EXERCISES

Which of the initial value problems in Exercises 1–6 are guaranteed a unique solution by the hypotheses of Theorem 7.16? Justify your answer.

1. $y' = 4 + y^2, \quad y(0) = 1$ **2.** $y' = \sqrt{y}, \quad y(4) = 0$

3. $y' = t \tan^{-1} y, \quad y(0) = 2$

4. $\omega' = \omega \sin \omega + s, \quad \omega(0) = -1$

5. $x' = \dfrac{t}{x+1}, \quad x(0) = 0$

6. $y' = \dfrac{1}{x} y + 2, \quad y(0) = 1$

For each differential equation in Exercises 7–8, perform each of the following tasks.

(i) Find the general solution of the differential equation. Sketch several members of the family of solutions portrayed by the general solution.

(ii) Show that there is no solution satisfying the given initial condition. Explain why this lack of solution does not contradict the existence theorem.

7. $ty' - y = t^2 \cos t, \quad y(0) = -3$

8. $ty' = 2y - t, \quad y(0) = 2$

9. Show that $y(t) = 0$ and $y(t) = t^3$ are both solutions of the initial value problem $y' = 3y^{2/3}$, where $y(0) = 0$. Explain why this fact does not contradict Theorem 7.16.

10. Show that $y(t) = 0$ and $y(t) = (1/16)t^4$ are both solutions of the initial value problem $y' = ty^{1/2}$, where $y(0) = 0$. Explain why this fact does not contradict Theorem 7.16.

In Exercises 11–16, use a numerical solver to sketch the solution of the given initial value problem.

(i) Where does your solver experience difficulty? Why? Use the image of your solution to estimate the interval of existence.

(ii) For 11–14 only, find an explicit solution; then use your formula to determine the interval of existence. How does it compare with the approximation found in part (i)?

11. $\dfrac{dy}{dt} = \dfrac{t}{y+1}, \quad y(2) = 0$

12. $\dfrac{dy}{dt} = \dfrac{t-2}{y+1}, \quad y(-1) = 1$

13. $\dfrac{dy}{dt} = \dfrac{1}{(t-1)(y+1)}, \quad y(0) = 1$

14. $\dfrac{dy}{dt} = \dfrac{1}{(t+2)(y-3)}, \quad y(0) = 1$

26. Is it possible to find a function $f(t, x)$ that is continuous and has continuous partial derivatives such that the functions $x_1(t) = \cos t$ and $x_2(t) = 1 - \sin t$ are both solutions to $x' = f(t, x)$ near $t = \pi/2$?

27. Suppose that x is a solution to the initial value problem

$$x' = x\cos^2 t \quad \text{and} \quad x(0) = 1.$$

Show that $x(t) > 0$ for all t for which x is defined.

28. Suppose that y is a solution to the initial value problem

$$y' = (y - 3)e^{\cos(ty)} \quad \text{and} \quad y(1) = 1.$$

Show that $y(t) < 3$ for all t for which y is defined.

29. Suppose that y is a solution to the initial value problem

$$y' = (y^2 - 1)e^{ty} \quad \text{and} \quad y(1) = 0.$$

Show that $-1 < y(t) < 1$ for all t for which y is defined.

30. Suppose that x is a solution to the initial value problem

$$x' = \frac{x^3 - x}{1 + t^2x^2} \quad \text{and} \quad x(0) = 1/2.$$

Show that $0 < x(t) < 1$ for all t for which x is defined.

31. Suppose that x is a solution to the initial value problem

$$x' = x - t^2 + 2t \quad \text{and} \quad x(0) = 1.$$

Show that $x(t) > t^2$ for all t for which x is defined.

32. Suppose that y is a solution to the initial value problem

$$y' = y^2 - \cos^2 t - \sin t \quad \text{and} \quad y(0) = 2.$$

Show that $y(t) > \cos t$ for all t for which y is defined.

In each of Exercises 15–22, an autonomous differential equation is given in the form $y' = f(y)$. Perform each of the following tasks without the aid of technology.

(i) Sketch a graph of $f(y)$.

(ii) Use the graph of f to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

(iii) Sketch the equilibrium solutions in the ty-plane. These equilibrium solutions divide the ty-plane into regions. Sketch at least one solution trajectory in each of these regions.

15. $y' = 2 - y$

16. $y' = 2y - 7$

17. $y' = (y + 1)(y - 4)$

18. $y' = 6 + y - y^2$

19. $y' = 9y - y^3$

20. $y' = (y + 1)(y^2 - 9)$

21. $y' = \sin y$

22. $y' = \cos 2y$

For each initial value problem presented in Exercises 23–26, perform each of the following tasks.

(i) Solve the initial value problem analytically.

(ii) Use the analytical solution from part (i) and the theory of limits to find the behavior of the function as $t \to +\infty$.

(iii) Without the aid of technology, use the theory of qualitative analysis presented in this section to predict the long-term behavior of the solution. Does your answer agree with that found in part (ii)? Which is the easier method?

23. $y' = 6 - y, \quad y(0) = 2$

24. $y' + 2y = 5, \quad y(0) = 0$

25. $y' = (1 + y)(5 - y), \quad y(0) = 2$

26. $y' = (3 + y)(1 - y), \quad y(0) = 2$