Homework 7

Status: Final (although there might be some typos).

> Please reach out to me via the CCLE forum for corrections and/or clarifications about statements found in this document.

Due date: Friday, May 22.

Regular exercises

Introduction to Graphs

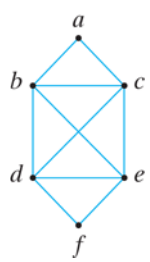
Section 8.1 in course textbook.

1. In a tournament, the *Spiders* beat the *Balrogs* once, the *Trolls* beat the *Wargs* once, the *Spiders* beat the *Trolls* twice, the *Balrogs* beat the *Wargs* once, and the *Balrogs* beat the *Spiders* once.

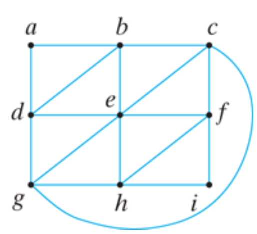
   In the following exercises, use a graph to model the tournament. The teams are the vertices. Describe the kind of graph used (e.g., undirected graph, directed graph, simple graph).

   i. There is an edge between teams if the teams played.
   ii. There is an edge from team t_i to team t_j if t_i beat t_j at least one time.
   iii. There is an edge between teams for each game played.
   iv. There is an edge from team t_i to team t_j for each victory of t_i over t_j.

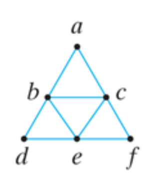
2. Show that each graph below has a path from a to a that passes through each edge exactly one time by finding such a path by inspection.



Graph 1



Graph 2

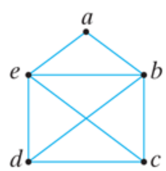
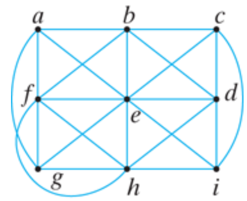
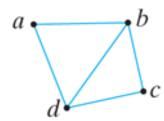


Graph 3

   1. Graph 1 (above)
   2. Graph 2 (above)

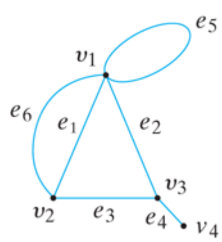
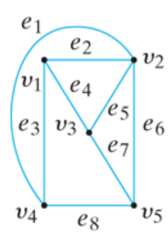
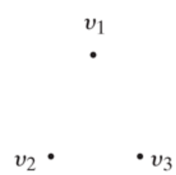
3. Graph 3 (above)

3. Explain why none of the graphs below has a path from a to a that passes through each edge exactly one time.

		
Graph 4	Graph 5	Graph 6

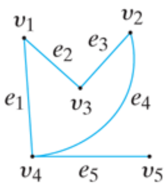
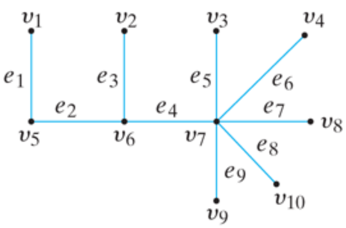
1. Graph 4 (above)
2. Graph 5 (above)
3. Graph 6 (above)

4. For each graph $G = (V, E)$ below, find V, E, all parallel edges, all loops, all isolated vertices, and tell whether G is a simple graph. Also, tell on which vertices edge e_1 is incident.

		
Graph 7	Graph 8	Graph 9

5. Find a formula for the number of edges in K_n.

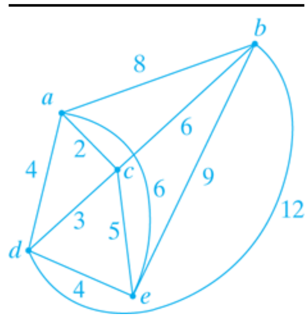
6. State which graphs below are bipartite graphs. If the graph is bipartite, specify the disjoint vertex sets.

	
Graph 10	Graph 11

i. Graph 7 (above)
ii. Graph 8 (above)
iii. Graph 10 (above)
iv. Graph 11 (above)

7. Find a formula for the number of edges in $K_{m,n}$.

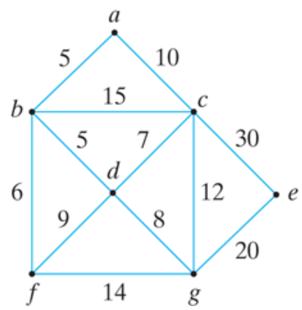
8. In the following exercises, find a path of minimum length from v to w in graph 12 (below) that passes through each vertex exactly one time.



Graph 12

i. $v = b$, $w = e$.
ii. $v = a$, $w = b$.
iii. $v = c$, $w = d$.

9. In the following graph the vertices represent cities and the numbers on the edges represent the costs of building the indicated roads. Find a least-expensive road system that connects all the cities.



