

Math 33A Sheet 6

Q1) $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$v_1 = \dots = v_m = 0$

$c_1(0) + \dots + c_m(0) = 0$

$\hookrightarrow c_1, \dots, c_m$ can be any real number

\hookrightarrow there are infinitely many solutions to

$c_1 v_1 + \dots + c_m v_m = 0$

$\hookrightarrow \vec{0}$ is linearly dependent

Q2) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \text{rref}(A)$

basis of $\text{im}(A) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}$
basis of $\text{im}(\text{rref}(A)) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Chapter 3.4

Ex 22) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

a) $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & | & 1 & 0 \\ 1 & -1 & | & 0 & 1 \end{bmatrix} \xrightarrow{-I}$

$\begin{bmatrix} 1 & 1 & | & 1 & 0 \\ 0 & -2 & | & -1 & 1 \end{bmatrix} \xrightarrow{\times -1/2}$

$\begin{bmatrix} 1 & 1 & | & 1 & 0 \\ 0 & 1 & | & 1/2 & -1/2 \end{bmatrix} \xrightarrow{-II}$

$\begin{bmatrix} 1 & 0 & | & 1/2 & 1/2 \\ 0 & 1 & | & 1/2 & -1/2 \end{bmatrix}$

$B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

$= \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

b) $\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 \Rightarrow T(\vec{x}) = A\vec{x} = 2c_1 \vec{v}_1$

$\downarrow \quad \downarrow$
 $\begin{bmatrix} \vec{x} \end{bmatrix}_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \xrightarrow{B} \begin{bmatrix} T(\vec{x}) \end{bmatrix}_B = \begin{bmatrix} 2c_1 \\ 0 \end{bmatrix}$

$B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

c) $B = [T(\vec{v}_1)]_B [T(\vec{v}_2)]_B$

$B = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

Ex 27) $A = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix} \xrightarrow{\times 1/4}$

$= \begin{bmatrix} 1 & 1/2 & -1 \\ 2 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-II}$

$= \begin{bmatrix} 1 & 1/2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$T(\vec{x}) = A\vec{x} \rightarrow$

$\times \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 + 1/2 c_2 + c_3 \\ 0 \\ 0 \end{bmatrix}$

$S = \begin{bmatrix} 2 & 0 & 1 & | & 1 & 0 & 0 \\ 1 & 2 & 0 & | & 0 & 1 & 0 \\ -2 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\times 1/2}$

$\begin{bmatrix} 1 & 0 & 1/2 & | & 1/2 & 0 & 0 \\ 1 & 2 & 0 & | & 0 & 1 & 0 \\ 0 & 1 & 2 & | & 1 & 0 & 1 \end{bmatrix} \xrightarrow{-I}$

$\begin{bmatrix} 1 & 0 & 1/2 & | & 1/2 & 0 & 0 \\ 0 & 2 & -1/2 & | & -1/2 & 1 & 0 \\ 0 & 1 & 2 & | & 1 & 0 & 1 \end{bmatrix} \xrightarrow{\times 1/2}$

$\begin{bmatrix} 1 & 0 & 1/2 & | & 1/2 & 0 & 0 \\ 0 & 1 & -1/4 & | & -1/4 & 1/2 & 0 \\ 0 & 1 & 2 & | & 1 & 0 & 1 \end{bmatrix} \xrightarrow{-II}$

$\begin{bmatrix} 1 & 0 & 1/2 & | & 1/2 & 0 & 0 \\ 0 & 1 & -1/4 & | & -1/4 & 1/2 & 0 \\ 0 & 0 & 9/4 & | & 5/4 & -1/2 & 1 \end{bmatrix} \xrightarrow{\times 4/9}$

$\begin{bmatrix} 1 & 0 & 1/2 & | & 1/2 & 0 & 0 \\ 0 & 1 & -1/4 & | & -1/4 & 1/2 & 0 \\ 0 & 0 & 1 & | & 5/9 & -2/9 & 4/9 \end{bmatrix} \xrightarrow{-1/2 III} \xrightarrow{+1/4 III}$

$\begin{bmatrix} 1 & 0 & 0 & | & 2/9 & 1/9 & -2/9 \\ 0 & 1 & 0 & | & -1/9 & 4/9 & 1/9 \\ 0 & 0 & 1 & | & 5/9 & -2/9 & 4/9 \end{bmatrix}$

$S^{-1} = \frac{1}{9} \begin{bmatrix} 2 & 1 & -2 \\ -1 & 4 & 1 \\ 5 & -2 & 4 \end{bmatrix}$

$B = \frac{1}{9} \begin{bmatrix} 2 & 1 & -2 \\ -1 & 4 & 1 \\ 5 & -2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ -2 & 1 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ -2 & 1 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Ex 32) $T(\vec{x}) = \vec{x} \times \vec{v}_3$

