20W-MATH33B-1 Midterm 1

CHARLES ZHANG

TOTAL POINTS

47 / 50

QUESTION 1

1 Q1 10 / 10

✓ **- 0 pts** Correct

- 3 pts Find the integrating factor or homogeneous solution.

- 2 pts Multiply whole equation by integrating factor

- 2 pts Integrate the equation

- 3 pts Solve the initial value problem

- 1 pts Minor computational mistake

- 2 pts Wrong sign in integrating factor/homogeneous solution

- 2 pts Find v' in variation of parameters

QUESTION 2

2 Q2 8 / 10

- 0 pts Correct

✓ **- 1 pts Solve explicitly for y.**

- 2 pts Algebra/integration error (see explanation/arrow)

- 1 pts Solutions are given by setting F(x,y) = C; you've just written out F(x,y).

✓ **- 1 pts Simplify by clearing out the natural logs and/or absolute values.**

- 2 pts Where is your undetermined constant C?

- 1 pts Rewrite v in terms of y and x.

QUESTION 3

Q3 15 pts

3.1 (a) 5 / 5

✓ **- 0 pts** Correct

- 2 pts correct idea

- 3 pts no explanation but right answer

- 1 pts miscellaneous mistakes

- 4 pts tried

3.2 (b) 10 / 10

✓ **- 0 pts** Correct

- 1 pts miscellaneous mistake

- 8 pts tried

- 6 pts used exactness

- 5 pts had the right idea

- 3 pts had the right idea and made a logical mistake

QUESTION 4

15 pts

4.1 a 4 / 5

+ 4 Point adjustment

e^t

4.2 b 3 / 3

✓ **- 0 pts** Correct

- 1 pts forget y<1 and y >2

- 3 pts blank answer

- 1 pts solutions can not cross each other

- 0.5 pts not dotted line. solution curve is continous

- 1 pts picture not correct.

- 2 pts where is the solution curves

- 0.5 pts 0<y<1 should be S-shape

- 2.5 pts not correct

- 1 pts 0<y<1 not correct

4.3 c 4 / 4

✓ **- 0 pts** Correct

- 4 pts blank

- 3 pts not a proof.

- 1 pts didn't check \partial f / \ partial y

- 0.5 pts \partial f / \ partial y wrong

- 2 pts Wrong theorem conditions.

- **- 2 pts** didn't check theorem condition
- **- 1 pts** didn't calculate partial derivative
- **- 1 pts** More detail
- **- 0.5 pts** minor mistake

4.4 d **3 / 3**

- ✓ **- 0 pts** Correct
- **- 3 pts** not correct
- **- 2 pts** with some reason

Midterm 1

Last Name: Zhang

First Name: Charles

Student ID: 305413659

Signature:

Section:

Tuesday:	Thursday:	
1A	1B	TA: YIH, SAMUEL
1C	1D	TA: KIM, BOHYUN
1E	1F	TA: BOSCHERT, NICHOLAS

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you.. Please make sure **your phone is silenced** and stowed where you cannot see it. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. You must **show your work** to receive credit. Please **circle or box your final answers.**

Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	15	
4	15	
Total:	50	

1. (10 points) Solve the initial value problem:

