20W-MATH33B-1 Final Exam

CHARLES ZHANG

TOTAL POINTS

91 / 100

QUESTION 1

1 Question 1 **10 / 10**

✓ **- 0 pts** Correct

- 1 pts minus sign error

- 2 pts need to be in the form of "y = " form

- 3 pts algebraic mistake need to be ln(x)

- 1 pts miscelleneous algebraic mistake

- 5 pts only the homogeneous equation substitution was right

- 2 pts the position of "+c" needs to be in denominator

- 8 pts tried

QUESTION 2

2 Question 2 **15 / 15**

✓ **- 0 pts** Correct

- 3 pts some portion particular solution wrong/not found correctly

- 1 pts miscellaneous mistake

- 6 pts answer for part (b) wrong

- 2 pts general solution for 2(a) incomplete

QUESTION 3

Question 3 15 pts

3.1 3 (a) **10 / 10**

✓ **- 0 pts** Correct

- 4 pts Miscomputed one eigenvector.

- 2 pts You found fundamental solutions, but forgot C_1 and C_2.

- 3 pts Miscomputed both eigenvectors.

- 1 pts Write down the general solution, not just fundamental solutions.

3.2 3(b) **5 / 5**

✓ **- 0 pts** Correct

- 1 pts Justification?

- 2 pts Eigenvectors graphed in incorrect quadrants.

- 2 pts Indicate direction travelled on solution curves.

- 2 pts The shape of your curves as t goes to infinity or - infinity is wrong

- 3 pts Draw solution curves in quadrants cut out by eigenvectors.

- 3 pts Draw the half-line solutions (the ones corresponding to the eigenvectors).

QUESTION 4

4 Question 4 **10 / 10**

✓ **- 0 pts** Correct

- 2 pts Your third fundamental solution is wrong/missing.

- 2 pts Your second fundamental solution is wrong/missing.

- 1 pts e^{2t} not e^t.

- 2 pts System of coefficients solved incorrectly/solution to IVP missing/incorrect.

- 2 pts Your first fundamental solution is wrong/missing.

QUESTION 5

5 Question 5 **15 / 15**

✓ **- 0 pts** Correct

- 2 pts Identify block matrices

- 3 pts Find eigenvalues for each block

- 3 pts Find (generalized) eigenvectors for each block

- 4 pts Construct solutions for each block

- 3 pts Combine solutions.

- 1 pts Minor calculation error

- 2 pts Moderate error in solution for one block

QUESTION 6

Question 6 15 pts

6.1 6(a) 5 / 5

✓ - **0 pts** Correct

- **2 pts** Knew to find zeros of RHS.
- **2 pts** Correctly found at least one infinite family of solutions.
- **1 pts** Found half of the solutions or made a computational mistake.

6.2 6(b) 5 / 5

✓ - **0 pts** Correct

- **2 pts** Included equilibria
- **3 pts** Solutions go in correct directions
- **1 pts** Violates uniqueness
- **1 pts** Small error

6.3 6(c) 5 / 5

✓ - **0 pts** Correct

- **2 pts** Invoke hypotheses of existence and uniqueness.
- **3 pts** Bound by equilibria
- **1 pts** Minor error
- **1 pts** Show it satisfies the hypotheses (show f and f' are continuous).
- **1 pts** State that there are arbitrarily large or small equilibria.
- **1 pts** Invoke existence and uniqueness.

QUESTION 7

Question 7 10 pts

7.1 7(a) 3 / 5

- **0 pts** Correct

✓ - **2 pts** **y_h /IF correct, didn't find y_p**

- **2 pts** minor mistake / gap
- **4 pts** Major mistake/gap
- **5 pts** blank
- **3 pts** y_h/IF minor mistake/not simplified, didn't find y_p
- **1 pts** y_h not simplifed
- **3 pts** y_h = ?
- **1 pts** typo

7.2 7(b) 2 / 3

- **0 pts** Correct:

✓ - **1 pts** **minor mistake (e.g. forget to say C can be anything), gap, logic flow not clear**

- **2 pts** some meaningful writings. not much detail provided, many gaps.
- **3 pts** nothing meaningful

since C can be any number.

