MATH 33A Midterm 1

CHARLES ZHANG

TOTAL POINTS

46 / 50

QUESTION 1

Problem 1 10 pts

1.1 2 / 2

+ **0 pts** incorrect/missing
✓ + **2 pts** correct answer
+ **1 pts** Characterisation in terms of being represented by a matrix

1.2 2 / 2

+ **0 pts** incorrect/missing
✓ + **1 pts** preservation of addition of vectors
✓ + **1 pts** preservation of scalar multiplication
+ **1 pts** Using characterization of being represented by a matrix
+ **1 pts** Stating that the matrix characterisation is an equivalent definition

1.3 3 / 3

✓ + **3 pts** Correct
+ **0 pts** Missing/incorrect

1.4 2 / 2

✓ + **2 pts** Correct
+ **0 pts** incorrect: rank is 2

1.5 1 / 1

✓ + **1 pts** Correct
+ **0 pts** Incorrect

QUESTION 2

10 pts

2.1 6 / 6

✓ - **0 pts** Correct
- **3 pts** T and S fail to commute
+ **1 pts** but gives definition of commuting [only given if 2 is given] [mutually exclusive with 5]
- **3 pts** S and U do commute
+ **1 pts** but gives definition of not commuting [only given if 4 is given] [mutually exclusive with 3]
- **6 pts** No credit
- **2 pts** S has changed

2.2 2 / 2

✓ + **1 pts** A correct (45 degree counterclockwise rotation + scaling by 1/sqrt(2)) - [note: it's OK to get the scaling factor incorrect, as long as some nontrivial scaling is done. Must mention or draw scaling].
✓ + **1 pts** B correct (reflection through x-axis / the line y = 0)
+ **0 pts** All incorrect / blank

2.3 2 / 2

✓ + **2 pts** Correct (up to possibly switching the two vectors)
+ **1 pts** Drew BA instead of AB [note that AB means B first, then A]
+ **1 pts** Computed the matrix AB correctly
+ **0 pts** Incorrect/blank

QUESTION 3

10 pts

3.1 4 / 4

✓ - **0 pts** Correct
- **1 pts** Point 1: Put into RREF with minor arithmetic error
- **2 pts** Point 1: Unable to put into RREF (Substantial error)
- **3 pts** Point 1: Unable to use Gauss-Jordan

elimination

- **1 pts** Point 2: Incorrectly reading off solutions from RREF

3.2 3 / 3

✓ **- 0 pts** Correct

- 1 pts Incorrect Answer

- 2 pts Incorrect Justification

3.3 3 / 3

✓ **- 0 pts** Correct

- 1 pts Incorrectly stated that no solutions was possible.

- 1 pts Incorrectly stated that one solution was possible.

- 2 pts Did not justify Answer

- 3 pts Incorrect

QUESTION 4

10 pts

4.1 1 / 2

+ 2 pts Correct

✓ **+ 1 pts Partial credit: The student states a theorem about invertible matrices but not the actual definition, e.g. that an invertible matrix row reduces to the identity. The student also could have attempted to give the actual definition but made some sort of small error.**

+ 0 pts Incorrect

4.2 0 / 2

+ 2 pts Correct: student gives either a non-invertible transformation T from R2 to R2 or a 2x2 non-invertible matrix

✓ **+ 0 pts Incorrect**

This is not a transformation from R2 to R2.

4.3 3 / 4

+ 4 pts Completely correct

✓ **+ 1 pts Full Credit Criterion 1: Student states that A is invertible**

✓ **+ 2 pts Full Credit Criterion 2: Student is able to preform row operations to reduce A down to the identity.**

+ 1 pts Full Credit Criterion 3: Student gives correct form of A^(-1)

+ 0 pts The student was unable to completely row reduce A correctly and did not determine that A was invertible.

4.4 2 / 2

✓ **+ 2 pts Correct: There are either infinitely many or no solutions**

+ 0 pts Incorrect

QUESTION 5

10 pts

5.1 2 / 2

✓ **- 0 pts Correct**

- 2 pts Incorrect

5.2 2 / 2

✓ **- 0 pts Correct**

- 2 pts Incorrect

5.3 2 / 2

✓ **+ 2 pts Correct**

+ 0 pts Incorrect

5.4 2 / 2

✓ **+ 2 pts Correct**

+ 0 pts Incorrect

5.5 2 / 2

✓ **+ 2 pts Correct (False)**

+ 0 pts Incorrect (True)

+ 0 pts Blank/no answer

125

Midterm 1 (Math 33A, Fall 2019)

Your Name: Charles Zhang

UCLA id: 305-413-659

Date: 10/21/19

The rules: You can answer using a pencil or ink pen. You are allowed to use only this paper, pencil or pen, and the scratch paper provided. You should not hand the scratch paper in. No calculators. No books, no notebooks, no notes, no mobile phones, no web access. You must write your name and UCLA id. You have exactly 50 minutes.

