Math 33B HW #2

4) $F(x,y) = \frac{1}{\sqrt{x^2+y^2}} = (x^2+y^2)^{-1/2}$

$$\frac{-2x\,dx}{2(x^2+y^2)^{3/2}} + \frac{-2y\,dy}{2(x^2+y^2)^{3/2}}$$

$$\boxed{\frac{-x\,dx - y\,dy}{(x^2+y^2)^{3/2}}}$$

6) $F(x,y) = \ln(xy) + x^2y^3$

$$\left(\frac{y}{xy} + 2xy^3\right)dx + \left(\frac{x}{xy} + 3y^2x^2\right)dy$$

$$\boxed{\left(\frac{1}{x} + 2xy^3\right)dx + \left(\frac{1}{y} + 3x^2y^2\right)dy}$$

8) $F(x,y) = \tan^{-1}\left(\frac{x}{y}\right) + y^4$

$$= \left(\frac{\frac{1}{y}}{\frac{x^2}{y^2}+1}\right)dx + \left(\frac{-\frac{x}{y^2}}{\frac{x^2}{y^2}+1} + 4y^3\right)dy$$

$$= \left(\frac{y}{x^2+y^2}\right)dx + \left(\frac{-x}{x^2+y^2} + 4y^3\right)dy$$

$$= \frac{y}{x^2+y^2}dx + \left(\frac{-x}{x^2+y^2} + \frac{4x^2y^3+4y^5}{x^2+y^2}\right)dy$$

$$= \boxed{\frac{y\,dx - x\,dy + 4x^2y^3\,dy + 4y^5\,dy}{x^2+y^2}}$$

10) $(1 - y\sin x)dx + \cos x\,dy = 0$

$P = 1 - y\sin x$, $Q = \cos x$

$\frac{\partial P}{\partial y} = -\sin x$, $\frac{\partial Q}{\partial x} = -\sin x$

equation is exact

$F(x,y) = \int (1 - y\sin x)dx$

$= x + y\cos x + \phi(y)$

$Q(x,y) = \frac{\partial F}{\partial y} = \cos x + \phi'(y)$

$\phi'(y) = 0$, $\phi(y) = C$

$$\boxed{x + y\cos x = C}$$

12) $\frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy = 0$

$P = \frac{x}{\sqrt{x^2+y^2}}$ $\quad Q = \frac{y}{\sqrt{x^2+y^2}}$

$\frac{\partial P}{\partial y} = \frac{-2xy}{(x^2+y^2)^{3/2}}$ $\quad \frac{\partial Q}{\partial x} = \frac{-2xy}{(x^2+y^2)^{3/2}}$

Equation is exact

$\int \frac{x}{\sqrt{x^2+y^2}}dx$ $\quad u = x^2+y^2,\ du = 2x$

$\frac{1}{2}\int \frac{du}{u^{1/2}} = \frac{1}{2}\int u^{-1/2}du$

$\sqrt{x^2+y^2} + \phi(y)$

$\frac{\partial F}{\partial y} = \frac{1}{2}(2y)\left(\frac{1}{\sqrt{x^2+y^2}}\right) + \phi'(y)$

$\frac{y}{\sqrt{x^2+y^2}} + \phi'(y) = Q(x,y)$

$\phi'(y) = 0$, $\phi(y) = C$

$$\boxed{\sqrt{x^2+y^2} = C}$$

14) $\frac{dy}{dx} = \frac{x}{x-y}$

$x\,dy - y\,dy = x\,dx$

$x\,dy - (x-y)dy$

$\frac{\partial P}{\partial y} = 0$ $\quad \frac{\partial Q}{\partial x} = -1$

$\boxed{\text{Not exact}}$

16) $\frac{2u}{u^2+v^2}du + \frac{2v}{u^2+v^2}dv = 0$

$P = \frac{2u}{u^2+v^2}$ $\quad Q = \frac{2v}{u^2+v^2}$

$\frac{\partial P}{\partial v} = \frac{4uv}{(u^2+v^2)^2}$ $\quad \frac{\partial Q}{\partial u} = \frac{4uv}{(u^2+v^2)^2}$

