Homework 6

Status: Final (although there might be some typos).

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Due date: Friday, May 15.

Regular exercises

Recurrence Relations

Section 7.1 in course textbook.

1. Find a recurrence relation and initial conditions that generate a sequence that begins with the given terms.
   i. 3, 7, 11, 15, …
   ii. 3, 6, 9, 15, 24, 39, …
   iii. 1, 1, 2, 4, 16, 128, 4096, …
2. Assume that a person invests \$2000 at 14 percent interest compounded annually. Let A_n represent the amount at the end of n years.
   i. Find a recurrence relation for the sequence $A_0, A_1, \ldots$
   ii. Find an initial condition for the sequence $A_0, A_1, \ldots$
   iii. Find A_1, A_2, and A_3.
   iv. Find an explicit formula for A_n.
3. Let S_n denote the number of n-bit strings that do not contain the pattern `000`. Find a recurrence relation and initial conditions for the sequence $\{S_n\}$.
4. For the following exercises, refer to the sequence S where S_n denotes the number of n-bit strings that do not contain the pattern `00`.
   i. Find a recurrence relation and initial conditions for the sequence $\{S_n\}$.
   ii. Show that $S_n = f_{n+2}$, $n = 1, 2, \cdots$, where f denotes the Fibonacci sequence.
5. Write explicit solutions for the Tower of Hanoi puzzle for $n = 3, 4$.
6. Suppose that we have n dollars and that each day we buy either orange juice (\$1), milk (\$2), or beer (\$2). If R_n is the number of ways of spending all the money, show that $R_n = R_{n-1} + 2R_{n-2}$.

Note: Order is taken into account. For example, there are 11 ways to spend \$4: MB, BM, OOM, OOB, OMO, OBO, MOO, BOO, OOOO, MM, BB.

7. The sequence $g_1, g_2, ...$ is defined by the recurrence relation

$$g_n = g_{n-1} + g_{n-2} + 1, \qquad n \geq 3,$$

and initial conditions $g_1 = 1$, $g_2 = 3$. By using mathematical induction or otherwise, show that

$$g_n = 2f_{n+1} - 1, \qquad n \geq 1,$$

where $f_1, f_2, ...$ is the Fibonacci sequence.

8. Define the sequence $t_1, t2, ...$ by the recurrence relation

$$t_n = t_{n-1}t_{n-2}, \qquad n \geq 3,$$

and initial conditions $t_1 = 1$, $t_2 = 2$. What is wrong with the following *"proof"* that $t_n = 1$ for all $n \geq 1$?

Basis Step: For $n = 1$, we have $t_1 = 1$; thus, the Basis Step is verified.

Inductive Step: Assume that $t_k = 1$ for $k < n$. We must prove that $t_n = 1$. Now

$$\begin{aligned} t_n &= t_{n-1}t_{n-2} \\ &= 1 \cdot 1 \qquad \text{by the inductive assumption} \\ &= 1. \end{aligned}$$

The Inductive Step is complete.

9. The Lucas sequence $L_1, L_2, ...$ (named after Édouard Lucas, the inventor of the Tower of Hanoi puzzle) is defined by the recurrence relation $L_n = L_{n-1} + L_{n-2}$, $n \geq 3$, and the initial conditions $L_1 = 1$, $L_2 = 3$.

   i. Find the values of L_3, L_4, and L_5.
   ii. Show that

$$L_{n+2} = f_{n+1} + f_{n+3}, \qquad n \geq 1,$$

where $f_1, f_2, ...$ denotes the Fibonacci sequence.

Solving Recurrence Relations

Section 7.2 in course textbook.

10. For the recurrence relations below, first determine the order of the relation, then determine whether or not each of them is

    - linear, or
    - linear homogeneous, or
    - linear with constant coefficients, or
    - linear homogeneous with constant coefficients.

    i. $a_n = -3a_{n-1}$
    ii. $a_n = 2na_{n-2} - a_{n-1}$
    iii. $a_n = 2na_{n-1}$
    iv. $a_n = a_{n-1} + n$
    v. $a_n = a_{n-1} + 1 + 2^{n-1}$

11. Solve the given recurrence relation for the initial conditions given.

    i. $a_n = 2na_{n-1}$; $a_0 = 1$.
    ii. $a_n = 6a_{n-1} - 8a_{n-2}$; $a_0 = 1$, $a_1 = 0$.
    iii. The Lucas sequence (see exercise 9).

