Three-Body Problem and Lagrangian Points

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Abstract

This paper mainly studies the problems of celestial mechanics. The three-body problem is a hot topic in society recently, from which we can extract many models of physics and astronomy. The three-body problem mainly refers to the chaotic system formed by three planets in the universe under the action of gravity. This problem has been widely studied by scientists. This article will discuss this issue extensively and accurately through the python simulation method. At the same time, we will also study its extended issues, such as the Lagrangian point of the three-body problem and celestial resonance.

1 Introduction and Basic model

1.1 Basic concepts and models of aerodynamics

For isolated celestial bodies and planets, the gravitational force they receive is undoubtedly the biggest influencing factor, and we do not consider the situation of general relativity here. Then for many celestial bodies, the gravitational force of their interaction is the main factor to change their motion state. We can know the formula of universal gravitation, and can use it to find the relationship of acceleration.

$$\mathbf{F}_{21} = -G\frac{m_1 m_2}{|\mathbf{r}_{21}|^2}\hat{\mathbf{r}}_{21} = m\vec{a} \tag{1}$$

Through the above formula, we decompose the horizontal and vertical directions in the Cartesian coordinate system, and we can get a set of motion equations, which is obviously a partial differential equation. Theoretically speaking, after obtaining this set of equations, we can calculate the orbit of celestial bodies as long as we input the initial conditions.

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -GM\frac{x}{(x^2+y^2)^{3/2}} \tag{2}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -GM\frac{y}{(x^2+y^2)^{3/2}} \tag{3}$$

Now that we have obtained the above formula, we might as well choose a few planets in the solar system for a short test. We perform calculations through python code, and use the method of Bulirsch-Stoer to solve and fit differential equations. Our known conditions are shown in the table below, we only need to know the initial position of the planet and its initial velocity to know its orbit. Here we choose the speed of its perihelion, and then we use python to plot.[Code name: final 1.1]

Planet	Mass(kg)	Perihelion(m)	Perihelion speed(m/s)	Orbital period(days)
Mercury	3.30×10^{23}	4.60×10^{10}	57200	88.0
Earth	5.97×10^{24}	1.471×10^{11}	30300	365.2
Mars	6.42×10^{23}	2.067×10^{11}	26400	687.0

After we run the code, we can get the following image, and we can also get the exact period number through the operation of the code. Meanwhile,we can get the orbital period compared with the table numbers. Because we directly use the comparison, all numbers are integer. Our numbers are 87.0, 364.0 and 686.0. Compare with the values in the table in question. We can know that the answer we calculated is not very accurate, because we only get the integer part, but compared with the data in the table, we have tried to approximate it as much as possible.

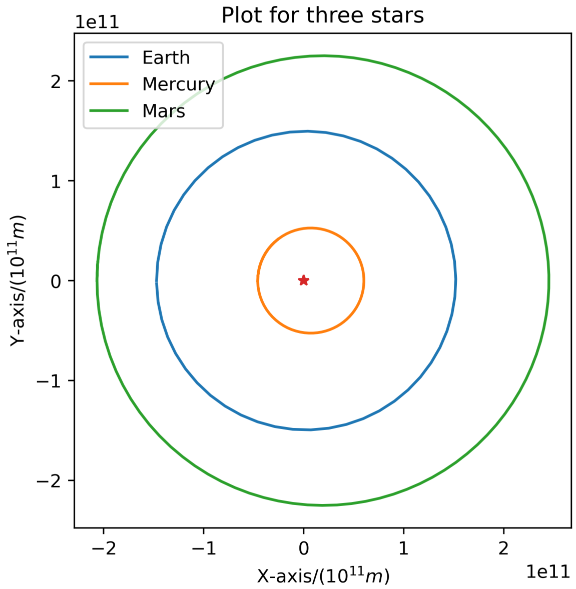


Figure 1: This is the plot for these three stars

1.2 Lagrange point

"Lagrangian points" are some special solutions in the "three-body problem". The "Three-Body Problem" is about three mass points, and only considers the problem of their trajectory under the action of universal gravitation. However, mathematicians have found that for the general three-body problem, the trajectory of particle motion is too complicated, presenting a "chaotic" state. But under some special conditions, the three-body problem can have a very simple solution, that is, the three particles present a simple periodic trajectory.