Points:

Problem 1: ____/10

Problem 2: ____/10

Problem 3: ____/10

Problem 4: ____/10

Problem 5: ____/10

Total: __________ (out of 50)

Problem 1 (10 points in total)

Consider the map $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

for any vector $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ in $\mathbb{R}^3$.

1. (2 points) Write down the definition of linear transformation.

Solution: A transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ such that

$T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$ and $T(k\vec{v}) = kT(\vec{v})$.

2. (2 points) Show that the map T is a linear transformation.

Solution:

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ $\quad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

$T(\vec{v}) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ $\quad T(\vec{w}) = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$T(\vec{v}+\vec{w}) = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$= T(\vec{v}) + T(\vec{w})$ ✓

$T(k\vec{v}) = \begin{bmatrix} kv_1 \\ kv_2 \end{bmatrix} = k\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} =$

$kT(\vec{v})$ ✓

3. (3 points) Write down the matrix A such that $T(v) = Av$ for all vectors v in $\mathbb{R}^3$.

Solution:

$v \to 3 \times 1$

$A = \begin{bmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{bmatrix}$

$= \left[T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) \; T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) \; T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) \right]$

$\boxed{A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}$

4. (2 points) Compute the rank of A.

Solution: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \text{rref}(A)$

2 leading 1s

$\boxed{\text{rank}(A) = 2}$

5. (1 point) Let B be any matrix of the same size as A. Can B have rank larger than A?

Solution:

$\boxed{\text{No}}$

rank $\leq n$

rank $\leq m$

Problem 2 (10 points in total)

1. (6 points) Give an example of three linear transformations $T: \mathbb{R}^2 \to \mathbb{R}^2$ and $S: \mathbb{R}^2 \to \mathbb{R}^2$ and $U: \mathbb{R}^2 \to \mathbb{R}^2$ such that the matrices that represent T and S commute, while the matrices that represent S and U do not commute. (Recall: we say that an $n \times m$ matrix A represents a linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ if $T(v) = Av$ for all $v \in \mathbb{R}^m$.)

Solution:

$T: \mathbb{R}^2 \to \mathbb{R}^2$ with matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, representing a counterclockwise $90°$ rotation

$S: \mathbb{R}^2 \to \mathbb{R}^2$ with matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, representing a clockwise $90°$ rotation.

$U: \mathbb{R}^2 \to \mathbb{R}^2$ with matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, representing a reflection over the x-axis.

$x = x-y$

$y = x+y$

$0.5x - 0.5y$

$0.5 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x-y \\ x+y \end{bmatrix}$

0, 1

1, 0

1, 1

$x = 0$

$y = 0$

2. (4 points in total) Let $A = \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

- (2 points) Give a geometric interpretation of the transformations represented by A and B. (In words, or using a drawing.)

Solution:

The transformation represented by A is a scaling down by $\sqrt{2}$ and a 45° counterclockwise rotation.

The transformation represented by B is a reflection over the x-axis.

$\frac{1}{2}$

$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

$\sqrt{2}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

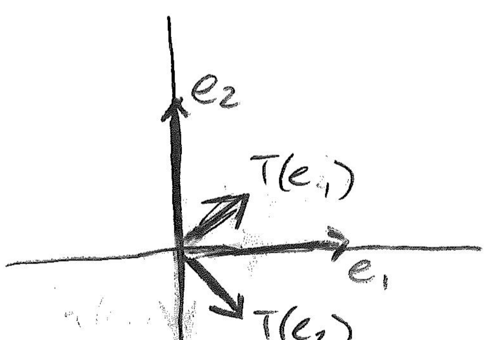
$\frac{\sqrt{2}}{2}$

- (2 points) Draw the images of the standard basis unit vectors of $\mathbb{R}^2$ under the linear transformation represented by AB. $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Solution:

$AB = \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$



Problem 3 (10 points in total)

Consider the following system of three linear equations in the variables x_1, x_2, x_3, x_4:

$$\begin{aligned} 2x_2 + x_4 &= 1 \\ x_1 + x_3 &= 1 \\ x_4 &= 1 \end{aligned}$$

1. (4 points) Solve the system using the Gauss-Jordan elimination algorithm.

