In Exercises 21 through 26, find a redundant column vector of the given matrix A, and write it as a linear combination of preceding columns. Use this representation to write a nontrivial relation among the columns, and thus find a nonzero vector in the kernel of A. (This procedure is illustrated in Example 8.)

21. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

23. $\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

25. $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

26. $\begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix}$

Find a basis of the image of the matrices in Exercises 27 through 33.

27. $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

28. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

29. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

30. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 5 & 8 \end{bmatrix}$

39. Consider some linearly independent vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ in $\mathbb{R}^n$ and a vector $\vec{v}$ in $\mathbb{R}^n$ that is not contained in the span of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$. Are the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m, \vec{v}$ necessarily linearly independent? Justify your answer.

52. For which values of the constants a, b, c, d, e, and f are the following vectors linearly independent? Justify your answer.

$$\begin{bmatrix} a \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} b \\ c \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} d \\ e \\ f \\ 0 \end{bmatrix}$$

In Exercises 21 through 25, find the reduced row-echelon form of the given matrix A. Then find a basis of the image of A and a basis of the kernel of A.

21. $\begin{bmatrix} 1 & 3 & 9 \\ 4 & 5 & 8 \\ 7 & 6 & 3 \end{bmatrix}$

22. $\begin{bmatrix} 2 & 4 & 8 \\ 4 & 5 & 1 \\ 7 & 9 & 3 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 1 \end{bmatrix}$

28. For which value(s) of the constant k do the vectors below form a basis of $\mathbb{R}^4$?

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \\ 4 \\ k \end{bmatrix}$$

29. Find a basis of the subspace of $\mathbb{R}^3$ defined by the equation

$$2x_1 + 3x_2 + x_3 = 0.$$

31. Let V be the subspace of $\mathbb{R}^4$ defined by the equation

$$x_1 - x_2 + 2x_3 + 4x_4 = 0.$$

Find a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^4$ such that $\ker(T) = \{\vec{0}\}$ and $\operatorname{im}(T) = V$. Describe T by its matrix A.

36. Can you find a 3×3 matrix A such that $\mathrm{im}(A) = \ker(A)$? Explain.

39. We are told that a certain 5×5 matrix A can be written as

$$A = BC,$$

where B is a 5×4 matrix and C is 4×5. Explain how you know that A is not invertible.