"Lagrangian points" are the simplest five periodic solutions of the three-body problem. What they have in common is that the relative positions of the three mass points are stationary, that is to say, looking at the other two mass points from one mass point seems to be stationary. We first study the Lagrangian point between the sun and the earth, and we make some assumptions: first, the sun is approximately stationary, second, the earth's orbit is approximately circular, and finally the added satellites will not affect the sun and the earth. We can calculate their positions through mathematical formulas.

For the Sun-Earth system, there are five Lagrangian points in total. Among them, three Lagrange points are located on the line connecting the sun and the earth, which we can obtain by the following formula.

$$\frac{M_1}{(R+r)^2} + \frac{M_2}{r^2} = \frac{M_1}{R^2} + \frac{r\,(M_1+M_2)}{R^3}$$

$$r \approx R\sqrt[3]{\frac{M_2}{3M_1}}$$

We have calculated two of the Lagrangian points, L1 and L2. r is the distance from the earth. L3 is located directly behind the sun, symmetrical to the Earth.

There are also two Lagrange points, L4 and L5, which are located at the other vertex of the equilateral triangle with the sun-earth line as one side. These two points are the balance point.We can use the geometric method to prove it simply, which only needs to be proved geometrically through the synthesis of forces and the concept of the center of mass, as shown in Fig (2).

We next simulate it in python, and we plot the potential energy function of its rotating reference frame. First, we construct a rotating reference system. At this time, for the satellite, there will be a centrifugal potential energy. We are constructing the gravitational potential energy function of the sun and the earth. We reflect them on a 3D graph, for which the potential energy varies with position. Their bump and saddle points correspond to Lagrangian points, their equilibrium positions. In the

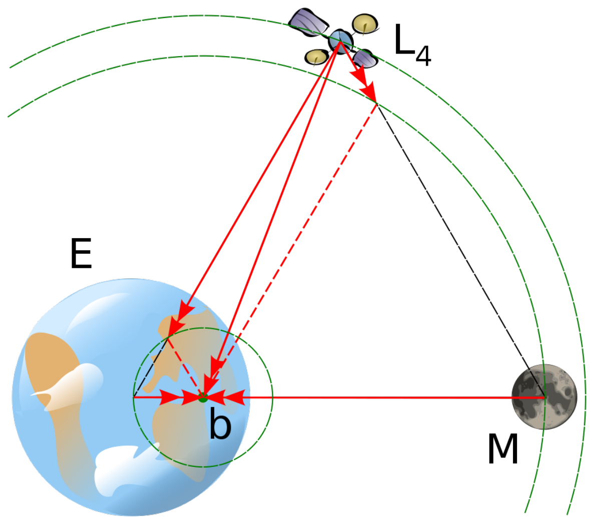
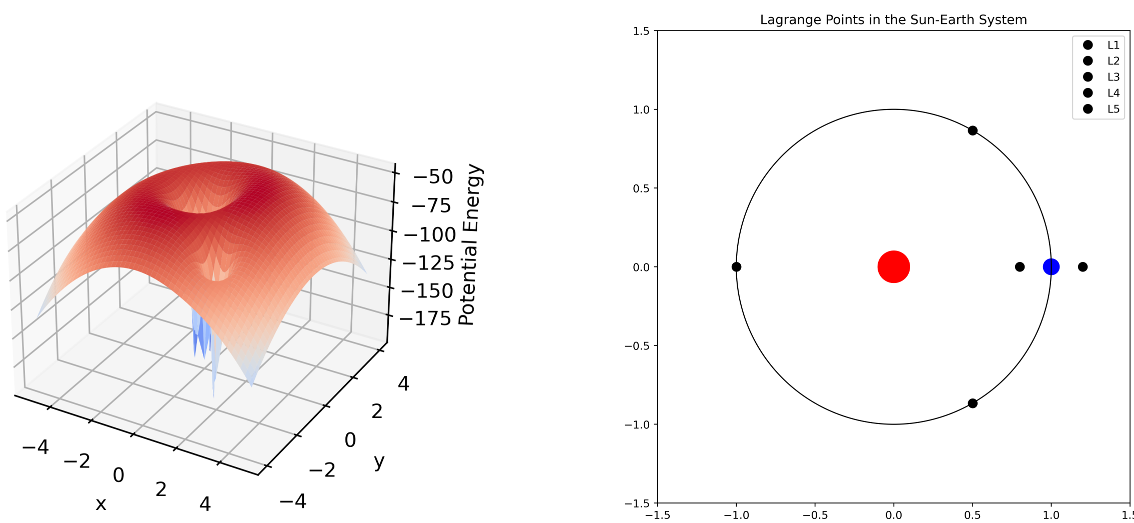