7.3 7(c) 0 / 2

- **0 pts** Correct
- **2 pts** wrong

✓ - **1 pts** **didn't put in normal form y' = F(y,t) = 1/ (t^2 - a^2) y + ...**

✓ - **1 pts** **didn't check/state that \partial F/ \partial y = 1/(t^2 - a^2) or calculation is wrong**

- **1 pts** Gap

QUESTION 8

8 Question 8 6 / 10

- **0 pts** Correct
- **4 pts** gap: did not verify (A-aI)^2 = 0
- **4 pts** minor mistake
- **2 pts** lack essential detail / some typos

✓ - **4 pts** **based on your flow, you didn't use math induction to give a proof for A^n**

- **8 pts** Major mistake

1. (10 points) Solve the homogeneous equation (Your final answer should be in $y = f(x, C)$ form, e.g $y = \frac{1}{C+x}$):

$$(-xy + y^2)dx + x^2 dy = 0.$$

$y = vx$, $dy = v\partial x + x\partial v$

$(-x(vx) + (vx)^2)\partial x + x^2(v\partial x + x\partial v) = 0$

$(-vx^2 + v^2x^2)\partial x + vx^2\partial x + x^3\partial v = 0$

$-vx^2\partial x + v^2x^2\partial x + vx^2\partial x + x^3\partial v = 0$

$v^2x^2\partial x + x^3\partial v = 0$

$v^2\partial x + x\partial v = 0$

$v^2\partial x = -x\partial v$

$\frac{\partial x}{x} = -\frac{\partial v}{v^2}$

$C + \ln|x| = -\int v^{-2}\partial v$

$C + \ln|x| = \frac{1}{v}$

$v = \frac{y}{x}$

$C + \ln|x| = \frac{x}{y}$

$\boxed{y = \frac{x}{C + \ln|x|}}$

1 Question 1 **10 / 10**

✓ **- 0 pts** **Correct**

- 1 pts minus sign error

- 2 pts need to be in the form of "y = " form

- 3 pts algebraic mistake need to be ln(x)

- 1 pts miscelleneous algebraic mistake

- 5 pts only the homogeneous equation substitution was right

- 2 pts the position of "+c" needs to be in denominator

- 8 pts tried

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2. (a) (5 points) Find the general solution to the differential equation:

$$y'' - 2y' + y = 0$$

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\lambda = 1$

$y_1(t) = e^t,\ y_2(t) = te^t$

$\boxed{y(t) = c_1 e^t + c_2 t e^t}$

(b) (10 points) Find a particular solution to the differential equation (Hint: split forcing term into two parts, check the table in P172 of your textbook):

$$y'' - 2y' + y = e^t(t+1) + e^t \sin t.$$

$y'' - 2y' + y = e^t(t+1)$

$e^{rt}(P(t))$

$y(t) = e^t(At+B)$ and $y(t) = te^t(At+B)$

result in exceptional cases

$y_1(t) = t^2 e^t (At+B) = e^t(At^3 + Bt^2)$

$y_1'(t) = e^t(At^3 + Bt^2) + e^t(3At^2 + 2Bt)$

$y_1''(t) = e^t(At^3+Bt^2) + e^t(3At^2+2Bt) + e^t(3At^2+2Bt) + e^t(6At+2B)$

$\cancel{(At^3+Bt^2)} + \cancel{2(3At^2+2Bt)} + (6At+2B) - \cancel{2(At^3+Bt^2)} - \cancel{2(3At^2+2Bt)} + \cancel{(At^3+Bt^2)} = t+1$

$6At + 2B = t+1$

$6A = 1 \quad 2B = 1$

$A = 1/6 \quad B = 1/2$

$y(t) = t^2 e^t\left(\frac{1}{6}t + \frac{1}{2}\right)$

$y'' - 2y' + y = e^t \sin t$

$e^{rt}[a\cos wt + b\sin wt]$

$y_2(t) = e^t(a\cos t + b\sin t)$

$y_2'(t) = e^t(a\cos t + b\sin t) + e^t(-a\sin t + b\cos t)$

$y_2''(t) = e^t(a\cos t + b\sin t) + 2e^t(-a\sin t + b\cos t) + e^t(-a\cos t - b\sin t)$

$\cancel{(a\cos t + b\sin t)} + 2\cancel{(-a\sin t + b\cos t)} + (-a\cos t - b\sin t) - 2\cancel{(a\cos t + b\sin t)} - 2\cancel{(-a\sin t + b\cos t)} + \cancel{(a\cos t + b\sin t)} = \sin t$

$-a\cos t - b\sin t = \sin t$

$a = 0,\ b = -1$

$y_2(t) = e^t(-\sin t)$

$\boxed{y_p(t) = e^t\left(\frac{1}{6}t^3 + \frac{1}{2}t^2 - \sin t\right)}$

2 Question 2 **15 / 15**

✓ **- 0 pts** Correct

- 3 pts some portion particular solution wrong/not found correctly

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- 6 pts answer for part (b) wrong