Equation is exact

$F(x,y) = \int \frac{2u}{u^2+v^2}du$ $\quad x = u^2+v^2,\ dx = 2u$

$= \int \frac{dx}{x} = \ln x + \phi(v)$

$= \ln(u^2+v^2) + \phi(v)$

$\frac{\partial F}{\partial v} = \frac{2v}{u^2+v^2} + \phi'(v) = Q(x,y)$

$\phi'(v) = 0$, $\phi(v) = C$

$$\boxed{\ln(u^2+v^2) = C}$$

18) $\frac{dy}{du} = \frac{2 - y/u}{\ln u}$

$\ln u \, dy = (2 - \frac{y}{u})\,du$

$(\ln u)\,dy + (\frac{y}{u} - 2)\,du = 0$

$P = \ln u \quad Q = \frac{y}{u} - 2$

$\frac{\partial P}{\partial u} = \frac{1}{u} \quad \frac{\partial Q}{\partial y} = \frac{1}{u}$

Equation is exact

$F(y,u) = \int (\frac{y}{u} - 2)\,du$

$= y \ln u - 2u + \phi(y)$

$\frac{\partial F}{\partial y} = \ln u + \phi'(y) = \ln u$

$\phi'(y) = 0, \; \phi(y) = C$

$F(y,u) = \boxed{y \ln u - 2u = C}$

20) $2xy^2 + 4x^3 + 2x^2 y \frac{dy}{dx} = 0$

$2xy^2 + 4x^3 = -2x^2 y \frac{dy}{dx}$

$(2xy^2 + 4x^3)\,dx + 2x^2 y \, dy = 0$

$P = 2xy^2 + 4x^3 \quad Q = 2x^2 y$

$\frac{\partial P}{\partial y} = 4xy \quad \frac{\partial Q}{\partial x} = 4xy$

Equation is exact

$F(x,y) = \int (2xy^2 + 4x^3)\,dx$

$= x^2y^2 + x^4 + \phi(y)$

$\frac{\partial F}{\partial y} = 2x^2 y + \phi'(y) = Q$

$\phi'(y) = 0, \; \phi(y) = C$

$\boxed{x^2y^2 + x^4 = C}$

22) $(y^2 - xy)\,dx + x^2\,dy = 0, \; M(x,y) = \frac{1}{xy^2}$

$(\frac{1}{x} - \frac{1}{y})\,dx + \frac{x}{y^2}\,dy = 0$

$\frac{\partial P}{\partial y} = \frac{1}{y^2} \quad \frac{\partial Q}{\partial x} = \frac{1}{y^2}$

$F(x,y) = \int (\frac{1}{x} - \frac{1}{y})\,dx$

$= \ln x - \frac{x}{y} + \phi(y)$

$\frac{\partial F}{\partial y} = \frac{x}{y^2} + \phi'(y)$

$\phi'(y) = 0, \; \phi(y) = C$

$\boxed{\ln x - \frac{x}{y} = C}$

24) $3(y+1)\,dx - 2x\,dy = 0, \; M(x,y) = \frac{y+1}{x^4}$

$\frac{3}{x^4}(y+1)^2\,dx - \frac{2(y+1)}{x^3} = 0$

$P = \frac{3(y+1)^2}{x^4} \quad Q = \frac{-2(y+1)}{x^4}$

$\frac{\partial P}{\partial y} = \frac{3}{x^4}(2)(y+1) \quad \frac{\partial Q}{\partial x} = \frac{6(y+1)}{x^4}$

