9/27 Lecture: Linear Equations and Systems

- Linear Algebra: study of (systems) of linear equations
  - $a_1x_1 + \dots + a_nx_n = b$
  - $a_1, a_2, \dots, a_n, b$ are real numbers/coefficients
  - $x_1, x_2, \dots, x_m$ are variables

geometry motivated
- lines and planes in linear equations
- geometric transformations as linear maps
- studying points of intersection of lines/planes as solving systems of linear eq.s

- Applications:
  - Statistics (orthonormal basis, QR decomp., method of least squares)
  - Dynamic Systems and Linear Diff. Eq.s (eigenvectors, eigenfunctions)
  - Scientific Computing

I. Linear Equations

i) in 2 variables: $\boxed{ax+by=c}$, a, b, c are real #s called coefficients
x and y are variables

- Solution to is an assignment of variables that makes both sides equal
  - it is a tuple $(x_0, y_0) \in \mathbb{R}^2$ such that $ax_0 + by_0 = c$
  - e.g. $x - 2y = 3$, then $(1, -1)$ is a solution
- Solution set of a system is the set of all possible solutions.
- Geometric interpretation
  - if a and $b \neq 0$, then the solution set forms a line
    - $a = 0$, $b \neq 0$ $\rightarrow y = \frac{c}{b}$
    - $a \neq 0$, $b = 0$ $\rightarrow x = \frac{c}{a}$
    - $a \neq 0$, $b \neq 0$ $\rightarrow y = -\frac{a}{b}x + \frac{c}{b}$
      - y-int of $\frac{c}{b}$, m of $-\frac{a}{b}$

ii) in 3 variables: $ax + by + cz = d$ $\mathbb{R}^3$
  - solution set of given by all points in the plane that satisfies it
- Geometric interpretation
  - if a, b, and $c \neq 0$, the solution set forms a plane in $\mathbb{R}^3$

iii) General geometric interpretation
  - if $a_1, a_2, \dots, a_m$ are not all 0, the solution set forms a hyperplane in $\mathbb{R}^m$

Systems of linear equations

↳ system of 2 equations in 2 variables

↳ $ax+by=c$

$dx+ey=f$

↳ solution set are all points $(x_0, y_0) \in \mathbb{R}^2$ that satisfy both eq.s

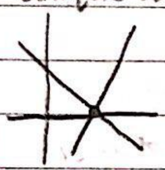
↳ $ax_0+by_0=c$ and $dx_0+ey_0=f$

↳ geometric interpretation

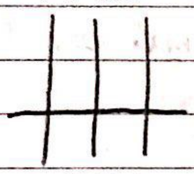
↳ point of intersection between the 2 lines defined by the linear equations

↳ 3 possibilities

↳ unique solution

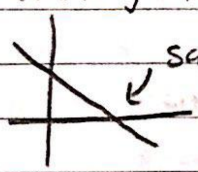


↳ no solution



no intersection, parallel lines

↳ infinitely many solutions



same line

9/30 Lecture: Systems of Linear Equations

- Systems of 2 equations: $\boxed{\begin{array}{l} ax+by=c \\ dx+ey=f \end{array}}$ ⊛

- Solution set of ⊛: all points $(x_0, y_0) \in \mathbb{R}^2$ such that $ax_0+by_0=c$ AND $dx_0+ey_0=f$
  - ↳ Geometric interpretation: the solution set consists of all points of intersection of the 2 lines defined by the 2 equations
    - ↳ 3 situations: unique solution, no solution, infinitely many solutions

$$\begin{pmatrix} x-y=5 \\ 2x+y=3 \end{pmatrix} \quad \begin{pmatrix} 3x+6y=4 \\ x+2y=1 \end{pmatrix} \begin{pmatrix} -x+2y=-3 \\ \frac{1}{2}x-y=\frac{3}{2} \end{pmatrix}$$

In general: system w/ n equations in m variables

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1m}x_m = b_1$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nm}x_m = b_n$$

Solution set: all points $(x_1, \dots, x_m) \in \mathbb{R}^m$ that satisfy all equations

- Geometric interpretation $(m=3)$
  - ↳ solution set defined by all points of intersection of the planes defined by the n equations

- Gauss-Jordan Elimination
  - ↳ Goal: to perform operations on the system such that we don't change the solution set and that we can easily read the solutions
    - ↳ Elementary operations on systems of 1 eq.
      - ↳ Multiplying/dividing both sides by the same #
      - ↳ Adding/subtracting the same # from both sides
    - ↳ Elementary operations on systems of multiple eqs
      - ↳ Multiplying/dividing any eq. by a non-zero number
      - ↳ Adding/subtracting a # from both sides of an eq.
      - ↳ Adding/subtracting a nonzero multiple of an eq. to another eq.
        - ↳ The points that satisfy eq. II are such that we are adding the same # to both sides of eq. I
      - ↳ Interchanging any 2 eq.s

· Matrix Notation

↳ A system of n linear eq.s in m variables can always be written as follows:

$$\begin{array}{l} a_{11}x_1 + \dots a_{1m}x_m = b_1 \\ \vdots \\ a_{n1}x_1 + \dots a_{nm}x_m + b_n \end{array} \rightarrow \overbrace{\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1m} & b_1 \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} & b_n \end{array}\right]}^{\text{columns}} \left.\vphantom{\begin{array}{c} a \\ \vdots \\ a \end{array}}\right\} \text{rows}$$

"augmented matrix of coefficients of the system"

↳ elementary row operations

↳ operations on a matrix that don't change the solution set

↳ multiply/divide any row by a nonzero number

↳ add/subtract a multiple of a row to another row

↳ swap any 2 rows

• Reduced row-echelon form (RREF)

↳ the 1st nonzero entry in every row is a 1 (pivot/leading 1)

↳ columns that contain pivots must have a 0 in every other entry

↳ if a row contains a pivot, each row above must have pivots that are to the left of it

10/2 Lecture: Gauss-Jordan Elimination

- Gauss-Jordan Elimination
  - ↳ A system of n equations and m variables can be written:

$$a_{11}x_1 + \dots + a_{1m}x_m = b_1$$
$$\vdots$$
$$a_{n1}x_1 + \dots a_{nm}x_m = b_n$$

  - ↳ In matrix notation: $A \cdot x = b$

where $A = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \\ a_{n1} & & a_{nm} \end{bmatrix}$

↳ matrix of coefficients of the system

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$ $\quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

↳ Augmented matrix of coefficients:

$$A|b = \left[\begin{array}{ccc|c} a_{11} & \dots & a_{1m} & b_1 \\ \vdots & & & \vdots \\ a_{n1} & \dots & a_{nm} & b_n \end{array}\right]$$

↳ Goal: perform operations on $A|b$ / A such that

↳ solution set of system is unchanged

↳ elementary row operations

↳ solutions can be easily read off

↳ reduced row-echelon form (RREF)

- Reduced row-echelon form of a matrix A:
  - ↳ the first non-zero entry of any row is a 1 (leading 1 or pivot)
  - ↳ if a column contains a pivot, it is the only non-zero entry in that column
  - ↳ if a row contains a pivot, all rows above it contain pivots to the left
  - ↳ the augmented coeff. matrix $A|b$ is in RREF if A is
  - ↳ Ex)

$$A|b = \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} x_1 + 2x_3 + x_4 = 1 \\ x_2 + x_4 = 0 \\ x_5 = 2 \end{array}$$

↳ we distinguish between

↳ leading variables: those that correspond to columns w/ pivots

↳ free variables: those that don't correspond to columns w/ pivots

↳ in example: x_1, x_2, and x_5 are leading, x_3 and x_4 are free

↳ easy to read solutions because:

↳ coefficient of every leading variable is 1

↳ leading variables only appear in one row/equation

↳ leading variables appear in order

↳ in example: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2t-r+1 \\ -r \\ t \\ r \\ 2 \end{bmatrix}$ with t and $r = \mathbb{R}$

- Gauss-Jordan elimination algorithm

↳ 1. write down the augmented matrix $A|b$

↳ 2. proceed thru the rows of $A|b$ from top to bottom

↳ 3. multiply rows w/ leading, non-zero entries by $\frac{1}{\text{entry}}$ (i.e. if its a 3, multiply by $\frac{1}{3}$)

↳ 4. transform entries in columns w/ leading 1s into 0s by adding appropriate multiples of the row w/ the leading 1 to the other rows

↳ 5. Repeat for all rows

↳ 6. Swap rows to order pivots correctly

↳ if a row of the form $[0 \dots 0 | k]$ and $k \neq 0$, the system has no solution

↳ # of solutions of a system of linear equations:

↳ $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{c} x_1 + 2x_2 = 0 \\ x_3 = 0 \\ 0 = 1 \end{array}$ $\boxed{\text{no solution}}$

$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{c} x_1 + 2x_2 = 1 \\ x_3 = 2 \end{array}$

$\downarrow$

$x_1 = -2x_2 + 1$

$x_3 = 2$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t+1 \\ t \\ 2 \end{bmatrix}$

$\boxed{\text{infinitely many solutions}}$

- Rank of a matrix = # of pivots

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\boxed{\text{unique solution}}$

10/4 Lecture Matrix Algebra

- Example. $2x_1 + 2x_2 = 0$
  $x_2 + x_3 = 4$

$$\left[\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 4 \end{array}\right] \times \tfrac{1}{2} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 4 \end{array}\right] -\mathrm{II} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -4 \\ 0 & 1 & 1 & 4 \end{array}\right]$$

$x_1 - x_3 = -4$
$x_2 + x_3 = 4$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t-4 \\ 4-t \\ t \end{bmatrix}}$$

- # of solutions:
  - ↳ In general, a system is:
    - ↳ inconsistent - there are no solutions, RREF has a row $[0 \ldots 0 | k], k \neq 0$
    - ↳ consistent - there is ≥ 1 solution
      - ↳ unique solution - all variables are leading
      - ↳ infinite solutions - at least 1 free variable
- Rank of a matrix = # of leading variables in the matrix's RREF
- Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{rank of } A = ? \quad \begin{array}{l} -4(\mathrm{I}) \\ -7(\mathrm{I}) \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \times -\tfrac{1}{3} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -6 & -12 \end{bmatrix} \begin{array}{l} -2(\mathrm{II}) \\ \\ +6(\mathrm{II}) \end{array} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{rank}(A) = 2$$

- What can we say about the solutions of a system with A as its coefficient matrix?
  - ↳ if $b_3 = 0$, there are infinitely many solutions since x_3 is free
  - ↳ if $b_3 \neq 0$, there are no solutions
  - ↳ the system cannot have a unique solution
  - ↳ rank < # of variables / rank < m, system cannot have a unique solution
- Rank and solutions of a system:
  - ↳ Given a system $Ax = b$ where A is an $n \times m$ matrix (n eq.s, m variables)
    - ↳ rank $(A) \leq n \rightarrow$ at most 1 pivot per column / per row
    - ↳ rank $(A) \leq m$
    - ↳ If the system is inconsistent, rank $(A) < n$ (must be an all 0 row)

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↳ If the system has a unique solution, $\text{rank}(A) = m$

↳ If the system has infinitely many solutions, $\text{rank}(A) < m$

- Contrapositive - "If P, then Q" -> "If not Q, then not P"
  ↳ equivalent statements

10/7 Lecture: Matrix Algebra

- Linear Transformations:
  - ↳ ex) in geometry, transformations of the plane
    - ↳ reflection about the y-axis
    - ↳ scaling by a #
    - ↳ rotation by an angle
  - ↳ called linear transformations because
    - ↳ $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$ for all vectors $\vec{v}, \vec{w}$ in the plane
    - ↳ $T(k\vec{v}) = kT(\vec{v})$ for all scalars k and vectors $\vec{v}$
  - ↳ in general, linear transformations $R^m \to R^n$
    - ↳ exactly those that can be represented by a matrix
- Matrix Algebra:
  - ↳ Size of a matrix: # of rows / columns of a matrix
    - ↳ n rows and m columns → n×m matrix
    - ↳ Matricies w/ only one row or one column = row vector and column vector, respectively
    - ↳ a matrix w/ the same # of rows and columns = square matrix
      - ↳ ex) $\begin{bmatrix} 2 & 1 & 0 \\ -3 & 4 & 1 \end{bmatrix}$ - 2×3 matrix $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ - column vector

        $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ - row vector $\begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}$ - square matrix
- Operations On Matricies
  - ↳ addition of 2 matricies A and B of the same size
    - ↳ $\begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \\ a_{n1} & \dots & a_{nm} \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1m} \\ & & \\ b_{n1} & \dots & b_{nm} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & \dots & a_{1m}+b_{1m} \\ & & \\ a_{n1}+b_{n1} & \dots & a_{nm}+b_{nm} \end{bmatrix}$
  - ↳ subtraction - similarly defined for matricies of the same size
  - ↳ properties of addition/subtraction:
    - ↳ commutativity: $A+B = B+A$
    - ↳ associativity: $(A+B)+C = A+(B+C)$
  - ↳ ex) $\begin{bmatrix} 1 & 0 & 3 \\ 4 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 \\ -3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 9 & 7 \end{bmatrix}$

↳ multiplication of A and B

↳ only defined when # of columns of A = # of rows of B

↳ A: $n \times m$ matrix, B: $m \times p$ matrix

↳ $$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{i1} & \cdots & a_{im} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \\ b_{m1} & \cdots & b_{mj} & \cdots & b_{mp} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1p} \\ & c_{ij} & \\ c_{n1} & \cdots & c_{np} \end{bmatrix}$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{im}b_{mj} = \sum_{k=1}^{m} a_{ik}b_{kj}$$

↳ ex) $$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 4 \end{bmatrix}$$

• Dot product

↳ $\vec{v} \cdot \vec{w} = v_1w_1 + \dots + v_nw_n$, $\vec{v}$ and $\vec{w}$ have equal # of components

↳ c_{ij} is the dot product of the ith row of A and the jth column of B

↳ Dot product in matrix notation

↳ $$\vec{v} \cdot \vec{w} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

↳ An $n \times m$ matrix A and column vector x w/ m entries

↳ $$A = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}, \quad Ax = \begin{bmatrix} r_1 \cdot x \\ \vdots \\ r_n \cdot x \end{bmatrix}$$

↳ ex) $$\begin{bmatrix} 1 & 0 & -2 \\ 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_3 \\ 3x_1 + 4x_3 \end{bmatrix}$$

↳ in general, we can write an arbitrary system of linear eq,s as

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, \quad Ax = b$$

• Properties of Matrix Multiplication

↳ not commutative in general ($AB \neq BA$)

10/9 Lecture: Linear Transformations

- Matrix Multiplication
  - ↳ A : $n \times m$ matrix
  - ↳ B : $m \times p$ matrix
  - ↳ AB : $n \times p$ matrix
  - ↳ $A = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ ← row vectors $B = [b_1, \ldots, b_p]$ column vectors $AB = \begin{bmatrix} a_1 \cdot b_1 & a_1 \cdot b_2 & \ldots & a_1 \cdot b_p \\ a_2 \cdot b_1 & \ldots & & \\ \vdots & & & \\ a_n b_1 & \ldots & & a_n b_p \end{bmatrix}$
    - ↳ Entry in the ith row and jth column is $a_i \cdot b_j$
- Multiplication of a matrix with a scalar
  - ↳ Given $k \in \mathbb{R}$ and A is an $n \times m$ matrix
    - ↳ $kA = \begin{bmatrix} ka_{11} & \ldots & ka_{1m} \\ \vdots & & \\ ka_{n1} & \ldots & ka_{nm} \end{bmatrix}$
- Properties of matrix multiplication
  - ↳ Not commutative : $AB \neq BA$
  - ↳ Associative : $A(BC) = (AB)C$ for and $n \times m$ matrix A, $m \times p$ matrix B, and $p \times q$ matrix C
  - ↳ Distributive : $A(C+D) = AC + AD$ for all $n \times m$ matrices C and D and $m \times p$ matrix A
  - ↳ For any scalar $k \in \mathbb{R}$, $n \times m$ matrix A, $m \times p$ matrix B
    - ↳ $(kA)B = A(kB) = k(AB)$
  - ↳ Multiplying w/ the identity matrix
    - ↳ $\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ → $n \times n$ Matrix
      $= I_n$ → notation for an identity matrix
    - ↳ For any $n \times m$ Matrix A :
      - ↳ $I_n A = A = A I_m$
        - ↳ The identity matrix is the neutral element for matrix products
- Linear Transformations
  - ↳ given 2 sets x and y, a map (or function) $f : x \to y$ is a rule that associates every element of x to a unique element of y

