

# Verification of Formal Languages

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- Implementation
- Theorems

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- Definition
- Implementation
- Theorems

# Formal Languages

## Definition (Alphabet)

An *alphabet*  $A$  is a finite set of symbols  $\{a_1, a_2, a_3, \dots, a_n\}$

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## Definition (Words)

A *string* (or *word*) is a sequence of symbols. The length of a word is the number of symbols in the sequence.

## Definition (Formal language)

Let  $A^i$  denote the set of the words created from the symbols of  $A$  that has length  $i$ .

Let  $A^*$  denote  $A^0 \cup A^1 \cup \dots$ , which in other words is the set of finite sequences created by the symbols of the alphabet.

We call the set  $L$  a *formal language* if  $L \subseteq A^*$

- **union** ( $L_1 \cup L_2$ )
- **subtraction** ( $L_1 \setminus L_2$ )
- **inclusion** ( $L_1 \subseteq L_2$ )

# Operations

- **union** ( $L_1 \cup L_2$ )
- **subtraction** ( $L_1 \setminus L_2$ )
- **inclusion** ( $L_1 \subseteq L_2$ )
- **concatenation** ( $L_1 \cdot L_2$ )

## Definition (Concatenation)

Let  $L_1 \subseteq A^*$  and  $L_2 \subseteq A^*$  be two languages. The concatenation of two languages is  $L_1 \cdot L_2 = \{u_1u_2 \mid u_1 \in L_1, u_2 \in L_2\}$



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- **power** ( $L^i$ )

## Definition (Power of languages)

let  $L^0 = \{\epsilon\}$  and  $L^{i+1} = L \cdot L^i$

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- **power** ( $L^i$ )
- **Kleene star** ( $L^*$ )

## Definition (Kleene star)

$$L^* = \{w_1 \dots w_n \mid n \geq 0, \forall i \in [1, n]. w_i \in L\}$$

## Lemma

$$L^* = \bigcup_n L^n$$

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- **concatenation** ( $L_1 \cdot L_2$ )
- **power** ( $L^i$ )
- **Kleene star** ( $L^*$ )
- complement (with respect to  $A^*$ )
- intersection ( $L_1 \cap L_2$ )

Last two are not part of the implementation

## Definition (Empty language)

*Empty language* is a language that does not contain any word, so  $L_0 = \emptyset$

## Definition (Unit language)

*Unit language* is a language, that contains only one word,  $\epsilon$ . So in other words,  $L_\epsilon = \{\epsilon\}$

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  - $\text{len}(w)$  can be represented as `w.size`
  - indexing can be represented using the indexing operator of lists,
  - range indexing can be implemented combining `take` and `drop`
  - concatenation of words  $w_1$  and  $w_2$  can be expressed as `w1 ++ w2`

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- Languages  $\rightarrow$  ???

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- Languages  $\rightarrow$ 
  - Set [Words]
  - Words  $\rightarrow$  Boolean
  - Unique (and ordered) List [Words]
  - List [Words]



# Languages as Set

```
case class Lang[T](set: Set[List[T]]) {  
  def concat(that: Lang[T]): Lang[T] = ???  
  
  def ++(that: Lang[T]): Lang[T] =  
    Lang[T](this.set ++ that.set)  
  
  def contains(word: List[T]): Boolean =  
    set.contains(word)  
  [...]  
}
```

# Languages as Function

```
case class Lang[T](f: List[T] => Boolean) {  
  def concat(that: Lang[T]): Lang[T] =  
    Lang[T](l => !forall( (i: BigInt) => !(  
      i <= l.size && i >= 0 &&  
        this.f(l.take(i)) && that.f(l.drop(i))  
    )))  
  
  def ++(that: Lang[T]): Lang[T] =  
    Lang[T](w => this.f(w) || that.f(w))  
  
  def == (that: Lang[T]): Boolean =  
    forall((x:List[T]) =>  this.f(x) == that.f(x))  
  
  def contains(word: List[T]): Boolean = f(word)  
}
```

# Languages as List (non-unique)

```
case class Lang[T](list: List[List[T]]) {  
  def concat(that: Lang[T]): Lang[T] =  
    Lang[T](concatLists(this.list, that.list))  
  
  def ++(that: Lang[T]): Lang[T] =  
    Lang[T](this.list ++ that.list)  
  
  def == (that: Lang[T]): Boolean =  
    (this.list.content == that.list.content)  
  
  def contains(word: List[T]): Boolean =  
    list.contains(word)  
  
  [...]  
}
```

