

# 1/6 Lecture : Introduction to Differential Equations

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• OH: Wed: 1pm - 2:30pm

• Textbook: Differential Equations 2nd Edition - Polking, Bogges and Arnold

• Normal Form: form with the highest order on the left side

$$\hookrightarrow f^{(n)} = \phi(t, f, f', \dots, f^{(n-1)})$$

$$\hookrightarrow f \cdot f' - 1 = 0 \rightarrow f' = \frac{1}{f}$$

$$\hookrightarrow (f')^2 + f = 0 \rightarrow f' = \sqrt{-f} \text{ or } f' = -\sqrt{-f}$$

• Solving Diff. Eq.s

$$\hookrightarrow y' = y$$

$\hookrightarrow y = e^x, y = 0, y = ce^x \rightarrow$  family of solutions  
particular solution  $\hookrightarrow$  general solution

$$\hookrightarrow y'' + 2y' + 3y = 0$$

$$\hookrightarrow y = c_1 f_1 + c_2 f_2$$

• Initial Value Problem

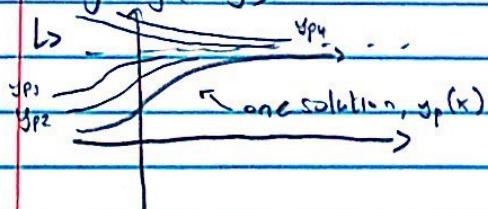
$\hookrightarrow$  diff. eq. + initial condition  $\rightarrow$  usually gives a unique solution

$$\hookrightarrow \text{Ex}) y' = y, y(0) = -1$$

$$\hookrightarrow y = -e^x$$

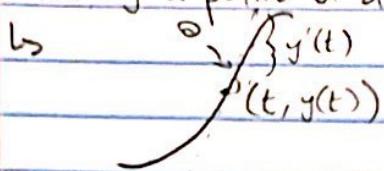
• Picture of Solutions

$$\hookrightarrow y' = y(1-y)$$



## 1/8 Lecture: Separable Variables

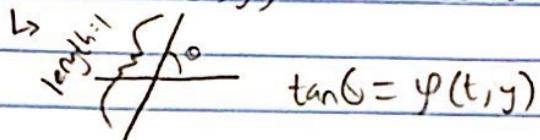
- Taking a point of a solution



↳ slope of tangent line:  $y'(t)$ ,  $\theta = \tan^{-1}(y'(t))$

$$\hookrightarrow y' = \varphi(t, y) = y(1-y)$$

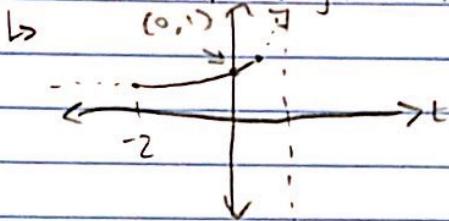
- For each  $(t, y)$  we can construct a line segment



↳ these are tangent to the solutions

$$\text{Ex: } y' = y^2, y(0) = 1$$

↳ Assume we know  $y(t) = \frac{1}{1-t}, (-2, \frac{1}{2})$  is the solution



↳ extend the interval from  $(-\infty, 1)$

↳ largest possible interval for a continuous solution - interval of existence

↳ contains an initial condition

### Separable Variables

$$\hookrightarrow y' = g(t) f(y) = \frac{dy}{dt}$$

$$\int \frac{dy}{f(y)} = \int g(t) dt$$

$$F(y) = G(t) + C$$

$$\hookrightarrow \text{Ex: } y' = ty^2$$

$$\int \frac{dy}{y^2} = \int t dt \rightarrow \text{loses solution } y=0$$

$$-\frac{1}{y} = \frac{1}{2}t^2 + C \rightarrow \text{implicit form}$$

$$\boxed{y = -\frac{2}{t^2 + C}} \rightarrow \text{explicit form}$$

$$\hookrightarrow \lim_{t \rightarrow \infty} \frac{-2}{t^2 + C} = 0$$

$$\hookrightarrow \text{Ex: } y' = y(1-y)$$

$$\frac{dy}{y(1-y)} = dt = \int \frac{1}{(1-y)} dy + \int \frac{1}{y} dy = \ln|1-y| - \ln|1+y| =$$

$$t + C = \ln \left| \frac{1}{1-y} \right| \rightarrow e^{t+C} = \left| \frac{y}{1-y} \right| \xrightarrow{\substack{t > 0 \\ y > 0}} k e^t = \left| \frac{y}{1-y} \right| \rightarrow \boxed{y = \frac{C e^t}{1 + C e^t}}$$

# 1/10 Lecture: 1st Order Linear Diff Eq.

Definition:  $y' = a(t) \cdot y + f(t)$  — forcing term

↪  $C'$  contains all differentiable functions

↪  $\{f | f, f' \text{ contains}\} f, g \in C'$ , then  $c_1 f + c_2 g \in C'$

↪  $y' - a(t) \cdot y = f$

$$(\frac{d}{dt} - a(t)) y = f \rightarrow A\vec{y} = \vec{f}$$

linear transformation from  $C' \rightarrow C'$

$$\hookrightarrow A\vec{y} = \vec{0}$$

↪ Solution: the kernel/null space  $\rightarrow$  still a linear space

$$\hookrightarrow A\vec{y}_p = \vec{b}$$

↪ ① Find one solution

$$A\vec{y}_p = \vec{b}$$

$$\hookrightarrow ② A\vec{y} = \vec{b}$$

$$A(\vec{y} - \vec{y}_p) = \vec{b} - \vec{b} = \vec{0} \rightarrow \vec{y} - \vec{y}_p \in \text{Ker}(A)$$

$$\vec{y} = \vec{y}_p + \text{Ker}(A)$$

$$\hookrightarrow \text{Ker}(A) = (\frac{d}{dt} - a(t)) \vec{y} = \vec{0}$$

$$= y' - a(t)y = \vec{0}$$

$$y' = a(t)y$$

$$\int \frac{dy}{y} = \int a(t) dt$$

$$\ln|y| = A(t) + C$$

$$|y| = e^{A(t)} e^C$$

$$|y| = k e^{A(t)}, k > 0$$

$$y = k e^{A(t)} \text{ or } y = -k e^{A(t)}$$

$$y = C e^{A(t)}, C \in \mathbb{R} \rightarrow 1\text{-D linear space}$$

↪  $y_h = e^{A(t)}$   $\rightarrow$  homogeneous part to the diff eq.  $y' = a(t)y + f$  (soln to  $y' = a(t)y$ )

$$\hookrightarrow y' = a(t)y + f$$

↪ Solution set is  $C y_h + y_p$ , where  $y_p$  is a particular solution to

$$y' = a(t)y + f$$

• How to find a particular solution  $\vec{y}_p$

↪ ① Integrating factors

② Variation of parameters

$$\hookrightarrow ① y' - a(t)y = f$$

$$\overbrace{=}^{(u(t)y)'} \quad$$

$$= u'y + u'y' = y' - a(t) \cdot y$$

$$(u + a(t))y + (u + 1)y' = 0 \times$$

$$\begin{aligned} \Rightarrow u(t)(y' - a(t)y) &= uf \\ uy' - ua(t)y &= uy' + u'y \\ -ua(t)y &= u'y \\ (ua(t) + u)y &= 0 \\ \text{always } ua(t) + u' &= 0 \\ \Rightarrow u &= e^{\int a(t) dt} \end{aligned}$$

$$\begin{aligned} \Rightarrow uy &= \int u f dt \\ y &= \frac{1}{u} \int u f dt \\ &= \frac{1}{u} (A(t) + C) \\ &= \frac{a(t)}{u} + \frac{C}{u} \leftarrow C \cdot y_h \end{aligned}$$

## Y13 Lecture: Solving for $y_p$

- Summary

↳  $y' = a(x)y + f \rightarrow$  1st order linear diff. eq.