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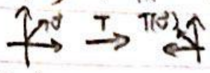
↳ x = domain of f, y = codomain of f

↳ Def: a map $T:\mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation iff:

↳ $T(v+w) = T(v) + T(w)$ for all $\vec{v},\vec{w} \in \mathbb{R}^m$

↳ $T(kv) = kT(v)$ for all scalars k and all vectors $\vec{v} \in \mathbb{R}^m$

↳ Ex) $T:\mathbb{R}^2 \to \mathbb{R}^2$: reflection about the y-axis



$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ $T(\vec{v}) = \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix}$

↳ let $v, w \in \mathbb{R}^m$ be arbitrary

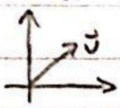
↳ $T(\vec{v}+\vec{w}) = \begin{bmatrix} -v_1 - w_1 \\ v_2 + w_2 \end{bmatrix} = \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} -w_1 \\ w_2 \end{bmatrix} = T(\vec{v}) + T(\vec{w})$

↳ let $k \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^m$ be arbitrary

↳ $T(k\vec{v}) = \begin{bmatrix} -kv_1 \\ kv_2 \end{bmatrix} = \begin{bmatrix} k(-v_1) \\ k(v_2) \end{bmatrix} = k\begin{bmatrix} -v_1 \\ v_2 \end{bmatrix} = kT(\vec{v})$

↳ T is a linear transformation

↳ Ex) $T:\mathbb{R}^2 \to \mathbb{R}^2$: scaling by $1/2$



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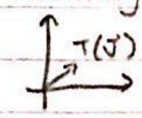
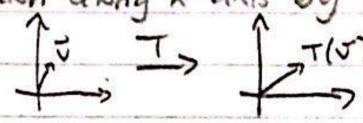


$T(\vec{v}) = \begin{bmatrix} \frac{1}{2}v_1 \\ \frac{1}{2}v_2 \end{bmatrix}$ → check if linear transformation

↳ Ex) $T:\mathbb{R}^2 \to \mathbb{R}^2$: translation along x-axis by 2

$T(\vec{v}) = \begin{bmatrix} v_1 + 2 \\ v_2 \end{bmatrix}$



↳ $T(\vec{v}+\vec{w}) = \begin{bmatrix} v_1 + w_1 + 2 \\ v_2 + w_2 \end{bmatrix} \neq T(\vec{v}) + T(\vec{w}) = \begin{bmatrix} v_1 + w_1 + 4 \\ v_2 + w_2 \end{bmatrix}$

↳ T is not a linear transformation

↳ Note: any linear transformation has to send the origin to itself

· Linear transformations and matricies

↳ proposition: a map $T:\mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation iff

↳ there exists an $n \times m$ matrix A such that $T(\vec{v}) = A\vec{v}$ for all vectors $\vec{v} \in \mathbb{R}^m$

↳ suppose that $T(\vec{v}) = A\vec{v}$ for all $\vec{v}$

↳ we have $T(\vec{v}+\vec{w}) = A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w} = T(\vec{v}) + T(\vec{w})$ ✓

↳ $T(k\vec{v}) = A(k\vec{v}) = kA\vec{v} = kT(\vec{v})$ ✓

↳ conversely, let T be a linear transformation, find an $n \times m$ matrix A so $T(\vec{v}) = A\vec{v}$

↳ $e_i = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ ← ith entry ← ith standard basis vector of $\mathbb{R}^n$

$A = \begin{bmatrix} T(e_1) & \dots & T(e_n) \end{bmatrix}$ ← $A\vec{v} = T(\vec{v})$

10/11 Lecture Linear Transformations

- Midterm #1 on Ch.1 nd Ch.2 (10/21)
- $f: X \to Y$ map (or function), X = domain, Y = codomain
  - ↳ $f(X) \subseteq Y$: image of f
  - ↳ $x \in X$, then $f(x) = y$ is the image of x under f
- Linear transformations and matricies
  - ↳ Proposition: a map $T: \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation iff there exists an $n \times m$ matrix A such that $T(\vec{v}) = A\vec{v}$ for all $\vec{v} \in \mathbb{R}^m$
  - ↳ Assume that $T: \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation
    - ↳ Let $e_i := \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ ← ith entry, ← called the ith standard basis vector of $\mathbb{R}^m$
    - ↳ Ex) in $\mathbb{R}^3$: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
  - ↳ For any $\vec{v} \in \mathbb{R}^m$:
    - ↳ $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} = v_1\vec{e}_1 + v_2\vec{e}_2 + \dots + \underbrace{v_m}_{\text{scalar}}\underbrace{\vec{e}_m}_{\text{vector}}$
    - ↳ Ex) $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
    - ↳ We say that $\vec{v}$ is a "linear combination" of the standard basis vectors
      - ↳ $T(\vec{v}) = T(v_1\vec{e}_1 + v_2\vec{e}_2 + \dots v_m\vec{e}_m) = T(v_1\vec{e}_1) + \dots + T(v_m\vec{e}_m)$

$$= v_1 T(\vec{e}_1) + \dots + v_m T(\vec{e}_m)$$

$$= v_1 \begin{bmatrix} T(\vec{e}_1)_1 \\ \vdots \\ T(\vec{e}_1)_n \end{bmatrix} + \dots + v_m \begin{bmatrix} T(\vec{e}_m)_1 \\ \vdots \\ T(\vec{e}_m)_n \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} | & & | \\ T(\vec{e}_1) & \dots & T(\vec{e}_m) \\ | & & | \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}}_{\vec{v}}$$

↳ A is such that $A\vec{v} = T(\vec{v})$ for all $\vec{v} \in \mathbb{R}^m$

↳ Ex) Find the matrix corresponding to a scaling by $1/2$ of the plane

↳ $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(\vec{v}) = \begin{bmatrix} \frac{1}{2}v_1 \\ \frac{1}{2}v_2 \end{bmatrix}$ ← writing the map 1st, then matrix

↳

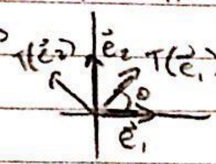
The matrix is $\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$

Transformation of $\vec{e}_1$

↳ Ex) What is the matrix corresponding to the rotation through an angle θ of the plane

writing the matrix 1st, then map

↳ $T: \mathbb{R}^2 \to \mathbb{R}^2$

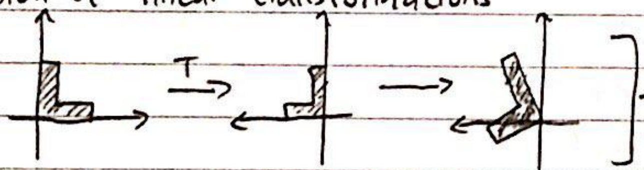
↳ 

$T(\vec{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ → Matrix $= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

↳ $T: \mathbb{R}^2 \to \mathbb{R}^2$ is $T(\vec{v}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1\cos\theta - v_2\sin\theta \\ v_1\sin\theta + v_2\cos\theta \end{bmatrix}$

· Composition of linear transformations

↳ Ex)



Composing 2 linear transformations

↳ Hint: What would happen if you 1st rotate, then reflect?

↳ Not the same end result or same?

↳ Recall: given 2 maps $f: x \to y$, $g: y \to z$ the composition of f with g is the map $g \circ f: x \to z$, such that $g \circ f(x) = g(f(x))$

↳ $S: \mathbb{R}^p \to \mathbb{R}^m$ and $T: \mathbb{R}^m \to \mathbb{R}^n$ are linear transformations

↳ The composition of S with T is the map given by $T \circ S: \mathbb{R}^p \to \mathbb{R}^n$

$\vec{v} \to T(S(\vec{v}))$

↳ Note: the composition of 2 linear transformations is also a linear transformation

↳ Matrix of the composition is the product of the matrices of the 2 linear transformations

↳ Let B be an $n \times m$ matrix such that $T(\vec{v}) = B\vec{v}$

A be an $m \times p$ " " " $S(\vec{v}) = A\vec{v}$

↳ $T \circ S(\vec{v}) = BA\vec{v}$

10/14 Lecture: Inverses

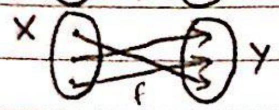
- The inverse of a linear transformation:
  - ↳ Recall: invertible functions, a function $f: X \to Y$ is said to be invertible if for every $y \in Y$ there is exactly one $x \in X$ such that $f(x) = y$
    - ↳ If f is invertible, the inverse of f is the function $f^{-1}: Y \to X$, such that $f^{-1}(y) = x$ iff $f(x) = y$
    - ↳ Ex)

not invertible, $f(x) = y_1$ has no unique solution

    - ↳ Ex)

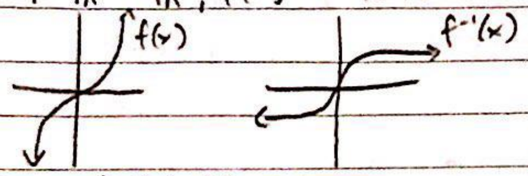
not invertible, $f(x) = y_3$ has no solution

    - ↳ Ex)



invertible, every $y \in Y$ has 1 $x \in X$ such that $f(x) = y$

    - ↳ Ex) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is invertible and $f^{-1}(x) = \sqrt[3]{x}$



  - ↳ Properties of $f^{-1}: Y \to X$
    - ↳ $f^{-1}(f(x)) = x$ for all $x \in X$
    - ↳ $f(f^{-1}(y)) = y$ for all $y \in Y$
    - ↳ f^{-1} is unique
  - ↳ Definition: a linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ is invertible if it is invertible as a function
    - ↳ Properties:
      - ↳ The inverse of a linear transformation is also a linear transformation
      - ↳ If $T: \mathbb{R}^m \to \mathbb{R}^n$ is an invertible linear transformation, then $n = m$
    - ↳ A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible $\to$ $T(\vec{v}) = A\vec{v}$ for all $\vec{v} \in \mathbb{R}^n$
      - ↳ The equation $T(\vec{v}) = \vec{w}$ has a unique solution $\vec{v} \in \mathbb{R}^n$ for all $\vec{w} \in \mathbb{R}^n$
      - ↳ The system $A\vec{v} = \vec{w}$ has a unique solution $\vec{v} \in \mathbb{R}^n$ for all $\vec{w} \in \mathbb{R}^n$
      - ↳ The rank of A = # of columns of $A = n$
        - ↳ Conversely, if rank$(A) = n$, then:
          - ↳ $\mathrm{RREF}(A) = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ → unique solution

↳ Therefore the system $A\vec{v}=\vec{w}$ has exactly 1 solution

↳ All together

↳ Proposition: A linear transformation $T:\mathbb{R}^n \to \mathbb{R}^n$ with $T(\vec{v})=A\vec{v}$ for all $\vec{v}\in\mathbb{R}^n$ is invertible iff $\text{rank}(A)=n$

↳ Ex) $T:\mathbb{R}^2\to\mathbb{R}^2$, $T(\vec{v})=\begin{bmatrix} 2v_1 \\ v_1+2v_2 \end{bmatrix}$, $A=\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

↳ Is T invertible? Does the system $A\vec{v}=\vec{w}$ have a unique solution?

↳ $\left[\begin{array}{cc|c} 2 & 0 & w_1 \\ 1 & 2 & w_2 \end{array}\right] \times \frac{1}{2} = \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2}w_1 \\ 1 & 2 & w_2 \end{array}\right] - (I) = \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2}w_1 \\ 0 & 2 & -\frac{1}{2}w_1+w_2 \end{array}\right] \times \frac{1}{2} =$

$\left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2}w_1 \\ 0 & 1 & -\frac{1}{4}w_1+\frac{1}{2}w_2 \end{array}\right]$

$\text{rank}(A)=2$, there is a unique solution and T is invertible

↳ This solution is $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}w_1 \\ -\frac{1}{4}w_1+\frac{1}{2}w_2 \end{bmatrix}$

↳ $T^{-1}:\mathbb{R}^2\to\mathbb{R}^2$ is $T(\vec{w})=\begin{bmatrix} \frac{1}{2}w_1 \\ -\frac{1}{4}w_1+\frac{1}{2}w_2 \end{bmatrix}$

↳ Check: $T^{-1}(T(\vec{v})) \stackrel{?}{=} \vec{v}$

$T^{-1}\left(\begin{bmatrix} 2v_1 \\ v_1+2v_2 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2}(2v_1) \\ -\frac{1}{4}(2v_1)+\frac{1}{2}(v_1+2v_2) \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ ✓

↳ Check: $T(T^{-1}(\vec{v})) \stackrel{?}{=} \vec{v}$

↳ Definition: A square $n\times n$ matrix A is invertible if $T:\mathbb{R}^n\to\mathbb{R}^n$ with $T(\vec{v})=A\vec{v}$ for all $\vec{v}\in\mathbb{R}^n$ is invertible

↳ We denote by A^{-1} the matrix representing T^{-1} and we call it the inverse matrix of A

↳ Note: $AA^{-1}=I_n=A^{-1}A$, A^{-1} is unique

↳ Ex) Is $A=\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ invertible? Solve by putting A in RREF

↳ $\text{rank}(A)=2$ ∴ A is invertible

↳ $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ → solve, put in RREF

↳ $\left[\begin{array}{cc|c} 1 & 0 & 3w_1-w_2 \\ 0 & 1 & -2w_1+w_2 \end{array}\right]$

$A^{-1}=\begin{bmatrix} T^{-1}(e_1) & T^{-1}(e_2) \end{bmatrix}$

$=\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

↳ Unique solution: $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3w_1-w_2 \\ -2w_1+w_2 \end{bmatrix}$

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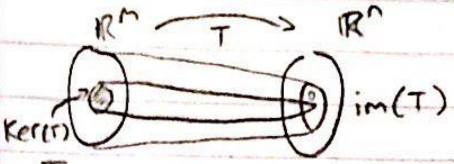
10/16 Lecture. Images and Kernels

· Subspaces of $\mathbb{R}^2$ and their dimensions

↳ $T: \mathbb{R}^m \to \mathbb{R}^n$



· Image of a linear transformation

↳ Recall: image of a function $f: X \to Y$ is the subset of Y defined as follows: $im(f) = \{y \in Y$, there exists some $x \in X$ such that $y = f(x)\}$

↳ Ex) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$: $im(f) = \{y \in \mathbb{R}, y > 0\}$

↳ Ex) $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

↳ $im(T)$ consists of all vectors of the form $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + v_2\begin{bmatrix} 3 \\ 6 \end{bmatrix}$
$= v_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3v_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

↳ $im(T)$ is all scalar multiples of the column vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

↳ Ex) $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

↳ $im(T)$ consists of all vectors of the form $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + v_2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

↳ $im(T)$ consists of a plane in $\mathbb{R}^3$ that passes through the column vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and the origin

↳ Definition: Let $v_1, \dots, v_m$ be vectors in $\mathbb{R}^n$. A vector w in $\mathbb{R}^n$ is a <u>linear combination</u> of these vectors if there are scalars $c_1, \dots, c_m$ so that $w = c_1v_1 + \dots + c_mv_m$

↳ A set of all possible linear combinations for the vectors is called their <u>span</u> → $span(v_1, \dots, v_m) = \{c_1v_1 + \dots + c_mv_m : c_1, \dots, c_m \in \mathbb{R}\}$

↳ Proposition: $T: \mathbb{R}^m \to \mathbb{R}^n$ with matrix A, $T(\vec{v}) = A\vec{v}\ \forall \vec{v} \in \mathbb{R}^m$ ($\forall$ = for all)

↳ Then, $im(T) = span(w_1, \dots, w_m)$, where $w_1, \dots, w_m$ are vectors corresponding to the columns of A

↳ $T(\vec{v}) = A\vec{v} = \begin{bmatrix} w_1 & \cdots & w_m \end{bmatrix}\begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} = v_1w_1 + \dots + v_mw_m$

↳ Properties:

↳ The zero vector $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ in $\mathbb{R}^n$ is in the image

↳ The image is closed under addition

↳ for any 2 vectors in the image, their sum is also in the image

↳ The image is closed under scalar multiplication

↳ im(T) is closed under taking linear combinations

· Kernel of a linear transformation

↳ Recall. The set of zeros of a function $f: \mathbb{R} \to \mathbb{R}$ is the set of solutions of $f(x)=x$, $zeros(f) = \{x \in \mathbb{R} : f(x) = 0\}$

