Math 33A Sheet 8

Chapter 5.3

Ex 1)* $\begin{bmatrix} 0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix}$

$\vec{v}_1 \cdot \vec{v}_2 = 0.48 + 0.48 = 0.96 \neq 0$

$\boxed{\text{Not orthogonal}}$

Ex 5) $3A$

$A = [\vec{v}_1, \vec{v}_2, \vec{v}_3]$

$\|\vec{v}_1\| = 1$

$3A = [3\vec{v}_1, 3\vec{v}_2, 3\vec{v}_3]$

$\|\vec{v}_1\| = 3$

$\boxed{\text{not orthogonal}}$

Ex 7) AB

$A = [\vec{v}_1\ \vec{v}_2 \ldots \vec{v}_n]$

$B = [\vec{w}_1\ \vec{w}_2 \ldots \vec{w}_n]$

$\boxed{\text{Orthogonal, product of 2 orthogonal matrices is orthogonal}}$

Ex 15) AB

$(AB)^T = B^T A^T = BA$

$AB \neq BA$

$\boxed{\text{Not symmetlic}}$

Ex 17) B^{-1}

$(B^T)^{-1} = (B^{-1})^T$

$B = B^T$

$(B^{-1})^T = (B^T)^{-1} = B^{-1}$

$\boxed{\text{Symmetlic}}$

Ex 23) $A - A^T$

$(A - A^T)^T = A^T - A$

$A - A^T \neq A^T - A$

$\boxed{\text{not symmetlic}}$

Ex 29)* Orthogonal matlices preserve dot product and length

$\theta_i = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right)$

$= \arccos\left(\frac{L(\vec{v}) \cdot L(\vec{w})}{\|L(\vec{v})\|\|L(\vec{w})\|}\right)$

$= \theta_f$ because all values are the same

$\therefore \theta_i = \theta_f$

The converse doesn't need to be true, a scaling preserves angles, but not length, and is therefore not orthogonal

Ex 30) $\vec{0}$ is the only vector that has length of zero, therefore $\boxed{\ker(L) = \{\vec{0}\}}$

The image's dimension must be equal to $\boxed{m}$ because of rank-nullity

Since the dimension of the image/rank must be m, $\boxed{m \leq n}$

The columns of A must be linearly independent because $\ker(L) = \ker(A) = \vec{0}$

$\boxed{A^T A = I_m}$ because the $\ker(A) = \vec{0}$, therefore A is invertible

AA^T is the matrix of an orthogonal projection

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix}$$

Ex32) a) $A^TA = I_m$

$AA^T \stackrel{?}{=} I_n$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

$AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I_3$

$\boxed{\text{Not true}}$

b) $A^{-1}A = I_n = AA^{-1}$

$\therefore A^TA = I_n = AA^T$

$\boxed{\text{Always true}}$

Chapter 6.1

Ex6)* $\begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

$\begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{matrix} 6 & 0 \\ 5 & 4 \\ 3 & 2 \end{matrix}$

$\det(A) = 24 + 0 + 0 - 0 - 0 - 0$

$= \boxed{24}$ $\boxed{\text{invertible}}$

Ex8) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{matrix} 1 & 2 \\ 1 & 1 \\ 3 & 2 \end{matrix}$

$\det(A) = 1 + 6 + 6 - 2 - 2 - 9$

$= \boxed{0}$ $\boxed{\text{not invertible}}$

Chapter 6.2

Ex8)* $\begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix} \times \frac{1}{2}$ $k_1 = 2$

$\begin{array}{c c} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix} & \begin{matrix} \\ -3I \\ -4I \\ -5I \\ -6I \end{matrix} \end{array}$

$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ $s = 4$

$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$\det(B) = 1$

$\det(A) = (-1)^4 (2) \det(B)$

$= 1(2)(1)$

$= \boxed{2}$

Ex16) $\det \begin{bmatrix} 6\vec{v}_1 + 2\vec{v}_4 \\ \vec{v}_2 \\ \vec{v}_3 \\ 3\vec{v}_1 + \vec{v}_4 \end{bmatrix}$ $k_1 = \frac{1}{6}$

