

Math 33A Sheet 4

Chapter 3.1

Ex 2) $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

$$\text{rref}([1 \ 2 \ 3 \ | \ 0]) =$$

$$x_1 = -2x_2 - 3x_3$$

$$x_2 = t \quad x_3 = s$$

$$\begin{bmatrix} -2t - 3s \\ t \\ s \end{bmatrix}$$

$$= t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$$\ker(A) = \text{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right)$$

Ex 14) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

$$\text{rref}(A) = \begin{bmatrix} 1 & 2 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$x_1 + 2x_2 + 3x_3 = 0$$

$$x_2 = -\frac{1}{2}x_1, \quad x_3 = -\frac{1}{3}x_1$$

all variables dependent on each other

$$\ker(A) = \text{span}\left(\begin{bmatrix} 1 \\ -1/2 \\ -1/3 \end{bmatrix}\right)$$

Ex 17) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$T(\vec{v}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$v_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + v_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$\text{img}(T)$ is the plane spanned by $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ in \mathbb{R}^2

Ex 3) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$0 = 0$$

$$\ker(A) = \mathbb{R}^2$$

Ex 8)* $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$\text{rref}(A) =$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$x_1 + x_2 + x_3 = 0$$

$$x_1 = -x_2 - x_3$$

$$x_2 = t, \quad x_3 = s$$

$$\begin{bmatrix} -t - s \\ t \\ s \end{bmatrix}$$

$$= t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\ker(A) = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$$

Ex 19)* $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & -8 \end{bmatrix}$

$$T(\vec{v}) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & -8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

$$= v_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + v_2 \begin{bmatrix} 2 \\ -4 \end{bmatrix} + v_3 \begin{bmatrix} 3 \\ -6 \end{bmatrix} + v_4 \begin{bmatrix} 4 \\ -8 \end{bmatrix}$$

$\text{img}(T)$ is the line spanned by $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ in \mathbb{R}^2

Ex 24) Orthogonal projection onto the plane $x+2y+3z=0$ in \mathbb{R}^3

The image consists of all points in the function's target space, therefore

$$\text{img}(T) = x+2y+3z=0 \text{ in } \mathbb{R}^3$$

A projection of a vector perpendicular to the plane makes $A\vec{v}=0$, therefore

$$\ker(T) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$$

Ex 34) $\ker(T) = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}\right)$ in \mathbb{R}^3

$$t \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} -t \\ t \\ 2t \end{bmatrix}$$

$$x_1 = -x_2, x_1 = 2x_3$$

$$x_1 + x_2 = 0, x_1 - 2x_3 = 0$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T(\vec{v}) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \vec{v}$$

Ex 4) $\text{span}(\vec{v}_1, \dots, \vec{v}_n) = \text{img}(\vec{v}_1, \dots, \vec{v}_n)$

the image is a subspace, therefore the $\text{span}(\vec{v}_1, \dots, \vec{v}_n)$ must be a subspace

$$\text{Ex 8) } \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} - 2(I)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix} \times -1$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$-1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is a lin. combo. of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$\text{Ex 10) } \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is redundant, they are linearly dependant

Chapter 3.2

$$\text{Ex 1)* } W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x+y+z=1 \right\}$$

$$0+0+0 \neq 1$$

Not a subspace of \mathbb{R}^3 , doesn't contain the zero vector of \mathbb{R}^3

$$\text{Ex 15) } \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} - 2(I)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix} \times -1$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$-\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is redundant, linearly dependant

$$\text{Ex 22)} \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} - 2(I)$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \vec{0}$$

$$\boxed{\ker(A) = \text{span} \left(\begin{bmatrix} 3 \\ 6 \end{bmatrix} \right)}$$

$$\text{Ex 24)} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ is redundant

$$2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \vec{0}$$

$$2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{0}$$

$$\boxed{\ker(A) = \text{span} \left(\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right)}$$

$$\text{Ex 29)} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - 4(I)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} \times -\frac{1}{3}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} - 2(II)$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$-1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\boxed{\text{basis of } A \text{ is } \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

$$\text{Ex 30)}^* \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix} - I$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 2 & 6 \end{bmatrix} - 2(II)$$

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & -2 \end{bmatrix} + \frac{1}{2}(III)$$

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 3(III)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$$\boxed{\text{basis of } A \text{ is } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}}$$

Q1) Image.

$$\vec{v}_1 = T(\vec{w}_1), \vec{v}_2 = T(\vec{w}_2)$$

$$\vec{v}_1 + \vec{v}_2 = T(\vec{w}_1) + T(\vec{w}_2) = T(\vec{w}_1 + \vec{w}_2)$$

\hookrightarrow closed under addition \checkmark

$$\vec{v}_1 = T(\vec{w}_1)$$

$$k\vec{v}_1 = kT(\vec{w}_1) = T(k\vec{w}_1)$$

\hookrightarrow closed under multiplication \checkmark

$$\vec{0} = A(\vec{0}) = T(\vec{0})$$

\hookrightarrow contains the zero vector \checkmark

Kernel:

$$T(\vec{v}) + T(\vec{w}) = T(\vec{v} + \vec{w})$$

$$T(\vec{0}) + T(\vec{0}) = T(\vec{0} + \vec{0}) = \vec{0}$$

\hookrightarrow closed under addition

$$kT(\vec{v}) = T(k\vec{v})$$

$$kT(\vec{0}) = T(k\vec{0}) = \vec{0}$$

\hookrightarrow closed under multiplication

The kernel is all zeros of the linear transformation

\hookrightarrow contains the origin