↳ Ex) $f(x) = \sin(x)$, $zeros(f) = \{k\pi, k \in \mathbb{R}\}$

↳ Ex) $f(x) = x^2$, $zeros(f) = \{0\}$

↳ Notation: origin denoted by 0 in $\mathbb{R}^n$

↳ Definition: $T: \mathbb{R}^m \to \mathbb{R}^n$, the kernal of T is the solution set of the linear system given by $A\vec{v} = 0$, denoted by $\ker(T)$

↳ $\ker(T) = \{\vec{v} \in \mathbb{R}^m : A\vec{v} = 0\} = \{\vec{v} \in \mathbb{R}^m : T(\vec{v}) = 0\} \subseteq \mathbb{R}^m$ ($\subseteq$ = subset of)

↳ Properties:

↳ $0 \in \ker(T)$

↳ if $\vec{v}, \vec{w} \in \ker(T)$, then $\vec{v} + \vec{w} \in \ker(T)$

↳ if $\vec{v} \in \ker(T)$ and $k \in \mathbb{R}$, then $k\vec{v} \in \mathbb{R}$

↳ Ex) $T: \mathbb{R}^3 \to \mathbb{R}^2$ with $T(\vec{v}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \vec{v}$, Find $\ker(T)$

↳ Solve the system $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

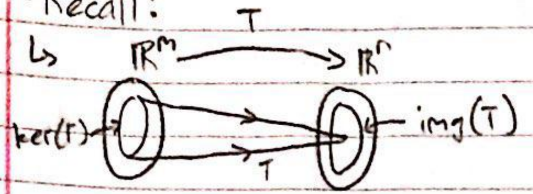
$$RREF\left(\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} t \in \mathbb{R} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\ker(T) = span\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right)$$

10/18 Lecture: Subspaces

- Recall:

↳ 

$\ker(T) = \{\vec{v} \in \mathbb{R}^m : T(\vec{v}) = 0\} \subseteq \mathbb{R}^m$
$= \{\vec{v} \in \mathbb{R}^m : A\vec{v} = 0\}$

$\operatorname{img}(T) = \{\vec{w} \in \mathbb{R}^n : \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \text{ in } \mathbb{R}^m\} \subseteq \mathbb{R}^n$
$= \{\vec{w} \in \mathbb{R}^n : \vec{w} = A\vec{v} \text{ for some } \vec{v} \text{ in } \mathbb{R}^m\}$

- Definition: Let A be an $n \times m$ matrix and $T: \mathbb{R}^m \to \mathbb{R}^n$ with $T(\vec{v}) = A\vec{v}$
  - ↳ The kernel of A is the kernel of T, denoted ker(A)
  - ↳ The image of A is the image of T, denoted img(A)
  - ↳ Ex) If $\ker(A) = \{0\}$, what can we say about A
    - ↳ Kernel is the solution of $A\vec{v} = 0$, A must have exactly 1 solution, therefore rank(A) = # of columns
- Proposition: For any $n \times m$ matrix A, $\ker(A) = \{0\} \longleftrightarrow \operatorname{rank}(A) = m$
  - ↳ Ex) Let A be an $n \times n$ invertible matrix
    - ↳ ker(A) = ?
      - ↳ Recall: A invertible $\longleftrightarrow$ $A\vec{v} = \vec{w}$ has a unique solution $\vec{v} \in \mathbb{R}^n$ for every $\vec{w} \in \mathbb{R}^n$
      - ↳ Therefore, $A\vec{v} = \vec{0}$ has 1 solution: $\ker(A) = \{0\}$
    - ↳ img(A) = ?
      - ↳ For every $\vec{w} \in \mathbb{R}^n$ there is a $\vec{v} \in \mathbb{R}^n$ such that $\vec{w} = A\vec{v}$
      - ↳ $\operatorname{img}(A) = \mathbb{R}^n$
- Characterization of invertible matricies
  - ↳ For any $n \times n$ matrix A, the following are equivalent:
    - ↳ A is invertible
    - ↳ The linear system $A\vec{v} = \vec{w}$ has a unique solution $\vec{v} \in \mathbb{R}^n$ for every $\vec{w} \in \mathbb{R}^n$
    - ↳ $\operatorname{rref}(A) = I_n$
    - ↳ $\operatorname{rank}(A) = n$, $\ker(A) = \{0\}$, $\operatorname{img}(A) = \mathbb{R}^n$
- Definition: a subset W of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if it:
  - ↳ contains 0
  - ↳ is closed under addition
  - ↳ is closed under scalar multiplication

↳ Ex) Let $T: \mathbb{R}^m \to \mathbb{R}^n$ with $T(\vec{v}) = A\vec{v}$, $\ker(T) = \ker(A)$ is a subspace of $\mathbb{R}^m$
$\operatorname{img}(T) = \operatorname{img}(A)$ " " " " $\mathbb{R}^n$

↳ Remark: $\mathbb{R}^n$ is a vector space, every subspace of a vector space is a vector space

↳ Ex) $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x \geq 0, y \geq 0 \right\}$, is W a subspace of $\mathbb{R}^2$?

↳ no, not closed under scalar multiplication

↳ Ex) What are all possible subspaces in $\mathbb{R}^2$

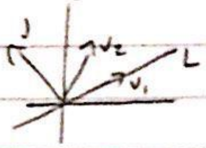
↳ any line passing through the origin

↳ $W = \operatorname{span}(\vec{v})$, where $\vec{v} \in \mathbb{R}$

↳ the origin, $W = \{0\}$

↳ $\mathbb{R}^2$, $W = \mathbb{R}^2$

⟶ Suppose W is a subspace different from $\{0\}$ and not a line passing through the origin, let $0 \neq \vec{v}_1 \in W$



$L = \operatorname{span}(\vec{v}_1)$ is a subset of W, different from W

Let $v_2 \in W$, and v_2 does not lie on W

↳ Let $\vec{v}$ be any vector in $\mathbb{R}^2$, then there are scalars c_1, c_2 such that $c_1 v_1 + c_2 v_2 = \vec{v}$, $\vec{v} \in W$, $W = \mathbb{R}^2$

10/25 Lecture: Linear Dependency

- Recall: $\vec{v}_1 \dots \vec{v}_m$ in $\mathbb{R}^n$ are linearly dependent when there are non-trivial relations between them
  - ↳ there are not all zero numbers $c_1 \dots c_m$ such that $c_1v_1 + \dots + c_mv_m = 0$
  - ↳ contrapositive: $\vec{v}_1 \dots \vec{v}_m$ in $\mathbb{R}^n$ are linearly independent, then the only relationship between the vectors are trivial
    - ↳ if $c_1v_1 + \dots + c_mv_m = 0$, then $c_1 = \dots = c_m = 0$
  - ↳ Ex) $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$
    - ↳ linearly dependent because $\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = 0$
  - ↳ Ex) $\vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{w}_3 = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix}$
    - ↳ linearly independent
  - ↳ Ex) Suppose that the column vectors of an $n \times m$ matrix A are linearly independent, what is $\ker(A)$?
    - ↳ Let $a_1 \dots a_m$ be the columns of A
      - ↳ $a_1 \dots a_m$ are linearly independent, the only solution to $c_1a_1 + \dots + c_ma_m = 0$ is the trivial solution $(c_1 = \dots = c_m = 0)$
      - ↳ To find the kernel of A, solve the system $A\vec{v} = 0 \to v_1a_1 + \dots v_ma_m = 0$
      - ↳ $\therefore v_1 = \dots v_m = 0$
      - ↳ $\ker(A) = \{0\}$
    - ↳ Summary: the vectors in $\ker(A)$ correspond to the linear relations among the column vectors of A
      - ↳ if the column vectors of A are linearly independent, $\ker(A) = \{0\}$
- Finding basis for kernel and image
  - ↳ Ex) $A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & 8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}$ Find a basis for i) $\ker(A)$ and ii) $\operatorname{im}(A)$
    - ↳ $\operatorname{RREF}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B \to \ker(A) = \ker(B)$, $A\vec{v} = 0$ same as $B\vec{v} = 0$
    - ↳ Solutions are: $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} -2s-3t+4r \\ s \\ 4t-5r \\ t \\ r \end{bmatrix}$, $s, t, r \in \mathbb{R}$

↳
$$= s\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix} + t\begin{bmatrix}-3\\0\\4\\1\\0\end{bmatrix} + r\begin{bmatrix}4\\0\\-5\\0\\1\end{bmatrix} = s\vec{w}_1 + t\vec{w}_2 + r\vec{w}_3$$

↳ Claim: $\vec{w}_1, \vec{v}_2, \vec{w}_3$ form a basis for $\ker(A)$

↳ the vectors span the kernal

↳ the vectors are linearly independent

↳ basis for $\text{im}(A)$

↳ we need to find the redundant columns of A

↳ denote the columns of A by $a_1, \dots, a_5$ → same w/ B

↳ redundant columns in B → columns w/out pivots

↳ key observation: b_i is redundant when a_i is redundant

↳ given by a_1 and a_3

10/30 Lecture: Dimensions of Subspaces

· Recall: given any $n \times m$ matrix A

↳ basis for $\ker(A)$ → solve the system $A\vec{v} = \vec{0}$

↳ solution is of the form $\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = s_{11} \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\vec{b}_1} + s_{12} \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\vec{b}_2} + \dots + s_{1k} \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\vec{b}_k}$

↳ The vectors $\vec{b}_1, \vec{b}_2, \dots, \vec{b}_k$ form a basis for $\ker(A)$

↳ basis for $\text{im}(A)$ → given by column vectors of A that correspond to column vectors w/ pivots in $\text{rref}(A)$

↳ important remark: $\ker(A) = \ker(\text{rref}(A))$, but $\text{im}(A) \neq \text{im}(\text{rref}(A))$ in general

· Dimension of subspaces of $\mathbb{R}^n$

↳ Recall: we had seen that all possible subspaces of $\mathbb{R}^2$ are:

↳ $\{0\}$ → basis: ϕ

↳ lines through the origin → basis: $0 \neq \vec{v}$ in $\mathbb{R}^2$

↳ $\mathbb{R}^2$ → basis: any pair of vectors $\vec{v}, \vec{u}$ in $\mathbb{R}^2$ that are linearly independent

↳ General fact: Let V be a subspace of $\mathbb{R}^n$. Then any basis of V has the same # of vectors

↳ Definition: Let V be a subspace of $\mathbb{R}^n$. The # of vectors in a basis of V is called the <u>dimension of V</u>, and denoted by $\dim(V)$

↳ Ex) Lines thru the origin in $\mathbb{R}^2$ are 1-dimensional

↳ Ex) For any $n \times m$ matrix A, $\dim(\text{im}(A)) = \text{rank}(A)$, $\dim(\ker(A)) =$ # of free variables $=$ # of total variables $-$ # of leading variables $= m - \text{rank}(A)$

· Rank-Nullity Theorem: Let A be any $n \times m$ matrix. Then $\dim(\text{im}(A)) + \dim(\ker(A)) = m$

↳ $\dim(\ker(A)) =$ nullity of the matrix A

↳ in other words: $\text{rank}(A) +$ nullity of $A = m$

↳ Geometric interpretation:

↳ let $T: \mathbb{R}^m \to \mathbb{R}^n$ w/ $T(\vec{v}) = A\vec{v}$

↳ nullity of A counts the # of dimensions that collapse as we perform T

↳ rank of A counts the # of dimensions that survive T

· Basis for $\mathbb{R}^n$

↳ Recall: the vectors $\vec{e}_1, \dots, \vec{e}_n$ in $\mathbb{R}^n$ are called standard basis vectors of $\mathbb{R}^n$ and form a basis for $\mathbb{R}^n$

↳ they span $\mathbb{R}^n$

↳ they are linearly independent

↳ Given n vectors in $\mathbb{R}^n$: $\vec{b}_1, \ldots, \vec{b}_n$, how can we tell whether they form a basis for $\mathbb{R}^n$? ⊕

· Characterization of basis: unique representation

↳ Let V be any subspace of $\mathbb{R}^n$. Then $\vec{b}_1, \ldots, \vec{b}_n$ is a basis for V

↳ every vector $\vec{v}$ in V can be written uniquely as a linear combination of $\vec{b}_1, \ldots, \vec{b}_n$

↳ in other words: if there is a unique solution to $c_1\vec{b}_1 + \ldots + c_m\vec{b}_m = \vec{v}$

↳ in relation to ⊕ → $\vec{b}_1, \ldots, \vec{b}_n$ form a basis → every vector $\vec{w} \in \mathbb{R}^n$ can be written uniquely as $\vec{w} = c_1\vec{b}_1 + \ldots + c_n\vec{b}_n$

↳ $= \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

↳ we know this system has a unique solution in $\mathbb{R}^n$ for any $\vec{w} \in \mathbb{R}^n$ iff $[b_1 \cdots b_n]$ is invertible

↳ Summary: Let $b_1, \ldots, b_n$ be any vectors in $\mathbb{R}^n$, they form a basis for $\mathbb{R}^n$ iff $[b_1 \cdots b_n]$ is invertible

11/1 Lecture: Coordinate Systems

- Coordinates

↳ Ex) $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $V = \text{span}(\vec{v}_1, \vec{v}_2)$

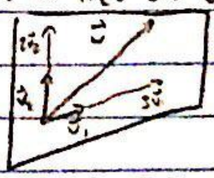
↳ Let $\vec{w} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$, does $\vec{w}$ lie on V

↳ $\vec{w} \in V$ if there are scalars c_1 and c_2 and $w = c_1\vec{v}_1 + c_2\vec{v}_2$

↳ same as solving the system $\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 1 & 3 & 9 \end{array}\right] = A|w$

↳ $\text{rref}(A|w) = \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \rightarrow$ unique solution: $c_1 = 3$, $c_2 = 2$

↳ ∴ w lies on the plane, $w = 3\vec{v}_1 + 2v_2$

↳ 

Interpretation: the lines spanned by $\vec{v}_1$ and $\vec{v}_2$ are axes for a coordinate system on V, $w = (3,2)$

↳ $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\} \rightarrow [w]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

- Definition

↳ Let V be a subspace of $\mathbb{R}^n$ with basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\}$. We know that each vector $\vec{v}$ in V can be expressed as a unique linear combination $\vec{v} = c_1\vec{v}_1 + \dots + c_m\vec{v}_m$

↳ We call the numbers $c_1, \dots, c_m$ the $\mathcal{B}$ coordinates of $\vec{v}$ and we write:

$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$

↳ Let $S = [\vec{v}_1 \cdots \vec{v}_m]$, then $\vec{v} = S[v]_{\mathcal{B}}$

↳ This gives us a linear transformation $\mathbb{R}^m \rightarrow \mathbb{R}^n$ / $[\vec{v}]_{\mathcal{B}} \longmapsto S[\vec{v}]_{\mathcal{B}} = \vec{v}$

- Coordinate systems on $\mathbb{R}^n$

↳ Given a basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$, Let $S = [\vec{v}_1 \cdots \vec{v}_n]$

↳ There is a linear transformation $\mathbb{R}^n \rightarrow \mathbb{R}^n$ / $[\vec{v}]_{\mathcal{B}} \longmapsto S[\vec{v}]_{\mathcal{B}} = \vec{v}$

↳ Since $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent, S is invertible

↳ $\mathbb{R}^n \underset{S^{-1}}{\overset{S}{\rightleftarrows}} \mathbb{R}^n \rightarrow [\vec{v}]_{\mathcal{B}} \mapsto S\vec{v} \rightarrow [\vec{v}]_{\mathcal{B}} = S^{-1}\vec{v} \longleftarrow \vec{v}$

↳ Consequence: coordinates satisfy linearity

↳ i) $[\vec{v} + \vec{w}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{B}} + [\vec{w}]_{\mathcal{B}}$ $\forall \vec{v}, \vec{w} \in \mathbb{R}^n$

↳ ii) $[k\vec{v}]_{\mathcal{B}} = k[\vec{v}]_{\mathcal{B}}$ $\forall k \in \mathbb{R}, \vec{v} \in \mathbb{R}^n$

↳ Ex) $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$ basis of $\mathbb{R}^2$

↳ Let $\vec{v} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$, what is $[\vec{v}]_{\mathcal{B}}$?

