LECTURE 12 : STRUCTURE - TEXTURE DECOMPOSITION + OPTIMIZATION

Rudin - Osher - Fatemi (ROF) algorithm :

Model : $I = I_s + I_t$

Goal :

I Input | I_s Structure Layer | I_t Texture Layer

Objective function :

$$I_s^* = \underset{\{I_s\}}{\arg\min} \sum_x \left(I_s(x) - I(x) \right)^2 + \lambda \, |\nabla I_s(x)|_2$$

where : $$|\nabla I_s(x)|_2 = \sqrt{\left(\frac{\partial}{\partial x} I_s\right)^2 + \left(\frac{\partial}{\partial y} I_s\right)^2}$$

$$= \sqrt{I_{s_x}^2 + I_{s_y}^2}$$

Or : $$I_s = \underset{\{I_s\}}{\arg\min} \; J(I_s)$$

where : $$J(I_s) = \iint \left(I_s(x) - I(x)\right)^2 + \lambda \, |\nabla I_s|_2 \, dx\, dy$$

$$\iint |\nabla I_s|_2 \, dx\, dy \longrightarrow$$ L2 - Total Variation (L2-TV)

Q: How to minimize $J(I_s)$?

A: ROF's "Non-linear total variation based noise removal alg.", 1992.

$$E(I_s) = (I_s(x) - I(x))^2 + \lambda \underbrace{|\nabla I_s(x)|_2}_{= \sqrt{I_{sx}^2 + I_{sy}^2}}$$

$$J(I_s) = \iint E(I_s)\, dx\, dy$$

Using the Euler-Lagrange equation:

$$\nabla J = \frac{\partial E}{\partial I_s} - \frac{\partial}{\partial x}\frac{\partial E}{\partial I_{sx}} - \frac{\partial}{\partial y}\frac{\partial E}{\partial I_{sy}} = 0$$

$$\frac{\partial E}{\partial I_s} = 2(I_s(x) - I(x)) \quad ; \quad \frac{\partial E}{\partial I_{sx}} = \frac{\lambda}{2}\frac{2 I_{sx}}{\sqrt{I_{sx}^2 + I_{sy}^2}}$$

$$\nabla J = 2(I_s(x) - I(x)) - \frac{\partial}{\partial x}\frac{\lambda I_{sx}(x)}{\sqrt{I_{sx}^2(x) + I_{sy}^2(x)}} - \frac{\partial}{\partial y}\frac{\lambda I_{sy}(x)}{\sqrt{I_{sx}^2(x) + I_{sy}^2(x)}} = 0$$

Gradient descent:

$$I_s^{new} = I_s^{old} - \alpha \nabla J(I_s)\Big|_{I_s = I_s^{old}}$$

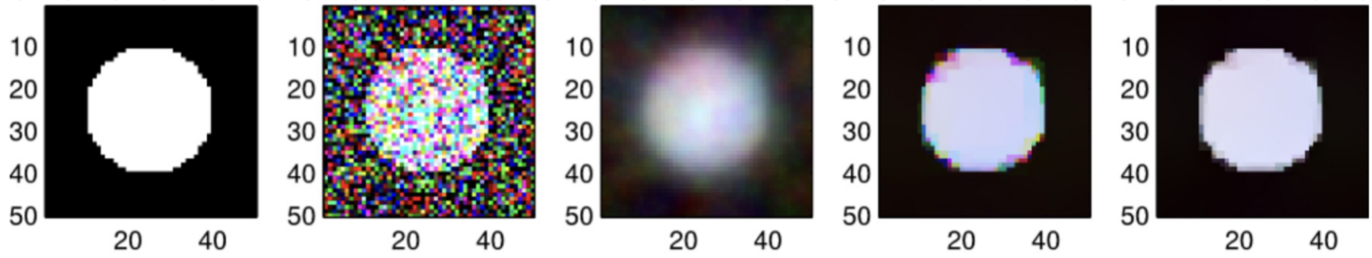
$$= I_s^{old}(x) - \alpha\left[2(I_s^{old}(x) - I(x)) - \frac{\partial}{\partial x}\frac{\lambda I_{sx}^{old}(x)}{\sqrt{I_{sx}^2(x) + I_{sy}^2(x)}} - \frac{\partial}{\partial y}\frac{\lambda I_{sy}^{old}(x)}{\sqrt{I_{sx}^2(x) + I_{sy}^2(x)}}\right]$$

$$= (1 - 2\alpha) I_s^{old} + 2\alpha I + \frac{\partial}{\partial x}\frac{\lambda I_{sx}^{old}}{\sqrt{I_{sx}^2 + I_{sy}^2}} + \frac{\partial}{\partial y}\frac{\lambda I_{sy}^{old}}{\sqrt{I_{sx}^2 + I_{sy}^2}}$$

Initialization: $I_s^{init} = I$

Q : Why don't we use $\|\nabla I_s(\bar{x})\|_2^2$, which is easy to optimize?

