# 1 Overview

The final exam is cumulative and will cover Chapters 1, 2, 3, 5.1-3, 6 (but not the Laplace expansion, nor Cramer's rule), 7.1-3 (excluding the parts on dynamical systems), and 8.1, as well as Sheets 1–9. Your review should, therefore, include the information on the study guides for each of the first two midterms. Here are some more practice problems from the textbook for the material covered after the second midterm:
Chapter 5.3 Ex. 2, 8
Chapter 6.1 Ex. 12
Chapter 6.2 Ex. 6, 13
Chapter 6.3 Ex. 1
Chapter 7.1 Ex. 21
Chapter 7.2 Ex. 9
Chapter 7.3 Ex. 2, 9
Chapter 8.1 Ex. 4

Below are some additional practice problems. The actual final exam will consist of problems that are related to the ones below. The types of questions in the final will be similar to that of the second midterm.

# 2 Practice problems

1. Solve the following system of linear equations using Gauss-Jordan. Does the system have exactly one, infinitely many or no solutions?

$$\begin{aligned} x_1 + 3x_3 + x_4 &= 1 \\ 2x_1 + x_2 - x_3 + 4x_4 &= 2 \\ x_1 - x_2 &= 0 \end{aligned}$$

2. For the following matrix $A$, compute its RREF, and find a basis for the kernel of $A$ and the image of $A$.
$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & -2 & 3 & 1 \\ 1 & -1 & 2 & 1 \end{bmatrix}.$$

3. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation that sends any vector $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$ in $\mathbb{R}^4$

to the vector $\begin{bmatrix} v_1 \\ v_2 + v_3 \\ 0 \end{bmatrix}$. Find the matrix $A$ that represents $T$. What is the rank of $A$?

4. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^5$ and $T'\colon \mathbb{R}^5 \to \mathbb{R}^4$ be two linear transformations. What possible values can the ranks of $T \circ T'$ and $T' \circ T$ have?

5. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation represented by the matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$.

   Consider the basis $\mathcal{B}$ of $\mathbb{R}^2$ given by the vectors $v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and s $v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

   - Compute the $\mathcal{B}$-coordinates of the vectors $T(e_1)$ and $T(e_2)$.
   - Compute the $\mathcal{B}$-matrix of $T$.
   - Let $B$ be the $\mathcal{B}$-matrix of $T$. Are $A$ and $B$ similar? If yes, give the invertible matrix $S$ such that $AS = SB$.

6. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the plane $V$ given by the equation $x_1 + x_2 + x_3 = 0$.

   - Find a basis for $V$.
   - Find a basis for the orthogonal complement of $V$.
   - Find the matrix $A$ that represents $T$.

7. Find the QR decomposition of the following matrix.

$$A = \begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

8. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the scaling by 2. Let $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the counterclockwise rotation by an angle $\theta$.

   - Make a drawing illustrating the image of the standard basis vectors under the composition $S \circ T$.
   - Is $S \circ T$ invertible? If yes, give the matrix representing the inverse linear transformation.

9. Let $A$ be an $3 \times 3$ matrix representing the reflection about the line $L$ spanned by the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in $\mathbb{R}^3$. Use geometric arguments to find the eigenvalues of $A$ and their corresponding geometric multiplicities. Is $A$ diagonalisable?

10. Consider the matrix
$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
    - Without computing eigenvalues or eigenvectors: why is $A$ diagonalisable?
    - Find the eigenvalues of $A$.
    - Find an orthogonal matrix $S$ such that $S^{-1}AS$ is a diagonal matrix.
    - Compute $A^5$.

11. Let $A$ and $B$ be two $3 \times 3$ matrices. If $\det A = \det B$, then which of the following are true?
    - The matrices $A$ and $B$ have the same rank.
    - The matrices $A$ and $B$ are similar.
    - The matrices describe the same linear transformation under different choices of basis for $\mathbb{R}^3$.
    - The parallelepipeds spanned by the vectors $Ae_1, Ae_2, Ae_3$ and $Be_1, Be_2, Be_3$ have the same volume.