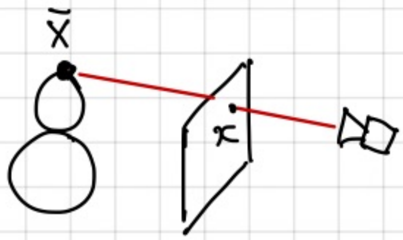
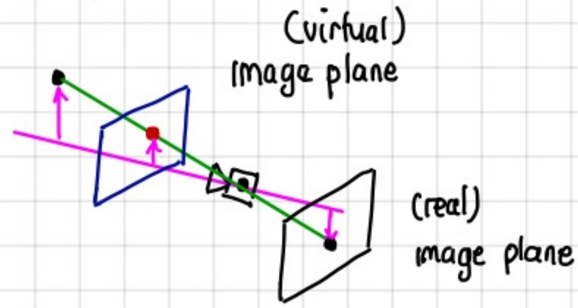
LECTURE 7: CAMERA GEOMETRIC PROPERTIES

[i] Image Formation

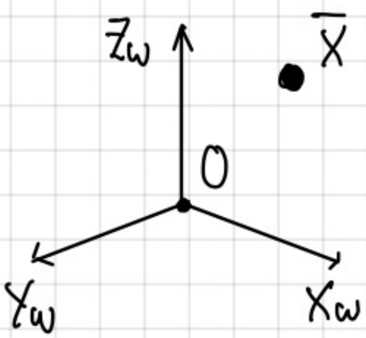
What is the correlation between a 3D point in the real world, $\bar{X}$, & its corresponding 2D point on an image, $\bar{x}$?



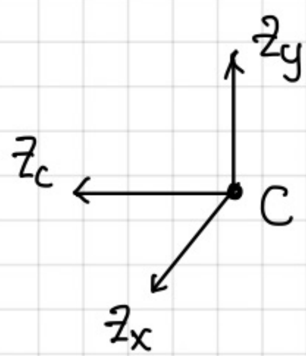


Coordinate Systems

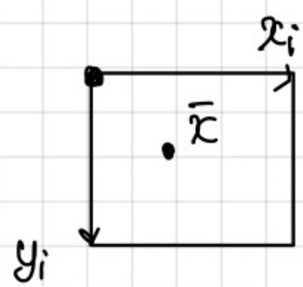
① World coordinate system



② Camera coordinate system



③ Image coordinate system



$$\bar{x} = \mathbb{P}\bar{X}$$

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \mathbb{P} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} \rightarrow \boxed{\text{What is } \mathbb{P}\text{?}}$$

Homogeneous coordinates

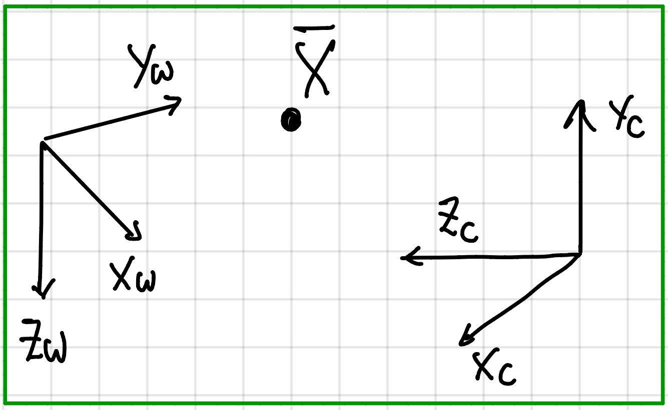
[2] Camera Matrix ($\mathbb{P}$)

$$\bar{x} = \mathbb{P}\,\bar{\bar{X}}$$

3×1 $\quad$ 3×4 $\quad$ 4×1

What matrix is $\mathbb{P}$?

(1) From the world to the camera:



$$\boxed{\bar{X}_c = \mathbb{T}\,\mathbb{R}\,\bar{X}_w}$$

$\bar{X}_w$ = the coordinates of $\bar{X}$ in the world coordinates.

$\bar{X}_c$ = the coordinates of $\bar{X}$ in the camera coordinates.