Figure 2: This is the plot for L4 and L5

figure, the overall arc represents the centrifugal potential energy surface, and the two holes represent the gravitational potential energy surface, thus forming such a figure. We know that the point of tangency between the potential energy surface and the horizontal plane is the Lagrangian point. (For the convenience of data visualization and calculation, we did not use the true parameters and ratios between the sun and the earth)[Code name: final 1.2][Code name: final 1.3]



((a)) 3D plot of the potential field

((b)) The five points between sun and earth

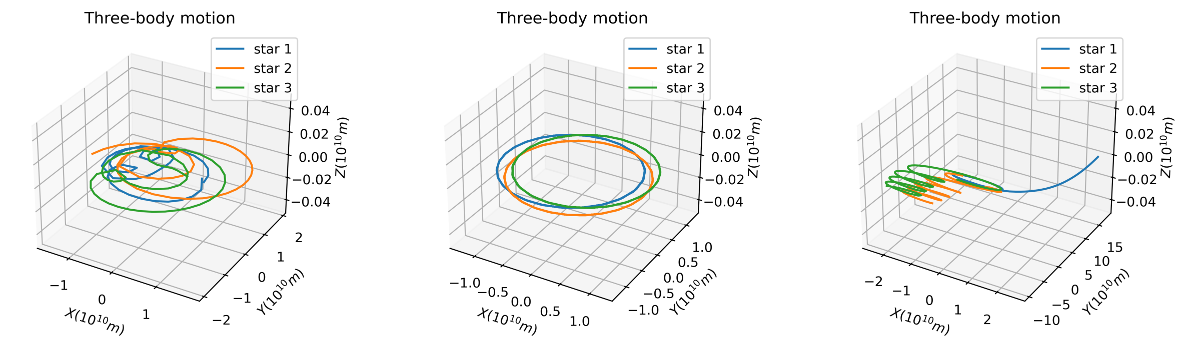
Figure 3: Lagrange Points

2 Main Research Content

2.1 Three Body System

The three-body problem is a relatively hot topic in recent years. Writer Liu Cixin introduced a universe model composed of three stars in his science fiction novel The Three-Body Problem, and described its mysteries. From the perspective of physics, we know that the three-body system is a system composed of three stars. With the difference in initial velocity and position, its motion state is also very different. The three-body system is a chaotic system, which is difficult to solve from a mathematical point of view. With the efforts of scientists in recent years, the three-body model has been solved through python computer simulation.

Similarly, we input different parameters for computer simulation. We assume that the three stars are three suns, and then give different parameters, and then solve the differential equation to draw the image. Since this world is a three-dimensional world, the form of the three-body movement is also three-dimensional.[[Code name: final 2.1]



((a)) Condition 2 ((b)) Condition 3 ((c)) Condition 4

Figure 4: Three body system plots

The second picture above draws a three-body problem with a stable orbit, which requires extremely strict conditions. We have chosen the following stable conditions to form a three-star surround system.

Different initial conditions will greatly affect its final answer, according to the principle of chaos. We can only guarantee the accuracy of its orbit in very small cases

$$\overrightarrow{r_1} = (-1, 0, 0)$$
$$\overrightarrow{r_2} = \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}, 0\right)$$
$$\overrightarrow{r_3} = \left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}, 0\right)$$
$$\overrightarrow{v_1} = (0, v_0, 0)$$
$$\overrightarrow{v_2} = \left(\frac{-\sqrt{3}v_0}{2}, \frac{-v_0}{2}, 0\right)$$
$$\overrightarrow{v_3} = \left(\frac{\sqrt{3}v_0}{2}, \frac{-v_0}{2}, 0\right)$$

For the other two pictures, we have slightly oscillated the conditions in 2, and then their motion patterns have changed significantly, which is the problem caused by the chaotic system of the three-body problem. As long as it is slightly perturbed, its motion state will change greatly.

