Math 61 HW#3

1. $\{1,2,3,4\}$

i) Reflexive, symmetric, not transitive

$\boxed{R=\{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1),(2,3),(3,2)\}}$

ii) Not reflexive, not symmetric, transitive

$\boxed{R=\{(1,2),(2,3),(1,3)\}}$

iii) Symmetric and anti-symmetric

$\boxed{R=\{(1,1),(2,2)\}}$

2. i) R transitive → R^{-1} transitive

$R=\{(x,y),(y,z),(x,z)\}$

$R^{-1}=\{(z,x),(z,y),(y,x)\}$

$\boxed{\text{true}}$

ii) R reflexive → R^{-1} reflexive

$R=\{(x,x),(y,y),(z,z)\}$

$R^{-1}=\{(x,x),(y,y),(z,z)\}=R$

$\boxed{\text{true}}$

iii) R symmetric → R^{-1} symmetric

by definition $(x,y)\in R \to (y,x)\in R$

$R^{-1} \to (x,y)\to(y,x),\ (y,x)\to(x,y)$

$\boxed{\text{true}}$

iv) R anti symmetric → R anti-symmetric

$(x,y)\in R$ and $(y,x)\in R$ implies $x=y$

$R^{-1} \to (y,x)\in R$ and $(x,y)\in R$, implies $y=x$

$\boxed{\text{true}}$

3. i) 1 element $\{x\}$

$\boxed{2^1-1=1\text{ relation}}$

ii) 2 element $\{x,y\}$

$2^4-1=\boxed{15\text{ relations}}$

iii) 3 elements $\{x,y,z\}$

$2^9-1=\boxed{255\text{ relations}}$

4. Symmetry and transitivity do not guarantee x is related to a y, they only provide conditions that are true if x is related to some y. Therefore, (x,y) may not exist.

5. $\{1,2,3,4,5\}$

i) Yes: $\{1,3\},\{2\},\{4\},\{5\}$

ii) No: 1 isn't related to 4

iii) Yes: $\{1,3,5\},\{4\},\{2\}$

6. i) $\boxed{\text{Yes}}$ everyone is the same height as themselves (reflexive), if A is the same height as B, B is the same height as A (symmetry), and if A is the same as B, B is the same as C, then A is the same as C (transitivity)

ii) $\boxed{\text{No}}$, people can't be taller than themselves (reflexivity not satisfied)

iii) $\boxed{\text{Yes}}$

7. $X=\{1,2,3,4\}$

i) $\{1,2\},\{3,4\}$

$\boxed{R:\{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1),(3,4),(4,3)\}}$

ii) $\{1\},\{2,4\},\{3\}$

$\boxed{R=\{(1,1),(2,2),(3,3),(4,4),(2,4),(4,2)\}}$

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8. $X=\{1,2,3,4,5\}$, $Y=\{3,4\}$, $C=\{1,3\}$

$R=\{P(x) \mid (A,B)\in R$ if $A\cup Y=B\cup Y\}$

↳ Reflexivity: $A\cup Y=A\cup Y$?

The union of A and Y is equal to itself

↳ Reflexivity satisfied

↳ Symmetry: $A\cup Y=B\cup Y \Rightarrow B\cup Y=A\cup Y$?

If the union of A and Y is the same as the union of B and Y, the union of B and Y is the same as the union of A and Y

↳ Symmetry satisfied

↳ Transitive: $A\cup Y=B\cup Y$, $B\cup Y=C\cup Y$, $A\cup Y=C\cup Y$?

Same logic as symmetry gives that $A\cup Y=C\cup Y$

↳ Transitive satisfied

↳ $\boxed{R \text{ is an equivalence relation}}$

9. $C\cup Y=\{1,3,4\}$

↳ contains 1, cannot contain 2 or 5

$\boxed{[C]=\{\{1,3\},\{1\},\{1,4\},\{1,3,4\}\}}$

10 $\boxed{\text{The relation must only have 1 element}}$

11 $X=\{1,2,3,4,5,6,7,8,9,10\}$

$R=\{X\times X \mid ((a,b),(c,d))\in R$ if $a+d=b+c\}$

↳ Reflexivity: $a+d=d+a$?

$a+d$ is equal to $d+a$ → reflexive

↳ Symmetric: if $a+d=b+c$, $c+b=d+a$

$a+d=b+c \to c+b=b+c \to c+b=a+d \to$

$d+a=a+d \to c+b=d+a$ ✓ → symmetric

↳ Transitive: if $a+d=b+c$, $c+f=d+e$, $\Rightarrow a+f=b+e$

$c+f=d+e \to c-d=e-f$, $a+d=b+c \to c-d=a-b \to$

$e-f=a-b \to a+f=b+e \to$ ✓ transitive

↳ $\boxed{R \text{ is an equivalence relation}}$

12. $R_1\cap R_2=R$

if $(x,y)\in R$, $(x,y)\in R_1\cap R_2$

↳ Reflexive: xRx?

