Math 33A Sheet 3

Chapter 2.1

Ex.2) $y_1 = 2x_2,\ y_2 = 3x_3,\ y_3 = x_1$

$\vec{v} = \langle 2x_2, 3x_3, x_1 \rangle$

$T(\vec{v}) = A\vec{v}$

$T(\langle x_1, x_2, x_3 \rangle) = A(\langle x_1, x_2, x_3 \rangle)$

$$\begin{bmatrix} 2x_2 \\ 3x_3 \\ x_1 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix}$$

$\boxed{\text{Linear}}$

Ex.4)* $y_1 = 9x_1 + 3x_2 - 3x_3$

$y_2 = 2x_1 - 9x_2 + x_3$

$y_3 = 4x_1 - 9x_2 - 2x_3$

$y_4 = 5x_1 + x_2 + 5x_3$

$$\underset{4\times 1}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}} = \underset{4\times 3}{A} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}$$

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 4 \\ 5 \end{bmatrix} \qquad T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -2 \\ 5 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ -9 \\ 1 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 9 & 3 & -3 \\ 2 & -9 & 1 \\ 4 & -9 & -2 \\ 5 & 1 & 5 \end{bmatrix}}$$

Ex.5)

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -13 \\ 17 \end{bmatrix}$$

$$A = \begin{bmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 7 & 6 & -13 \\ 11 & 9 & 17 \end{bmatrix}}$$

Chapter 2.2

Ex.2)* Theorem 2.2.3:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$60°$:

$$\begin{bmatrix} \cos(\frac{\pi}{3}) & -\sin(\frac{\pi}{3}) \\ \sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3}) \end{bmatrix}$$

$$\boxed{\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}}$$

Ex.20) Reflection about x-z plane

$$\therefore\ T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x \\ -y \\ z \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Chapter 1.3

Ex.2)* $\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$

$3\times 2 \quad 2\times 2$ ✓

$$\begin{bmatrix} 1(3)+(-1)(1) & 1(2)+(-1)(0) \\ 0(3)+2(1) & 0(2)+2(0) \\ 2(3)+1(1) & 2(2)+1(0) \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 7 & 4 \end{bmatrix}}$$

Ex.3) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$2\times 3 \quad 2\times 2$ ✗

Cannot be performed, # of columns of 1st matrix ≠ # of rows of 2nd

Ex.7) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

$3\times 3 \quad 3\times 3$ ✓

$$\begin{bmatrix} 1(1)+0(3)+(-1)(2) & 1(2)+0(2)+(-1)(1) & 1(3)+0(1)+(-1)(3) \\ 0(1)+1(3)+1(2) & 0(2)+1(2)+1(1) & 0(3)+1(1)+1(3) \\ 1(1)+(-1)(3)+(-2)(2) & 1(2)+(-1)(2)+(-2)(1) & 1(3)+(-1)(1)+(-2)(3) \end{bmatrix}$$

$$\boxed{\begin{bmatrix} -1 & 1 & 0 \\ 5 & 3 & 4 \\ -6 & -2 & -4 \end{bmatrix}}$$

Ex.17) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$B = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$

$AB = \begin{bmatrix} x_1 & x_2 \\ 2x_3 & 2x_4 \end{bmatrix}$

$BA = \begin{bmatrix} x_1 & 2x_2 \\ x_3 & 2x_4 \end{bmatrix}$

$x_1 = x_1 \rightarrow x_1 = \mathbb{R}$

$x_2 = 2x_2 \rightarrow x_2 = 0$

$2x_3 = x_3 \rightarrow x_3 = 0$

$2x_4 = 2x_4 \rightarrow x_4 = \mathbb{R}$

$$\boxed{\begin{bmatrix} x_1 & 0 \\ 0 & x_4 \end{bmatrix}, \ x_1, x_4 \in \mathbb{R}}$$

Ex.55) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$2\times 2 \quad 2\times 2 \quad 2\times 2$

$X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$

$\therefore x_1 + 2x_3 = 0 \quad I = III$

$x_2 + 2x_4 = 0. \quad II = IV$

$2x_1 + 4x_3 = 0$

$2x_2 + 4x_4 = 0$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{array}\right]$$

let $x_3 = t$, $x_4 = s$

$$\boxed{X = \begin{bmatrix} -2t & -2s \\ t & s \end{bmatrix}, \ t, s \in \mathbb{R}}$$

Chapter 2.4

Ex.1) $\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix} \times \frac{1}{2}$

$\begin{bmatrix} 1 & \frac{3}{2} \\ 5 & 8 \end{bmatrix} -5(\text{I})$

$\begin{bmatrix} 1 & \frac{3}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{matrix} -3(\text{II}) \\ \times 2 \end{matrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\boxed{\text{Invertible}}$

$\left| \begin{matrix} 2x_1 + 3x_2 = y_1 \\ 5x_1 + 8x_2 = y_2 \end{matrix} \right| \times \frac{1}{2}$

$\left| \begin{matrix} x_1 + \frac{3}{2}x_2 = \frac{1}{2}y_1 \\ 5x_1 + 8x_2 = y_2 \end{matrix} \right| -5(\text{I})$

$\left| \begin{matrix} x_1 + \frac{3}{2}x_2 = \frac{1}{2}y_1 \\ \frac{1}{2}x_2 = -\frac{5}{2}y_1 + y_2 \end{matrix} \right| \begin{matrix} -3(\text{II}) \\ \times 2 \end{matrix}$

