MATH 33A Midterm 2

CHARLES ZHANG

TOTAL POINTS

41 / 43

QUESTION 1

1 Problem 1 7 / 7

✓ **- 0 pts** Correct

- 2 pts Incorrect orthonormal basis for the x-z plane

- 2 pts Incorrect Projection Formula

- 1 pts Incorrect Projection. It should be (1,0,3)

- 2 pts Incorrect projection vector drawing

QUESTION 2

2 Problem 2 10 / 10

+ 1 pts Transforms v1 correctly [no credit if T is applied, as this is the wrong vector]

+ 1 pts Transforms 2v1 correctly [no credit if T is applied, as this is the wrong vector]

+ 3 pts Transforms e2 correctly [no credit if T is applied, as this is the wrong vector]

- 1 pts Computational error

+ 2 pts Transforms Te_2 instead of e_2 [need to specifically find the B-coordinates of Te_2, which are (7/5, -3/5) - no points for computing Ae_2, A^{-1}e_2, Be_2, or B^{-1}e_2]

✓ **+ 5 pts All transforms correct**

+ 3 pts Used S instead of S^{-1} for the transforms

✓ **+ 5 pts Finds B-matrix correctly**

- 1 pts Computational error

+ 2 pts Gives formula for B

+ 0 pts No credit

QUESTION 3

3 Problem 3 6 / 6

✓ **- 0 pts** Correct

- 2 pts Incorrect solutions to the matrix. That is, finding the kernel.

- 2 pts Incorrect basis for the kernel.

- 2 pts Incorrect basis for the image.

QUESTION 4

4 Problem 4 10 / 10

✓ **+ 10 pts** Correct

+ 5 pts Rubric Criterion 1: Correctly Calculates Q

+ 5 pts Rubric Criterion 2: Correctly Calculates R

+ 3 pts Potential Partial Credit: Student makes a small mistake in the calculation of R or Q, i.e. gets one entry incorrect.

+ 0 pts Incorrect/ insufficient progress

QUESTION 5

Problem 5 10 pts

5.1 2 / 2

✓ **+ 2 pts Correct, since the columns vectors span a subspace of R^4.**

+ 0 pts Wrong, if they were, they would span a 5-dimensional subspace of a 4-dimensional space!

+ 0 pts no answer

5.2 2 / 2

✓ **+ 2 pts Correct, as 5=dim(V)+dim(V_orth)**

+ 0 pts Wrong. For that to be true you would need to have R^n with n an even number instead of R^5

5.3 2 / 2

✓ **+ 2 pts Correct, since we can take S to be the indentity matrix, and then AS=SA**

+ 0 pts Wrong

+ 0 pts no answer

5.4 0 / 2

+ 2 pts Correct, since by the rank-nullity theorem the dimension of the kernel and image sum up to 5. The image can be at most 3-dimensional, thus the kernel has to be at least 2-dimensional.

✓ + **0 pts** **Wrong. For this to be true the image would need to be a 4-dimensional subspace of a 3-dimensional space.**

+ **0 pts** no answer

5.5 2 / 2

✓ + **2 pts** **Correct, since the kernel of the orthogonal projection is the orthogonal complement of V.**

+ **0 pts** Wrong

+ **0 pts** no answer

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Midterm 2 (Math 33A, Fall 2019)

Your Name: Charles Zhang

UCLA id: 305413659

Date: 11/18/19

The rules: You can answer using a pencil or ink pen. You are allowed to use only this paper, pencil or pen, and the scratch paper provided. You should not hand the scratch paper in. No calculators. No books, no notebooks, no notes, no mobile phones, no web access. You must write your name and UCLA id. You have exactly 50 minutes.

Points:

Problem 1: ____/7

Problem 2: ____/10

Problem 3: ____/6

Problem 4: ____/10

Problem 5: ____/10

Total: __________ (out of 43)

Problem 1 (7 points) ✓

Let V be the x-z plane in $\mathbb{R}^3$.

What is the orthogonal projection of $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto V?

You should (i) compute the projection $\text{proj}_V(v)$ and (ii) make a drawing of $\text{proj}_V(v)$.

Solution:

i) $V = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$

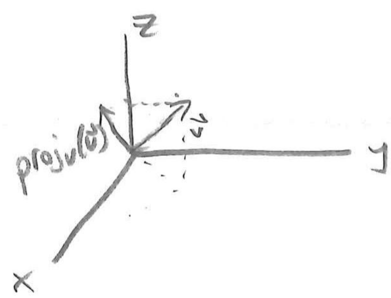
$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\text{proj}_V(\vec{v}) = (\vec{u}_1 \cdot \vec{v})\vec{u}_1 + (\vec{u}_2 \cdot \vec{v})\vec{u}_2$

$= \vec{u}_1 + 3\vec{u}_2$

$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}}$

ii)



Solution:

Problem 2 (10 points)

Let $\mathcal{B}$ be the basis of $\mathbb{R}^2$ given by the vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and

$$v_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Find the $\mathcal{B}$-coordinates of the vectors v_1, $2v_1$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

Find the $\mathcal{B}$-matrix of T.