↳ solution of  $Cy_n + y_p$

↳ solve for  $y_p$  by:

↳ integrating factors:  $\rightarrow e^{\int -a(t)dt}$

↳ Ex)  $y' = y + e^{-t}$

$$IF = e^{\int -a(t)dt} = e^{-t}$$

$$y' - y = e^{-t}$$

$$e^{-t}y' - e^{-t}y = e^{-2t}$$

$$(e^{-t}y)' = e^{-2t}$$

$$e^{-t}y = -\frac{1}{2}e^{-2t} + C$$

$$\boxed{y = -\frac{1}{2}e^{-t} + (e^t)}$$

↳ Ex)  $(1+t^2)y' + 4ty = \frac{1}{1+t^2}, y(0)=2$

$$(1+t^2)y' = \frac{1}{1+t^2} - 4ty$$

$$y' = \frac{1}{(1+t^2)^2} - \frac{4t}{1+t^2}$$

$$y' = -\frac{4t}{1+t^2}y + \frac{1}{(1+t^2)^2}$$

$$a(t) = -\frac{4t}{1+t^2}$$

$$IF = e^{\int -a(t)dt}$$

$$= e^{\int \frac{4t}{1+t^2} dt}$$

$$u = t^2, du = 2t dt$$

$$= e^{\int \frac{2du}{u}}$$

$$= e^{\ln(1+t^2)^2}$$

$$= (1+t^2)^2$$

$$(1+t^2)^2(y' + \frac{4t}{1+t^2}y) = (1+t^2)^2 \left(\frac{1}{(1+t^2)^2}\right)$$

$$(1+t^2)^2 y' = 1$$

$$(1+t^2)^2 y = t + C$$

$$y = \frac{t+C}{(1+t^2)^2}$$

$$C=2$$

$$\boxed{y = \frac{t+2}{(1+t^2)^2}}$$

• Variation of Parameter  $\rightarrow$  Duhamel's Principle

$$\Rightarrow y = (C + \frac{y_p}{y_h})y_h$$

$$= u(x) \cdot y_h(x)$$

$$(uy_h)' = a(x)(uy_h) + f$$

$$u'y_n + u'y_n' = auy_n + f$$

$$u'y_n + u(y_n' - a y_n) = f \leftarrow y_n' = a y_n$$

$$u'y_n = f$$

$$u' = \frac{f}{y_n}$$

$$u = \int \frac{f}{y_n} + C$$

↳ Must solve for  $y_n$  first:  $y_n = e^{\int a(t) dt}$

$$\text{↳ Ex)} (1+t^2)y' + 4ty = \frac{1}{1+t^2}$$

$$y' + \frac{4t}{(1+t^2)}y = \frac{1}{(1+t^2)^2} \leftarrow f(t)$$

$$y' + \frac{4t}{(1+t^2)}y = 0 \rightarrow \text{get homogeneous part}$$

$$y' = -\frac{4t}{(1+t^2)}y$$

$$dy/y = -\frac{4t}{(1+t^2)}dt$$

$$\ln y = -2 \ln(1+t^2)$$

$$y_n = \frac{1}{(1+t^2)^2}$$

$$y = Cy_n$$

$$y' = ay_n + f$$

$$y = u(x)y_n$$

$$(uy_n)' = auy_n + f$$

$$u' = \frac{f}{y_n}$$

$$u' = \frac{(1+t^2)^2}{(1+t^2)^2}$$

$$u' = 1$$

$$u = t + C$$

## 11.15 Lecture: Exact Equations

• Review:

↳  $x(t), y(t), w(x, y)$

$$\mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

$$t \rightarrow (x(t), y(t)) \rightarrow w(x(t), y(t))$$

$$\frac{\partial}{\partial t} w(x(t), y(t)) = \frac{\partial w}{\partial t}$$

$$\frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t}$$

$$\stackrel{x'}{\overbrace{x}} \quad \stackrel{y'}{\overbrace{y}}$$

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy$$

$$\cdot y' = \varphi(x, y)$$

$$\frac{dy}{dx} = \varphi(x, y)$$

$$dy - \varphi(x, y) dx = 0$$

$$P(x, y) dx + Q(x, y) dy = 0$$

$$\cdot \text{Ex: } (x^2 + y^2) dx + 2xy dy = 0, \text{ let } w(x, y) = \frac{1}{3}x^3 + xy^2$$

↳ The total derivative of  $w(x, y)$  is the equation

$$dw = (x^2 + y^2) dx + 2xy dy = 0$$

$$w = C$$

$$w(x, y) = w(x_0, y_0)$$

$$w(x_1, y_1) - w(x_0, y_0) = \frac{\partial w}{\partial x} |_{(x-x_0)}$$

•  $P(x, y) dx + Q(x, y) dy$  is exact iff there exists  $w(x, y)$  such that  $\frac{\partial w}{\partial x} = P(x, y)$

$$\text{and } \frac{\partial w}{\partial y} = Q(x, y)$$

• Theorem: If  $P, Q, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$  are continuous and  $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ , then the differential equation is exact

$$\hookrightarrow (x^2 + y^2) dx + 2xy dy = 0$$

$$\frac{\partial P}{\partial y} = 2y, \frac{\partial Q}{\partial x} = 2y$$

• How to find  $w$  when diff. eq. is exact

$$\hookrightarrow P(x, y) = \frac{\partial w}{\partial x}$$

$$\int \frac{\partial w}{\partial x} dx = \int P(x, y) dx$$

↳ Let  $H$  be the antiderivative of  $P(x, y)$

$$\hookrightarrow \text{let } D = w(x, y) - H(x, y)$$

$$\frac{\partial D}{\partial x} = 0 = \frac{\partial w}{\partial x} - \frac{\partial H}{\partial x} = P(x, y) - P(x, y)$$

$$D(x_1, y_0) - D(x_0, y_0) = \frac{\partial D}{\partial y} |_{(x_1-x_0)} = 0$$

$$\hookrightarrow D(x, y) = \sigma(y) = w - H$$

$$w = \sigma(y) + H(x, y)$$

$$\frac{\partial w}{\partial y} = Q(x, y) = \frac{\partial}{\partial y} (\sigma(y) + H(x, y)) = \sigma'(y) + \frac{\partial H}{\partial y}(x, y) = Q(x, y)$$

$$\sigma'(y) = Q(x, y) - \frac{\partial H}{\partial y}(x, y)$$

$$\sigma(y) = \int (Q - \frac{\partial H}{\partial y}) dy + C$$

$$\frac{\partial w}{\partial x} = x^2 + y^2$$

$$w = \frac{1}{3}x^3 + xy^2 + \sigma(y)$$

$$\frac{\partial w}{\partial y} = 2xy + \sigma'(y) = 2xy = Q(x, y)$$

$$\sigma'(y) = 0$$

$$\sigma(y) = C$$

$$w(x, y) = \frac{1}{3}x^3 + xy^2 + C$$

$$\boxed{\frac{1}{3}x^3 + xy^2 = C}$$

# 1/17 Lecture: Integrating Factors of Exact Equations

$$\cdot P(x,y)dx + Q(x,y)dy = 0$$

• Theorem: there exists a  $H(x,y)$  such that  $H(x,y)P(x,y) + H(x,y)Q(x,y) = 0$  is exact

