

9/27 Lecture: Linear Equations and Systems

- Linear Algebra: Study of (systems) of linear equations \rightarrow

$\hookrightarrow a_1x_1 + \dots + a_nx_n = b$

geometry motivated

$\hookrightarrow a_1, a_2, \dots, a_n, b$ are real numbers / coefficients \rightarrow lines and planes in

$\hookrightarrow x_1, x_2, \dots, x_m$ are variables

linear equations

\hookrightarrow geometric transformations

as linear maps

• Applications:

\hookrightarrow Statistics (orthonormal basis, QR decompr., method of least squares)

\hookrightarrow studying points of intersection of lines/planes as solving systems of linear eq.s

\hookrightarrow Dynamic Systems and Linear Diff. Eq.s (eigenvalues, eigenvectors)

\hookrightarrow Scientific Computing

I. Linear Equations

- i) in 2 variables: $|ax+by=c|$, a, b, c are real #s called coefficients

x and y are variables

\cdot Solution to $|ax+by=c|$ is an assignment of variables that makes both sides equal

\hookrightarrow it is a tuple $(x_0, y_0) \in \mathbb{R}^2$ such that $ax_0+by_0=c$

\hookrightarrow e.g. $x-2y=3$, then $(1, -1)$ is a solution

\cdot Solution set of a system is the set of all possible solutions

• Geometric interpretation

\hookrightarrow if a and $b \neq 0$, then the solution set forms a line

$$\hookrightarrow \begin{cases} a=0 \\ b \neq 0 \end{cases} \rightarrow y = \frac{c}{b}$$

$$\hookrightarrow \begin{cases} a \neq 0 \\ b=0 \end{cases} \rightarrow x = \frac{c}{a}$$

$$\hookrightarrow \begin{cases} a \neq 0 \\ b \neq 0 \end{cases} \rightarrow y = -\frac{a}{b}x + \frac{c}{b}$$

\hookrightarrow y-int of $\frac{c}{b}$, m of $-\frac{a}{b}$

$\hookrightarrow \mathbb{R}^3$

- ii) in 3 variables: $|ax+by+cz=d|$

\hookrightarrow solution set of $|ax+by+cz=d|$ given by all points in the plane that satisfies it

• Geometric interpretation

\hookrightarrow if a, b , and $c \neq 0$, the solution set forms a plane in \mathbb{R}^3

- iii) General geometric interpretation

\hookrightarrow if a_1, a_2, \dots, a_n are not all 0, the solution set forms a hyperplane

in \mathbb{R}^m

Systems of linear equations

↳ System of 2 equations in 2 variables

$$ax+by=c$$

$$dx+ey=f$$

↳ solution set are all points $(x_0, y_0) \in \mathbb{R}^2$ that satisfy both eqs.

$$ax_0+by_0=c \text{ and } dx_0+ey_0=f$$

↳ geometric interpretation

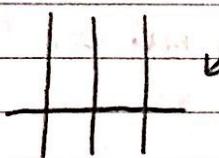
↳ point of intersection between the 2 lines defined by the linear equations

↳ 3 possibilities

↳ unique solution

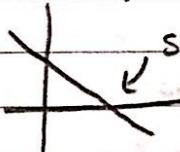


↳ no solution



↳ no intersection, parallel lines

↳ infinitely many solutions



↳ same line

9/30 Lecture: Systems of Linear Equations

- Systems of 2 equations:

$$\begin{array}{l} ax + by = c \\ dx + ey = f \end{array} \quad \textcircled{2}$$

• Solution set of $\textcircled{2}$: all points $(x_0, y_0) \in \mathbb{R}^2$ such that $ax_0 + by_0 = c$ AND $dx_0 + ey_0 = f$

↳ Geometric interpretation: the solution set consists of all points of intersection of the 2 lines defined by the 2 equations

↳ 3 situations: unique solution, no solution, infinitely many solutions

$$\begin{array}{l} x - y = 5 \\ 2x + y = 3 \end{array} \quad \begin{array}{l} 3x + 6y = 4 \\ x + 2y = 1 \end{array} \quad \begin{array}{l} -x + 2y = -3 \\ \frac{1}{2}x - y = \frac{3}{2} \end{array}$$

In general: system w/ n equations in m variables

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1m}x_m = b_1$$

:

:

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nm}x_m = b_n$$

Solution set: all points $(x_1, \dots, x_m) \in \mathbb{R}^m$ that satisfy all equations

- Geometric interpretation ($m=3$)

↳ solution set defined by all points of intersection of the planes defined by the n equations

- Gauss-Jordan Elimination

↳ Goal: to perform operations on the system such that we don't change the solution set and that we can easily read the solutions

↳ Elementary operations on systems of 1 eq.

↳ Multiplying/dividing both sides by the same #

↳ Adding/subtracting the same # from both sides

↳ Elementary operations on systems of multiple eqs

↳ Multiplying/dividing any eq. by a non-zero number

↳ Adding/subtracting a # from both sides of an eq.

↳ Adding/subtracting a nonzero multiple of an eq. to another eq.

↳ The points that satisfy eq. II are such that we are adding the same # to both sides of eq. I

↳ Interchanging any 2 eqs

• Matrix Notation

↳ A system of n linear eq's in m variables can always be written as follows:

$$\begin{aligned} a_{11}x_1 + \dots + a_{1m}x_m &= b_1 \\ \vdots \\ a_{n1}x_1 + \dots + a_{nm}x_m &= b_n \end{aligned}$$

$$\rightarrow \left[\begin{array}{ccc|c} a_{11} & a_{12} & \dots & a_{1m} & b_1 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} & b_n \end{array} \right] \text{rows}$$

"augmented matrix of coefficients
of the system"

↳ elementary row operations

↳ operations on a matrix that don't change the solution set

↳ multiply/divide any row by a nonzero number

↳ add/subtract a multiple of a row to another row

↳ swap any 2 rows

• Reduced row-echelon form (RREF)

↳ the 1st nonzero entry in every row is a 1 (pivot/leading 1)

↳ columns that contain pivots must have a 0 in every other entry

↳ if a row contains a pivot, each row above must have pivots that are to the left of it

10/2 Lecture: Gauss-Jordan Elimination

Gauss-Jordan Elimination

↳ A system of n equations and m variables can be written:

$$a_{11}x_1 + \dots + a_{1m}x_m = b_1$$

⋮

$$a_{n1}x_1 + \dots + a_{nm}x_n = b_n$$

↳ In matrix notation: $A \cdot x = b$

where $A = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \\ a_{n1} & \dots & a_{nm} \end{bmatrix}$

↳ matrix of coefficients of the system

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

↳ Augmented matrix of coefficients:

$$A|b = \left[\begin{array}{ccc|c} a_{11} & \dots & a_{1m} & b_1 \\ \vdots & & & \vdots \\ a_{n1} & \dots & a_{nm} & b_n \end{array} \right]$$

↳ Goal: perform operations on $A|b / A$ such that

↳ solution set of system is unchanged

↳ elementary row operations

↳ solutions can be easily read off

↳ reduced row-echelon form (RREF)

• Reduced row-echelon form of a matrix A :

↳ the first non-zero entry of any row is a 1 (leading 1 or pivot)

↳ if a column contains a pivot, it is the only non-zero entry in that column

↳ if a row contains a pivot, all rows above it contain pivots to the left

↳ the augmented coeff. matrix $A|b$ is in RREF if A is

↳ Ex)
$$A|b = \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \begin{aligned} x_1 + 2x_3 + x_4 &= 1 \\ x_2 + x_4 &= 0 \\ x_5 &= 2 \end{aligned}$$

- ↳ We distinguish between
 - ↳ leading variables: those that correspond to columns w/ pivots
 - ↳ free variables: those that don't correspond to columns w/ pivots
 - ↳ in example: x_1, x_2 , and x_5 are leading, x_3 and x_4 are free
- ↳ easy to read solutions because:
 - ↳ coefficient of every leading variable is 1
 - ↳ leading variables only appear in one row/equation
 - ↳ leading variables appear in order
 - ↳ in example: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2t - r + 1 \\ -r \\ t \\ r \\ 2 \end{bmatrix}$ with t and $r \in \mathbb{R}$

Gauss-Jordan elimination algorithm

1. write down the augmented matrix $A|b$
2. proceed thru the rows of $A|b$ from top to bottom
3. multiply rows w/ leading, non-zero entries by $\frac{1}{\text{entry}}$ (i.e. if its a 3, multiply by $\frac{1}{3}$)
4. transform entries in columns w/ leading 1s into 0s by adding appropriate multiples of the row w/ the leading 1 to the other rows
5. Repeat for all rows
6. Swap rows to order pivots correctly
 - ↳ if a row of the form $[0 \dots 0 | k]$ and $k \neq 0$, the system has no solution

↳ # of solutions of a system of linear equations:

$$\begin{array}{c} \hookrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\substack{x_1+2x_2=0 \\ x_3=0 \\ 0=1}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\substack{x_1+2x_2=1 \\ x_3=2 \\ x_4=-2x_2+1}} \end{array}$$

no solution Rank of a matrix = # of pivots $x_3=2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right] \xrightarrow{\substack{x_1=1 \\ x_2=2 \\ x_3=3}} \left[\begin{array}{c|c} x_1 & 1 \\ x_2 & 2 \\ x_3 & 3 \end{array} \right]$$

Unique solution

$$\left[\begin{array}{c|c} x_1 & -2t+1 \\ x_2 & t \\ x_3 & 2 \end{array} \right]$$

infinitely many solutions

10/4 Lecture Matrix Algebra

- Example: $2x_1 + 2x_2 = 0$

$$x_2 + x_3 = 4$$

$$\left[\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 4 \end{array} \right] \xrightarrow{\cdot \frac{1}{2}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 4 \end{array} \right] \xrightarrow{-\text{II}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -4 \\ 0 & 1 & 1 & 4 \end{array} \right]$$

$$x_1 - x_3 = -4$$

$$x_2 + x_3 = 4$$

$$\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right] = \left[\begin{array}{c} t-4 \\ 4-t \\ t \end{array} \right]$$

- # of solutions:

↳ In general, a system is:

↳ inconsistent - there are no solutions, RREF has a row $[0 \dots 0 | k], k \neq 0$

↳ consistent - there is ≥ 1 solution

↳ unique solution - all variables are leading

↳ infinite solutions - at least 1 free variable

- Rank of a matrix = # of leading variables in the matrix's RREF

- Example:

$$A = \left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array} \right] \quad \text{rank of } A = ?$$

- H(I)

- 7(I)

$$\left[\begin{array}{ccc} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{array} \right] \xrightarrow{-\frac{1}{3}} \left[\begin{array}{ccc} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -6 & -12 \end{array} \right] \xrightarrow{+6(\text{II})} \left[\begin{array}{ccc} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right] \quad \text{rank}(A) = 2$$

- What can we say about the solutions of a system with A as its coefficient matrix?

↳ if $b_3 = 0$, there are infinitely many solutions since x_3 is free

↳ if $b_3 \neq 0$, there are no solutions

↳ the system cannot have a unique solution

↳ rank < # of variables / rank < m, system cannot have a unique solution

- Rank and solutions of a system:

↳ Given a system $Ax = b$ where A is an $n \times m$ matrix (n eqs, m variables)

↳ $\text{rank}(A) \leq n \rightarrow$ at most 1 pivot per column / per row

↳ $\text{rank}(A) \leq n \rightarrow$

↳ If the system is inconsistent, $\text{rank}(A) < n$ (must be an all 0 row)

- ↳ If the system has a unique solution, $\text{rank}(A) = m$
- ↳ If the system has infinitely many solutions, $\text{rank}(A) < m$
- Contrapositive - "If P, then Q" \rightarrow "If not Q, then not P"
- ↳ equivalent statements

10/7 Lecture: Matrix Algebra

• Linear Transformations:

↳ ex) in geometry, transformations of the plane

↳ reflection about the y-axis

↳ scaling by a #

↳ rotation by an angle

↳ called linear transformations because

↳ $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for all vectors \vec{v}, \vec{w} in the plane

↳ $T(k\vec{v}) = kT(\vec{v})$ for all scalars k and vectors \vec{v}

↳ in general, linear transformations $R^m \rightarrow R^n$

↳ exactly those that can be represented by a matrix

• Matrix Algebra:

↳ Size of a matrix: # of rows / columns of a matrix

↳ n rows and m columns $\rightarrow n \times m$ matrix

↳ Matrices w/ only one row or one column = row vector and column vector, respectively

↳ a matrix w/ the same # of rows and columns = square matrix

↳ ex) $\begin{bmatrix} 2 & 1 & 0 \\ -3 & 4 & 1 \end{bmatrix}$ - 2×3 matrix $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ - column vector

$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ - row vector $\begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}$ - square matrix

Operations On Matrices

↳ addition of 2 matrices A and B of the same size

$$\begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nm} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & \dots & a_{1m} + b_{1m} \\ \vdots & & \vdots \\ a_{n1} + b_{n1} & \dots & a_{nm} + b_{nm} \end{bmatrix}$$

↳ subtraction - similarly defined for matrices of the same size

↳ properties of addition / subtraction:

↳ commutativity: $A + B = B + A$

↳ associativity: $(A + B) + C = A + (B + C)$

↳ ex) $\begin{bmatrix} 1 & 0 & 3 \\ 4 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 \\ -3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 9 & 7 \end{bmatrix}$

↳ multiplication of A and B

↳ only defined when # of columns of A = # of rows of B

↳ A: n × m matrix, B: m × p matrix

$$\begin{matrix} \hookrightarrow & \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix} & \begin{bmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \dots & b_{nj} & \dots & b_{np} \end{bmatrix} & = & \begin{bmatrix} c_{11} & \dots & c_{1p} \\ \vdots & & \vdots \\ c_{n1} & \dots & c_{np} \end{bmatrix} \end{matrix}$$

$c_{ij} = a_{1j}b_{1j} + a_{2j}b_{2j} + \dots + a_{nj}b_{nj}$
 $= \sum_{k=1}^n a_{kj}b_{kj}$

$$\hookrightarrow \text{ex}) \quad \begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 4 \end{bmatrix}$$

• Dot product

↳ $\vec{v} \cdot \vec{w} = v_1w_1 + \dots + v_nw_n$, \vec{v} and \vec{w} have equal # of components

↳ C_{ij} is the dot product of the ith row of A and the jth column of B

↳ Dot product in matrix notation

$$\hookrightarrow \vec{v} \cdot \vec{w} = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

↳ An n × m matrix A and column vector x w/ m entries

$$\hookrightarrow A = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}, Ax = \begin{bmatrix} r_1 \cdot x \\ \vdots \\ r_n \cdot x \end{bmatrix}$$

$$\hookrightarrow \text{ex}) \quad \begin{bmatrix} 1 & 0 & -2 \\ 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_3 \\ 3x_1 + 4x_3 \end{bmatrix}$$

↳ in general, we can write an arbitrary system of linear eqs as

$$\begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, Ax = b$$

• Properties of Matrix Multiplication

↳ not commutative in general ($AB \neq BA$)

10/9 Lecture: Linear Transformations

• Matrix Multiplication

↳ A: $n \times m$ matrix

↳ B: $m \times p$ matrix

↳ AB: $n \times p$ matrix

↳ $A = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ ↳ row vectors

$$A = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad B = [b_1 \dots b_p] \quad \text{column vectors} \rightarrow$$

$$AB = \begin{bmatrix} a_1 \cdot b_1 & a_1 \cdot b_2 & \dots & a_1 \cdot b_p \\ a_2 \cdot b_1 & \dots & & \\ \vdots & & & \\ a_n \cdot b_1 & \dots & a_n \cdot b_p \end{bmatrix}$$

↳ Entry in the i th row and j th column is $a_i \cdot b_j$

• Multiplication of a matrix with a scalar

↳ Given $k \in \mathbb{R}$ and A is an $n \times m$ matrix

$$kA = \begin{bmatrix} ka_{11} & \dots & ka_{1m} \\ \vdots & & \\ ka_{n1} & \dots & ka_{nm} \end{bmatrix}$$

• Properties of matrix multiplication

↳ Not commutative: $AB \neq BA$

↳ Associative: $A(BC) = (AB)C$ for all $n \times m$ matrix A, $m \times p$ matrix B, and $p \times q$ matrix C

↳ Distributive: $A(c + D) = AC + AD$ for all $n \times m$ matrices C and D and $m \times p$ matrix A

↳ For any scalar $k \in \mathbb{R}$, $n \times m$ matrix A, $m \times p$ matrix B

$$(kA)B = A(kB) = k(AB)$$

↳ Multiplying w/ the identity matrix

$$\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \rightarrow n \times n \text{ Matrix} \\ = I_n \rightarrow \text{notation for an identity matrix}$$

↳ For any $n \times m$ matrix A:

$$I_n A = A = A I_m$$

↳ The identity matrix is the neutral element for matrix products

• Linear Transformations

↳ given 2 sets x and y , a map (or function) $f: x \rightarrow y$ is a rule that associates every element of x to a unique element of y

