Each of Exercises 10–15 provides a general solution of $\mathbf{y}' = A\mathbf{y}$, for some A. Without the help of a computer or a calculator, sketch the half-line solutions generated by each exponential term of the solution. Then, sketch a rough approximation of a solution in each region determined by the half-line solutions. Use arrows to indicate the direction of motion on all solutions. Classify the equilibrium point as a saddle, a nodal sink, or a nodal source.

10. $\mathbf{y}(t) = C_1 e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

11. $\mathbf{y}(t) = C_1 e^{t}\begin{pmatrix} -1 \\ -2 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

12. $\mathbf{y}(t) = C_1 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

13. $\mathbf{y}(t) = C_1 e^{-3t}\begin{pmatrix} -4 \\ 1 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

14. $\mathbf{y}(t) = C_1 e^{-t}\begin{pmatrix} -5 \\ 2 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

15. $\mathbf{y}(t) = C_1 e^{3t}\begin{pmatrix} 4 \\ 1 \end{pmatrix} + C_2 e^{t}\begin{pmatrix} 1 \\ 5 \end{pmatrix}$

In Exercises 16–19, verify that the equilibrium point at the origin is a center by showing that the real parts of the system's complex eigenvalues are zero. In each case, calculate and sketch the vector generated by the right-hand side of the system at the point $(1, 0)$. Use this to help sketch the elliptic solution trajectory for the system passing through the point

$(1, 0)$. Draw arrows on the solution, indicating the direction of motion. Use your numerical solver to check your result.

16. $\mathbf{y}' = \begin{pmatrix} -4 & 8 \\ -4 & 4 \end{pmatrix} \mathbf{y}$

17. $\mathbf{y}' = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \mathbf{y}$

18. $\mathbf{y}' = \begin{pmatrix} 2 & 2 \\ -4 & -2 \end{pmatrix} \mathbf{y}$

19. $\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{y}$

In Exercises 1–12, classify the equilibrium point of the system $\mathbf{y}' = A\mathbf{y}$ based on the position of (T, D) in the trace-determinant plane. Sketch the phase portrait by hand. Verify your result by creating a phase portrait with your numerical solver.

1. $A = \begin{pmatrix} 8 & 20 \\ -4 & -8 \end{pmatrix}$

2. $A = \begin{pmatrix} -16 & 9 \\ -18 & 11 \end{pmatrix}$

3. $A = \begin{pmatrix} 2 & -4 \\ 8 & -6 \end{pmatrix}$

4. $A = \begin{pmatrix} 8 & 3 \\ -6 & -1 \end{pmatrix}$

5. $A = \begin{pmatrix} -11 & -5 \\ 10 & 4 \end{pmatrix}$

6. $A = \begin{pmatrix} 6 & -5 \\ 10 & -4 \end{pmatrix}$

7. $A = \begin{pmatrix} -7 & 10 \\ -5 & 8 \end{pmatrix}$

8. $A = \begin{pmatrix} 4 & 3 \\ -15 & -8 \end{pmatrix}$

9. $A = \begin{pmatrix} 3 & 2 \\ -4 & -1 \end{pmatrix}$

10. $A = \begin{pmatrix} -5 & 2 \\ -6 & 2 \end{pmatrix}$

Use Definition 6.5 to calculate e^A for the matrices in Exercises 1–4.

1. $A = \begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix}$

2. $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

4. $A = \begin{pmatrix} -2 & 1 & -3 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

5. Suppose that the matrix A satisfies $A^2 = \alpha A$, where $\alpha \neq 0$.

(a) Use Definition 6.5 to show that

$$e^{tA} = I + \frac{e^{\alpha t} - 1}{\alpha} A.$$

(b) Use part (a) to compute e^{tA} for

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

6. There are many important series in mathematics, such as the exponential series. For example,

$$\cos t = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} = 1 - \frac{t^2}{2!} - \frac{t^4}{4!} + \cdots \quad \text{and}$$

$$\sin t = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots .$$

Use these infinite series together with Definition 6.5 to show that

$$e^{t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

7. Use the result of Exercise 6 to show that if

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

then

$$e^{tA} = e^{at} \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix}.$$

Hint: $A = aI + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

8. If

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix},$$

find e^{tA}. *Hint*: See the hint for Exercise 7.

9. Let

$$A = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}.$$

(a) Show that $AB \neq BA$.

(b) Evaluate e^{A+B}. *Hint*: This is a simple computation if you use Exercise 7.

(c) Use Definition 6.5 to evaluate e^A and e^B. Use these results to compute e^Ae^B and compare this with the result found in part (b). What have you learned from this exercise?

