A PROOF

A.1 Proof of Proposition 2

Proof. Since $s_i \in \mathbb{Z}$, $\|s_i - s_i^*\| \le \delta$, $\bar{f}(s_i^*) \ge \bar{f}(s_i)$ equals to $\bar{f}(s_i+1) \ge \bar{f}(s_i)$, $\bar{f}(s_i-1) \ge \bar{f}(s_i)$. Suppose there exists a $s_i' \notin C_i$ and $\bar{f}(s_i+1) \ge \bar{f}(s_i)$, $\bar{f}(s_i'-1) \ge \bar{f}(s_i')$, we discuss the value range of s_i' as follows. For simplicity, we denote s' as s in the following discussion.

(1) If $s_i > \sup(C_i)$, then $\Delta y_{i+j\times m} - s_i > 0$ and $s_i > 0$,

$$\bar{f}(s_i) = \log_2(s_i) + \sum_{j=0}^{\lfloor \frac{n-i}{m} \rfloor} \log_2\left(\Delta y_{i+j\times m} - s_i\right) + 2\times\left(\lfloor \frac{n-i}{m} \rfloor + 2\right).$$

Taking the derivative of this function yields:

$$\bar{f}'(s_i) = \frac{\ln(2)}{s_i} + \sum_{j=0}^{\lfloor \frac{n-i}{m} \rfloor} \frac{\ln(2)}{\Delta y_{i+j\times m} - s_i}.$$

For $s_i > \sup(C_i)$, it always holds that $\bar{f}'(s_i) > 0$, and thus $\bar{f}(s_i)$ is monotonically decreasing. That is, for any s_i, $\bar{f}(s_i+1) < \bar{f}(s_i)$, $\bar{f}(s_i-1) > \bar{f}(s_i)$. This is a contradiction, since there should exist a s_i where $\bar{f}(s_i+1) \ge \bar{f}(s_i)$, $\bar{f}(s_i-1) \ge \bar{f}(s_i)$.

(2) If $s_i < \inf(C_i)$, then $\Delta y_{i+j\times m} - s_i < 0$ and $s_i < 0$,

$$\begin{aligned}\bar{f}(s_i) = &\log_2(-2s_i - 1) + \sum_{j=0}^{\lfloor \frac{n-i}{m} \rfloor} \log_2\left(-2\times(\Delta y_{i+j\times m} - s_i) - 1\right) \\ &+ \lfloor \frac{n-i}{m} \rfloor + 2.\end{aligned}$$

Taking the derivative of this function yields:

$$\bar{f}'(s_i) = \frac{2\times\ln(2)}{2s_i + 1} + \sum_{j=0}^{\lfloor \frac{n-i}{m} \rfloor} \frac{2\times\ln(2)}{2\times(\Delta y_{i+j\times m} - s_i) + 1}.$$

For $s_i < \inf(C_i)$, it always holds that $\bar{f}'(s_i) > 0$, and thus $\bar{f}(s_i)$ is monotonically increasing. That is, for any s_i, $\bar{f}(s_i+1) > \bar{f}(s_i)$, $\bar{f}(s_i-1) < \bar{f}(s_i)$. This is a contradiction, since there should exist a $s_i \notin \{\Delta y_{i+j\times m} \mid j = 0, 1, \cdots, \lfloor \frac{n-i}{m} \rfloor\} \cup \{0\}$ and $\bar{f}(s_i+1) \ge \bar{f}(s_i)$, $\bar{f}(s_i-1) \ge \bar{f}(s_i)$.

(3) If $\inf(C_i) < s_i < \sup(C_i)$, without loss of generality, we assume $0, \Delta y_{i+0\times m}, \cdots, \Delta y_{i+k\times m} < s_i < \Delta y_{i+(k+1)\times m}, \cdots, \Delta y_{i+\lfloor \frac{n-i}{m} \rfloor\times m}$, then

$$\begin{aligned}\bar{f}(s_i) = &\log_2(s_i) + \log_2(s_i - \Delta y_{i+0\times m}) + \cdots + \log_2(s_i - \Delta y_{i+k\times m}) \\ &+ \log_2(-2\times(s_i - \Delta y_{i+(k+1)\times m}) - 1) + \cdots \\ &+ \log_2(-2\times(s_i - \Delta y_{i+\lfloor \frac{n-i}{m} \rfloor\times m}) - 1).\end{aligned}$$