↳  $P, Q$  defined on  $\mathbb{R}^2$

↳ How to find  $H(x,y)$ ? There's no general way

↳  $H$  is an integrating factor

↳ If we can find  $H(x)$

$$\hookrightarrow HPdx + HQdy = 0$$

$$\frac{\partial}{\partial y}(HP) = -\frac{\partial}{\partial x}(HQ)$$

$$P \frac{\partial}{\partial y}H(x) + \frac{\partial}{\partial y}PH = \frac{\partial}{\partial x}HQ + H \frac{\partial Q}{\partial x}$$

$$0 + H\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = H'Q$$

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{H'(x)}{H(x)}$$

↳ If  $h(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$  is a function of  $x$  only. Then  $e^{\int h(x)dx}$  is an IF

↳  $g(y) = \frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$  is a function of  $y$  only. Then  $e^{-\int g(y)dy}$  is an IF

$$\hookrightarrow -xydy + f(x)g(y)xydx = 0$$

↳ check if separable

$$\hookrightarrow \text{Ex: } (xy-2)dx + (x^2-xy)dy = 0$$

↳ Step 1: check if separable  $\rightarrow$  not

↳ Step 2:  $H(x)$  or  $H(y)$ ?

$$\hookrightarrow x - 2x + y = y - x \rightarrow \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$$

$$\frac{1}{x^2-xy} \cdot y - x = -\frac{1}{x} \rightarrow \text{function of } x$$

$$H(x) = e^{\int -\frac{1}{x} dx} = \frac{1}{x}$$

$$H(x) = \frac{1}{x} \text{ or } -\frac{1}{x}$$

$$(y - \frac{2}{x})dx + (x - y)dy = 0$$

$$w = \int y - \frac{2}{x} dx = yx - 2\ln|x| + \varphi(y)$$

$$\frac{\partial w}{\partial y} = x + \varphi'(y) = x - y$$

$$\varphi'(y) = -y \rightarrow \varphi(y) = -\frac{1}{2}y^2 + C$$

$$yx - 2\ln|x| - \frac{1}{2}y^2 = C$$

## 1122 Lecture: Homogeneous Equations

- $P(x,y) \rightarrow P(tx,ty) = t^n P(x,y)$

↳ Homogeneous function of degree  $n$ :

↳  $x^2 + y^2 \rightarrow t^2 x^2 + t^2 y^2 \rightarrow t^2 (xy)$ , deg. 2

↳  $e^{y/x} \rightarrow e^{t y/t x} \rightarrow e^{y/x} \rightarrow \text{deg. 0}$

-  $P(x,y) dx + Q(x,y) dy \rightarrow P$  and  $Q$  are homogeneous of same degree

• How to solve:

↳ Let  $v(x) = \frac{y(x)}{x} \rightarrow y = vx$

↳ Ex)  $(x^2 + y^2) dx + xy dy = 0$

$$(x^2 + v^2 x^2) dx + vx^2 d(vx) = 0$$

$$d(vx) = dv(x) + vdx$$

$$(x^2 + v^2 x^2) dx + vx^2 (x dv + v dx) = 0$$

$$(x^2 + v^2 x^2 + v^2 x^2) dx + vx^3 dv = 0$$

$$x^2 (1 + 2v^2) dx + x^2 (xv) dv = 0$$

$$(1 + 2v^2) dx + vx dv = 0 \rightarrow \text{guaranteed to be separable}$$

$$\frac{dx}{x} + \frac{v}{1+2v^2} dv = 0$$

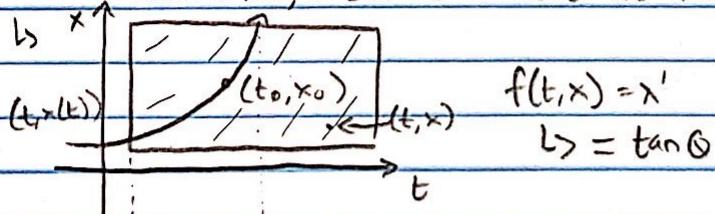
• Existence/Uniqueness

↳ Theorem: Consider IVP,  $x' = f(t,x)$ ,  $x(t_0) = x_0$

↳ If  $f(t,x)$  is continuous on  $R$  (a rectangle), then  $(t_0, x_0) \in R$

↳ Then, IVP has a solution defined on an interval contains  $t_0$ .

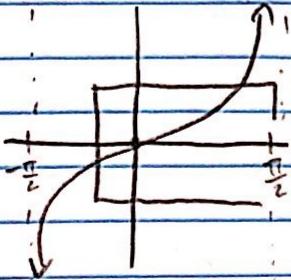
Furthermore, it can be extended at least until  $(t, x(t))$  leaves  $R$



existence this interval

↳ Ex)  $x' = 1 + x^2$ ,  $x(0) = 0$

$$x = \tan t$$



## 1/24 Lecture: Uniqueness

• E/U

$$\hookrightarrow y' = y^{1/3}, y(0) = 0$$

$y_1 = 0$  is a solution

$$y_2 = \begin{cases} 0, & t \leq 0 \\ (\frac{2}{3}t)^{3/2}, & t > 0 \end{cases}$$

$\hookrightarrow$  Theorem:  $f(t, x)$  and  $\frac{\partial f}{\partial x}(x, t)$  are continuous on a rectangle  $R$

(closed) containing  $(t_0, x_0)$  in the  $x$ -plane, then the IVP  $x' = f(t, x)$ ,  
 $x(t_0) = x_0$  has a unique solution inside  $R$

$\hookrightarrow$  Ex)  $x' = tx$   $f(t, x) = tx$ ,  $\frac{\partial f}{\partial x} = t \rightarrow$  continuous on  $\mathbb{R}^2$

$\hookrightarrow$  There is a unique solution

$\hookrightarrow$  Ex)  $y' = \sin(y-x) + 1$ ,  $y(0) = 1$

Prove that  $y(x) > x$  for all  $x \in \mathbb{R}$

$y = x$  is a solution

$\frac{\partial f}{\partial y}$  and  $f$  are continuous on  $\mathbb{R}^2$

$\hookrightarrow$  The solution through whatever I.C. is unique  $\rightarrow$  solution curves do not intersect with each other

$\hookrightarrow$  let  $y(x)$  be the solution through  $(0, 1)$ , then  $(x, y(x))$  does not intersect  $(x_0, x)$

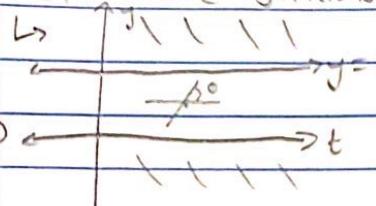
## Lecture 1/27: Autonomous Equations

•  $y' = f(y) \rightarrow$  one variable

↳ Ex)  $y' = y(1-y) \rightarrow y(t) = 0, 1$

↳ If  $y_0$  is a zero of  $f(y)$ , then  $y(t) = y_0$  is a solution  $\rightarrow$  equilibrium

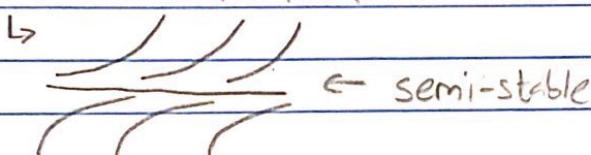
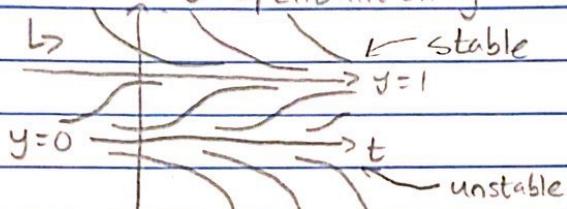
• Sketch the solutions of the autonomous diff. eq.

