1 Overview

The final exam is cumulative and will cover Chapters 1, 2, 3, 5.1-3, 6 (but not the Laplace expansion, nor Cramer's rule), 7.1-3 (excluding the parts on dynamical systems), and 8.1, as well as Sheets 1–9. Your review should, therefore, include the information on the study guides for each of the first two midterms. Here are some more practice problems from the textbook for the material covered after the second midterm:
Chapter 5.3 Ex. 2, 8
Chapter 6.1 Ex. 12
Chapter 6.2 Ex. 6, 13
Chapter 6.3 Ex. 1
Chapter 7.1 Ex. 21
Chapter 7.2 Ex. 9
Chapter 7.3 Ex. 2, 9
Chapter 8.1 Ex. 4

Below are some additional practice problems. The actual final exam will consist of problems that are related to the ones below. The types of questions in the final will be similar to that of the second midterm.

2 Practice problems

1. Solve the following system of linear equations using Gauss-Jordan. Does the system have exactly one, infinitely many or no solutions?

$$\begin{aligned} x_1 + 3x_3 + x_4 &= 1 \\ 2x_1 + x_2 - x_3 + 4x_4 &= 2 \\ x_1 - x_2 &= 0 \end{aligned}$$

Solution: The augmented coefficient matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 1 \\ 2 & 1 & -1 & 4 & 2 \\ 1 & -1 & 0 & 0 & 0 \end{array}\right] = A|b\,.$$

We put this in RREF using elementary row operations and obtain

$$\mathrm{RREF}(A|b) = \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{13}{10} & \frac{13}{10} \\ 0 & 1 & 0 & \frac{27}{10} & \frac{7}{10} \\ 0 & 0 & 1 & -\frac{1}{10} & \frac{1}{10} \end{array}\right].$$

Thus the system has infinitely many solutions, of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{13}{10}t + \frac{13}{10} \\ -\frac{27}{10}t + \frac{7}{10} \\ \frac{1}{10}t + \frac{1}{10} \\ t \end{bmatrix}$$

for $t \in \mathbb{R}$ arbitrary.

2. For the following matrix A, compute its RREF, and find a basis for the kernel of A and the image of A.

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & -2 & 3 & 1 \\ 1 & -1 & 2 & 1 \end{bmatrix}.$$

Solution: We use Gauss-Jordan to put A in RREF, and we get

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first and second columns of RREF(A) are the ones containing pivots, so a basis for the image of A is given by the first and second columns of A, so by the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ -2 \\ -1 \end{bmatrix}.$$

The solutions to $Av = 0$ are given by vectors of the form

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} -r - t \\ r \\ r \\ t \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

for $r, t \in \mathbb{R}$ arbitrary, so we have that a basis for the kernel is given by the vectors

$$\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

3. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation that sends any vector $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$ in $\mathbb{R}^4$ to the vector $\begin{bmatrix} v_1 \\ v_2 + v_3 \\ 0 \end{bmatrix}$. Find the matrix A that represents T. What is the rank of A?

   Solution: We know that
$$A = \begin{bmatrix} | & | & | & | \\ T(e_1) & T(e_2) & T(e_3) & T(e_4) \\ | & | & | & | \end{bmatrix}$$
   We have $T(e_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $T(e_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $T(e_3) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $T(e_4) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Thus
$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
   The matrix A is already in RREF. It has two pivots, and thus its rank is 2.

4. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^5$ and $T'\colon \mathbb{R}^5 \to \mathbb{R}^4$ be two linear transformations. What possible values can the ranks of $T \circ T'$ and $T' \circ T$ have?

   Solution: Both compositions can have rank $0, 1, 2, 3$ or 4. Note that $T \circ T'\colon \mathbb{R}^5 \to \mathbb{R}^5$ can't have rank 5, since $T'\colon \mathbb{R}^5 \to \mathbb{R}^4$ can have rank at most 4, and the rank of $T \circ T'$ is smaller or equal to the rank of T'.

5. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation represented by the matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$.

   Consider the basis $\mathcal{B}$ of $\mathbb{R}^2$ given by the vectors $v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and s $v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

   - Compute the $\mathcal{B}$-coordinates of the vectors $T(e_1)$ and $T(e_2)$.
   - Compute the $\mathcal{B}$-matrix of T.
   - Let B be the $\mathcal{B}$-matrix of T. Are A and B similar? If yes, give the invertible matrix S such that $AS = SB$.