# Languages as List (non-unique)

```
def concatLists(l1: List[List[T]],
                l2: List[List[T]]): List[List[T]] =
  l2 match {
    case Nil() => Nil[List[T]]()
    case Cons(x,xs) =>
      appendToAll(l1,x)++combineLists(l1,xs)
  }
```

```
def appendToAll( l: List[List[T]],
                 suffix: List[T] ):List[List[T]] = l match {
  case Nil() => Nil[List[T]]()
  case Cons(x,xs) =>
    (x ++ suffix)::appendToAll(xs, suffix)
}
```

# Comparison

	Sets	Lists - 1	Lists - 2	Functions
Unique	Yes	Yes	<b>No</b>	Yes
Iterable	No	<u>Yes</u>	<u>Yes</u>	No
Infinity	No	No	No	<u>Yes</u>
Equality	Trivial	Content =	Content =	$\forall \text{ expr}$
Concat	<b>???</b>	<u>S.I. 2x</u>	<u>S.I. 2x</u>	<b>Complex</b>
Contain	Trivial	Trivial	Trivial	Trivial
Set ops ( $\cup$ , $\cap$ , ...)	Trivial	<b>Uniqueness</b>	Trivial	Trivial
Lemmas	Trivial	Trivial	<b>About cont</b>	Trivial

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## Theorem

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Let  $n \in \mathbb{N}$ . The  $n$ 'th close of the language can be defined as  $L^{(n)} = \bigcup_{i=0}^n L^i$ .



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## Definition (Close)

Let  $n \in \mathbb{N}$ . The  $n$ 'th close of the language can be defined as  
$$L^{(n)} = \bigcup_{i=0}^n L^i.$$

## Lemma

If  $w \in L^*$  then  $\exists n \in \mathbb{N}$ . such that  $w \in L^{(n)}$

# Theorems and Lemmas About Languages

## Theorem (Left Distributivity)

*For any languages  $L_1, L_2, L_3$  we have  $(L_1 \cup L_2) \cdot L_3 == L_1 \cdot L_3 \cup L_2 \cdot L_3$ .*

## Theorem (Right Distributivity)

*For any languages  $L_1, L_2, L_3$  we have  $(L_1 \cup L_2) \cdot L_3 == L_1 \cdot L_3 \cup L_2 \cdot L_3$ .*

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Same operation performed on the same languages.



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## Lemma

*Let  $L_1, L_2, L_3$  be three languages over the same alphabet, and let  $L_1 = L_2$ .  
Then  $L_1 \cdot L_3 = L_2 \cdot L_3$*

We can also state the lemma for the other case, where the left hand side operator is fixed.

## Lemma

*Let  $L_1, L_2, L_3$  be three languages over the same alphabet, and let  $L_2 = L_3$ .  
Then  $L_1 \cdot L_2 = L_1 \cdot L_3$*

# Associativity

## Theorem (Associativity)

*For every  $L_1, L_2, L_3 \subseteq A^*$  we have  $(L_1 \cdot L_2) \cdot L_3 = L_1 \cdot (L_2 \cdot L_3)$*

Proof.



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## Proof.

①  $((\{hd\} \cup tl) \cdot L_2) \cdot L_3 =$  applying the distributive law



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## Proof.

- 1  $((\{hd\} \cup tl) \cdot L_2) \cdot L_3 = \text{applying the distributive law}$
- 2  $((\{hd\} \cdot L_2) \cup (tl \cdot L_2)) \cdot L_3 = \text{applying the distributive law}$



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- ④  $((\{hd\} \cdot L_2) \cdot L_3) \cup (tl \cdot (L_2 \cdot L_3)) =$  cheating



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- ④  $((\{hd\} \cdot L_2) \cdot L_3) \cup (tl \cdot (L_2 \cdot L_3)) =$  cheating
- ⑤  $(\{hd\} \cdot (L_2 \cdot L_3)) \cup (tl \cdot (L_2 \cdot L_3)) =$  applying the distributive law





## Theorem (Associativity)

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- ⑤  $(\{hd\} \cdot (L_2 \cdot L_3)) \cup (tl \cdot (L_2 \cdot L_3)) =$  applying the distributive law
- ⑥  $(\{hd\} \cup tl) \cdot (L_2 \cdot L_3)$



# Cheating?

## Lemma

*For any word  $w$  and languages  $L_1$  and  $L_2$  we have*  
$$(\{w\} \cdot L_1) \cdot L_2 = \{w\} \cdot (L_1 \cdot L_2)$$

## Proof.

Applying induction once more... ☐

# Theorems About Power of Languages

Definition of the power operation only states  $L^0 = \{\epsilon\}$  explicitly.

