EXERCISES 1.1

GOAL *Set up and solve systems with as many as three linear equations with three unknowns, and interpret the equations and their solutions geometrically.*

In Exercises 1 through 10, find all solutions of the linear systems using elimination as discussed in this section. Then check your solutions.

1. $\begin{vmatrix} x + 2y = 1 \\ 2x + 3y = 1 \end{vmatrix}$

2. $\begin{vmatrix} 4x + 3y = 2 \\ 7x + 5y = 3 \end{vmatrix}$

3. $\begin{vmatrix} 2x + 4y = 3 \\ 3x + 6y = 2 \end{vmatrix}$

4. $\begin{vmatrix} 2x + 4y = 2 \\ 3x + 6y = 3 \end{vmatrix}$

5. $\begin{vmatrix} 2x + 3y = 0 \\ 4x + 5y = 0 \end{vmatrix}$

6. $\begin{vmatrix} x + 2y + 3z = 8 \\ x + 3y + 3z = 10 \\ x + 2y + 4z = 9 \end{vmatrix}$

7. $\begin{vmatrix} x + 2y + 3z = 1 \\ x + 3y + 4z = 3 \\ x + 4y + 5z = 4 \end{vmatrix}$

8. $\begin{vmatrix} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 10z = 0 \end{vmatrix}$

9. $\begin{vmatrix} x + 2y + 3z = 1 \\ 3x + 2y + z = 1 \\ 7x + 2y - 3z = 1 \end{vmatrix}$

10. $\begin{vmatrix} x + 2y + 3z = 1 \\ 2x + 4y + 7z = 2 \\ 3x + 7y + 11z = 8 \end{vmatrix}$

In Exercises 11 through 13, find all solutions of the linear systems. Represent your solutions graphically, as intersections of lines in the x–y-plane.

11. $\begin{vmatrix} x - 2y = 2 \\ 3x + 5y = 17 \end{vmatrix}$

12. $\begin{vmatrix} x - 2y = 3 \\ 2x - 4y = 6 \end{vmatrix}$

13. $\begin{vmatrix} x - 2y = 3 \\ 2x - 4y = 8 \end{vmatrix}$

In Exercises 14 through 16, find all solutions of the linear systems. Describe your solutions in terms of intersecting planes. You need not sketch these planes.

14. $\begin{vmatrix} x + 4y + z = 0 \\ 4x + 13y + 7z = 0 \\ 7x + 22y + 13z = 1 \end{vmatrix}$

15. $\begin{vmatrix} x + y - z = 0 \\ 4x - y + 5z = 0 \\ 6x + y + 4z = 0 \end{vmatrix}$

16. $\begin{vmatrix} x + 4y + z = 0 \\ 4x + 13y + 7z = 0 \\ 7x + 22y + 13z = 0 \end{vmatrix}$

17. Find all solutions of the linear system

$$\begin{vmatrix} x + 2y = a \\ 3x + 5y = b \end{vmatrix},$$

where a and b are arbitrary constants.

18. Find all solutions of the linear system

$$\begin{vmatrix} x + 2y + 3z = a \\ x + 3y + 8z = b \\ x + 2y + 2z = c \end{vmatrix},$$

where a, b, and c are arbitrary constants.

19. Consider the linear system

$$\begin{vmatrix} x + y - z = -2 \\ 3x - 5y + 13z = 18 \\ x - 2y + 5z = k \end{vmatrix},$$

where k is an arbitrary number.

a. For which value(s) of k does this system have one or infinitely many solutions?

b. For each value of k you found in part a, how many solutions does the system have?

c. Find all solutions for each value of k.

20. Consider the linear system

$$\begin{vmatrix} x + y - z = 2 \\ x + 2y + z = 3 \\ x + y + (k^2 - 5)z = k \end{vmatrix},$$

where k is an arbitrary constant. For which value(s) of k does this system have a unique solution? For which value(s) of k does the system have infinitely many solutions? For which value(s) of k is the system inconsistent?

21. The sums of any two of three real numbers are 24, 28, and 30. Find these three numbers.

22. Emile and Gertrude are brother and sister. Emile has twice as many sisters as brothers, and Gertrude has just as many brothers as sisters. How many children are there in this family?

12. $$\left|\begin{array}{rcl} 2x_1 \quad\quad - 3x_3 \quad\quad + 7x_5 + 7x_6 &=& 0 \\ -2x_1 + x_2 + 6x_3 \quad\quad - 6x_5 - 12x_6 &=& 0 \\ x_2 - 3x_3 \quad\quad + x_5 + 5x_6 &=& 0 \\ -2x_2 \quad\quad + x_4 + x_5 + x_6 &=& 0 \\ 2x_1 + x_2 - 3x_3 \quad\quad + 8x_5 + 7x_6 &=& 0 \end{array}\right|$$

Solve the linear systems in Exercises 13 through 17. You may use technology.