   Solution:

- We have
$$T(e_1) = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = v_2$$
and
$$T(e_2) = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} = -v_1\,.$$
Thus, we have that $[T(e_1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $[T(e_2)]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.
- The $\mathcal{B}$-matrix of T is
$$B = \begin{bmatrix} | & | \\ [T(v_1)]_{\mathcal{B}} & [T(v_2)]_{\mathcal{B}} \\ | & | \end{bmatrix}$$
We have that
$$T(v_1) = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \end{bmatrix}$$
and
$$T(v_2) = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}\,.$$
To find the $\mathcal{B}$-coordinates of $T(v_1)$ we solve the system $\left[\begin{array}{cc|c} -3 & 1 & 0 \\ 1 & 3 & -10 \end{array}\right]$ and we obtain $[T(v_1]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$.
Similarly, for $T(v_2)$ we solve the system $\left[\begin{array}{cc|c} -3 & 1 & 10 \\ 1 & 3 & 0 \end{array}\right]$ and obtain $[T(v_2]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.
Thus the $\mathcal{B}$-matrix of T is
$$B = \begin{bmatrix} -1 & -3 \\ -3 & 1 \end{bmatrix}\,.$$
- Yes, and $S = \begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$.

6. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the plane V given by the equation $x_1 + x_2 + x_3 = 0$.
   - Find a basis for V.
   - Find a basis for the orthogonal complement of V.

- Find the matrix A that represents T.

Solution:

- The vectors in V are given by the solutions to the equation $x_1 + x_2 + x_3 = 0$, thus to find a basis for V we solve the system $Ax = 0$ with coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

The solutions are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, thus a basis for V is given by the vectors $v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

- A vector $w \in \mathbb{R}^3$ is in the orthogonal complement of V if and only if $w \cdot v_1 = 0$ and $w \cdot v_2 = 0$, where v_1, v_2 are the two vectors in the basis for V we found in the previous point. Thus the vectors in $V^\perp$ are the solutions to the system $Ax = 0$ with coefficient matrix A given by

$$A = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Solving this system we find that a basis for $V^\perp$ is given by the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

- We first use Gram-Schmidt to obtain an orthonormal basis for V given by $u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $u_2 = \frac{\sqrt{6}}{3} \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}$.

Thus

$$\text{proj}_V(e_1) = -1/2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - 1/3 \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix} = 1/3 \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

and similarly we find

$$\text{proj}_V(e_2) = 0 + 2/3 \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix} = 1/3 \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

and

$$\mathrm{proj}_V(e_3) = 1/2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - 1/3 \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix} = 1/3 \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$$

Then the matrix for the orthogonal projection is

$$A = 1/3 \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

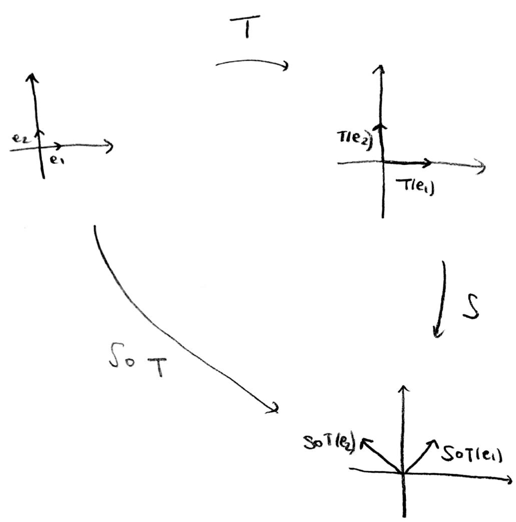
7. Find the QR decomposition of the following matrix.

$$A = \begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

8. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the scaling by 2. Let $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the counterclockwise rotation by an angle θ.

   - Make a drawing illustrating the image of the standard basis vectors under the composition $S \circ T$.
   - Is $S \circ T$ invertible? If yes, give the matrix representing the inverse linear transformation.

   Solution:



Yes, $S \circ T$ is invertible. The matrix of the inverse linear transformation is given by taking the product of the matrix representing a rotation by $2\pi - \theta$ with the matrix representing a scaling by $1/2$. The rotation is represented by the matrix

$$\begin{bmatrix} \cos(2\pi - \theta) & -\sin(2\pi - \theta) \\ \sin(2\pi - \theta) & \cos(2\pi - \theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

and we thus have that the inverse linear transformation is represented by

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} 1/2\cos(\theta) & 1/2\sin(\theta) \\ -1/2\sin(\theta) & 1/2\cos(\theta) \end{bmatrix}.$$

9. Let A be an 3×3 matrix representing the reflection about the line L spanned by the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in $\mathbb{R}^3$. Use geometric arguments to find the eigenvalues of A and their corresponding geometric multiplicities. Is A diagonalisable?