↳ Solution: solve the system: $\begin{bmatrix} 10 \\ 10 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} \Rightarrow [\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

↳ Let $[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, what is $\vec{w}$?

↳ By definition of $\mathcal{B}$-coordinates, we know: $\vec{w} = 2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$

- The matrix of a linear transformation

↳ Recall: for any lin. transformation $T: \mathbb{R}^n \to \mathbb{R}^n$, there exists an $n \times n$ matrix A such that $T(\vec{v}) = A\vec{v}$ for all $v \in \mathbb{R}^n$

↳ We constructed A as $A = [T(\vec{e}_1) \ldots T(\vec{e}_n)]$

↳ Theorem: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, and a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ of $\mathbb{R}^n$

↳
$$\begin{tikzcd} \vec{v} \arrow[r, "A"] & T(\vec{v}) \\ {[\vec{v}]_{\mathcal{B}}} \arrow[u, "S"] \arrow[r, "B"] & {[T(\vec{v})]_{\mathcal{B}}} \arrow[u, "S"'] \end{tikzcd} \qquad S = [\vec{b}_1 \ldots \vec{b}_n]$$

↳ There exists a unique $n \times n$ matrix B such that $[T(\vec{v})]_{\mathcal{B}} = B[\vec{v}]_{\mathcal{B}}$

↳ Proof: Let $\vec{v} \in \mathbb{R}^n$ be arbitrary, let $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

↳ This means $\vec{v} = c_1\vec{b}_1 + \ldots + c_n\vec{b}_n$, $[T(\vec{v})]_{\mathcal{B}} = [T(c_1\vec{b}_1 + \ldots + c_n\vec{b}_n)]_{\mathcal{B}} =$
$[c_1 T(\vec{b}_1) + \ldots + c_n T(\vec{b}_n)]_{\mathcal{B}} = c_1[T(\vec{b}_1)]_{\mathcal{B}} + \ldots + c_n[T(\vec{b}_n)]_{\mathcal{B}} =$
$\begin{bmatrix} [T(\vec{b}_1)]_{\mathcal{B}} & \cdots & [T(\vec{b}_n)]_{\mathcal{B}} \end{bmatrix} = B \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

11/4 Lecture: $\mathcal{B}$-matricies and Orthogonality

- The matrix of a linear transformation

↳ $T: \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{B} = \{b_1, \dots, b_n\}$

↳
$$\begin{array}{lccc} \text{standard coordinates} & \vec{v} & \xrightarrow{A} & T(\vec{v}) \\ & S\uparrow & & \uparrow S \\ \mathcal{B}\text{-coordinates} & [\vec{v}]_{\mathcal{B}} & \xrightarrow[B]{} & [T(\vec{v})]_{\mathcal{B}} \end{array}$$

$S = [b_1 \dots b_n]$

$S[\vec{v}]_{\mathcal{B}} = \vec{v}$

$A = [T(e_1) \dots T(e_n)]$

$B = [[T(b_1)]_{\mathcal{B}} \dots [T(b_n)]_{\mathcal{B}}]$

↳ A: standard matrix of T, B: $\mathcal{B}$-matrix of T

↳ Any 2 paths in the diagram give the same result, the diagram "commutes"

↳ $SB[\vec{v}]_{\mathcal{B}} = AS[\vec{v}]_{\mathcal{B}} = T(\vec{v})$

- Standard vs. $\mathcal{B}$-matrix

↳ $AS = SB \to S^{-1}AS = B \to A = SBS^{-1}$

- Definition: Let A and B be $n \times n$ matricies. We say that <u>A is similar to B</u> if there exists an invertible matrix S such that $AS = SB$ or $B = S^{-1}AS$
- Exercise: show that similarity is an equivalence relation for any $n \times n$ matricies A and B

↳ i) A is similar to A (reflexivity)

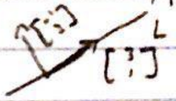
↳ ii) if A is similar to B, B is similar to A (symmetry)

↳ iii) if A is similar to B, B is similar to C, then A is similar to C (transitivity)

- What is this good for?

↳ Given a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$, there might be a basis $\mathcal{B}$ in which T has a very simple form

↳ Ex) $T: \mathbb{R}^2 \to \mathbb{R}^2$, projection onto the line $L = \text{span}\left(\begin{bmatrix}3\\1\end{bmatrix}\right)$

*? ↳ 

$\mathcal{B} = \left\{\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}-1\\3\end{bmatrix}\right\}$, $\mathcal{B}$-matrix of T is $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$

- Definition: An $n \times n$ matrix A is <u>diagonal</u> if $a_{ij} = 0$ when $i \neq j$
- Orthogonality

↳ <u>Recall</u>: Definition: 2 vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are <u>orthogonal</u> (perpendicular) if their dot product is zero

↳ Let V be a subspace of $\mathbb{R}^n$

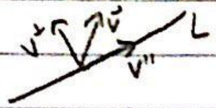
↳ A vector $\vec{v}$ in $\mathbb{R}^n$ is <u>orthogonal to the subspace</u> if it is orthogonal to all vectors in the subspace

↳ Note given a basis $b_1, \dots, b_m$ of V, then a vector $\vec{v}$ in $\mathbb{R}^n$ is orthogonal to V when it is orthogonal to the vectors in the basis

↳ Let $\vec{w}$ be any vector in V, $\vec{w} = c_1 b_1 + \dots + c_m b_m$, $\vec{v} \cdot \vec{w} = \vec{v} \cdot (c_1 b_1 + \dots + c_m v_m)$

↳ $= c_1(\vec{v} \cdot \vec{b}_1) + c_2(\vec{v} \cdot \vec{b}_2) + \dots + c_m(\vec{v} \cdot \vec{b}_m) = 0$

↳ Ex) Given a line L in $\mathbb{R}^2$, and a vector $\vec{v}$ in $\mathbb{R}^2$

↳ 

$\vec{v} = \vec{v}^{\parallel} + \vec{v}^{\perp}$ → parallel to L + orthogonal to L

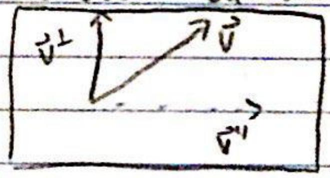
↳ this decomposition is unique

↳ $v^{\parallel}$ is the orthogonal projection of $\vec{v}$ onto L

· Orthogonal projection onto a subspace

↳ Let V be a subspace of $\mathbb{R}^n$, let $\vec{v}$ be any vector in $\mathbb{R}^n$, then $\vec{v} = \vec{v}^{\parallel} + \vec{v}^{\perp}$, where $\vec{v}^{\parallel}$ is in V, $v^{\perp}$ is orthogonal to V

↳ these vectors are unique

↳ 

V → The vector $\vec{v}^{\parallel}$ is the orthogonal projection of $\vec{v}$ onto V, denoted by $proj_V(\vec{v})$

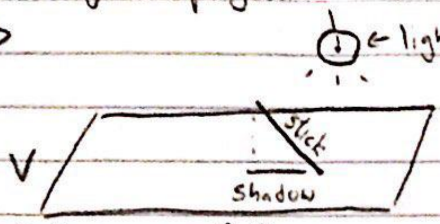
↳ The transformation $\mathbb{R}^n \to \mathbb{R}^n$ given by $T(\vec{v}) = proj_V(\vec{v})$ is linear

↳ Definition: The length of a vector $\vec{v}$ in $\mathbb{R}^n$ is $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + \dots + v_n^2}$, a vector $\vec{v}$ in $\mathbb{R}^n$ is a unit vector if $\|\vec{v}\| = 1$, vectors $\vec{u}_1, \dots, \vec{u}_m$ in $\mathbb{R}^n$ are orthonormal if $\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ for all i, j

11/6 Lecture: Orthogonal Projections

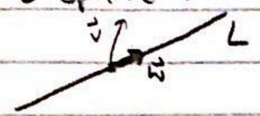
· Orthogonal projection onto a subspace V of $\mathbb{R}^n$

↳



the projection is the shadow if the vector is the stick

↳ Ex) $L = \text{span}(\vec{w})$



$\vec{v} = v^{\perp} + \vec{w}$ $\vec{w} = \text{proj}_L(\vec{v})$

↳ Let V be any subspace of $\mathbb{R}^n$, any vector in $\mathbb{R}^n$ can be written as a sum, $\vec{v} = \vec{v}^{\perp} + \vec{v}^{\parallel}$, where $\vec{v}^{\perp}$ is orthogonal to V and $v^{\parallel}$ is a vector in V

↳ these vectors are unique

↳ we call $\vec{v}^{\parallel}$ the <u>orthogonal projection</u> of $\vec{v}$ onto V and denote it by $\text{proj}_V(\vec{v})$

↳ the transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ that sends any $\vec{v} \in \mathbb{R}^n$ to $\text{proj}_V(\vec{v})$ is linear

· Orthonormality

↳ Ex) $e_1, \dots, e_m$ in $\mathbb{R}^n$, with $1 \le m \le n$ are orthonormal

↳ Ex) Let θ be any real number, then $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ are orthonormal

· Properties of Orthonormal Vectors

↳ i) orthonormal vectors are linearly independent

ii) orthonormal vectors $\vec{u}_1, \dots, \vec{u}_n$ in $\mathbb{R}^n$ form a basis of $\mathbb{R}^n$

↳ Proof of i): Let $\vec{u}_1, \dots, \vec{u}_m$ in $\mathbb{R}^n$ be orthonormal

↳ To show: if there are $c_1, \dots, c_m \in \mathbb{R}$ such that $c_1\vec{u}_1 + \dots + c_m\vec{u}_m = 0$, then $c_1 = \dots = c_m = 0$

↳ Let $\vec{u}_i$ be arbitrary, $(c_1\vec{u}_1 + \dots + c_m\vec{u}_m) \cdot \vec{u}_i = 0 \cdot u_i = 0$

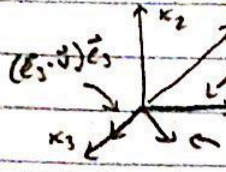
↳ $c_1\underbrace{(\vec{u}_1 \cdot \vec{u}_i)}_{0} + \dots + c_i\underbrace{(\vec{u}_i \cdot \vec{u}_i)}_{1} + \dots c_m\underbrace{(\vec{u}_m \cdot \vec{u}_i)}_{0} = 0$

↳ $c_i \cdot 1 = 0 \to c_i = 0$

· Formula for orthogonal projection

↳ Let V be a subspace of $\mathbb{R}^n$, with an orthonormal basis $\vec{u}_1, \dots, \vec{u}_m$, then for all $\vec{v}$ in $\mathbb{R}^n \to \text{proj}_V(\vec{v}) = (\vec{u}_1 \cdot \vec{v})\vec{u}_1 + \dots + (\vec{u}_m \cdot \vec{v})\vec{u}_m$

- Recall: $L = \text{span}(\vec{w})$ in $\mathbb{R}^2$ for any $\vec{v}$ in $\mathbb{R}^2$. $\text{proj}_L(\vec{v}) = \left(\frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}$
  - ↳ Note: $(\vec{u}_i \cdot \vec{v})\vec{u}_i$ is the orthogonal projection of $\vec{v}$ onto the line spanned by $\vec{u}_i$
  - ↳ Ex) Let V be the $x_1 - x_3$ plane in $\mathbb{R}^3$

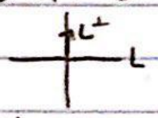


$\text{proj}_V(\vec{v}) = (\vec{e}_1 \cdot \vec{v})\vec{e}_1 + (\vec{e}_3 \cdot \vec{v})\vec{e}_3$

  - ↳ Ex) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$, $V = \text{im}(A)$
    - ↳ subspace of $\mathbb{R}^4$
    - ↳ Let $\vec{v} \in \mathbb{R}^4$ be $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 7 \end{bmatrix}$, find $\text{proj}_V(\vec{v})$
      - ↳ Solution: The 2 column vectors of A are linearly independent, so they form a basis, and are also orthogonal
        - ↳ We get an orthonormal basis of V by dividing each vector by its length
          - ↳ $\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$
          - ↳ $\text{proj}_V(\vec{v}) = (\vec{u}_1 \cdot \vec{v})\vec{u}_1 + (\vec{u}_2 \cdot \vec{v})\vec{u}_2 = 6\vec{u}_1 + 2\vec{u}_2 = \begin{bmatrix} 4 \\ 2 \\ 2 \\ 4 \end{bmatrix}$
            - ↳ can check using $\vec{v} - \text{proj}_V(\vec{v})$ is orthogonal to $\vec{u}_1$ and $\vec{u}_2$

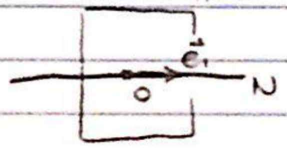
11/8 Lecture: Orthogonal Complements

- Midterm 2: Chapter 3, Chapter 5.1 - 5.2, Sheets 4, 5, 6, 7
- If $V = \mathbb{R}^n$, then the formula for the orthogonal projection
  - Theorem: Let $\vec{u}_1, \dots \vec{u}_n$ be an orthonormal basis of $\mathbb{R}^n$. Then for all $\vec{v} \in \mathbb{R}^n$: $\vec{v} = (\vec{u}_1 \cdot \vec{v})\vec{u}_1 + (\vec{u}_2 \cdot \vec{v})\vec{u}_2 + \dots + (\vec{u}_n \cdot \vec{v})\vec{u}_n = \text{proj}_{\mathbb{R}^n}(\vec{v})$
  - Note: in general, given a basis $\vec{b}_1, \dots, \vec{b}_n$ of $\mathbb{R}^n$, we know that for any $\vec{v} \in \mathbb{R}^n$, there exists unique numbers such that $\vec{v} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$
    - In general we have to solve a system to find $c_1, \dots, c_n$. If the basis is orthonormal, we have $c_i = \vec{b}_i \cdot \vec{v}$
- Let V be a subspace of $\mathbb{R}^n$, $T: \mathbb{R}^n \to \mathbb{R}^n$ be $T(\vec{v}) = \text{proj}_V(\vec{v})$
  - We have.
    - i) $\text{im}(T) = V$
    - ii) $\ker(T) = \{$all vectors in $\mathbb{R}^n$ orthogonal to $\vec{v}\}$
  - Definition: Let V be a subspace of $\mathbb{R}^n$, the <u>orthogonal complement</u> of V is the set of all vectors in $\mathbb{R}^n$ that are orthogonal to V, denoted by $V^\perp$
    - $V^\perp = \{\vec{w}$ in $\mathbb{R}^n$ such that $\vec{w} \cdot \vec{v} = 0$ for all vectors $\vec{v}$ in V
  - Ex) L a line in $\mathbb{R}^2$



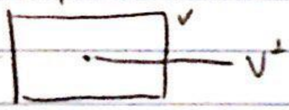
$\dim(L) + \dim(L^\perp) = 1 + 1 = 2$

  - Ex) N a line in $\mathbb{R}^3$, $N = \text{span}(\vec{e}_1)$, $N^\perp = \ker(\text{proj}_N)$



$\dim(N) + \dim(N^\perp) = 1 + 2 = 3$

  - Ex) V a plane in $\mathbb{R}^3$



$\dim(V) + \dim(V^\perp) = 2 + 1 = 3$

- Properties of the orthogonal complement
  - Let V be a subspace of $\mathbb{R}^n$
    - $V^\perp$ is a subspace of $\mathbb{R}^n \to V^\perp$ is $\ker(\text{proj}_V)$
    - $\dim(V) + \dim(V^\perp) = n \to$ consequence of the rank-nullity theorem
      - $V = \text{im}(\text{proj}_V)$, $V^\perp = \ker(\text{proj}_V)$
    - $V \cap V^\perp = \{0\}$
      - Suppose $\vec{v} \in \mathbb{R}^n$ is in both V and $V^\perp$, V has to be orthogonal to itself, therefore $\vec{v}$ would be $\vec{0}$
    - $(V^\perp)^\perp = V$

- Given any 2 sets A and B, subsets of a set C
  - $A \cap B$ = intersection of A and B = $\{x \in C$ such that $x \in A$ and $x \in B\}$
- Gram-Schmidt algorithm
  - Given a basis $\vec{v}_1, \ldots, \vec{v}_m$ of a subspace V, we want to find an orthonormal basis $\vec{u}_1, \ldots, \vec{u}_m$
    - Ex) Let $V = \text{span}(\vec{w})$ where $\vec{w} (\neq 0)$ is a vector on $\mathbb{R}^n$, then $\vec{u}_1 = \frac{\vec{w}}{\|\vec{w}\|}$ is an orthonormal basis of V
    - Ex) $V = \text{span}(\vec{v}_1, \vec{v}_2)$, $\vec{v}_1$ and $\vec{v}_2$ are linearly independent