Solution:

$\mathcal{B}$-coordinates of $\vec{v}_1$:

$$S = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \qquad S^{-1} = \begin{bmatrix} 3/5 & 1/5 \\ -2/5 & 1/5 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] -2\mathrm{I}$$

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 5 & -2 & 1 \end{array}\right] \times 1/5$$

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & -2/5 & 1/5 \end{array}\right] + \mathrm{II}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 3/5 & 1/5 \\ 0 & 1 & -2/5 & 1/5 \end{array}\right]$$

$$\begin{bmatrix} 3/5 & 1/5 \\ -2/5 & 1/5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3/5 \\ -2/5 \end{bmatrix} + \begin{bmatrix} 2/5 \\ 2/5 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

Solution:

B-coordinates of $2\vec{v}_1$:

$$\begin{bmatrix} 3/5 & 1/5 \\ -2/5 & 1/5 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 6/5 \\ -4/5 \end{bmatrix} + \begin{bmatrix} 4/5 \\ 4/5 \end{bmatrix}$$

$$\boxed{[2\vec{v}_1]_B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}}$$

B-coordinates of $\vec{e}_2$:

$$\begin{bmatrix} 3/5 & 1/5 \\ -2/5 & 1/5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\boxed{[\vec{e}_2]_B = \begin{bmatrix} 1/5 \\ 1/5 \end{bmatrix}}$$

Check:

$1/5\,\vec{v}_1 + 1/5\,\vec{v}_2$

$$\frac{1}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix} + \begin{bmatrix} -1/5 \\ 3/5 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \checkmark$$

B-matrix of T:

$$B = S^{-1}AS$$

$$B = \begin{bmatrix} 3/5 & 1/5 \\ -2/5 & 1/5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/5 & 7/5 \\ -4/5 & -3/5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

$$\boxed{B = \begin{bmatrix} 3 & 4 \\ -2 & -1 \end{bmatrix}}$$

Check:

$$B = \begin{bmatrix} [T(\vec{b}_1)]_B & [T(\vec{b}_2)]_B \end{bmatrix}$$

$$T(\vec{b}_1) = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \to 3\vec{v}_1 - 2\vec{v}_2$$

$$T(\vec{b}_2) = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \to 4\vec{v}_1 - \vec{v}_2$$

$$3\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \checkmark$$

$$4\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \checkmark$$

Problem 3 (6 points)

Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Find a basis for the image of A and a basis for the kernel of A.

Solution:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ -2I \\ +2I \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{rref}(A)$$

1st column has a pivot

$$\boxed{\therefore \text{basis for } \mathrm{im}(A) = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}}$$

x_2 and x_3 are free

$x_2 = t$, $x_3 = s$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ t \\ s \end{bmatrix}$$

$$= t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\text{basis for } \ker(A) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$$

Solution:

Problem 4 (10 points)

Compute the QR decomposition of the following matrix.

$$M = \begin{bmatrix} -1 & -1 \\ 1 & 3 \\ -1 & -1 \\ 1 & 3 \end{bmatrix}$$

Solution:

$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 3 \end{bmatrix},$

$\|\vec{v}_1\| = 2$

$\vec{u}_1 = \frac{\vec{v}_1}{2} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$

$\vec{v}_2^{\perp} = \vec{v}_2 - \mathrm{proj}_V(\vec{v}_2)$

$= \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$

$\vec{u}_1 \cdot \vec{v}_2 = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ -1 \\ 3 \end{bmatrix} = \frac{1}{2}(8) = 4$

$\vec{v}_2^{\perp} = \vec{v}_2 - 4\vec{u}_1$

$= \begin{bmatrix} -1 \\ 3 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \\ -2 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$\|\vec{v}_2^{\perp}\| = 2$

$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{2} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$M = QR$

$$\boxed{\begin{bmatrix} -1 & -1 \\ 1 & 3 \\ -1 & -1 \\ 1 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \\ -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}}_{R}}$$

Solution: Check:

$$\underbrace{\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \\ -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}}_{4\times 2} \underbrace{\begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}}_{2\times 2} \to 4\times 2 \checkmark$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 3 \\ -1 & -1 \\ 1 & 3 \end{bmatrix} = M \checkmark$$

linear independent -> ker = 0

$\therefore$ rank = 5

cannot be true, must be linearly dependent

Problem 5 (10 points total; 2 points each)

Answer the following questions with true or false.

0 1. Let A be an arbitrary 4×5 matrix (i.e., with 4 rows and 5 columns). The column vectors of A cannot be linearly independent.

True

$\begin{bmatrix} & \\ & \end{bmatrix}$

2. There exists a subspace V of $\mathbb{R}^5$ so that V and its orthogonal complement $V^\perp$ have the same dimension.

False

3. Any square matrix A is similar to itself. Reflexivity

True

4. There exists a linear transformation $T\colon \mathbb{R}^5 \to \mathbb{R}^3$ with kernel of dimension 1. → 3×5

True

~~dim(ker)~~

nullity = 1

rank = 2

5. Let V be a subspace of $\mathbb{R}^n$, and $T\colon \mathbb{R}^n \to \mathbb{R}^n$ the orthogonal projection onto V. Let v be a vector in the kernel of T. Then $v \cdot w = 0$ for all vectors w in V.

True

image of $\text{proj}_V(\vec{v}) = V$

kernel of T -> all vectors perpendicular to V

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Use this sheet of paper if you need additional space.