

HW2 .. Due 10/18 .

2.6 : 4. 6. 8. 10. 12. 20. 24. 28. 30., 34.  
36. 38. 40.

2.7 : 4. 6. 8. 14. 20, 30. 32.

Lecture: Homogeneous Equ. (2.7)

Def: A function  $G(x,y)$  is homogeneous of degree  $n$ .

if  $G(tx, ty) = t^n G(x, y)$ . ( $G(x, y) = x^n (G(1, \frac{y}{x}))$ )  
for all  $t > 0$ ,  $x, y \neq 0$ .

Def: A DE  $P dx + Q dy = 0$  is homogeneous if  $P, Q$  are  
homogenous of the same degree!

Q: How to solve homogeneous DE?

→ degree 2.

Example:  $(x^2 + y^2) dx + xy dy = 0$ .

Step 1: set  $v(x) = \frac{y}{x}$ , change DE to a DE of  $v(x)$ .

$$y = vx \Rightarrow dy = (dv) \cdot x + v \cdot dx.$$

$$(x^2 + y^2) = x^2 (1 + v^2) = x^2 (1 + v^2)$$

$$xy = x^2 \cdot \frac{y}{x} = x^2 v$$

$$x^2 (1 + v^2) dx + x^2 v (dx \cdot x + v \cdot dx)$$

$$x^2 (1 + v^2 + v^2) dx + x^3 v dv = 0.$$

$$\Leftrightarrow (1 + 2v^2) dx + x v dv = 0$$

Step 2:

⇒ Separable Case..?

$$\frac{dx}{x} + \frac{v}{1+2v^2} dv = 0$$

$$\ln|x| + \int \frac{v}{1+2v^2} dv = 0$$

$$\ln|x| + \frac{1}{2} \int \frac{dv^2}{1+2v^2} \neq 0$$

$$= \ln|x| + \frac{1}{4} \int \frac{d(2v^2)}{1+2v^2} = 0$$

$$= \ln|x| + \frac{1}{4} \ln |1+2v^2| = C$$

position

$$4\ln|x| + \ln|1+2v^2| = C$$

$$x^4 \cdot (1+2v^2) = e^C = C$$

$$\Rightarrow x^4 + 2x^2v^2 = C$$

( It is always Separable Variable Case ! )

Ex2: Q37.

$$(3x+y) dx + x dy = 0 \quad (\text{Homogeneous Eq})$$

$$\text{let } v \quad \frac{y}{x} = v \quad \Rightarrow y = xv \\ dy = x dv + v dx$$

$$x(3+v) dx + x(x dv + v dx) = 0$$

$$(3+v+v) dx + x dv = 0$$

$$\frac{dx}{x} + \frac{dv}{3+v} = 0 \quad \checkmark v \left( \frac{\frac{1+v^2}{v^2-2} - 1}{v} \right)$$

$$v \left( \frac{\frac{1+v^2}{v^2-2} - 1}{v} \right).$$

$$(Q38): \frac{dy}{dx} = \frac{y(x^2+xy^2)}{xy^2-2x^3}$$

Homogeneous of degree 0 ✓

Fact. P. Q terms of same degree.

$\Leftrightarrow -\frac{P}{Q}$  is homogeneous of degree 0.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dx \cdot u}{du \cdot dx} = \frac{\frac{y}{x}(1 + \frac{y^2}{x})}{(\frac{y}{x})^2 - 2} = \frac{u(1+u^2)}{u^2-2} \\ \frac{u \cdot du + x \cdot du}{dx} &= \frac{u(1+u^2)}{u^2-2} \\ \frac{x \cdot \frac{du}{dx}}{u^2-2} &= \frac{u(1+u^2)}{u^2-2} - u \\ \frac{(u^2-2)}{u} \cdot \frac{du}{dx} &= \frac{u(1+u^2)}{u^2-2} - u \\ \frac{du}{dx} &= \frac{u(2u^2-1)}{u^2-2} \cdot \frac{1}{x}\end{aligned}$$