A : This quadratic L2-norm generates blurry outputs, while the TV prior generate sharper edges.

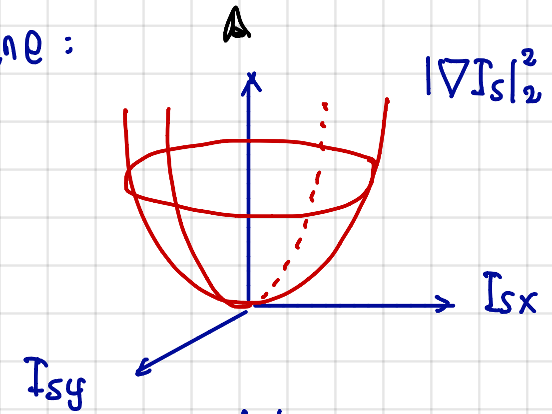


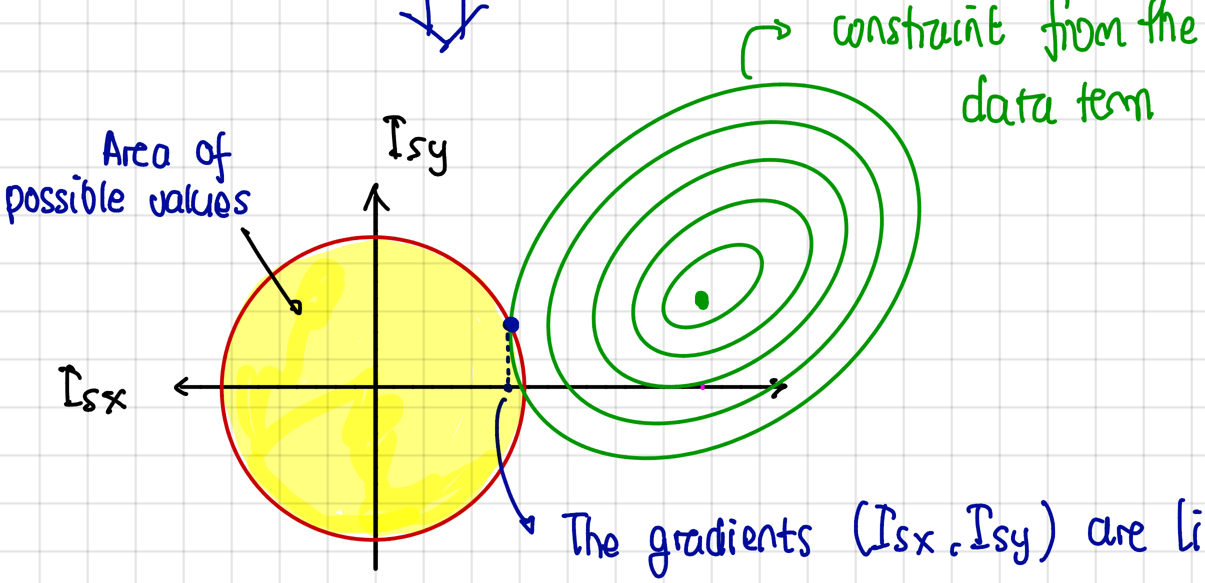
clean image | noisy image (input) | Quadratic L2-norm | L1-TV | L2-TV

Q : Why does the quadratic L2-norm regularization cause the recovered image to be blurry?

A : Quadratic L2 norm: $|\nabla I_s|_2^2 = I_{sx}^2 + I_{sy}^2$

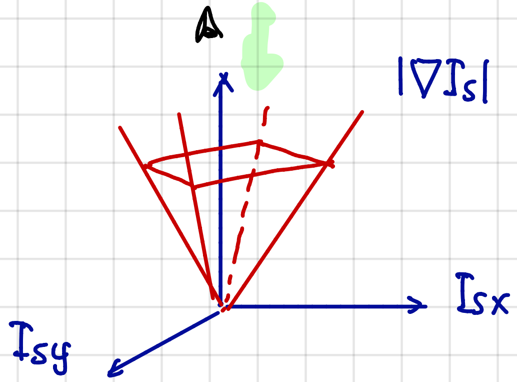
Imagine:

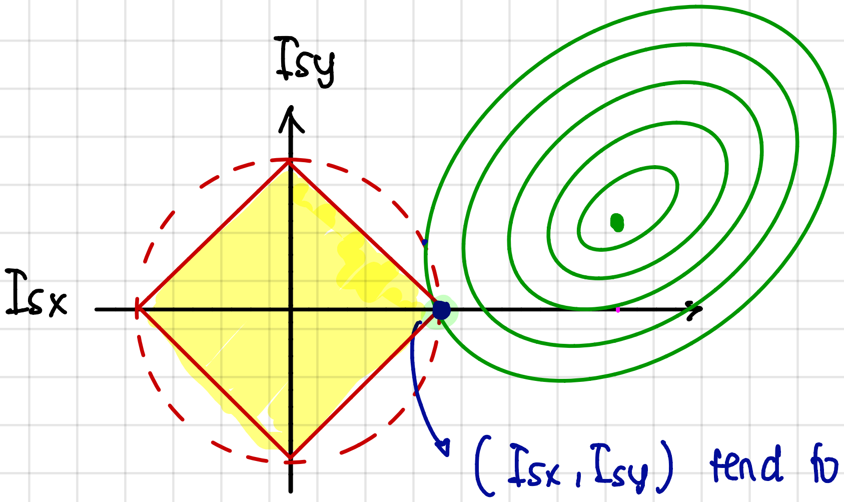




The gradients (I_{sx}, I_{sy}) are likely not zero.

L1-norm:





Note: If (I_{sx}, I_{sy}) are mostly zero (sparse), the values of I_s will be sharper.

L2-TV $\left(|\nabla I_s|_2\right)$ is not exactly L1 norm, but sparser than that of L2-norm $\left(|\nabla I_s|_2^2\right)$.

GRADUATED NON-CONVEXITY

Problem: Robust objective functions are not convex

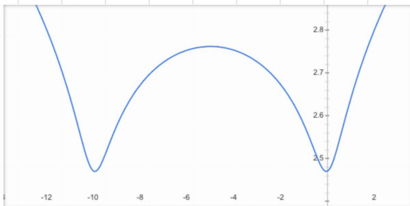
e.g. Lorentzian:

$$f(x) = \log\left(1 + \frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right)$$

Charbonnier:

$$f(x) = (x^2 + \varepsilon^2)^a \quad ; \quad a < 0.5$$

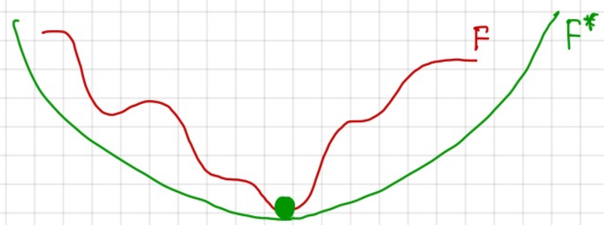
Non-convex: Many local minima cause the optimization to be trapped.



e.g: $[(x+10)^2 + \varepsilon^2]^{0.1} + [x^2 + \varepsilon^2]^{0.1}$

Solution: Graduate Non-Convexity (GNC)

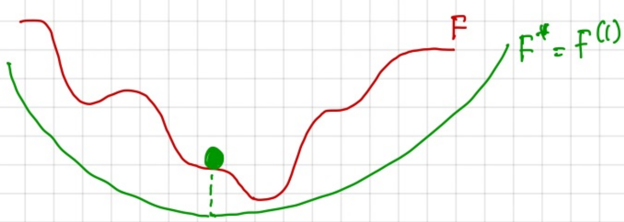
Ideal case:



The approximation function F^* provides the correct minimum.

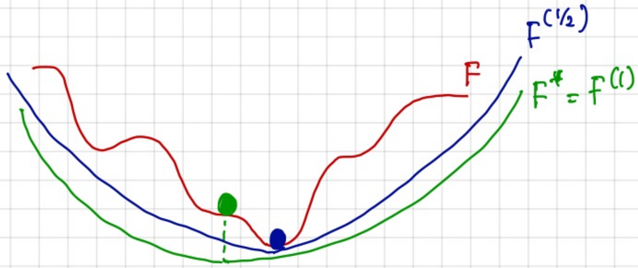
Practically:

(1)



The approximation doesn't directly indicate the minimum but the approximated function $F^{(1)}$ is convex and generates the initial value of Δu.

(2)



Since $F^{(1/2)}$ gets closer to the actual function F, and is convex, it can generate a better value of ΔU.

GNC for optical flow:

$$E_c(u,v) = \lambda E_Q(u,v) + (1-\lambda)\underbrace{E(u,v)}_{\text{Original non-convex function}}$$

where:

- $E(u,v) = f(I_x u + I_y v + I_t) + \alpha\left[f(\nabla u) + f(\nabla v)\right]$

  $f(x)$ is a non-convex function, yet <u>differentiable</u>

- $E_Q(u,v) = (I_x u + I_y v + I_t)^2 + \alpha\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)$

  ↳ Convex function.

$$\boxed{J(u,v) = \iint \left[\lambda E_Q(u,v) + (1-\lambda) E(u,v)\right] dx\, dy}$$

- λ is 1 in the beginning of the iteration, producing $(u,v)_1$
- λ is set to 0.5 in the second iteration, the initial $(u,v)_{init} = (u,v)_1$ producing $(u,v)_2$
- λ is set to 0 in the third iteration, with initial $(u,v)_2$, & producing $(u,v)_{final}$.