$T(\vec{v}_1) = -\vec{v}_2$

$T(\vec{v}_2) = \vec{v}_1$

$T(\vec{v}_3) = 0$

$[T(\vec{v}_1)]_B = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

$[T(\vec{v}_2)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$[T(\vec{v}_3)]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Rotation $\frac{\pi}{2}$ clockwise

Ex 37) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \vec{v}_1$

$\vec{v}_2 \perp \vec{v}_1$

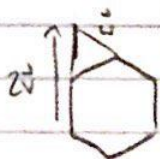
$\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\therefore T(\vec{x}) = c_1 \vec{v}_1, \vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2$

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

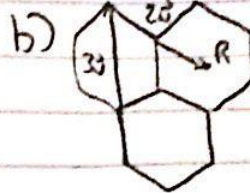
Ex 50) a) \vec{v}



$[\vec{ow}]_B = 2\vec{v} + \vec{w}$



$[\vec{ow}]_B = \vec{v} + 2\vec{w}$



Center of a tile

c) $\begin{bmatrix} 1 & 7 \\ 4 & 7 \end{bmatrix}$

odd # \vec{v} = from upper vertex

odd # \vec{w} = at a vertex
vertex

Ex 61) $T(\vec{x}) = \begin{bmatrix} -5 & -9 \\ 4 & 7 \end{bmatrix} \vec{x}, B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2$

$[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \xrightarrow{B} [T(\vec{x})]_B = \begin{bmatrix} c_1 + c_2 \\ c_2 \end{bmatrix}$

$\begin{bmatrix} -5 & -9 \\ 4 & 7 \end{bmatrix} \xrightarrow{S} \begin{bmatrix} c_1 + c_2 \\ c_2 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = [T(\vec{v}_1)]_B \rightarrow A \vec{v}_1$

$\begin{bmatrix} -5 & -9 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 & -9 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$-5a - 9b = 1$

$4a + 7b = 0$

$-a - \frac{7}{4}b = \frac{1}{4}$

$-4a - \frac{7}{4}b = \frac{1}{4}$

$-\frac{1}{4}b = \frac{1}{4}$

$b = -4, a = 7$

$\begin{bmatrix} -5 & -9 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$-5c - 9d = 1$

$4c + 7d = 1$

$-4c - \frac{7}{4}d = \frac{1}{4}$

$-\frac{1}{4}d = \frac{1}{4}$

$d = -1, c = 16$

$\begin{bmatrix} 7 \\ -4 \end{bmatrix}, \begin{bmatrix} 16 \\ -1 \end{bmatrix}$

Chapter 5.1

Ex 1)* $\vec{v} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$

$$\|\vec{v}\| = \sqrt{7^2 + 11^2} = \sqrt{170}$$

Ex 4) $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

$$\theta = \arccos\left(\frac{18}{\sqrt{2} \sqrt{170}}\right)$$

$$\theta = \arccos\left(\frac{18}{\sqrt{340}}\right)$$

Ex 10) $\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ k \\ 1 \end{bmatrix}$

$$2(1) + 3k + 4(1) = 0$$

$$6 + 3k = 0$$

$$k = -2$$

Ex 16) $\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$

$$\vec{u}_4 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$x_1 + x_2 + x_3 + x_4 = 0$$

$$x_1 + x_2 - x_3 - x_4 = 0$$

$$x_1 - x_2 + x_3 - x_4 = 0$$

$$-2x_3 - 2x_4 = 0$$

$$-2x_2 - 2x_4 = 0$$

$$x_3 + x_4 = 0$$

$$x_2 + x_4 = 0$$

$$x_1 - x_4 = 0$$

$$x_4 = t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ -t \\ t \end{bmatrix}$$

$|t|/2 \leftarrow$ unit vector

$$\vec{u}_4 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \text{ or } \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

Ex 17)* $W = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}\right)$

$U^\perp \rightarrow$ all vectors perpendicular to U

\therefore find perpendicular to basis of U

$$0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \text{ and } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = 0$$

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$

$$5x_1 + 6x_2 + 7x_3 + 8x_4 = 0$$

$$-4x_2 - 8x_3 - 12x_4 = 0$$

$$x_2 + 2x_3 + 3x_4 = 0$$

$$x_1 - x_3 - 2x_4 = 0$$

$$x_3 = t, x_4 = s$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t + 2s \\ -2t - 3s \\ t \\ s \end{bmatrix}$$

$$= t \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Ex 28)* $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\text{proj}_W(\vec{x}) = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \dots + (\vec{u}_3 \cdot \vec{x})\vec{u}_3$$

$$\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \left(\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \right) + \dots$$

$$= \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 1/4 \\ -1/4 \\ -1/4 \end{bmatrix} + \begin{bmatrix} 1/4 \\ -1/4 \\ 1/4 \\ -1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 3/4 \\ 1/4 \\ -1/4 \\ 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 3/4 \\ 1/4 \\ -1/4 \\ 1/4 \end{bmatrix}$$