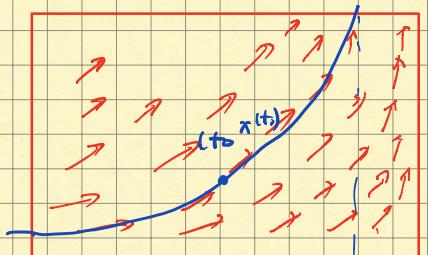
$\frac{3u^5}{u^2-2}$

### Lecture: Existence and Uniqueness.

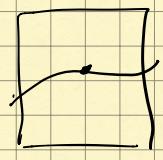
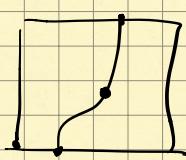
Thm: (Existence). Suppose the function  $f(t, x)$  is defined and continuous on Rect R in the  $tx$ -plane. Then, given a point  $(t_0, x_0) \in R$ , the initial value problem.

$$x' = f(t, x) \quad \text{and} \quad x(t_0) = x_0$$

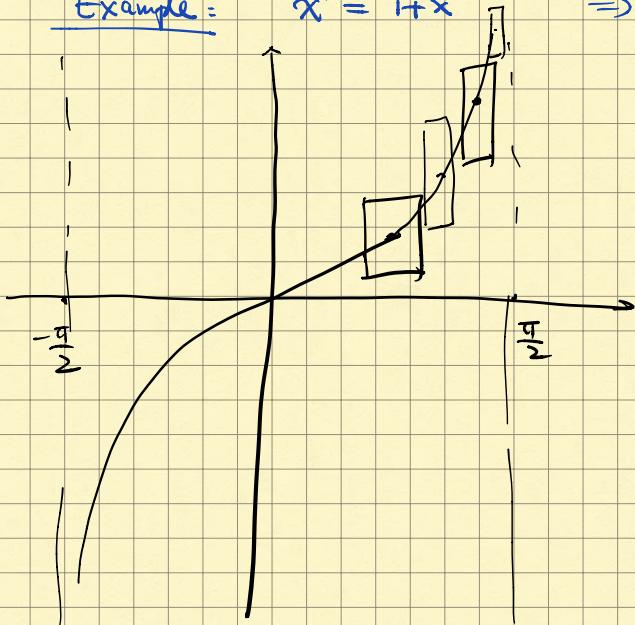
has a solution  $x(t)$  defined in an interval containing  $t_0$ . Furthermore, the solution will be defined at least until  $t \rightarrow (t, x(t))$  leaves the rectangle.



$\exists$  at least exists on  $[a, b]$



Example:  $x' = 1+x^2$   $x(0)=0 \Rightarrow x = \tan t$

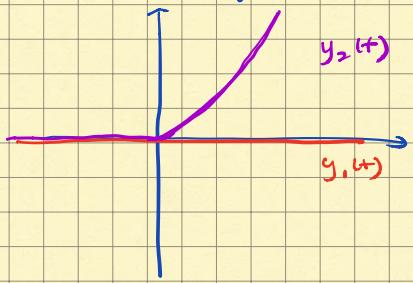


Lecture Uniqueness: 2.7. (cont.)

Example:  $y' = y^{\frac{1}{3}}$   $y(0) = 0$

①  $y_1(t) \equiv 0$  is a solution.

②  $y_2(t) = \begin{cases} (\frac{2}{3}t)^{\frac{3}{2}} & t \geq 0 \\ 0 & t \leq 0 \end{cases}$  is also a solution.



In this case, IVP DOES NOT have a unique solution. We have to add more restrictions.

(Sufficient Condition)

Thm: Suppose,  $f(t, x)$ ,  $\frac{\partial f}{\partial x}(t, x)$  both continuous on a rectangle  $R$  in the  $t-x$ -plane. ( $[a, b] \times [c, d]$ ) bounded. Closed Rectangle!

Suppose  $(t_0, x_0) \in R$  and that the solution

$x' = f(t, x)$ ,  $y' = f(t, y)$  are two solutions  
satisfies.  $x(t_0) = y(t_0) = x_0$ , Then as long as  
 $(t, x(t))$ , and  $(t, y(t))$  stay in  $R$ . we have

$$\underline{x(t) = y(t)}.$$

Back to the example

$$\boxed{f(t, y) = y^{\frac{1}{3}} \quad \frac{\partial f}{\partial y} = \frac{1}{3} y^{-\frac{2}{3}} \quad \text{Not cont when } y=0}$$

Proof = (Not in the textbook).

$$(\Leftrightarrow |\frac{\partial f}{\partial x}| \leq M \text{ on } R)$$

Fact =  $\frac{\partial f}{\partial x}$  continuous on  $R \Rightarrow \frac{\partial f}{\partial x}$  is bdd. on  $R$ .

