}\,dv = 0$
17. $\dfrac{dr}{ds} = \dfrac{\ln s}{r/s - 2s}$
18. $\dfrac{dy}{du} = \dfrac{2 - y/u}{\ln u}$
19. $\sin 2t\,dx + (2x\cos 2t - 2t)\,dt = 0$
20. $2xy^2 + 4x^3 + 2x^2 y\dfrac{dy}{dx} = 0$
21. $(2r + \ln y)\,dr + ry\,dy = 0$

In Exercises 22–25, the equations are not exact. However, if you multiply by the given integrating factor, then you can solve the resulting exact equation.

22. $(y^2 - xy)\,dx + x^2\,dy = 0, \quad \mu(x, y) = \dfrac{1}{xy^2}$
23. $(x^2y^2 - 1)y\,dx + (1 + x^2y^2)x\,dy = 0, \quad \mu(x, y) = \dfrac{1}{xy}$
24. $3(y + 1)\,dx - 2x\,dy = 0, \quad \mu(x, y) = \dfrac{y + 1}{x^4}$
25. $(x^2 + y^2 - x)\,dx - y\,dy = 0, \quad \mu(x, y) = \dfrac{1}{x^2 + y^2}$
26. Suppose that $y\,dx + (x^2 y - x)\,dy = 0$ has an integrating factor that is a function of x alone [i.e., $\mu = \mu(x)$]. Find the integrating factor and use it to solve the differential equation.
27. Suppose that $(xy - 1)\,dx + (x^2 - xy)\,dy = 0$ has an integrating factor that is a function of x alone [i.e., $\mu = \mu(x)$]. Find the integrating factor and use it to solve the differential equation.
28. Suppose that $2y\,dx + (x + y)\,dy = 0$ has an integrating factor that is a function of y alone [i.e., $\mu = \mu(y)$]. Find the integrating factor and use it to solve the differential equation.
29. Suppose that $(y^2 + 2xy)\,dx - x^2\,dy = 0$ has an integrating factor that is a function of y alone [i.e., $\mu = \mu(y)$]. Find the integrating factor and use it to solve the differential equation.
30. Consider the differential equation $2y\,dx + 3x\,dx = 0$. Determine conditions on a and b so that $\mu(x, y) = x^a y^b$ is an integrating factor. Find a particular integrating factor and use it to solve the differential equation.

The equations in Exercises 31–34 each have the form $P(x, y)\,dx + Q(x, y)\,dy = 0$. In each case, show that P and Q are homogeneous of the same degree. State that degree.

31. $(x + y)\,dx + (x - y)\,dy = 0$
32. $(x^2 - xy - y^2)\,dx + 4xy\,dy = 0$
33. $\left(x - \sqrt{x^2 + y^2}\right) dx - y\,dy = 0$
34. $(\ln x - \ln y)\,dx + dy = 0$

Find the general solution of each homogeneous equation in Exercises 35–39.

35. $(x^2 + y^2)\,dx - 2xy\,dy = 0$
36. $(x + y)\,dx + (y - x)\,dy = 0$
37. $(3x + y)\,dx + x\,dy = 0$
38. $\dfrac{dy}{dx} = \dfrac{y(x^2 + y^2)}{xy^2 - 2x^3}$
39. $x^2 y' = 2y^2 - x^2$
40. $(y + 2xe^{-y/x})\,dx - x\,dy = 0$
41. In Figure 8, a goose starts in flight a miles due east of its nest. Assume that the goose maintains constant flight speed (relative to the air) so that it is always flying directly toward its nest. The wind is blowing due north at w miles per hour. Figure 8 shows a coordinate frame with the nest at $(0, 0)$ and the goose at (x, y). It is easily seen (but you should verify it yourself) that

$$\frac{dx}{dt} = -v_0 \cos\theta,$$
$$\frac{dy}{dt} = w - v_0 \sin\theta.$$

(a) Show that

$$\frac{dy}{dx} = \frac{y - k\sqrt{x^2 + y^2}}{x}, \tag{6.43}$$

where $k = w/v_0$, the ratio of the wind speed to the speed of the goose.