↳ 

$$\tan \theta = y' = f(y)$$

$y=0$  

↳ Q) only dependant on  $y$



↳ determine sign of  $f(y)$  in each interval, then sketch the graph

↳ Given I.C.  $y(0) = 1/2$

↳  $y(t)$  is a solution, what is  $\lim_{t \rightarrow \infty} y(t) = ?$ ,  $\lim_{t \rightarrow -\infty} y(t) = ?$

↳ 1st, sketch the solutions, then follow the curve

$$\lim_{t \rightarrow \infty} = 1, \lim_{t \rightarrow -\infty} = 0$$

↳ Show that  $0 < y(t) < 1$  for all  $t \in (-\infty, \infty)$

↳ there are solutions at 0 and 1, therefore  $y(t)$  cannot intersect with them

↳ proof:  $U/E \rightarrow f(y) = y(1-y)$  and  $\frac{dy}{dt} = f'(y) = 1-2y$  is continuous,  
so the solutions to the IVP are unique

because  $y(0)=0$  and  $y(t)=1$  are 2 equilibrium solutions,  $(0, 1/2)$   
is a point between the 2 solutions

Therefore, the solution going through  $(0, 1/2)$  will not intersect  
with  $y(t)=0$  or  $y(t)=1$

In other words  $0 < y(t) < 1$  for all  $t \in (-\infty, \infty)$

• Review → Integrating Factor formulas provided

↳ 1st Order Diff. Eq.

↳ First, put question in normal form

↳ Integrating Factor or Variation of Parameter

↳ Ex)  $y' = \frac{1}{x}y + \sqrt{x}$ ,  $y(1) = 0$

$$y = 2x^{3/2} - 2x$$

↳ Interval of existence:  $(0, \infty)$

↳ includes both solution and original diff. eq.

↳ Homogeneous Equation:

↳ Ex)  $(y^2 + 2xy)dx - x^2dy = 0$

$$y = xv, dy = xdv + vdx$$

$$(v^2x^2 + 2x^2v)dx - x^2(xdv + vdx) = 0$$

$$(v^2 + 2v) + (-x^2v - x^2) = 0$$

$$v^2 + 2v = x^2dv + vdx$$

$$\frac{dv}{v^2 + 2v} = \frac{1}{x^2}dx$$

$$\frac{1}{v(v+1)} = \frac{A}{v} + \frac{B}{v+1}$$

$$1 = Av + A + Bv$$

$$A = 1, B = -1$$

$$\frac{1}{v(v+1)} = \frac{1}{v} - \frac{1}{v+1}$$

$$\frac{dx}{x} = \left(\frac{1}{v} - \frac{1}{v+1}\right)dv$$

$$\ln|x| = \ln|v| - \ln|v+1| + C$$

$$\ln|x| = \ln\left|\frac{v}{v+1}\right| + C$$

$$x = \frac{v}{v+1} + C$$

$$x = \frac{y/x}{y/x+1} + C$$

$$\frac{y/x}{y/x+1} = C$$

$$\frac{x^2 + xy}{y} = C$$

$$x^2 + xy = Cy$$

$$\boxed{x(x+y) = Cy}$$

↳ Exact Equations:

$$\rightarrow Pdx + Qdy = 0$$

↳ Suppose there is an I.F. of either  $H(x)$  or  $H(y)$  form, then solve the eq.

## 1/29 Lecture: Derivative of Solutions

• Ex)  $y' = f_1(t, y)$ ,  $y(t_0) = y_0$

$$y'(t_0) = f_1(t_0, y_0)$$

$$y''(t_0) = \frac{\partial}{\partial t} f_1(t, y)(t_0, y_0)$$

$$= \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial y} \frac{\partial y}{\partial t}$$

$$= \left( \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial y} f_1 \right)(t_0, y_0)$$

↳ Ex)  $y' = t + y$

$$y'' = 1 + 1(t+y)$$

$$y''' = 1 + t + y$$

• Higher Derivatives:

↳  $y^{(n-1)}(t) = f_{n-1}(t, y)$

$$\frac{\partial}{\partial t} y^{(n-1)}(t) = \frac{\partial f_{n-1}}{\partial t} + \frac{\partial f_{n-1}}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$y^{(n)}(t) = \frac{\partial f_{n-1}}{\partial t} + \frac{\partial f_{n-1}}{\partial y} f_1$$

↳ If we have  $t_0, y_0$ , then we know all  $y^{(n)}(t_0)$

• Recall: for a function  $f(t)$

↳  $y(t) = y(t_0) + \frac{y'(t_0)}{1!}(t-t_0) + \frac{y''(t_0)}{2!}(t-t_0)^2 + \dots + \frac{y^{(n)}(t_0)}{n!}(t-t_0)^n$

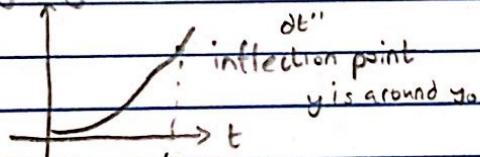
↳ Analytic Function:  $f(t) = \text{Taylor Series of } f(t)$

↳ If  $f_1$  is analytic  $\rightarrow y(t)$  is analytic

↳ Consequently, if we know  $y^{(n)}(t_0)$ , then,  $y(t) = \sum_{n=0}^{\infty} \frac{y^{(n)}(t_0)}{n!}(t-t_0)^n$

↳  $y = \# \text{ of daily infected patients}$

$$y' = y(k-y)$$



$$y'' = (k-2y)y'$$

$$y'' = 0 \text{ iff } y = \frac{1}{2}k$$

• Second Order Differential Equations

↳  $y'' = f(t, y, y')$

↳ I.C.  $y(t_0) = y_0, y'(t_0) = y_1$

↳  $y' = a(t)y + f$

↳  $y'' + p(t)y' + q(t)y = f$

↳ If  $p(t), q(t), f(t)$  are continuous on  $t \in (a, b)$ , then there exists a unique solution to the I.V.P.