Suppose  $x(t)$ ,  $y(t)$  are two solutions of the IVP.

$$\text{let } h(t) = x(t) - y(t),$$

$$h'(t) = x'(t) - y'(t) = f(t, x) - f(t, y).$$

$$\begin{aligned} M.V.T &= \frac{\partial f}{\partial x}(t, \xi(t)) \cdot (x(t) - y(t)) \\ &= \frac{\partial f}{\partial x}(t, \xi(t)) \cdot h(t). \end{aligned}$$

$$|\frac{h'}{h}| = \left| \frac{\partial f}{\partial x}(t, \xi(t)) \right| \leq M.$$

$$\left| \int_{t_0}^t \frac{h'}{h} dt \right| \leq \int_{t_0}^t \left| \frac{h'}{h} \right| dt \leq M \cdot (t - t_0)$$

$$|\ln|h| - \ln|h_0| | \leq M \cdot (t - t_0)$$

$$\Leftrightarrow |\ln \left| \frac{h}{h_0} \right| | \leq M \cdot (t - t_0)$$

$$\left| \frac{h}{h_0} \right| \leq R^{M(t-t_0)}$$

$$|x(t) - y(t)| \leq |x(t_0) - y(t_0)| \cdot e^{M(t-t_0)}$$

as  $x_0 = y_0 \Rightarrow x(t) = y(t)$ . on  $\mathbb{R}$ .

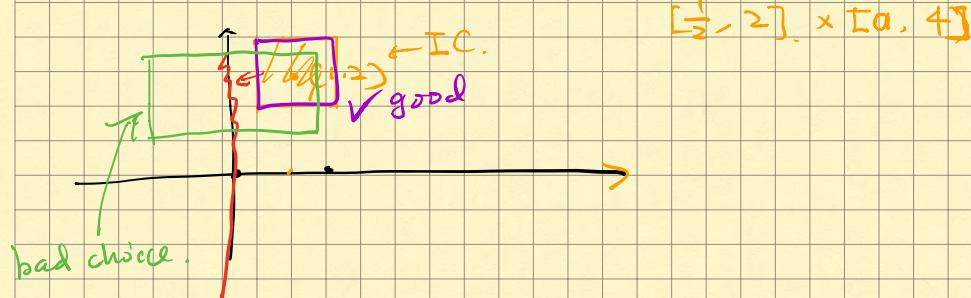
Example = Check Existence and Uniqueness of the IVP:

$$\underline{t y' = y + 3t^2}, \quad y(1)=2.$$

Q: Is there a solution? If so, is the solution unique?

$$A: \underline{y' = \frac{1}{t}(y + 3t^2)} \stackrel{f(t,y)}{=} \text{cont when } t \neq 0$$

$$\frac{\partial f}{\partial y} = \frac{1}{t} \text{ cont when } t \neq 0$$



Example = Suppose  $y(t)$  satisfies DE:

$$y' = (y-1) \cdot \cos(y+t)$$

- ① If  $y(t_0) = 1$ , prove that  $y(t) \equiv 1$

② Suppose  $y(0) = 2$ , prove that  $y(t) > 1$  for all  $t$

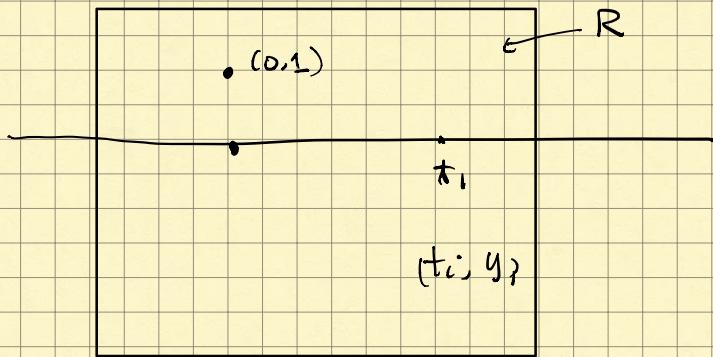
$$\frac{\partial}{\partial y} f(t, y) = -\cos(y-t) + (-y) \cdot y \cdot \sin(y-t)$$

cont on  $(-\infty, +\infty) \times (-\infty, +\infty)$

(cont on ANY Rectangle)