- ↳ $x = \text{domain of } f, y = \text{codomain of } f$
- ↳ Def: a map $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation iff:
 - ↳ $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for all $\vec{v}, \vec{w} \in \mathbb{R}^n$
 - ↳ $T(k\vec{v}) = kT(\vec{v})$ for all scalars k and all vectors $\vec{v} \in \mathbb{R}^m$
 - ↳ Ex) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$: reflection about the y -axis
 $\vec{v} \xrightarrow{T} \vec{v}'$ $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ $T(\vec{v}) = \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix}$
 - ↳ Let $\vec{v}, \vec{w} \in \mathbb{R}^m$ be arbitrary
 $T(\vec{v} + \vec{w}) = \begin{bmatrix} -v_1 - w_1 \\ v_2 + w_2 \end{bmatrix} = \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} -w_1 \\ w_2 \end{bmatrix} = T(\vec{v}) + T(\vec{w})$
 - ↳ Let $k \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^m$ be arbitrary
 $T(k\vec{v}) = \begin{bmatrix} -kv_1 \\ kv_2 \end{bmatrix} = \begin{bmatrix} k(-v_1) \\ k(v_2) \end{bmatrix} = k \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix} = kT(\vec{v})$
 - ↳ T is a linear transformation
- ↳ Ex) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$: scaling by $1/2$
 $\vec{v} \xrightarrow{T} \vec{v}'$ $T(\vec{v}) = \begin{bmatrix} \frac{1}{2}v_1 \\ \frac{1}{2}v_2 \end{bmatrix}$ → check if linear transformation
- ↳ Ex) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$: translation along x -axis by 2
 $T(\vec{v}) = \begin{bmatrix} v_1 + 2 \\ v_2 \end{bmatrix}$ $\vec{v} \xrightarrow{T} \vec{v}'$ $T(\vec{v}') = \begin{bmatrix} v_1 + 4 \\ v_2 \end{bmatrix}$
 $T(\vec{v} + \vec{w}) = \begin{bmatrix} v_1 + w_1 + 2 \\ v_2 + w_2 \end{bmatrix} \neq T(\vec{v}) + T(\vec{w}) = \begin{bmatrix} v_1 + w_1 + 4 \\ v_2 + w_2 \end{bmatrix}$
- ↳ T is not a linear transformation
- ↳ Note: any linear transformation has to send the origin to itself
- Linear transformations and matrices
- ↳ proposition: a map $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a linear transformation iff
 - ↳ there exists an $n \times m$ matrix A such that $T(\vec{v}) = A\vec{v}$ for all vectors $\vec{v} \in \mathbb{R}^m$
 - ↳ suppose that $T(\vec{v}) = A\vec{v}$ for all \vec{v}
 - ↳ we have $T(\vec{v} + \vec{w}) = A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w} = T(\vec{v}) + T(\vec{w}) \checkmark$
 - ↳ $T(k\vec{v}) = A(k\vec{v}) = kA\vec{v} = kT(\vec{v}) \checkmark$
 - ↳ conversely, let T be a linear transformation, find an $n \times m$ matrix A so $T(\vec{v}) = A\vec{v}$

$e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}$ ← i th entry
 \vec{e}_i ← i th standard basis vector of \mathbb{R}^n

$A = \begin{bmatrix} T(e_1) & \dots & T(e_n) \end{bmatrix} \leftarrow A\vec{v} = T(\vec{v})$

10/11 Lecture: Linear Transformations

• Midterm #1 on Ch. 1 and Ch. 2 (10/21)

• $f: x \rightarrow y$ map (or function), $x = \text{domain}$, $y = \text{codomain}$
 $\hookrightarrow f(x) \subseteq y$: image of f

$\hookrightarrow x \in X$, then $f(x) = y$ is the image of x under f

• Linear transformations and matrices

\hookrightarrow Proposition: a map $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation iff there exists an $n \times m$ matrix A such that $T(\vec{v}) = A\vec{v}$ for all $\vec{v} \in \mathbb{R}^m$

\hookrightarrow Assume that $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation

\hookrightarrow let $e_i := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}$ \hookrightarrow i th entry

\hookrightarrow called the i th standard basis vector of \mathbb{R}^m

\hookrightarrow Ex) in \mathbb{R}^3 :

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

\hookrightarrow For any $\vec{v} \in \mathbb{R}^m$:

$$\hookrightarrow \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} = v_1 \vec{e}_1 + v_2 \vec{e}_2 + \dots + v_m \vec{e}_m \quad \begin{matrix} \text{scalar} \\ \hookrightarrow \end{matrix} \quad \begin{matrix} \text{vector} \\ \hookrightarrow \end{matrix}$$

$$\hookrightarrow \text{Ex)} \quad \vec{v} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

\hookrightarrow We say that \vec{v} is a 'linear combination' of the standard basis vectors

$$\hookrightarrow T(\vec{v}) = T(v_1 \vec{e}_1 + v_2 \vec{e}_2 + \dots + v_m \vec{e}_m) = T(v_1 \vec{e}_1) + \dots + T(v_m \vec{e}_m)$$

$$= v_1 T(\vec{e}_1) + \dots + v_m T(\vec{e}_m)$$

$$= v_1 \begin{bmatrix} T(\vec{e}_1)_1 \\ \vdots \\ T(\vec{e}_1)_n \end{bmatrix} + \dots + v_m \begin{bmatrix} T(\vec{e}_m)_1 \\ \vdots \\ T(\vec{e}_m)_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & 1 \\ & \ddots & \\ T(\vec{e}_1)_1 & \dots & T(\vec{e}_m)_1 \\ 1 & & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$$

$A \qquad \qquad \qquad \vec{v}$

$\hookrightarrow A$ is such that $A\vec{v} = T(\vec{v})$ for all $\vec{v} \in \mathbb{R}^m$

↳ Ex) Find the matrix corresponding to a scaling by $\frac{1}{2}$ of the plane

↳ $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T(\vec{v}) = \begin{bmatrix} \frac{1}{2}v_1 \\ \frac{1}{2}v_2 \end{bmatrix}$ ← writing the map 1st, then matrix

↳

The matrix is $\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

Transformation of \vec{e}_1

↳ Ex) What is the matrix corresponding to the rotation through an angle θ of the plane

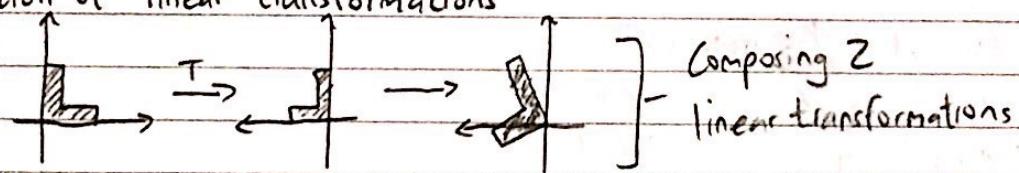
↳ $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

↳ $T(\vec{e}_2) = \vec{e}_2 - T(\vec{e}_1)$, $T(\vec{e}_1) = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$ → Matrix = $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

↳ $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is $T(\vec{v}) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \cos \theta - v_2 \sin \theta \\ v_1 \sin \theta + v_2 \cos \theta \end{bmatrix}$

• Composition of linear transformations

↳ Ex)



↳ Hint: What would happen if you 1st rotate, then reflect?

↳ Not the same end result or same?

↳ Recall: given 2 maps $f: x \rightarrow y$, $g: y \rightarrow z$ the composition of f with g is the map $gof: x \rightarrow z$, such that $gof(x) = g(f(x))$

↳ $S: \mathbb{R}^p \rightarrow \mathbb{R}^m$ and $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ are linear transformations

↳ The composition of S with T is the map given by $T \circ S: \mathbb{R}^p \rightarrow \mathbb{R}^n$

$$v \rightarrow T(S(v))$$

↳ Note: the composition of 2 linear transformations is also a linear transformation

↳ Matrix of the composition is the product of the matrices of the 2 linear transformations

↳ Let B be an $n \times m$ matrix such that $T(\vec{v}) = B\vec{v}$

A be an $m \times p$ matrix such that $S(\vec{v}) = A\vec{v}$

↳ $T \circ S(\vec{v}) = BA\vec{v}$

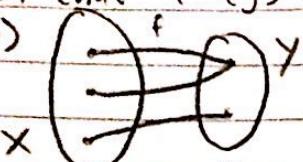
10/14 Lecture: Inverses

- The inverse of a linear transformation:

\hookrightarrow Recall: invertible functions, a function $f: X \rightarrow Y$ is said to be invertible if for every $y \in Y$ there is exactly one $x \in X$ such that $f(x) = y$

\hookrightarrow If f is invertible, the inverse of f is the function $f^{-1}: Y \rightarrow X$, such that $f^{-1}(y) = x$ iff $f(x) = y$

\hookrightarrow Ex)



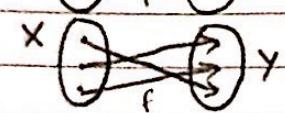
not invertible, $f(x) = y$, has no unique solution

\hookrightarrow Ex)



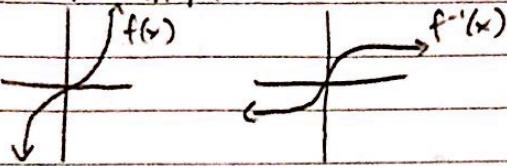
not invertible, $f(x) = y_3$ has no solution

\hookrightarrow Ex)



invertible, every $y \in Y$ has 1 $x \in X$ such that $f(x) = y$

\hookrightarrow Ex) $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$ is invertible and $f^{-1}(x) = \sqrt[3]{x}$



\hookrightarrow Properties of $f^{-1}: Y \rightarrow X$

$\hookrightarrow f^{-1}(f(x)) = x$ for all $x \in X$

$\hookrightarrow f(f^{-1}(y)) = y$ for all $y \in Y$

$\hookrightarrow f^{-1}$ is unique

\hookrightarrow Definition: a linear transformation $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is invertible if it is invertible as a function

\hookrightarrow Properties:

\hookrightarrow The inverse of a linear transformation is also a linear transformation

\hookrightarrow If $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is an invertible linear transformation, then $n=m$

\hookrightarrow A linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is invertible $\Rightarrow T(\vec{v}) = \vec{w}$ for all $\vec{v} \in \mathbb{R}^n$

\hookrightarrow The equation $T(\vec{v}) = \vec{w}$ has a unique solution $\vec{v} \in \mathbb{R}^n$ for all $\vec{w} \in \mathbb{R}^n$

\hookrightarrow The system $A\vec{v} = \vec{w}$ has a unique solution $\vec{v} \in \mathbb{R}^n$ for all $\vec{w} \in \mathbb{R}^n$

\hookrightarrow The rank of $A = \#$ of columns of $A = n$

\hookrightarrow Conversely, if $\text{rank}(A) = n$, then:

$$\hookrightarrow \text{RREF}(A) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \text{unique solution}$$

↳ Therefore the system $A\vec{v} = \vec{w}$ has exactly 1 solution

↳ $\Rightarrow A$ is invertible

↳ Proposition: A linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $T(\vec{v}) = A\vec{v}$ for all $\vec{v} \in \mathbb{R}^n$ is invertible iff $\text{rank}(A) = n$

↳ Ex) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T(\vec{v}) = \begin{bmatrix} 2v_1 \\ v_1 + 2v_2 \end{bmatrix}$, $A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

↳ Is T invertible? Does the system $A\vec{v} = \vec{w}$ have a unique solution?

$$\begin{aligned} \Leftrightarrow \left[\begin{array}{cc|c} 2 & 0 & v_1 \\ 1 & 2 & w_2 \end{array} \right] \times \frac{1}{2} &= \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2}w_1 \\ 1 & 2 & w_2 \end{array} \right] - (I) = \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2}w_1 \\ 0 & 2 & -\frac{1}{2}w_1 + w_2 \end{array} \right] \times \frac{1}{2} = \\ &\left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2}w_1 \\ 0 & 1 & -\frac{1}{4}w_1 + \frac{1}{2}w_2 \end{array} \right] \end{aligned}$$

$\text{rank}(A) = 2$, there is a unique solution and T is invertible

$$\Leftrightarrow \text{This solution is } \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}w_1 \\ -\frac{1}{4}w_1 + \frac{1}{2}w_2 \end{bmatrix}$$

$$\Leftrightarrow T^{-1}: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ is } T(\vec{v}) = \begin{bmatrix} \frac{1}{2}v_1 \\ -\frac{1}{4}v_1 + \frac{1}{2}v_2 \end{bmatrix}$$

$$\Leftrightarrow \text{Check: } T^{-1}(T(\vec{v})) = \vec{v}$$

$$T^{-1}\left(\begin{bmatrix} 2v_1 \\ v_1 + 2v_2 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2}(2v_1) \\ -\frac{1}{4}(2v_1) + \frac{1}{2}(v_1 + 2v_2) \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \checkmark$$

$$\Leftrightarrow \text{Check: } T(T^{-1}(\vec{v})) = \vec{v}$$

↳ Definition: A square $n \times n$ matrix A is invertible if $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $T(\vec{v}) = A\vec{v}$ for all $\vec{v} \in \mathbb{R}^n$ is invertible

↳ We denote by A^{-1} the matrix representing T^{-1} and we call it the inverse matrix of A

↳ Note: $AA^{-1} = I_n = A^{-1}A$, A^{-1} is unique

↳ Ex) Is $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ invertible? Solve by putting A in RREF

↳ $\text{rank}(A) = 2 \therefore A$ is invertible

$$\Leftrightarrow \left[\begin{array}{cc|c} 1 & 1 & v_1 \\ 2 & 3 & v_2 \end{array} \right] = \left[\begin{array}{cc|c} 1 & 1 & w_1 \\ 0 & 1 & w_2 \end{array} \right] \rightarrow \text{Solve, put in RREF}$$

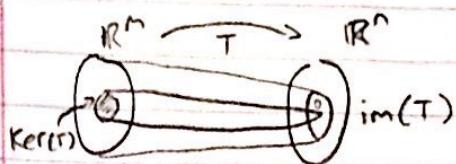
$$\Leftrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3w_1 - w_2 \\ 0 & 1 & -2w_1 + w_2 \end{array} \right] \quad A^{-1} = \left[T^{-1}(e_1) \ T^{-1}(e_2) \right]$$

$$\Leftrightarrow \text{Unique solution: } \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3w_1 - w_2 \\ -2w_1 + w_2 \end{bmatrix} \quad = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

10/16 Lecture: Images and Kernels

- Subspaces of \mathbb{R}^n and their dimensions

$\hookrightarrow T: \mathbb{R}^m \rightarrow \mathbb{R}^n$



- Image of a linear transformation

\hookrightarrow Recall: image of a function $f: X \rightarrow Y$ is the subset of Y defined as follows: $im(f) = \{y \in Y \mid \text{there exists some } x \in X \text{ such that } y = f(x)\}$

\hookrightarrow Ex) $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = e^x$: $im(f) = \{y \in \mathbb{R}, y > 0\}$

\hookrightarrow Ex) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$\hookrightarrow im(T)$ consists of all vectors of the form $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + v_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3v_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\hookrightarrow im(T)$ is all scalar multiples of the column vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

\hookrightarrow Ex) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $T\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$\hookrightarrow im(T)$ consists of all vectors of the form $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + v_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\hookrightarrow im(T)$ consists of a plane in \mathbb{R}^3 that passes through the column vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and the origin

\hookrightarrow Definition: Let v_1, \dots, v_m be vectors in \mathbb{R}^n . A vector w in \mathbb{R}^n is a linear combination of these vectors if there are scalars c_1, \dots, c_m so that $w = c_1 v_1 + \dots + c_m v_m$

\hookrightarrow A set of all possible linear combinations for the vectors is called their span $\rightarrow \text{span}(v_1, \dots, v_m) = \{c_1 v_1 + \dots + c_m v_m \mid c_1, \dots, c_m \in \mathbb{R}\}$

\hookrightarrow Proposition: $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ with matrix A , $T(\vec{v}) = A\vec{v} \forall \vec{v} \in \mathbb{R}^m$ (\forall = for all)

\hookrightarrow Then, $im(T) = \text{span}(w_1, \dots, w_m)$, where w_1, \dots, w_m are vectors corresponding to the columns of A

$$\hookrightarrow T(\vec{v}) = A\vec{v} = [w_1 \dots w_m] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = v_1 w_1 + \dots + v_m w_m$$

\hookrightarrow Properties:

\hookrightarrow The zero vector $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ in \mathbb{R}^n is in the image