↳  $y'' = -py' - qy + f(t)$

↳  $\frac{\partial F}{\partial y} = -q, \frac{\partial F}{\partial y'} = -p$

## 2/3 Lecture: Solution Spaces

$$y'' + py' + qy = f$$

$$\hookrightarrow \left( \frac{\partial}{\partial t} \frac{\partial}{\partial t} + p(t) \frac{\partial}{\partial t} + q(t) \right) y = f$$

↪ 1st: solve homogeneous part

$$\hookrightarrow \left( \frac{\partial}{\partial t} \frac{\partial}{\partial t} + p(t) \frac{\partial}{\partial t} + q(t) \right) y = 0 \rightarrow \text{find } \ker(A)$$

↪ 2nd: find particular solution  $y_p$

↪ Solution space =  $\ker(A) + y_p$

↪ Goal: prove  $\dim(\ker(A)) = 2$

↪ Def:  $u(t)$  and  $v(t)$  are functions on  $(\alpha, \beta)$ , they are linearly dependent iff  $u(t) = cv(t)$  or  $v(t) = cu(t)$  for some  $c \in \mathbb{R}$

↪ Wronskian  $W(u, v)$

$$\hookrightarrow W(u, v) = \begin{vmatrix} u(t) & u'(t) \\ v(t) & v'(t) \end{vmatrix} = uv' - u'v$$

↪ Proposition:  $u$  and  $v$  are linearly dependent  $\Leftrightarrow W(u, v) = 0$  on  $(\alpha, \beta)$

$$\hookrightarrow uv' = u'v$$

$$\frac{v'}{v} = \frac{u'}{u} \rightarrow \ln|v| = \ln|u| + C \rightarrow \ln|\frac{v}{u}| = C$$

$$\frac{v}{u} = e^C \rightarrow \frac{v}{u} = K \rightarrow v = Ku$$

↪ Suppose  $u, v$ , are solutions to the homogeneous part ( $y'' + py' + qy = 0$ ), then  $u, v$ , linearly independent / dependent  $\Leftrightarrow W(u, v) \neq 0 / = 0$  on  $(\alpha, \beta)$

↪ If  $W(u, v) = 0$  at  $t_0$ , then  $W(u, v) = 0$  on  $(\alpha, \beta)$

$$\left| \begin{matrix} u(t_0) & v(t_0) \\ u'(t_0) & v'(t_0) \end{matrix} \right| = 0 \Leftrightarrow \left( \begin{matrix} u(t_0) \\ u'(t_0) \end{matrix} \right) = C \left( \begin{matrix} v(t_0) \\ v'(t_0) \end{matrix} \right) \text{ or } \left( \begin{matrix} v(t_0) \\ v'(t_0) \end{matrix} \right) = C \left( \begin{matrix} u(t_0) \\ u'(t_0) \end{matrix} \right)$$

↪ Let  $y_1(t) = u(t)$  and  $y_2(t) = cv(t) \rightarrow y_1$  and  $y_2$  satisfy the same I.C.

↪ By existence/uniqueness  $\rightarrow y_1 = y_2$  on  $(\alpha, \beta)$

↪ Theorem: Suppose  $y_1$  and  $y_2$  are 2 linearly independent solutions to

$y'' + py' + qy = 0$ , then all solutions are linear combinations of  $y_1$  and  $y_2$

$$\hookrightarrow c_1 y_1 + c_2 y_2 = \text{span}\{y_1, y_2\}$$

↪ Because  $y_1$  and  $y_2$  are linearly independent, we can find  $c_1, c_2$  such that

$c_1 y_1 + c_2 y_2$  satisfies some initial condition  $(y_0, y_1)$

$$\hookrightarrow c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$$

$$c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_1$$

## 215 Lecture: Constant Coefficient 2nd Order Diff. Eq.

$y'' + p(x)y' + q(x)y = f \rightarrow y'' + py' + qy = f$ , where  $p/q$  are constants

↳ Step 1: Solve Homogeneous part

$$y_1, y_2 \rightarrow C_1 y_1 + C_2 y_2$$

↳ Step 2: Get a particular solution

↳ Method 1: Undetermined Coefficient

↳ Fast/easy to calculate, requirement on  $f$

↳ Method 2: Variation of Parameter

↳ No requirements, relatively hard to compute

$$\text{↳ Ex: } x^2 + xp + q = f$$

$$\text{↳ } x^2 + xp + q = 0 \rightarrow \gamma_1, \gamma_2$$

$$(\frac{\partial}{\partial t} \frac{\partial}{\partial t} + p \frac{\partial}{\partial t} + q) = 0$$

$$(\frac{\partial}{\partial t} - \gamma_1)(\frac{\partial}{\partial t} - \gamma_2) = 0$$

$$\ker(BA) = \ker(\frac{\partial}{\partial t} - \gamma_1) + \ker(\frac{\partial}{\partial t} - \gamma_2)$$

$$\frac{\partial}{\partial t} y - \gamma_1 y = 0, Ce^{\gamma_1 t} = y_1$$

$$\frac{\partial}{\partial t} y - \gamma_2 y = 0, Ce^{\gamma_2 t} = y_2$$

$$y = C_1 e^{\gamma_1 t} + C_2 e^{\gamma_2 t}$$

$$\hookrightarrow \gamma_1, \gamma_2 \in \mathbb{R} \rightarrow e^{\gamma_1 t}, e^{\gamma_2 t}$$

$$\hookrightarrow \gamma_1 = \alpha + \beta i, \gamma_2 = \alpha - \beta i \rightarrow e^{\alpha t} \cos \beta t, e^{\alpha t} \sin \beta t$$

$$\hookrightarrow \gamma_1 = \gamma_2 = \alpha \rightarrow e^{\alpha t}, te^{\alpha t}$$

$$\text{↳ Ex: } 8y'' + 2y' - y = 0, y(-1) = 1, y'(-1) = 0$$

$$\gamma_1 = \frac{1}{4}, \gamma_2 = -\frac{1}{2}$$

$$e^{1/4t} = y_1, y_2 = e^{-1/2t}$$

$$y = C_1 e^{1/4t} + C_2 e^{-1/2t} \rightarrow 1 = C_1 e^{-1/4} + C_2 e^{1/2}$$

$$y' = \frac{1}{4}C_1 e^{1/4t} - \frac{1}{2}C_2 e^{-1/2t} \rightarrow 0 = \frac{1}{4}C_1 e^{-1/4} - \frac{1}{2}C_2 e^{1/2}$$

$$C_1 = \frac{2}{3}e^{1/4}, C_2 = \frac{1}{3}e^{-1/2}$$

## 2/7 Lecture: Undetermined Coefficient

•  $y'' + py' + qy = f$

↳ Restrictions on  $f$ :

↳ 1)  $f = Ce^{at}$

2)  $f = A\cos rt + B\sin rt$

3)  $f = \text{polynomial}$

• Ex)  $y'' - y' - 2y = 2e^{-2t}$

↳ Step 1:  $\lambda^2 - \lambda - 2 = 0$

$\lambda = -1, 2$

$y_1 = e^{-t}, y_2 = e^{2t}$

↳ Step 2: find  $y_p$

↳ in form  $y = Ce^{-2t} \rightarrow C$  is the undetermined coefficient

$y_p' = -2Ce^{-2t}, y_p'' = 4Ce^{-2t}$

$$4Ce^{-2t} - (-2Ce^{-2t}) - 2Ce^{-2t} = 2e^{-2t}$$

$$4Ce^{-2t} = 2e^{-2t}$$

$$C = \frac{1}{2}$$

$$y_p = \frac{1}{2}e^{-2t}$$

↳  $y = C_1 e^{-t} + C_2 e^{2t} + \frac{1}{2} e^{-2t}$

• Ex)  $y'' - y' - 2y = e^{2t}$

↳  $y_1 = e^{-t}, y_2 = e^{2t}$

↳ Solution: Let  $y_p = Cte^{2t}$ , solve same way as above

• Ex)  $y'' - 2y' + y = e^t$

$$\lambda^2 - 2\lambda + 1 = 0$$

$\lambda = 1$ , repeated  $\rightarrow y_1 = e^t, y_2 = te^t$

↳  $y_p = t^2 e^t$

• Ex)  $y'' + 2y' - 3y = 5\sin 3t + 0\cos 3t$

$$\lambda^2 + 2\lambda - 3 = 0$$

$$(\lambda + 3)(\lambda - 1) = 0$$

$$\lambda = -3, 1$$

$y_p = Acos 3t + B\sin 3t$

$y_p' = -3A\sin 3t + 3B\cos 3t$

$y_p'' = -9A\cos 3t - 9B\sin 3t$

$$(-9A\cos 3t - 9B\sin 3t) + 2(-3A\sin 3t + 3B\cos 3t) - 3(A\cos 3t + B\sin 3t)$$

$$(-12A + 6B)\cos 3t + (-6A - 12B)\sin 3t = 5\sin 3t + 0\cos 3t$$

$$A = -1/6, B = -1/3$$

$y_p = -\frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t$

$$\begin{aligned} & iy'' + 2iy' - 3iy = 5i \sin 3t \\ & + x'' + 2x' - 3x = 5 \cos 3t \\ & = (x+iy)'' + 2(x+iy)' - 3(x+iy) = 5(\cos 3t + i \sin 3t) \end{aligned}$$