      $\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$, $\vec{v}_2 = \text{proj}_{(\text{span}(\vec{u}_1))}(\vec{v}_2) = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = v_2^{\perp}$

      $\vec{u}_2 = \frac{v_2^{\perp}}{\|\vec{v}_2^{\perp}\|}$
  - In general: V is a subspace of $\mathbb{R}^n$, with basis $\vec{v}_1, \ldots, \vec{v}_m$
    - $\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$
    - $\vec{v}_i^{\perp} := \vec{v}_i - \underbrace{[(\vec{u}_1 \cdot \vec{v}_i)\vec{u}_1 + \ldots + (\vec{u}_{i-1} \cdot \vec{v}_i)\vec{u}_{i-1}]}_{\text{orthogonal projection of } \vec{v} \text{ onto } \text{span}(\vec{u}_1, \ldots, \vec{u}_{i-1})}$
    - $\vec{u}_i = \frac{\vec{v}_i^{\perp}}{\|\vec{v}_i^{\perp}\|}$
    - Then, $\vec{u}_1, \ldots, \vec{u}_m$ is an orthonormal basis of V
  - Ex) $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 7 \\ -8 \end{bmatrix}$, $V = \text{span}(\vec{v}_1, \vec{v}_2)$, a plane in $\mathbb{R}^3$

    $\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{\vec{v}_1}{3} = \frac{1}{3}\vec{v}_1$ $\quad L = \text{span}(\vec{v}_1)$

    $\text{proj}_L(\vec{v}_2) = (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = [\frac{1}{3}(27)]\vec{u}_1 = 3\vec{v}_1$

    $\vec{v}_2^{\perp} = \vec{v}_2 - 3\vec{v}_1 = \begin{bmatrix} -4 \\ 4 \\ -2 \end{bmatrix}$

    $\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{6} = \frac{1}{6}\vec{v}_2$

    $$\boxed{\vec{u}_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1/3 \\ 7/6 \\ -4/3 \end{bmatrix}}$$

11/13 Lecture: QR Factorization

- Recall: Gram-Schmidt Algorithm
  - ↳ input: a subspace V of $\mathbb{R}^n$ and a basis $\vec{b}_1, \ldots, \vec{b}_m$ of V
  - ↳ output: orthonormal basis $\vec{u}_1, \ldots, \vec{u}_m$ of V
  - ↳ Let $\mathcal{B} = \{b_1, \ldots, b_m\}$, $\mathcal{U} = \{\vec{u}_1, \ldots, \vec{u}_m\}$
  - ↳ Recall: Let $S_\mathcal{B} = [\vec{b}_1 \ldots \vec{b}_m]$, then for every $\vec{v}$ in V we have
    - ↳ $S_\mathcal{B}[\vec{v}]_\mathcal{B} = \vec{v}$
    - ↳ Let $S_\mathcal{U} = [\vec{u}_1 \ldots \vec{u}_m]$
    - ↳ $\mathcal{B}$ coordinates

$$[\vec{v}]_\mathcal{B} \xrightarrow{S_\mathcal{B}} \underset{\text{standard}}{\vec{v}} \xleftarrow{S_\mathcal{U}} [\vec{v}]_\mathcal{U}$$

→ This means, $S_\mathcal{B}[\vec{v}]_\mathcal{B} = \vec{v} = S_\mathcal{U}[\vec{v}]_\mathcal{U}$

$$[\vec{b}_i] = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \} m \text{ rows} \\ \leftarrow i\text{th entry} \end{array}$$

- ↳ In particular, for any vector $\vec{b}_i$ in $\mathcal{B}$, $S_\mathcal{B}[\vec{b}_i] = i$th column of $S_\mathcal{B} = S_\mathcal{U}[\vec{b}_i]_\mathcal{U}$
- ↳

$$\underbrace{\begin{bmatrix} | & & | \\ \vec{b}_1 & \cdots & \vec{b}_m \\ | & & | \end{bmatrix}}_{n\times m \text{ matrix } S_\mathcal{B}} \underbrace{\begin{bmatrix} | & & | \\ [\vec{b}_1]_\mathcal{B} & \cdots & [\vec{b}_m]_\mathcal{B} \\ | & & | \end{bmatrix}}_{m\times m \text{ matrix } I_m} = \underbrace{\begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix}}_{n\times m \text{ matrix } Q} \underbrace{\begin{bmatrix} | & & | \\ [\vec{b}_1]_\mathcal{U} & \cdots & [\vec{b}_m]_\mathcal{U} \\ | & & | \end{bmatrix}}_{m\times m \text{ matrix } R}$$

- QR Factorization - Let M be any $n\times m$ matrix with linearly independent column vectors, then there exists matrices Q and R where Q is an $n\times m$ matrix with orthonormal column vectors, and R is an $m\times m$ upper-diagonal matrix
  - ↳ Upper diagonal - form $\begin{bmatrix} \ddots & * \\ 0 & \ddots \end{bmatrix}$, where the diagonal entries are positive
- ↳ Q and R are unique
- ↳ Proof: Existence of Q and R
  - ↳ Apply Gram Schmidt to the column vectors of M
- ↳ Proof: Properties of R
  - ↳ Let $\vec{b}_1, \ldots, \vec{b}_m$ be the column vectors of M, and $\vec{u}_1, \ldots, \vec{u}_m$ be the column vectors of Q
  - ↳ Then:

$$R = \begin{bmatrix} | & & | \\ [\vec{b}_1]_\mathcal{U} & \cdots & [\vec{b}_m]_\mathcal{U} \\ | & & | \end{bmatrix}, \text{ where } \mathcal{U} = \{\vec{u}_1, \ldots, \vec{u}_m$$

- ↳ Recall: $\vec{u}_1 = \frac{\vec{b}_1}{\|\vec{b}_1\|} \rightarrow [\vec{b}_1]_\mathcal{U} = \|\vec{b}_1\| \vec{u}_1$

$\vec{u}_2 = \frac{\vec{b}_2^\perp}{\|\vec{b}_2^\perp\|}$, where $\vec{b}_2^\perp = \vec{b}_2 - (\vec{u}_1 \cdot \vec{b}_2)\vec{u}_1 \rightarrow [\vec{b}_2]_\mathcal{U} = (\vec{u}_1 \cdot \vec{b}_2)\vec{u}_1 + \|\vec{b}_2^\perp\| \vec{u}_2$

↳ $R = \begin{bmatrix} \|\vec{b}_1\| & \vec{u}_1\cdot\vec{b}_2 & \cdots & \vec{u}_1\cdot\vec{b}_m \\ 0 & \|\vec{b}_2^\perp\| & \cdots & \vec{u}_2\cdot\vec{b}_m \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \|\vec{b}_m^\perp\| \end{bmatrix}$

↳ Ex) Find the QR Factorization of $M = \begin{bmatrix} 2 & 2 \\ 1 & 7 \\ -2 & 8 \end{bmatrix}$

↳ $\vec{b}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 2 \\ 7 \\ -8 \end{bmatrix}$

↳ $M = \begin{bmatrix} | & | \\ \vec{b}_1 & \vec{b}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix} \begin{bmatrix} \|\vec{b}_1\| & \vec{u}_1\cdot\vec{b}_2 \\ 0 & \|\vec{b}_2^\perp\| \end{bmatrix}$

↳ 1st column of R, then 1st of Q, then 2nd of R, then 2nd of Q

↳ $\|\vec{b}_1\| = 3$

↳ $\vec{u}_1 = \frac{\vec{b}_1}{\|\vec{b}_1\|} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

↳ $\vec{u}_1 \cdot \vec{b}_2 = 9$

↳ $\vec{b}_2^\perp = \begin{bmatrix} -4 \\ 4 \\ -2 \end{bmatrix}$

↳ $\|\vec{b}_2^\perp\| = 6$

↳ $\vec{u}_2 = \frac{\vec{b}_2^\perp}{6} = \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$

↳ $Q = \frac{1}{3}\begin{bmatrix} 2 & -2 \\ 1 & 2 \\ -2 & -1 \end{bmatrix}$, $R = \begin{bmatrix} 3 & 9 \\ 0 & 6 \end{bmatrix}$

• Orthogonal transformations

↳ Definition: A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if it preserves length of vectors, i.e. if $\|T(\vec{v})\| = \|\vec{v}\|$ for all $\vec{v}$ in $\mathbb{R}^n$

↳ Note: Any rotation and reflection in $\mathbb{R}^2$ is an orthogonal transformation

↳ Ex) $T: \mathbb{R}^2 \to \mathbb{R}^2$ counterclockwise rotation through angle θ

↳ For all $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in $\mathbb{R}^2$: $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \cos\theta v_1 - \sin\theta v_2 \\ \sin\theta v_1 + \cos\theta v_2 \end{bmatrix}$

↳ $\|T(\vec{v})\|^2 = (\cos\theta v_1 - \sin\theta v_2)^2 + (\sin\theta v_1 + \cos\theta v_2)^2 = v_1^2 + v_2^2 = \|v\|^2$

↳ T is orthogonal

11/15 Lecture: Orthogonal Transformations

- Recall: $T: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if $\|T(\vec{x})\| = \|\vec{x}\|$ for all vectors in $\mathbb{R}^n$

↳ Ex) rotations and reflections in $\mathbb{R}^2$

↳ Remark: If T preserves length, then it also preserves angles

↳ If the angle changes, the length of the line that connects the terminal ends of the vectors changes

↳ Used to create the matrix A for T

↳ Start with the standard basis $\vec{e}_1, \dots, \vec{e}_n$ and T orthogonal

↳ $A = \begin{bmatrix} T(\vec{e}_1) & \dots & T(\vec{e}_n) \end{bmatrix}$, $\|\vec{e}_i\| = 1 = \|T(\vec{e}_i)\|$

↳ if $i \neq j$, $\vec{e}_i \cdot \vec{e}_j = 0$ $(\vec{e}_i \perp \vec{e}_j)$

↳ since T is orthogonal, $T(\vec{e}_i) \perp T(\vec{e}_j) = T(\vec{e}_i) \cdot T(\vec{e}_j) = 0$

↳ $T(\vec{e}_1), \dots, T(\vec{e}_n)$ forms an orthonormal basis

- Definition: An $n \times n$ matrix is <u>orthogonal</u> if its columns form an orthonormal basis

- $B = \begin{bmatrix} 1 & 0 \\ 2 & 24 \\ 3 & 18 \end{bmatrix}$, $B^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 24 & 18 \end{bmatrix}$ → transpose

- $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ → $\vec{v}^T = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$

  $\vec{v}^T \vec{w}$ → matrix multiplication → $\vec{v} \cdot \vec{w}$

↳ If $\vec{x}$ and $\vec{y}$ are column n-vectors, then $\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}$

- $A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$, is orthogonal, is invertible

  $= \begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix}$, $\vec{u}_1 \cdot \vec{u}_1 = 1$, $\vec{u}_1 \cdot \vec{u}_2 = 0$, $\vec{u}_1 \cdot \vec{u}_3 = 0$, etc.

↳ $\vec{u}_1^T A = \vec{u}_1^T \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

↳ $\vec{u}_2^T A = \vec{u}_2^T \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$

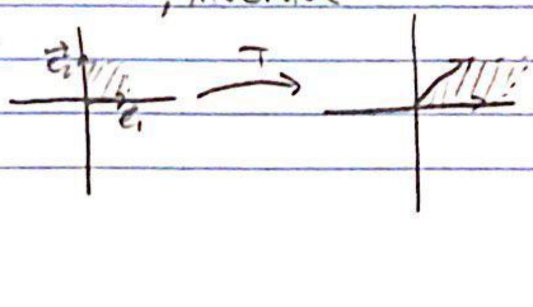
↳ $\underbrace{\begin{bmatrix} - \vec{u}_1 - \\ - \vec{u}_2 - \\ - \vec{u}_3 - \end{bmatrix}}_{A^{-1}} \underbrace{\begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I_n}$

↳ $A^{-1} = A^T$ when A is orthogonal, A^T is also orthogonal

↳ Orthogonal matrices' inverses are their transpose, vice versa

- $T: \mathbb{R}^2 \to \mathbb{R}^2$, invertible

↳



↳ What is the new area of the basis (parallelograms)

↳ Say T has matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $T(\vec{e}_1) = \begin{bmatrix} a \\ c \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} b \\ d \end{bmatrix}$

↳ $A = \|T(\vec{e}_1)\| \|T(\vec{e}_2)\| |\sin\theta| = |ad - bc|$

↳ Ex) T is a reflection across the x-axis

↳ $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

↳ when the angle between $T(\vec{e}_1)$ and $T(\vec{e}_2)$ exceeds π is exactly when $\sin\theta < 0$

↳ we care about $ad - bc$ and we think of it as a signed area

↳ Remark. A is any 2×2 matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $ad - bc \neq 0$ when A is invertible

↳ Ex) $A = \begin{bmatrix} a & \lambda a \\ c & \lambda c \end{bmatrix}$, $a\lambda c - \lambda ac = 0$

· Definition: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we define its <u>determinant</u> to be $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

↳ Facts:

↳ $\det(A)$ is a signed area

↳ $\det(A) \neq 0$ when A is invertible

↳ if $\det(A) \neq 0$, $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

11/20 Lecture: Determinants

- 2×2 determinants

↳ Given a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we call the number $ad-bc$ the determinant of A and denote it by $\det(A)$

↳ $\det(A) \neq 0$ when A is invertible

- Determinant of arbitrary square matrices

↳ Let $A = \begin{bmatrix} a_{11} & \cdots & \\ \vdots & \ddots & \\ & & a_{33} \end{bmatrix}$ and denote the column vectors of A by $\vec{v}_1, \vec{v}_2, \vec{v}_3$

↳ A is not invertible $\longleftrightarrow$ the image is not $\mathbb{R}^3 \longleftrightarrow$ im(A) is at most 2-D

↳ Consider the cross product $\vec{v}_2 \times \vec{v}_3$

↳ if $\vec{v}_2 \times \vec{v}_3 \neq 0$, then they are linearly independent and thus im(A) = 2-D

↳ Thus $\vec{v}_1$ is $\perp$ to $\vec{v}_2$ and $\vec{v}_3 \rightarrow \vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3) = 0$

↳ if $\vec{v}_2 \times \vec{v}_3 = 0$, then $\vec{v}_2 = c\vec{v}_3$ for some real # c, and $\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3) = 0$

↳ Thus if A is not invertible, then $\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3) = 0$

↳ Conversely, if A is invertible, then $\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3) \neq 0$

↳ Definition: The number $\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3)$ is the determinant of a 3×3 matrix A

- Sarrus' Rule:

↳ Mnemonic for the determinant of a 3×3 matrix A, write the 1st 2 columns of A to the right of A, then multiply the entries along the 6 diagonals

↳ Add the products from top left to bottom right and subtract the products from bottom left to bottom right

↳ Ex) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \rightarrow \det A = 24$

↳ Ex) Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be 3 column vectors in $\mathbb{R}^3$, what is the relationship between $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$ and $B = \begin{bmatrix} \vec{v}_1 & \vec{v}_3 & \vec{v}_2 \end{bmatrix}$?

