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Abstract **Keywords**

1 Introduction

The minimum broadcast time consists of a set of communication nodes with a subset of source nodes. The task is to disseminate a signal to every node in a shortest possible time while abiding by communication rules. An *informed* node is a node that has received the signal. Otherwise, a node is *non-informed*. At the beginning, the set of informed nodes is exactly the set of sources. An informed node u can send the signal to a non-informed node v if u and v are located within a communication vicinity of each other.

The continuous time is divided into discrete time steps. At each time step, every informed node can forward the signal to at most one non-informed neighbor. The number of informed nodes at some time step can therefore be up to double the number of informed nodes at the previous time step.

2 Network Model and Notation

The communication network is represented by an undirected unweighted graph $G = (V, E)$ and a subset of nodes $S \subseteq V$. Broadcasting is defined as a sequence of sets $S = V_0 \subseteq \cdots \subseteq V_k = V$ where each V_i represents the nodes informed after time step i, $0 \leq i \leq k$. For each node $v \in V_i \setminus V_{i-1}$, there exists a single

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node $p(v) \in V_{i-1}$ adjacent to v, which forwarded the signal to v. Also, for every two distinct nodes $u, v \in V_i \setminus V_{i-1}$ we have $p(u) \neq p(v)$. The optimization problem in question is defined as follows:

Problem 1 Given $G = (V, E)$ and $S \subseteq V$, find a shortest sequence $S = V_0 \subseteq \cdots \subseteq V_k = V$ and a mapping $p : V_i \to V_{i-1}$, such that for each $v \in V \setminus S$: $\{v, p(v)\} \in E$, and for each $\{u, v\} \subseteq V_i \setminus V_{i-1} : p(u) \neq p(v)$.

For convenience, we define the set $A = \{(i, j), (j, i) : \{i, j\} \in E\}$ that consists of all arcs that can be derived by directing edges in E. The set of neighbors of $v \in V$ in G is denoted as $N(v)$.

3 Exact methods

Problem 1 can be formulated as an integer linear program and solved to optimality. In this section, we present two different modeling approaches.

3.1 Broadcast time model

The first studied model is a straightforward formulation of the problem. Consider variables

$$x_{ij}^t = \begin{cases} 1, & \text{if } j \in V \setminus S \text{ becomes informed in time } t \text{ and } p(j) = i, \\ 0, & \text{otherwise,} \end{cases}$$

and a variable c representing the number of necessary time steps. The worst case scenario is when G is a path $v_1, \ldots, v_n$ with $S = \{v_1\}$. In such an instance, the necessary number of time steps is $n - 1$, which gives a trivial upper bound on c. Problem 1 is then formulated as follows:

$$
\begin{aligned}
& \min c && && \text{(1a)}\\
& \sum_{i\in N(s)} x^1_{si} \le 1 && s \in S, && \text{(1b)}\\
& \sum_{t=1}^{n-1} \sum_{j\in N(i)} x^t_{ji} = 1 && i \in V \setminus S, && \text{(1c)}\\
& \sum_{j\in N(i)} x^t_{ij} \le 1 && t = 1,\dots,n-1, i \in V, && \text{(1d)}\\
& \sum_{j\in N(i)} x^t_{ij} \le \sum_{u=1}^{t-1} \sum_{k\in N(i)\setminus\{j\}} x^u_{ki} && i \in V \setminus S, t = 2,\dots,n-1, && \text{(1e)}\\
& \sum_{t=1}^{n-1} t \sum_{j\in N(i)} x^t_{ij} \le c && i \in V, && \text{(1f)}\\
& x^1_{ij} = 0 && (i,j) \in A, i \notin S, && \text{(1g)}\\
& && x \in \{0,1\}^{A\times V}, c \in \{1,\dots,n-1\}. && \text{(1h)}
\end{aligned}
$$

Constraints (1b) indicate that source nodes transmit in the first time step. By (1c), all non-source nodes have a single predecessor. Constraints (1d) enforce that each node transmits the signal to at most one neighbor at each time step. The requirement that only informed nodes can relay a signal is modeled by (1e). The maximum time step at which any transmission takes place is captured by (1f), and finally, (1g) states that a node that is not a source never transmits in the first time step.