$$x^2y' + 2xy + 1 = 0, y(1) = 0$$

$y' = a(t)y + f$

$y' = a(x)y + f$

$x^2y' = -2xy - 1$

$y' = -\frac{2}{x}y - \frac{1}{x^2}$

$\underline{y_h}$

$y' = -\frac{2}{x}y$

$\frac{dy}{dx} = -\frac{2}{x}y$

$\frac{dy}{y} = -\frac{2}{x}dx$

$\ln|y| = -2\ln|x|$

$\ln|y| = \ln\frac{1}{x^2}$

$y_h = \frac{1}{x^2}$

$\underline{y_p}$

$a(x) = -\frac{2}{x}$

IF: $e^{\int -a(x)dx}$

IF: $e^{\int 2/x\, dx}$

IF: $e^{2\int 1/x\, dx}$

IF: $e^{2\ln x}$

IF: $e^{\ln x^2}$

IF: x^2

$x^2y' = x^2(-\frac{2}{x})y - 1$

$x^2y' - x^2(-\frac{2}{x})y = -1 \rightarrow -x^2(-\frac{2}{x})y = 2xy$ ✓

$(x^2y)' = -1 \rightarrow x^2y' + 2xy$ ✓

$x^2y = -x$

$y_p = -\frac{1}{x}$

$y = \frac{c}{x^2} - \frac{1}{x}$

$0 = \frac{c}{1} - 1$

$c = 1$

$$\boxed{y = \frac{1}{x^2} - \frac{1}{x}}$$

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2. (10 points) Solve the homogeneous equation:

$$(3x + 2y)dx + xdy = 0.$$

$y = vx, dy = vdx + xdv$

$(3x + 2vx)dx + x(vdx + xdv) = 0$

$3xdx + 2vxdx + vxdx + x^2dv = 0$

$3xdx + 3vxdx + x^2dv = 0$

$x(3 + 3v)dx + x(x)dv = 0$

$(3+3v)dx + xdv = 0$

$(3+3v)dx = -xdv$

$\frac{dx}{x} = \frac{-dv}{3+3v}$

$\ln|x| = -\int \frac{dv}{3+3v}$

$u = 3+3v, \; du = 3dv$

$\ln|x| = -\frac{1}{3}\int \frac{du}{u}$

$\ln|x| = -\frac{1}{3}\ln|3+3v| + C$

$C = \ln|x| + \frac{1}{3}\ln|3+3v|$

$C = \ln|x| + \frac{1}{3}\ln|3 + 3(\frac{y}{x})|$

$C = \ln|x| + \frac{1}{3}\ln|3 + \frac{3y}{x}|$

$\boxed{C = \ln|x| + \frac{1}{3}\ln\left|\frac{3x+3y}{x}\right|}$

3. Consider the following differential equation:

$$(5x^3 + 2y^2)dx + 2yxdy = 0$$

(a) (5 points) The above differential equation has a one-variable integrating factor (i.e. $\mu(x)$ or $\mu(y)$) Find the intergrating factor.

$P = 5x^3 + 2y^2 \qquad Q = 2yx$

$\frac{\partial P}{\partial y} = 4y \qquad \frac{\partial Q}{\partial x} = 2y$

$\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 2y$

$h(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{2yx}(2y) = \frac{1}{x}$

$M(x) = e^{\int h(x)\partial x} = e^{\int 1/x \, dx} = e^{\ln x}$

$\boxed{M(x) = x}$

(b) (10 points) Find the general solutions to the above differential equations.

$x(5x^3 + 2y^2)\,dx + 2yx^2 dy = 0$

$\frac{\partial P}{\partial y} = 4xy \qquad \frac{\partial Q}{\partial x} = 4xy$ ✓

$F(x,y) = \int P\partial x = \int (5x^4 + 2xy^2)\,dx = x^5 + x^2y^2 + \phi(y)$

$Q = \frac{\partial F}{\partial y} = 2x^2y + \phi'(y) = 2x^2y$

$\phi'(y) = 0, \ \phi(y) = C$

$F(x,y) = \boxed{x^5 + x^2y^2 = C}$

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4. Consider the autonomous equation:

$$y' = (y-1)(y-2)$$

(a) (5 points) Find the general solutions $y(t)$ to the above differential equations.

$\frac{dy}{dt} = (y-1)(y-2) \to$ zeros : $\boxed{y(t) \equiv 1,\ y(t) \equiv 2}$ Equilibrium Sol.