- 2 pts general solution for 2(a) incomplete

3. (a) (10 points) Find the general solution ($y_{\text{general}} = C_1 y_1(t) + C_2 y_2(t)$) to the following 2×2 system $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}$$

✓

$$\begin{bmatrix} -2-\lambda & -2 \\ 2 & 3-\lambda \end{bmatrix}$$

$(-2-\lambda)(3-\lambda) - (-4) = 0$

$-6-\lambda+\lambda^2+4=0$

$(\lambda^2-\lambda-2)=0$

$(\lambda-2)(\lambda+1)=0$

$\lambda_1 = 2, \lambda_2 = -1$

$\begin{bmatrix} -2-2 & -2 \\ 2 & 3-2 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 2 & 1 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\begin{bmatrix} -2-(-1) & -2 \\ 2 & 3-(-1) \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & 4 \end{bmatrix}$

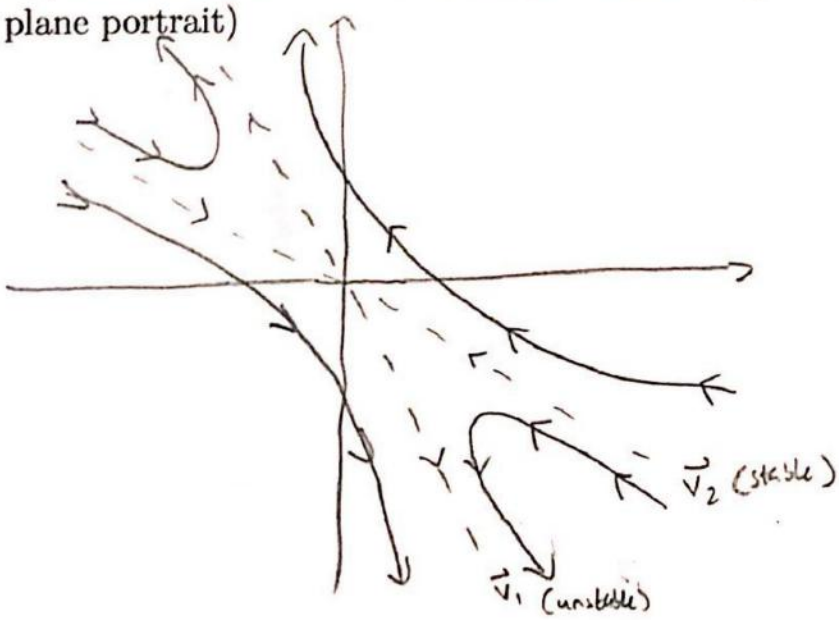
$\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$\boxed{y(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

saddle

(b) (5 points) Sketch the solutions on the phase plane. (i.e. Draw the phase plane portrait)

✓



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3.1 3 (a) **10 / 10**

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- 4 pts Miscomputed one eigenvector.

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3. (a) (10 points) Find the general solution ($y_{\text{general}} = C_1 y_1(t) + C_2 y_2(t)$) to the following 2×2 system $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}$$

$$\begin{bmatrix} -2-\lambda & -2 \\ 2 & 3-\lambda \end{bmatrix}$$

$$(-2-\lambda)(3-\lambda)-(-4)=0$$

$$-6-\lambda+\lambda^2+4=0$$

$$(\lambda^2-\lambda-2)=0$$

$$(\lambda-2)(\lambda+1)=0$$

$$\lambda_1=2,\ \lambda_2=-1$$

$$\begin{bmatrix} -2-2 & -2 \\ 2 & 3-2 \end{bmatrix}=\begin{bmatrix} -4 & -2 \\ 2 & 1 \end{bmatrix}$$