$\hookrightarrow = e^{3it}$

$$z''(t) + 2z'(t) - 3z(t) = 5e^{3it} \rightarrow \text{not a root}$$

$$z_p = Ce^{3it}$$

$$z'_p = 3iCe^{3it}$$

$$z''_p = -9Ce^{3it}$$

$$(-9C + 6iC - 3C)e^{3it} = 5e^{3it}$$

$$(-12C + 6iC) = 5$$

$$C = \frac{5}{-12+6i} = -\frac{1}{6}(2+i)$$

$$z_p = -\frac{1}{6}(2+i)e^{3it} = x_p + y_p$$

$$y_p = \operatorname{Im}(z_p)$$

$$= -\frac{1}{6}(2+i)(\cos 3t + i \sin 3t)$$

$$= -\frac{1}{6}((2\cos 3t - \sin 3t) + (\cos 3t + 2\sin 3t)i)$$

## 2/10 Lecture: Variation of Parameter

$$\cdot y'' + py' + qy = f, \rightarrow y_p$$

$$y'' + py' + qy = f_1 \rightarrow y_{p1}$$

$$y'' + py' + qy = f_2 \rightarrow y_{p2}$$

$$\hookrightarrow \text{Ex) } y'' - y' - 2y = e^{-2t} - 3e^{-t}$$

$$f_1: y_{p1} = \frac{1}{4}e^{-2t} \quad [y_p = y_{p1} + y_{p2} = \frac{1}{4}e^{-2t} + te^{-t}]$$

$$f_2: y_{p2} = te^{-t}$$

$$\cdot y'' + p(t)y' + q(t)y = f(t)$$

$\hookrightarrow$  Suppose we have 2 linearly independent solutions to  $y'' + p(t)y' + q(t)y = 0$ ,

$$y_1(t), y_2(t)$$

$$\hookrightarrow y_h = c_1 y_1(t) + c_2 y_2(t)$$

$$\hookrightarrow y_p = v_1(t)y_1(t) + v_2(t)y_2(t)$$

$$\hookrightarrow v_1(t) = \int \frac{-y_2(t)f(t)}{y_1y_2' - y_1'y_2} = \int \frac{-y_2f}{W(t)}$$

$$v_2(t) = \int \frac{y_1(t)f(t)}{y_1y_2' - y_1'y_2} = \int \frac{y_1f}{W(t)}$$

$$\hookrightarrow \text{Ex) } y'' + y = \tan t \rightarrow \text{1st consider undetermined coefficient}$$

$\hookrightarrow$  Step 1.  $y_h$

$$\lambda^2 + 1 = 0 \rightarrow \lambda = \pm i$$

$$y_1 = \cos t, y_2 = \sin t$$

$\hookrightarrow$  Step 2:  $W(t)$

$$\cos t(\cos t) + \sin t(\sin t)$$

$$= 1$$

$\hookrightarrow$  Step 3:  $v(t)$

$$v_1 = \int -\sin t (\tan t)$$

$$= \int -\frac{\sin^2 t}{\cos} = \int \frac{\cos^2 t - 1}{\cos}$$

$$= \int \cos t dt - \int \frac{1}{\cos} dt$$

$$= \sin t + \ln |\sec t - \tan t|$$

$$\hookrightarrow \text{Ex) } t^2 y'' + 3t y' - 3y = \frac{1}{t}$$

$$y_1 = t, y_2 = t^{-3}$$

$$W(t) = -3t^{-2} - t^{-2} = -4t^{-2}$$

$$v_1 = \int \frac{-t^{-2}(\frac{1}{t})}{-4t^{-2}} = \frac{1}{4} \int \frac{1}{t^3} dt$$

$$= -\frac{1}{8}t^{-2}$$

$$v_2 = \int \frac{t(\frac{1}{t^3})}{-4t^{-2}} = -\frac{1}{4} \int t dt$$

$$= -\frac{1}{8}t^2$$

$$y_p = -\frac{1}{8}t^{-1} - \frac{1}{8}t^{-1} = \boxed{-\frac{1}{8t}}$$

## 2/12 Lecture: Linear Systems

- $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, A(t) = \begin{bmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{bmatrix}, f(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$

$\hookrightarrow \vec{x}'(t) = A(t) \cdot \vec{x}(t) + f(t)$

$\hookrightarrow$  Theorem: Solution space for  $\vec{x}' = A\vec{x}$  is  $c_1\vec{x}_1(t) + \dots + c_n\vec{x}_n(t)$

$\hookrightarrow \vec{x}_1(t), \dots, \vec{x}_n(t)$  are linearly independent as  $n \times 1$  vectors

•  $2 \times 2$  systems

$\hookrightarrow A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} ax_1(t) \\ bx_2(t) \end{bmatrix}$$

$$x_1(t) = e^{at}, x_2(t) = e^{bt}$$

$$\boxed{\vec{x}(t) = \begin{bmatrix} e^{at} \\ e^{bt} \end{bmatrix}}$$

• Eigenvalue/Eigenvector

$\hookrightarrow A\vec{v} = \lambda\vec{v}, \vec{v} \neq 0$

$\hookrightarrow$  Suppose  $e^{\lambda t}\vec{v} = \vec{x}(t)$

$$\frac{d}{dt}(e^{\lambda t}\vec{v}) = A(e^{\lambda t}\vec{v}) = e^{\lambda t}A\vec{v} = \lambda e^{\lambda t}\vec{v}$$

$$\frac{d}{dt}e^{\lambda t} \cdot \vec{v} + e^{\lambda t}(\frac{d}{dt}\vec{v}) \leftarrow 0$$

$$\lambda e^{\lambda t}\vec{v} = \lambda e^{\lambda t}\vec{v} \checkmark$$

• Goal: find 2 linearly independent solutions to  $\vec{x}' = A\vec{x}$

$\hookrightarrow$  Case 1:  $A$  has 2 real eigenvalues,  $\lambda_1 \neq \lambda_2$

$$\vec{x}^1 = e^{\lambda_1 t}\vec{v}_1, \vec{x}^2 = e^{\lambda_2 t}\vec{v}_2$$

$$\vec{x}^1 = \begin{bmatrix} -5 & 1 \\ -2 & -2 \end{bmatrix} \vec{v}$$

$$A\vec{v} = \lambda\vec{v} \rightarrow (A - \lambda I)\vec{v} \rightarrow \ker(A - \lambda I) \neq 0 \rightarrow \det(A - \lambda I) = 0$$

