Homework 1

Status: Final (although there might be some typos).

> Please reach out to me via the CCLE forum for corrections and/or clarifications about statements found in this document.

Due date: Wednesday, April 8.

Regular exercises

Mathematical Induction

For the problems below, use induction to verify that each equation is true for every positive integer n.

1. $1 + 3 + 5 + \cdots + (2n-1) = n^2$.
2. $1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
3. $\frac{1}{2^2-1} + \frac{1}{3^2-1} + \cdots + \frac{1}{(n+1)^2-1} = \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$.

Use induction to prove the following statements:

4. $7^n - 1$ is divisible by 6, for all $n \geq 1$.
5. $11^n - 6$ is divisible by 5, for all $n \geq 1$.

Use induction to verify the following inequalities:

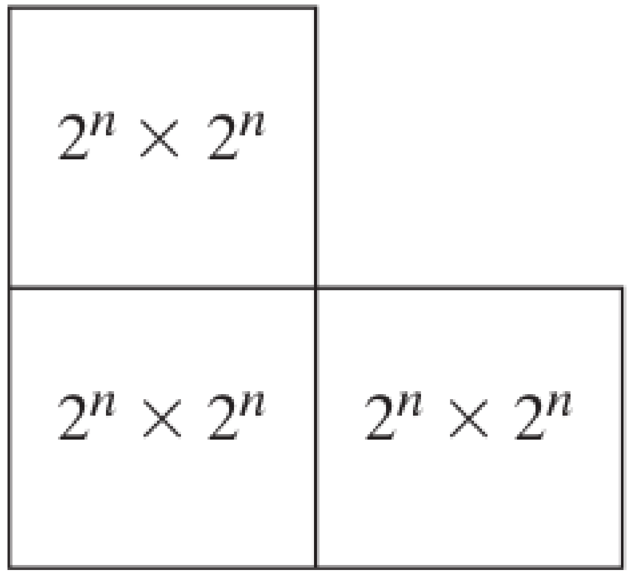
6. $\frac{1}{2n} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$, for $n = 1, 2, \ldots$
7. $2n + 1 \leq 2^n$, for $n = 3, 4, \ldots$
8. $2^n \geq n^2$, for $n = 4, 5, \ldots$
9. Use the geometric sum formula to prove that

$$r^0 + r^1 + \cdots + r^n < \frac{1}{1-r}$$

for all $n \geq 0$ and $0 < r < 1$.

This exercise should help you complete *Example 3* from our mathematical induction lecture.

10. A $2^n \times 2^n$ *L-shape*, $n \geq 0$, is a figure of the form



with no missing squares. Show that any $2^n \times 2^n$ L-shape can be tiled with trominoes.

Sets & functions

11. Let the *universal set*[1] be the set $U = \{1, 2, 3, \ldots, 10\}$. Let $A = \{1, 4, 7, 10\}$, $B = \{1, 2, 3, 4, 5\}$, and $C = \{2, 4, 6, 8\}$. List the elements of each set.

    a) $A \cup B - C$
    b) $\overline{A \cap B} \cup C$
    c) $(A \cup B) - (C - B)$

12. Answer the following questions:

    a) What is the cardinality of $\emptyset$?
    b) What is the cardinality of $\{\emptyset\}$?
    c) What is the cardinality of $\{a, b, a, c\}$?
    d) What is the cardinality of $\{\{a\}, \{a, b\}, \{a, c\}, a, b\}$?

13. Carefully show that $A \neq B$.

    a) $A = \{1, 2\}$, $B = \{x \mid x^3 - 2x^2 - x + 2 = 0\}$.

[1] The *universe*, or *universal set*, is the set of all elements under discussion for possible membership in a set.

b) $A = \{1, 3, 5\}$, $B = \{n \in \mathbb{Z} \mid n > 0 \text{ and } n^2 - 1 \leq n\}$.

14. Carefully show that A is not a subset of B.

    a) $A = \{1, 2, 3\}$, $B = \{1, 2\}$.
    b) $A = \{1, 2, 3\}$, $B = \emptyset$.

15. A television poll of 151 persons found that 68 watched *Law and Disorder*; 61 watched *Twenty-five*; 52 watched *The Tenors*; 16 watched both *Law and Disorder* and *Twenty-five*; 25 watched both *Law and Disorder* and *The Tenors*; 19 watched both *Twenty-five* and *The Tenors*; and 26 watched none of these shows. How many persons watched all three shows?

16. Let $X = \{1, 2\}$, $Y = \{a\}$, and $Z = \{\alpha, \beta\}$. List the elements of each of the following sets.

    a) $X \times Y \times Z$
    b) $X \times X \times X$
    c) $Z \times Y \times X$

17. Determine whether each set below is a function from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$. If it is a function, find its domain and range, draw its arrow diagram, and determine if it is one-to-one, onto, or both.

    a) $\{(1, a), (2, a), (3, c), (4, b)\}$
    b) $\{(1, c), (2, a), (3, b), (4, c), (2, d)\}$
    c) $\{(1, c), (2, d), (3, a), (4, b)\}$

18. Determine whether each function below is one-to-one, onto, or both. Prove your answers. The domain of each function is the set of all integers. The codomain of each function is also the set of all integers.

    a) $f(n) = n + 1$
    b) $f(n) = n^2 - 1$
    c) $f(n) = n^3$

19. Write the definition of *one-to-one* using logical notation (*i,e.,* use $\forall$, $\exists$, etc.).

20. Write the definition of *onto* using logical notation (*i,e.,* use $\forall$, $\exists$, etc.).

Miscellaneous exercises

- Use induction to prove the following identity

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

- A *3D-septomino* is a three-dimensional $2 \times 2 \times 2$ cube with one $1 \times 1 \times 1$ corner cube removed. A deficient cube is a $k \times k \times k$ cube with one $1 \times 1 \times 1$ cube removed.

  Prove that a $2^n \times 2^n \times 2^n$ deficient cube can be tiled by 3D-septominoes.

- Let $\mathcal{P}(X)$ denote the *power set* of X. Answer the following questions:
  - List the members of $\mathcal{P}(\{a, b\})$. Which are proper subsets of $\{a, b\}$?
  - If X has 10 members, how many members does $\mathcal{P}(X)$ have? How many proper subsets does X have?
  - If X has n members, how many members does $\mathcal{P}(X)$ have? How many proper subsets does X have?

- Let $\mathbb{N}$ denote the set of natural numbers. Prove that the function f from $\mathbb{N} \times \mathbb{N}$ defined by $f(m, n) = 2^m 3^n$ is *one-to-one* but not *onto*.

- Use *De Morgan's laws of logic* to negate the definition of *one-to-one*.

- Use *De Morgan's laws of logic* to negate the definition of *onto*.

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