EXERCISES 3.1

GOAL *Use the concepts of the image and the kernel of a linear transformation (or a matrix). Express the image and the kernel of any matrix as the span of some vectors. Use kernel and image to determine whether a matrix is invertible.*

For each matrix A in Exercises 1 through 13, find vectors that span the kernel of A. Use paper and pencil.

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

3. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

7. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

For each matrix A in Exercises 14 through 16, find vectors that span the image of A. Give as few vectors as possible. Use paper and pencil.

14. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

15. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

16. $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$

For each matrix A in Exercises 17 through 22, describe the image of the transformation $T(\vec{x}) = A\vec{x}$ geometrically (as a line, plane, etc. in $\mathbb{R}^2$ or $\mathbb{R}^3$).

17. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

18. $A = \begin{bmatrix} 1 & 4 \\ 3 & 12 \end{bmatrix}$

19. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & -8 \end{bmatrix}$

Describe the images and kernels of the transformations in Exercises 23 through 25 geometrically.

23. Reflection about the line $y = x/3$ in $\mathbb{R}^2$

24. Orthogonal projection onto the plane $x + 2y + 3z = 0$ in $\mathbb{R}^3$

34. Give an example of a linear transformation whose kernel is the line spanned by

$$\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

in $\mathbb{R}^3$.

EXERCISES 3.2

GOAL *Check whether or not a subset of $\mathbb{R}^n$ is a subspace. Apply the concept of linear independence (in terms of Definition 3.2.3, Theorem 3.2.7, and Theorem 3.2.8). Apply the concept of a basis, both in terms of Definition 3.2.3 and in terms of Theorem 3.2.10.*

Which of the sets W in Exercises 1 through 3 are subspaces of $\mathbb{R}^3$?

1. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 1 \right\}$

2. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \leq y \leq z \right\}$

3. $W = \left\{ \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix} : x, y, z \text{ arbitrary constants} \right\}$

4. Consider the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ in $\mathbb{R}^n$. Is span $(\vec{v}_1, \ldots, \vec{v}_m)$ necessarily a subspace of $\mathbb{R}^n$? Justify your answer.

5. Give a geometrical description of all subspaces of $\mathbb{R}^3$. Justify your answer.

6. Consider two subspaces V and W of $\mathbb{R}^n$.

   a. Is the intersection $V \cap W$ necessarily a subspace of $\mathbb{R}^n$?

   b. Is the union $V \cup W$ necessarily a subspace of $\mathbb{R}^n$?

7. Consider a nonempty subset W of $\mathbb{R}^n$ that is closed under addition and under scalar multiplication. Is W necessarily a subspace of $\mathbb{R}^n$? Explain.

8. Find a nontrivial relation among the following vectors:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

In Exercises 10 through 20, use paper and pencil to identify the redundant vectors. Thus determine whether the given vectors are linearly independent.

10. $\begin{bmatrix} 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

11. $\begin{bmatrix} 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 11 \\ 7 \end{bmatrix}$

12. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

13. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

14. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$

15. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

In Exercises 21 through 26, find a redundant column vector of the given matrix A, and write it as a linear combination of preceding columns. Use this representation to write a nontrivial relation among the columns, and thus find a nonzero vector in the kernel of A. (This procedure is illustrated in Example 8.)

21. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

23. $\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

25. $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

26. $\begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix}$

Find a basis of the image of the matrices in Exercises 27 through 33.

27. $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

28. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

29. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

30. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 5 & 8 \end{bmatrix}$