In Exercises 13–16, show, by direct substitution, that the given functions $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation. Then verify, again by direct substitution, that any linear combination $C_y y_1(t) + C_2 y_2(t)$ of the two given solutions is also a solution.

13. $y'' - y' - 6y = 0, \quad y_1(t) = e^{3t}, \quad y_2(t) = e^{-2t}$
14. $y'' + 4y = 0, \quad y_1(t) = \cos 2t, \quad y_2(t) = \sin 2t$
15. $y'' - 2y' + 2y = 0, \quad y_1(t) = e^t \cos t, \quad y_2(t) = e^t \sin t$
16. $y'' + 4y' + 4y = 0, \quad y_1(t) = e^{-2t}, \quad y_2(t) = te^{-2t}$

In Exercises 17–20, use Definition 1.22 to explain why $y_1(t)$ and $y_2(t)$ are linearly independent solutions of the given differential equation. In addition, calculate the Wronskian and use it to explain the independence of the given solutions.

17. $y'' - y' - 2y = 0, \quad y_1(t) = e^{-t}, \quad y_2(t) = e^{2t}$
18. $y'' + 9y = 0, \quad y_1(t) = \cos 3t, \quad y_2(t) = \sin 3t$
19. $y'' + 4y' + 13y = 0, \quad y_1(t) = e^{-2t} \cos 3t, \quad y_2(t) = e^{-2t} \sin 3t$
20. $y'' + 6y' + 9y = 0, \quad y_1(t) = e^{-3t}, \quad y_2(t) = te^{-3t}$
21. Show that the functions

$$y_1(t) = t^2 \qquad \text{and} \qquad y_2(t) = t|t|$$

are linearly independent on $(-\infty, +\infty)$. Next, show that the Wronskian of the two functions is identically zero on the interval $(-\infty, +\infty)$. Why doesn't this result contradict Proposition 1.27?

22. Show that $y_1(t) = e^t$ and $y_2(t) = e^{-3t}$ form a fundamental set of solutions for $y'' + 2y' - 3y = 0$, then find a solution satisfying $y(0) = 1$ and $y'(0) = -2$.
23. Show that $y_1(t) = \cos 4t$ and $y_2(t) = \sin 4t$ form a fundamental set of solutions for $y'' + 16y = 0$, then find a solution satisfying $y(0) = 2$ and $y'(0) = -1$.
24. Show that $y_1(t) = e^{-t} \cos 2t$ and $y_2(t) = e^{-t} \sin 2t$ form a fundamental set of solutions for $y'' + 2y' + 5y = 0$, then find a solution satisfying $y(0) = -1$ and $y'(0) = 0$.

The equations in Exercises 1–8 have distinct, real, characteristic roots. Find the general solution in each case.

1. $y'' - y' - 2y = 0$
2. $2y'' - 3y' - 2y = 0$
3. $y'' + 5y' + 6y = 0$
4. $y'' + y' - 12y = 0$
5. $2y'' - y' - y = 0$
6. $6y'' + y' - y = 0$
7. $3y'' - 2y' - y = 0$
8. $6y'' + 5y' - 6y = 0$

The equations in Exercises 9–16 have complex characteristic roots. Find the general solution in each case.

9. $y'' + y = 0$
10. $y'' + 4y = 0$
11. $y'' + 4y' + 5y = 0$
12. $y'' + 2y' + 17y = 0$
13. $y'' + 2y = 0$
14. $y'' + 2y' + 3y = 0$
15. $y'' - 2y' + 4y = 0$
16. $y'' + 2y' + 2y = 0$

The equations in Exercises 17–24 have repeated, real, characteristic roots. Find the general solution in each case.

17. $y'' - 4y' + 4y = 0$
18. $y'' - 6y' + 9y = 0$
19. $4y'' + 4y' + y = 0$
20. $4y'' + 12y' + 9y = 0$
21. $16y'' + 8y' + y = 0$
22. $y'' + 4y' + 4y = 0$
23. $16y'' + 24y' + 9y = 0$
24. $y'' + 8y' + 16y = 0$