   Solution: The only vectors $v \in \mathbb{R}^3$ for which $Av = \lambda v$ for some $\lambda \in \mathbb{R}$ are the vectors lying on the line L, for which we have $\lambda = 1$, and the vectors in the orthogonal complement of L, for which $\lambda = -1$. Thus, we have that 1 and -1 are the two eigenvalues of A. We have that $E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$ and $E_{-1} = L^\perp = \text{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right)$. Thus $\text{gemu}(1) = 1$ and $\text{gemu}(-1) = 2$. If we choose bases for E_1 and E_{-1} we obtain a basis of $\mathbb{R}^3$ consisting of eigenvectors, thus A is diagonalisable. (Note that in this particular question it doesn't really matter by which vector the line L is spanned, because to derive the geometric multiplicty of E_{-1} one could simply use the fact that the orthogonal complement of L has dimension 2, by the rank-nullity theorem.)

10. Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

   - Without computing eigenvalues or eigenvectors: why is A diagonalisable?
   - Find the eigenvalues of A.
   - Find an orthogonal matrix S such that $S^{-1}AS$ is a diagonal matrix.
   - Compute A^5.

   Solution:

   - A is diagonalisable because it is symmetric.

- We have

$$f_A(\lambda) = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = (1-\lambda)\lambda^2 - (1-\lambda)$$
$$= -\lambda^3 + \lambda^2 + \lambda - 1$$
$$= (-1)(\lambda-1)^2(\lambda+1)$$

  The roots of $f_A(\lambda)$ are $\lambda_1 = 1$ with multiplicity 2 and $\lambda_2 = -1$ with multiplicity 1. Therefore A has eigenvalues 1 and -1.

- We first compute bases for the eigenspaces E_1 and E_{-1} and then use Gram-Schmidt on these bases to get an orthogonal matrix S.

  We have

$$E_1 = \ker(A - I_3) = \ker\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right).$$

  So a basis for E_1 is given by the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

  We have

$$E_{-1} = \ker(A + I_3) = \ker\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right).$$

  So a basis for E_{-1} is given by the vector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

  Now we use Gram-Schmidt to obtain orthonormal bases for each eigenspace:

  For E_1 we have that $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are orthogonal, so we only need to divide the vectors by their length. We get that $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ give an orthonormal basis of E_1.

Similarly, for E_{-1} we have the orthonormal basis given by the vector $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}$.

Thus

$$S=\begin{bmatrix}0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ 1 & 0 & 0\\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}$$

is orthogonal and we have

$$\begin{bmatrix}0 & 1 & 0\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ 1 & 0 & 0\\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}.$$

- We know that if A and B are similar matrices, then there exists S invertible with $A=SBS^{-1}$, and so $A^5=SB^5S^{-1}$. Here

$$B^5=\begin{bmatrix}1^5 & 0 & 0\\ 0 & 1^5 & 0\\ 0 & 0 & (-1)^5\end{bmatrix}=B\,.$$

so $A^5=SBS^{-1}=A$.

11. Let A and B be two 3×3 matrices. If $\det A=\det B$, then which of the following are true?

- The matrices A and B have the same rank. **Solution:** False. Consider the two matrices $A=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$. They have both determinant equal to 0, but A has rank 2 while B has rank 1. (Note that if two matrices have both non-zero determinant, then they are invertible, and thus have full rank. So it is true that if $\det(A)=\det(B)\neq 0$, then $\operatorname{rank}(A)=\operatorname{rank}(B)$.)
- The matrices A and B are similar. **Solution:** False, consider for example two 3×3 matrices A and B, where A is the matrix with all entries equal to 0, while B is the matrix representing the orthogonal projection onto a line in $\mathbb{R}^3$. These matrices have both determinant 0. However A and B can't be similar, as similar matrices must have the same rank, and the rank of A is 0, while the rank of B is 1.
- The matrices describe the same linear transformation under different choices of basis for $\mathbb{R}^3$. **Solution:** False, as the matrices would be similar; see previous point.

- The parallelepipeds spanned by the vectors Ae_1, Ae_2, Ae_3 and Be_1, Be_2, Be_3 have the same volume. **Solution:** True, as the volume of those parallelepipeds is given by computing the determinants of A, respectively B.