\hookrightarrow The image is closed under addition

\hookrightarrow for any 2 vectors in the image, their sum is also in the image

\hookrightarrow The image is closed under scalar multiplication

↳ $\text{im}(T)$ is closed under taking linear combinations

• Kernel of a linear transformation

↳ Recall. The set of zeros of a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is the set of solutions

of $f(x) = 0$, $\text{zeros}(f) = \{x \in \mathbb{R} : f(x) = 0\}$

↳ Ex) $f(x) = \sin(x)$, $\text{zeros}(f) = \{k\pi, k \in \mathbb{Z}\}$

↳ Ex) $f(x) = x^2$, $\text{zeros}(f) = \{0\}$

↳ Notation: origin denoted by 0 in \mathbb{R}^n

↳ Definition: $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$, the kernel of T is the solution set of the linear system given by $A\vec{v} = 0$, denoted by $\ker(T)$

↳ $\ker(T) = \{\vec{v} \in \mathbb{R}^m : A\vec{v} = 0\} = \{\vec{v} \in \mathbb{R}^m : T(\vec{v}) = 0\} \subseteq \mathbb{R}^m$ (\subseteq = subset of)

↳ Properties:

↳ $0 \in \ker(T)$

↳ if $\vec{v}, \vec{w} \in \ker(T)$, then $\vec{v} + \vec{w} \in \ker(T)$

↳ if $\vec{v} \in \ker(T)$ and $k \in \mathbb{R}$, then $k\vec{v} \in \ker(T)$

↳ Ex) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ with $T(\vec{v}) = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \vec{v}$, Find $\ker(T)$

↳ Solve the system $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

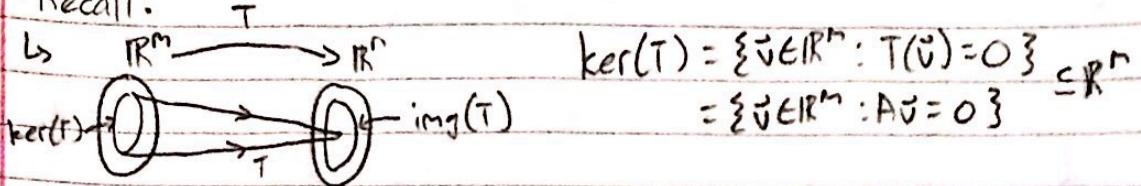
$$\text{RREF} \left(\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \right) = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right]$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\ker(T) = \text{span} \left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right)$$

10/18 Lecture: Subspaces

• Recall:



$$\text{img}(T) = \{\vec{w} \in \mathbb{R}^n : \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \in \mathbb{R}^m\} \subseteq \mathbb{R}^n$$

$$= \{\vec{w} \in \mathbb{R}^n : \vec{w} = A\vec{v} \text{ for some } \vec{v} \in \mathbb{R}^m\}$$

• Definition: Let A be an $n \times m$ matrix and $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ with $T(\vec{v}) = A\vec{v}$

↳ The kernel of A is the kernel of T , denoted $\text{ker}(A)$

↳ The image of A is the image of T , denoted $\text{img}(A)$

↳ Ex) If $\text{ker}(A) = \{\vec{0}\}$, what can we say about A

↳ Kernel is the solution of $A\vec{v} = \vec{0}$, A must have exactly 1 solution, therefore $\text{rank}(A) = \# \text{ of columns}$

• Proposition: For any $n \times m$ matrix A , $\text{ker}(A) = \{\vec{0}\} \iff \text{rank}(A) = m$

↳ Ex) Let A be an $n \times n$ invertible matrix

↳ $\text{ker}(A) = ?$

↳ Recall: A invertible $\iff A\vec{v} = \vec{w}$ has a unique solution $\vec{v} \in \mathbb{R}^n$ for every $\vec{w} \in \mathbb{R}^n$

↳ Therefore, $A\vec{v} = \vec{0}$ has 1 solution: $\text{ker}(A) = \{\vec{0}\}$

↳ $\text{img}(A) = ?$

↳ For every $\vec{w} \in \mathbb{R}^n$ there is a $\vec{v} \in \mathbb{R}^n$ such that $\vec{w} = A\vec{v}$

↳ $\text{img}(A) = \mathbb{R}^n$

• Characterization of invertible matrices

↳ For any $n \times n$ matrix A , the following are equivalent:

↳ A is invertible

↳ The linear system $A\vec{v} = \vec{w}$ has a unique solution $\vec{v} \in \mathbb{R}^n$ for every $\vec{w} \in \mathbb{R}^n$

↳ $\text{rref}(A) = I_n$

↳ $\text{rank}(A) = n$, $\text{ker}(A) = \{\vec{0}\}$, $\text{img}(A) = \mathbb{R}^n$

• Definition: a subset V of \mathbb{R}^n is a subspace of \mathbb{R}^n if it:

↳ contains $\vec{0}$

↳ is closed under addition

↳ is closed under scalar multiplication

↳ Ex) Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $T(\vec{v}) = A\vec{v}$, $\ker(T) = \ker(A)$ is a subspace of \mathbb{R}^n
 $\text{img}(T) = \text{img}(A) \subset \mathbb{R}^m$

↳ Remark: \mathbb{R}^n is a vector space, every subspace of a vector space is a vector space

↳ Ex) $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x \geq 0, y \geq 0 \right\}$, is W a subspace of \mathbb{R}^2 ?

↳ no, not closed under scalar multiplication

↳ Ex) What are all possible subspaces in \mathbb{R}^2

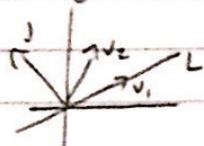
↳ any line passing through the origin

↳ $W = \text{span}(\vec{v})$, where $\vec{v} \in \mathbb{R}^2$

↳ the origin, $W = \{0\}$

↳ \mathbb{R}^2 , $W = \mathbb{R}^2$

↳ Suppose W is a subspace different from $\{0\}$ and not a line passing through the origin, let $0 \neq \vec{v} \in W$



$L = \text{span}(\vec{v}_1)$ is a subset of W , different from W

Let $\vec{v}_2 \in W$, and \vec{v}_2 does not lie on L

↳ Let \vec{v} be any vector in \mathbb{R}^2 , then there are scalars c_1, c_2 such that
 $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{v}$, $\vec{v} \in W$, $W = \mathbb{R}^2$

10/25 Lecture: Linear Dependency

• Recall: $\vec{v}_1, \dots, \vec{v}_m$ in \mathbb{R}^n are linearly dependent when there are non-trivial relations between them

↳ there are not all zero numbers c_1, \dots, c_m such that $c_1\vec{v}_1 + \dots + c_m\vec{v}_m = 0$

↳ contrapositive: $\vec{v}_1, \dots, \vec{v}_n$ in \mathbb{R}^n are linearly independent, then the only relationship between the vectors are trivial

↳ if $c_1\vec{v}_1 + \dots + c_n\vec{v}_n = 0$, then $c_1 = \dots = c_n = 0$

↳ Ex) $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

↳ linearly dependent because $\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = 0$

↳ Ex) $\vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{w}_3 = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix}$

↳ linearly independent

↳ Ex) Suppose that the column vectors of an $n \times m$ matrix A are linearly independent, what is $\ker(A)$?

↳ Let a_1, \dots, a_m be the columns of A

↳ a_1, \dots, a_m are linearly independent, the only solution to $c_1a_1 + \dots + c_ma_m = 0$ is the trivial solution ($c_1 = \dots = c_m = 0$)

↳ To find the kernel of A , solve the system $A\vec{v} = 0 \rightarrow v_1a_1 + \dots + v_ma_m = 0$

↳ $\therefore v_1 = \dots = v_m = 0$

↳ $\ker(A) = \{\vec{0}\}$

↳ Summary: the vectors in $\ker(A)$ correspond to the linear relations among the column vectors of A

↳ if the column vectors of A are linearly independent, $\ker(A) = \{\vec{0}\}$

• Finding basis for kernel and image

↳ Ex) $A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & 8 & 9 \\ 3 & 6 & 1 & 5 & 7 \end{bmatrix}$ Find a basis for i) $\ker(A)$ and ii) $\text{im}(A)$

↳ $\text{RREF}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B \rightarrow \ker(A) = \ker(B), A\vec{v} = 0 \text{ same as } B\vec{v} = 0$

↳ Solutions are: $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} -2s-3t+4r \\ s \\ t \\ 4t-5r \\ r \end{bmatrix}, s, t, r \in \mathbb{R}$

$$\hookrightarrow = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} 4 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} = s \vec{w}_1 + t \vec{w}_2 + r \vec{w}_3$$

\hookrightarrow Claim: $\vec{w}_1, \vec{w}_2, \vec{w}_3$ form a basis for $\ker(A)$

\hookrightarrow the vectors span the kernel

\hookrightarrow the vectors are linearly independent

\hookrightarrow basis for $\text{im}(A)$

\hookrightarrow we need to find the redundant columns of A

\hookrightarrow denote the columns of A by $a_1, \dots, a_5 \rightarrow$ same w/ B

\hookrightarrow redundant columns in $B \rightarrow$ columns w/o pivot

\hookrightarrow key observation: b_1 is redundant when a_1 is redundant

\hookrightarrow given by a_1 and a_3

10/30 Lecture: Dimensions of Subspaces

• Recall: given any $n \times m$ matrix A

↳ basis for $\ker(A) \rightarrow$ solve the system $A\vec{v} = \vec{0}$

↳ solution is of the form $\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = s_{11} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} + s_{12} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + s_{1k} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = s_{11} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} + s_{12} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + s_{1k} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= s_{11} \vec{b}_1 + s_{12} \vec{b}_2 + \dots + s_{1k} \vec{b}_k$$

↳ The vectors $\vec{b}_1, \vec{b}_2, \dots, \vec{b}_k$ form a basis for $\ker(A)$

↳ basis for $\text{im}(A) \rightarrow$ given by column vectors of A that correspond to column vectors w/ pivots in rref(A)

↳ important remark: $\ker(A) = \ker(\text{rref}(A))$, but $\text{im}(A) \neq \text{im}(\text{rref}(A))$ in general

• Dimension of subspaces of \mathbb{R}^n

↳ Recall: we had seen that all possible subspaces of \mathbb{R}^2 are:

↳ $\mathbb{R}^3 \rightarrow$ basis: \emptyset

↳ lines through the origin \rightarrow basis: $0 + \vec{v}$ in \mathbb{R}^2

↳ $\mathbb{R}^2 \rightarrow$ basis: any pair of vectors \vec{v}, \vec{w} in \mathbb{R}^2 that are linearly independent

↳ General fact: Let V be a subspace of \mathbb{R}^n . Then any basis of V has the same # of vectors

↳ Definition: Let V be a subspace of \mathbb{R}^n . The # of vectors in a basis of V is called the dimension of V , and denoted by $\dim(V)$

↳ Ex) Lines thru the origin in \mathbb{R}^2 are 1-dimensional

↳ Ex) For any $n \times m$ matrix A , $\dim(\text{im}(A)) = \text{rank}(A)$, $\dim(\ker(A)) = \# \text{ of free variables} = \# \text{ of total variables} - \# \text{ of leading variables} = n - \text{rank}(A)$

• Rank-Nullity Theorem: Let A be any $n \times m$ matrix. Then $\dim(\text{im}(A)) + \dim(\ker(A)) = m$

↳ $\dim(\ker(A)) = \text{nullity of the matrix } A$

↳ in other words: $\text{rank}(A) + \text{nullity of } A = m$

↳ Geometric interpretation:

↳ let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ w/ $T(\vec{v}) = A\vec{v}$

↳ nullity of A counts the # of dimensions that collapse as we perform T

↳ rank of A counts the # of dimensions that survive T

• Basis for \mathbb{R}^n

↳ Recall: the vectors $\vec{e}_1, \dots, \vec{e}_n$ in \mathbb{R}^n are called standard basis vectors of \mathbb{R}^n and form a basis for \mathbb{R}^n

↳ they span \mathbb{R}^n

↳ they are linearly independent

↳ Given n vectors in \mathbb{R}^n : $\vec{b}_1, \dots, \vec{b}_n$, how can we tell whether they form a basis for \mathbb{R}^n ? \oplus

• Characterisation of basis: unique representation

↳ Let V be any subspace of \mathbb{R}^n . Then $\vec{b}_1, \dots, \vec{b}_n$ is a basis for V

↳ every vector \vec{v} in V can be written uniquely as a linear combination of $\vec{b}_1, \dots, \vec{b}_n$

↳ in other words: if there is a unique solution to $c_1\vec{b}_1 + \dots + c_n\vec{b}_n = \vec{v}$

↳ in relation to $\oplus \rightarrow \vec{b}_1, \dots, \vec{b}_n$ form a basis \rightarrow every vector $\vec{w} \in \mathbb{R}^n$ can be written uniquely as $\vec{w} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$

$$\Leftrightarrow \begin{bmatrix} \vec{b}_1 & \dots & \vec{b}_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

↳ We know this system has a unique solution in \mathbb{R}^n for any $\vec{w} \in \mathbb{R}^n$ iff $[\vec{b}_1, \dots, \vec{b}_n]$ is invertible

↳ Summary: Let b_1, \dots, b_n be any vectors in \mathbb{R}^n , they form a basis for \mathbb{R}^n iff $[b_1, \dots, b_n]$ is invertible

11/1 Lecture: Coordinate Systems

• Coordinates

↳ Ex)

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, V = \text{span}(\vec{v}_1, \vec{v}_2)$$

↳

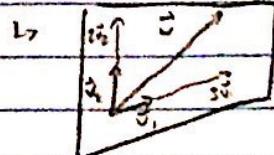
$$\text{Let } \vec{w} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}, \text{ does } \vec{w} \text{ lie on } V$$

↳ $\vec{w} \in V$ if there are scalars c_1 and c_2 and $w = c_1\vec{v}_1 + c_2\vec{v}_2$

$$\text{↳ same as solving the system } \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 1 & 3 & 9 \end{array} \right] = A \setminus w$$

$$\text{↳ rref}(A \setminus w) = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right] \rightarrow \text{unique solution: } c_1 = 3, c_2 = 2$$

↳ ∴ w lies on the plane, $w = 3\vec{v}_1 + 2\vec{v}_2$



Interpretation: the lines spanned by \vec{v}_1 and \vec{v}_2 are axes for a coordinate system on V , $w = (3, 2)$

$$\text{↳ } B = \{\vec{v}_1, \vec{v}_2\} \rightarrow [w]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

• Definition

↳ Let V be a subspace of \mathbb{R}^n with basis $B = \{\vec{v}_1, \dots, \vec{v}_m\}$. We know that each vector \vec{v} in V can be expressed as a unique linear combination $\vec{v} = c_1\vec{v}_1 + \dots + c_m\vec{v}_m$

↳ We call the numbers c_1, \dots, c_m the B coordinates of \vec{v} and we write:

$$[\vec{v}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$$

↳ Let $S = [\vec{v}_1 \dots \vec{v}_m]$, then $\vec{v} = S[\vec{v}]_B$

↳ This gives us a linear transformation $\mathbb{R}^n \rightarrow \mathbb{R}^m$ / $[\vec{v}]_B \mapsto S[\vec{v}]_B = \vec{v}$

• Coordinate systems on \mathbb{R}^n

↳ Given a basis $B = \{\vec{v}_1, \dots, \vec{v}_n\}$, let $S = [\vec{v}_1 \dots \vec{v}_n]$

↳ There is a linear transformation $\mathbb{R}^n \rightarrow \mathbb{R}^n$ / $[\vec{v}]_B \mapsto S[\vec{v}]_B = \vec{v}$

↳ Since $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent, S is invertible

$$\underbrace{\mathbb{R}^n \xrightarrow[S]{S^{-1}} \mathbb{R}^n}_{S^{-1}} \xrightarrow{S} [\vec{v}]_B \xrightarrow{S^{-1}} S\vec{v} \rightarrow [\vec{v}]_B = S^{-1}\vec{v} \leftarrow \vec{v}$$

↳ Consequence: coordinates satisfy linearity

$$\text{i.) } [\vec{v} + \vec{w}]_B = [\vec{v}]_B + [\vec{w}]_B \quad \forall \vec{v}, \vec{w} \in \mathbb{R}^n$$

$$\text{ii.) } [k\vec{v}]_B = k[\vec{v}]_B \quad \forall k \in \mathbb{R}, \vec{v} \in \mathbb{R}^n$$

Ex) $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ basis of \mathbb{R}^2

Let $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, what is $[\vec{v}]_B$?

Solution: solve the system: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow [\vec{v}]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Let $[\vec{w}]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, what is \vec{w} ?

