Math 33A Sheet 8

Chapter 5.3

Ex 1)* $\begin{bmatrix} 0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix}$

$\vec{v}_1 \cdot \vec{v}_2 = 0.48 + 0.48 = 0.96 \neq 0$

Not orthogonal

Ex 5) 3A

$A = [\vec{v}_1, \vec{v}_2, \vec{v}_3]$

$\|\vec{v}_1\| = 1$

$3A = [3\vec{v}_1, 3\vec{v}_2, 3\vec{v}_3]$

$\|3\vec{v}_1\| = 3$

not orthogonal

Ex 7) AB

$A = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n]$

$B = [\vec{w}_1 \ \vec{w}_2 \ \dots \ \vec{w}_n]$

Orthogonal, product of 2 orthogonal matrices is orthogonal

Ex 15) AB

$(AB)^T = B^T A^T = BA$

$AB \neq BA$

Not symmetric

Ex 17) B^{-1}

$(B^T)^{-1} = (B^{-1})^T$

$B = B^T$

$(B^{-1})^T = (B^T)^{-1} = B^{-1}$

Symmetric

Ex 23) $A - A^T$

$(A - A^T)^T = A^T - A$

$A - A^T \neq A^T - A$

not symmetric

Ex 29)* Orthogonal matrices preserve dot product and length

$\theta_i = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right)$

$= \arccos\left(\frac{L(\vec{v}) \cdot L(\vec{w})}{\|L(\vec{v})\|\|L(\vec{w})\|}\right)$

$= \theta_f$ because all values are the same

$\therefore \theta_i = \theta_f$

The converse doesn't need to be true, a scaling preserves angles, but not length, and is therefore not orthogonal

Ex 30) $\vec{0}$ is the only vector that has length of zero, therefore $\ker(L) = \{0\}$

The image's dimension must be equal to m because of rank-nullity

Since the dimension of the image/rank must be m, $m \leq n$

The columns of A must be linearly independent because $\ker(L) = \ker(A) = \vec{0}$

$A^T A = I_m$ because the $\ker(A) = \vec{0}$, therefore A is invertible

AA^T is the matrix of an orthogonal projection

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix}$$

Ex32) a) $A^TA = I_n$

$AA^T \stackrel{?}{=} I_n$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

$AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I_3$

Not true

b) $A^{-1}A = I_n = AA^{-1}$

$\therefore A^TA = I_n = AA^T$

Always true

Chapter 6.1

Ex6) $\begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

$\begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{matrix} 6 & 0 \\ 5 & 4 \\ 3 & 2 \end{matrix}$

$\det(A) = 24 + 0 + 0 - 0 - 0 - 0$

$= \boxed{24}$ invertible

Ex8) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{matrix} 1 & 2 \\ 1 & 1 \\ 3 & 2 \end{matrix}$

$\det(A) = 1 + 6 + 6 - 2 - 2 - 9$

$= \boxed{0}$ not invertible

Chapter 6.2

Ex8) $\begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}$ $\times \frac{1}{2}$ $k_1 = 2$

$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}$ $-3I$, $-4I$, $-5I$, $-6I$

$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ $s = 4$

$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$\det(B) = 1$

$\det(A) = (-1)^4(2)\det(B)$

$= 1(2)(1)$

$= \boxed{2}$

Ex16) $\det \begin{bmatrix} 6\vec{v}_1 + 2\vec{v}_4 \\ \vec{v}_2 \\ \vec{v}_3 \\ 3\vec{v}_1 + \vec{v}_4 \end{bmatrix}$ $k_1 = \frac{1}{6}$

$\det(A) = 8$

$\det(A) = (-1)^s(k_1)(\det(B))$

$8 = 1(\frac{1}{6})\det(B)$

$\det(B) = 48$

Ex59) $\det(A) = \det(B)$

$A \stackrel{?}{=} S^{-1}BS$

$A = I_2 \rightarrow \det(A) = 1$

$B = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} \rightarrow \det(B) = 1$