$\det(A) = 8$

$\det(A) = (-1)^s (k_1)(\det(B))$

$8 = 1(\frac{1}{6}) \det(B)$

$\boxed{\det(B) = 48}$

Ex59) $\det(A) = \det(B)$

$A \stackrel{?}{=} S^{-1}BS$

$A = I_2 \rightarrow \det(A) = 1$

$B = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \rightarrow \det(B) = 1$

Identity matrix is only similar to itself

$\boxed{\text{False}}$

Chapter 6.1

Ex2) $\begin{bmatrix} 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 8 \\ 2 \end{bmatrix}$

$A = \frac{1}{2}\det\begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}$

$= \frac{1}{2}(|6-56|) = \frac{1}{2}(50)$

$= \boxed{25}$

Ex6)* $\begin{bmatrix} a_1 \\ a_2 \\ 1 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ 1 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \\ 1 \end{bmatrix}$

$A_1 = \frac{1}{6}\left|\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix}\right|$

$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix} \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ 1 & 1 \end{matrix}$

$a_1b_2 + b_1c_2 + c_1a_2 - b_1a_2 - a_1c_2 - c_1b_2$

$A_1 = \frac{1}{6}\left|a_1b_2 + b_1c_2 + a_2c_1 - a_2b_1 - a_1c_2 - b_2c_1\right|$

$A_2 = \frac{1}{2}\left|\det\begin{bmatrix} b_1-a_1 & c_1-a_1 \\ b_2-a_2 & c_2-a_2 \end{bmatrix}\right|$

$\det\begin{bmatrix} b_1-a_1 & c_1-a_1 \\ b_2-a_2 & c_2-a_2 \end{bmatrix}$

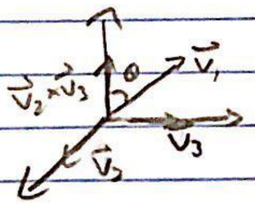
$b_1c_2 + \cancel{a_1a_2} - a_1c_2 - a_2b_1 - (b_2c_1 + \cancel{a_1a_2} - a_2c_1 - a_1b_2)$

$= a_1b_2 + b_1c_2 + a_2c_1 - a_2b_1 - a_1c_2 - b_2c_1$

$A_2 = \frac{1}{2}\left|a_1b_2 + b_1c_2 + a_2c_1 - a_2b_1 - a_1c_2 - b_2c_1\right|$

$\boxed{A_1 = \frac{1}{3}A_2}$

Ex19)



$\det(A) = \vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3)$

↳ positive only when θ between $\vec{v}_1$ and $\vec{v}_2 \times \vec{v}_3$ is $> \frac{\pi}{2}$

Ex20) $T(\vec{x}) = A\vec{x}$

~~x is~~ some 3×3 matrix B that's positively oriented

$T(\vec{x}) = AB$

$\det(AB) > 0$

$\det(B) > 0$

$\det(A) \times \det(B) = \det(AB)$

$\therefore$ $\det(A) \times +\text{number} = +\text{number}$

$\boxed{\therefore \det(A) \text{ must be positive}}$

Q1) $A = [T(\vec{e}_1)\ T(\vec{e}_2)]$

↳ put into another basis

↳ vector that spans the line, vector $\perp$ to first vector

$m = \frac{y}{x}$

$x = 1 \quad m = y$

$\vec{b}_1 = \begin{bmatrix} 1 \\ m \end{bmatrix}$

$m = -\frac{x}{y}, \; y = 1$

$x = -m$

$\vec{b}_2 = \begin{bmatrix} -m \\ 1 \end{bmatrix}$

$S = \begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix}$

$[T(\vec{x})]_B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\vec{x}$

$\begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & m \\ m & -1 \end{bmatrix}$

$T(\vec{x}) = \begin{bmatrix} 1 & m \\ m & -1 \end{bmatrix}\vec{x}$

$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & m \\ m & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$= v_1\begin{bmatrix} 1 \\ m \end{bmatrix} + v_2\begin{bmatrix} m \\ -1 \end{bmatrix} = \begin{bmatrix} v_1 + v_2m \\ v_1m - v_2 \end{bmatrix}$

$(v_1^2 + 2v_1v_2m + v_2^2m^2 + v_1^2m^2 - 2v_1v_2m + v_2^2)$

$= (v_1^2 + v_2^2m^2 + v_1^2m^2 + v_2^2)$

Well, that didn't go right :(