By definition of B -coordinates, we know: $\vec{w} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

The matrix of a linear transformation

Recall: for any lin. transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$, there exists an $n \times m$ matrix

A such that $T(\vec{v}) = A\vec{v}$ for all $\vec{v} \in \mathbb{R}^n$

We constructed A as $A = [T(e_1) \dots T(e_n)]$

Theorem: Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation, and a basis

$B = \{b_1, \dots, b_n\}$ of \mathbb{R}^n

$\vec{v} \xrightarrow{\sim} T(\vec{v})$

$$\begin{array}{ccc} S & \uparrow & S \\ \vec{v} & \xrightarrow{\quad B \quad} & [T(\vec{v})]_B \\ S & \uparrow & S \\ [v]_B & \xrightarrow{\quad B \quad} & [T(v)]_B \end{array}$$

There exists a unique $n \times n$ matrix B such that $[T(\vec{v})]_B = B[\vec{v}]_B$

Proof: Let $\vec{v} \in \mathbb{R}^n$ be arbitrary, let $[\vec{v}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

This means $\vec{v} = c_1 \vec{b}_1 + \dots + c_n \vec{b}_n$, $[T(\vec{v})]_B = [T(c_1 \vec{b}_1 + \dots + c_n \vec{b}_n)]_B =$

$[c_1 T(\vec{b}_1) + \dots + c_n T(\vec{b}_n)]_B = c_1 [T(\vec{b}_1)]_B + \dots + c_n [T(\vec{b}_n)]_B =$

$$[T(c_1)]_B + \dots + [T(c_n)]_B = B \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

11/4 Lecture: \mathbb{B} -matrices and Orthogonality

- The matrix of a linear transformation

$\hookrightarrow T: \mathbb{R}^n \rightarrow \mathbb{R}^n, B = \{b_1, \dots, b_n\}$

\hookrightarrow standard coordinates $\vec{v} \xrightarrow{\quad A \quad} T(\vec{v})$

$$S = [b_1 \dots b_n]$$

$$S[\vec{v}]_B = \vec{v}$$

\hookrightarrow B -coordinates $\vec{v} \xrightarrow[B]{\quad} [T(\vec{v})]_B$

$$A = [T(e_1) \dots T(e_n)]$$

$$B = [T(b_1) \dots T(b_n)]_B$$

$\hookrightarrow A$: standard matrix of T , B : B -matrix of T

\hookrightarrow Any 2 paths in the diagram give the same result, the diagram "commutes"

$$\hookrightarrow SB[\vec{v}]_B = AS[\vec{v}]_B = T(\vec{v})$$

- Standard vs. B -matrix

$$\hookrightarrow AS = SB \rightarrow S^{-1}AS = B \rightarrow A = SBS^{-1}$$

• Definition: Let A and B be $n \times n$ matrices. We say that A is similar to B if there exists an invertible matrix S such that $AS = SB$ or $B = S^{-1}AS$

• Exercise: Show that similarity is an equivalence relation for any $n \times n$ matrices A and B

\hookrightarrow i) A is similar to A (reflexivity)

\hookrightarrow ii) if A is similar to B , B is similar to A (symmetry)

\hookrightarrow iii) if A is similar to B , B is similar to C , then A is similar to C (transitivity)

• What is this good for?

\hookrightarrow Given a linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$, there might be a basis B in which T has a very simple form

\hookrightarrow Ex) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, projection onto the line $L = \text{span}([1; 1])$

\hookrightarrow $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$, B -matrix of T is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

• Definition: An $n \times n$ matrix A is diagonal if $a_{ij} = 0$ when $i \neq j$

• Orthogonality

\hookrightarrow Recall: Definition. 2 vectors \vec{v} and \vec{w} in \mathbb{R}^n are orthogonal (perpendicular)

if their dot product is zero

\hookrightarrow let V be a subspace of \mathbb{R}^n

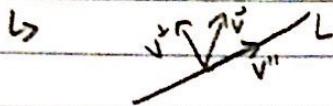
\hookrightarrow A vector \vec{v} in \mathbb{R}^n is orthogonal to the subspace if it is orthogonal to all vectors in the subspace

\hookrightarrow Note given a basis b_1, \dots, b_m of V , then a vector \vec{v} in \mathbb{R}^n is orthogonal to V when it is orthogonal to the vectors in the basis

↳ Let \vec{w} be any vector in V , $\vec{w} = c_1 \vec{b}_1 + \dots + c_m \vec{b}_m$, $\vec{v} \cdot \vec{w} = \vec{v} \cdot (c_1 \vec{b}_1 + \dots + c_m \vec{b}_m)$

$$\hookrightarrow c_1(\vec{v} \cdot \vec{b}_1) + c_2(\vec{v} \cdot \vec{b}_2) + \dots + c_m(\vec{v} \cdot \vec{b}_m) = 0$$

↳ Ex) Given a line L in \mathbb{R}^2 , and a vector \vec{v} in \mathbb{R}^2



$$\vec{v} = \vec{v}'' + \vec{v}^\perp \rightarrow \text{parallel to } L + \text{orthogonal to } L$$

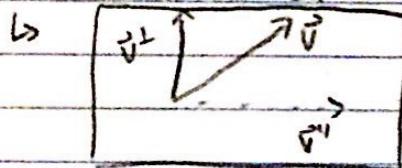
↳ this decomposition is unique

↳ \vec{v}'' is the orthogonal projection of \vec{v} onto L

• Orthogonal projection onto a subspace

↳ Let V be a subspace of \mathbb{R}^n , let \vec{v} be any vector in \mathbb{R}^n , then $\vec{v} = \vec{v}'' + \vec{v}^\perp$, where \vec{v}'' is in V , \vec{v}^\perp is orthogonal to V

↳ these vectors are unique



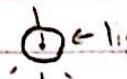
$\vec{v}'' \rightarrow V \rightarrow$ The vector \vec{v}'' is the orthogonal projection of \vec{v} onto V , denoted by $\text{proj}_V(\vec{v})$

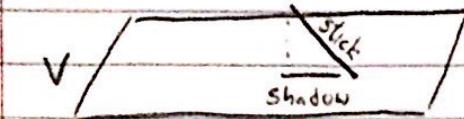
↳ The transformation $\mathbb{R}^n \rightarrow \mathbb{R}^n$ given by $T(\vec{v}) = \text{proj}_V(\vec{v})$ is linear

↳ Definition: The length of a vector \vec{v} in \mathbb{R}^n is $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + \dots + v_n^2}$, a vector \vec{v} in \mathbb{R}^n is a unit vector if $\|\vec{v}\| = 1$, vectors $\vec{u}_1, \dots, \vec{u}_n$ in \mathbb{R}^n are orthonormal if $\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$ for all i, j

11/6 Lecture: Orthogonal Projections

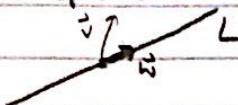
- Orthogonal projection onto a subspace V of \mathbb{R}^n

↳  light



the projection is the shadow if the vector is the stick

↳ Ex) $L = \text{span}(\vec{w})$



$$\vec{v} = \vec{v}^\perp + \vec{w} \quad \vec{w} = \text{proj}_L(\vec{v})$$

↳ Let V be any subspace of \mathbb{R}^n , any vector in \mathbb{R}^n can be written as a sum, $\vec{v} = \vec{v}^\perp + \vec{v}''$, where \vec{v}^\perp is orthogonal to V and \vec{v}'' is a vector in V

↳ these vectors are unique

↳ we call \vec{v}'' the orthogonal projection of \vec{v} onto V and denote it by $\text{proj}_V(\vec{v})$

↳ the transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ that sends any $\vec{v} \in \mathbb{R}^n$ to $\text{proj}_V(\vec{v})$ is linear

• Orthonormality

↳ Ex) e_1, \dots, e_n in \mathbb{R}^n , with $1 \leq m \leq n$ are orthonormal

↳ Ex) Let α be any real number, then $\begin{bmatrix} \cos \alpha \\ \sin \alpha \end{bmatrix}$ and $\begin{bmatrix} -\sin \alpha \\ \cos \alpha \end{bmatrix}$ are orthonormal

• Properties of Orthonormal Vectors

↳ i) orthonormal vectors are linearly independent

ii) orthonormal vectors $\vec{u}_1, \dots, \vec{u}_m$ in \mathbb{R}^n form a basis of \mathbb{R}^n

↳ Proof of i): Let $\vec{u}_1, \dots, \vec{u}_m$ in \mathbb{R}^n be orthonormal

↳ To show: if there are $c_1, \dots, c_m \in \mathbb{R}$ such that $c_1\vec{u}_1 + \dots + c_m\vec{u}_m = 0$, then

$$c_1 = \dots = c_m = 0$$

↳ let \vec{u}_i be arbitrary, $(c_1\vec{u}_1 + \dots + c_m\vec{u}_m) \cdot \vec{u}_i = 0 \cdot \vec{u}_i = 0$

$$\underbrace{c_1(\vec{u}_1 \cdot \vec{u}_i)}_{0} + \dots + \underbrace{c_i(\vec{u}_i \cdot \vec{u}_i)}_1 + \dots + \underbrace{c_m(\vec{u}_m \cdot \vec{u}_i)}_0 = 0$$

$$\Rightarrow c_i \cdot 1 = 0 \rightarrow c_i = 0$$

• Formula for orthogonal projection

↳ Let V be a subspace of \mathbb{R}^n , with an orthonormal basis $\vec{u}_1, \dots, \vec{u}_m$, then

for all \vec{v} in $\mathbb{R}^n \rightarrow \text{proj}_V(\vec{v}) = (\vec{u}_1 \cdot \vec{v})\vec{u}_1 + \dots + (\vec{u}_m \cdot \vec{v})\vec{u}_m$

Recall: $L = \text{span}(\vec{w})$ in \mathbb{R}^2 for any \vec{v} in \mathbb{R}^2 : $\text{proj}_L(\vec{v}) = \left(\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \vec{w}$

↳ Note: $(\vec{u}_1, \vec{v})\vec{u}_1$ is the orthogonal projection of \vec{v} onto the line spanned by \vec{u}_1 .

↳ Ex) Let V be the $x_1 - x_3$ plane in \mathbb{R}^3

$$\text{proj}_V(\vec{v}) = (\vec{e}_1 \cdot \vec{v}) \vec{e}_1 + (\vec{e}_3 \cdot \vec{v}) \vec{e}_3$$

↳ Ex) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $V = \text{im}(A)$

↳ Let $\vec{v} \in \mathbb{R}^4$ be $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, find $\text{proj}_V(\vec{v})$

↳ Solution: The 2 column vectors of A are linearly independent, so they form a basis, and are also orthogonal

↳ We get an orthonormal basis of V by dividing each vector by its length

$$\hookrightarrow \vec{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\hookrightarrow \text{proj}_V(\vec{v}) = (\vec{u}_1 \cdot \vec{v}) \vec{u}_1 + (\vec{u}_2 \cdot \vec{v}) \vec{u}_2 = 6\vec{u}_1 + 2\vec{u}_2 = \begin{bmatrix} 4 \\ 2 \\ 2 \\ 4 \end{bmatrix}$$

↳ can check using $\vec{v} - \text{proj}_V(\vec{v})$ is orthogonal to \vec{u}_1 and \vec{u}_2

11/8 Lecture: Orthogonal Complements

• Midterm 2: Chapter 3, Chapter 5.1 - 5.2, Sheets 4, 5, 6, 7

• If $V = \mathbb{R}^n$, then the formula for the orthogonal projection

↳ Theorem: Let $\vec{u}_1, \dots, \vec{u}_n$ be an orthonormal basis of \mathbb{R}^n . Then for all $\vec{v} \in \mathbb{R}^n$: $\vec{v} = (\vec{u}_1 \cdot \vec{v}) \vec{u}_1 + (\vec{u}_2 \cdot \vec{v}) \vec{u}_2 + \dots + (\vec{u}_n \cdot \vec{v}) \vec{u}_n = \text{proj}_{\mathbb{R}^n}(\vec{v})$

↳ Note: in general, given a basis $\vec{b}_1, \dots, \vec{b}_n$ of \mathbb{R}^n , we know that for any $\vec{v} \in \mathbb{R}^n$, there exists unique numbers such that $\vec{v} = c_1 \vec{b}_1 + \dots + c_n \vec{b}_n$

↳ In general we have to solve a system to find c_1, \dots, c_n . If the basis is orthonormal, we have $c_i = \vec{b}_i \cdot \vec{v}$

• Let V be a subspace of \mathbb{R}^n , $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be $T(\vec{v}) = \text{proj}_V(\vec{v})$

↳ We have:

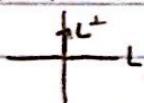
↳ i) $\text{im}(T) = V$

ii) $\ker(T) = \{\text{all vectors in } \mathbb{R}^n \text{ orthogonal to } V\}$

↳ Definition: Let V be a subspace of \mathbb{R}^n , the orthogonal complement of V is the set of all vectors in \mathbb{R}^n that are orthogonal to V , denoted by V^\perp

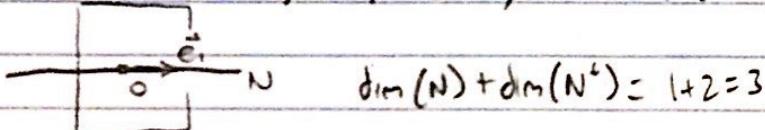
↳ $V^\perp = \{\vec{w} \in \mathbb{R}^n \text{ such that } \vec{w} \cdot \vec{v} = 0 \text{ for all vectors } \vec{v} \text{ in } V\}$

↳ Ex) L a line in \mathbb{R}^2



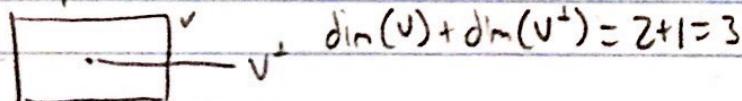
$$\dim(L) + \dim(L^\perp) = 1 + 1 = 2$$

↳ Ex) N a line in \mathbb{R}^3 , $N = \text{span}(\vec{e}_1)$, $N^\perp = \ker(\text{proj}_N)$



$$\dim(N) + \dim(N^\perp) = 1 + 2 = 3$$

↳ Ex) V a plane in \mathbb{R}^3



$$\dim(V) + \dim(V^\perp) = 2 + 1 = 3$$

• Properties of the orthogonal complement

↳ Let V be a subspace of \mathbb{R}^n

↳ V^\perp is a subspace of $\mathbb{R}^n \rightarrow V^\perp$ is $\ker(\text{proj}_V)$

↳ $\dim(V) + \dim(V^\perp) = n \rightarrow$ consequence of the rank-nullity theorem

↳ $V = \text{im}(\text{proj}_V)$, $V^\perp = \ker(\text{proj}_V)$

↳ $V \cap V^\perp = \{0\}$

↳ Suppose $\vec{v} \in \mathbb{R}^n$ is in both V and V^\perp , V has to be orthogonal to itself, therefore \vec{v} would be $\vec{0}$

↳ $(V^\perp)^\perp = V$

• Given any 2 sets A and B, subsets of a set C

↳ $A \cap B = \text{intersection of } A \text{ and } B = \{x \in C \text{ such that } x \in A \text{ and } x \in B\}$

• Gram-Schmidt algorithm

↳ Given a basis $\vec{v}_1, \dots, \vec{v}_m$ of a subspace V, we want to find an orthonormal basis $\vec{u}_1, \dots, \vec{u}_m$

↳ Ex) Let $V = \text{span}(\vec{w})$ where $\vec{w} (\neq 0)$ is a vector on \mathbb{R}^n , then $\vec{u}_1 = \frac{\vec{w}}{\|\vec{w}\|}$ is an orthonormal basis of V

↳ Ex) $V = \text{span}(\vec{v}_1, \vec{v}_2)$, \vec{v}_1 and \vec{v}_2 are linearly independent

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}, \vec{v}_2 = \text{proj}_{(\text{span}(\vec{v}_1))}(\vec{v}_2) = \vec{v}_2 - (\vec{v}_1 \cdot \vec{v}_2)\vec{u}_1 = \vec{v}_2^\perp$$

$$\vec{v}_2^\perp = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

↳ In general: V is a subspace of \mathbb{R}^n , with basis $\vec{v}_1, \dots, \vec{v}_m$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\vec{v}_1^\perp := \vec{v}_1 - \underbrace{[(\vec{v}_1 \cdot \vec{v}_1)\vec{u}_1 + \dots + (\vec{v}_{i-1} \cdot \vec{v}_1)\vec{u}_{i-1}]}_{\text{Orthogonal projection of } \vec{v}_1 \text{ onto }} \text{span}(\vec{u}_1, \dots, \vec{u}_{i-1})$$

Orthogonal projection of \vec{v}_1 onto

$$\text{span}(\vec{u}_1, \dots, \vec{u}_{i-1})$$

$$\vec{u}_i = \frac{\vec{v}_i}{\|\vec{v}_i\|}$$

↳ Then, $\vec{u}_1, \dots, \vec{u}_m$ is an orthonormal basis of V

↳ Ex) $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 7 \\ -8 \end{bmatrix}, V = \text{span}(\vec{v}_1, \vec{v}_2)$, a plane in \mathbb{R}^3

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{\vec{v}_1}{\sqrt{5}} = \frac{1}{\sqrt{5}}\vec{v}_1, L = \text{span}(\vec{v}_1)$$

$$\text{proj}_L(\vec{v}_2) = (\vec{v}_2 \cdot \vec{v}_1)\vec{u}_1 = \left[\frac{1}{\sqrt{5}}(27)\right]\vec{u}_1 = 3\vec{u}_1$$

$$\vec{v}_2^\perp = \vec{v}_2 - 3\vec{u}_1 = \begin{bmatrix} -4 \\ 4 \\ -2 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^\perp}{\|\vec{v}_2^\perp\|} = \frac{1}{6}\vec{v}_2^\perp$$

$$\boxed{\vec{u}_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1/\sqrt{5} \\ 7/\sqrt{5} \\ -4/\sqrt{5} \end{bmatrix}}$$