Solution:

$$\left|\begin{aligned} 2x_2 + x_4 &= 1 \\ x_1 + x_3 &= 1 \\ x_4 &= 1 \end{aligned}\right|$$

$$\left|\begin{aligned} x_1 + x_3 &= 1 \\ 2x_2 + x_4 &= 1 \\ x_4 &= 1 \end{aligned}\right| - E.3$$

$$\left|\begin{aligned} x_1 + x_3 &= 1 \\ 2x_2 &= 0 \\ x_4 &= 1 \end{aligned}\right| \times \tfrac{1}{2}$$

Let $x_3 = t$

$$\boxed{\begin{aligned} x_1 &= 1 - t \\ x_2 &= 0 \\ x_3 &= t \\ x_4 &= 1 \end{aligned} \quad , \; t \in \mathbb{R}}$$

2. (3 points) Let b_1, b_2, b_3 be arbitrary real numbers. How many solutions does the system

$$\begin{aligned} 2x_2 + x_4 &= b_1 \\ x_1 + x_3 &= b_2 \\ x_4 &= b_3 \end{aligned}$$

have?

Solution:

$$\left[\begin{array}{cccc|c} 0 & 2 & 0 & 1 & b_1 \\ 1 & 0 & 1 & 0 & b_2 \\ 0 & 0 & 0 & 1 & b_3 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & b_2 \\ 0 & 2 & 0 & 1 & b_1 \\ 0 & 0 & 0 & 1 & b_3 \end{array}\right] - \mathrm{III}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & b_2 \\ 0 & 2 & 0 & 0 & b_1 - b_3 \\ 0 & 0 & 0 & 1 & b_3 \end{array}\right]$$

infinitly many solutions, x_3 is free

3. (3 points) Let A be any $n \times n$ matrix. Is there always a sequence of elementary row operations that transforms the identity matrix I_n into A? You should motivate your answer.

Solution: No, A could have a row of zeroes, which would be impossible to transform into, as multiplying a row by zero is not an elementary row operation.

Problem 4 (10 points in total)

1. (2 points) Write down the definition of invertible matrix.

Solution: An $n \times n$ matrix A such that rank(A) is equal to n.

2. (2 points) Give an example of a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that the matrix that represents T is not invertible.

Solution:

$T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is not invertible, as the matrix A isn't square.

3. (4 points) Consider the following matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Is A invertible? If yes, compute its inverse.

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -2\mathrm{I} \\ -3\mathrm{I} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -5 & -2 & 1 & 0 \\ 0 & -5 & -7 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \times -1 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & -1 & 0 \\ 0 & -5 & -7 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} -2\mathrm{II} \\ \\ +5\mathrm{II} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -7 & -3 & 2 & 0 \\ 0 & 1 & 5 & 2 & -1 & 0 \\ 0 & 0 & 18 & 7 & -5 & 1 \end{array}\right] \begin{array}{l} \\ \\ \times \frac{1}{18} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -7 & -3 & 2 & 0 \\ 0 & 1 & 5 & 2 & -1 & 0 \\ 0 & 0 & 1 & \frac{7}{18} & -\frac{5}{18} & \frac{1}{18} \end{array}\right] \begin{array}{l} +7\mathrm{III} \\ -5\mathrm{III} \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{18} & -\frac{1}{18} & \frac{7}{18} \\ 0 & 1 & 0 & \frac{22}{18} & \frac{7}{18} & -\frac{5}{18} \\ 0 & 0 & 1 & \frac{7}{18} & -\frac{5}{18} & \frac{1}{18} \end{array}\right]$$

Yes, A is invertible

$$\boxed{A^{-1} = \frac{1}{18}\begin{bmatrix} -5 & -1 & 7 \\ 22 & 7 & -5 \\ 7 & -5 & 1 \end{bmatrix}}$$

4. (2 points) Let A be an $n \times n$ matrix. Assume that A is not invertible. How many solutions does the system $Ax = b$ have?

Solution: A has either no solution or infinitely many solutions since $\operatorname{rank}(A) \neq n$.

Problem 5 (10 points total; 2 points each)

Answer the following questions with true or false.

1. Any 4×3 matrix with rank equal to 3 is invertible.

False

2. Let θ and η be any two angles with $\theta \neq \eta$. Let $T_\theta \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the counterclockwise rotation in $\mathbb{R}^2$ through θ, and $T_\eta \colon \mathbb{R}^2 \to \mathbb{R}^2$ the counterclockwise rotation in $\mathbb{R}^2$ through η. Then $T_\eta \circ T_\theta = T_\theta \circ T_\eta$.

True

3. There exists a real number a for which the following matrix is in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & a & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

False

4. The transformation $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1^2 + 2v_1 + 1 \\ v_1 + v_2 \end{bmatrix}$ is linear.

False

5. A linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if for every $v \in \mathbb{R}^n$ there exists a unique $w \in \mathbb{R}^n$ such that $T(v) = w$.

False

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Use this sheet of paper if you need additional space.