$\left| \begin{matrix} x_1 = 8y_1 - 3y_2 \\ x_2 = -5y_1 + 2y_2 \end{matrix} \right|$

$\boxed{A^{-1} = \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}}$

Ex.11) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} -(\text{III})$

$\boxed{\text{Invertible}}$

$\left| \begin{matrix} x_1 + x_3 = y_1 \\ x_2 = y_2 \\ x_3 = y_3 \end{matrix} \right| -(\text{III})$

$\left| \begin{matrix} x_1 = y_1 - y_3 \\ x_2 = y_2 \\ x_3 = y_3 \end{matrix} \right|$

$\boxed{A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$

Ex.12) $\begin{bmatrix} 2 & 5 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix} \begin{matrix} \times \frac{1}{2} \\ \\ \\ -2(\text{III}) \end{matrix}$

$\begin{bmatrix} 1 & \frac{5}{2} & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ -(\text{I}) \\ -2(\text{IV}) \\ \\ \end{matrix}$

$\begin{bmatrix} 1 & \frac{5}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} -5(\text{II}) \\ \times 2 \\ \\ \\ \end{matrix}$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\boxed{\text{Invertible}}$

$\left| \begin{matrix} 2x_1 + 5x_2 = \frac{1}{2}y_1 \\ x_1 + 3x_2 = y_2 \end{matrix} \right| \times \frac{1}{2}$ $\left| \begin{matrix} x_3 + 2x_4 = y_3 \\ 2x_3 + 5x_4 = y_4 \end{matrix} \right| -2(\text{I})$

$\left| \begin{matrix} x_1 + \frac{5}{2}x_2 = \frac{1}{2}y_1 \\ x_1 + 3x_2 = y_2 \end{matrix} \right| -(\text{I})$ $\left| \begin{matrix} x_3 + 2x_4 = y_3 \\ x_4 = y_4 - 2y_3 \end{matrix} \right| -2(\text{II})$

$\left| \begin{matrix} x_1 + \frac{5}{2}x_2 = \frac{1}{2}y_1 \\ \frac{1}{2}x_2 = -\frac{1}{2}y_1 + y_2 \end{matrix} \right| \begin{matrix} -5(\text{II}) \\ \times 2 \end{matrix}$ $\left| \begin{matrix} x_3 = 5y_3 - 2y_4 \\ x_4 = -2y_3 + y_4 \end{matrix} \right|$

$\left| \begin{matrix} x_1 = 3y_1 - 5y_2 \\ x_2 = -y_1 + 2y_2 \end{matrix} \right|$

$\boxed{A^{-1} = \begin{bmatrix} 3 & -5 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix}}$

Ex. 29)*

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{bmatrix} \begin{matrix} \\ -(\mathrm{I}) \\ -(\mathrm{I}) \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & k-1 \\ 0 & 3 & k^2-1 \end{bmatrix} \begin{matrix} -(\mathrm{II}) \\ \\ -3(\mathrm{II}) \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -k+2 \\ 0 & 1 & k-1 \\ 0 & 0 & k^2-3k+2 \end{bmatrix} \begin{matrix} \\ \\ \times \frac{1}{k^2-3k+2} \end{matrix}$$

$k^2-3k+2=0$

$(k-2)(k-1)$

$k \neq 2, 1$

$$\begin{bmatrix} 1 & 0 & -k+2 \\ 0 & 1 & k-1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{\text{k can be any real number other than 1 and 2}}$

Q1) $T(\vec{v}) = A\vec{v}$

Conditions: $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$

$T(k\vec{v}) = kT(\vec{v})$

$T(\vec{0}_m) = \vec{0}_n$

$\vec{v} = \vec{0}_m \quad T(\vec{v}) = \vec{0}_n$

$$T(\vec{v}+\vec{w}) = \begin{bmatrix} 0+w_1 \\ \vdots \\ 0+w_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

$= T(\vec{v}) + T(\vec{w})$ ✓

✓ $T(k\vec{0}_m) = kT(\vec{0}_m) = 0$

Q2) Composition. $S \circ T(\vec{v})$

$S \circ T(\vec{v}+\vec{w}) = S \circ T(\vec{v}) + S \circ T(\vec{w})$?

$S \circ T(\vec{v}+\vec{w}) = S(T(\vec{v}+\vec{w}))$

$T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$

$= S(T(\vec{v}) + T(\vec{w}))$

$= S(T(\vec{v})) + S(T(\vec{w}))$

$= S \circ T(\vec{v}) + S \circ T(\vec{w})$ ✓

$S \circ T(k\vec{v}) = kS \circ T(\vec{v})$?

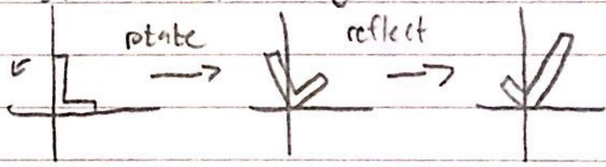
$S(T(k\vec{v})) = S(kT(\vec{v}))$

$= kS(T(\vec{v}))$

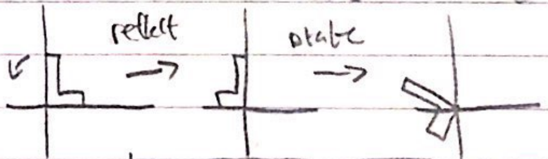
$= kS \circ T(\vec{v})$ ✓

Q3) If the matricies do not commute, then the transformations cannot be done in the reverse order.

An example of this is rotating and then reflecting:



vs. reflecting, then rotating:



$\boxed{\text{not commutative}}$