11/13 Lecture: QR Factorization

• Recall: Gram-Schmidt Algorithm

↳ input: a subspace V of \mathbb{R}^n and a basis $\vec{b}_1, \dots, \vec{b}_m$ of V

↳ output: orthonormal basis $\vec{u}_1, \dots, \vec{u}_m$ of V

↳ Let $B = \{\vec{b}_1, \dots, \vec{b}_m\}$, $U = \{\vec{u}_1, \dots, \vec{u}_m\}$

↳ Recall: Let $S_B = [\vec{b}_1 \dots \vec{b}_m]$, then for every \vec{v} in V we have

$$S_B[\vec{v}]_B = \vec{v}$$

$$\hookrightarrow S_B[\vec{v}]_B = [\vec{u}_1 \dots \vec{u}_m]$$

↳ B coordinates

$$[\vec{v}]_B \in S_B \quad [\vec{v}]_U \rightarrow \text{This means, } S_B[\vec{v}]_B = \vec{v} = S_U[\vec{v}]_U$$

$$\hookrightarrow \vec{v} \in S_U$$

Standard

$$[\vec{b}_i] = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{\text{m rows}} \text{ith entry}$$

↳ In particular, for any vector \vec{b}_i in B , $S_B[\vec{b}_i] = \text{ith column of } S_B = S_U[\vec{b}_i]_U$

$$\hookrightarrow \begin{bmatrix} 1 & 1 \\ \vec{b}_1 & \dots & \vec{b}_m \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ [\vec{b}_1]_B & \dots & [\vec{b}_m]_B \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \vec{u}_1 & \dots & \vec{u}_m \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ [\vec{b}_1]_U & \dots & [\vec{b}_m]_U \\ 1 & 1 \end{bmatrix}$$

$n \times m$ matrix S_B

$m \times m$ matrix I_m

$n \times n$ matrix Q

$m \times m$ matrix R

• QR Factorization - Let M be any $n \times m$ matrix with linearly independent column vectors, then there exists matrices Q and R where Q is an $n \times m$ matrix with orthonormal column vectors, and R is an $m \times m$ upper-diagonal matrix

↳ Upper diagonal - form $\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$, where the diagonal entries are positive

↳ Q and R are unique

↳ Proof: Existence of Q and R

↳ Apply Gram-Schmidt to the column vectors of M

↳ Proof: Properties of R

↳ Let $\vec{b}_1, \dots, \vec{b}_m$ be the column vectors of M , and $\vec{u}_1, \dots, \vec{u}_m$ be the column vectors of Q

↳ Then:

$$R = \begin{bmatrix} 1 & & \\ [\vec{b}_1]_U & \dots & [\vec{b}_m]_U \\ 1 & 1 \end{bmatrix}, \text{ where } U = \{\vec{u}_1, \dots, \vec{u}_m\}$$

$$\hookrightarrow \text{Recall: } \vec{u}_1 = \frac{\vec{b}_1}{\|\vec{b}_1\|} \rightarrow [\vec{b}_1]_U = \|\vec{b}_1\| \vec{u}_1$$

$$\vec{u}_2 = \frac{\vec{b}_2 - (\vec{u}_1 \cdot \vec{b}_2) \vec{u}_1}{\|\vec{b}_2 - (\vec{u}_1 \cdot \vec{b}_2) \vec{u}_1\|}, \text{ where } \vec{b}_2^\perp = \vec{b}_2 - (\vec{u}_1 \cdot \vec{b}_2) \vec{u}_1 \rightarrow [\vec{b}_2]_U = (\vec{u}_1 \cdot \vec{b}_2) \vec{u}_1 + \|\vec{b}_2^\perp\| \vec{u}_2$$

$$\hookrightarrow R = \begin{bmatrix} \|\vec{b}_1\| \vec{u}_1 \cdot \vec{b}_2 & \cdots & \vec{u}_1 \cdot \vec{b}_m \\ 0 & \|\vec{b}_2\| & \cdots & \vec{u}_2 \cdot \vec{b}_m \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \|\vec{b}_m\| \end{bmatrix}$$

\hookrightarrow Ex) Find the QR Factorization of $m = \begin{bmatrix} 2 & 2 \\ 1 & 7 \\ -2 & 8 \end{bmatrix}$

$$\hookrightarrow \vec{b}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 2 \\ 7 \\ -8 \end{bmatrix}$$

$$\hookrightarrow m = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \|\vec{b}_1\| \vec{u}_1 \cdot \vec{b}_2 \\ 0 \quad \|\vec{b}_2\| \end{bmatrix}$$

\hookrightarrow 1st column of R, then 1st of Q, then 2nd of R, then 2nd of Q

$$\hookrightarrow \|\vec{b}_1\| = \sqrt{9} = 3$$

$$\hookrightarrow \vec{u}_1 = \frac{\vec{b}_1}{\|\vec{b}_1\|} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$\hookrightarrow \vec{u}_1 \cdot \vec{b}_2 = 9$$

$$\hookrightarrow \vec{b}_2^\perp = \begin{bmatrix} -4 \\ -4 \\ 2 \end{bmatrix}$$

$$\hookrightarrow \|\vec{b}_2^\perp\| = 6$$

$$\hookrightarrow \vec{u}_2 = \frac{\vec{b}_2^\perp}{\|\vec{b}_2^\perp\|} = \frac{1}{2} \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$$\hookrightarrow Q = \frac{1}{3} \begin{bmatrix} 2 & -2 \\ 1 & 2 \\ -2 & -1 \end{bmatrix}, R = \begin{bmatrix} 3 & 9 \\ 0 & 6 \end{bmatrix}$$

Orthogonal transformations

\hookrightarrow Definition: A linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is orthogonal if it preserves length of vectors, i.e. if $\|T(\vec{v})\| = \|\vec{v}\|$ for all \vec{v} in \mathbb{R}^n

\hookrightarrow Note: Any rotation and reflection in \mathbb{R}^2 is an orthogonal transformation

\hookrightarrow Ex) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ counterclockwise rotation through angle θ

$$\hookrightarrow \text{For all } \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \text{ in } \mathbb{R}^2: T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \cos \theta v_1 - \sin \theta v_2 \\ \sin \theta v_1 + \cos \theta v_2 \end{bmatrix}$$

$$\hookrightarrow \|T(\vec{v})\|^2 = (\cos \theta v_1 - \sin \theta v_2)^2 + (\sin \theta v_1 + \cos \theta v_2)^2 = v_1^2 + v_2^2 = \|\vec{v}\|^2$$

$\hookrightarrow T$ is orthogonal

11/15 Lecture: Orthogonal Transformations

• Recall: $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is orthogonal if $\|T(\vec{x})\| = \|\vec{x}\|$ for all vectors in \mathbb{R}^n

↳ Ex) rotations and reflections in \mathbb{R}^2

↳ Remark: If T preserves length, then it also preserves angles

↳ If the angle changes, the length of the line that connects the terminal ends of the vectors changes

↳ Used to create the matrix A for T

↳ Start with the standard basis $\vec{e}_1, \dots, \vec{e}_n$ and T orthogonal

$$\hookrightarrow A = [T(\vec{e}_1) \dots T(\vec{e}_n)], \|\vec{e}_i\| = 1 = \|T(\vec{e}_i)\|$$

↳ if $i \neq j$, $\vec{e}_i \cdot \vec{e}_j = 0$ ($\vec{e}_i \perp \vec{e}_j$)

↳ since T is orthogonal, $T(\vec{e}_i) \perp T(\vec{e}_j) = T(\vec{e}_i) \cdot T(\vec{e}_j) = 0$

↳ $T(\vec{e}_1), \dots, T(\vec{e}_n)$ forms an orthonormal basis

• Definition: An $n \times n$ matrix is orthogonal if its columns form an orthonormal basis

$$\cdot B = \begin{bmatrix} 1 & 0 \\ 2 & 24 \\ 3 & 18 \end{bmatrix}, B^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 24 & 18 \end{bmatrix} \rightarrow \text{transpose}$$

$$\cdot \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \rightarrow \vec{v}^T = [v_1, v_2, v_3] \quad \vec{v}^T \vec{v} \rightarrow \text{matrix multiplication} \rightarrow \vec{v} \cdot \vec{v}$$

↳ If \vec{x} and \vec{y} are column n -vectors, then $\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}$

$$\cdot A = \begin{bmatrix} \sqrt{15} & \sqrt{15} & \sqrt{15} \\ \sqrt{15} & 0 & -\sqrt{15} \\ \sqrt{15} & -\sqrt{15} & \sqrt{15} \end{bmatrix}, \text{ is orthogonal, +s invertible}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ 1 & 1 & 1 \end{bmatrix}, \vec{u}_1 \cdot \vec{u}_1 = 1, \vec{u}_1 \cdot \vec{u}_2 = 0, \vec{u}_1 \cdot \vec{u}_3 = 0, \text{etc.}$$

$$\hookrightarrow \vec{u}_1^T A = \vec{u}_1^T \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix} = [1 \ 0 \ 0]$$

$$\hookrightarrow \vec{u}_2^T A = \vec{u}_2^T \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix} = [0 \ 1 \ 0]$$

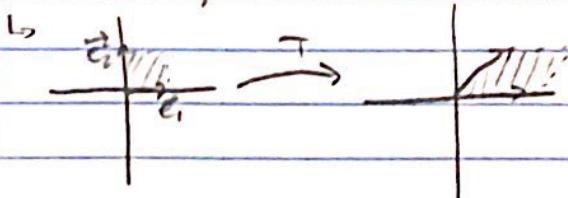
$$\hookrightarrow \begin{bmatrix} -\vec{u}_1 \\ \vec{u}_2 \\ -\vec{u}_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} \quad A \quad I_n$$

↳ $A^{-1} = A^T$ when A is orthogonal, A^T is also orthogonal

↳ Orthogonal matrices' inverses are their transpose, vice versa

• $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, invertible



↳ What is the new area of the basis (parallelograms)

↳ Say T has matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $T(\vec{e}_1) = \begin{bmatrix} a \\ c \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} b \\ d \end{bmatrix}$

$$\hookrightarrow A = \|T(\vec{e}_1)\| \|T(\vec{e}_2)\| |\sin\theta| = |ad - bc|$$

↳ Ex) T is a reflection across the x -axis

$$\hookrightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

↳ When the angle between $T(\vec{e}_1)$ and $T(\vec{e}_2)$ exceeds π is exactly when $\sin\theta < 0$

↳ We care about $ad - bc$ and we think of it as a signed area

↳ Remark: A is any 2×2 matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $ad - bc \neq 0$ when A is invertible

$$\hookrightarrow \text{Ex)} A = \begin{bmatrix} a & \lambda a \\ c & \lambda c \end{bmatrix}, ad - bc = 0$$

• Definition: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we define its determinant to be $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

↳ Facts:

↳ $\det(A)$ is a signed area

↳ $\det(A) \neq 0$ when A is invertible

$$\hookrightarrow \text{if } \det(A) \neq 0, A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

11/20 Lecture: Determinants

- 2×2 determinants:

↳ Given a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we call the number $ad - bc$ the determinant of A and denote it by $\det(A)$

↳ $\det(A) \neq 0$ when A is invertible

- Determinant of arbitrary square matrices

↳ Let $A = \begin{bmatrix} a_{11} & \dots & \dots \\ \vdots & \ddots & a_{33} \end{bmatrix}$ and denote the column vectors of A by $\vec{v}_1, \vec{v}_2, \vec{v}_3$

↳ A is not invertible \Leftrightarrow the image is not $\mathbb{R}^3 \Leftrightarrow \text{im}(A)$ is at most 2-D

↳ Consider the cross product $\vec{v}_2 \times \vec{v}_3$

↳ if $\vec{v}_2 \times \vec{v}_3 \neq 0$, then they are linearly independent and thus $\text{im}(A) = 2\text{-D}$

↳ Thus \vec{v}_1 is \perp to \vec{v}_2 and $\vec{v}_3 \rightarrow \vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3) = 0$

↳ if $\vec{v}_2 \times \vec{v}_3 = 0$, then $\vec{v}_2 = c\vec{v}_3$ for some real # c , and $\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3) = 0$

↳ Thus if A is not invertible, then $\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3) = 0$

↳ Conversely, if A is invertible, then $\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3) \neq 0$

↳ Definition: The number $\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3)$ is the determinant of a 3×3 matrix A

• Sarrus' Rule:

↳ Mnemonic for the determinant of a 3×3 matrix A , write the 1st 2 columns of A to the right of A , then multiply the entries along the 6 diagonals

↳ Add the products from top left to bottom right and subtract the products from bottom left to bottom right

↳ Ex) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \rightarrow \det A = 24$

↳ Ex) Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be 3 column vectors in \mathbb{R}^3 , what is the relationship

between $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$ and $B = \begin{bmatrix} \vec{v}_1 & \vec{v}_3 & \vec{v}_2 \end{bmatrix}$?

↳ $\det(A) = -\det(B)$

↳ Alternating property of the determinant - swapping any 2 columns changes the determinant's sign

• The determinant of an $n \times n$ matrix

↳ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is invertible, but Sarrus' Rule gives 0

↳ Definition - a pattern in an $n \times n$ matrix A is an unordered sequence of $n!$ entries of A such that no 2 entries in the sequence have the same row and column

↳ How many patterns are there in an $n \times n$ matrix A?

↳ There are $n!$ patterns

↳ The sign is related to the alternating property of the determinant

↳ Definition - an inversion in a pattern of an $n \times n$ matrix A is a number in P which lies above and to the right of another # of P

↳ Definition: Sign of a pattern = $(-1)^{\# \text{ of inversions}}$

↳ $\det(A) = \sum \text{Sign}(P) \prod(P)$

• Upper diagonal matrix $\rightarrow \det = \text{all the } \# \text{ s on the main diagonal}$

↳ Determinant of a diagonal matrix is the product of the diagonal entries

• Determinants and elementary row operations

↳ $\begin{bmatrix} a & b & c \\ k & l & m \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \det = \frac{1}{k} \det$

↳ $\begin{bmatrix} c & d \\ 0 & b \end{bmatrix} \rightarrow \det = -\det$

↳ $\begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix} \rightarrow \det = \det$

• Using Gauss-Jordan

↳ If we achieve RREF by swapping, s times and dividing by k_1, \dots, k_r ,

↳ $\det(A) = (-1)^s k_1 \dots k_r \det(\text{RREF}(A))$

↳ Ex) $A = \begin{bmatrix} 0 & 7 & 5 & 3 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix} \rightarrow B = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

↳ $\det(A) = (-1)^2 \det(B) = 14$

• Invertibility and determinant

↳ If A is invertible, then $\det(\text{RREF}(A)) = 1 \rightarrow \det(A) = (-1)^s k_1 \dots k_r \neq 0$

11/22 Lecture: Properties of the Determinant

Properties of determinants

$\hookrightarrow A$ and B are any $n \times n$ matrices

$$\hookrightarrow \det(AB) = \det(A)\det(B)$$

\hookrightarrow If A and B are similar matrices, then $\det(A) = \det(B)$

\hookrightarrow Explanation: $AS = SB$ for similar matrices for an invertible matrix S

$$\det(AS) = \det(A)\det(S) = \det(S)\det(B)$$

\hookrightarrow since S is invertible, $\det(S) \neq 0 \therefore \det(A) = \det(B)$

\hookrightarrow Converse is not true

\hookrightarrow If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$

\hookrightarrow Explanation: $AA^{-1} = I_n = A^{-1}A$

$$\det(A)\det(A^{-1}) = \det(I_n) = 1$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$\hookrightarrow \det(A^T) = \det(A)$

\hookrightarrow Explanation: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, consider a pattern in A (3,4,8) and the corresponding pattern in A^T , the pattern still holds w/ the same # of inversions

$$A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Recall: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\det(A) = \sum \text{sign}(P) \prod(P)$$

Theorem: Let A be an $n \times n$ matrix

$\hookrightarrow A$ is orthogonal $\rightarrow A^T A = I_n \rightarrow A^{-1} = A^T$

\hookrightarrow Proof: $A = \begin{bmatrix} 1 & \dots & n \end{bmatrix}$, then $A^T = \begin{bmatrix} -a_1 & - \\ \vdots & \vdots \\ -a_n & - \end{bmatrix}$

$$A^T A = \begin{bmatrix} -a_1 & - \\ -a_2 & - \\ \vdots & \vdots \\ -a_n & - \end{bmatrix} \begin{bmatrix} 1 & 1 \\ a_1 & \dots & a_n \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a_1 a_1 a_1 \dots a_n a_n \\ \vdots \\ a_n a_n \dots a_n a_n \end{bmatrix}$$