This is similar to the case in real numbers where  $a^0 = 1$ .

Can we have a rule like  $a^1 = a$

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**Lemma (First power of languages)**

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## Lemma (First power of languages)

*For any language  $L$  we have  $L^1 = L$ .*

## Proof.

$L^1 =$  (by definition)

$L \cdot (L^0) =$  (by definition)

$L \cdot \{\epsilon\} =$  (by unit concatenation lemma)

$L$



# Theorems About Power of Languages

We also know that  $1^n = 1$ , but what about languages?

## Lemma (Power of unit language)

*For any  $i \in \mathbb{N}$  we have  $\{\epsilon\}^i = \{\epsilon\}$*

## Proof.

Applying the definition, induction on  $i$  and after the unit combination lemma, we get:  $\{\epsilon\}^i = \{\epsilon\} \cdot \{\epsilon\}^{i-1} = \{\epsilon\} \cdot \{\epsilon\} = \{\epsilon\}$  □

# Theorems About Power of Languages

The decision that power unfolds to the left was ad-hoc

It could have been defined the other way

## Theorem (Power definition equality)

*For all language  $L$  and  $i \in \mathbb{N}$  we have  $L^i = L \cdot L^{i-1} = L^{i-1} \cdot L$*

## Proof.

- case  $i = 0$ : trivial
- case  $i = 1$ : trivial (applying first power lemma)
- case  $i > 1$ : apply induction on  $i$ , or simply unfold and apply associativity  $i - 2$  times



# Theorems About Power of Languages

## Lemma (Language to the sum)

*For any language  $L$  and numbers  $a, b \in \mathbb{N}$  we have  $L^{a+b} = (L^b) \cdot (L^a)$ .*

## Proof.

Apply induction on  $a$  (or  $b$ )





# Theorems About Close of Languages

## Lemma (Close of Empty Language)

*For every  $i \in \mathbb{N}$  we have  $\emptyset^{(i)} = \{\epsilon\}$*

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## Lemma (Close Order)

$\forall L. L^{(i)} \subseteq L^{(j)} \text{ iff. } i \leq j.$

And so on...

# Theorems About Close of Languages

We would also like to define something similar to "Language to the sum"

## Lemma

*For every language  $L$  and  $a, b \in \mathbb{N}$  we have  $L^{(a+b)} = L^{(a)} \cdot L^{(b)}$*

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Does it even hold?

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Does it even hold?

Yes, but it would be hard to prove, and something weaker will be enough

## Lemma

*For every language  $L$  and  $a, b \in \mathbb{N}$  we have  $L^{(a)} \cdot L^{(b)} \subseteq L^{(a+b)}$*

# Regular Expressions

# "Official" definition

A regular expression can contain the following constants:

- The empty language  $\emptyset$ .
- The unit language  $\{\epsilon\}$  (denoted by simply  $\epsilon$ )
- A language of one word  $\{w\}$  (denoted by  $w$ )

It defines the following operations.

- **Concatenation** of sets of words (denoted by  $\cdot$  or sequentiality)
- **Union** of sets of words (denoted by  $|$ )
- **Kleene star** (repetition) of a set of words (denoted by  $*$ )

## Example

$(abc)^*d(e|f)$ , matches for example "abcde", "abcabcdf" and "de"



# Implementation

How to represent RegExes in Scala?

Instead of these constant use any language as building block, but limit their number to 2

Have a function

`eval[T](l1: Lang[T], l2: Lang[T]): Lang[T]` to evaluate the value

Use case classes

```
sealed abstract class Regex {  
  //evaluate the regular expression to a language  
  def eval[T](l1: Lang[T], l2: Lang[T]): Lang[T]  
}
```

# Case Classes

```
case class L1() extends RegEx {  
  override def eval[T](...): Lang[T] = 11  
}  
  
case class L2() extends RegEx {  
  override def eval[T](...): Lang[T] = 12  
}  
  
case class Union(l:RegEx, r:RegEx) extends RegEx {  
  override def eval[T](...): Lang[T] =  
    l.eval(...) ++ r.eval(...)  
}  
  
case class Conc(l:RegEx, r:RegEx) extends RegEx {  
  override def eval[T](...): Lang[T] =  
    l.eval(...) concat r.eval(...)  
}
```