↳ $\det(A) = -\det(B)$

↳ Alternating property of the determinant - swapping any 2 columns changes the determinant's sign

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- The determinant of an $n \times n$ matrix
  - $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ is invertible, but Sarrus' Rule gives 0
  - Definition - a pattern in an $n \times n$ matrix A is an unordered sequence of n of entries of A such that no 2 entries in the sequence have the same row and column
    - How many patterns are there in an $n \times n$ matrix A?
      - There are $n!$ patterns
    - The sign is related to the alternating property of the determinant
      - Definition - an inversion in a pattern of an $n \times n$ matrix A is a number in P which lies above and to the right of another # of P
  - Definition: Sign of a pattern $= (-1)^{\# \text{ of inversions}}$
    - $\det(A) = \sum \text{sign}(P)\, \text{prod}(P)$
- Upper diagonal matrix -> det = all the #s on the main diagonal
  - Determinant of a diagonal matrix is the product of the diagonal entries
- Determinants and elementary row operations
  - $\begin{bmatrix} a/k & b/k \\ c & d \end{bmatrix} \to \det = \frac{1}{k}\det$
  - $\begin{bmatrix} c & d \\ a & b \end{bmatrix} \to \det = -\det$
  - $\begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix} \to \det = \det$
- Using Gauss-Jordan
  - If we achieve RREF by swapping s times and dividing by $k_1, \dots, k_r$
    - $\det(A) = (-1)^s k_1 \dots k_r \det(RREF(A))$
  - Ex) $A = \begin{bmatrix} 0 & 7 & 5 & 3 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \to B = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$
    - $\det(A) = (-1)^2 \det(B) = 14$
- Invertibility and determinant
  - If A is invertible, then $\det(RREF(A)) = 1 \to \det(A) = (-1)^s k_1 \dots k_r \neq 0$

11/22 Lecture: Properties of the Determinant

- Properties of determinants
  - ↳ A and B are any $n \times n$ matrices
    - ↳ $\det(AB) = \det(A)\det(B)$
    - ↳ If A and B are similar matrices, then $\det(A) = \det(B)$
      - ↳ Explanation: $AS = SB$ for similar matrices for an invertible matrix S
        - $\det(AS) = \det(A)\det(S) = \det(S)\det(B)$
        - ↳ since S is invertible, $\det(S) \neq 0 \therefore \det(A) = \det(B)$
      - ↳ Converse is not true
    - ↳ If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$
      - ↳ Explanation. $AA^{-1} = I_n = A^{-1}A$

$$\det(A)\det(A^{-1}) = \det(I_n) = 1$$
$$\det(A^{-1}) = \frac{1}{\det(A)}$$

- ↳ $\det(A^T) = \det(A)$
  - ↳ Explanation: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, consider a pattern in A $(3,4,8)$ and the corresponding pattern in A^T, the pattern still holds w/ the same # of inversions

$$A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

- Recall: $A = \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right]\begin{array}{cc} a_{11} & a_{22} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{array}$

$$\det(A) = \sum_P \operatorname{sign}(P)\operatorname{prod}(P)$$

- Theorem: Let A be an $n \times n$ matrix
  - ↳ A is orthogonal $\rightarrow A^TA = I_n \rightarrow A^{-1} = A^T$
  - ↳ Proof: $A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}$, then $A^T = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_n^T & - \end{bmatrix}$

$$A^TA = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_n^T & - \end{bmatrix}\begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} = \begin{bmatrix} a_1 \cdot a_1 & a_1 \cdot a_2 & \cdots & a_1 \cdot a_n \\ \vdots & & & \vdots \\ a_n \cdot a_1 & \cdots & \cdots & a_n \cdot a_n \end{bmatrix}$$

  - ↳ $A^TA = I_n \rightarrow a_i \cdot a_j = 1$ if $i = j$, 0 if $j \neq i$ $\rightarrow$ A is orthogonal
- ↳ Consequence: $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ orthogonal preserves the dot product for all $\vec{v}, \vec{w}$ in $\mathbb{R}^n$: $T(\vec{v}) \cdot T(\vec{w}) = \vec{v} \cdot \vec{w}$
  - ↳ Explanation: Let A be the orthogonal matrix that represents T
    - ↳ $T(\vec{v}) \cdot T(\vec{w}) = A\vec{v} \cdot A\vec{w} = (A\vec{v})^T(A\vec{w}) \rightarrow (AB)^T = B^TA^T$
    - ↳ $\vec{v}^TA^TA\vec{w} = \vec{v}^T\vec{w} = \vec{v} \cdot \vec{w}$

• Proposition: Let A be an orthogonal matrix, then $\det(A) = \pm 1$

↳ Proof: If $A^T A = I_n$

↳ $\det(A^T A) = \det(I_n) = \det(A^T)\det(A) = 1$

↳ $\det(A)\det(A) = 1 \quad \therefore \det(A) = \pm 1$

↳ Ex) $\det\begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix} = 1 \Rightarrow$ rotation in $\mathbb{R}^2$

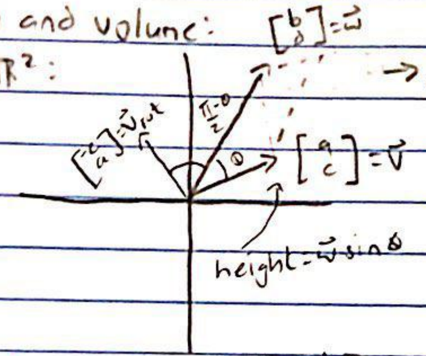
$\det\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -1 \rightarrow$ reflection about x-axis

• Geometric interpretations of the determinant

↳ 1st absolute value

↳ As area and volume:

↳ In $\mathbb{R}^2$:



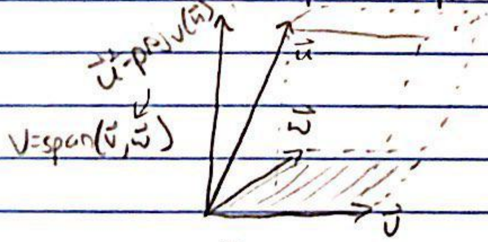
→ area of the parallelogram is $|\det(A)|$

$\det(A) = ad - bc = \begin{bmatrix} -c \\ a \end{bmatrix} \cdot \begin{bmatrix} b \\ d \end{bmatrix} = \|\vec{v}_{rot}\|\|\vec{w}\|\cos(\frac{\pi}{2} - \theta) = \underbrace{\|\vec{v}\|}_{\text{base}}\underbrace{\|\vec{w}\|\sin\theta}_{\text{height}}$

↳ Alternatively, with QR-decomposition

↳ $A = QR \rightarrow |\det(A)| = \underbrace{|\det(Q)|}_{1}\ \underbrace{|\det(R)|}_{\underbrace{\|\vec{v}\|\|\vec{w}^\perp\|}_{\text{base} \times \text{height}}}$

↳ Volume of a parallelepided in $\mathbb{R}^3$:

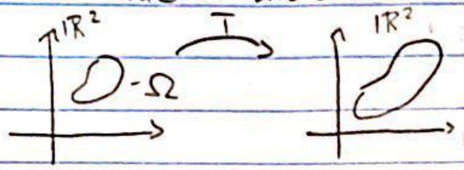


Volume = A of base × height

$A = \begin{bmatrix} | & | & | \\ \vec{v} & \vec{w} & \vec{u} \\ | & | & | \end{bmatrix}$, $A = QR \rightarrow \det(A) = \det(Q)\det(R) = \underbrace{\|\vec{v}\|\|\vec{w}^\perp\|}_{\text{base}}\underbrace{\|\vec{u}^\perp\|}_{\text{height}}$

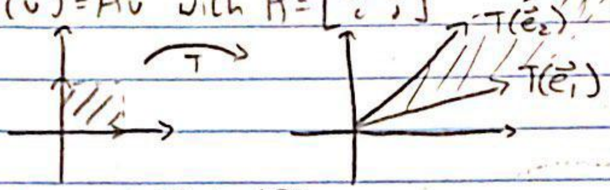
11/25 Lecture: Expansion Factor

- Revision Lecture. Thurs. 12/5, 6-8pm Boelter 3400
- Absolute value of the determinant as a scaling factor

↳ 

↳ $\dfrac{\text{area of } T(\Omega)}{\text{area of } \Omega}$ → expansion factor of T

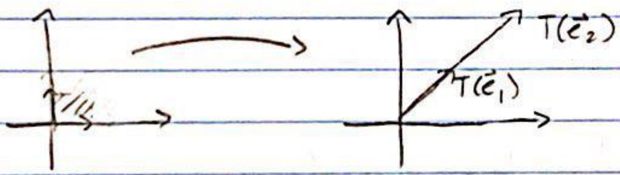
↳ $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(\vec{v}) = A\vec{v}$ with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

↳ Ex) $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ 

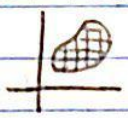
← area of parallelogram given by det(A) = 10

↳ expansion factor of T is 10

↳ Ex) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$



↳ det(A) = 0 → expansion factor = 0

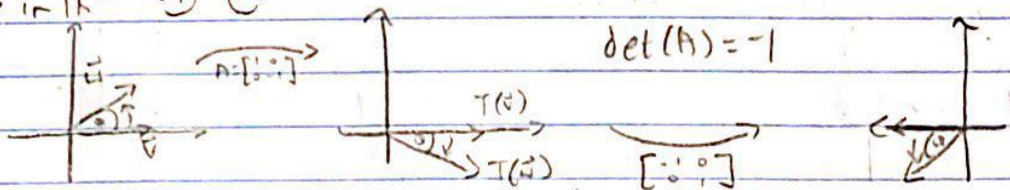
- For an arbitrary $\Omega \subseteq \mathbb{R}^2$:  use estimation by squares for the area of Ω

↳ The expansion factor on each square is |det(A)|

↳ Choosing smaller and smaller squares, integration gives that the expansion factor of T on Ω is |det(A)|

- Sign of the determinant

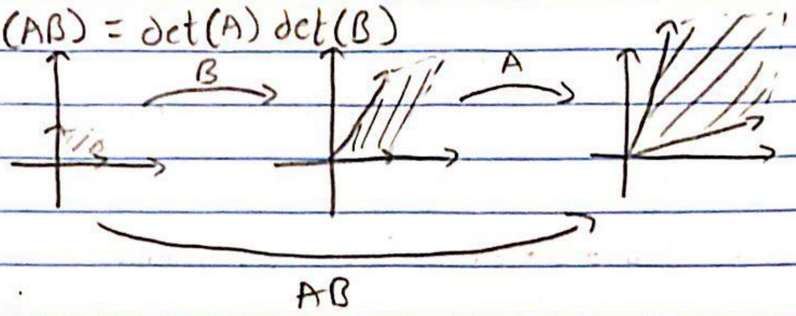
↳ in $\mathbb{R}^2$ ⊕ ⊖

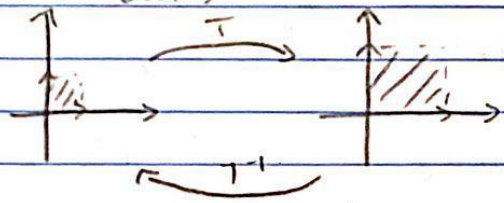


↳ In general: det(A) = 1 when the linear transformation preserves orientation and det(A) = -1 when it reverses orientation

- Recall: We have seen that if A is an orthogonal matrix, then det(A) = ±1

↳ In $\mathbb{R}^2$, reflections have determinant -1, rotations have +1

- Properties of determinants revisited:
  - ↳ $\det(AB) = \det(A)\det(B)$
    - ↳ 

    - ↳ in terms of scaling factors, scaling factor of AB = (scaling factor of A)(scaling factor of B)
  - ↳ $\det(A^{-1}) = \frac{1}{\det(A)}$
    - ↳ 

      - ↳ The expansion factor of T^{-1} is 1/expansion factor of T
- Eigenvalues and eigenvectors
  - ↳ It is often easier to work with diagonal matrices
    - ↳ Ex) $\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = A$, $B = \begin{bmatrix} \sim \\ \sim \\ \sim \\ \sim \end{bmatrix}$ ← non-zero values
      - ↳ Compute rank, basis for the kernel, determinant, A^6 / B^6
        - ↳ For A. rank(A) = 3 → # of non-zero entries on the diagonal
          det(A) = 0 → given by product of diagonal entries
          basis for ker(A) = $\vec{e}_2$ → 2nd column has a free variable
        - ↳ For B: 1st put in rref to find rank, determinant, kernel, but cannot find the powers easily
          - ↳ You can take powers of diagonal entries
  - ↳ Goal: if a matrix is not diagonal, find a diagonal matrix that is similar to it
- Definition: $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation represented by matrix A, then T or A is <u>diagonalizable</u> if there exists a basis $\mathcal{B} = \{\vec{v}_1, \ldots \vec{v}_n\}$ so that the $\mathcal{B}$ matrix is diagonal
  - ↳ for all $\vec{v}_1, \ldots, \vec{v}_n$: $A\vec{v}_i = \lambda\vec{v}_i$ for some $\lambda \in \mathbb{R}$
- Definition: A non-zero vector in $\mathbb{R}^n$ is an <u>eigenvector</u> of T (or A) if $A\vec{v} = \lambda\vec{v}$ for some $\lambda \in \mathbb{R}$, where λ is an <u>eigenvalue</u> of T (or A)
  - ↳ A basis $\vec{v}_1, \ldots, \vec{v}_n$ of eigenvectors is called an <u>eigenbasis</u>

Recall from last time:

given an $n \times n$ matrix A we want to find an $n \times n$ matrix B that is diagonal and similar to it, or in other words, a basis $v_1, \dots, v_n$ of $\mathbb{R}^n$ such that

$$A v_i = \lambda_i v_i \qquad (i=1,\dots,n)$$

for some $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

Definition

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation represented by a matrix A.

A nonzero vector $v \in \mathbb{R}^n$ is called eigenvector of A (or T) if $Av = \lambda v$ for some $\lambda \in \mathbb{R}$.

In this case λ is called eigenvalue of A (or T).

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A basis $v_1, \dots, v_n$ of $\mathbb{R}^n$ consisting of eigenvectors of A (or T) is called <u>eigenbasis</u> for A (or T).

<u>Proposition</u>

A square matrix A $^{n \times n}$ is diagonalisable

$$\Updownarrow$$

there exists a basis of eigenvectors for A.

<u>Proof</u>

Let $v_1, \dots, v_n$ be a basis of eigenvectors and let $\lambda_1, \dots, \lambda_n$ be scalars s.t.

$$Av_i = \lambda_i v_i \quad \text{for } i = 1, \dots, n.$$

Then $B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$, $S = \begin{bmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{bmatrix}$

and $AS = SB$.

$\square$

Examples

1) Reflections in $\mathbb{R}^2$

- First: reflection about x-axis.

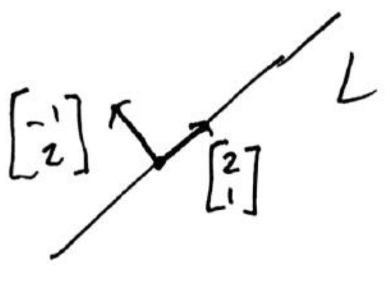
$T:\mathbb{R}^2 \to \mathbb{R}^2$, $T(v) = Av$ with $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\left.\begin{array}{l} Ae_1 = e_1 \\ Ae_2 = -e_2 \end{array}\right\} \Rightarrow \lambda_1 = 1 \text{ and } \lambda_2 = -1 \text{ are the eigenvalues of } A.$$

- Reflection about an arbitrary line L in $\mathbb{R}^2$

$T: \mathbb{R}^2 \to \mathbb{R}^2$

e.g. $L = \operatorname{span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$



$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = -\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\Rightarrow$ $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ are eigenvectors of T with eigenvalues 1 and -1 respectively.

$$S = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A = SBS^{-1}.$$

2) <u>Orthogonal transformations</u>

In general, 1 and -1 are the only eigen= values of an orthogonal transformation:

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal transforma= tion with matrix A, then $\|T(v)\| = \|v\|$ for all $v \in \mathbb{R}^n$ by definition.

Let v be an eigenvector of A with associated eigenvalue λ, so that $Av = \lambda v$.

Then $\|v\| = \|Av\| = \|\lambda v\| = |\lambda|\,\|v\|$

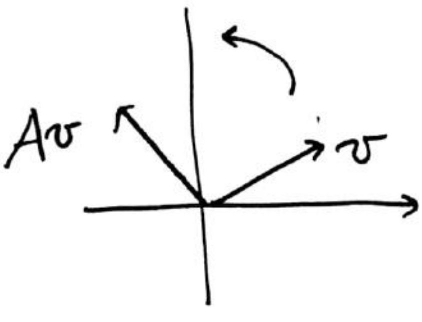
$$\Rightarrow \lambda = \pm 1.$$

3) <u>Rotation in $\mathbb{R}^2$</u>

e.g. $T: \mathbb{R}^2 \to \mathbb{R}^2$ with $T(v) = Av = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} v$

rotation through $\pi/2$ in counterclockwise direction.

Is A diagonalisable?



If v is any non-zero vector in $\mathbb{R}^2$, then Av fails to be parallel to v (it's perpendicular).