In order to simplify, we tried to use the initial conditions in the question, and we got a simple harmonic vibration model, in which one star reaches equilibrium under the gravitational force of the other two stars and remains motionless; the other two stars use this star As a symmetrical point, it moves along a straight line and maintains the form of simple harmonic vibration. As the Fig.5 showed

After analyzing the three-body problem, I found the following text on the Internet by searching for literature, and quoted the pictures generated by it.Through the cited literature, we can know that we can describe the three-body problem in the form of a moving graph. So you only need to run the code to generate the animation. Since the animation cannot be displayed on the PDF, please read the code to run the code by yourself.

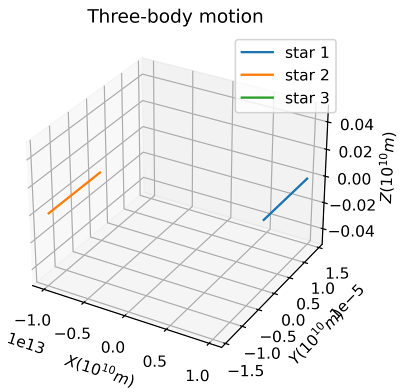


Figure 5: Plot for the condition given in the problem

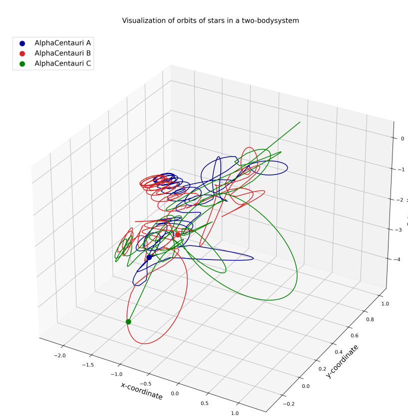


Figure 6: Plot quoted from the Internet

2.2 The Lagrangian points in the stable three-body problem

In 2.1 we introduced the three-body problem, under selected special conditions, we can get a stable and regular orbit. Let's call it the Samsung system. Its characteristic is that these three stars can rotate stably, and their revolution periods are the same, and the positions of their centers of mass are the centers of their orbits.[Code name: final 2.2]

We can express it through a simple formula calculation

$$F_{ab} = -G\frac{MaMb}{{R_{ab}}^2}$$
$$F_{12} = F_{13} = F_{23} = -\sqrt{3}\cdot G\cdot\frac{M^2}{R^2}$$

At this time, for one of the planets, we can understand that the force it receives comes from the resultant force produced by the other two, and the angular velocity of rotation can be obtained through the resultant force between the three.

$$\begin{aligned} F &= m\cdot a \qquad = m\omega^2 r = \sqrt{3}\cdot G\cdot\frac{m^2}{R^2} \\ R &= \sqrt{3}r \\ \omega &= \sqrt{\frac{\sqrt{3}GM}{3r^3}} \end{aligned}$$

We have found out the state of the three-body problem. For the requirements in the question, we can conduct the following analysis. If we want to ensure that the four planets have the same angular velocity after the earth joins the three-star system, we need to ensure that the earth is at the position of the Lagrangian point of this system, and we need to study how many Lagrangian points there are in total.

We can also use the potential energy image drawing and analysis method, we draw a potential energy model of a three-body system. When drawing the image, we did not take the real parameters, we changed its position and potential energy proportionally to make the image more visualized.As the figure7 shows.

We can accurately find the concave and convex poles of its potential energy and their saddle points through the right picture of Fig6. These points are the Lagrangian points of the orbit, which can be known from the depth of the color. We found a total of seven points to exist. One of the points is located at the center of mass of the three-body system, and the satellite at this point does not revolve around. So this doesn't fit the point. In addition, according to the potential energy image, this point belongs to the standard unstable equilibrium point.

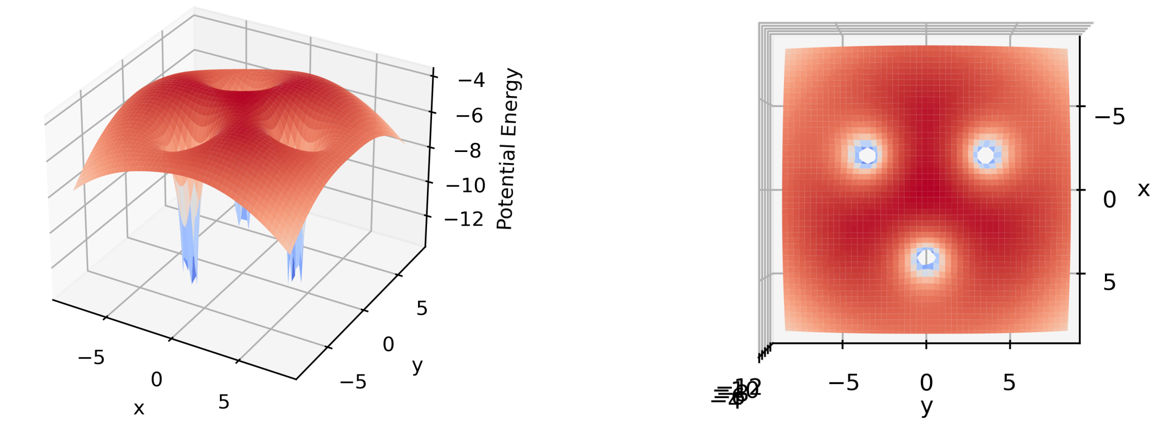
Through the drawing of the image, we can intuitively find the points that meet the meaning of the question, but we can also obtain their results through theoretical calculations.

