17. Find a basis for $W^{\perp}$, where

$$W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \right).$$

19. For a line L in $\mathbb{R}^2$, draw a sketch to interpret the following transformations geometrically:

a. $T(\vec{x}) = \vec{x} - \text{proj}_L \vec{x}$

b. $T(\vec{x}) = \vec{x} - 2\text{proj}_L \vec{x}$

c. $T(\vec{x}) = 2\text{proj}_L \vec{x} - \vec{x}$

EXERCISES 5.2

GOAL *Perform the Gram–Schmidt process, and thus find the QR factorization of a matrix.*

Using paper and pencil, perform the Gram–Schmidt process on the sequences of vectors given in Exercises 1 through 14.

1. $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

2. $\begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}$

3. $\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 25 \\ 0 \\ -25 \end{bmatrix}$

4. $\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 25 \\ 0 \\ -25 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$

5. $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$

6. $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$

7. $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix}$

8. $\begin{bmatrix} 5 \\ 4 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 7 \\ -2 \end{bmatrix}$

9. $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ -5 \\ 3 \end{bmatrix}$

10. $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 6 \\ 4 \end{bmatrix}$

Using paper and pencil, find the QR factorizations of the matrices in Exercises 15 through 28. Compare with Exercises 1 through 14.

15. $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

16. $\begin{bmatrix} 6 & 2 \\ 3 & -6 \\ 2 & 3 \end{bmatrix}$

29. Perform the Gram–Schmidt process on the following basis of $\mathbb{R}^2$:

$$\vec{v}_1 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 7 \end{bmatrix}.$$

Illustrate your work with sketches, as in Figures 1 through 3 of this section.

30. Consider two linearly independent vectors $\vec{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} c \\ d \end{bmatrix}$ in $\mathbb{R}^2$. Draw sketches (as in Figures 1 through 3 of this section) to illustrate the Gram–Schmidt process for $\vec{v}_1, \vec{v}_2$. You need not perform the process algebraically.

31. Perform the Gram–Schmidt process on the following basis of $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} b \\ c \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} d \\ e \\ f \end{bmatrix}.$$

Here, a, c, and f are positive constants, and the other constants are arbitrary. Illustrate your work with a sketch, as in Figure 4.

32. Find an orthonormal basis of the plane

$$x_1 + x_2 + x_3 = 0.$$

33. Find an orthonormal basis of the kernel of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

36. Consider the matrix

$$M = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 5 \\ 0 & -4 & 6 \\ 0 & 0 & 7 \end{bmatrix}.$$

Find the QR factorization of M.

37. Consider the matrix

$$M = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 0 & 5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Find the QR factorization of M.

38. Find the QR factorization of

$$A = \begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

39. Find an orthonormal basis $\vec{u}_1, \vec{u}_2, \vec{u}_3$ of $\mathbb{R}^3$ such that

$$\operatorname{span}(\vec{u}_1) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$$

and

$$\operatorname{span}(\vec{u}_1, \vec{u}_2) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right).$$