$F(x,y) = \int \frac{3(y+1)^2}{x^4}\,dx$

$= 3(y+1)^2 \int \frac{1}{x^4}\,dx$

$= -(y+1)^2 \frac{1}{x^3} + \phi(y)$

$\frac{\partial F}{\partial y} = -2(y+1)(\frac{1}{x^3}) + \phi'(y)$

$\phi'(y) = 0, \; \phi(y) = C$

$\boxed{-\frac{(y+1)^2}{x^3} = C}$

28) $2y\,dx + (x+y)\,dy = 0$

$P = 2y, \; Q = x + y$

$\frac{\partial P}{\partial y} = 2, \; \frac{\partial Q}{\partial x} = 1$

$h(y) = \frac{1}{2y}$

$M(y) = e^{-\int 1/2y \, dy}$

$= e^{-1/2 \ln y}$

$= \frac{1}{\sqrt{y}}$

$\frac{2y}{\sqrt{y}}\,dx + \frac{(x+y)}{\sqrt{y}}\,dy = 0$

$P = \frac{2y}{\sqrt{y}}, \; Q = \frac{(x+y)}{\sqrt{y}}$

$\frac{\partial P}{\partial y} = \frac{1}{\sqrt{y}}, \; \frac{\partial Q}{\partial x} = \frac{1}{\sqrt{y}}$

$F(x,y) = \int \frac{2y}{\sqrt{y}} = \int 2y^{1/2}\,dy$

$F(x,y) = \frac{4}{3} y^{3/2} + \phi(y)$

$\frac{\partial F}{\partial y} = 2y^{1/2} + \phi'(y) = C$

$\phi'(y) + 2\sqrt{y} = xy^{-1/2} + y^{1/2}$

$\phi'(y) = -y^{1/2} + xy^{-1/2}$

$= \int -y^{1/2} + xy^{-1/2}\,dy$

$-\frac{2}{3}y^{3/2} + 2xy^{1/2} + C$

$\boxed{\frac{2}{3}y^{3/2} + 2xy^{1/2} = C}$

30) $2y\,dx + 3x\,dx = 0$

$P = 2y$, $Q = 3x$

$\frac{\partial P}{\partial y} = 2$ $\frac{\partial Q}{\partial x} = 3$

$x^a y^b$

$P = 2y x^a y^b$ $Q = 3x x^a y^b$

$P = 2x^a y^{b+1}$ $Q = 3x^{a+1} y^b$

$\frac{\partial P}{\partial y} = 2(b+1)x^a y^b$ $\frac{\partial Q}{\partial x} = 3(a+1)x^a y^b$

$2(b+1) = 3(a+1)$

$2b+2 = 3a+3$

$2b = 3a+1$

$b = \frac{3a+1}{2} x$

$P = (2y+3x)dx$ $Q = 0$

$2y\,dx + 3x\,dy = 0$

$M(x,y) = x^a b^y$

$P = 2y$, $Q = 3x$

$\frac{\partial P}{\partial y} = 2$, $\frac{\partial Q}{\partial x} = 3$

$P = 2y x^a y^b$ $Q = 3x x^a y^b$

$P = 2x^a y^{b+1}$, $Q = 3x^{a+1} y^b$

$\frac{\partial P}{\partial y} = 2(b+1)x^a y^b$ $\frac{\partial Q}{\partial x} = 3(a+1)x^a y^b$