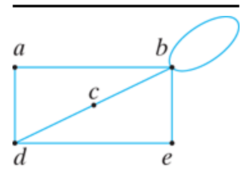
Graph 13

10. Let $\mathcal{G}$ denote the set of simple graphs $G = (V, E)$, where $V = \{1, 2, \cdots, n\}$ for some $n \in \mathbf{N}$. Define a function f from $\mathcal{G}$ to $\mathbf{N} \cup \{0\}$ by the rule $f(G) = |E|$. Is f one-to-one? Is f onto? Explain.

Paths and Cycles

Section 8.2 in course textbook.

11. In the following exercises, tell whether the given path in the graph 14 is
    - A simple path
    - A cycle
    - A simple cycle



Graph 14

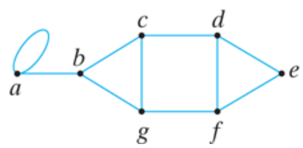
   i. (a, d, c, d, e)
   ii. (e, d, c, b)
   iii. (b, c, d, e, b, b)
   iv. $(b, c, d, a, b, e, d, c, b)$
   v. (d, c, d)

12. In the following exercises, draw a graph having the given properties or explain why no such graph exists.

i. Six vertices each of degree 3.
ii. Four vertices each of degree 1.
iii. Five vertices each of degree 3.
iv. Six vertices; four edges.
v. Four edges; four vertices having degrees 1, 2, 3, 4.
vi. Simple graph; six vertices having degrees 1, 2, 3, 4, 5, 5.

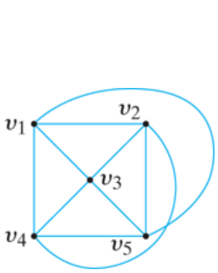
13. For graph 15 (below), find all

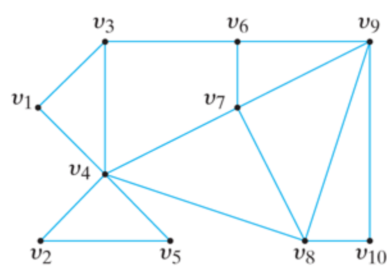
- the simple cycles; and
- the simple paths from a to e.



Graph 15

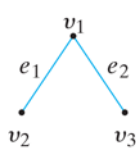
14. Find the degree of each vertex for the following graphs.

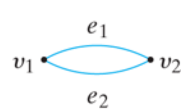


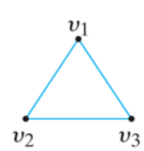


Graph 16 Graph 17

15. In the following exercises, find all subgraphs having at least one vertex of the specified graph.





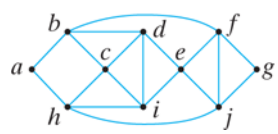


Graph 18 Graph 19 Graph 20

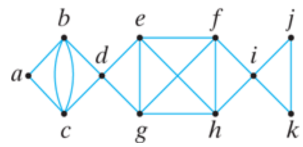
i. Graph 18 (above)

ii. Graph 19 (above)
iii. Graph 20 (above)

16. In the following exercises, decide whether the specified graph has an Euler cycle. If it does, exhibit one.



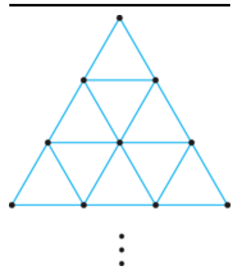
Graph 21



Graph 22

i. Graph 21 (above)
ii. Graph 22 (above)

17. The following graph is continued to an arbitrary, finite depth. Does the graph contain an Euler cycle? If the answer is yes, describe one.



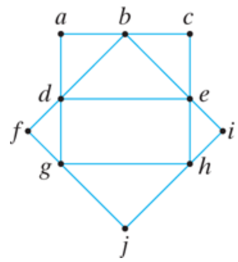
Graph 23

18. When does the complete

i. graph K_n contain an Euler cycle?
ii. bipartite graph $K_{m,n}$ contain an Euler cycle?

19. A sports conference has 11 teams. It was proposed that each team play precisely one game against each of exactly nine other conference teams. Prove that this proposal is impossible to implement.

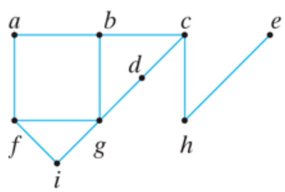
20. For graph 24 (below), find a path with no repeated edges from d to e containing all the edges.



Graph 24

Miscellaneous exercises

- Let G be a connected graph with four vertices v_1, v_2, v_3, and v_4 of odd degree. Show that there are paths with no repeated edges from v_1 to v_2 and from v_3 to v_4 such that every edge in G is in exactly one of the paths.
- Illustrate the previous exercise using the following graph.



Graph 25

- Tell whether each assertion is true or false. If false, give a counterexample and if true, prove it.
  - Let G be a graph and let v and w be distinct vertices. If there is a path from v to w, there is a simple path from v to w.
  - If a graph contains a cycle that includes all the edges, the cycle is an Euler cycle.
- Let G be a connected graph. Suppose that an edge e is in a cycle. Show that G with e removed is still connected.
- Show that the maximum number of edges in a simple, disconnected graph with n vertices is $(n-1)(n-2)/2$.

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