EXERCISES 7.1

GOAL *Apply the concepts of eigenvalues, eigenvectors, eigenbases, and diagonalization. Use eigenvectors to analyze discrete dynamical systems.*

In Exercises 1 through 4, let A be an invertible $n \times n$ matrix and $\vec{v}$ an eigenvector of A with associated eigenvalue λ.

1. Is $\vec{v}$ an eigenvector of A^3? If so, what is the eigenvalue?

2. Is $\vec{v}$ an eigenvector of A^{-1}? If so, what is the eigenvalue?

3. Is $\vec{v}$ an eigenvector of $A + 2I_n$? If so, what is the eigenvalue?

4. Is $\vec{v}$ an eigenvector of $7A$? If so, what is the eigenvalue?

5. If a vector $\vec{v}$ is an eigenvector of both A and B, is $\vec{v}$ necessarily an eigenvector of $A + B$?

6. If a vector $\vec{v}$ is an eigenvector of both A and B, is $\vec{v}$ necessarily an eigenvector of AB?

7. If $\vec{v}$ is an eigenvector of the $n \times n$ matrix A with associated eigenvalue λ, what can you say about

$$\ker(A - \lambda I_n)?$$

Is the matrix $A - \lambda I_n$ invertible?

8. Find all 2×2 matrices for which $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector with associated eigenvalue 5.

9. Find all 2×2 matrices for which $\vec{e}_1$ is an eigenvector.

10. Find all 2×2 matrices for which $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector with associated eigenvalue 5.

11. Find all 2×2 matrices for which $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector with associated eigenvalue -1.

12. Consider the matrix $A = \begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix}$. Show that 2 and 4 are eigenvalues of A and find all corresponding eigenvectors. Find an eigenbasis for A and thus diagonalize A.

13. Show that 4 is an eigenvalue of $A = \begin{bmatrix} -6 & 6 \\ -15 & 13 \end{bmatrix}$ and find all corresponding eigenvectors.

14. Find all 4×4 matrices for which $\vec{e}_2$ is an eigenvector.

Arguing geometrically, find all eigenvectors and eigenvalues of the linear transformations in Exercises 15 through 22. In each case, find an eigenbasis if you can, and thus determine whether the given transformation is diagonalizable.

15. Reflection about a line L in $\mathbb{R}^2$

16. Rotation through an angle of 180° in $\mathbb{R}^2$

17. Counterclockwise rotation through an angle of 45° followed by a scaling by 2 in $\mathbb{R}^2$

18. Reflection about a plane V in $\mathbb{R}^3$

37. Consider the matrix

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}.$$

a. Use the geometric interpretation of this transformation as a reflection combined with a scaling to find the eigenvalues of A.

b. Find an eigenbasis for A.

c. Diagonalize A.

For each of the matrices in Exercises 1 through 13, find all real eigenvalues, with their algebraic multiplicities. Show your work. Do not use technology.

1. $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 2 & 1 & 2 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 4 \\ -1 & 4 \end{bmatrix}$

5. $\begin{bmatrix} 11 & -15 \\ 6 & -7 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

7. I_3

8. $\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$

9. $\begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 7 \\ 0 & 0 & 2 \end{bmatrix}$

10. $\begin{bmatrix} -3 & 0 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 3 \end{bmatrix}$

11. $\begin{bmatrix} 5 & 1 & -5 \\ 2 & 1 & 0 \\ 8 & 2 & -7 \end{bmatrix}$

12. $\begin{bmatrix} 2 & -2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 2 & -3 \end{bmatrix}$

13. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

14. Consider a 4×4 matrix $A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$, where B, C, and D are 2×2 matrices. What is the relationship among the eigenvalues of A, B, C, and D?

15. Consider the matrix $A = \begin{bmatrix} 1 & k \\ 1 & 1 \end{bmatrix}$, where k is an arbitrary constant. For which values of k does A have two distinct real eigenvalues? When is there no real eigenvalue?

16. Consider the matrix $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, where a, b, and c are nonzero constants. For which values of a, b, and c does A have two distinct eigenvalues?

EXERCISES 7.3

GOAL *For a given eigenvalue, find a basis of the associated eigenspace. Use the geometric multiplicities of the eigenvalues to determine whether a matrix is diagonalizable.*

For each of the matrices A in Exercises 1 through 20, find all (real) eigenvalues. Then find a basis of each eigenspace, and diagonalize A, if you can. Do not use technology.

1. $\begin{bmatrix} 7 & 8 \\ 0 & 9 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 6 & 3 \\ 2 & 7 \end{bmatrix}$

4. $\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 4 & 5 \\ -2 & -2 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

For which values of constants a, b, and c are the matrices in Exercises 40 through 50 diagonalizable?

40. $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

41. $\begin{bmatrix} 1 & a \\ 0 & 2 \end{bmatrix}$

42. $\begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}$

43. $\begin{bmatrix} 1 & 1 \\ a & 1 \end{bmatrix}$

44. $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

45. $\begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 3 \end{bmatrix}$

46. $\begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 1 \end{bmatrix}$

47. $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$

48. $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & a \\ 0 & 1 & 0 \end{bmatrix}$

EXERCISES 8.1

GOAL *Find orthonormal eigenbases for symmetric matrices. Apply the spectral theorem.*

For each of the matrices in Exercises 1 through 6, find an orthonormal eigenbasis. Do not use technology.

1. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$

For each of the matrices A in Exercises 7 through 11, find an orthogonal matrix S and a diagonal matrix D such that $S^{-1}AS = D$. Do not use technology.

7. $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

8. $A = \begin{bmatrix} 3 & 3 \\ 3 & -5 \end{bmatrix}$

12. Let L from $\mathbb{R}^3$ to $\mathbb{R}^3$ be the reflection about the line spanned by

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

a. Find an orthonormal eigenbasis $\mathfrak{B}$ for L.

b. Find the matrix B of L with respect to $\mathfrak{B}$.

c. Find the matrix A of L with respect to the standard basis of $\mathbb{R}^3$.