Homework 3

Status: Final

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Due date: Friday, April 24.

Regular exercises

Relations

Section 3.3 in course textbook.

1. Give examples of relations on $\{1, 2, 3, 4\}$ having the properties listed below:
   i. Reflexive, symmetric, and not transitive.
   ii. Not reflexive, not symmetric, and transitive
   iii. Symmetric and anti-symmetric.
2. Determine whether each statement below true or false. If the statement is true, prove it; otherwise, give a counterexample.
   i. If R is transitive, then R^{-1} is transitive.
   ii. If R is reflexive, then R^{-1} is reflexive.
   iii. If R is symmetric, then R^{-1} is symmetric.
   iv. If R is symmetric, then R^{-1} is anti-symmetric.
3. How many non-empty relations are there on a set that contains
   i. 1 element?
   ii. 2 elements?
   iii. 3 elements?
4. What is wrong with the following argument, which supposedly shows that any relation R on X that is symmetric and transitive is reflexive?

   > *Let $x \in X$. Using symmetry, we have (x, y) and (y, x) both in R. Since $(x, y), (y, x) \in R$, by transitivity we have $(x, x) \in R$. Therefore, R is reflexive.*

Equivalence Relations

Section 3.4 in course textbook.

5. Determine whether the given relation is an equivalence relation on $\{1, 2, 3, 4, 5\}$. If the relation is an equivalence relation, list the equivalence classes.

   i. $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 1)\}$
   ii. $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 1), (3, 4), (4, 3)\}$
   iii. $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 5), (5, 1), (3, 5), (5, 3), (1, 3), (3, 1)\}$

6. Determine whether the given relation is an equivalence relation on the set of all people.

   i. $\{(x, y) \,|\, x$ and y are the same height$\}$
   ii. $\{(x, y) \,|\, x$ is taller than $y\}$
   iii. $\{(x, y) \,|\, x$ and y have the same color hair$\}$

7. Let $X = \{1, 2, 3, 4\}$. List the members of the equivalence relation on X obtained via the partitions below.

   i. Partition: $\{1, 2\}$, $\{3, 4\}$.
   ii. Partition: $\{1\}$, $\{2, 4\}$, $\{3\}$.

8. Let $X = \{1, 2, 3, 4, 5\}$, $Y = \{3, 4\}$, and $C = \{1, 3\}$. Define the relation R on $\mathcal{P}(X)$, the set of all subsets of X, as $(A, B) \in R$ if $A \cup Y = B \cup Y$. Show that R is an equivalence relation.

9. Let X, Y, C, and R as in problem 8. List the elements of $[C]$, the equivalence class containing C.

10. If an equivalence relation has only one equivalence class, what must the relation look like?

11. Let $X = \{1, 2, \ldots, 10\}$. Define a relation R on $X \times X$ by $\big((a, b), (c, d)\big) \in R$ if $a + d = b + c$. Show that R is an equivalence relation on $X \times X$.

12. Let R_1 and R_2 be equivalence relations on X. Show that $R_1 \cap R_2$ is an equivalence relation on X.

13. Let f be a function from X to Y. Define a relation R on X by $(x, y) \in R$ if $f(x) = f(y)$. Show that R is an equivalence relation on X.

14. Let R be an equivalence relation on a set A. Define a function f from A to the set of equivalence classes of A by the rule $f(x) = [x]$. When do we have $f(x) = f(y)$?

15. If X and Y are sets, we define the set X to be equivalent to the set Y if there is a one-to-one, onto function from X to Y. Show that this set equivalence is an equivalence

relation.

Matrices of Relations

Section 3.5 in course textbook.

16. For each of the relations R below, find the matrix of the relation R from X to Y relative to the orderings given.
    i. $R = \{(1,\delta),(2,\alpha),(2,\gamma),(3,\beta),(3,\gamma)\}$; ordering of $X : 1,2,3$; ordering of Y : $\alpha,\beta,\gamma,\delta$.
    ii. $R = \{(1,\delta),(2,\alpha),(2,\gamma),(3,\beta),(3,\gamma)\}$; ordering of $X : 3,2,1$; ordering of Y : $\gamma,\beta,\alpha,\delta$.
17. For each of the relations R below, find the matrix of the relation R on X relative to the ordering given.
    i. $R = \{(1,2),(2,3),(3,4),(4,5)\}$; ordering of $X : 1,2,3,4,5$.
    ii. $R = \{(x,y) \mid x < y\}$; ordering of $X : 1,2,3,4$.
18. Let $X = \{1,2,3\}$, $Y = \{x,y\}$, and $Z = \{a,b,c\}$. Consider the relations $R_1 : X \to Y$, $R_2 : Y \to Z$, given by

$$R_1 = \{(1,x),(1,y),(2,x),(3,x)\}; \qquad R_2 = \{(x,b),(y,b),(y,a),(y,c)\};$$

as well as the orderings:

$$X : 1,2,3; \qquad Y : x,y; \qquad Z : a,b,c.$$

    i. Find the matrix A_1 of the relation R_1 (relative to the given orderings).
    ii. Find the matrix A_2 of the relation R_2 (relative to the given orderings).
    iii. Find the matrix product A_1A_2.
    iv. Use the previous result to find the matrix of the relation $R_2 \circ R_1$.
    v. Use the previous result to find the relation $R_2 \circ R_1$ (as a set of ordered pairs).
19. Let $X = Y = \{2,3,4,5\}$, and $Z = \{1,2,3,4\}$. Consider the relations $R_1 : X \to Y$, $R_2 : Y \to Z$, given by

$$R_1 = \{(x,y) \mid x \text{ divides } y\}; \qquad R_2 = \{(y,z) \mid y > z\};$$

as well as the orderings:

$$X : 5,4,3,2; \qquad Y : 5,4,3,2; \qquad Z : 4,3,2,1.$$

    i. Find the matrix A_1 of the relation R_1 (relative to the given orderings).

ii. Find the matrix A_2 of the relation R_2 (relative to the given orderings).
iii. Find the matrix product $A_1 A_2$.
iv. Use the previous result to find the matrix of the relation $R_2 \circ R_1$.
v. Use the previous result to find the relation $R_2 \circ R_1$ (as a set of ordered pairs).

20. How can we quickly determine whether a relation R is a function by examining the matrix of R (relative to some ordering)?

Miscellaneous exercises

- How many non-empty relations are there on a set that contains 2020 elements?
- Define a relation on $\mathbf{N} \times \mathbf{N}$ by $\big((p,q),(r,s)\big) \in R$ if $ps - qr = 0$. Show that R is an equivalence relation on $\mathbf{N} \times \mathbf{N}$.

  i. What is the equivalence class of $(1,1)$?
  ii. What is the equivalence class of $(1,2)$?
  iii. What can you say about the set $\mathbf{Q}^+$ of positive rational numbers?

  Hint:

  Try using the notation $\frac{p}{q}$ to represent the ordered pair (p,q).

- Let $\mathbf{Q}$ be the set of rational numbers (*i.e.,* the set of all fractions). Let $\{a_n\} \subseteq \mathbf{Q}$ be a sequence of rational numbers. We say that $\{a_n\}$ is a **Cauchy sequence** if for each $\epsilon > 0$, there exists $N \in \mathbf{N}$, such that

$$d(a_n, a_m) < \epsilon, \text{ whenever } n, m \geq N;$$

  where the *distance function* $d : \mathbf{Q} \times \mathbf{Q}$ is given by $d(p,q) = |p - q|$.

  Note:

  Convergent sequences in $\mathbf{Q}$ are Cauchy sequences. However, the opposite is not always true. For example, the sequence

$$\begin{array}{llll} a_1 = 3, & a_2 = 3.1, & a_3 = 3.14, & a_4 = 3.141, \\ a_5 = 3.1415, & a_6 = 3.14159, & a_7 = 3.141592, & \cdots \end{array}$$

  is a Cauchy Sequence that does not converge in $\mathbf{Q}$. It converges in $\mathbf{R}$ to π.

i. Let $\mathcal{C}_{\mathbf{Q}}$ be the set of all Cauchy sequences in $\mathbf{Q}$. Define a relation on $\mathcal{C}_{\mathbf{Q}}$ by

$$(\{a_n\}, \{b_n\}) \in R, \text{ if } \lim_{n\to\infty} |a_n - b_n| = 0.$$

Prove that R is an equivalence relation.

ii. Let $\mathbf{Q}^*$ be the set of equivalence classes of $\mathcal{C}_{\mathbf{Q}}$ obtained via R. Show that $\mathbf{Q}$ can be *embedded* into $\mathbf{Q}^*$. That is, show there exists a *one-to-one* function $f : \mathbf{Q} \to \mathbf{Q}^*$.

Hint:

Are constant sequences Cauchy sequences? If so, try $f(q) = [\{q_n\}]$, where $q_n = q$, for $n \geq 1$.

Fun fact:

The space $\mathbf{Q}^*$ can be turned into a *metric space*. That is, a function $d^* : \mathbf{Q}^* \times \mathbf{Q}^* \to \mathbf{R}$ can be defined in such a way that it represents the *distance* between two equivalence classes. With this *structure* in place, *Cauchy sequences* in $\mathbf{Q}^*$, are defined by replacing the distance function d (above) with d^*. Unlike $\mathbf{Q}$, in this space $\mathbf{Q}^*$ all Cauchy sequences are now convergent. As it turns out, $\mathbf{Q}^*$ is equivalent to the set of real numbers $\mathbf{R}$.

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