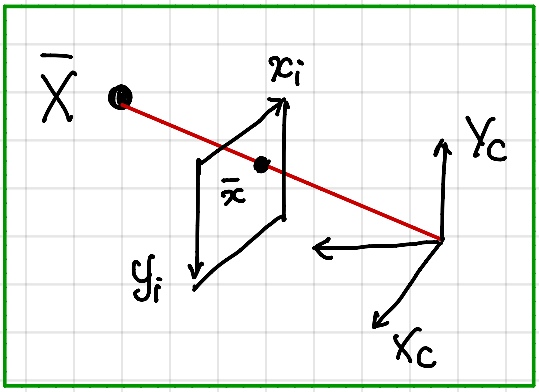
$\mathbb{R}$ = a rotation matrix that rotates the world coordinates of $\bar{X}$ to the camera coordinates.

$\mathbb{T}$ = a translation matrix that translates the world coordinates of $\bar{X}_w$ to the camera coordinates

$\Downarrow$

What matrices are $\mathbb{R}$ & $\mathbb{T}$ exactly?

(2) From the camera to the image:



$$\boxed{\bar{x} = \mathbb{K}\,\bar{X}_c}$$

$\mathbb{K}$: ① to scale down
② to translate from the center of the image to the top left of the image

$$\mathbb{K} = \underbrace{\begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{to translate}} \underbrace{\begin{bmatrix} f_x & 0 & 0 \\ 0 & f_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{to scale down}}$$

$$\mathbb{K} = \begin{bmatrix} f_x & 0 & x_0 \\ 0 & f_y & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\rightarrow$ the center of the image

$$f_x \approx f_y$$

$$\bar{X}_c = \mathbb{R}\,\mathbb{T}\,\bar{X}_w$$

where :

$$\mathbb{T} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad \mathbb{R} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we can write :

$$\underbrace{\mathbb{T}\,\mathbb{R}}_{4\times4} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & T_x \\ R_{21} & R_{22} & R_{23} & T_y \\ R_{31} & R_{32} & R_{33} & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{c|c} \mathbb{R} & \bar{t} \\ \hline 0 & 1 \end{array}\right]$$

$$\underbrace{\bar{X}_c}_{4\times1} = \underbrace{\mathbb{T}\,\mathbb{R}}_{4\times4}\,\underbrace{\bar{X}_w}_{4\times1} = \left[\begin{array}{c|c} \mathbb{R} & \bar{t} \\ \hline 0 & 1 \end{array}\right] \bar{X}_w$$

$$\underbrace{\bar{X}_c}_{3\times1} = \underbrace{[\mathbb{R} \mid \bar{t}]}_{3\times4}\,\underbrace{\bar{X}_w}_{4\times1}$$

$\bar{X}_c$ is in the inhomogeneous coordinates

$\bar{X}_w$ is in the homogeneous coordinates

$\Downarrow$

Hence :

$$\bar{x} = \mathbb{P}\,\bar{X}_w = \mathbb{K}\,\bar{X}_c$$

$$= \underbrace{\mathbb{K}}_{}\,\underbrace{[\mathbb{R} \mid \bar{t}]}_{}\,\bar{X}_w$$

Intrinsic parameters (Internal camera parameters) | Extrinsic parameters (External camera parameters)

$\Downarrow$

Camera geometric calibration / camera resectioning :

To estimate the values of $\mathbb{K}, \mathbb{R}, \bar{t}$.

[4] Forward & Backward Projection

Forward Projection

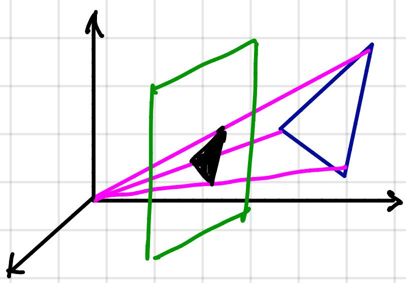


Image plane

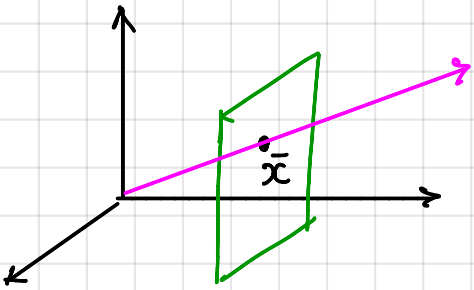
$$\bar{x} = \mathbb{P}\bar{X}$$
$$\underset{3\times1}{\bar{x}} \quad \underset{3\times4}{\mathbb{P}} \quad \underset{4\times1}{\bar{X}}$$

Note: $\bar{x} = \begin{bmatrix} x_i \\ y_i \\ w \end{bmatrix}$, hence

don't forget to normalize:

$$\bar{x} = \begin{bmatrix} x_i/w \\ y_i/w \\ 1 \end{bmatrix}$$

Backward Projection



Q: Given a point $\bar{x}$ on an image, can we recover $\bar{X}$, even if we know $\mathbb{P}$?