$$\begin{aligned} E_{\text{potential}} &= -\frac{GMm}{r} \\ E_{\text{force}} &= -\frac{1}{2}m\omega^2 r^2 \end{aligned}$$

$$U = -\frac{1}{2}w^2\left(x^2+y^2\right) - G\cdot\frac{M}{\left((x-R)^2+y^2\right)^{\frac{1}{2}}} - G\frac{M}{\left((x+0.5R)^2+\left(y+\frac{1}{2}\sqrt{3}R\right)^2\right)^{\frac{1}{2}}} - G\frac{M}{\left((x+0.5R)^2+\left(y-\frac{1}{2}\sqrt{3}R\right)^2\right)^{\frac{1}{2}}}$$

We just need to keep $\frac{dU}{dx}$ and $\frac{dU}{dy}$ all zero, we can find the Lagrange points. we will use python to calculate it. However, the direct solution is too troublesome, so we choose to simplify and analyze it. Due to its 120-degree symmetry, we know that the remaining 6 Lagrangian points must be distributed according to this symmetric shape. We first determine the first group of Lagrangian points, which are located on the line between the midpoint of the two planets and the third planet.

$$-\frac{1}{2}\omega^2 r^2 - \frac{GM}{r+R} - \frac{2GM}{\sqrt{\left(1-\frac{1}{2}R\right)^2+\frac{3}{4}R^2}}$$



((a)) 3D plot of Three Body System potential energy

((b)) The energy plot with viewed from the top

Figure 7: Potential Energy Picture

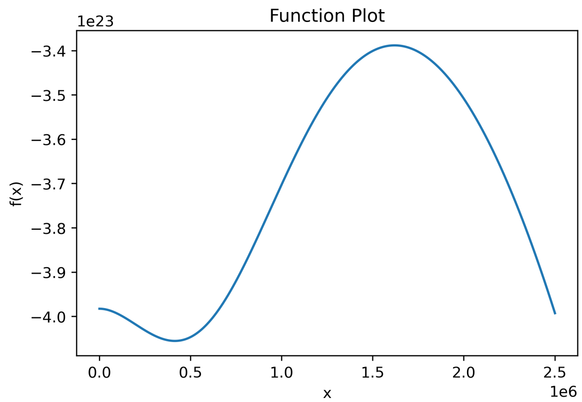


Figure 8: The potential energy plot for the model

We can directly draw its picture with accurate numbers and we get such a picture. The minimum point is the Lagrange point with the meaning of the minimum and maximum potential energy. As the figure8 shows.[Code name: final 2.3]

What we can know from the symmetry is that every planet has a similar curve, which means that there are a total of 6 points where the derivative of the potential energy is zero, and these are the Lagrangian points. We know through the study of mechanics that the positive or negative of its second order derivative can judge its stability or not. From this, we know that among the six points, three groups of points are stable, and the other three groups are unstable. And the stable point is located inside the circle formed by the three stars, and the unstable point is located outside.

Then we construct these points through geometric relations.We choose approximate values for construction, and we find that the black points inside the circle are unstable points, and the black points outside the circle are stable points.[Code name: final 2.4]

We found that this answer is the same as the conclusion mentioned in the three bodies of science fiction, there are a total of six Lagrangian points. If the earth is at these six points, it can ensure that these four stars rotate at the same angular velocity, thus forming a stable system. For these six points, the inner three points are stable, which means that even if the earth has a slight disturbance in this position, it will not easily break away. As for the three points outside the circle, if the earth is in these positions, if the earth is disturbed, it may cause the destruction of the entire system, and thus the earth will break away from this system.