$$\vec{v}_1=\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

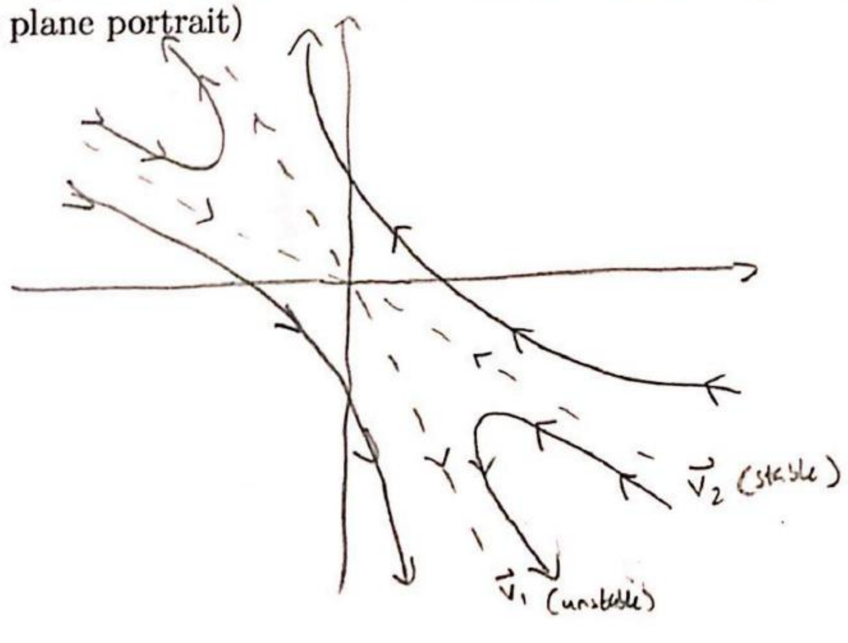
$$\begin{bmatrix} -2-(-1) & -2 \\ 2 & 3-(-1) \end{bmatrix}=\begin{bmatrix} -1 & -2 \\ 2 & 4 \end{bmatrix}$$

$$\vec{v}_2=\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\boxed{y(t)=c_1e^{2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}+c_2e^{-t}\begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

saddle

(b) (5 points) Sketch the solutions on the phase plane. (i.e. Draw the phase plane portrait)



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3.2 3(b) **5 / 5**

- ✓ **- 0 pts** Correct
- **- 1 pts** Justification?
- **- 2 pts** Eigenvectors graphed in incorrect quadrants.
- **- 2 pts** Indicate direction travelled on solution curves.
- **- 2 pts** The shape of your curves as t goes to infinity or - infinity is wrong
- **- 3 pts** Draw solution curves in quadrants cut out by eigenvectors.
- **- 3 pts** Draw the half-line solutions (the ones corresponding to the eigenvectors).

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4. (10 points) Find the solution $\mathbf{y}(t)$ to the following 3×3 system with given initial condition $\mathbf{y}(0) = (2, -2, 1)^T$:

$$\mathbf{y}' = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix} \mathbf{y}$$

(Hint: Use rational zero theorem to find integer roots of the characteristic polynomial)

$$\left| \begin{matrix} 3-\lambda & 1 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & 1 & 2-\lambda \end{matrix} \right| \begin{matrix} 3-\lambda & 1 \\ -1 & 1-\lambda \\ 1 & 1 \end{matrix}$$

$(3-\lambda)(1-\lambda)(2-\lambda) - 2 - [-(2-\lambda) - (3-\lambda) + (1-\lambda)] = 0$

$(3 - 4\lambda + \lambda^2)(2-\lambda) - 2 - [-2 + \lambda - 3 + \lambda + 1 - \lambda] = 0$

$(6 - 3\lambda - 8\lambda + 4\lambda^2 + 2\lambda^2 - \lambda^3 - 2 + 2 - \lambda + 3 - \lambda - 1 + \lambda) = 0$

$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$

$p: \pm 1, \pm 2, \pm 4, \pm 8, \; q = 1$

* cannot be a neg #

$p/q = 1, 2, 4, 8$

$\lambda = 2 \rightarrow (\lambda - 2)^3 = 0$

check
$(\lambda-2)^3$
$(\lambda^2 - 4\lambda + 4)(\lambda - 2)$
$\lambda^3 - 2\lambda^2 - 4\lambda^2 + 8\lambda +$
$4\lambda - 8$ ✓

$A - \lambda I = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$

$\text{rref}(A - \lambda I) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$x + y = 0$
$z = 0$

$\vec{v}_1 = e^{2t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$(A - \lambda I)^2 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\vec{v}_2 = e^{2t} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$

$\vec{v}_2 = e^{2t} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right)$

$(A - \lambda I)^3 = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\vec{v}_3 = e^{2t} \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) = e^{2t} \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right)$

$y(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 e^{2t} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right) + c_3 e^{2t} \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right)$

$y(0) = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$c_1 = 2, \; c_2 = 1, \; c_3 = 0$

$$\boxed{y(t) = 2e^{2t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + e^{2t} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right)}$$

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4 Question 4 **10 / 10**

✓ **- 0 pts** **Correct**

- **- 2 pts** Your third fundamental solution is wrong/missing.
- **- 2 pts** Your second fundamental solution is wrong/missing.
- **- 1 pts** e^{2t} not e^t.
- **- 2 pts** System of coefficients solved incorrectly/solution to IVP missing/incorrect.
- **- 2 pts** Your first fundamental solution is wrong/missing.