12. Assume that the horse population of *The Kingdom of Rohan*[1] is 0 at time $n = 0$. Suppose that at time n, $100n$ horses are introduced into the kingdom and that the population increases 20 percent each year. Write a recurrence relation and an initial condition that define the horse population at time n and then solve the recurrence relation.

    Note: The following formula may be of use:

$$\sum_{k=1}^{n-1} kq^{k-1} = \frac{(n-1)q^n - nq^{n-1} + 1}{(q-1)^2} \tag{1}$$

13. Solve the recurrence relation

$$\sqrt{a_n} = \sqrt{a_{n-1}} + 2\sqrt{a_{n-2}}$$

with initial conditions $a_0 = a_1 = 1$ by making the substitution $b_n = \sqrt{a_n}$.

[1] Of course this is a rather silly assumption. As pretty much everybody knows, *Rohan* (or *Rochand*) is the home of the *Rohirrim*, and the meaning of the word is *Horse-country*. Thus, it would be impossible to have a population of 0 horses if the name was already established.

14. Solve the recurrence relation

$$a_n = \sqrt{\frac{a_{n-2}}{a_{n-1}}}$$

with initial conditions $a_0 = 8$, $a_1 = \frac{1}{2\sqrt{2}}$ by taking the logarithm of both sides and making the substitution $b_n = \log a_n$.

   Note: Here $\log x$ stands for the *natural logarithm*; but any other logarithm will work.

15. The equation

$$a_n = f(n)a_{n-1} + g(n)a_{n-2}$$

is called a *second-order, linear homogeneous* recurrence relation. The coefficients $f(n)$ and $g(n)$ are not necessarily constant. Show that if S and T are solutions of the equation above, then $\alpha S + \beta T$ is also a solution, for any choice of the numbers α and β.

Miscellaneous exercises

- Use your knowledge of combinations to establish the following *well-known* identity

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

   Hint:

   Write $(a+b)^n = (a+b)(a+b)\cdots(a+b)$ and realize that any term in the algebraic expression that results from developing the right hand side consists of exactly n factors. Some of them (say k of them) will be a's while the rest will be b's. Then count the number of terms in every possible grouping.

- By choosing specific values of a and b in the previous problem, prove the following identities:

   i. $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$
   ii. $2^n\binom{n}{0} - 2^{n-1}\binom{n}{1} + 2^{n-2}\binom{n}{2} + \cdots + (-1)^n\binom{n}{n} = 1$
   iii. $\binom{n}{0} - 2\binom{n}{1} + 4\binom{n}{2} - 8\binom{n}{3} + \cdots + (-2)^n\binom{n}{n} = \begin{cases} 1 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$

- For the following exercises, refer to the sequence $S_1, S_2, \ldots$ where S_n denotes the number of n-bit strings that do not contain the pattern `010`.

i. Compute S_1, S_2, S_3, and S_4.

ii. By considering the number of n-bit strings that do not contain the pattern `010` that have no leading `0`'s (i.e., that begin with `1`); that have one leading `0` (i.e., that begin `01`); that have two leading `0`'s; and so on, derive the recurrence relation

$$S_n = S_{n-1} + S_{n-3} + S_{n-4} + S_{n-5} + \cdots + S_1 + 3.$$

iii. By replacing n by $n-1$ in the equation above, write a formula for S_{n-1}. Subtract the formula for S_{n-1} from the formula for S_n and use the result to derive the recurrence relation

$$S_n = 2S_{n-1} - S_{n-2} + S_{n-3}.$$

- Suppose that both roots of

  $$t^2 - c_1 t - c_2 = 0$$

  are equal to r, and suppose that a_n satisfies

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2}, \qquad a_0 = C_0, \qquad a_1 = C_1.$$

  Show that there exist constants α and β such that

  $$a_n = \alpha r^n + \beta n r^n, \qquad n = 0, 1, \ldots$$

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