Solutions for Midterm 2 (Math 33A, Fall 2019)

Problem 1 (7 points in total)

Let V be the x-z plane in $\mathbb{R}^3$.

What is the orthogonal projection of $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto V?

You should (i) compute the projection $\text{proj}_V(v)$ and (ii) make a drawing of $\text{proj}_V(v)$.

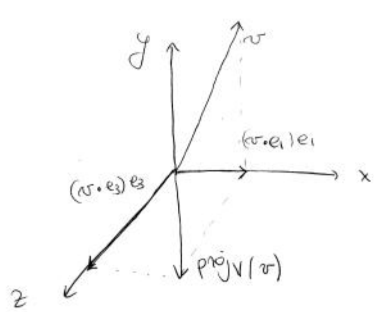
Solution:

An orthonormal basis for V is given by the standard basis vectors e_1 and e_3.

We have

$$\begin{aligned} \text{proj}_V(v) &= (v \cdot e_1)e_1 + (v \cdot e_3)e_3 \\ &= e_1 + 3e_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \end{aligned}$$

Drawing:



Problem 2 (10 points in total)

Let $\mathcal{B}$ be the basis of $\mathbb{R}^2$ given by the vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and

$$v_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Find the $\mathcal{B}$-coordinates of the vectors v_1, $2v_1$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

Find the $\mathcal{B}$-matrix of T.

Solution:

We have

$$[v_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$[2v_1]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

To find the $\mathcal{B}$-coordinates for e_2 we solve the following linear system:

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 3 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 5 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 1/5 \\ 0 & 1 & 1/5 \end{array}\right].$$

And thus we get

$$[e_2]_{\mathcal{B}} = \begin{bmatrix} 1/5 \\ 1/5 \end{bmatrix}$$

The $\mathcal{B}$-matrix of T is

$$B = \begin{bmatrix} | & | \\ [T(v_1)]_{\mathcal{B}} & [T(v_2)]_{\mathcal{B}} \\ | & | \end{bmatrix}$$

We have $T(v_1) = Av_1 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ and $T(v_2) = Av_2 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$. We note that $T(v_1) = 5e_1$ and $T(v_2) = 5e_1 + 5e_2$. Hence, to find the $\mathcal{B}$-coordinates of $T(v_1)$ and $T(v_2)$, we siply have to compute the $\mathcal{B}$-coordinates of e_1 similarly as we have done for e_2, and then we can use the fact that $\mathcal{B}$-coordinates are linear.

We have

$$[e_1]_{\mathcal{B}} = \begin{bmatrix} 3/5 \\ -2/5 \end{bmatrix}$$

Thus

$$[T(v_1)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

and

$$[T(v_2)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

Thus we get

$$B = \begin{bmatrix} 3 & 4 \\ -2 & -1 \end{bmatrix}.$$

Problem 3 (6 points in total)

Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Find a basis for the image of A and a basis for the kernel of A.

Solution:

We first put the matrix in RREF: $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \rightsquigarrow \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The solutions of the system $Ax = 0$ are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} r \\ t \\ r \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Thus, a basis for the kernel of A is given by the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

A basis for the image of A is given by the column vectors of A corresponding to the column vectors of RREF(A) that contain pivots. Thus, a basis for the image of A is given by the vector

$$\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}.$$

Problem 4 (10 points in total)

Compute the QR decomposition of the following matrix.

$$M = \begin{bmatrix} -1 & -1 \\ 1 & 3 \\ -1 & -1 \\ 1 & 3 \end{bmatrix}$$

Solution:

Denote the column vectors of the matrix by v_1 and v_2. We first use the Gram-Schmidt algorithm to compute an orthonormal basis $\mathcal{U} = \{u_1, u_2\}$ for the subspace of $\mathbb{R}^4$ spanned by v_1 and v_2:

$$u_1 = \frac{v_1}{\| v_1 \|} = \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$u_2 = \frac{v_2^{\perp}}{\| v_2^{\perp} \|}$$

where $v_2^{\perp} = v_2 - (v_2 \cdot u_1)u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Thus $u_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$. We thus have the matrix Q:

$$Q = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \\ -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

To compute the matrix R, we recall that

$$R = \begin{bmatrix} | & | \\ [v_1]_{\mathcal{U}} & [v_2]_{\mathcal{U}} \\ | & | \end{bmatrix}$$

We have $v_1 = \| v_1 \| u_1 = 2u_1$ and $v_2 = (v_2 \cdot u_1) + \| v_2^{\perp} \| u_2 = 4u_1 + 2u_2$. Thus

$$R = \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}.$$

Problem 5 (10 points total; 2 points each)

Answer the following questions with true or false.

1. Let A be an arbitrary 4×5 matrix (i.e., with 4 rows and 5 columns). The column vectors of A cannot be linearly independent.

   _________TRUE_________

2. There exists a subspace V of $\mathbb{R}^5$ so that V and its orthogonal complement $V^{\perp}$ have the same dimension.

   _________FALSE_________

3. Any square matrix A is similar to itself.

   _________TRUE_________

4. There exists a linear transformation $T\colon \mathbb{R}^5 \to \mathbb{R}^3$ with kernel of dimension 1.

   _________FALSE_________

5. Let V be any subspace of $\mathbb{R}^n$, and $T\colon \mathbb{R}^n \to \mathbb{R}^n$ the orthogonal projection onto V. Let v be a vector in the kernel of T. Then $v \cdot w = 0$ for all vectors w in V.

   _________TRUE_________