5. (15 points) Find the general solution(fundamental set) $\mathbf{y}(t)$ to the following 6×6 system:

$$\mathbf{y}' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} \mathbf{y}$$

(Hint : This is a block matrix. Try find a 1 by 1, 3 by 3, and 2 by 2 block.)

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$y' = Ay$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & -1 & -\lambda \end{vmatrix} \begin{matrix} -\lambda & 0 \\ 0 & 2-\lambda \\ 0 & -1 \end{matrix}$$

$= \lambda^2(2-\lambda) - \lambda$

$= 2\lambda^2 - \lambda^3 - \lambda$

$\lambda^3 - 2\lambda^2 + \lambda = 0$

$\lambda(\lambda^2 - 2\lambda + 1) = 0$

$\lambda(\lambda - 1)^2 = 0$

$\lambda_1 = 0, \lambda_2 = 1$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix} \to \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \to \vec{v}_2 = e^t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$(A - \lambda_2 I)^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}^2 =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{v}_3 = e^t\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$\vec{v}_3 = e^t\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right)$$

$$= e^t \begin{bmatrix} 0 \\ t \\ 1-t \end{bmatrix}$$

$B = [1]$

$\det(B - \lambda I) = 1 - \lambda$

$\lambda = 1$

$y' = [1]y$

$y = [e^t]$

$\vec{v} = [e^t]$

$$C = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

$y' = Cy$

$$y' = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} y$$

$$\vec{y}_1 = \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix}, \vec{y}_2 = \begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix} = \begin{bmatrix} -2\sin 2t \\ 2\cos 2t \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix} = \begin{bmatrix} -2\cos 2t \\ -2\sin 2t \end{bmatrix} \checkmark$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ e^t \\ -e^t \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ te^t \\ e^t(1-t) \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ e^t \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cos 2t \\ \sin 2t \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2\cos 2t \\ -2\sin 2t \end{bmatrix} \right.$$

5 Question 5 **15 / 15**

- ✓ **- 0 pts** **Correct**
- **- 2 pts** Identify block matrices
- **- 3 pts** Find eigenvalues for each block
- **- 3 pts** Find (generalized) eigenvectors for each block
- **- 4 pts** Construct solutions for each block
- **- 3 pts** Combine solutions.
- **- 1 pts** Minor calculation error
- **- 2 pts** Moderate error in solution for one block

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6. Consider the autonomous equation:

$$y' = \sin y + \cos y$$

(a) (5 points) Find all equilibria of the differential equation.

$0 = \sin y + \cos y$

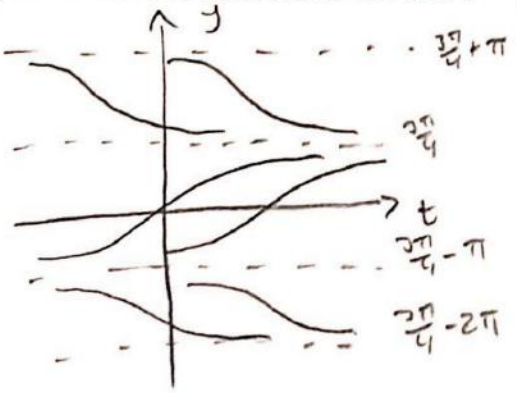
$-\sin y = \cos y$

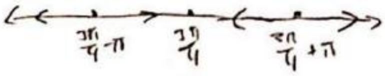
$-\tan y = 1$

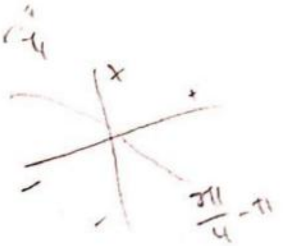
$\tan y = -1$

$\boxed{y = \frac{3\pi}{4} + n\pi, \text{ where } n \text{ is an integer}}$

(b) (5 points) Sketch the solutions on the $t - y$ plane.







