Math 33A Sheet 9

Chapter 7.1

Ex1) $A\vec{v} = \lambda\vec{v}$

$A^3\vec{v} = \lambda^3\vec{v}$

$\boxed{\text{Yes, the eigenvalue is } \lambda^3}$

Ex2) $A\vec{v} = \lambda\vec{v}$

$A^{-1}A = I_n$

$A^{-1}A\vec{v} = A^{-1}\lambda\vec{v}$

$\vec{v} = A^{-1}\lambda\vec{v}$

$A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$

$\boxed{\text{Yes, the eigenvalue is } \frac{1}{\lambda},\ \lambda \neq 0}$

Ex4) $A\vec{v} = \lambda\vec{v}$

$7A\vec{v} = 7\lambda\vec{v}$

$\boxed{\text{Yes, the eigenvalue is } 7\lambda}$

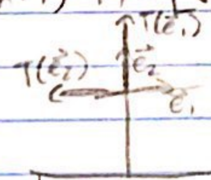
Ex16) $T(\vec{v}) = -\vec{v}$

$\boxed{\vec{v} \in \mathbb{R}^2 \text{ are eigenvectors}}$

$\boxed{\lambda = -1 \text{ is the eigenvalue}}$

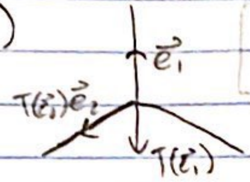
$\boxed{\text{The transformation is diagonalisable}}$

Ex17) $A = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$



$\boxed{\text{no eigenvalues / eigenvectors, } T \text{ is not diagonalisable}}$

Ex18)



Eigenvalues are 1 and -1

Eigenvectors are $\vec{v} \in V$ and $\vec{v} \in V^\perp$

T is diagonalisable

Ex37)* $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

a) rotation = orthogonal matrix = eigenvalues of ± 1

scaling $\times 5$ -> eigenvalues $\boxed{\pm 5}$

b) $\ker(A - (5)\lambda)$

$= \ker\left(\begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}\right)$

$= \text{span}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\ker(A - (-5)\lambda)$

$= \ker\left(\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\right)$

$= \text{span}\left(\begin{bmatrix} 1/2 \\ -1 \end{bmatrix}\right)$

$\boxed{\text{eigenbasis: } \begin{bmatrix} 2 & 1/2 \\ 1 & -1 \end{bmatrix}}$

c) $\boxed{D = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}}$

Chapter 7.2

Ex1) $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

$\begin{bmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix}$

$(1-\lambda)(3-\lambda) = 0$

$\boxed{\lambda = 1, 3}$

$\boxed{\text{almu}(1) = 1}$

$\boxed{\text{almu}(3) = 1}$

Ex 13) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} \begin{matrix} -\lambda & 1 \\ 0 & -\lambda \\ 1 & 0 \end{matrix}$$

$-\lambda^3 + 1 = 0$

$-\lambda^3 = -1$

$\lambda^3 = 1$

$\boxed{\text{eigenvalue is } 1 \quad \text{almu}(1) = 3}$

Ex16)* $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$\begin{bmatrix} a-\lambda & b \\ b & c-\lambda \end{bmatrix}$

$(a-\lambda)(c-\lambda) - b^2 = 0$

$(a-\lambda)(c-\lambda) = b^2$

$ac - c\lambda - a\lambda + \lambda^2 = b^2$

$ac - (a+c)\lambda + \lambda^2 = b^2$

$\lambda^2 - (a+c)\lambda + (ac - b^2) = 0$

$$\lambda = \frac{a+c \pm \sqrt{a^2+2ac+c^2-4ac+4b^2}}{2}$$

$$\lambda = \frac{a+c \pm \sqrt{(a-c)^2+4b^2}}{2}$$

$\boxed{a \neq c \text{ or } b \neq 0}$

Chapter 7.3

Ex1) $\begin{bmatrix} 7 & 8 \\ 0 & 9 \end{bmatrix}$

$(7-\lambda)(9-\lambda) = 0$

$\lambda = 7, 9$

$\ker\left(\begin{bmatrix} 0 & 8 \\ 0 & 2 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

$\ker\left(\begin{bmatrix} -2 & 8 \\ 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right)$

eigenbasis $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\boxed{B = \begin{bmatrix} 7 & 0 \\ 0 & 9 \end{bmatrix}}$

Ex4) $\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

$\begin{bmatrix} -\lambda & -1 \\ 1 & 2-\lambda \end{bmatrix}$

$-\lambda(2-\lambda) - (-1) = 0$

$-2\lambda + \lambda^2 + 1 = 0$

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda-1)(\lambda-1) = 0$

$\lambda = 1$

$\ker\left(\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

eigenbasis: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\text{gemu}(1) \neq n$

$\boxed{\text{not diagonalisable}}$

Ex 12)* $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{matrix} 1-\lambda & 1 \\ 0 & 1-\lambda \\ 0 & 0 \end{matrix}$$

$(1-\lambda)^3 = 0$

$\lambda = 1$

$\ker\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$

eigenbasis: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\text{gemu} \neq n$

$\boxed{\text{not diagonalisable}}$

Ex 47) $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$

$\lambda = 1$

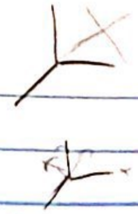
$\ker\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right)$

$ax_2 - bx_3 = 0$

$cx_3 = 0$

x_2 and $x_3 \neq \vec{0}$

$\boxed{a = b = c = 0}$



Chapter 8.1

Ex1) $\begin{bmatrix}1&0\\0&2\end{bmatrix}$

$(1-\lambda)(2-\lambda)=0$

$\lambda=1,2$

$\ker\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)=\operatorname{span}\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$