## 2/14 Lecture: 2nd Order Linear Systems

- Case 2:  $\lambda = \alpha + \beta i, \bar{\lambda} = \alpha - \beta i$

$$\hookrightarrow (A - \lambda I) \vec{v} = 0$$

$$e^{\lambda t} \vec{v} = e^{(\alpha+i\beta)t} \vec{v} = \vec{z}'(t)$$

$$\vec{z}'(t) = e^{\bar{\lambda}t} (\vec{v}) = (e^{\bar{\lambda}t} \vec{v})$$

$$\vec{z} = c_1 \vec{z}' + c_2 \vec{z}''$$

$\vec{z}'(t) = \vec{u}(t) + \vec{w}(t)i$  - real/imaginary parts are linearly independent

$$\therefore \vec{y} = c_1 \vec{u}(t) + c_2 \vec{w}(t)$$

$$\hookrightarrow \text{Ex) } \vec{y}' = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \vec{y}$$

$$\hookrightarrow (\lambda - \alpha)(\lambda - 2) + 2 = 0$$

$$\lambda^2 - 2\lambda + 2 = 0$$

$$\lambda = 1 \pm i$$

$$\lambda = 1+i, \bar{\lambda} = 1-i$$

$$(A - \lambda I) \vec{v} = 0$$

$$\begin{bmatrix} 1+i & -1 \\ -2 & -1+i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, a, b \in \mathbb{C}$$

$$(1+i)a - b = 0$$

$$\text{let } a = 1, b = 1+i$$

$$\vec{v} = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

$$e^{(1+i)t} \begin{bmatrix} 1 \\ 1+i \end{bmatrix} = \vec{y}'(t)$$

$$e^{it} \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

$$e^{t(\cos t + i \sin t)} \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

$$\begin{bmatrix} e^{t(\cos t + i \sin t)} \\ e^{t(\cos t + i \sin t)} i \end{bmatrix}$$

$$\boxed{\vec{y} = c_1 \begin{bmatrix} e^{t(\cos t + i \sin t)} \\ e^{t(\cos t + i \sin t)} i \end{bmatrix} + c_2 \begin{bmatrix} e^{t \sin t} \\ e^{t \sin t} i \end{bmatrix}}$$

- Case 3:  $\lambda_1 = \lambda_2$

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}, \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & \alpha \\ 0 & \lambda \end{bmatrix}$$

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \vec{y}' = A\vec{y}$$

$$\hookrightarrow \begin{bmatrix} x'_1(t) \\ x'_2(t) \end{bmatrix} = \begin{bmatrix} \gamma x'_1(t) \\ \gamma x'_2(t) \end{bmatrix}$$

$$x_1 = c_1 e^{\gamma t}, x_2 = c_2 e^{\gamma t}$$

$$c_1 e^{\gamma t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, c_2 e^{\gamma t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\hookrightarrow A = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix}$$

$\hookrightarrow$  Theorem: suppose  $\vec{v}_1$  is the only eigenvector for  $A$ ,  $\vec{w} (\neq 0)$  is a vector linearly independent to  $\vec{v}_1$

$$(A - \gamma I)^2 = 0$$

$$(A - \gamma I)(A - \gamma I)\vec{w} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(A - \gamma I)\vec{w} \in \ker(A - \gamma I), \dim(\ker(A - \gamma I)) = 1$$

$$(A - \gamma I)\vec{w} = k (\neq 0) \vec{v}_1$$

$$v_2 = \frac{1}{k} \cdot \vec{w}, (A - \gamma I)v_2 = \vec{v}_1$$

Fundamental set  $\left\{ e^{\gamma t} \vec{v}_1, e^{\gamma t} (\vec{v}_2 + t \vec{v}_1) \right\}$

$$\hookrightarrow (E) \quad A = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$$

$\gamma = -2$  repeated

$$A - \gamma I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 9 \\ b \end{bmatrix} = 0$$

$$\text{eigenvector } \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{v}_1$$

$$\text{let } \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{v}_1(t)$$

$$e^{-2t} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \vec{x}_2(t)$$

## 2/21 Lecture: U/E of Linear Systems

- $\vec{x}'(t) = A(t)\vec{x}(t) + \vec{f}(t)$

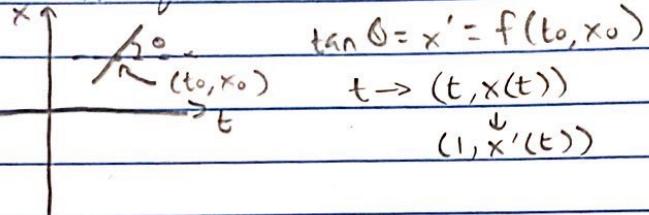
↳ When  $\vec{f}(t) = \vec{0}$ , this is homogeneous

• Theorem: Suppose  $A(t), \vec{f}(t)$  is continuous on  $(\alpha, \beta)$

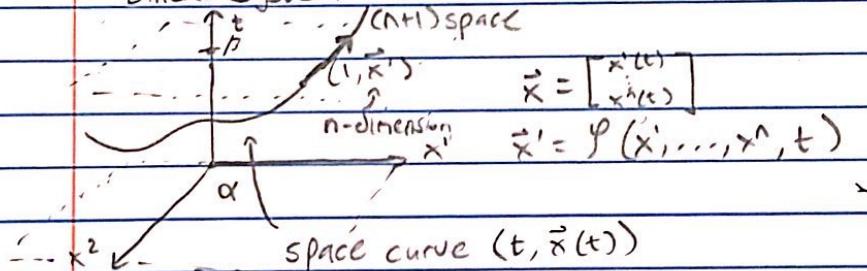
↳ Then for an IVP:  $\vec{x}' = A\vec{x}(t) + \vec{f}(t)$ ,  $\vec{x}(t_0) = \vec{v}$ , it has a "unique" solution on  $(\alpha, \beta)$

• Directional Field:

↳ 1st order diff. eq.



↳ Linear System



• If  $\vec{x}_1(t)$  and  $\vec{x}_2(t)$  are solutions to  $\vec{x}' = A(t)\vec{x} + \vec{f}(t)$ , then  $\vec{x}_1 - \vec{x}_2$  is also a solution

↳ If  $\vec{x}_1(t)$  and  $\vec{x}_2(t)$  are solutions to  $\vec{x}' = A(t)\vec{x}(t)$ , then  $c_1\vec{x}_1 + c_2\vec{x}_2$  is still a solution

↳ Solution space for  $\vec{x}' = A(t)\vec{x} + \vec{f}(t)$ :  $\ker(\frac{\partial}{\partial t} - A(t)) + \vec{y}_p$

solutionspace for  $\vec{x}' = A\vec{x}$

$$\hookrightarrow \frac{\partial}{\partial t}(\vec{x}(t)) = A\vec{x} \rightarrow (\frac{\partial}{\partial t} - A(t))\vec{x} = \vec{0}$$

↳ If  $\vec{x}_1(t), \dots, \vec{x}_n(t)$  are solutions to  $\vec{x}' = A(t)\vec{x}$ , then  $\vec{x}_1, \dots, \vec{x}_n$  are linearly independent on  $(\alpha, \beta) \iff \vec{x}_1(t_0), \dots, \vec{x}_n(t_0)$  are linearly independent