Thus there are no eigen vectors and A is not diagonalisable.

Note that the same is true for any rotation in $\mathbb{R}^2$ through an angle θ, as long as $\theta \neq k\pi$, $k \in \mathbb{Z}$.

How to compute the eigenvalues of a matrix A

A an $n \times n$ matrix, $\lambda \in \mathbb{R}$.

λ is an eigenvalue of A $\iff$ there exists a non-zero vector $v \in \mathbb{R}^n$ s.t.:

$$Av = \lambda v$$

$\iff$ there exists $0 \neq v \in \mathbb{R}^n$ s.t.:

$$(A - \lambda I_n) v = 0$$

$\iff$ $\mathrm{Ker}(A - \lambda I_n) \neq \{0\}$

$\iff$ $A - \lambda I_n$ is not invertible

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this is called "characteristic equation of A" $\Longleftrightarrow$ $\boxed{\det(A - \lambda I_n) = 0}$ (*)

Thus, to find the eigenvalues of A, we have to find the solutions to (*).

Example

1) $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

$$\det(A - \lambda I_2) = \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} \right)$$

$$= (1-\lambda)(3-\lambda) - 8$$

$$= \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1)$$

$\Rightarrow$ eigenvalues of A: $\lambda_1 = 5$

$\lambda_2 = -1$

2) $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$

$\det(A - \lambda I_3) = (2-\lambda)(3-\lambda)(4-\lambda)$

$\Rightarrow$ eigenvalues of A: $\lambda_1 = 2$
$\lambda_2 = 3$
$\lambda_3 = 4$

Proposition

Let A be any upper (or lower) triangular $n \times n$ matrix. Then the eigenvalues of A are its diagonal entries.

Important fact: $\det(A - \lambda I_n)$ is a polynomial in λ of degree n, in other words:

$$\det(A - \lambda I_n) = c_n \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_0 \lambda^0$$

where $c_0, \dots, c_n \in \mathbb{R}$,

called characteristic polynomial of A and denoted by $f_A(\lambda)$.

What does this tell us about the eigenvalues of a matrix?

We know that a polynomial of degree n has at most n real roots. Thus, our $n \times n$ matrix A has at most n eigenvalues.

Summary : to find the eigenvalues of an $n \times n$ matrix A we need to find the ~~root~~ real roots of the polynomial

$$f_A(\lambda) = \det(A - \lambda I_n).$$

Next time we will see how to compute the corresponding eigenvectors.

12/2 Lecture: Multiplicities

- Thursday 6-8pm in Boelter
- Recall: to find eigenvalues of an $n \times n$ matrix A, we need to solve for the solutions of $\det(A - \lambda I_n) = 0 \rightarrow$ characteristic eq. of $A = f_A(\lambda)$
  - $\hookrightarrow = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0$ where $c_0, \dots, c_n \in \mathbb{R} \rightarrow$ characteristic polynomial of A
    - $\hookrightarrow$ the eigenvalues of A are the roots of this polynomial
    - $\hookrightarrow$ we have at most n eigenvalues (real)
- For polynomials, roots might have multiplicities greater than 1
  - $\hookrightarrow x^2 \rightarrow$ the root 0 has a multiplicity of 2
- Definition. Let A be an $n \times n$ matrix. The algebraic multiplicity of an eigenvalue of the matrix is the multiplicity of λ_0 as a root of $f_A(\lambda)$, denoted by $\text{almu}(\lambda_0)$
  - $\hookrightarrow$ In other words, if $\text{almu}(\lambda_0) = k$, then $f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda)$, where $g(\lambda)$ has degree $n-k$ and $g(\lambda_0) \neq 0$
- Ex) $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ eigenvalues: 0 w/ $\text{almu}(0) = 2$, 2 w/ $\text{almu}(2) = 1$, and -1 w/ $\text{almu}(-1) = 1$
- How to find eigenvectors of a matrix A
  - $\hookrightarrow$ If $0 \neq \vec{v} \in \mathbb{R}^n$ is an eigenvector of a matrix A with eigenvalue λ, then $A\vec{v} = \lambda\vec{v} \rightarrow (A - \lambda)\vec{v} = 0$
    - $\hookrightarrow$ Find $\ker(A - \lambda I_n)$
- Definition: Let λ be an eigenvalue of a $n \times n$ matrix A, then $\ker(A - \lambda I_n)$ is the eigenspace of λ denoted by E_λ
  - $\hookrightarrow$ The dimension of E_λ is called geometric multiplicity of λ, denoted by $\text{gemu}(\lambda)$
  - $\hookrightarrow$ Ex) $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ has eigenvalues 5 and -1
    - $\hookrightarrow$ The corresponding eigenspaces are $E_5 = \ker(A - 5I_2) = \ker\left(\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right)$ and $E_{-1} = \ker(A - (-1)I_2) = \ker\left(\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$
      - $\hookrightarrow \text{gemu}(E_5) = 1$, $\text{gemu}(E_{-1}) = 1$
      - $\hookrightarrow$ the vectors $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ form an eigenbasis for A
        - $\hookrightarrow A$ is diagonalisable, $S = \begin{bmatrix} 1/2 & 1 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$ and $AS = SB$

↳ Ex) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $f_A(\lambda) = (1-\lambda)(-\lambda)(1-\lambda)$, eigenvalues: 0 w/ almu=1, 1 w/ almu=2

↳ $E_1 = \ker(A - I_n) = \ker\left(\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$, gemu(1) = 1

↳ $E_0 = \ker(A) = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right)$, gemu(0) = 1

↳ A is not diagonisable because there are only 2 linearly independent eigenvectors

- Important fact. Let $\lambda_1, \dots, \lambda_m$ be different eigenvalues of a matrix A. Compute a basis for each eigenspace $E_{\lambda_1}, \dots, E_{\lambda_m}$ and concatenate these vectors $\vec{v}_1, \dots, \vec{v}_s$

↳ We have s vectors, where s is the sum of the dimensions of the eigenspaces (geometric multiplicities)

- Proposition: An $n \times n$ matrix is diagonalisable if the dimensions of the eigenspaces sum up to n
- Special case: If a matrix has all eigenvalues with almu=1, then it is diagonalisable

12/4 Lecture: Diagonalization

- Recall:

↳ introduction of algebraic multiplicity and geometric multiplicity

↳ Note: gemu of an eigenvalue ≤ almu of an eigenvalue

↳ $gemu(\lambda) \leq almu(\lambda)$

↳ an $n \times n$ matrix A is diagonalisable when $\sum_\lambda \dim(E_\lambda) = n$

↳ special case: if a matrix has n different eigenvalues, then it is diagonalisable

↳ $n \times n$ matrix A is diagonalisable when the matrix has n eigenvalues and $gemu(\lambda) = almu(\lambda)$ for all λ

↳ Ex)

$$A = \begin{bmatrix} 0 & & & & * \\ & 1 & & & \\ & & 1 & & \\ & & & 2 & \\ 0 & & & & 3 \end{bmatrix} \quad f_A(\lambda) = \det(A - \lambda I_5) = \lambda(1-\lambda)^2(2-\lambda)(3-\lambda)$$

↳ $\lambda = 0, 1, 2, 3 \rightarrow almu(1) = 2$

↳ Summary of diagonalisation

↳ Given an $n \times n$ matrix A, we say that A is diagonalisable if there exists an invertible matrix S such that $S^{-1}AS$ is diagonal

↳ 1) find the eigenvalues of A: the real roots of $f_A(\lambda) = \det(A - \lambda I_n)$

2) for each eigenvalue λ, compute its eigenspace $E_\lambda = \ker(A - \lambda I_n)$ find a basis for E_λ

3) A is diagonalisable when $\sum_\lambda E_\lambda = n$

↳ if A is diagonalisable, concatenate all bases for the eigenspaces to obtain $\vec{v}_1, \ldots, \vec{v}_n \rightarrow$ eigenbasis

↳ then,

$$S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}, \quad S^{-1}AS = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

- Ex) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, A has 2 eigenvalues: 0, $almu(0) = 2$ and 3, $almu(3) = 1$

$\lambda_1 = 0, \lambda_2 = 3$

$$S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}, \quad S^{-1}AS = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$$

- Question: for which $n \times n$ matrices A is there an orthogonal matrix S such that $S^{-1}AS$ is diagonal

↳ Notes:

↳ in this case, we would have an orthonormal basis of eigenvectors

↳ the eigenspaces corresponding to different eigenvalues are orthogonal

↳ Assume that $S^{-1}AS$ (orthogonal) $= B$ (diagonal)
  ↳ $A = SBS^{-1} = SBS^T$
   ↳ $A^T = (SBS^T)^T = (S^T)^T B^T S^T = SBS^T = A$
    ↳ $A = A^T \rightarrow$ symmetric matrices

• Definition: A square matrix A such that $A = A^T$ is symmetric
↳ Ex) All 2×2 symmetric matrices are of the form $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

• Theorem: Given an $n \times n$ matrix A, there exists S orthogonal such that $S^{-1}AS$ is diagonal iff A is symmetric

• 2 important facts
↳ any symmetric matrix is diagonalisable
↳ for any symmetric matrix, if λ_1 and λ_2 are different eigenvalues, then E_{λ_1} and E_{λ_2} are orthogonal

• General strategy: given a symmetric $n \times n$ matrix A, how do we find S orthogonal w/ $S^{-1}AS$ diagonal
↳ 1) Compute the eigenvalues of A
↳ 2) Compute corresponding eigenspaces of A
↳ 3) For each eigenspace, choose a basis and apply Gram-Schmidt to obtain an orthonormal basis
↳ 4) Concatenate all orthonormal bases for the eigenspaces and obtain an orthonormal basis of $\mathbb{R}^n$

Matrices and linear systems

A linear system with 2 equations in variables x_1, x_2, x_3, x_4, x_5:

$$x_2 + x_3 = 2$$

$$x_1 + x_2 + x_3 - x_4 = 0$$

Augmented coefficient matrix

$$\left[\begin{array}{ccccc|c} 0 & 1 & 1 & 0 & 0 & 2 \\ 1 & 1 & 1 & -1 & 0 & 0 \end{array}\right] = A|b$$

Put A|b in RREF using elementary row operations

From top to bottom

$$\left[\begin{array}{ccccc|c} 0 & 1 & 1 & 0 & 0 & 2 \\ 1 & 1 & 1 & -1 & 0 & 0 \end{array}\right]$$

swap I and II $\rightsquigarrow$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 2 \end{array}\right]$$

Substract II from I $\rightsquigarrow$ $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 0 & -2 \\ 0 & 1 & 1 & 0 & 0 & 2 \end{array}\right]$

Solve for leading variables:

$x_1 = x_4 - 2$

$x_2 = -x_3 + 2$

Write solution by plugging in arbitrary values for free variable

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} r-2 \\ -s+2 \\ s \\ r \\ t \end{bmatrix}$$ with $s, t, r \in \mathbb{R}$

$\rightsquigarrow$ infinitely many solutions.

Matrices as representing linear transformations

A function $T: \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation

$$\Updownarrow$$

there is an $n \times m$ matrix A such that for any $v \in \mathbb{R}^m$: $T(v) = Av$.

- **Given A, find T**

e.g. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, then

$$T: \mathbb{R}^3 \to \mathbb{R}^2$$

$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3 \quad T(v) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 + v_2 \\ v_1 + 2v_2 - v_3 \end{bmatrix}$

- **Given T, find A.** e.g. $T: \mathbb{R}^2 \to \mathbb{R}^3$

with $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 + 2v_2 \\ 0 \\ -v_1 \end{bmatrix}$ $\leadsto A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}$$

- Composition of linear transformations ⇝ multiplication of matrices

$T: \mathbb{R}^m \to \mathbb{R}^n$ repr. by $n \times m$ matrix A

$S: \mathbb{R}^n \to \mathbb{R}^p$ repr. by $m \times p$ matrix B

$S \circ T: \mathbb{R}^m \to \mathbb{R}^p$ repr. by BA

- Inverse linear transformation ⇝ inverse matrix.

$T: \mathbb{R}^n \to \mathbb{R}^n$ invertible $\iff \mathrm{im}(T) = \mathbb{R}^n$
$\iff \mathrm{rank}(T) = n$.
$(\iff \dim(\mathrm{im}\, T) = n$
$\iff \ker(T) = \{0\})$

To find inverse: put A in RREF:

$$[A \mid I_n] \overset{\text{Gauss-Jordan}}{\rightsquigarrow} [I_n \mid B]$$

B: inverse of A.

Image, kernel, bases, dimension

$$T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

$\cup|$ with $T(v) = Av$ $\cup|$

$\ker(T) = \ker(A)$

$\|$

$\{v \in \mathbb{R}^m : Av = 0\}$

$\operatorname{im}(T) = \operatorname{im}(A)$

$\|$

$\{w \in \mathbb{R}^n$: there is $v \in \mathbb{R}^m$ with $Av = w\}$

$\|$

$\{T(v) : v \in \mathbb{R}^m\}$

e.g. $T: \mathbb{R}^3 \to \mathbb{R}^4$ $\quad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\operatorname{im}(T)$ consists of all vectors of the form

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} v_1 + \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} v_2 + \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} v_3$$

$v_1, v_2, v_3 \in \mathbb{R}$

$\rightsquigarrow \operatorname{im}(T)$ is a plane in $\mathbb{R}^4$ passing through the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$ and the origin.

in general : $\operatorname{im}(T) = \operatorname{span}(a_1, \ldots, a_m)$
where $a_1, \ldots, a_m$ are the columns of A

What is the smallest amount of vectors that spans $\operatorname{im}(T)$?

$\rightsquigarrow$ basis : basis of a subspace V of $\mathbb{R}^n$ is a collection of vectors that spans V and such that this collection is minimal.

$\rightsquigarrow$ $\dim V$ = nr. of vectors in a basis for V.

To compute basis for $\operatorname{im} T$: • put A in RREF

• a basis is given by column vectors in A corresponding to columns in RREF A that contain pivots

" — $\ker T$: • solve system $Ax = 0$

• write solution set in vector form

e.g. $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\rightsquigarrow$ $RREF(A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

basis for $\text{im}(T)$: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$

basis for $\ker(T)$: solutions to $Ax = 0$

are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ 0 \\ t \end{bmatrix}, \ t \in \mathbb{R}$

$= t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ basis for $\ker(T)$

Rank-nullity

$T: \mathbb{R}^m \to \mathbb{R}^n$ repr. by A

rank (T) = dim (im T) = nr. of pivots in RREF(A)

nullity (T) = dim (ker T) = m − rank (T) = nr. of free variables in RREF(A).

$\leadsto$ m = rank (T) + nullity (T)

= dim (im T) + dim (ker T)

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Coordinates for a subspace V of $\mathbb{R}^n$

$\mathcal{B} = \{v_1, \dots, v_m\}$ a basis of $V \subseteq \mathbb{R}^n$

then there exist unique numbers $c_1, \dots, c_m$

s.t.

$$w = c_1 v_1 + \dots + c_m v_m \quad \forall\, w \in V.$$

these are called $\mathcal{B}$-coordinates of w

e.g. $V = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right)$

$$w = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & 4 \\ 0 & 0 & 0 \\ 1 & 0 & 3 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow [w]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Coordinates for $\mathbb{R}^n$

$\mathcal{B} = \{v_1, \ldots, v_n\}$ a basis of $\mathbb{R}^n$. Then there exist unique numbers $c_1, \ldots, c_n \in \mathbb{R}$ s.t.

$$w = c_1 v_1 + \ldots + c_n v_n$$

these are called $\mathcal{B}$-coordinates of w, written

$$[w]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Given $T: \mathbb{R}^n \to \mathbb{R}^n$, we can represent T in the $\mathcal{B}$-coordinates with the matrix

$$B = \begin{bmatrix} | & & | \\ [T(v_1)]_{\mathcal{B}} & \cdots & [T(v_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix}$$

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$$
\begin{tikzcd}
\mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^n \\
\mathbb{R}^n \arrow[u, "S"] \arrow[r, "B"'] & \mathbb{R}^n \arrow[u, "S"']
\end{tikzcd}
$$

standard coordinates

$\mathfrak{B}$ - coordinates

change of basis matrix $S = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$

we have: $\underbrace{S\,[w]_{\mathfrak{B}}}_{\overset{\shortparallel}{w}} = w$ for all $w \in \mathbb{R}^n$

and $AS[w]_{\mathfrak{B}} = SB[w]_{\mathfrak{B}}$

for all $w \in \mathbb{R}^n$

$\boxed{AS = SB}$ A and B are similar

Way to think about similar matrices:

they represent the same linear transformation under different choices of basis.