(c) (5 points) Prove that if $y(t)$ is a solution, then $y(t)$ is a bounded function. (In other words, given a solution $y(t)$, there exists m, M such that, $m < y(t) < M$ for all $t \in (-\infty, +\infty)$)

$f(t,y)$ is continuous for $(-\infty, \infty)$, and so is its derivative, $\frac{\partial f}{\partial y} = \cos y - \sin y$. Therefore, this function must satisfy uniqueness. This means that if $y(t)$ is a solution, then it cannot intersect with another solution. Therefore, since equilibrium solutions are defined by $y = \frac{3\pi}{4} + n\pi$ (there are infinitely many) $y(t)$ can always be bounded by equilibrium solutions if it is a non-equilibrium solution, or by trivial values if it is an equilibrium solution (ie. $\pi/2$ and π bound $y = \frac{3\pi}{4}$).

6.1 6(a) **5 / 5**

✓ **- 0 pts** Correct

- 2 pts Knew to find zeros of RHS.

- 2 pts Correctly found at least one infinite family of solutions.

- 1 pts Found half of the solutions or made a computational mistake.

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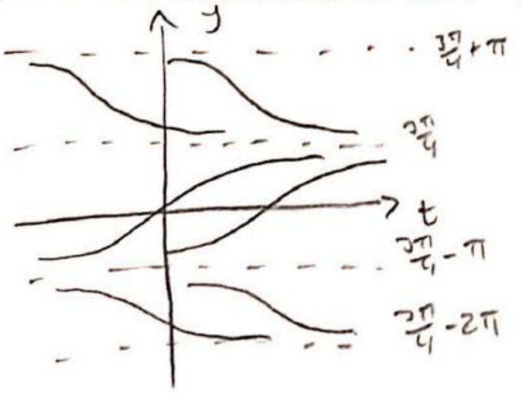

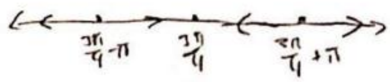

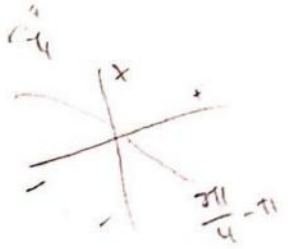

(c) (5 points) Prove that if $y(t)$ is a solution, then $y(t)$ is a bounded function. (In other words, given a solution $y(t)$, there exists m, M such that, $m < y(t) < M$ for all $t \in (-\infty, +\infty)$)

$f(t,y)$ is continuous for $(-\infty, \infty)$, and so is its derivative, $\frac{\partial f}{\partial y} = \cos y - \sin y$. Therefore, this function must satisfy uniqueness. This means that if $y(t)$ is a solution, then it cannot intersect with another solution. Therefore, since equilibrium solutions are defined by $y = \frac{3\pi}{4} + n\pi$ (there are infinitely many) $y(t)$ can always be bounded by equilibrium solutions if it is a non-equilibrium solution, or by trivial values if it is an equilibrium solution (ie. $\pi/2$ and π bound $y = \frac{3\pi}{4}$).

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6.2 6(b) **5 / 5**

✓ **- 0 pts** Correct

- 2 pts Included equilibria

- 3 pts Solutions go in correct directions

- 1 pts Violates uniqueness

- 1 pts Small error

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6. Consider the autonomous equation:

$$y' = \sin y + \cos y$$

(a) (5 points) Find all equilibria of the differential equation.

$0 = \sin y + \cos y$

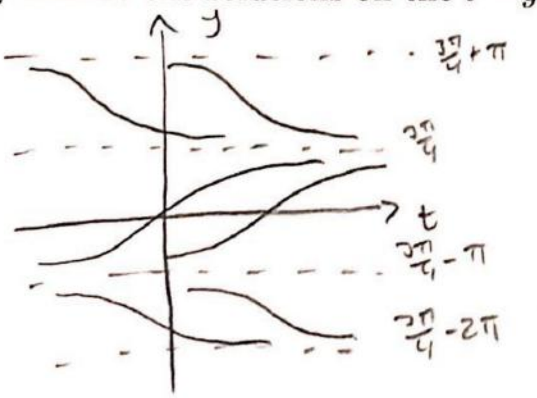
$-\sin y = \cos y$

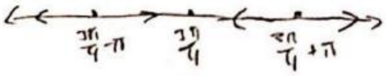
$-\tan y = 1$

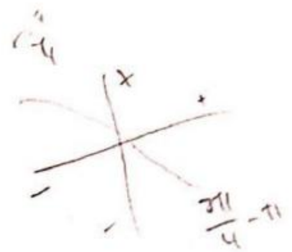
$\tan y = -1$

$\boxed{y = \frac{3\pi}{4} + n\pi, \text{ where } n \text{ is an integer}}$

(b) (5 points) Sketch the solutions on the $t - y$ plane.







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$f(t,y)$ is continuous for $(-\infty, \infty)$, and so is its derivative, $\frac{\partial f}{\partial y} = \cos y - \sin y$. Therefore, this function must satisfy uniqueness. This means that if $y(t)$ is a solution, then it cannot intersect with another solution. Therefore, since equilibrium solutions are defined by $y = \frac{3\pi}{4} + n\pi$ (there are infinitely many) $y(t)$ can always be bounded by equilibrium solutions if it is a non-equilibrium solution, or by trivial values if it is an equilibrium solution (ie. $\pi/2$ and π bound $y = \frac{3\pi}{4}$).

6.3 6(c) **5 / 5**

✓ **- 0 pts** **Correct**

- 2 pts Invoke hypotheses of existence and uniqueness.

- 3 pts Bound by equilibria

- 1 pts Minor error

- 1 pts Show it satisfies the hypotheses (show f and f' are continuous).

- 1 pts State that there are arbitrarily large or small equilibria.

- 1 pts Invoke existence and uniqueness.

7. Let a be a positive integer(it is a fixed unknown number). Consider the following differential equation:

$$(t^2-a^2)y' = y + t^2 - t - a^2.$$

(a) (5 points) Find the general solution to the above differential equation.

$(t^2-a^2)y' = y+t^2-t-a^2$

$y' = \frac{y+t^2-t-a^2}{(t^2-a^2)}$

$y' = \frac{1}{t^2-a^2}y - \frac{t}{(t^2-a^2)} + 1$

$a(t) = \frac{1}{t^2-a^2}$

$IF = e^{\int a(t)dt}$

$IF = e^{-\int \frac{1}{t^2-a^2}}$

$= e^{\ln\left|\frac{t-a}{t+a}\right|^{+2a}}$

$= \left|\frac{t-a}{t+a}\right|^{2a}$

$\left|\frac{t-a}{t+a}\right|^{2a} y' - \left|\frac{t-a}{t+a}\right|^{2a}\left(\frac{y}{t^2-a^2}\right) = -\frac{t}{(t^2-a^2)}\left(\frac{t-a}{t+a}\right)^{2a} + \left(\frac{t-a}{t+a}\right)^{2a}$

$\int\left(\left(\frac{t-a}{t+a}\right)^{2a} y\right)' = \int \frac{-t}{(t^2-a^2)}\left(\frac{t-a}{t+a}\right)^{2a} + \left(\frac{t-a}{t+a}\right)^{2a} dt$

$\left(\frac{t-a}{t+a}\right)^{2a} y = \int \frac{t^2-t-a^2}{(t^2-a^2)}\left(\frac{t-a}{t+a}\right)^{2a} dt$

I've tried every method that we've been taught, and I've gotten an impossible integral every time :(

$\boxed{\text{Assume: } y = C\left(\frac{t-a}{t+a}\right)^{\frac{1}{2a}} + a}$

$\frac{1}{t+a}C_1$

$b = \frac{1}{2a}C_1$

$C_1 = 2ab$

$\frac{1}{t+a}2ab$

(b) (3 points) Consider the above differential equation together with the initial condition $y(a) = b$ (initial value problem), where b is a real number. Prove that,

- if $b = a$, there are infinite many solution to the initial value problem. (i.e. go through the initial condition.)
- if $b \neq a$, there is no solutions to the initial value problem.

Assuming $y = C\left(\frac{t-a}{t+a}\right)^{\frac{1}{2a}} + a$

$y(a) = C\left(\frac{0}{2a}\right)^{\frac{1}{2a}} + a$

$y(a) = a \to$ if $b=a$, then $a=a \to$ infinite solutions

$y(a) = a \to$ if $b \neq a$ then $b=a$ is false $\to$ no solution

(c) (2 points) Does the above (weird) result contradict with the existence and uniqueness theorem? Why?

This doesn't contradict U/E because the function/its derivative is not continuous over the necessary interval.