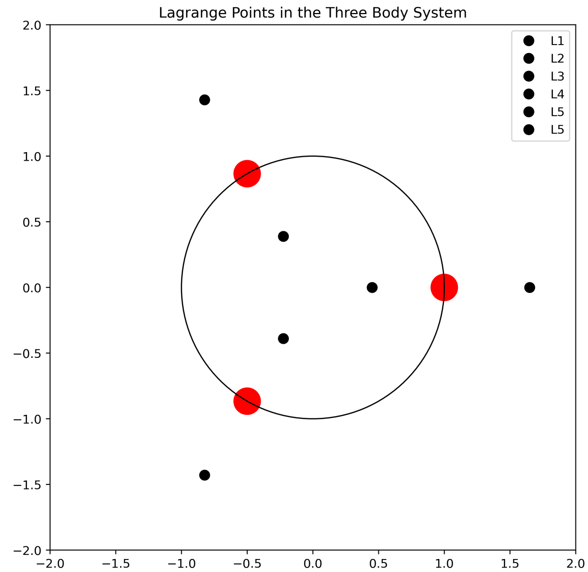


Figure 9: The potential energy plot for the model

2.3 The Lagrangian points between Jupiter and the sun

Regarding the problem of Lagrangian points, we have already touched on it in the introduction. It's just that we are studying the relationship between the earth and the sun in that one. For this question. We need to study the relationship between Jupiter and the sun, and it is clear that their basic concepts have not changed. We only need to change certain parameters to get the position of the new Lagrangian point.

For the planets around Jupiter. They will be stably distributed around Jupiter because of the effect of Lagrangian points. Under the action of gravity, the potential energy in space changes with position.We just need to simulate calculations with Python. We can get the distribution of the space midfield.We can clearly know. There are still 5 Lagrangian points in this system. 2 of them are stable and the other 3 are unstable.

In the previous introduction, we only performed analytical simulations. But no theoretical calculations were performed. Next, we will deduce it through theoretical calculation.

Then we use different ways to solve the equations, the most common way is using the Euler Lagrange Equation. And we will get such exact and accurate answers.

$$\mathrm{L}_1 = \left(\mathrm{R}\cdot\left(1-\sqrt[3]{\frac{\mu}{3}}\right), 0\right), \mu = \frac{\mathrm{M}_2}{\mathrm{M}_1+\mathrm{M}_2}$$

$$\mathrm{L}_2 = \left(\mathrm{R}\cdot\left(1+\sqrt[3]{\frac{\mu}{3}}\right), 0\right)$$

$$\mathrm{L}_3 = \left(-\mathrm{R}\cdot\left(1+\frac{5}{12}\right), 0\right)$$

$$\mathrm{L}_4 = \left(\frac{\mathrm{R}}{2}\frac{\mathrm{M}_1-\mathrm{M}_2}{\mathrm{M}_1+\mathrm{M}_2}, \frac{\sqrt{3}}{2}\mathrm{R}\right)$$

$$\mathrm{L}_5 = \left(\frac{\mathrm{R}}{2}\frac{\mathrm{M}_1-\mathrm{M}_2}{\mathrm{M}_1+\mathrm{M}_2}, -\frac{\sqrt{3}}{2}\mathrm{R}\right)$$

In these equations, the two points L4 and L5 correspond to the stable equilibrium points. This means that once a planet enters the 2 points of L4 and L5, it means that this planet will not leave.

Thus, for a two-body system of Jupiter and the sun, as long as the planets within the 2 points of L4L5 will not break away from the whole system, more and more stars will gather in this point. This leads to the formation of a planetary belt here, and these stars will rotate as Jupiter orbits the sun.We call the planetary belts on L4 and L5 the Trojan camp and the Greek camp, respectively.