$\frac{dy}{(y-1)(y-2)} = dt$

$\frac{A}{y-1} + \frac{B}{y-2} = dt$

$Ay - 2A + By - B = 1$

$A = -B$

$-2A - B = 1$

$A = -\frac{1}{3}$

$B = \frac{1}{3}$

$dy\left(-\frac{1}{3(y-1)} + \frac{1}{3(y-2)}\right) = dt$

$\frac{1}{3}\int\left(\frac{1}{y-2} - \frac{1}{y-1}\right)dy = \int dt$

$\frac{1}{3}\left[\ln|y-2| - \ln|y-1|\right] = t + C$

$\ln\left|\frac{y-2}{y-1}\right| = 3t + K$

$\frac{y-2}{y-1} = e^{3t+K} = Ae^{3t}$

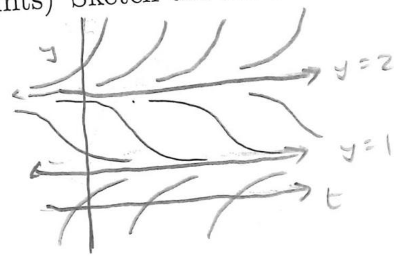
$y - 2 = Ae^{3t}y - Ae^{3t}$

$y - Ae^{3t}y = 2 - Ae^{3t}$

$y(1 - Ae^{3t}) = 2 - Ae^{3t}$

$\boxed{y(t) = \frac{2 - Ae^{3t}}{1 - Ae^{3t}}}$

(b) (3 points) Sketch the solutions on the $t - y$ plane.



(c) (4 points) Prove that if $y(t)$ is a solution and $y(0) = 1.9$, then $1 < y(t) < 2$ for all $t \in (-\infty, \infty)$

$f(t,y) = (y-1)(y-2)$ is a continuous function with regards to y on the given interval

$\frac{\partial f}{\partial y} = 2y - 3$ is also a continuous function with regards to y on the given interval

Therefore, the function satisfies uniqueness. Since $y \equiv 1$ and $y \equiv 2$ are equilibrium solutions, and $y(0) = 1.9$, which is between 2 solutions, the function $y(t)$ has to satisfy $1 < y(t) < 2$, as it cannot cross another solution.

(d) (3 points) Let $y(t)$ be the funtion in part(c). Calculate $\lim_{t\to+\infty} y(t)$.

2 is an unstable solution

$\therefore \boxed{\lim_{t\to\infty} y(t) = 1}$

$\lim_{t\to\infty} \frac{2 - Ae^{3t}}{1 - Ae^{3t}}$

$= \lim_{t\to\infty} \frac{-3Ae^{3t}}{-3Ae^{3t}} = 1$ ✓

Scratch Paper

$$y' = (y-1)(y-2)$$

$$\frac{\partial y}{\partial t} = y^2 - 3y + 2$$

$$\frac{\partial y}{(y-1)(y-2)} = \partial t$$

$$\frac{A}{y-1} + \frac{B}{y-2} = \frac{1}{(y-1)(y-2)}$$

$$Ay - 2A + By - B = 1$$

$$A = -B$$

$$-2A - B = 1$$

$$-2A - A = 1$$

$$A = -\frac{1}{3}$$

$$B = \frac{1}{3}$$

$$\int \left(\frac{1}{3(y-1)} + \frac{1}{3(y-2)} \right) \partial y = t$$

$$\frac{1}{3} \ln|y-1| + \frac{1}{3} \ln|y-2| = t$$

$$\ln|y-1| + \ln|y-2| = 3t$$

$$\ln|(y-1)(y-2)| = 3t$$

$(y$

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Some useful formulas, etc:

Integrating factor $u(x)$ of a 1st Order Linear DE $x' = ax + f$:

$$u(x) = e^{-\int a(t)dt}$$

Single variable integrating factor μ for $Pdx + Qdy = 0$

- If $h(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$,

$$\mu(x) = e^{\int h(x)dx}$$

- If $g(y) = \frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$,

$$\mu(y) = e^{-\int g(y)dy}$$