A: No!

2D image		3D world
a point	$\rightarrow$	a line
a line	$\rightarrow$	a plane
a plane	$\rightarrow$	a cone

$\Downarrow$

Backprojection: to find a set of points forming a ray passing through the camera center and $\bar{x}$.

Camera center

in the camera coord.:

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

in the world coord.:

$$\bar{C} = -R^{-1}\bar{t}$$
$$\underbrace{= -R^{T}\bar{t}}$$

see last page.

passing through $\bar{x}$:

$$\bar{x} = \mathbb{P}\bar{X}$$
$$\hat{X} = \mathbb{P}^{+}\bar{x}$$
$$\underset{4\times1}{\hat{X}} = (\underset{4\times3}{\mathbb{P}^{T}}\underset{3\times4}{\mathbb{P}})^{-1}\underset{4\times3}{\mathbb{P}^{T}}\underset{3\times1}{\bar{x}}$$

$\hat{X}$ & $\bar{X}$ are not the same point, but they both lie on the same line / ray.

Hence: $\bar{X} = \bar{C} + \lambda\,\mathbb{P}^{+}\bar{x}$

[5] Camera Calibration

Problem statement: Given N corresponding points $\bar{x}_n \leftrightarrow \bar{X}_n$ estimate $\mathbb{K}, \mathbb{R}, t$ such that $\bar{x}_n = \mathbb{P}\bar{X}_n$.

Solution:
① Compute $\mathbb{P}$
② Decompose $\mathbb{P}$ into $\mathbb{K}, \mathbb{R}$, and $\bar{t}$

$$\Downarrow$$

[6] Step 1: Compute $\mathbb{P}$

$$\bar{x} = \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

$$x_i = \frac{p_{11}X_w + p_{12}Y_w + p_{13}Z_w + p_{14}}{p_{31}X_w + p_{32}Y_w + p_{33}Z_w + p_{34}} \quad ; \quad y_i = \frac{p_{21}X_w + p_{22}Y_w + p_{23}Z_w + p_{24}}{p_{31}X_w + p_{32}Y_w + p_{33}Z_w + p_{34}}$$

$$\Leftrightarrow \quad x_i(p_{31}X_w + p_{32}Y_w + p_{33}Z_w + p_{34}) = p_{11}X_w + p_{12}Y_w + p_{13}Z_w + p_{14}$$
$$y_i(p_{31}X_w + p_{32}Y_w + p_{33}Z_w + p_{34}) = p_{21}X_w + p_{22}Y_w + p_{23}Z_w + p_{24}$$

$$p_{11}X_w + p_{12}Y_w + p_{13}Z_w + p_{14} + \emptyset p_{21} + \emptyset p_{22} + \emptyset p_{23} + \emptyset p_{24} - x_iX_w p_{31} - x_iY_w p_{32} - x_iZ_w p_{33} - x_i p_{34} = \emptyset$$
$$p_{11}\emptyset + p_{12}\emptyset + p_{13}\emptyset + p_{14}\emptyset + X_w p_{21} + Y_w p_{22} + Z_w p_{24} - y_iX_w p_{31} - y_iY_w p_{32} - y_iZ_w p_{33} - y_i p_{34} = \emptyset$$

Vectorization:
$$\begin{bmatrix} X_w & Y_w & Z_w & 1 & \emptyset & \emptyset & \emptyset & \emptyset & -x_iX_w & -x_iY_w & -x_iZ_w & -x_i \\ \emptyset & \emptyset & \emptyset & \emptyset & X_w & Y_w & Z_w & 1 & -y_iX_w & -y_iY_w & -y_iZ_w & -y_i \end{bmatrix} \bar{p} = 0$$

where:
$$\bar{p} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{21} & p_{22} & p_{23} & p_{24} & p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix}^T$$

Thus:
$$\underset{2\times12}{\mathbb{A}}\ \underset{12\times1}{\bar{p}} = \underset{2\times1}{\bar{0}} \quad ; \quad \bar{p} \neq 0 \text{ ; otherwise we end up with a trivial solution.}$$

[6] Step 1 (Compute $\mathbb{P}$): Homogeneous Linear System

$\underset{2\times 12}{A}\ \underset{12\times 1}{\bar{p}} = 0$ → Since 1 pair of $(\bar{x}_n, \bar{X}_n)$ provides 2 equations, to estimate $\bar{p}$, we need at least 6 pairs $(N \geq 6)$:

$$\underset{(2N\times 12)}{A}\ \underset{(12\times 1)}{\bar{p}} = 0 \Rightarrow \text{Homogeneous Linear System}$$

How to solve this?