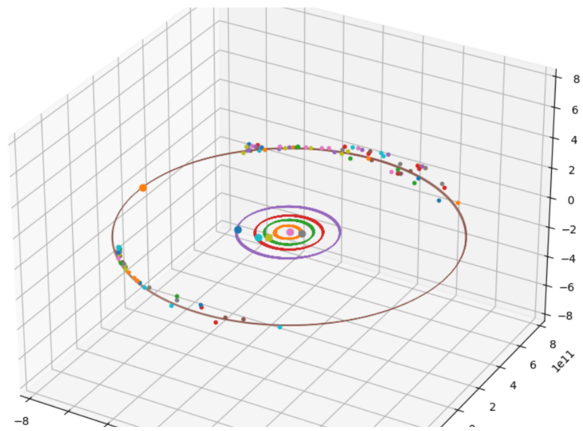


Figure 10: Trojan and Greece groups from the simulation

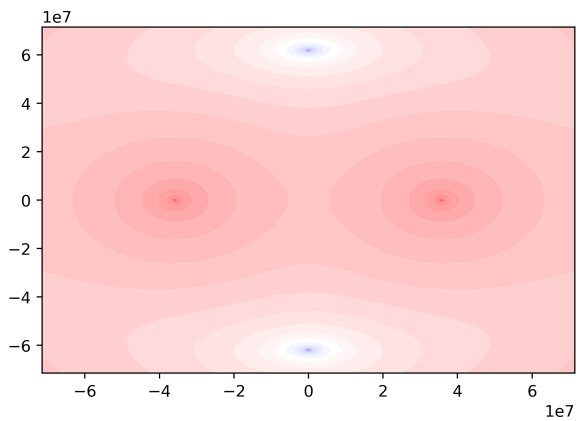


Figure 11: L4 and L5 positions in the potential energy plot

Then, let us plot it using python. And we will get an animation. We made some literature references and quoted the images given in them, which are the images formed by the 2 star groups of Jupiter.[Code name: final 2.5]

We still consider the 2 stable Lagrangian points back to Jupiter, the 2 points of L4 and L5. We draw Python through these two points. We can get the following picture.Since the weight condition assumes that Jupiter has the same mass as the Sun, the center of mass of this binary star system is at the midpoint of the line connecting the two.

Corresponding to the drawn figure are two red dots, which are Jupiter and the sun. This picture shows us the change of energy around Jupiter and the sun. We use the BWR method to draw, so we can know that their potential energy is shown by the depth of the image color. Therefore, we find that the two blue points are the binary star system, and the external energy point is the smallest point, and it is corresponding to these two particularly stable Lagrangian points.

And thus we complete the explanation of the Trojan and Greek constellations. Their existence and their stable motion in the system are due to the stability of the Lagrangian points. The 2 Lagrangian points of L4 and L5 will have such properties for any 2 binary star systems. So it is not difficult for us to find that Jupiter is flanked by two satellite groups before and after.[Code name: final 2.6]

2.4 Resonance of the Jupiter and Hildas group

Objects of the Hildas Group orbit the Sun in orbits characterized by semi-major axes of about 3.7 to 4.2 astronomical units (AU), orbital eccentricities of about 0.07 to 0.30, and inclinations of about 14 to 27 degrees. The number of members of this celestial population is estimated to be between 3000 and 5000, the largest of which is Hilda's star, which is about 170 kilometers in diameter. The formation of the Hildas group may be related to Jupiter's gravity. Under the influence of Jupiter's gravity, the celestial bodies of the Hildas group are gradually pushed to the state of orbital resonance, which leads to a 3:2 resonance relationship between their orbital semi-major axis and Jupiter. This resonance relationship can be maintained stably for a long time, so the celestial bodies of the Hildas group can move stably in this region.

The celestial bodies in the Hildas group are generally considered to be icy objects originating from the Kuiper belt, and they were gradually drifted to the main belt region by the gravitational disturbance of Jupiter after their formation. Compared with other groups of celestial bodies, the celestial bodies in the Hildas group have a larger orbital inclination, which means that they are likely to be thrown into the outer solar system by a large-scale impact event during the formation of the solar system, and then later During its evolution, it was captured by Jupiter's gravity into the main belt region.

Then we use a gif picture to show it, and please check it in the file.

3 Our conclusions

1.By analyzing the system of the solar system, we have accurately calculated the periods of some planets and their images.
2.We simulated the three-body problem with a computer and gave some specific images.
3.We conducted in-depth research on the problem of Lagrangian points, and came to the conclusion that the three-body system has 6 Lagrangian points. We used python to simulate the Lagrangian point problem between the sun and Jupiter, and simulated their planetary groups.
4.We briefly studied the problem of astral resonance.We know that some simple laws of planets and planetary groups are due to resonance. 5.We use the Python language to draw a large number of images to visualize the data in front of the readers, so that they can have a better understanding of celestial mechanics.

References

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