Math 33A Sheet 7

Chapter 5.1

Ex 17)* $W = \text{span}\left(\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}5\\6\\7\\8\end{bmatrix}\right)$

$W^\perp \to$ all vectors perpendicular to W/ basis of W

$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} \cdot \begin{bmatrix}1\\2\\3\\4\end{bmatrix} = 0$ and $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} \cdot \begin{bmatrix}5\\6\\7\\8\end{bmatrix} = 0$

$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$

$5x_1 + 6x_2 + 7x_3 + 8x_4 = 0$

$-4x_2 - 8x_3 - 12x_4 = 0$

$x_2 + 2x_3 + 3x_4 = 0$

$x_1 \quad - x_3 - 2x_4 = 0$

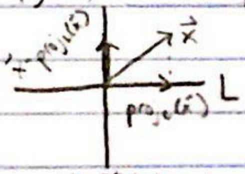
$x_3 = t, \; x_4 = s$

$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}t+2s\\-2t-3s\\t\\s\end{bmatrix}$

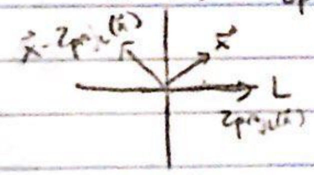
$= t\begin{bmatrix}1\\-2\\1\\0\end{bmatrix} + s\begin{bmatrix}2\\-3\\0\\1\end{bmatrix}$

$\boxed{\text{basis of } U^\perp = \begin{bmatrix}1\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}2\\-3\\0\\1\end{bmatrix}}$

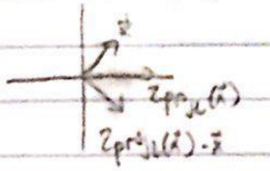
Ex 19) a) $T(\vec{x}) = \vec{x} - \text{proj}_L(\vec{x})$



b) $T(\vec{x}) = \vec{x} - 2\text{proj}_L(\vec{x})$



c) $T(\vec{x}) = 2\text{proj}_L(\vec{x}) - \vec{x}$



Chapter 5.2

Ex 1) $\begin{bmatrix}2\\1\\-2\end{bmatrix}$

$\vec{u}_1 = \frac{\vec{v}}{\|\vec{v}\|}$

$\vec{u}_1 = \frac{\vec{v}}{3} = \frac{1}{3}\vec{v}$

$\boxed{\vec{u}_1 = \begin{bmatrix}2/3\\1/3\\-2/3\end{bmatrix}}$

Ex 10)* $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}6\\4\\6\\4\end{bmatrix}$

$\boxed{\vec{u}_1 = \frac{\vec{v}_1}{2} = \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}}$

$L = \text{span}(\vec{v}_1)$

$\text{proj}_L(\vec{v}_2) = (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = 10\vec{u}_1 = 5\vec{v}_1$

$\vec{v}_2^\perp = \vec{v}_2 - 5\vec{v}_1 = \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$

$\vec{u}_2 = \frac{\vec{v}_2^\perp}{\|\vec{v}_2^\perp\|} = \frac{1}{2}v_2^\perp$

$\boxed{\vec{u}_2 = \begin{bmatrix}1/2\\-1/2\\1/2\\-1/2\end{bmatrix}}$

Ex 16) $\begin{bmatrix}6 & 2\\3 & -6\\2 & 3\end{bmatrix}$

$\vec{u}_1 = \frac{\vec{v}_1}{7} = \frac{1}{7}\begin{bmatrix}6\\3\\2\end{bmatrix}$

$\text{proj}_L(\vec{v}_2) = (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = (\frac{1}{7}(0))\vec{u}_1 = \vec{0}$

$\vec{v}_2^\perp = \vec{v}_2 = \begin{bmatrix}2\\-6\\3\end{bmatrix}$

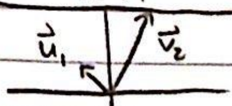
$\vec{u}_2 = \frac{\vec{v}_2^\perp}{\|\vec{v}_2^\perp\|} = \frac{\vec{v}_2^\perp}{7} = \frac{1}{7}\vec{v}_2^\perp$

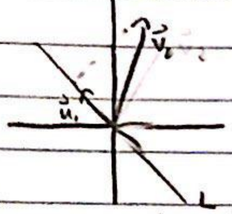
$\boxed{Q = \frac{1}{7}\begin{bmatrix}6 & 2\\3 & -6\\2 & 3\end{bmatrix}}$

$\boxed{R = \begin{bmatrix}7 & 0\\0 & 7\end{bmatrix}}$

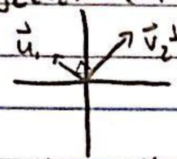
Ex29) $\vec{v}_1=\begin{bmatrix}-3\\4\end{bmatrix}, \vec{v}_2=\begin{bmatrix}1\\7\end{bmatrix}$

$$\boxed{\vec{u}_1=\frac{\vec{v}_1}{\|\vec{v}_1\|}=\frac{1}{5}\begin{bmatrix}-3\\4\end{bmatrix}}$$





