20S-MATH61-2 Midterm 2

CHARLES ZHANG

TOTAL POINTS

35 / 40

QUESTION 1

Question 1 10 pts

1.1 Part 1, subparts a) b) c) 6 / 6

✓ **- 0 pts Correct, or mostly correct.**

- **2 pts** Inconsistency in the problem interpretation
- **6 pts** grobgrobkladbischepidor
- **2 pts** Wrong reasoning to one of the subparts
- **0.5 pts** Minor arithmetic issue

1.2 Part 2 2 / 4

- **0 pts** Correct

✓ **- 2 pts Choosing 5 special friends who get 5 different beers**

- **4 pts** Incorrect or missing solution
- **0.5 pts** Minor arithmetic issue

QUESTION 2

2 Question 2 (all graphs) 10 / 10

✓ **- 0 pts Correct**

- **10 pts** Solved a different problem.
- **6 pts** One part done
- **3 pts** Two parts done

QUESTION 3

3 Question 3 10 / 10

✓ **- 0 pts Correct**

- **1 pts** Minor, clearly identifiable error
- **7 pts** Major Errors
- **3 pts** Correct approach, but critical errors
- **5 pts** Did not solve the recurrence
- **5 pts** Guessed the formula, without a proof of correctness

This is correct, but the answer can be simplified considerably. This is just \mu^n-2

QUESTION 4

Question 4 10 pts

4.1 Part 1 5 / 5

✓ **- 0 pts Correct**

- **2 pts** Wrong method to count the number of exams
- **1 pts** Minor error
- **4 pts** Bad probability argument

4.2 Part 2 2 / 5

- **0 pts** Correct
- **1 pts** Minor Error

✓ **- 3 pts Incomplete pigeonhole argument**

- **4 pts** Major errors
- **5 pts** No progress

You have the right idea, but you make too many assumptions about how the houses are placed. You should group the houses into sets of three and apply the pigeonhole principle to the problem of placing houses in those groups.

Q1D: Balrogs, Nazgul, Dragons, Spiders, Wargs, Trolls, Orcs, Crebain

Question seems ambiguous: pick 6 kinds of creatures or pick 6 creatures?

1a) $\frac{8}{1}\ \frac{7}{2}\ \frac{6}{3}\ \frac{5}{4}\ \frac{4}{5}\ \frac{3}{6}$

~~or~~

~~$\frac{8}{1}\ \frac{8}{2}\ \frac{8}{3}\ \frac{8}{4}\ \frac{8}{5}\ \frac{8}{6}$~~

~~There are 8^6 ways to choose 6 creatures~~

There are ${}_8P_6 = \frac{8!}{2!}$ ways to choose 6 kinds of creatures

1b) $\underbrace{{}_8C_6}_{\text{choose 6}} * \underbrace{{}_6P_3}_{\text{Pick 1st line}} * \underbrace{{}_3P_3}_{\text{Pick 2nd line}}$

There are ${}_8C_6 * {}_6P_3 * {}_3P_3 = {}_8P_6 = \frac{8!}{6!}$ ways to choose 6 and put them in 2 lines

* I'm assuming the questions are asking to pick 6 <u>kinds</u> of evil creatures

1c) Order doesn't matter

There are ${}_8C_6 = \frac{8!}{2!\,6!}$ ways to choose a six-creature squad

2) 13 drinks, must be at least 1 of each drink type (5)

$\underbrace{\frac{5}{1}\ \frac{4}{2}\ \frac{3}{3}\ \frac{2}{4}\ \frac{1}{5}}_{\text{make sure all types are chosen}}\ \underbrace{\frac{5}{6}\ \frac{5}{7}\ \frac{5}{8}\ \frac{5}{9}\ \frac{5}{10}\ \frac{5}{11}\ \frac{5}{12}\ \frac{5}{13}}_{\text{any type}}$

$\boxed{5! \times 5^8 \text{ ways}}$

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1.1 Part 1, subparts a) b) c) **6 / 6**

- ✓ **- 0 pts** **Correct, or mostly correct.**
- **- 2 pts** Inconsistency in the problem interpretation
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- **- 0.5 pts** Minor arithmetic issue

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~~or~~

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$5! \times 5^8$ ways

1.2 Part 2 **2 / 4**

- **- 0 pts** Correct
- ✓ **- 2 pts** Choosing 5 special friends who get 5 different beers
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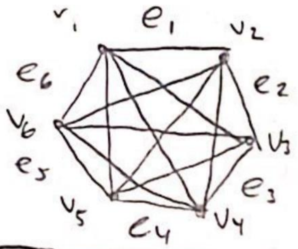
Q2C:

a) Hamilton cycle in K_6: → visit each vertex 1 time

K_6:

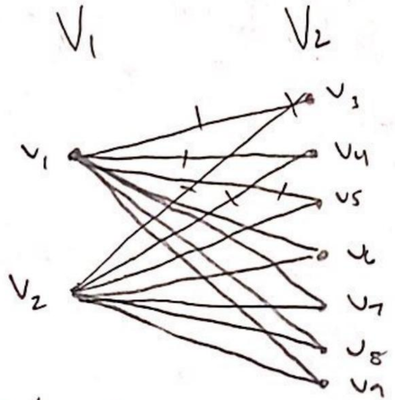


→ go around hexagon

$\boxed{\{v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_5, e_5, v_6, e_6, v_1\}}$

b) Euler path in $K_{2,7}$ → visit each edge 1 time

$K_{2,7}$:



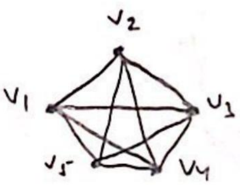
→ go from V_1 to 1st in V_2, then back to other in V_1, then 2nd in V_2, then 1st in V_1, etc., bounce back and forth

No parallel edges → edges are implicitly stated

$\boxed{\{v_1, v_3, v_2, v_4, v_1, v_5, v_2, v_6, v_1, v_7, v_2, v_8, v_1, v_9, v_2\}}$

c) Euler cycle in K_5 → visit each edge 1 time

K_5 :



→ Trace star, then pentagram

No parallel edges → edges are implicitly stated

$\boxed{\{v_1, v_3, v_5, v_2, v_4, v_1, v_2, v_3, v_4, v_5, v_1\}}$

2 Question 2 (all graphs) **10 / 10**

- ✓ **- 0 pts** Correct
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Q30: $a_{n+1} = (H+1)a_n - Ha_{n-1}$, $n \geq 1$

$a_0 = -1$, $a_1 = H-2$

$\hookrightarrow$ rewrite: $a_n = (H+1)a_{n-1} - Ha_{n-2}$

treating H as a constant:

$a_n - (H+1)a_{n-1} + Ha_{n-2} = 0 \rightarrow$ linear homogeneous w/ constant coeff.

$t^2 - (H+1)t + H = 0$

$$t = \frac{(H+1) \pm \sqrt{(H+1)^2 - 4H}}{2}$$

$$r_1 = \frac{(H+1) + \sqrt{(H+1)^2-4H}}{2}, \quad r_2 = \frac{(H+1) - \sqrt{(H+1)^2-4H}}{2}$$

assume $H = 2$

$r_1 = 2$, $r_2 = 1$

$a_n = br_1^n + dr_2^n$

$a_n = b(2^n) + d$

$a_0 = b + d = -1$

$a_1 = 2b + d = H-2 = 0$

$$\begin{array}{r} 2b+d=0 \\ b+d=-1 \\ \hline b=1 \\ d=-2 \end{array}$$

Check

works w/ $H = 2, 5$

$\downarrow$

$a_n = 2^n - 2(1)^n = r_1^n - 2r_2^n$

$$\boxed{a_n = \left(\frac{(H+1) + \sqrt{(H+1)^2-4H}}{2}\right)^n - 2\left(\frac{(H+1) - \sqrt{(H+1)^2-4H}}{2}\right)^n}$$

3 Question 3 **10 / 10**

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Q4A:

1) 3 questions, 4 possible answers

$4 \times 4 \times 4 = 64$ possible answer sheets

Since only 64 distinct answer sheets are possible, and there are 130 students in the class, it is impossible for there to not be 3 identical answer sheets Assuming each distinct answer sheet was submitted by exactly 2 students, there would still be 2 students that must have turned in an answer sheet. This is assuming every student completed the test. Effectively, there are 64 pigeonholes and 130 pigeons, therefore, at least 2 pigeonholes must have 3 or more pigeons.

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2) There are 50 possible numbers → 50 pigeonholes.
41 houses → 41 pigeons

Imagine the closest scenario that doesn't fulfill this requirement: the numbers are distributed so there are 4 consecutive #s, then skip 1. Each grouping of 5 numbers is a pigeonhole, so there are 10 pigeonholes. Each grouping of 4 houses is a pigeon, so there are 10 pigeons, plus 1 house. That house must fit in a pigeonhole, and, since it must also be assigned a number, it must create a group of 5 consecutive numbers.

4.1 Part 1 5 / 5

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4.2 Part 2 **2 / 5**

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- **- 1 pts** Minor Error
- ✓ **- 3 pts Incomplete pigeonhole argument**
- **- 4 pts** Major errors
- **- 5 pts** No progress

You have the right idea, but you make too many assumptions about how the houses are placed. You should group the houses into sets of three and apply the pigeonhole principle to the problem of placing houses in those groups.

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