$\ker\left(\begin{bmatrix}-1&0\\0&0\end{bmatrix}\right)=\operatorname{span}\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$

$\boxed{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}}$

Ex 8)* $A=\begin{bmatrix}3&3\\3&-5\end{bmatrix}$

$\begin{bmatrix}3-\lambda&3\\3&-5-\lambda\end{bmatrix}$

$(3-\lambda)(-5-\lambda)-9=0$

$-15+2\lambda+\lambda^2-9=0$

$\lambda^2+2\lambda-24=0$

$(\lambda+6)(\lambda-4)=0$

$\lambda=-6,4$

$\ker\left(\begin{bmatrix}9&3\\3&1\end{bmatrix}\right)=\ker\left(\begin{bmatrix}3&1\\0&0\end{bmatrix}\right)=\operatorname{span}\left(\begin{bmatrix}1\\-3\end{bmatrix}\right)$

$\ker\left(\begin{bmatrix}-1&3\\3&-9\end{bmatrix}\right)=\ker\left(\begin{bmatrix}1&-3\\0&0\end{bmatrix}\right)=\operatorname{span}\left(\begin{bmatrix}3\\1\end{bmatrix}\right)$

$\vec{v}_1\cdot\vec{v}_2=3-3=0 \rightarrow$ they're orthogonal

$\vec{u}_1=\frac{\vec{v}_1}{\|\vec{v}_1\|}=\frac{1}{\sqrt{10}}\begin{bmatrix}1\\-3\end{bmatrix}$

$\vec{u}_2=\frac{1}{\sqrt{10}}\begin{bmatrix}3\\1\end{bmatrix}$

$\boxed{S=\frac{1}{\sqrt{10}}\begin{bmatrix}1&3\\-3&1\end{bmatrix}}$

$\left[\begin{array}{cc|cc}1/\sqrt{10}&3/\sqrt{10}&1&0\\-3/\sqrt{10}&1/\sqrt{10}&0&1\end{array}\right]$

$\left[\begin{array}{cc|cc}1/\sqrt{10}&3/\sqrt{10}&1&0\\0&10/\sqrt{10}&+3&1\end{array}\right]$

$\left[\begin{array}{cc|cc}1&3&\sqrt{10}&0\\0&10&+3\sqrt{10}&\sqrt{10}\end{array}\right]$

$\left[\begin{array}{cc|cc}1&3&\sqrt{10}&0\\0&1&+3/\sqrt{10}&1/\sqrt{10}\end{array}\right]$

$\left[\begin{array}{cc|cc}1&0&1/\sqrt{10}&-3/\sqrt{10}\\0&1&3/\sqrt{10}&1/\sqrt{10}\end{array}\right]$

$S^{-1}=\begin{bmatrix}1&-3\\3&1\end{bmatrix}\frac{1}{\sqrt{10}}$

$D=S^{-1}AS$

$D=\begin{bmatrix}1&-3\\3&1\end{bmatrix}\begin{bmatrix}3&3\\3&-5\end{bmatrix}\begin{bmatrix}1&3\\-3&1\end{bmatrix}\frac{1}{10}$

$=\begin{bmatrix}-6&18\\12&4\end{bmatrix}\begin{bmatrix}1&3\\-3&1\end{bmatrix}\frac{1}{10}$

$=\frac{1}{10}\begin{bmatrix}-60&0\\0&40\end{bmatrix}$

$\boxed{D=\begin{bmatrix}-6&0\\0&4\end{bmatrix}}$

Ex12) $\vec{v}_1=\begin{bmatrix}1\\0\\2\end{bmatrix}$

a) need orthogonal vectors to $\vec{v}_1$

$\vec{v}_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$

$x_1+2x_3=0$

$x_1=-2x_3$

$x_3=1,\ x_1=-2$

$\vec{v}_3=\begin{bmatrix}-2\\0\\1\end{bmatrix}$

$\boxed{\vec{u}_1=\frac{1}{\sqrt{5}}\begin{bmatrix}1\\0\\2\end{bmatrix},\ \vec{u}_2=\begin{bmatrix}0\\1\\0\end{bmatrix},\ \vec{u}_3=\frac{1}{\sqrt{5}}\begin{bmatrix}-2\\0\\1\end{bmatrix}}$

b) $\begin{bmatrix}1/\sqrt{5}&0&-2/\sqrt{5}\\0&1&0\\2/\sqrt{5}&0&1/\sqrt{5}\end{bmatrix}$

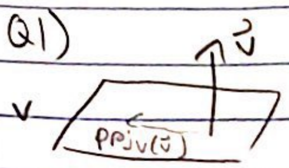
$B=\begin{bmatrix}T(\vec{b}_1)&T(\vec{b}_2)&T(\vec{b}_3)\end{bmatrix}$

$\boxed{B=\begin{bmatrix}1/\sqrt{5}&0&2/\sqrt{5}\\0&-1&0\\2/\sqrt{5}&0&-1/\sqrt{5}\end{bmatrix}}$

c) $A=\begin{bmatrix}T(\vec{e}_1)&T(\vec{e}_2)&T(\vec{e}_3)\end{bmatrix}$

$\boxed{A=\begin{bmatrix}1/\sqrt{5}&0&2/\sqrt{5}\\0&1&0\\2/\sqrt{5}&0&-1/\sqrt{5}\end{bmatrix}}$

Q1)



all projections are not parallel to $\vec{v}$ unless $\vec{v} \in V$

$\vec{v}$ must be in the subspace V

$\therefore$ $\boxed{\vec{v} \in V \text{ are eigenvectors}}$ ← length doesn't change

$\boxed{\text{eigenvalue} = 0, 1}$ ←

$\boxed{\text{diagonalisable?}}$

0 is the kernel of the transformation

1 is the eigenvalue for the remaining eigenvectors