# Dealing with Star

- Still can not represent infinite languages
- We could apply the close-trick
- Only a syntactic sugar for finite union of pows

```
case class Pow(r:Regex, n:BigInt) extends Regex {  
  override def eval[T](...): Lang[T] =  
    r.eval(...) ^ n  
}
```

# Theorems and Lemmas

## Theorem

*For every regular expression  $r$  defined over the languages  $L_1, L_2$ , if  $r$  evaluates to  $L$ , then  $L \subseteq (L_1 \cup L_2)^*$*

Still can not handle  $*$  properly

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Try to construct such  $i$  manually

# Finding exponent

Lets define a function `evalExp()`: `BigInt` such that

- for L1 and L2 it is 1



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## Lemma

*For every regular expression  $r$  defined over the languages  $L_1, L_2$ , if  $L$  is defined by  $r$ , if  $i = r.evalExp()$  then  $L \subseteq (L_1 \cup L_2)^{(i)}$*

# Proving suitability - Constants

Case  $L_1$  and  $L_2$  it is trivial because of the following lemma.

## Lemma

*For all languages  $L_1, L_2$  we have  $L_1 \subseteq (L_1 \cup L_2)$  and  $L_2 \subseteq (L_1 \cup L_2)$*

# Proving suitability - Union

Case **Union** we can say that  $(L_1 \cup L_2)^{(a)} \subseteq (L_1 \cup L_2)^{(\max(a,b))}$  and  $(L_1 \cup L_2)^{(b)} \subseteq (L_1 \cup L_2)^{(\max(a,b))}$  because of the "Close order lemma" ( $a \leq \max(a, b)$  and  $b \leq \max(a, b)$ )

$a, b$  are `r1.evalExp()` and `r2.evalExp()`

## Lemma (Distributivity of subset)

*Let  $L_1, L_2, L_3$  be three languages. If  $L_1 \subseteq L_3$  and  $L_2 \subseteq L_3$  then  $(L_1 \cup L_2) \subseteq L_3$*

## Lemma (Transitivity of subset)

*Let  $L_1, L_2, L_3$  be three languages. If  $L_1 \subseteq L_2$  and  $L_2 \subseteq L_3$  then  $L_1 \subseteq L_3$ .*

Applying this two we can prove the statement

Recall the following lemma:

## Lemma

*For every language  $L$  and  $a, b \in \mathbb{N}$  we have  $L^{(a)} \cdot L^{(b)} \subseteq L^{(a+b)}$*

Case `Conc` the proof is similar to the previous case, but now we use the lemma above. Since we only want to prove inclusion, the weaker (and proved) form of the lemma is sufficient.

# Proving suitability - Power

$\text{Pow}(r, n)$  can be expressed as:

$\text{Conc}(r, \text{Conc}(r, \text{Conc}(\dots, \text{Conc}(r, \text{Pow}(r, 1)) \dots)))$

Which is equivalent to:

$\text{Conc}(r, \text{Conc}(r, \text{Conc}(\dots, \text{Conc}(r, r) \dots)))$

So power is only a syntactic sugar in regular expressions

We can deduce it from the previous case, where we get  $\sum_{i=1}^n r.\text{evalExp}()$   
 $= n * r.\text{evalExp}()$



# Last missing piece

## Lemma

*For each regular expression  $r$  we have  $r.\text{evalExp}() > 0$ .*

## Proof.

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- In case of  $\text{Pow}(r, n)$ , we can apply induction. We know that  $n \leq 0$  and  $r.\text{evalExp}() \leq 0$ . The product of two non-negative numbers will be also non-negative.



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Metrics (excluding lemmas about regular expressions):

- 95 functions (methods and lemmas)
- 651 verified conditions
- 5127.148s of total time for verification (longest: 318.947s)
- 1936.337s of total time for checking termination

CPU: Dual-Core Intel® Core™ i5-4210U CPU @ 1.70GHz processor  
RAM: 8GB



# Conclusion and Future Work

## Summary:

- analyzed different language representations
- presented a formal language implementation using lists
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## Possible future work:

- Experiment with other language representation
- Finish incomplete lemmas about regular expressions
- Try to prove further statements about regular expressions
- Examine other abstractions using formal languages

# Questions?