10. If $A = PDP^{-1}$, prove that $e^{tA} = Pe^{tD}P^{-1}$.

Use the results of Exercise 53 of Section 9.1 and Exercise 10 to calculate e^{tA} for each matrix in Exercises 11–12.

11. $A = \begin{pmatrix} -2 & 6 \\ 0 & -1 \end{pmatrix}$ **12.** $A = \begin{pmatrix} -2 & 0 \\ -3 & -3 \end{pmatrix}$

13. Let A be a 2×2 matrix with a single eigenvalue λ of algebraic multiplicity 2 and geometric multiplicity 1. Prove that

$$e^{At} = e^{\lambda t}\left[I + (A - \lambda I)t\right].$$

In Exercises 14–17, each matrix has an eigenvalue of algebraic multiplicity 2 but geometric multiplicity 1. Use the technique of Exercise 13 to compute e^{tA}.

14. $A = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}$ **15.** $A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$

16. $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$ **17.** $A = \begin{pmatrix} -3 & -1 \\ 4 & 1 \end{pmatrix}$

Each of the matrices in Exercises 18–25 has only one eigenvalue λ. In each exercise, determine the smallest k such that $(A - \lambda I)^k = 0$. The use the fact that

$$e^{tA} = e^{\lambda t}\left[I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \cdots\right]$$

to compute e^{tA}.

18. $A = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & -1 \\ -2 & 4 & -3 \end{pmatrix}$ **19.** $A = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 2 & -2 \end{pmatrix}$

20. $A = \begin{pmatrix} -2 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & -4 \end{pmatrix}$ **21.** $A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 1 & -2 \end{pmatrix}$

22. $A = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{pmatrix}$

23. $A = \begin{pmatrix} -5 & 0 & -1 & 4 \\ -4 & 0 & 1 & 5 \\ 4 & -4 & -5 & -4 \\ 0 & -1 & -1 & -2 \end{pmatrix}$

24. $A = \begin{pmatrix} 0 & 4 & 5 & -2 \\ 1 & -5 & -7 & 3 \\ 0 & 2 & 3 & -1 \\ 3 & -10 & -13 & 6 \end{pmatrix}$

25. $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -9 & 4 & 1 & 4 \\ 13 & -3 & -1 & -5 \\ 2 & -1 & 0 & 0 \end{pmatrix}$

Do the following for each of the matrices in Exercises 26–33. Exercises 26–29 can be done by hand, but you should use a computer for the rest.

(i) Find the eigenvalues.

(ii) For each eigenvalue, find the algebraic and the geometric multiplicities.

(iii) For each eigenvalue λ, find the smallest integer k such that the dimension of the nullspace of $(A - \lambda I)^k$ is equal to the algebraic multiplicity.

(iv) For each eigenvalue λ, find q linearly independent generalized eigenvectors, where q is the algebraic multiplicity of λ.

(v) Verify that the collection of the generalized eigenvectors you find in part (iv) for all of the eigenvalues is linearly independent.

(vi) Find a fundamental set of solutions for the system $\mathbf{y}' = A\mathbf{y}$.

26. $A = \begin{pmatrix} -2 & 1 & -1 \\ 1 & -3 & 0 \\ 3 & -5 & 0 \end{pmatrix}$ **27.** $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & -2 \\ 0 & 0 & 2 \end{pmatrix}$

28. $A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 0 & 1 \end{pmatrix}$ **29.** $A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -5 & -1 \\ 0 & 4 & -1 \end{pmatrix}$

30. $A = \begin{pmatrix} 11 & -42 & 4 & 28 \\ -12 & 39 & -4 & -28 \\ 0 & 0 & -1 & 0 \\ -24 & 81 & -8 & -57 \end{pmatrix}$

31. $A = \begin{pmatrix} 18 & -7 & 24 & 24 \\ 15 & -8 & 20 & 16 \\ 0 & 0 & -1 & 0 \\ -12 & 4 & -15 & -17 \end{pmatrix}$

36. $\mathbf{y}' = \begin{pmatrix} 8 & 3 & 2 \\ 0 & 4 & 0 \\ -8 & -6 & 0 \end{pmatrix} \mathbf{y}$

37. $x' = -2x - 4y + 13z$
$y' = 5y - 4z$
$z' = y + z$

38. $x' = -x + 5y + 3z$
$y' = y + z$
$z' = -2y - 2z$

40. $\mathbf{x}' = \begin{pmatrix} -12 & -1 & 8 & 10 \\ -8 & 0 & -1 & 9 \\ 0 & 0 & 5 & 0 \\ -17 & -1 & 8 & 15 \end{pmatrix} \mathbf{x}$