Example: given w, how to find $[w]_\beta$?

β: $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ $\quad w = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 3 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\rightsquigarrow [w]_\beta = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

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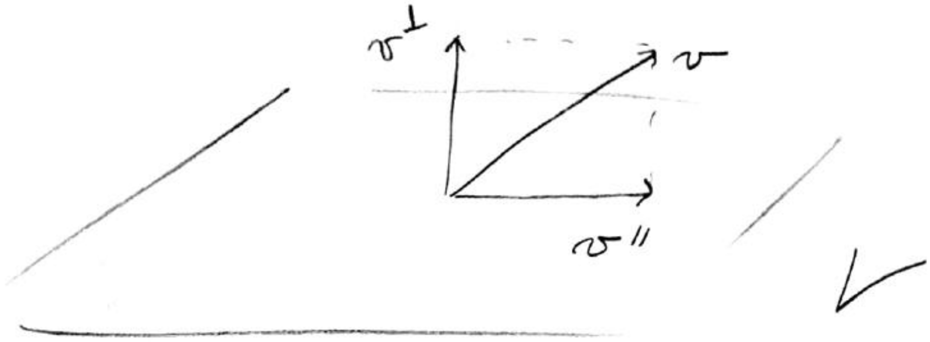
Orthogonal projection onto a subspace V of $\mathbb{R}^n$



Any $v \in \mathbb{R}^n$ can be uniquely written as $v = v^{\parallel} + v^{\perp}$

with $v^{\parallel} \in V$ $\quad v^{\perp}$ orthogonal to V.

$v^{\parallel} = \text{proj}_V(v)$ is called orthogonal projection of v onto V.

Formula for orthogonal projection.

$u_1, \dots, u_m$ ONB of V, then

$$\text{proj}_V(v) = (u_1 \cdot v)\, u_1 + \dots + (u_m \cdot v)\, u_m$$

$\forall\, v \in \mathbb{R}^n$

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$\text{proj}_V : \mathbb{R}^n \to \mathbb{R}^n$

$\text{im}(\text{proj}_V) = V$

$\Rightarrow \dim(V) + \dim(V^\perp) = n.$

$\text{Ker}(\text{proj}_V) = V^\perp$

$\hookrightarrow$ any subspace V of $\mathbb{R}^n$ can be realised as the kernel or image of a linear transf.

<u>Example</u>: Given V, how to find $V^\perp$?

$x \in V^\perp \iff x \cdot v = 0 \;\; \forall \; v \in V.$

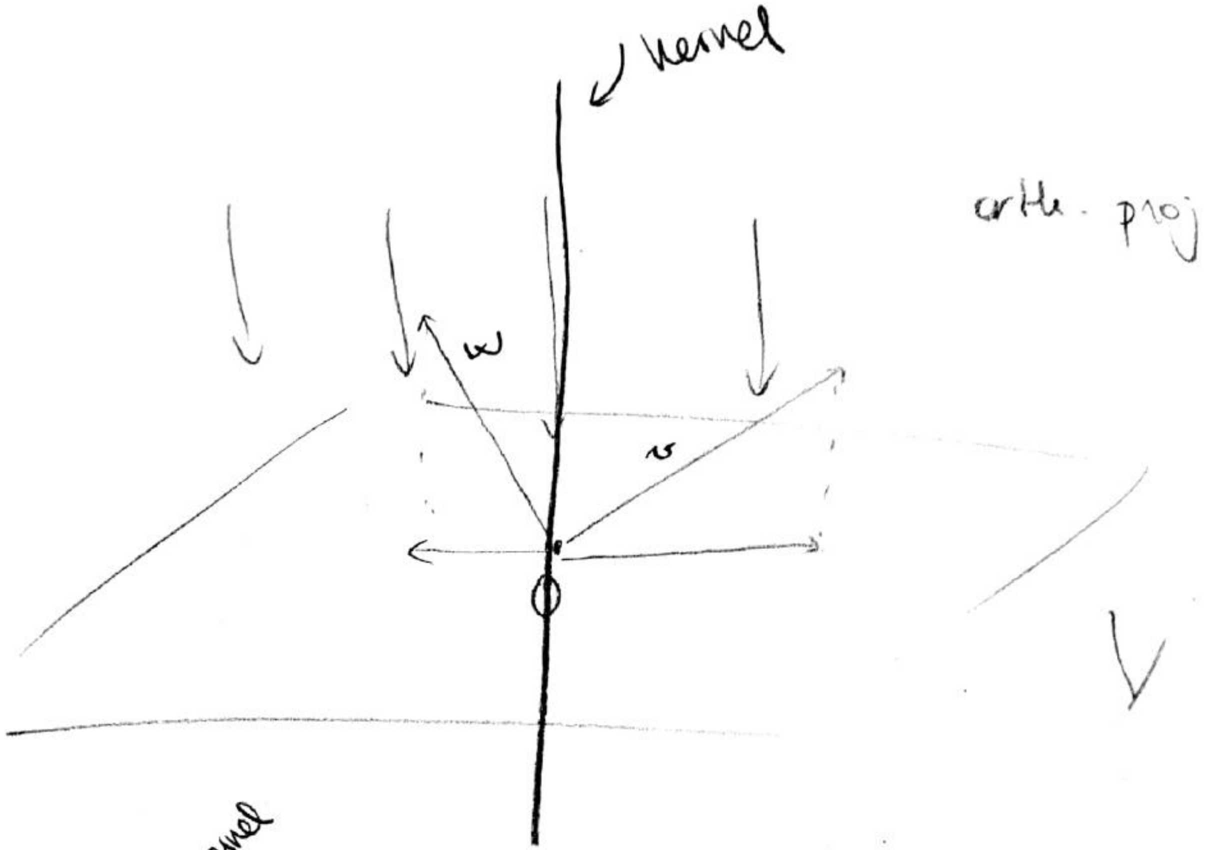
$\Updownarrow$

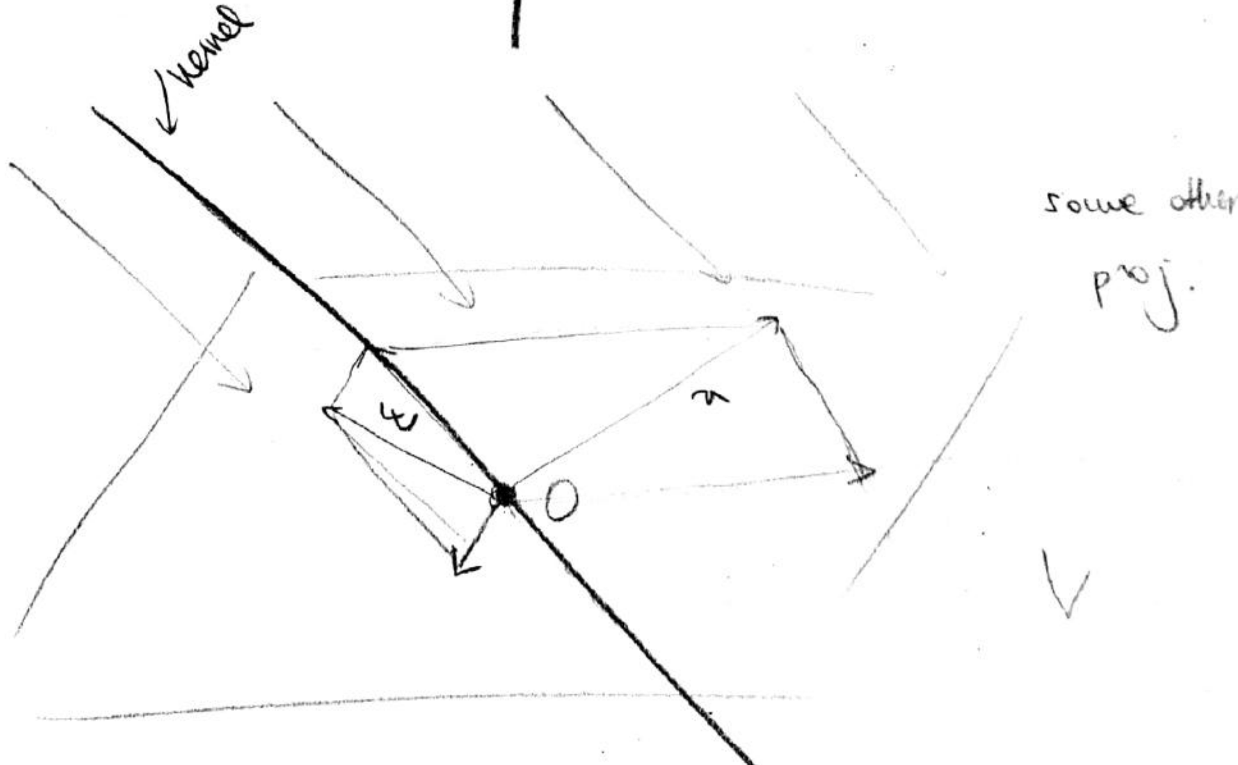
$x \cdot v_i = 0 \;\; \forall \; v_i$ vectors in a basis of V

Find solutions to

$$\rightsquigarrow \begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_m & - \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$$

<u>Note</u>: the reason that proj_V is called "orthogonal" is that it is the only projection onto V such that its image and kernel are orthogonal to each other


kernel
ortho. proj
w
v
O
V


kernel
some other proj.
w
u
O
V

Gram - Schmidt algo: given a basis $\mathcal{B} = \{b_1, \dots, b_m\}$ for a subspace V of $\mathbb{R}^n$, returns an ONB $\mathcal{U} = \{u_1, \dots, u_m\}$ of V.

application

QR factorization: any $n \times m$ matrix M w/ linearly independent column vectors can be written

$$M = Q R$$

$n \times m$ matrix with orthonormal columns

upper-triangular matrix w/ positive diagonal entries.

starting basis

ONB basis

$$\underbrace{\begin{bmatrix} | & & | \\ b_1 & \cdots & b_m \\ | & & | \end{bmatrix}}_{=M} \underbrace{\begin{bmatrix} | & & | \\ [b_1]_{\mathcal{B}} & \cdots & [b_m]_{\mathcal{B}} \\ | & & | \end{bmatrix}}_{= I_m} = \underbrace{\begin{bmatrix} | & & | \\ u_1 & \cdots & u_m \\ | & & | \end{bmatrix}}_{=Q} \underbrace{\begin{bmatrix} | & & | \\ [b_1]_{\mathcal{U}} & \cdots & [b_m]_{\mathcal{U}} \\ | & & | \end{bmatrix}}_{=R}$$

$n \times m$ matrix with lin. ind. columns

Orthogonal transformations : reflections, rotations and composition thereof.

(called like this because they preserve right angles)

important : $T: \mathbb{R}^n \to \mathbb{R}^n$ orthog $\Leftrightarrow$ $T(e_1), \dots, T(e_n)$ ONB of $\mathbb{R}^n$.

A orthogonal matrix $\Leftrightarrow$ columns of A form ONB of $\mathbb{R}^n$.

$\Leftrightarrow$ $A^{-1} = A^t$

- $\det(A) = \pm 1$

- eigenvalues of A if they exist : ± 1.

[Orthogonal projection is <u>not</u> an orthogonal transformation]

Determinants.

How to compute:

- Sarru's rule for 3×3 matrices
- $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$
- Using Gauss-Jordan for $n\times n$ matrices:

suppose that to go from A to RREF(A) we swap rows s times and divide various rows by the non-zero numbers $k_1, \dots, k_r$.

Then $\det(A) = (-1)^s k_1 \cdots k_r \det(\text{RREF}(A))$

$$= \begin{cases} (-1)^s k_1 \cdots k_r & \text{if } A \text{ invertible} \\ 0 & \text{, if } A \text{ not invertible.} \end{cases}$$

Properties: $\det(A) = 0 \iff A$ not invertible

$\det\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$ = product of diagonal entries.

Geometric interpretation:

- as volume and area
- as expansion factor.

Note: if A and B are similar, then

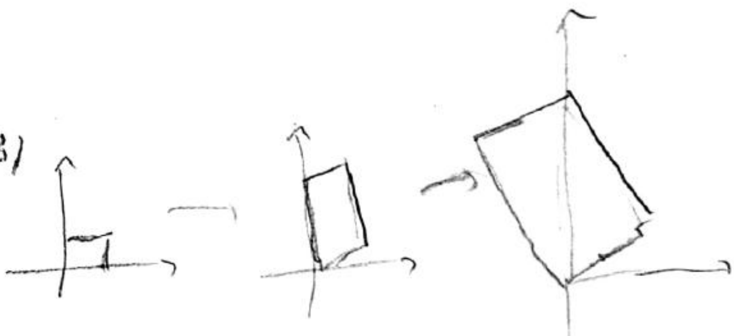
$$\det(A) = \det(B)$$

If $\det(A) = \det(B)$, are A and B similar?

No: take two different linear transformation with the same expansion factor.

$\rightsquigarrow$ geometric interpretation also helps to remember properties:

$\det(AB) = \det(A)\det(B)$



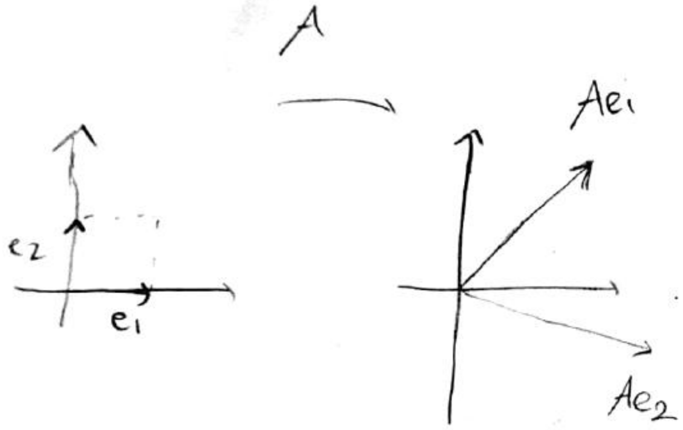
$\det(A) = 0 \iff A$ not inv'ble

$\det(A^{-1}) = \dfrac{1}{\det(A)}$

Eigenvalues and eigenvectors

geometrically

$T : \mathbb{R}^n \to \mathbb{R}^n$



are there vectors $v \in \mathbb{R}^n$ for which Av is parallel to v? So vectors whose direction is not changed by the linear transformation?

If we can find n linearly independent vectors like this, then the matrix of T in this basis will be diagonal.

algebraically

eigenvalues: roots of

$$f_A(\lambda) = \det(A - \lambda I_n)$$

eigenspaces: for λ an eigenvalue of A

$$E_\lambda = \mathrm{Ker}(A - \lambda I_n)$$

$\dim(E_\lambda) =$ geom. mult of λ

multiplicity of λ as a root of $f_A(\lambda)$ = alg. mult. of λ.

if there are n eigenvalues $\lambda_1, \dots, \lambda_n$ and $\mathrm{gemu}(\lambda) = \mathrm{almu}(\lambda)$ for each eigenvalue λ, then

$$S = \begin{bmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{bmatrix}$$ and

$$S^{-1}AS = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \ddots \ \lambda_n \end{bmatrix}$$

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Symmetric matrices : • are always diagonalisable

• we can find ONB basis of eigenvectors, so that S is an orthogonal matrix.

• eigenspaces are orthogonal to each other.

Examples : reflections, orthogonal projections.

Why should we care?

geometry : • insights into what the linear transformation does

• "uncoupling" of information

algebra. it's easier to work with diagonal matrices, and

if A and B similar, then

- $\text{rank}(A) = \text{rank}(B)$
- $\det(A) = \det(B)$ ~ $\det(A)$ "$\det(S)\det(B)\det(S^{-1})$"
- $A^k = SB^kS^{-1}$
- $\text{nullity}(A) = \text{nullity}(B)$

e.g $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ vs. $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x_1 \to y_1$, $x_2 \to y_2$ (crossed) $x_1 - y_1$, $x_2 - y_2$

(*) If S invertible, then $\text{rank}(AS) = \text{rank}(A)$

$= \text{rank}(SA)$

for any matrix A

Similarly, $\text{nullity}(AS) = \text{nullity}(SA) = \text{nullity}(A)$