Furthermore, we can derive:

$$\begin{aligned}\bar{f}(s_i+1) - \bar{f}(s_i) = &\log_2(1 + \frac{1}{s_i}) + \log_2(1 + \frac{1}{s_i - \Delta y_{i+0\times m}}) + \cdots \\ &+ \log_2(1 + \frac{1}{s_i - \Delta y_{i+k\times m}}) + \cdots \\ &+ \log_2(1 + \frac{1}{-2\times(s_i - \Delta y_{i+\lfloor \frac{n-i}{m} \rfloor\times m}) - 1})\end{aligned}$$

$$\begin{aligned}\bar{f}(s_i+1) - \bar{f}(s_i) = &\log_2(1 + \frac{1}{s_i})(1 + \frac{1}{s_i - \Delta y_{i+0\times m}})\cdots \\ &(1 + \frac{1}{s_i - \Delta y_{i+k\times m}})\cdots \\ &(1 + \frac{1}{-2\times(s_i - \Delta y_{i+\lfloor \frac{n-i}{m} \rfloor\times m}) - 1}).\end{aligned}$$

Similarly,

$$\begin{aligned}\bar{f}(s_i-1) - \bar{f}(s_i) = &\log_2(1 - \frac{1}{s_i}) + \log_2(1 - \frac{1}{s_i - \Delta y_{i+0\times m}}) + \cdots \\ &+ \log_2(1 - \frac{1}{s_i - \Delta y_{i+k\times m}}) + \cdots \\ &+ \log_2(1 - \frac{1}{-2\times(s_i - \Delta y_{i+\lfloor \frac{n-i}{m} \rfloor\times m}) - 1})\end{aligned}$$

$$\begin{aligned}\bar{f}(s_i-1) - \bar{f}(s_i) = &\log_2(1 - \frac{1}{s_i})(1 - \frac{1}{s_i - \Delta y_{i+0\times m}})\cdots \\ &(1 - \frac{1}{s_i - \Delta y_{i+k\times m}})\cdots \\ &(1 - \frac{1}{-2\times(s_i - \Delta y_{i+\lfloor \frac{n-i}{m} \rfloor\times m}) - 1}).\end{aligned}$$

Considering $\bar{f}(s_i+1) - \bar{f}(s_i) \ge 0$ and $\bar{f}(s_i-1) - \bar{f}(s_i) \ge 0$, then

$$\begin{aligned}&(1 + \frac{1}{s_i})(1 + \frac{1}{s_i - \Delta y_{i+0\times m}})\cdots(1 + \frac{1}{s_i - \Delta y_{i+k\times m}})\cdots \\ &(1 + \frac{1}{-2\times(s_i - \Delta y_{i+\lfloor \frac{n-i}{m} \rfloor\times m}) - 1}) \ge 1, \\ &(1 - \frac{1}{s_i})(1 - \frac{1}{s_i - \Delta y_{i+0\times m}})\cdots(1 - \frac{1}{s_i - \Delta y_{i+k\times m}})\cdots \\ &(1 - \frac{1}{-2\times(s_i - \Delta y_{i+\lfloor \frac{n-i}{m} \rfloor\times m}) - 1}) \ge 1.\end{aligned}$$

Multiplying the left-hand sides of these two equations yields:

$$\begin{aligned}&(1 - (\frac{1}{s_i})^2)(1 - (\frac{1}{s_i - \Delta y_{i+0\times m}})^2)\cdots(1 - (\frac{1}{s_i - \Delta y_{i+k\times m}})^2)\cdots \\ &(1 - (\frac{1}{-2\times(s_i - \Delta y_{i+\lfloor \frac{n-i}{m} \rfloor\times m}) - 1})^2) \le 1.\end{aligned}$$

This is a contradiction.

In summary, for any $s_i \in \mathbb{Z}$ such that there exists a δ, where for all s_i^* satisfying $\|s_i - s_i^*\| \le \delta$, $\bar{f}(s_i^*) \ge \bar{f}(s_i)$, it always holds that $s_i \in C_i$. □