① Suppose  $y(t_i) = y_i \neq 1$

Find Rectangle  $R$  contains  $(0, 1), (t_i, y_i)$



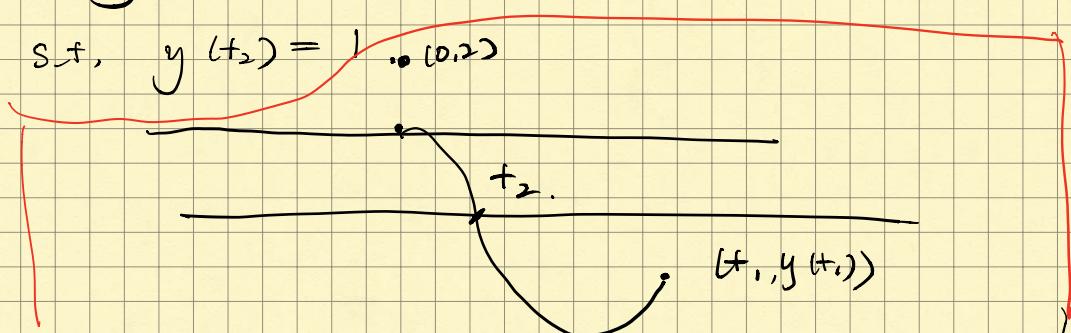
By Uniqueness IVP on  $R$  has unique solution.

$\Rightarrow y(t) = 1$  on  $R$ .  $\Rightarrow y(t_i) = 1$ .  $\therefore$

② Suppose  $y(t_p) \stackrel{=} {<} 1$ . ,  $y(t)$  cont

By IVT.  $\exists t_2 \in [0, t_1]$  or  $[t_1, 0]$

s.t.,  $y(t_2) = 1$   $\bullet (0, 2)$



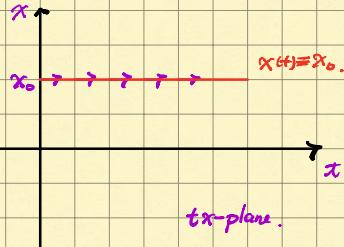
As before. if  $y(t_0) = 1$ . -  $y(t) \equiv 1$  is a solution  
and is unique.  $\Rightarrow y(t) = 1$  on any rectangle.  
( $\Rightarrow$  on  $\mathbb{R}$ )

Lecture

Autonomous Equation and Stability. (2.9)

Def: Autonomous DE  $\Leftrightarrow x' = f(x)$ . ( $x = x(t)$ )

Let  $x_0$  be a root of  $f(x)$ , i.e.  $f(x_0) = 0$ . Then  $x(t) \equiv x_0$  is a solution.



$x_0$  is called an equilibrium point.

$x(t) \equiv x_0$  is called an equilibrium solution.

Analyzing the solutions of the autonomous equation.

Example:  $y' = y(1-y)$ . (1) Sketch the direction fields.

(2) find Equilibrium.

The DE has two equilibria..  $y=0$  and  $y=1$

(3) Sketch the solutions.