(A) Estimate the initial value of $\bar{p}$ using SVD

(B) Refine $\bar{p}$ using least-squares

$$\boxed{\begin{array}{l} \text{minimize } |A\bar{p}| \\ \text{subject to } |\bar{p}| = 1 \end{array}}$$

→ $|A\bar{p}|$ means the magnitude of $A\bar{p}$.

This is a constraint to avoid the trivial solution of $\bar{p} = 0$.

$$A = \mathbb{U}\mathbb{D}\mathbb{V}^T$$

$\bar{p}$ is the last row of $\mathbb{V}^T$.

[7] Step 1.B: Refine $\mathbb{P}$ using least Squares

$$\mathbb{P}^* = \underset{\{\mathbb{P}\}}{\operatorname{argmin}}\, E(\mathbb{P}) = \underset{\{\mathbb{P}\}}{\operatorname{argmin}} \sum_n \left(\begin{bmatrix} x_n \\ y_n \\ 1 \end{bmatrix} - \mathbb{P} \begin{bmatrix} X_n \\ Y_n \\ Z_n \\ 1 \end{bmatrix} \right)^2$$

$$= \underset{\{\mathbb{P}\}}{\operatorname{argmin}} \sum_n \begin{pmatrix} x_n - p_{11}X_n - p_{12}Y_n - p_{13}Z_n - p_{14} \\ y_n - p_{21}X_n - p_{22}Y_n - p_{23}Z_n - p_{24} \\ 1 - p_{31}X_n - p_{32}Y_n - p_{33}Z_n - p_{34} \end{pmatrix}^2$$

Minimization:

$$\frac{\partial E(\mathbb{P})}{\partial p_{11}} = \sum_n \frac{\partial}{\partial p_{11}} \left(x_n - p_{11}X_n - p_{12}Y_n - p_{13}Z_n - p_{14} \right)^2 = 0$$

$$= 2 \sum_n \left(x_n - p_{11}X_n - p_{12}Y_n - p_{13}Z_n - p_{14} \right) (\cancel{-}X_n) = 0$$

This is not possible to be solved since p_{11} depends on p_{12}, p_{13} & p_{14} (which are unknown)

$\Downarrow$

Solution: Newton's method

$$x^{new} = x^{old} - \frac{\nabla E(x)}{\nabla\nabla E(x)} \quad ; \quad \begin{aligned} \nabla E(x) &= \partial E/\partial x \\ \nabla\nabla E(x) &= \partial^2 E/\partial x^2 \end{aligned}$$

$$p_{11}^{new} = p_{11}^{new} - \frac{\sum_n \left(x_n - p_{11}X_n - p_{12}Y_n - p_{13}Z_n - p_{14} \right)(-X_n)}{\sum_n X_n^2}$$

where: $\mathbb{P}^{init} = \mathbb{P}^{SVD}$; the iteration is done till converged $(|\bar{p}^{new} - \bar{p}^{old}| < \varepsilon)$

Better solution: Levenberg - Marquardt

$$x^{new} = x^{old} - \left(\mathbb{H}(x) + \mu \mathbb{I} \right)^{-1} \nabla E(x)$$

Textbook (Hartley · Zisserman): sect 7.2 page 181.