$(2b+2)x^a y^b = (3a+3)x^a y^b$

$2b+2 = 3a+3$

$2b = 3a+1$

$b = \frac{3}{2}a + \frac{1}{2}$

$a = 1$, $b = 2$

$M(x,y) = xy^2$

$2y(xy^2)dx + 3x(xy^2)dy = 0$

$2xy^3 dx + 3x^2y^2 dy = 0$

$P = 2xy^3$ $Q = 3x^2y^2$

$\frac{\partial P}{\partial y} = 6xy^2$ $\frac{\partial Q}{\partial x} = 6xy^2$

Eq. is exact ✓

$F(x,y) = \int 6xy^2\,dx$

$= 3x^2y^2 + \phi(y)$

$\frac{\partial F}{\partial y} = 6x^2y + \phi'(y) = Q(x,y)$

$6x^2y + \phi'(y) = 3x^2y^2$

$\phi'(y) = -6x^2y + 3x^2y^2$

$\phi(y) = \int(-6x^2y + 3x^2y^2)dy$

$= -3x^2y^2 + x^2y^3 + C$

$F(x,y) = 3x^2y^2 - 3x^2y^2 + x^2y^3 + C$

$\boxed{x^2y^3 = C}$

34) $(\ln x - \ln y)dx + dy = 0$

$P = (\ln x - \ln y)$ $Q = 1$

$P = \ln\frac{x}{y}$

$P(tx,ty) = \ln\frac{tx}{ty} = \ln\frac{x}{y}$

$Q(tx,ty) = 1$

$\boxed{\text{both homogenous of degree } 0}$

36) $(x+y)dx + (y-x)dx = 0$

$P = (x+y)$ $Q = y-x$

$P(tx,ty) = (tx+ty) = t(x+y)$ deg. 1

$Q(tx,ty) = (ty-tx) = t(y-x) \Rightarrow$ deg. 1

$y = xv$, $dy = v\,dx + x\,dv$

$(x+xv)dx + (xv-x)(v\,dx + x\,dv) = 0$

$(x+xv)dx + xv^2dx - xv\,dx + x^2v\,dv - x^2dv = 0$

$x\,dx + xv\,dx + xv^2dx - xv\,dx + x^2v\,dv - x^2dv = 0$

$x\,dx + xv^2dx + x^2v\,dv - x^2dv = 0$

$(1+v^2)dx + x(v-1)dv = 0$

$\frac{dx}{x} = -\frac{(v-1)}{1+v^2}dv$

$\int\frac{dx}{x} = \int\frac{1}{1+v^2}dv \int -\frac{v}{1+v^2}dv$

$\ln x + C = \arctan(v) - \frac{1}{2}\ln(1+v^2)$

$\boxed{\arctan\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) - \ln x = C}$

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38) $\frac{dy}{dx} = \frac{y(x^2+y^2)}{xy^2-2x^3}$

$y(x^2+y^2)dx = (xy^2-2x^3)dy$

$P = y(x^2+y^2)$ $Q = 2x^3-xy^2$

$\frac{\partial P}{\partial y} = x^2+3y^2$ $\frac{\partial Q}{\partial x} = 6x^2-y^2$

$P(tx,ty) = ty(t^2x^2+t^2y^2) \rightarrow$ deg. 3

$Q(tx,ty) = t^3(2x^3-xy^2) \rightarrow$ deg. 3

$y = vx$ $dy = vdx + xdv$

$y(x^2+y^2)dx + (2x^3-xy^2)dy = 0$

$(vx^3+v^3x^3)dx + (2x^3-v^2x^3)(vdx+xdv) = 0$

$(vx^3+v^3x^3)dx + 2vx^3dx - v^3x^3dx + 2x^4dv - v^2x^4dv = 0$

$3vx^3dx + (2x^4-v^2x^4)dv = 0$

$3vdx + x(2-v^2)dv = 0$

$3vdx = -x(2-v^2)dv$

$\frac{dx}{x} = \frac{v^2-2}{3v}$

$\frac{dx}{x} = \frac{v}{3} - \frac{2}{3v}$

$\ln x = \frac{1}{6}v^2 - \frac{2}{3}\ln v + C$

$6\ln x = \frac{y^2}{x^2} - 4\ln(\frac{y}{x}) + C$

$6\ln x = \frac{y^2}{x^2} - 4\ln(y) + 4\ln x + C$

$\boxed{2\ln x + 4\ln y - \frac{y^2}{x^2} = C}$

40) $(y+2xe^{-y/x})dx - xdy = 0$

both homogenous, deg. 1

$P = y+2xe^{-y/x}$ $Q = x$

$y = vx$, $dy = vdx + xdv$

$(vx+2xe^{-v})dx - x(vdx+xdv) = 0$

$vxdx + 2xe^{-v}dx - vxdx - x^2dv = 0$

$2xe^{-v}dx - x^2dv = 0$

$2e^{-v}dx - xdv = 0$

$\frac{dx}{x} = \frac{dv}{2e^{-v}}$

$\frac{dx}{x} = \frac{e^v}{2}dv$

$\ln x = \frac{1}{2}e^v + C$

$\ln x = \frac{1}{2}e^{y/x} + C$

$\boxed{\ln x - \frac{1}{2}e^{y/x} = C}$