## **EXERCISES 3.1**

**GOAL** Use the concepts of the image and the kernel of a linear transformation (or a matrix). Express the image and the kernel of any matrix as the span of some vectors. Use kernel and image to determine whether a matrix is invertible.

For each matrix A in Exercises 1 through 13, find vectors that span the kernel of A. Use paper and pencil.

**1.** 
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

**2.** 
$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$\mathbf{3.} \ \ A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

**4.** 
$$A = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$$

$$\mathbf{5.} \ A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$$

**6.** 
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

7. 
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\mathbf{8.} \ \ A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

For each matrix A in Exercises 14 through 16, find vectors that span the image of A. Give as few vectors as possible. Use paper and pencil.

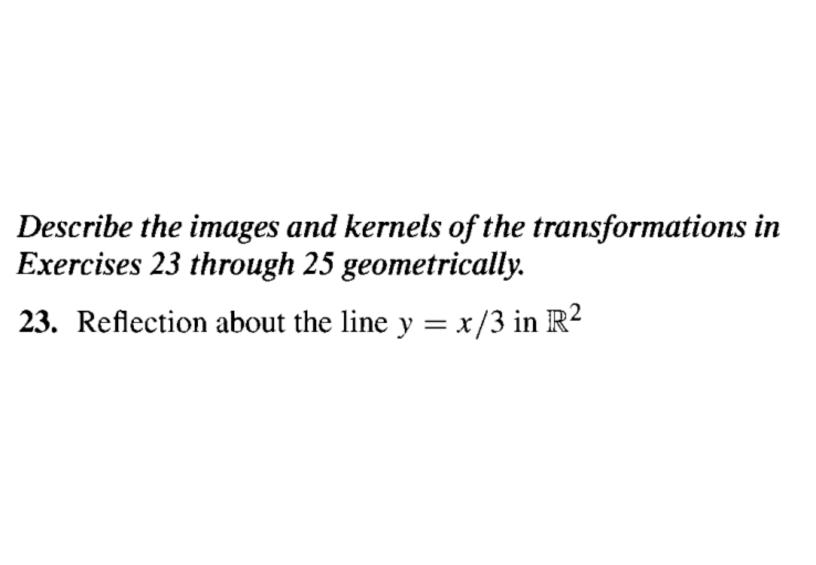
**14.** 
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

**15.** 
$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$
 **16.**  $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ 

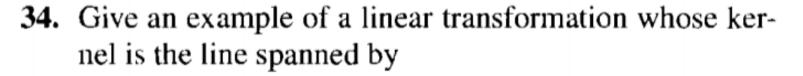
For each matrix A in Exercises 17 through 22, describe the image of the transformation  $T(\vec{x}) = A\vec{x}$  geometrically (as a line, plane, etc. in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ ).

**17.** 
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
 **18.**  $A = \begin{bmatrix} 1 & 4 \\ 3 & 12 \end{bmatrix}$ 

**19.** 
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & -8 \end{bmatrix}$$



24.	Orthogonal projection onto the plane $x + 2y + 3z = 0$
	in $\mathbb{R}^3$



$$\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

in  $\mathbb{R}^3$ .

## **EXERCISES 3.2**

**GOAL** Check whether or not a subset of  $\mathbb{R}^n$  is a subspace. Apply the concept of linear independence (in terms of Definition 3.2.3, Theorem 3.2.7, and Theorem 3.2.8). Apply the concept of a basis, both in terms of Definition 3.2.3 and in terms of Theorem 3.2.10.

Which of the sets W in Exercises 1 through 3 are subspaces of  $\mathbb{R}^3$ ?

$$1. W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 1 \right\}$$

$$2. W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \le y \le z \right\}$$

3. 
$$W = \left\{ \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix} : x, y, z \text{ arbitrary constants} \right\}$$

- **4.** Consider the vectors  $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$  in  $\mathbb{R}^n$ . Is span  $(\vec{v}_1, \ldots, \vec{v}_m)$  necessarily a subspace of  $\mathbb{R}^n$ ? Justify your answer.
- **5.** Give a geometrical description of all subspaces of  $\mathbb{R}^3$ . Justify your answer.
- **6.** Consider two subspaces V and W of  $\mathbb{R}^n$ .
  - **a.** Is the intersection  $V \cap W$  necessarily a subspace of  $\mathbb{R}^n$ ?
  - **b.** Is the union  $V \cup W$  necessarily a subspace of  $\mathbb{R}^n$ ?
- 7. Consider a nonempty subset W of  $\mathbb{R}^n$  that is closed under addition and under scalar multiplication. Is W necessarily a subspace of  $\mathbb{R}^n$ ? Explain.
- **8.** Find a nontrivial relation among the following vectors:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

In Exercises 10 through 20, use paper and pencil to identify the redundant vectors. Thus determine whether the given vectors are linearly independent.

$$10. \begin{bmatrix} 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

11. 
$$\begin{bmatrix} 7 \\ 11 \end{bmatrix}$$
,  $\begin{bmatrix} 11 \\ 7 \end{bmatrix}$ 

12. 
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
,  $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$ 

13. 
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
,  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ 

15. 
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
,  $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ ,  $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ 

In Exercises 21 through 26, find a redundant column vector of the given matrix A, and write it as a linear combination of preceding columns. Use this representation to write a nontrivial relation among the columns, and thus find a nonzero vector in the kernel of A. (This procedure is illustrated in Example 8.)

21. 
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
 22.  $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$  23.  $\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ 
24.  $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$  25.  $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ 

Find a basis of the image of the matrices in Exercises 27 through 33.

**27.** 
$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$
 **28.**  $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$  **29.**  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ 

30. 
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix}$$
 31. 
$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 7 \\ 5 & 8 \end{bmatrix}$$