$\hookrightarrow A^T A = I_n \rightarrow a_i \cdot a_j = 1 \text{ if } i=j, 0 \text{ if } j \neq i \rightarrow A$ is orthogonal

\hookrightarrow Consequence: $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ orthogonal preserves the dot product for all \vec{v}, \vec{w} in

$$\mathbb{R}^n: T(\vec{v}) \cdot T(\vec{w}) = \vec{v} \cdot \vec{w}$$

\hookrightarrow Explanation: Let A be the orthogonal matrix that represents T

$$\hookrightarrow T(\vec{v}) \cdot T(\vec{w}) = A\vec{v} \cdot A\vec{w} = (A\vec{v})^T (A\vec{w}) \rightarrow (AB)^T = B^T A^T$$

$$\hookrightarrow \vec{v}^T A^T A \vec{w} = \vec{v}^T \vec{w} = \vec{v} \cdot \vec{w}$$

• Proposition. Let A be an orthogonal matrix, then $\det(A) = \pm 1$

↳ Proof: If $A^T A = I_n$

$$\hookrightarrow \det(A^T A) = \det(I_n) = \det(A^T) \det(A) = 1$$

$$\hookrightarrow \det(A) \det(A) = 1 \therefore \det(A) = \pm 1$$

↳ Ex) $\det \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix} = 1 \Rightarrow \text{rotation in } \mathbb{R}^2$

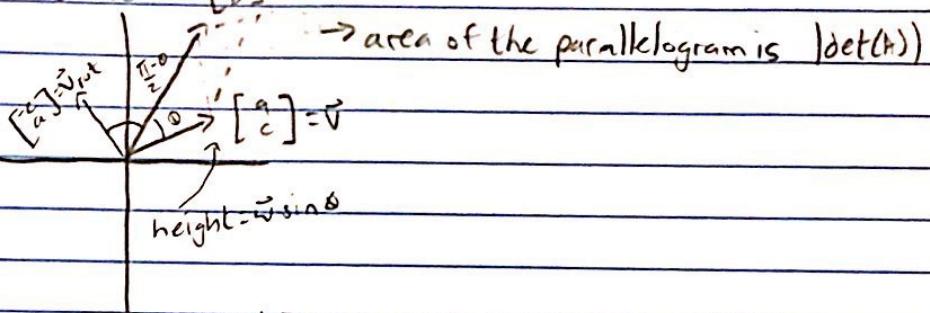
$$\det \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -1 \rightarrow \text{reflection about x-axis}$$

• Geometric interpretations of the determinant

↳ 1st absolute value

↳ As area and volume: $\begin{bmatrix} b \\ 0 \end{bmatrix} = \vec{v}$

↳ In \mathbb{R}^2 :



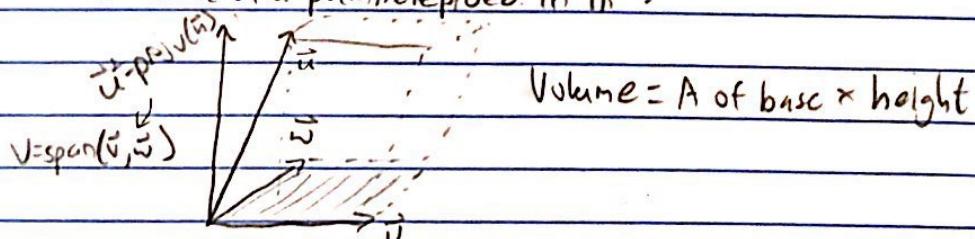
$$\det(A) = ad - bc = \begin{bmatrix} -c \\ a \end{bmatrix} \cdot \begin{bmatrix} b \\ 0 \end{bmatrix} = \|\vec{v}\| \|\vec{w}\| \cos(\frac{\pi}{2} - \theta) = \|\vec{v}\| \|\vec{w}\| \sin \theta$$

↳ Alternatively, with QR-decomposition

$$\hookrightarrow A = QR \rightarrow |\det(A)| = |\det(Q)| |\det(R)|$$

$$= \underbrace{\|\vec{v}\|}_{\text{base}} \underbrace{\|\vec{w}\|}_{\text{height}}$$

↳ Volume of a parallelepiped in \mathbb{R}^3 :

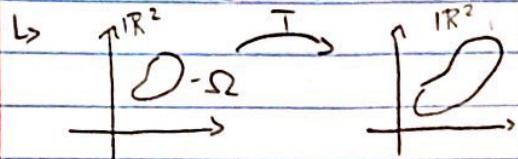


$$A = \begin{bmatrix} \vec{v} & \vec{w} & \vec{u} \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = QR \rightarrow \det(A) = \det(Q) \det(R) = \underbrace{\|\vec{v}\|}_{\text{base}} \underbrace{\|\vec{w}\|}_{\text{height}} \underbrace{\|\vec{u}\|}_{\text{height}}$$

11/25 Lecture: Expansion Factor

• Revision Lecture. Thurs. 12/5, 6-8pm Boelter 3400.

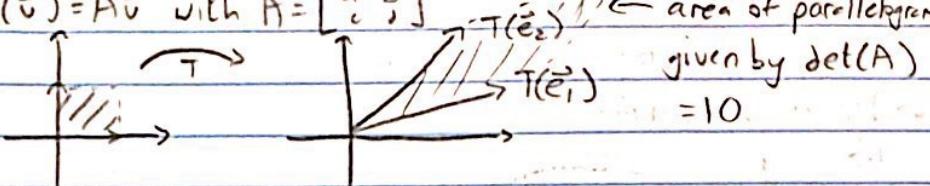
• Absolute value of the determinant as a scaling factor



↳ area of $T(\Omega)$ → expansion factor of T
area of Ω

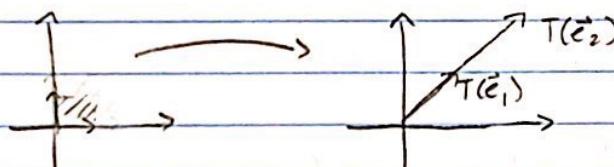
↳ $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2, T(\vec{v}) = A\vec{v}$ with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\text{Ex: } A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$



↳ expansion factor of T is 10

$$\text{Ex: } A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$



↳ $\det(A) = 0 \rightarrow \text{expansion factor} = 0$

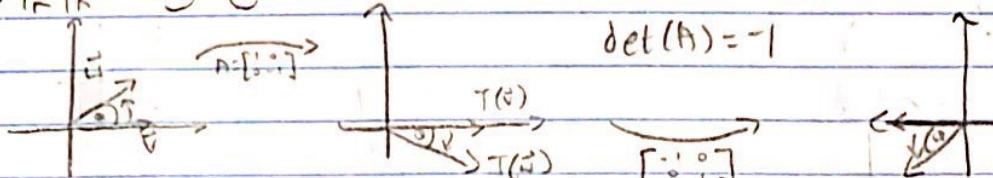
• For an arbitrary $\Omega \subseteq \mathbb{R}^2$: use estimation by squares for the area of Ω

↳ The expansion factor on each square is $|\det(A)|$

↳ Choosing smaller and smaller squares, integration gives that the expansion factor of T on Ω is $|\det(A)|$

• Sign of the determinant

↳ in \mathbb{R}^2 $\begin{cases} +1 & \text{if } \det(A) > 0 \\ -1 & \text{if } \det(A) < 0 \end{cases}$



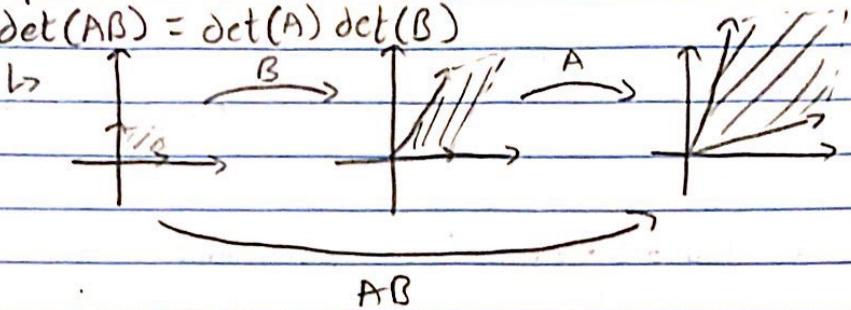
↳ In general: $\det(A) = 1$ when the linear transformation preserves orientation
and $\det(A) = -1$ when it reverses orientation

• Recall: We have seen that if A is an orthogonal matrix, then $\det(A) = \pm 1$

↳ In \mathbb{R}^2 , reflections have determinant -1 , rotations have $+1$

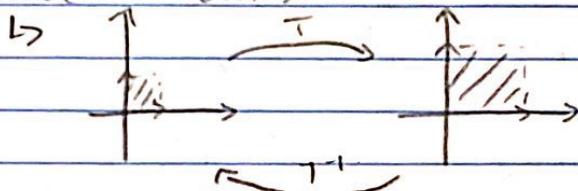
• Properties of determinants revisited:

↳ $\det(AB) = \det(A)\det(B)$



↳ in terms of scaling factors, scaling factor of AB = (scaling factor of A)
(scaling factor of B)

↳ $\det(A^{-1}) = \frac{1}{\det(A)}$



↳ The expansion factor of T^{-1} is $1/\text{expansion factor of } T$

• Eigenvalues and eigenvectors

↳ It is often easier to work with diagonal matrices

↳ Ex) $\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = A, B = \begin{bmatrix} \sim & & & \\ \sim & & & \\ \sim & & & \\ \sim & & & \end{bmatrix} \leftarrow \text{non-zero values}$

↳ Compute rank, basis for the kernel, determinant, A^6/B^6

↳ For A : $\text{rank}(A)=3 \rightarrow \# \text{ of non-zero entries on the diagonal}$

$\det(A)=0 \rightarrow$ given by product of diagonal entries

basis for $\ker(A)=\vec{e}_2 \rightarrow$ 2nd column has a free variable

↳ For B : 1st put in rref to find rank, determinant, kernel, but
cannot find the powers easily

↳ You can take powers of diagonal entries

↳ Goal: if a matrix is not diagonal, find a diagonal matrix that is similar to it

• Definition: $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear transformation represented by matrix A , then A is diagonalisable if there exists a basis $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ so that the B matrix is diagonal

↳ for all $\vec{v}_1, \dots, \vec{v}_n$: $A\vec{v}_i = \lambda_i \vec{v}_i$ for some $\lambda \in \mathbb{R}$

• Definition: A non-zero vector in \mathbb{R}^n is an eigenvector of T (or A) if $A\vec{v} = \lambda \vec{v}$ for some $\lambda \in \mathbb{R}$, where λ is an eigenvalue of T (or A)

↳ A basis $\vec{v}_1, \dots, \vec{v}_n$ of eigenvectors is called an eigenbasis

Recall from last time:

given an $n \times n$ matrix A we want to find an $n \times n$ matrix B that is diagonal and similar to it, or in other words, a basis v_1, \dots, v_n of \mathbb{R}^n such that

$$Av_i = \lambda_i v_i \quad (i=1, \dots, n)$$

for some $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

Definition

Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation represented by a matrix A .

A non-zero vector $v \in \mathbb{R}^n$ is called eigenvector of A (or T) if $Av = \lambda v$ for some $\lambda \in \mathbb{R}$.

In this case λ is called eigenvalue of A (or T).

A basis v_1, \dots, v_n of \mathbb{R}^n consisting of eigenvectors of A (or T) is called eigenbasis for A (or T). (2)

Proposition

A square matrix A is diagonalisable \Updownarrow

there exists a basis of eigenvectors for A .

Proof

Let v_1, \dots, v_n be a basis of eigenvectors and let $\lambda_1, \dots, \lambda_n$ be scalars s.t.

$$Av_i = \lambda_i v_i \quad \text{for } i=1, \dots, n.$$

Then $B = \begin{bmatrix} \lambda_1 & & \\ 0 & \ddots & \\ & & \lambda_n \end{bmatrix}$, $S = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}$

and $AS = SB$.

II

L3

Examples

1) Reflections in \mathbb{R}^2

- First: reflection about x-axis.

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2, T(v) = Av \text{ with } A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

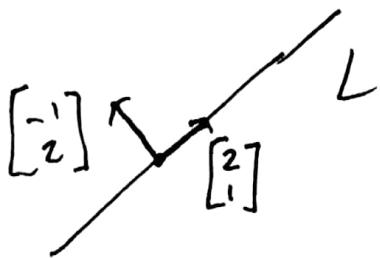
$$Ae_1 = e_1$$

$$\Rightarrow \lambda_1 = 1 \text{ and } \lambda_2 = -1$$

are the eigenvalues of A.

$$Ae_2 = -e_2$$

- Reflection about an arbitrary line L in \mathbb{R}^2
 $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$
e.g. $L = \text{span}(\begin{bmatrix} 2 \\ 1 \end{bmatrix})$



$$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = -\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

\Rightarrow $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ are eigenvectors
of T with eigenvalues 1 and -1
respectively.

$$S = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A = SBS^{-1}.$$

2) Orthogonal transformations

In general, 1 and -1 are the only eigenvalues of an orthogonal transformation:

Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an orthogonal transformation with matrix A , then $\|T(v)\| = \|v\|$ for all $v \in \mathbb{R}^n$ by definition.

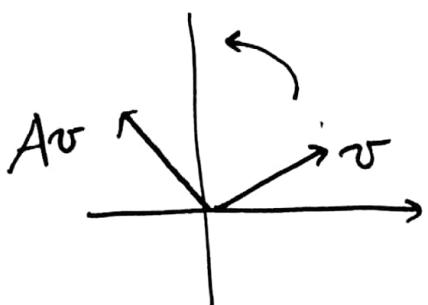
Let v be an eigenvector of A with associated eigenvalue λ , so that $Av = \lambda v$. L5

Then $\|v\| = \|Av\| = \|\lambda v\| = |\lambda| \|v\|$
 $\Rightarrow \lambda = \pm 1$.

3) Rotation in \mathbb{R}^2

e.g. $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $T(v) = Av = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}v$
 rotation through $\pi/2$ in counterclockwise direction.

Is A diagonalizable?



If v is any non-zero vector in \mathbb{R}^2 , then

Av fails to be parallel to v (it's perpendicular).

Thus there are no eigenvectors and A is not diagonalizable.

Note that the same is true for any
rotation $\overset{\text{in } \mathbb{R}^2}{\curvearrowleft}$ through an angle θ , as long as
 $\theta \neq k\pi$, $k \in \mathbb{Z}$. (6)

How to compute the eigenvalues of a matrix A

A an $n \times n$ matrix, $\lambda \in \mathbb{R}$.

λ is an eigenvalue of A \iff there exists a non-zero vector $v \in \mathbb{R}^n$ s.t.:

$$Av = \lambda v$$

\iff there exists $\neq 0 \neq v \in \mathbb{R}^n$ s.t.:

$$(A - \lambda I_n)v = 0$$

$\iff \text{Ker}(A - \lambda I_n) \neq \{0\}$

$\iff A - \lambda I_n$ is not invertible

L7

this is called
"characteristic
equation of A"

$$\Leftrightarrow \boxed{\det(A - \lambda I_n) = 0} \quad \textcircled{*}$$

Thus, to find the eigenvalues of A , we have to find the solutions to $\textcircled{*}$.

Example

i) $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

$$\begin{aligned} \det(A - \lambda I_2) &= \det \left(\begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} \right) \\ &= (1-\lambda)(3-\lambda) - 8 \\ &= \lambda^2 - 4\lambda - 5 = (\lambda-5)(\lambda+1) \end{aligned}$$

\Rightarrow eigenvalues of A : $\lambda_1 = 5$
 $\lambda_2 = -1$

$$2) \quad A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

18

$$\det(A - \lambda I_3) = (2-\lambda)(3-\lambda)(4-\lambda)$$

$$\Rightarrow \text{eigenvalues of } A: \quad \lambda_1 = 2 \\ \lambda_2 = 3 \\ \lambda_3 = 4$$

Proposition

Let A be any upper (or lower) triangular $n \times n$ matrix. Then the eigenvalues of A are its diagonal entries.

Important fact: $\det(A - \lambda I_n)$ is a L9
polynomial in λ of degree n , in other
words:

$$\det(A - \lambda I_n) = c_n \lambda^n + c_{n-1} \lambda^{n-1} + \dots + c_0$$

where $c_0, \dots, c_n \in \mathbb{R}$,

called characteristic polynomial of A

and denoted by $f_A(\lambda)$.

What does this tell us about the eigenvalues
of a matrix?

We know that a polynomial of degree n
has at most n real roots. Thus, our
 $n \times n$ matrix A has at most n eigenvalues.

Summary : to find the eigenvalues

of an $n \times n$ matrix A we

need to find the ~~real~~ real roots of the polynomial

$$f_A(\lambda) = \det(A - \lambda I_n).$$

Next time we will see how to compute the corresponding eigenvectors.