(4) Prove that if

$$y(0) = \frac{1}{2}$$

then  $0 < y(t) < 1$   
for all  $t \in (-\infty, \infty)$

Let's sketch the directional fields of  $y' = y(1-y) = f(y)$

$$y > 1, f(y) < 0$$

The dir

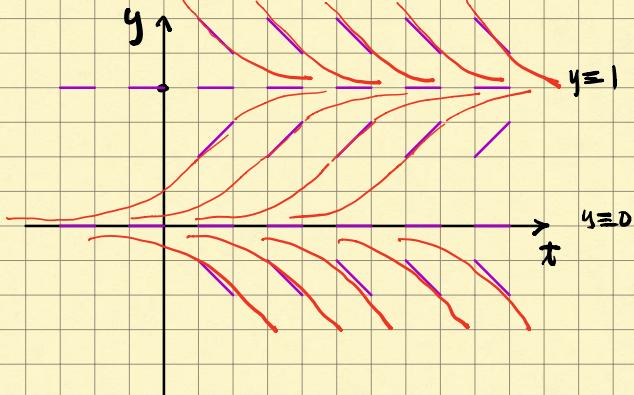
$$y = 1, f(y) = 0$$

$$\tan \theta = f(y)$$

$$0 < y < 1, f(y) > 0$$

$$y = 0, f(y) = 0$$

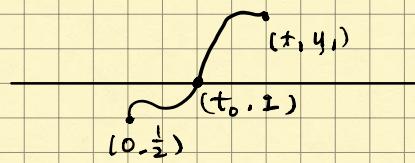
$$y < 0, f(y) < 0$$



two Equilibrium solutions.

Step 1 WLOG (Without loss of generality)

4) Proof = Suppose  $y(t_1) = y_1 > 1$ .  $t_1 > 0$

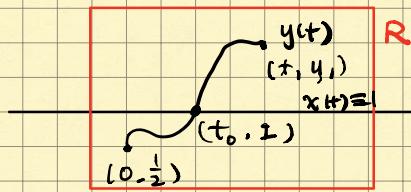


Note that, the solution of DE  
is a continuous function. (Actually  $D(y) \Rightarrow \text{cont}$ )  
 $\text{y}(0)$

$$\text{As } y(0) = \frac{1}{2} < 1$$

$$\text{and } y(t_1) = y_1 > 1.$$

By I.V.T, there exist  $t_0 \in (0, t_1)$  s.t  $y(t_0) = 1$ .



Step 2 =  $\exists$  Rectangle R contains  $(0, \frac{1}{2})$ ,  $(t_0, 1)$ ,  $(t_1, y_1)$

We know that  $\frac{dy}{dt} = 1 - 2y$  is continuous on R.

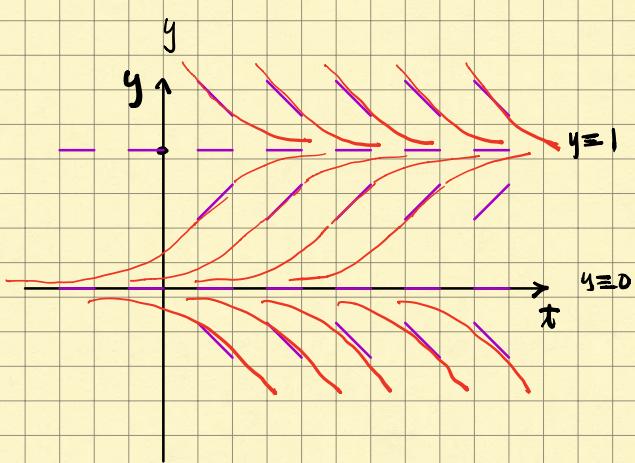
By Uniqueness Thm, the solution go through  $(t_0, 1)$  is unique!

However, we have two solutions  $y(t)$  and  $x(t) \equiv 1$ .

They are different, b/c.  $x(0) = 1$ .  $y(0) = \frac{1}{2}$ .