↳ Study space  $\ker(\frac{\partial}{\partial t} - A(t))$ :

$$\text{IUP}_i: \vec{x}_i = A\vec{x}, \vec{x}_i(t_0) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\text{IUP}_k: \vec{x}_k = A\vec{x}, \vec{x}_k(t_0) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \leftarrow k\text{th coord} = 1$$

$$\text{IUP}_n: \vec{x}_n = A\vec{x}, \vec{x}_n(t_0) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\hookrightarrow \text{for } \vec{x}(t_0) = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}; \vec{y} = k_1\vec{x}_1 + k_n\vec{x}_n = k_1\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + k_n\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

• Theorem: Suppose  $A(t), \vec{f}(t)$  is continuous on  $(\alpha, \beta)$ , then the solution space for  $\vec{x}' = A(t)\vec{x} + \vec{f}(t) = \text{span}(\vec{x}_1(t), \dots, \vec{x}_n(t)) + \vec{x}_p(t)$ , where  $\vec{x}_1, \dots, \vec{x}_n$  are solutions to  $\vec{x}' = A\vec{x}$

## 2126 Lecture: Phase Plane Portraits

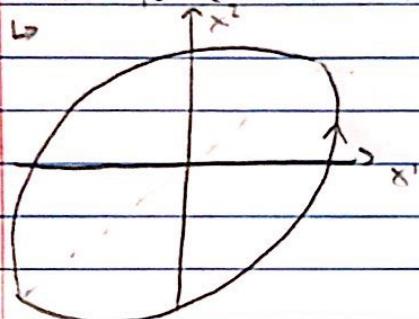
$\cdot A_{2 \times 2}$

$\cdot$  Case 2: Complex Root

$$\lambda = \alpha + \beta i, \vec{w} = \vec{v}_1 + \vec{v}_2 i$$

$$\bar{\lambda} = \alpha - \beta i, \vec{w} = \vec{v}_1 - \vec{v}_2 i$$

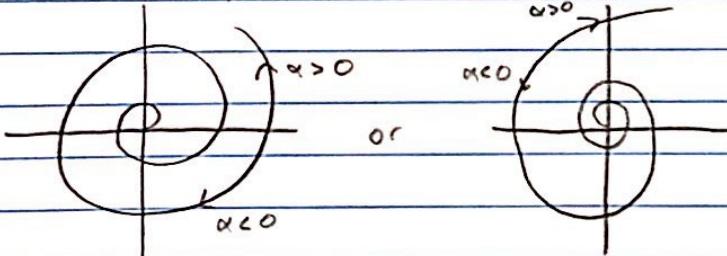
$$\hookrightarrow \vec{x}(t) = C_1 e^{\alpha t} (\cos \beta t \vec{v}_1 - \sin \beta t \vec{v}_2) + C_2 e^{\alpha t} (\sin \beta t \vec{v}_1 + \cos \beta t \vec{v}_2)$$



$\hookrightarrow$  Norm of  $\vec{x}(t)$

$$\frac{|\vec{x}(t)|}{\lim_{t \rightarrow \infty} |\vec{x}(t)|} = \begin{cases} +\infty, & \text{when } \alpha > 0 \\ 0, & \text{when } \alpha < 0 \end{cases}$$

$\hookrightarrow$



$\hookrightarrow$  Direction: pick  $\vec{v}_1$

$$t \rightarrow (t, x'(t), x''(t))$$

$$\text{direction: } (1, x''(t), x''(t))$$

$$\hookrightarrow \text{Project to Phase Plane } \frac{\partial}{\partial t} \begin{bmatrix} x'(t) \\ x''(t) \end{bmatrix} = A \begin{bmatrix} x'(t) \\ x''(t) \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \rightarrow \text{direction based on } a_{11}$$

$\hookrightarrow$  Consider  $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\vec{x}'(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{x}''(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= C_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_1 e^t \\ C_2 \end{bmatrix}$$

## 2/28 Lecture: Exp. of Matrix

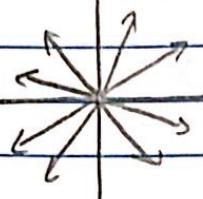
### • Repeated Root ( $\lambda \neq 0$ )

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\hookrightarrow c_1 e^{\lambda t} \vec{v}_1 + c_2 e^{\lambda t} \vec{v}_2$$



Defective Node



Star Node

### • Trace-Determinant Plane:

$$\hookrightarrow \lambda^2 + p\lambda + q = 0 \rightarrow \text{characteristic polynomial}$$

$$\hookrightarrow p = -\text{tr}(A) = a_{11} + a_{22} = -T$$

$$\hookrightarrow q = \det(A) = a_{11}a_{22} - a_{12}a_{21} = D$$

$$\hookrightarrow \begin{array}{c} \textcircled{1} \quad \textcircled{2} \quad \textcircled{3} \quad \textcircled{4} \\ \text{U} \quad \text{D} \quad \text{+} \quad \text{-} \\ \textcircled{1} \quad \textcircled{2} \quad \textcircled{3} \quad \textcircled{4} \end{array} / T^2 - 4D$$

$$\begin{array}{c} \text{---} \\ \text{I} \text{ I} \text{ I} \text{ I} \text{ I} \text{ I} \\ \text{---} \end{array} \quad \text{Saddle}$$

Generic

$\hookrightarrow T=0, D>0 \rightarrow \text{Non-generic}$

$\hookrightarrow \textcircled{1}$ : Nodal Sink,  $\textcircled{2}$ : Spiral Sink,  $\textcircled{3}$ : Spiral Source,  $\textcircled{4}$ : Nodal Source

$\hookrightarrow$  Generic

$\hookrightarrow$  Repeated Root when  $T^2 - 4D = 0$

### • Exp of Matrix

$$\hookrightarrow 1st: \mathbf{x}' = A\mathbf{x}, \mathbf{x}(t) = e^{At}$$

$$n\text{-dim } \vec{x}' = A\vec{x}, \vec{x} = e^{At}\vec{v}$$

$$\hookrightarrow e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots + \frac{t^n}{n!}$$

$$\hookrightarrow \text{Def: } e^A = I^n + A + \frac{A \cdot A}{2!} + \dots + \frac{A^n}{n!}$$

$\hookrightarrow$  Well-defined:  $(e^{At})_{ij}$  should be a series, and the sum of the series must converge

$$\hookrightarrow \text{Def: } \|A\| \text{ (Norm of matrix)} = \max \{|\lambda_{ij}| \} \rightarrow \left\| \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \right\| = |a_{11}| + |a_{12}| + \dots + |a_{1n}| \leq M$$

$$\hookrightarrow |a_{ij}| \leq M \quad (1 \leq i, j \leq n), \quad |(A \cdot A)_{ij}| = \left[ \begin{array}{ccc} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{array} \right] \left[ \begin{array}{ccc} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{array} \right] = |a_{11}a_{1j} + \dots + a_{nj}a_{nj}| \leq |a_{11}|a_{1j} + \dots + |a_{nj}|a_{nj} \leq n \cdot M^2$$

$$|(A^k)_{ij}| \leq (n \cdot M)^{k-1} \cdot M = \frac{n \cdot M^k}{k!}$$

$$|(e^A)_{ij}| \leq 1 + \frac{1}{1!}M + \dots + \frac{1}{k!} \frac{(n \cdot M)^k}{k!} \rightarrow \text{converges by Ratio Test}$$

$$\hookrightarrow e^{tA} = I + tA + \frac{(tA)^2}{2!} + \dots + \frac{(tA)^n}{n!}, \quad \frac{d}{dt}(e^{tA}) = \frac{d}{dt} \left( \sum_{k=0}^n \frac{(tA)^k}{k!} \right) = Ae^{tA}$$