12/2 Lecture: Multiplicities

• Thursday 6-8pm in Boelter

• Recall: to find eigenvalues of an $n \times n$ matrix A , we need to solve for the solutions of $\det(A - \lambda I_n) = 0 \rightarrow$ characteristic eq. of $A = f_A(\lambda)$
 $\hookrightarrow = c_n \lambda^n + c_{n-1} \lambda^{n-1} + \dots + c_1 \lambda + c_0$ where $c_0, \dots, c_n \in \mathbb{R} \rightarrow$ characteristic polynomial of A

\hookrightarrow the eigenvalues of A are the roots of this polynomial

\hookrightarrow we have at most n eigenvalues (real)

• For polynomials, roots might have multiplicities greater than 1

$\hookrightarrow \lambda^2 \rightarrow$ the root 0 has a multiplicity of 2

• Definition: Let A be an $n \times n$ matrix. The algebraic multiplicity of an eigenvalue of the matrix is the multiplicity of λ_0 as a root of $f_A(\lambda)$, denoted by $\text{alm}_A(\lambda_0)$

\hookrightarrow In other words, if $\text{alm}_A(\lambda_0) = k$, then $f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda)$, where $g(\lambda)$ has degree $n-k$ and $g(\lambda_0) \neq 0$

• Ex) $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ eigenvalues: 0 w/ $\text{alm}_A(0) = 2$, 2 w/ $\text{alm}_A(2) = 1$, and -1 w/ $\text{alm}_A(-1) = 1$

• How to find eigenvectors of a matrix A

\hookrightarrow If $0 \neq \vec{v} \in \mathbb{R}^n$ is an eigenvector of a matrix A with eigenvalue λ , then $A\vec{v} = \lambda\vec{v} \rightarrow (A - \lambda I_n)\vec{v} = 0$

\hookrightarrow Find $\ker(A - \lambda I_n)$

• Definition: Let λ be an eigenvalue of a $n \times n$ matrix A , then $\ker(A - \lambda I_n)$ is the eigenspace of λ denoted by E_λ

\hookrightarrow The dimension of E_λ is called geometric multiplicity of λ , denoted by $\text{gemu}(\lambda)$

\hookrightarrow Ex) $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ has eigenvalues 5 and -1

\hookrightarrow The corresponding eigenspaces are $E_5 = \ker(A - 5I_2) = \ker\left(\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ and $E_{-1} = \ker(A - (-1)I_2) = \ker\left(\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$

$\hookrightarrow \text{gemu}(E_5) = 1, \text{gemu}(E_{-1}) = 1$

\hookrightarrow the vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ form an eigensbasis for A

$\hookrightarrow A$ is diagonalisable, $S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$ and $AS = SB$

$$\hookrightarrow E_x) A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \det(A) = (1-\lambda)(-\lambda)(1-\lambda), \text{ eigenvalues: } 0 \text{ with } \text{algebraic multiplicity } 1, \\ 1 \text{ with } \text{algebraic multiplicity } 2$$

$$\hookrightarrow E_1 = \ker(A - I_n) = \ker\left(\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right), \text{geometric multiplicity } 1$$

$$\hookrightarrow E_0 = \ker(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), \text{geometric multiplicity } 1$$

\hookrightarrow A is not diagonalisable because there are only 2 linearly independent eigenvectors

• Important fact. Let $\lambda_1, \dots, \lambda_m$ be different eigenvalues of a matrix

A. Compute a basis for each eigenspace $E_{\lambda_1}, \dots, E_{\lambda_m}$ and concatenate these vectors $\vec{v}_1, \dots, \vec{v}_s$

\hookrightarrow We have s vectors, where s is the sum of the dimensions of the eigenspaces (geometric multiplicities)

• Proposition: An $n \times n$ matrix is diagonalisable if the dimensions of the eigenspaces sum up to n

• Special case: If a matrix has all eigenvalues with $\text{algebraic multiplicity } 1$, then it is diagonalisable

12/4 Lecture: Diagonalization

• Recall:

↳ introduction of algebraic multiplicity and geometric multiplicity

↳ Note: $\text{genu}(\lambda) \leq \text{almu}(\lambda)$

↳ $\text{genu}(\lambda) \leq \text{almu}(\lambda)$

↳ an $n \times n$ matrix A is diagonalisable when $\sum \dim(E_\lambda) = n$

↳ special case: if a matrix has n different eigenvalues, then it is diagonalisable

↳ $n \times n$ matrix A is diagonalisable when the matrix has n eigenvalues and $\text{genu}(\lambda) = \text{almu}(\lambda)$ for all λ

↳ Ex)

$$A = \begin{bmatrix} 0 & 1 & * \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{aligned} f_A(\lambda) &= \det(A - \lambda I_3) \\ &= \lambda(1-\lambda)^2(2-\lambda)(3-\lambda) \end{aligned}$$

↳ $\lambda = 0, 1, 2, 3 \rightarrow \text{almu}(1) = 2$

↳ Summary of diagonalisation

↳ Given an $n \times n$ matrix A , we say that A is diagonalisable if there exists an invertible matrix S such that $S^{-1}AS$ is diagonal

↳ 1) find the eigenvalues of A : the real roots of $f_A(\lambda) = \det(A - \lambda I_n)$

2) for each eigenvalue λ , compute its eigenspace $E_\lambda = \ker(A - \lambda I_n)$
find a basis for E_λ

3) A is diagonalisable when $\sum \dim(E_\lambda) = n$

↳ if A is diagonalisable, concatenate all bases for the eigenspaces
to obtain $\vec{v}_1, \dots, \vec{v}_n \rightarrow$ eigenbasis

↳ then,

$$S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix}, S^{-1}AS = \begin{bmatrix} \lambda_1 & & & \\ 0 & \ddots & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

• Ex) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, A has 2 eigenvalues: $0, \text{almu}(0) = 2$ and $3, \text{almu}(3) = 1$

$$\lambda_1 = 0, \lambda_2 = 3$$

$$S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}, S^{-1}AS = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$$

• Question: for which $n \times n$ matrices A is there an orthogonal matrix S such that $S^{-1}AS$ is diagonal

↳ Notes:

↳ in this case, we could have an orthonormal basis of eigenvectors

↳ the eigenspaces corresponding to different eigenvalues are orthogonal

↳ Assume that $S^{-1}AS$ (orthogonal) = B (diagonal)

$$\hookrightarrow A = SBS^{-1} = SBS^T$$

$$\hookrightarrow A^T = (SBS^T)^T = (S^T)^T B^T S^T = SBS^T = A$$

$\hookrightarrow A = A^T \rightarrow$ symmetric matrices

• Definition: A square matrix A such that $A = A^T$ is symmetric

↳ Ex) All 2×2 symmetric matrices are of the form $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

• Theorem: Given an $n \times n$ matrix A , there exists S orthogonal such that $S^{-1}AS$ is diagonal iff A is symmetric

• 2 important facts

↳ any symmetric matrix is diagonalisable

↳ for any symmetric matrix, if λ_1 and λ_2 are different eigenvalues, then E_{λ_1} and E_{λ_2} are orthogonal

• General strategy: given a symmetric $n \times n$ matrix A , how do we find S orthogonal w/ $S^{-1}AS$ diagonal

↳ 1) Compute the eigenvalues of A

↳ 2) Compute corresponding eigenspaces of A

↳ 3) For each eigenspace, choose a basis and apply Gram-Schmidt to obtain an orthonormal basis

↳ 4) Concatenate all orthonormal bases for the eigenspaces and obtain an orthonormal basis of \mathbb{R}^n

Matrices and linear systems

A linear system with 2 equations in variables x_1, x_2, x_3, x_4, x_5 :

$$x_2 + x_3 = 2$$

$$x_1 + x_2 + x_3 - x_4 = 0$$

Augmented coefficient matrix

$$\left[\begin{array}{ccccc|c} 0 & 1 & 1 & 0 & 0 & 2 \\ 1 & 1 & 1 & -1 & 0 & 0 \end{array} \right] = A | b$$

Put $A | b$ in RREF using elementary row operations

From top to bottom

$$\left[\begin{array}{ccccc|c} 0 & 1 & 1 & 0 & 0 & 2 \\ 1 & 1 & 1 & -1 & 0 & 0 \end{array} \right]$$

swap I and II

\rightsquigarrow

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 2 \end{array} \right]$$

Subtract

II from I



$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 0 & -2 \\ 0 & 1 & 1 & 0 & 0 & 2 \end{array} \right]$$

Solve for leading variables:

$$x_1 = x_4 - 2$$

$$x_2 = -x_3 + 2$$

Write solution by plugging in arbitrary values for free variable

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} r-2 \\ -s+2 \\ s \\ r \\ t \end{bmatrix}$$

with $s, t, r \in \mathbb{R}$

→ infinitely many solutions.

Matrices as representing linear transformation

function

A $\sqrt{T: \mathbb{R}^m \rightarrow \mathbb{R}^n}$ is a linear transformation



there is an $n \times m$ matrix A such that

for any $v \in \mathbb{R}^m : T(v) = Av.$

Given A , find T .

e.g. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, then

$$T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3 \quad T(v) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 + v_2 \\ v_1 + 2v_2 - v_3 \end{bmatrix}$$

Given T , find A.

e.g. $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$

with $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 + 2v_2 \\ 0 \\ -v_1 \end{bmatrix} \Rightarrow A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}$$

• Composition of linear transformations \leadsto multiplication of matrices

$T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ repr. by $n \times n$ matrix A

$S: \mathbb{R}^n \rightarrow \mathbb{R}^p$ repr. by $m \times p$ matrix B

$S \circ T: \mathbb{R}^m \rightarrow \mathbb{R}^p$ repr. by BA

• Inverse linear transformation \leadsto inverse matrix.
 $\Leftrightarrow \text{im}(T) = \mathbb{R}^n$

$T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ invertible $\Leftrightarrow \text{rank}(T) = n$.

($\Leftrightarrow \dim(\text{im } T) = n$)
 $\Leftrightarrow \text{ker}(T) = \{0\}$)

To find inverse : put A in RREF:

$$[A \mid I_n] \xrightarrow{\text{Gauss-Jordan}} [I_n \mid B]$$

inverse of A .

Image, kernel, bases, dimension

$$T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

VI with $T(v) = Av$ VI

$$\text{Ker}(T) = \text{Ker}(A)$$

"

$$\{v \in \mathbb{R}^m : Av = 0\}$$

$$\text{im}(T) = \text{im}(A)$$

"

$\{w \in \mathbb{R}^n : \text{there is } v \in \mathbb{R}^m \text{ with } Av = w\}$

$$\{T(v) : v \in \mathbb{R}^m\}$$

e.g. $T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\text{im}(T)$ consists of all vectors of the form

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} v_1 + \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} v_2 + \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} v_3$$

$v_1, v_2, v_3 \in \mathbb{R}$

so $\text{im}(T)$ is a plane in \mathbb{R}^4 passing through the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$ and the origin.

in general : $\text{im}(T) = \text{span}(\mathbf{a}_1, \dots, \mathbf{a}_m)$
where $\mathbf{a}_1, \dots, \mathbf{a}_m$ are the columns of A

What is the smallest amount of vectors that spans $\text{im}(T)$?

as basis : basis of a subspace V of \mathbb{R}^n
is a collection of vectors that spans V
and such that this collection is minimal.

$\leadsto \dim V = \text{nr. of vectors in a basis for } V.$

To compute basis for $\text{im } T$: • put A in RREF

- a basis is given by column vectors in A corresponding to columns in RREF A that contain pivots

— $\text{Ker } T$: • solve system $A\mathbf{x} = \mathbf{0}$

- write solution set in vector form

e.g. $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\rightsquigarrow \text{REF}(A) = \left[\begin{array}{ccc|cc} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$

basis for $\text{im}(T)$: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$

basis for $\text{ker}(T)$: solutions to $Ax = 0$

are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ 0 \\ t \end{bmatrix}, t \in \mathbb{R}$$

$$= t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \text{ basis for } \text{ker}(T)$$

Rank - nullity

$T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ repn. by A

$\text{rank}(T) = \dim(\text{im } T) = \text{nr. of pivots in RREF}(A)$

$\text{nullity}(T) = \dim(\text{Ker } T) = m - \text{rank}(T) = \text{nr. of free variables in RREF}(A).$

$$\text{~m)~} m = \text{rank}(T) + \text{nullity}(T)$$

$$= \dim(\text{im } T) + \dim(\text{Ker } T)$$

Coordinates for a subspace V of \mathbb{R}^n

$\mathcal{B} = \{v_1, \dots, v_m\}$ a basis of $V \subseteq \mathbb{R}^n$

then there exist unique numbers c_1, \dots, c_m
s.t

$$w = c_1v_1 + \dots + c_mv_m \quad \forall w \in V.$$

these are called \mathcal{B} -coordinates of w .

e.g. $V = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right)$

$$w = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & 4 \\ 0 & 0 & 0 \\ 1 & 0 & 3 \end{array} \right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array} \right]$$

$$\Rightarrow [w]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Coordinates for \mathbb{R}^n

$\mathcal{B} = \{v_1, \dots, v_n\}$ a basis of \mathbb{R}^n . Then there exist unique numbers $c_1, c_2, \dots, c_n \in \mathbb{R}$ s.t.

$$w = c_1 v_1 + \dots + c_n v_n$$

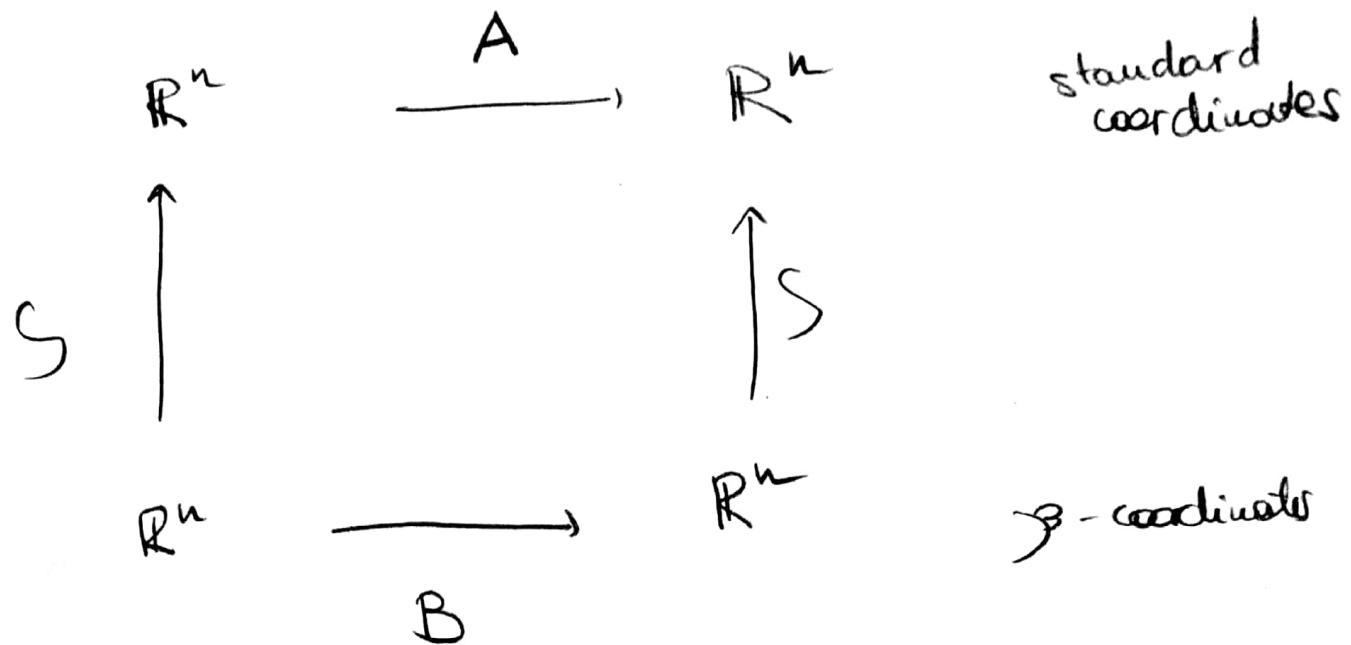
these are called \mathcal{B} -coordinates of w ,

written

$$[w]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Given $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$, we can represent T in the \mathcal{B} -coordinates with the matrix

$$\mathcal{B} = \begin{bmatrix} & & & & & & & \\ & [T(v_1)]_{\mathcal{B}} & \cdots & [T(v_n)]_{\mathcal{B}} & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix}$$



change of basis matrix

$$S = \begin{bmatrix} 1 & \dots & 1 \\ v_1 & \dots & v_n \\ 1 & \dots & 1 \end{bmatrix}$$

We have:

$$\underbrace{S}_{\text{and}} \underbrace{[w]_{\mathcal{g}}}_{\text{"}} = w \quad \text{for all } w \in \mathbb{R}^n$$

and

for all $w \in \mathbb{R}^n$

$$AS[w]_{\mathcal{g}} = S B[w]_{\mathcal{g}}$$

$$\boxed{AS = SB}$$

A and B are similar

Way to think about similar matrices:

they represent the same linear transformation under different choices of basis.