Identity matrix is only similar to itself

False

Chapter 6.1

Ex 2) $\begin{bmatrix}3\\7\end{bmatrix}, \begin{bmatrix}5\\2\end{bmatrix}$

$A = \frac{1}{2}\det\begin{bmatrix}3&5\\7&2\end{bmatrix}$

$= \frac{1}{2}(16-56) = \frac{1}{2}(50)$

$= \boxed{25}$

Ex 6)* $\begin{bmatrix}a_1\\a_2\\1\end{bmatrix}, \begin{bmatrix}b_1\\b_2\\1\end{bmatrix}, \begin{bmatrix}c_1\\c_2\\1\end{bmatrix}$

$A_1 = \frac{1}{6}\det\left|\begin{bmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\1&1&1\end{bmatrix}\right|$

$\begin{bmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\1&1&1\end{bmatrix}\begin{matrix}a_1&b_1\\a_2&b_2\\1&1\end{matrix}$

$a_1b_2 + b_1c_2 + c_1a_2 - b_1a_2 - a_1c_2 - c_1b_2$

$A_1 = \frac{1}{6}\left|a_1b_2 + b_1c_2 + a_2c_1 - a_2b_1 - a_1c_2 - b_2c_1\right|$

$A_2 = \frac{1}{2}\left|\det\begin{bmatrix}b_1-a_1&c_1-a_1\\b_2-a_2&c_2-a_2\end{bmatrix}\right|$

$\det\begin{bmatrix}b_1-a_1&c_1-a_1\\b_2-a_2&c_2-a_2\end{bmatrix}$

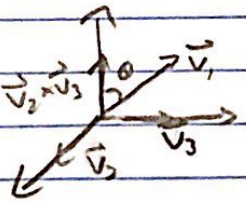
$b_1c_2 + a_1a_2 - a_1c_2 - a_2b_1 - (b_2c_1 + a_1a_2 - a_2c_1 - a_1b_2)$

$= -a_1b_2 + b_1c_2 + a_2c_1 - a_2b_1 - a_1c_2 - b_2c_1$

$A_2 = \frac{1}{2}\left|a_1b_2 + b_1c_2 + a_2c_1 - a_2b_1 - a_1c_2 - b_2c_1\right|$

$\boxed{A_1 = \frac{1}{3}A_2}$

Ex 19)



$\det(A) = \vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3)$

↳ positive only when θ between $\vec{v}_1$ and $\vec{v}_2 \times \vec{v}_3$ is $< \frac{\pi}{2}$

Ex 20) $T(\vec{x}) = A\vec{x}$

x is some 3×3 matrix B that is positively oriented

$T(\vec{x}) = AB$

$\det(AB) > 0$

$\det(B) > 0$

$\det(A) \times \det(B) = \det(AB)$

$\therefore$ det(#) × number = + number

$\therefore$ det(A) must be positive

Q1) $A = [T(\vec{e}_1)\ T(\vec{e}_2)]$

[put into another basis

↳ vector that spans the line, vector ⊥ to first vector

$m = \frac{y}{x}$

$x = 1 \quad m = y$

$\vec{b}_1 = \begin{bmatrix}1\\m\end{bmatrix}$

$m = -\frac{x}{y} \rightarrow y = 1$

$x = -m$

$\vec{b}_2 = \begin{bmatrix}-m\\1\end{bmatrix}$

$S = \begin{bmatrix}1&-m\\m&1\end{bmatrix}$

$[T(\vec{x})]_B = \begin{bmatrix}1&0\\0&-1\end{bmatrix}\vec{x}$

$\begin{bmatrix}1&-m\\m&1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix} = \begin{bmatrix}1&m\\m&-1\end{bmatrix}$

$T(\vec{x}) = \begin{bmatrix}1&m\\m&-1\end{bmatrix}\vec{x}$

$T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right) = \begin{bmatrix}1&m\\m&-1\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}$

$= v_1\begin{bmatrix}1\\m\end{bmatrix} + v_2\begin{bmatrix}m\\-1\end{bmatrix} = \begin{bmatrix}v_1 + v_2m\\v_1m - v_2\end{bmatrix}$

$(v_1^2 + 2v_1v_2m + v_2^2m^2 + v_1^2m^2 - 2v_1v_2m + v_2^2)$

$= (v_1^2 + v_2^2m^2 + v_1^2m^2 + v_2^2)$

Well, that didn't go right :(