↳ xR_1x exists

xR_2x exists

∴ x is in the $R_1\cap R_2$

xRx ✓ → reflexive

↳ Symmetric: $xRy \Rightarrow yRx$?

$xR_1y \Rightarrow yR_1x$

$xR_2y \Rightarrow yR_2x$

if x is related to y, then it exists in $R_1\cap R_2$ ✓

↳ symmetric

↳ Transitive: xRy, $yRz \Rightarrow xRz$?

$xR_1yR_1z \Rightarrow xR_1z$

$xR_2yR_2z \Rightarrow xR_2z$

xR_1z and xR_2z exist →

xRz exists

↳ transitive

↳ $\boxed{R_1\cap R_2 \text{ is an equivalence relation}}$

13. $f: X\to Y$, $R=\{f \mid (x,y)\in R$ if $f(x)=f(y)\}$

↳ reflexive: $f(x)=f(x)$?

a function maps each x to exactly 1 y

↳ $f(x)=f(x)$ ✓ → reflexive

↳ symmetric: $f(x)=f(y) \Rightarrow f(y)=f(x)$?

a function can map an x and a y to the same number

↳ if they are the same (in the relation), then ✓ symmetry

↳ transitive: $f(x)=f(y)$, $f(y)=f(z)$, $f(x)=f(z)$?

↳ $f(x)=f(y)=f(z)$ ✓

↳ transitive ✓

$\boxed{R \text{ is an equivalence relation}}$

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14. $f(x) = [x]$

If $f(x)=f(y) \Rightarrow f(x)=[y]$

$\boxed{\text{When } x \text{ and } y \text{ are in the same equivalence class}}$

15. $f: X \to Y$ is 1-to-1 and onto

↳ Reflexive: if $f(x)=y$, $y=f(x)$

↳ equality is reflexive ✓

↳ Symmetric: $f(a)=b$, $f(b)=a$

↳ equality is reflexive

↳ Transitive: $f(a)=b$, $f(b)=c$, $f(a)=c$

↳ equality is transitive

↳ $\boxed{\text{is an equivalence relation}}$

16 $R = \{(1,\delta),(2,\alpha),(2,\gamma),(3,\beta),(3,\gamma)\}$

i) $X: 1,2,3$, $Y: \alpha,\beta,\gamma,\delta$

$$\begin{array}{c|c} & \alpha\ \beta\ \gamma\ \delta \\ \begin{array}{c}1\\2\\3\end{array} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

ii) $X: 3,2,1$, $Y: \gamma,\beta,\alpha,\delta$

$$\begin{array}{c|c} & \gamma\ \beta\ \alpha\ \delta \\ \begin{array}{c}3\\2\\1\end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

17 i) $R = \{(1,2),(2,3),(3,4),(4,5)\}$

$X: 1,2,3,4,5$

$$\begin{array}{c|c} & 1\ \ 2\ \ 3\ \ 4\ \ 5 \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

ii) $R = \{(x,y) \mid x<y\}$

$X: 1,2,3,4$

$$\begin{array}{c|c} & 1\ \ 2\ \ 3\ \ 4 \\ \begin{array}{c}1\\2\\3\\4\end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

18. $R_1 = \{(1,x),(1,y),(2,x),(3,x)\}$

$R_2 = \{(x,b),(y,b),(y,a),(y,c)\}$

$X: 1,2,3$, $Y: x,y$, $Z: a,b,c$

i)
$$\begin{array}{c|c} & x\ \ y \\ \begin{array}{c}1\\2\\3\end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \end{array} = A_1$$

ii)
$$\begin{array}{c|c} & a\ \ b\ \ c \\ \begin{array}{c}x\\y\end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array} = A_2$$

iii) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

iv)
$$\begin{array}{c|c} & a\ \ b\ \ c \\ \begin{array}{c}1\\2\\3\end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array} = R_2 \circ R_1$$

v) $\boxed{R_2 \circ R_1 = \{(1,a),(1,b),(1,c),(2,b),(3,b)\}}$

19. $R_1 = \{(x,y) \mid x \text{ divides } y\}$

$R_2 = \{(y,z) \mid y > z\}$

X: 5, 4, 3, 2

Y: 5, 4, 3, 2

Z: 4, 3, 2, 1

i)
$$\begin{array}{c} \\ 5 \\ 4 \\ 3 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 5 & 4 & 3 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array} = A_1$$

ii)
$$\begin{array}{c} \\ 5 \\ 4 \\ 3 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 3 & 2 & 1 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} = A_2$$

iii)
$$A_1 A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1 A_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

iv)
$$\begin{array}{c} \\ 5 \\ 4 \\ 3 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 3 & 2 & 1 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array} = R_2 \circ R_1$$

v) $\boxed{R_2 \circ R_1 = \{(5,4), (5,3), (5,2), (5,1), (4,3), (4,2), (4,1), (3,2), (3,1), (2,3), (2,2), (2,1)\}}$

20. Each row in the matrix must contain exactly 1 one.