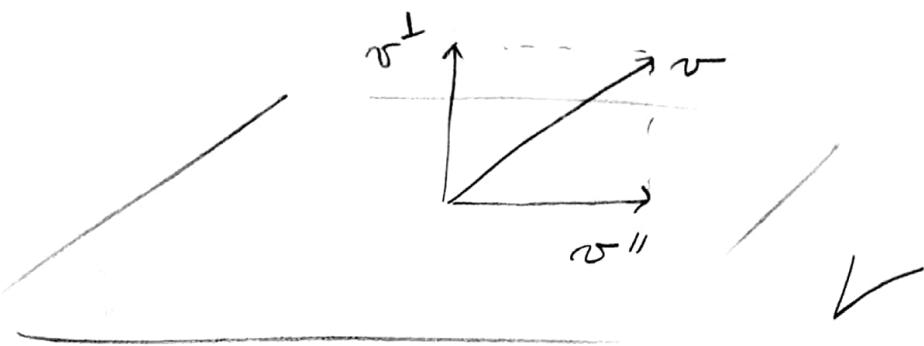
Example: given ω , how to find $[\omega]_{\beta}$?

B: $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ $\omega = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 3 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$\rightsquigarrow [\omega]_{\beta} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

Orthogonal projection onto a subspace V of \mathbb{R}^n



Any $v \in \mathbb{R}^n$ can be uniquely written as $v = v'' + v^\perp$ with $v'' \in V$ v^\perp orthogonal to V .

$v'' = \text{proj}_V(v)$ is called orthogonal projection of v onto V .

Formula for orthogonal projection.

u_1, \dots, u_m ONB of V , then

$$\text{proj}_V(v) = (u_1 \cdot v)u_1 + \dots + (u_m \cdot v)u_m$$

$\forall v \in \mathbb{R}^n$

$$\text{proj}_V : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$\text{im } (\text{proj}_V) = V$$

$$\Rightarrow \dim(V) + \dim(V^\perp) = n.$$

$$\text{Ker } (\text{proj}_V) = V^\perp$$

↪ any subspace V of \mathbb{R}^n
can be realized as
the Kernel or
image of a
linear transf.

Example:

Given V , how to find V^\perp ?

$$x \in V^\perp \iff x \cdot v = 0 \quad \forall v \in V.$$

↑↑

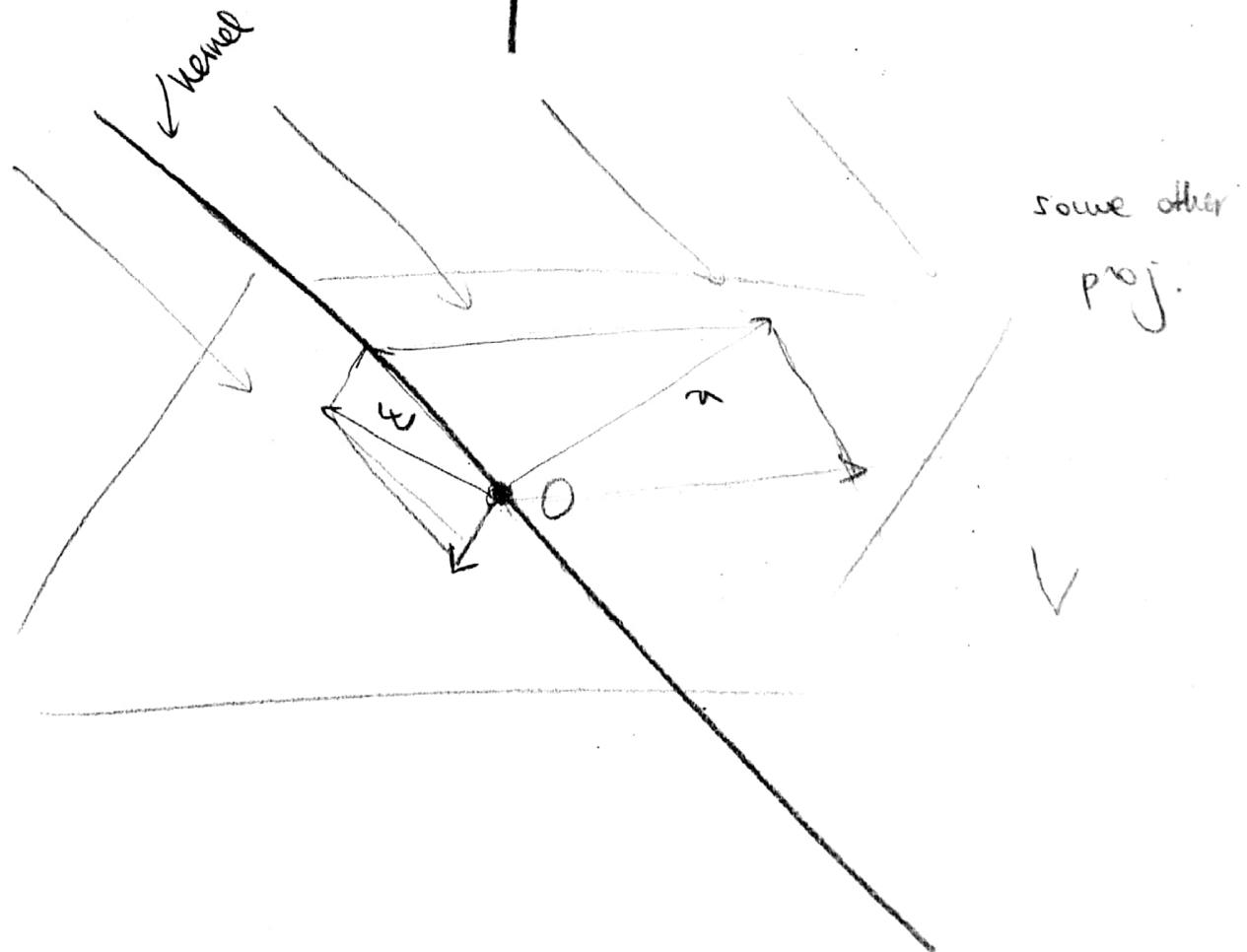
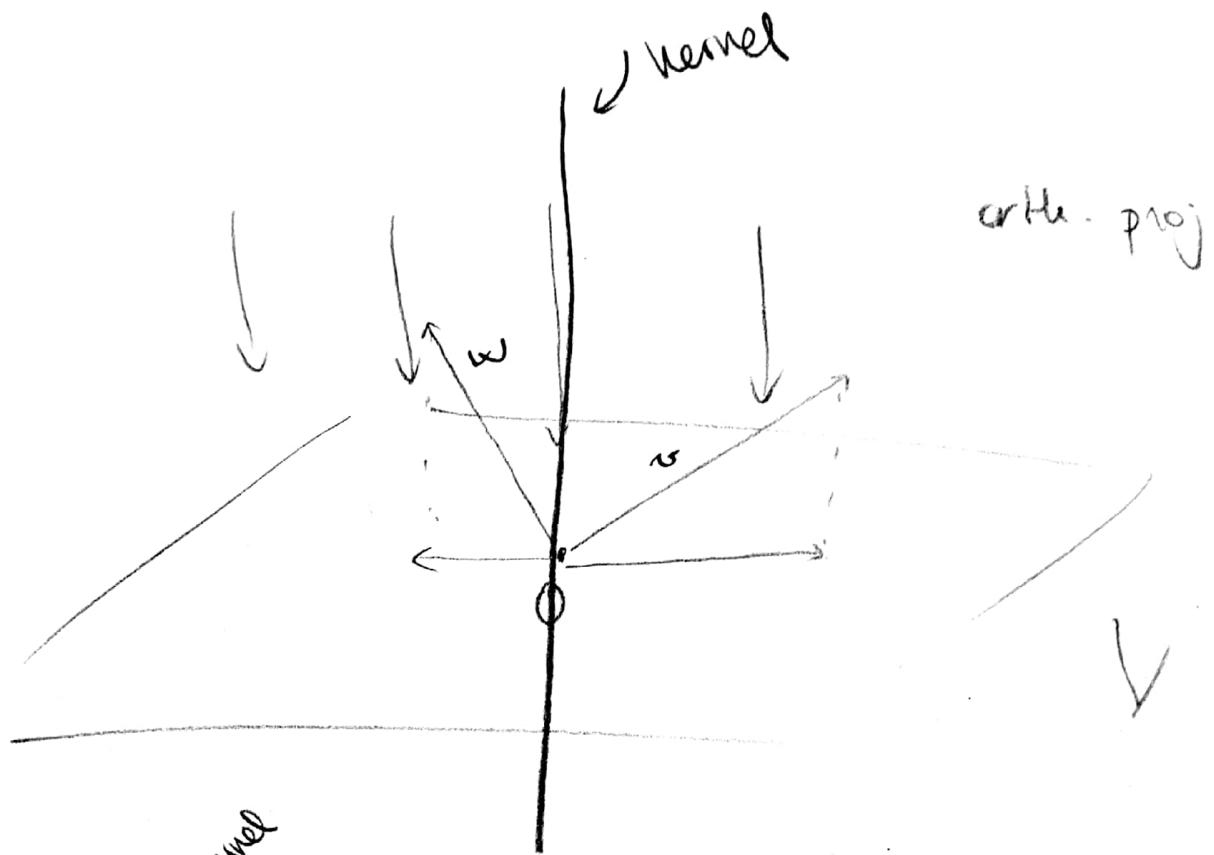
$$x \cdot v_i = 0 \quad \forall v_i \text{ vectors in a basis of } V$$

Find solutions to

$$\sim \begin{bmatrix} -v_1 & - \\ \vdots & \vdots \\ -v_m & - \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$$

↓

Note: the reason that proj_V is called "orthogonal"
is that it is the ^{only} projection onto V such that
its image and kernel are orthogonal to each other



Gram-Schmidt algo.: given a basis $\mathcal{B} = \{b_1, \dots, b_m\}$

for a subspace V of \mathbb{R}^n , returns an ONB $\mathcal{U} = \{u_1, \dots, u_m\}$ of V .

↓ application

QR Factorization

any $n \times m$ matrix M w/

linearly independent column vectors can be written

$$M = Q R$$

$n \times m$ matrix
with orthonormal
columns

upper-triangular
matrix w/
positive diagonal
entries.

starting basis

ONB basis

$$\begin{bmatrix} 1 & & \\ b_1 & \dots & b_m \\ 1 & & 1 \end{bmatrix}$$

$n \times m$ matrix
with lin. ind.
columns

$$\begin{bmatrix} 1 & & \\ [b_1]_2 & \dots & [b_m]_2 \\ 1 & & 1 \end{bmatrix}$$

$$= I_m$$

$$\begin{bmatrix} 1 & & \\ u_1 & \dots & u_m \\ 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ [b_1]_2 & \dots & [b_m]_2 \\ 1 & & 1 \end{bmatrix}$$

Q

R

Orthogonal transformations : reflections, rotations
and composition thereof.

(called like this because they preserve right angles)

important : $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ orthog $\Leftrightarrow T(e_1), \dots, T(e_n)$
ONB of \mathbb{R}^n .

A orthogonal matrix \Leftrightarrow columns of
 A form ONB of \mathbb{R}^n .

$$\Leftrightarrow A^{-1} = A^t$$

- $\det(A) = \pm 1$
- eigenvalues of A if they exist : ± 1 .

[Orthogonal projection is not an orthogonal transformation]

Determinants.

How to compute :

- Sarrus' rule for 3×3 matrices
- $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$
- Using Gauss-Jordan for $n \times n$ matrices:
Suppose that to go from A to $RREF(A)$ we swap rows s times and divide various rows by the non-zero numbers k_1, \dots, k_r .

Then $\det(A) = (-1)^s k_1 \dots k_r \det(RREF(A))$

$$= \begin{cases} (-1)^s k_1 \dots k_r & \text{if } A \text{ invertible} \\ 0 & \text{if } A \text{ not invertible.} \end{cases}$$

Properties: $\det(A) = 0 \Leftrightarrow A \text{ not invertible}$

$$\det \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = \text{product of diagonal entries.}$$

Geometric interpretation:

- as volume and area
- as expansion factor.

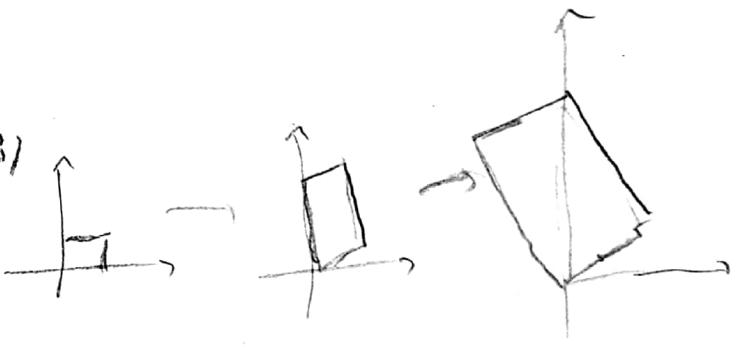
Note: if A and B are similar, then
 $\det(A) = \det(B)$

If $\det(A) = \det(B)$, are A and B similar?

No: take two different linear transformation with the same expansion factor.

Its geometric interpretation also helps to remember properties:

$$\det(AB) = \det(A) \det(B)$$



$$\det(A) = 0 \Leftrightarrow A \text{ not inv'ble}$$



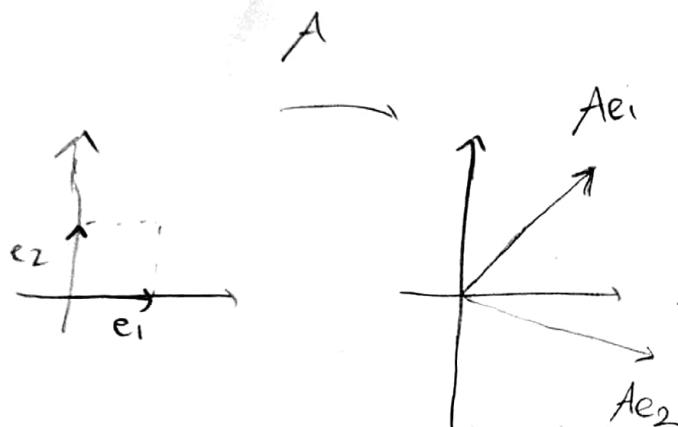
$$\det(A) = \frac{1}{\det(A)}$$



Eigenvalues and eigenvectors

geometrically

$$T: \mathbb{R}^n \rightarrow \mathbb{R}^n$$



are there vectors $v \in \mathbb{R}^n$

for which Av is parallel to v ? So vectors whose direction is not changed by the linear transformation?

If we can find n linearly independent vectors like this, then the matrix of T in this basis will be diagonal.

\ algebraically

eigenvalues: roots of
 $f_A(\lambda) = \det(A - \lambda I_n)$

eigen spaces: for λ one eigenvalue of A
 $E_\lambda = \text{Ker}(A - \lambda I_n)$

$\dim(E_\lambda) = \text{geom. mult. of } \lambda$

multiplicity of λ as a root of $f_A(\lambda) = \text{alg. mult. of } \lambda$.

if there are n eigenvalues $\lambda_1, \dots, \lambda_n$ and $\text{geom. mult. } \lambda_i = \text{alg. mult. } \lambda_i$ for each eigenvalue λ_i , then

$$S = \begin{bmatrix} v_1 & \dots & v_n \\ 0 & \dots & 0 \\ \vdots & & \vdots \end{bmatrix} \text{ and } S^{-1}AS = \begin{bmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_n & \\ & & & 0 \end{bmatrix}$$

- Symmetric matrices :
- are always diagonalisable
 - we can find ONB basis of eigenvectors, so that S is an orthogonal matrix.
 - eigenspaces are orthogonal to each other.

Examples : reflections, orthogonal projections.

Why should we care?

geometry : insights into what the linear transformation does

"uncoupling" of information

$$\text{e.g. } \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{vs. } \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{array}{c} x_1 \\ x_2 \end{array} \xrightarrow{\quad} \begin{array}{c} y_1 \\ y_2 \end{array}$$

$$\begin{array}{l} x_1 - y_1 \\ x_2 - y_2 \end{array}$$

algebra : it's easier to work with diagonal matrices, and

if A and B similar, then

- $\text{rank}(A) = \text{rank}(B)$
- $\det(A) = \det(B) \rightsquigarrow \det(A)$
- $A^k = S B^k S^{-1}$
- $\text{nullity}(A) = \text{nullity}(B)$

$$A = Q \Lambda Q^{-1} \quad \text{or} \quad A = P D P^{-1}$$

④ If S invertible, then $\text{rank}(AS) = \text{rank}(A)$
 $= \text{rank}(SA)$

for any matrix A

Similarly, nullity $(AS) = \text{nullity}(SA) = \text{nullity}(A)$