Steps:

1. Extract $\mathbb{M}$ (a 3x3 matrix) from the first 3x3 submatrix of $\mathbb{P}$.
2. Factor $\mathbb{M}$ into $\mathbb{K}\mathbb{R}$ using RQ decomposition

$$\mathbb{M} = \mathbb{K}\mathbb{R}$$

; recall $\mathbb{K}$ is a triangular matrix

Example :

$$A = \mathbb{Q}\mathbb{R} \rightarrow \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -0.89 & -0.45 \\ -0.45 & 0.89 \end{bmatrix} \begin{bmatrix} -2.24 & -2.24 \\ 0 & 0 \end{bmatrix}$$

$$A = \mathbb{R}\mathbb{Q} \rightarrow \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2.83 \\ 0 & -1.41 \end{bmatrix} \begin{bmatrix} -0.71 & 0.71 \\ -0.71 & -0.71 \end{bmatrix}$$

3. $$\bar{t} = \mathbb{K}^{-1} \begin{pmatrix} P_{14} \\ P_{24} \\ P_{34} \end{pmatrix} \rightarrow \text{why ?}$$

Because : $\mathbb{K}^{-1}\mathbb{P} = [\mathbb{R} \mid t]$

$$\mathbb{K}^{-1} \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & t_x \\ R_{21} & R_{22} & R_{23} & t_y \\ R_{31} & R_{32} & R_{33} & t_z \end{bmatrix}$$

[9] Intrinsic Camera Properties, $\mathbb{K}$: Skew Parameters

Previously we define: $\mathbb{K} = \begin{bmatrix} f_x & 0 & x_0 \\ 0 & f_y & y_0 \\ 0 & 0 & 1 \end{bmatrix}$

A more common definition:

$$\mathbb{K} = \begin{bmatrix} f_x & s & x_0 \\ 0 & f_y & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

where:

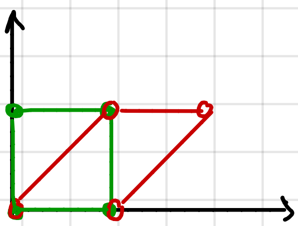
s = the skew parameter

Three operations in $\mathbb{K}$:

$$\mathbb{K} = \underbrace{\begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{translation}} \underbrace{\begin{bmatrix} f_x & 0 & 0 \\ 0 & f_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{scaling}} \underbrace{\begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{shearing}}$$

Shearing example:

$$\begin{array}{c} \quad\; s \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$



Notes:

1. In most cameras, $s = 0$
2. Why does the skew happen horizontally? It's rare that images are skewed vertically.

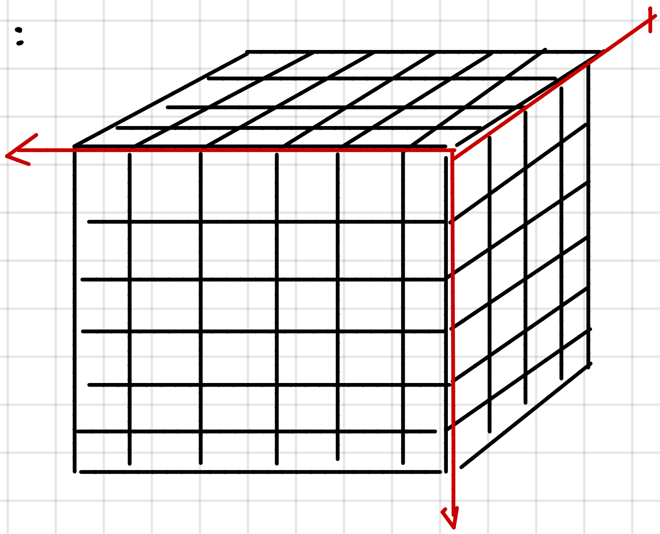
Practical Notes on Calibration

1. Q: How to obtain the corresponding points: $\bar{x}_n$ and $\bar{X}_n$?

   A: Use either a 2D planar surface or a 3D box with identifiable patterns (= a checker pattern)

$\Downarrow$

3D calibration box:



2. Q: Why $\bar{c} = -\mathbb{R}^T\bar{t}$?

   A: We know that: $[\mathbb{R}|\bar{t}]\,\bar{c} = 0$,

which means when we transform $\bar{c}$, the center of the camera in the world coordinate system, onto the camera coordinate system, it should be at the origin, $\bar{0}$.

$$[\underbrace{\mathbb{R}}_{4\times 3}|\underbrace{\bar{t}}_{4\times 1}]\,\underbrace{\bar{c}}_{4\times 1} = 0 \quad ; \quad \mathbb{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbb{R}\bar{c} + \bar{t} = 0 \quad ; \quad \mathbb{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbb{R}\bar{c} = -\bar{t}$$

$$\left.\begin{aligned} \bar{c} &= \mathbb{R}^{-1}\bar{t} \\ &= \mathbb{R}^{T}\bar{t} \end{aligned}\right\} \text{since } \mathbb{R} \text{ is an orthonormal matrix.}$$