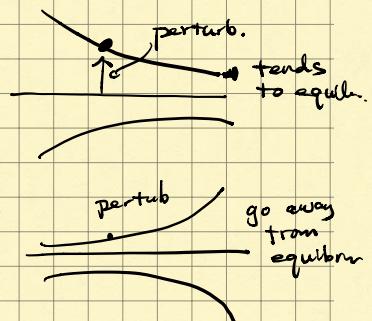
$$x(t_1) = 1 \quad y(t_1) = y_1 > 1$$

### Classification of Equilibrium

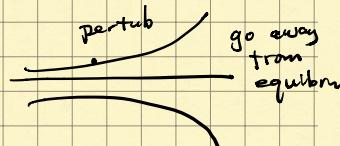


$$y' = f'(y)$$

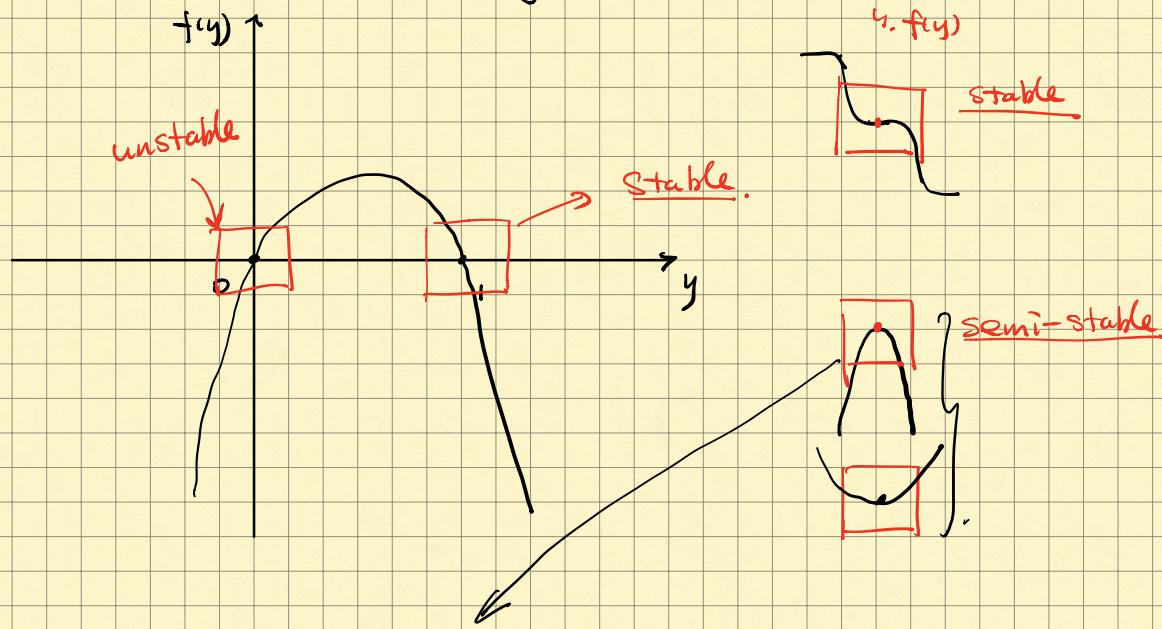
stable :



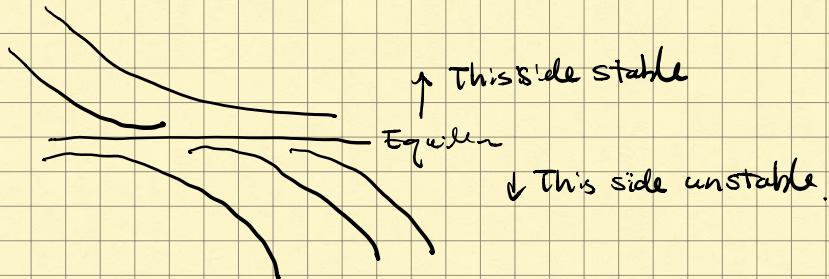
unstable :



Also, we can check the graph of fig)



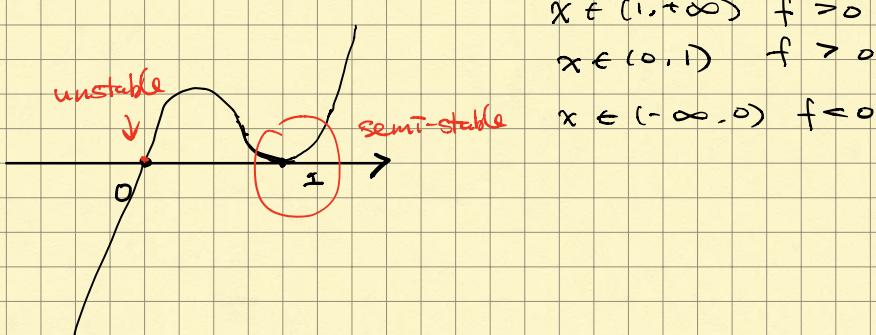
Solution



Example = Analyze the solutions of

$$x' = x^3 - 2x^2 + x = x(x-1)^2 = f(x)$$

Step 1 = Draw graph of  $f(x)$



Step 2:

