Final Exam

Last Name: ______________________________

First Name: ______________________________

Student ID: ______________________________

Signature: ______________________________

Section:

Tuesday:	Thursday:	
1A	1B	TA: Khang Huynh
1C	1D	TA: Eli Sadovnik
1E	1F	TA: Jason Snyder

Instructions: Do not open this exam until instructed to do so. You will have 180 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. You must **show your work** to receive credit. Please circle or box your final answers.

Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	10	
4	15	
5	15	
6	15	
7	15	
8	10	
Total:	100	

1. (10 points) Solve the initial value problem:

$$(1+t^2)y' + 4ty = \frac{1}{t^2+1}, y(0) = 2$$

IF

it can differed by a constant

$$\frac{t+2}{(t^2+1)^2}.$$

$$y' = \left(-\frac{4t}{1+t^2}\right)y + \frac{1}{(t^2+1)^2}.$$

$$\parallel$$

$$a(x)$$

$$\text{IF}: \quad e^{\int -a(x)\,dx} = e^{\int \frac{4t}{1+t^2}\,dt}.$$

$$= e^{\int \frac{2\cdot d(t^2)}{1+(t^2)}}$$

$$= e^{2\cdot \ln(1+t^2)}$$

$$= (1+t^2)^2.$$

$$(1+t^2)^2 y' + 4t(1+t^2)y = 1$$

$$(1+t^2)^2 y = t + C.$$

$$y = \frac{t+C}{(1+t^2)^2}. \qquad y(0)=2$$

$$\Rightarrow y = \frac{t+2}{(1+t^2)^2}.$$

2. (a) (2 points) Find the general solution $y_h = C_1y_1 + C_2y_2$ to the differential equation:

$$y'' - 3y' + 2y = 0$$

(b) (8 points) Use undetermined coefficient or variation of parameters, find the general solution to the differential equations

$$y'' - 3y' + 2y = e^t + \sin t.$$

$C_1e^t + C_2e^{2t} - e^t \cdot t + \frac{1}{10}\sin t + \frac{3}{10}\cos t$

3. (10 points) Solve the differential equation:

$$(3x^2 + y^2 + 2xy)dx + (3y^2 + x^2 + 2xy)dy = 0$$

$$x^3 + y^3 + x^2y + xy^2 = C.$$

$f(x,y) =$ $\geq 1.$

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4. Consider the autonomous equation:

$$y' = y\sin(y)$$

(a) (4 points) Find the equalibrium solutions of the above differential equations.

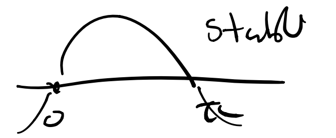
$y = k\pi$

(b) (5 points) Determine the stability of the equalibrium solutions.

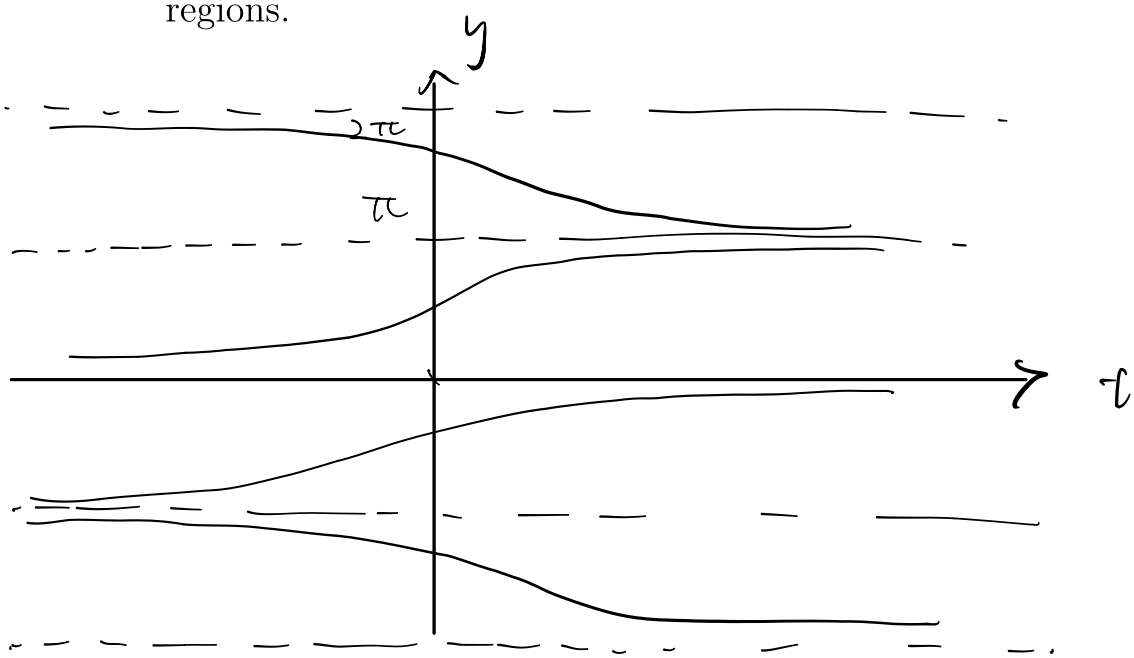
$y=0$ semi-stable.

$(2k+1)\pi$ stable.

$2k\pi$ $k\neq 0$, unstable.



(c) (6 points) Sketch the solutions in the following rectangle region: $R = \{(t,y) | -2\pi < y < 2\pi, -5 < t < 5\}$. These equalibrium divide the R into several regions, sketch at least one solution in each of these regions.



5. Let A be the following 2×2 matrix:

$$A = \begin{pmatrix} -4 & 1 \\ 2 & -5 \end{pmatrix}$$

(a) (5 points) Find the general solution $\mathbf{y}(t)$ to the 2×2 system $\mathbf{y}' = A\mathbf{y}$.

$$\begin{pmatrix} -4-\lambda & 1 \\ 2 & -5-\lambda \end{pmatrix}$$

$$(\lambda+4)(\lambda+5) - 2$$

$$\lambda^2 + 9\lambda + 18$$

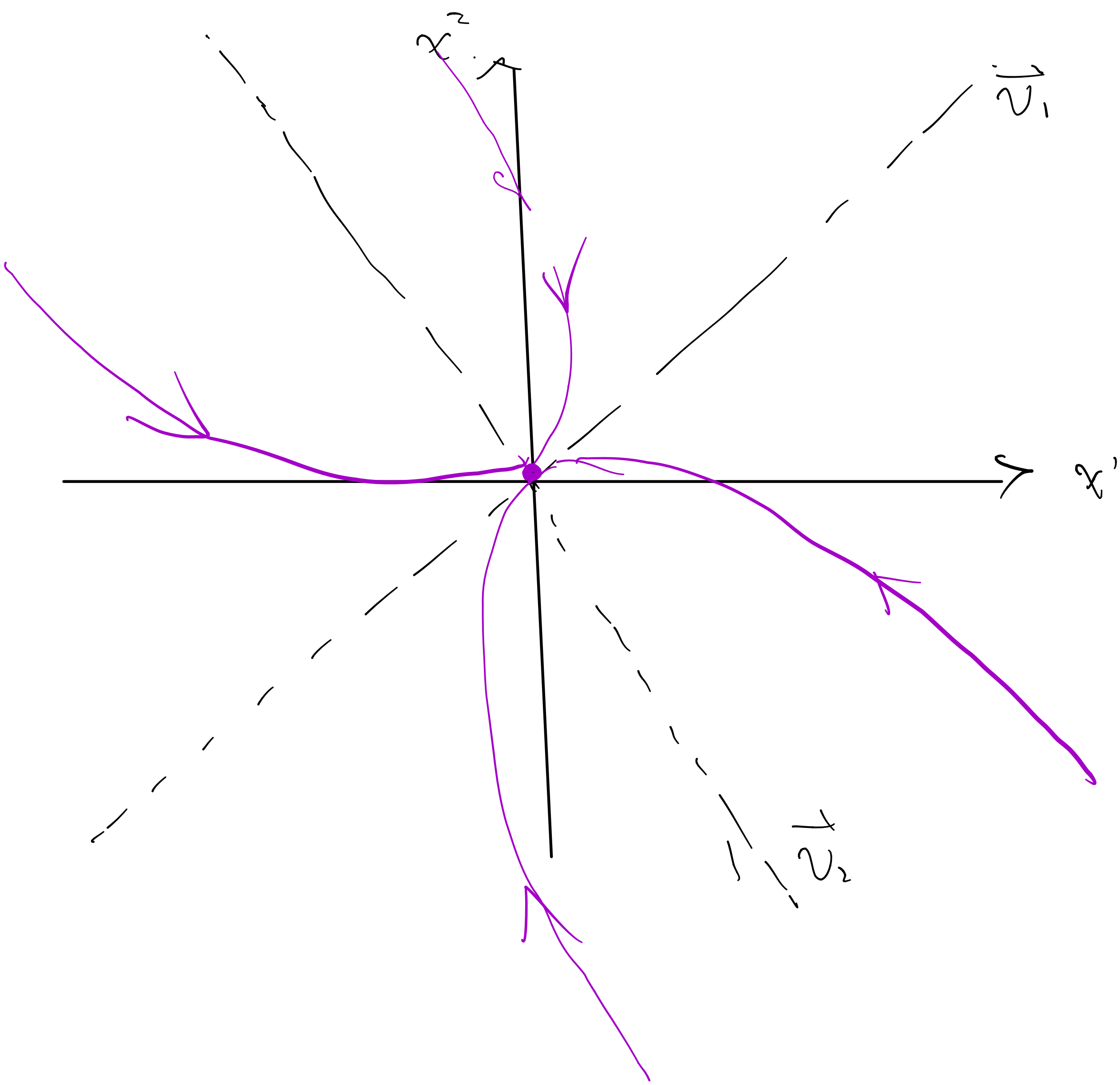
$$(\lambda+3)(\lambda+6) = 0$$

$$y(t) = c_1 e^{-6t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Question 5 continues on the next page...

(b) (2 points) State the type of equilibrium (i.e. The type of Phase Portraits: Saddle, Nodal Source/Sink, Spiral Source/Sink or Center ,etc.)

(c) (8 points) Sketch the phase portraits (you also have to show the direction of the non-equilibrium solution curves on phase plane).



6. (15 points) Find the solution $\mathbf{y}(t)$ to the following 3×3 system with given initial condition $\mathbf{y}(0) = (-1, 1, -1)^T$:

$$\mathbf{y}' = \begin{pmatrix} -1 & -4 & -4 \\ 2 & 5 & 4 \\ -1 & -2 & -1 \end{pmatrix} \mathbf{y}$$

(Hint: Use rational zero theorem to find integer roots of the characteristic polynomial)

$$\begin{vmatrix} -1-\lambda & -4 & -4 \\ 2 & 5-\lambda & 4 \\ -1 & -2 & -1-\lambda \end{vmatrix} \qquad (\lambda-1)^3.$$

$\lambda = 1$

$$x_1 = e^{t}\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -4 & -4 \\ 2 & 4 & 4 \\ -1 & -2 & -2 \end{pmatrix} = A - \lambda I. \qquad x_2 = e^{t}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$a + 2b + 2c = 0.$ $\qquad \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ are two eigenvalue.

$$(A - \lambda I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ is lin indep}$$

$$\vec{x}_3 = e^{t}\left(\vec{v}_3 + t(A-\lambda I)\vec{v}_3\right)$$

$$= e^{t}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}\right)$$

$$C_1\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + C_2\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} c_3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$C_1 = 1 \quad C_2 = -1 \quad C_3 = -1$

$$\begin{pmatrix} 2t - 1 \\ -2t + 1 \\ t - 1 \end{pmatrix} e^{t}$$

$-1 + 2t$
$1 - 2t$
$-1 + t$

7. (15 points) Find the general solution(fundamental set) $\mathbf{y}(t)$ to the following 5×5 system:

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \mathbf{y}$$

(Hint: Use rational zero theorem to find integer roots of the characteristic polynomial, Characteristic polynomial of the matrix is $\lambda^5 - \lambda^4 - \lambda + 1$. This is a block matrix. You can also think about the method we talked in the last lecture.)

Solve:

① $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ and. ② $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$x' = Ax$ $\qquad$ $y' = By$

① $\det(A - \lambda I)$

$= \det \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & -1 \\ 0 & 0 & 1-\lambda \end{pmatrix}$

$= (1-\lambda)(\lambda^2 - 1)$

$= -(\lambda+1)(\lambda-1)^2$.

$\Rightarrow \lambda_1 = -1, \quad \lambda_2 = 1$.

1) when $\lambda_1 = -1$.

$(A - \lambda_1 I) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow[\text{Reduction}]{\text{Row}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$a + b + c = 0$ $\qquad$ let $a = 1$. $b = -1$, $c = 0$.

$2c = 0$

$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\qquad$ $\vec{x}_1 = e^{-t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ -e^{-t} \\ 0 \end{pmatrix}$

2) when $\lambda_2 = 1$.