$proj_L(\vec{v}_2)=(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1=5\vec{u}_1=\vec{v}_1$



$\vec{v}_2-\vec{v}_1=\begin{bmatrix}4\\3\end{bmatrix}=\vec{v}_2^{\perp}$

$$\boxed{\vec{u}_2=\frac{1}{5}\vec{v}_2^{\perp}=\frac{1}{5}\begin{bmatrix}4\\3\end{bmatrix}}$$

Ex 32) $x_1+x_2+x_3=0$

$x_1=-x_2-x_3$

$x_1=-2 \quad x_2=2, x_3=0$

$x_1=-2 \quad x_2=0, x_3=2$

$\text{span}\left(\begin{bmatrix}-2\\2\\0\end{bmatrix},\begin{bmatrix}-2\\0\\2\end{bmatrix}\right)$

$$\boxed{\vec{u}_1=\frac{1}{2}\vec{v}_1=\begin{bmatrix}-1\\1\\0\end{bmatrix}}$$

$proj_L(\vec{v}_2)=(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1=(2)\vec{u}_1=\vec{v}_1$

$\vec{v}_2^{\perp}=\vec{v}_2-\vec{v}_1=\begin{bmatrix}0\\-2\\2\end{bmatrix}$

$$\boxed{\vec{u}_2=\frac{1}{2}\vec{v}_2^{\perp}=\begin{bmatrix}0\\-1\\1\end{bmatrix}}$$

Ex33)* $A=\begin{bmatrix}1&1&1&1\\1&-1&-1&1\end{bmatrix}$

$=\begin{bmatrix}1&1&1&1\\0&-2&-2&0\end{bmatrix}$

$=\begin{bmatrix}1&1&1&1\\0&1&1&0\end{bmatrix}$

$=\begin{bmatrix}1&0&0&1\\0&1&1&0\end{bmatrix}$

$x_1+x_4=0$

$x_2+x_3=0$

$x_4=s, x_3=t$

$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-s\\-t\\t\\s\end{bmatrix}=t\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}+s\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$

$\vec{v}_1=\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}, \vec{v}_2=\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}$

$$\boxed{\vec{u}_1=\frac{v_1}{\sqrt{2}}=\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}}$$

$proj_L(\vec{v}_2)=(\vec{u}_1\cdot\vec{v}_2)\vec{u}_1=0$

$\vec{v}_2^{\perp}=\vec{v}_2$

$$\boxed{\vec{u}_2=\frac{\vec{v}_2^{\perp}}{\sqrt{2}}=\frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}}$$

Ex 36) $M = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 5 \\ 0 & -4 & 6 \\ 0 & 0 & 7 \end{bmatrix}$

↳ Q columns must be orthonormal ✓

↳ R diagonal must be positive ✗

↳ R must be upper diagonal ✓

Change signs of Q col. #2 and R row #2

↳ $Q = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}, R = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & -6 \\ 0 & 0 & 7 \end{bmatrix}$

$$\boxed{M = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & -6 \\ 0 & 0 & 7 \end{bmatrix}}$$

Ex 38)* $A = \begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Columns are orthogonal

$\vec{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$$\boxed{Q = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}}$$

Ex 39)* $\text{span}(\vec{u}_1) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$

$\text{span}(\vec{u}_1, \vec{u}_2) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right)$

$\vec{V}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{V}_2^{\perp} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$$\boxed{\begin{aligned} \vec{u}_1 &= \frac{V_1}{\sqrt{14}} = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ \vec{u}_2 &= \frac{\vec{V}_2^{\perp}}{\sqrt{3}} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \end{aligned}}$$

$\vec{u}_1 \times \vec{u}_2 = \vec{u}_3$ → cross product is orthonormal

$$= \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1/\sqrt{14} & 2/\sqrt{14} & 3/\sqrt{14} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix} = \frac{1}{\sqrt{42}}\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{42}}\left[\vec{i}\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}\right]$$

$$= \frac{1}{\sqrt{42}}\left[\vec{i}(-5) - \vec{j}(-4) + \vec{k}(-1)\right]$$

$$\boxed{\vec{u}_3 = -\frac{1}{\sqrt{42}}\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}}$$

Q1) i) $A \stackrel{?}{=} S^{-1}AS$

let $S = I_n$

↳ invertible

$A = I_n A I_n$

↳ $A = A$ ✓

↳ I_n is a neutral element

ii) $A = S^{-1}BS$ ✓

$B = S^{-1}AS$?

$B = S^{-1}(S^{-1}BS)S$

$B = S^{-1}S^{-1}BSS$

$B = B$ ✓

iii) $A = S^{-1}BS$

$B = S^{-1}CS$

∴ $A \stackrel{?}{=} S^{-1}(S^{-1}CS)S$

$A = S^{-1}S^{-1}CSS$

$A = C$ ✓