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$$(-y - t^2 + t + a^2)\,\partial t + (t^2 - a^2)\,dy = 0$$

$$P = -y - t^2 + t + a^2 \qquad Q = t^2 - a^2$$

$$\frac{\partial P}{\partial y} = -1 \qquad \frac{\partial Q}{\partial t} = 2t$$

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial t} = -1 - 2t = -(2t+1)$$

$$\frac{1}{t^2-a^2}(-2t+1) \leftarrow \text{only in terms of } t$$

$$IF: e^{\int f(t)} = e^{\int \frac{-2t+1}{t^2-a^2}\partial t}$$

$$= e^{\int \frac{-2t}{t^2-a^2} + \int \frac{1}{t^2-a^2}}$$

$$= e^{-\ln|t^2-a^2| + \ln\left|\frac{t-a}{t+a}\right|^{1/2a}}$$

$$= e^{\ln|t^2-a^2|^{-1}}\, e^{\ln\left|\frac{t-a}{t+a}\right|^{1/2a}}$$

$$\frac{1}{t^2-a^2}\left(\frac{t-a}{t+a}\right)^{1/2a} = IF$$

$$P = \frac{1}{t^2-a^2}\left(\frac{t-a}{t+a}\right)^{1/2a}(-y - t^2 + t + a^2)$$

$$Q = \cancel{\frac{1}{t^2-a^2}(t^2-a^2)}\left(\frac{t-a}{t+a}\right)^{1/2a}$$

$$\int Q\,dy = \int \left(\frac{t-a}{t+a}\right)^{1/2a} dy =$$

$$\left(\frac{(t-a)^{1/2a}}{(t+a)^{1/2a}}\right) y + \phi(t) = F(t,y)$$

$$f(x) = y(t-a)^{1/2a}, \quad f'(x) = \frac{y}{2a}(t-a)^{1/2a-1}$$

$$g(x) = (t+a)^{1/2a}, \quad g'(x) = \frac{1}{2a}(t+a)^{\frac{1}{2a}-1}$$

$$\frac{\partial F}{\partial t} = \frac{\frac{y}{2a}(t+a)^{1/2a}(t-a)^{\frac{1}{2a}-1} - \frac{y}{2a}(t-a)^{1/2a}(t+a)^{\frac{1}{2a}-1}}{(t+a)^{1/a}} + \phi'(t)$$

AWESOME

Pretending that $y = c\left(\frac{t-a}{t+a}\right)^{\frac{1}{2a}} + a$, from now on

7.1 7(a) **3 / 5**

- **- 0 pts** Correct
- ✓ **- 2 pts** **y_h /IF correct, didn't find y_p**
- **- 2 pts** minor mistake / gap
- **- 4 pts** Major mistake/gap
- **- 5 pts** blank
- **- 3 pts** y_h/IF minor mistake/not simplified, didn't find y_p
- **- 1 pts** y_h not simplifed
- **- 3 pts** y_h = ?
- **- 1 pts** typo

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(c) (2 points) Does the above (weird) result contradict with the existence and uniqueness theorem? Why?

This doesn't contradict U/E because the function/its derivative is not continuous over the necessary interval.

7.2 7(b) **2 / 3**

- **- 0 pts** Correct:
- ✓ **- 1 pts minor mistake (e.g. forget to say C can be anything), gap, logic flow not clear**
- **- 2 pts** some meaningful writings. not much detail provided, many gaps.
- **- 3 pts** nothing meaningful

since C can be any number.

7. Let a be a positive integer(it is a fixed unknown number). Consider the following differential equation:

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(a) (5 points) Find the general solution to the above differential equation.

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(c) (2 points) Does the above (weird) result contradict with the existence and uniqueness theorem? Why?

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7.3 7(c) **0 / 2**

- **- 0 pts** Correct
- **- 2 pts** wrong
- ✓ **- 1 pts didn't put in normal form y' = F(y,t) = 1/ (t^2 - a^2) y + ...**
- ✓ **- 1 pts didn't check/state that \partial F/ \partial y = 1/(t^2 - a^2) or calculation is wrong**
- **- 1 pts** Gap

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8. (10 points) Calculate e^{tA}, where $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ ← never truncates

(Hint: Use truncation formula) ?

$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \cdots$

$A^2 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}$

$A^3 = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}$

$A^4 = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^4 & 4a^3b \\ 0 & a^4 \end{bmatrix}$

$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^n}{n!}A^n$

$$\boxed{I + t\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix} + \cdots + \frac{t^n}{n!}\begin{bmatrix} a^n & n(a^{n-1}b) \\ 0 & a^n \end{bmatrix}}$$

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8 Question 8 **6 / 10**

- **- 0 pts** Correct
- **- 4 pts** gap: did not verify (A-aI)^2 = 0
- **- 4 pts** minor mistake
- **- 2 pts** lack essential detail / some typos
- ✓ **- 4 pts** **based on your flow, you didn't use math induction to give a proof for A^n**
- **- 8 pts** Major mistake