$(A - \lambda_2 I) = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{cases} -a + b = 0. \\ a - b - c = 0. \end{cases}$ $\qquad$ $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is a solution. $\in \operatorname{Ker}(A - \lambda_2 I)$

$(A - \lambda_2 I)^2 = \begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$2a - 2b - c = 0$

$\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ is a solution.

$\vec{x}_2 = e^{t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\vec{x}_3 = e^{t} e^{t(A - \lambda_2 I)} \vec{v}_3$

$= e^{t} \left(\vec{v}_3 + t(A - \lambda_2 I)\vec{v}_3 \right)$

$= e^{t} \left(\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \right)$

② $\vec{y}_1 = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ $\qquad$ $\vec{y}_2 = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$

Fundamental set. from x_1 $\rightarrow$ $\begin{pmatrix} e^{-t} \\ -e^{-t} \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} e^{t} \\ e^{t} \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} e^{t}(1-t) \\ -te^{t} \\ 2e^{t} \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos t \\ \sin t \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\sin t \\ \cos t \end{pmatrix}$

8. (10 points) Let $y(x)$ be the solution to the initial value problem:

$$y' = \sin(y - x) + 1, y(0) = 1.$$

Prove that $y(x) > x$ for all $x \in \mathbb{R}$. (Hint: This differential equation has no equalibrium, but you can guess a non-constant solution of $y' = \sin(y - x) + 1$, then apply existence and uniqueness theorem.)

$y' = F(x, y) = \sin(y - x) + 1.$

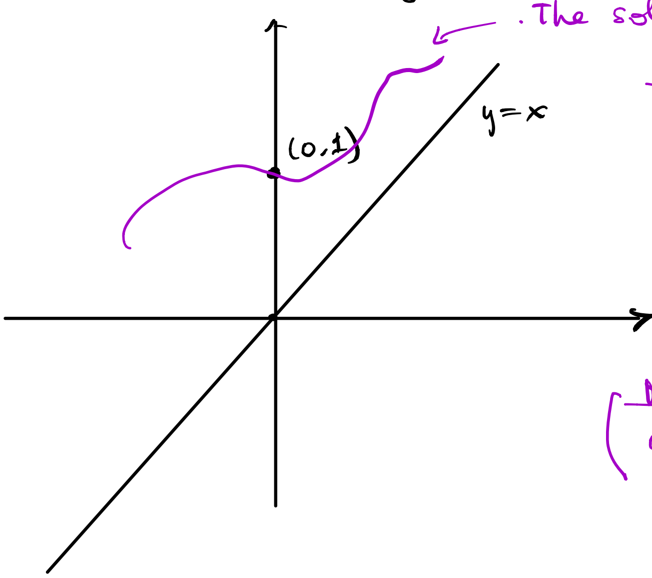
① – Since $F(x, y)$ is continuous, and $\frac{\partial F}{\partial y} = \cos(y - x)$ is also continuous on $\mathbb{R}^2$.

– By U/E thm, the sol (which go through certain initial condition) exist and unique.

② let $y(x) = x$, $y' = \frac{dy}{dx} = 1$.

b/c $y' = 1 = \sin(\underset{\uparrow \atop y(x)}{x} - x) + 1$, Hence, $y(x) = x$ is a solution.

. The solution (which goes through $(0,1)$)



Method 1:

As $(0,1)$ is above $y = x$.

the sol goes through $(0,1)$ should not cross $y = x$.

Hence, $y(x) > x$ for all $x \in \mathbb{R}$.

Method 2:

or alternatively, you can use I.V.P

Here, suppose $y(x_0) \leq x_0$ for some $x_0 \in \mathbb{R}$.

By I.V.P, you can find two curve cross at some pt. ($\Leftrightarrow$ sol not unique.) ↯.

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Scratch Paper

Some useful formulas, etc:

Integrating factor $u(x)$ of a 1st Order Linear DE $x' = ax + f$:

$$u(x) = e^{-\int a(t)dt}$$

Single variable integrating factor μ for $Pdx + Qdy = 0$

- If $h = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$,
$$\mu(x) = e^{\int h(x)dx}$$
- If $g = \frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$,
$$\mu(x) = e^{-\int g(x)dx}$$

Variation of Parameters, (2nd Order Differential Equations)

$$v_1(x) = -\int \frac{1}{W} y_2(x) f(x) \mathrm{d}x$$

$$v_2(x) = \int \frac{1}{W} y_1(x) f(x) \mathrm{d}x$$