13. $$\left|\begin{array}{l} 3x + 11y + 19z = -2 \\ 7x + 23y + 39z = 10 \\ -4x - 3y - 2z = 6 \end{array}\right|$$

14. $$\left|\begin{array}{l} 3x + 6y + 14z = 22 \\ 7x + 14y + 30z = 46 \\ 4x + 8y + 7z = 6 \end{array}\right|$$

15. $$\left|\begin{array}{l} 3x + 5y + 3z = 25 \\ 7x + 9y + 19z = 65 \\ -4x + 5y + 11z = 5 \end{array}\right|$$

16. $$\left|\begin{array}{l} 3x_1 + 6x_2 + 9x_3 + 5x_4 + 25x_5 = 53 \\ 7x_1 + 14x_2 + 21x_3 + 9x_4 + 53x_5 = 105 \\ -4x_1 - 8x_2 - 12x_3 + 5x_4 - 10x_5 = 11 \end{array}\right|$$

17. $$\left|\begin{array}{l} 2x_1 + 4x_2 + 3x_3 + 5x_4 + 6x_5 = 37 \\ 4x_1 + 8x_2 + 7x_3 + 5x_4 + 2x_5 = 74 \\ -2x_1 - 4x_2 + 3x_3 + 4x_4 - 5x_5 = 20 \\ x_1 + 2x_2 + 2x_3 - x_4 + 2x_5 = 26 \\ 5x_1 - 10x_2 + 4x_3 + 6x_4 + 4x_5 = 24 \end{array}\right|$$

18. Determine which of the matrices below are in reduced row-echelon form:

a. $\begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ d. $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \end{bmatrix}$

19. Find all 4×1 matrices in reduced row-echelon form.

20. For which values of a, b, c, d, and e is the following matrix in reduced row-echelon form?

$$A = \begin{bmatrix} 0 & a & 2 & 1 & b \\ 0 & 0 & 0 & c & d \\ 0 & 0 & e & 0 & 0 \end{bmatrix}$$

21. For which values of a, b, c, d, and e is the following matrix in reduced row-echelon form?

$$A = \begin{bmatrix} 1 & a & b & 3 & 0 & -2 \\ 0 & 0 & c & 1 & d & 3 \\ 0 & e & 0 & 0 & 1 & 1 \end{bmatrix}$$

22. We say that two $n \times m$ matrices in reduced row-echelon form are of the same type if they contain the same number of leading 1's in the same positions. For example,

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are of the same type. How many types of 2×2 matrices in reduced row-echelon form are there?

23. How many types of 3×2 matrices in reduced row-echelon form are there? See Exercise 22.

24. How many types of 2×3 matrices in reduced row-echelon form are there? See Exercise 22.

25. Suppose you apply Gauss–Jordan elimination to a matrix. Explain how you can be sure that the resulting matrix is in reduced row-echelon form.

26. Suppose matrix A is transformed into matrix B by means of an elementary row operation. Is there an elementary row operation that transforms B into A? Explain.

27. Suppose matrix A is transformed into matrix B by a sequence of elementary row operations. Is there a sequence of elementary row operations that transforms B into A? Explain your answer. See Exercise 26.

28. Consider an $n \times m$ matrix A. Can you transform $\text{rref}(A)$ into A by a sequence of elementary row operations? See Exercise 27.

29. Is there a sequence of elementary row operations that transforms

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{into} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}?$$

Explain.

30. Suppose you subtract a multiple of an equation in a system from another equation in the system. Explain why the two systems (before and after this operation) have the same solutions.

31. *Balancing a chemical reaction.* Consider the chemical reaction

$$a\ NO_2 + b\ H_2O \to c\ HNO_2 + d\ HNO_3,$$

where a, b, c, and d are unknown positive integers. The reaction must be balanced; that is, the number of atoms of each element must be the same before and after the reaction. For example, because the number of oxygen atoms must remain the same,

$$2a + b = 2c + 3d.$$

While there are many possible values for a, b, c, and d that balance the reaction, it is customary to use the smallest possible positive integers. Balance this reaction.

32. Find the polynomial of degree 3 [a polynomial of the form $f(t) = a + bt + ct^2 + dt^3$] whose graph goes through the points $(0, 1)$, $(1, 0)$, $(-1, 0)$, and $(2, -15)$. Sketch the graph of this cubic.

33. Find the polynomial of degree 4 whose graph goes through the points $(1, 1)$, $(2, -1)$, $(3, -59)$, $(-1, 5)$, and $(-2, -29)$. Graph this polynomial.