3.2 Binomial tree model

The following method solves Problem 1 by solving a sequence of decision problems:

Problem 2 Given $G = (V, E)$, $S \subseteq V$ and $k \in \mathbb{N}$, is there a sequence $S = V_0 \subseteq \cdots \subseteq V_k = V$ and a mapping $p : V_i \to V_{i-1}$, such that for each $v \in V \setminus S$: $\{v, p(v)\} \in E$ and for each $\{u, v\} \subseteq V_i \setminus V_{i-1} : p(u) \neq p(v)$?

For a deadline k, at most $|S| \cdot 2^k$ nodes can be informed within k steps. This can be achieved when arcs along which signals initiated at sources $s \in S$ are transmitted form binomial trees B^k of order k rooted at s. Hence, if there is a partition of G into $|S|$ truncated binomial trees of order at most k rooted at sources, then (G, S, k) is a YES instance of Problem 2. Finding a partition of G into $|S|$ truncated binomial trees can be equivalently formulated as finding a partition of G into $|S|$ (complete) binomial trees in G'' constructed from G as follows: Let $\alpha := |S| \cdot 2^k - |V|$, and let $K_\alpha = (V_\alpha, E_\alpha)$ be a complete graph on α

nodes. Each node in K_α is connected to every node $v \in V$ in the original graph G. Thus, $G'' = (V'', E'')$ with $V'' = V \cup V_\alpha$ and $E'' = E \cup E_\alpha \cup \{\{u, v\} : u \in V \wedge v \in V_\alpha\}$. The set of arcs A'' is constructed by creating two arcs of opposite orientation for each edge, but arcs with orientation from V_α to V are excluded. Formally, $A'' = A \cup \{(u, v), (v, u) : \{u, v\} \in E_\alpha\} \cup \{(u, v) : u \in V \wedge v \in V_\alpha\}$.

Proposition 1 *For an instance (G, S, k) of Problem 2, all nodes in G become informed within k time steps iff there exists a partition of G'' into $|S|$ oriented binomial trees $B^k_1, \ldots, B^k_{|S|}$ of order k rooted at sources in S, such that for each $B^k_i = (V_i, A_i)$ we have that $A_i \cap \{(u, v) : u \in V_\alpha \wedge v \in V_i\} = \emptyset$.*
Proof □

Let $I = \{1, \ldots, 2^k\}$. The ILP model based on partition into binomial trees uses only one type of variables

$$y^v_{is} = \begin{cases} 1, & \text{if node } v \text{ is the } i\text{-th node of the binomial tree rooted at } s \in S, \\ 0, & \text{otherwise,} \end{cases}$$

where $v \in V''$ and $i \in I$. The use of these variables requires a systematic numbering of nodes of a binomial tree $B^k = (V^k, E^k)$, so that a node number determines a unique position in B^k. I.e., we need a bijective mapping $\beta : V^k \to I$. A suitable mapping β assigns values increasingly with decreasing outgoing degree. If there is an ambiguity, a node whose parent has a lower number is assigned a lower number. This mapping is defined recursively as

$$\beta(v) = \begin{cases} 1, & \text{if } v \in S \text{ is a root of } B^k, \\ \beta(p(v)) + 2^{k - deg^+(v) - 1}, & \text{otherwise.} \end{cases} \tag{2}$$

It is also useful to define the set $C(i)$ of β-values of children of node v with $\beta(v) = i$ in B^k:

$$C(i) = \{2^l + i : l = \lceil \log_2 i \rceil, \ldots, k - 1\}. \tag{3}$$

With these definitions it is straightforward to specify constraints that enforce desired values for y-variables. An obvious weakness of this approach is that the number of nodes increases to $|V''| = \mathcal{O}(n|S|)$, and the dimension of variables is thus $\mathcal{O}(n^2|S|^2)$. However, once a suitable partition is found, the arcs of binomial trees contained in K_α can be diversely shuffled while preserving the layour of binomial trees in G. Instead of adding the entire complete graph K_α, a single node v_0 with a loop (v_0, v_0) is connected as an apex to the original G. Let us denote this graph as $G' = (V', E')$, where $V' = V \cup \{v_0\}$, $E' = E \cup \{\{u, v_0\} : u \in V\}$. The arc set is then analogously defined as $A' = A \cup \{(u, v_0) : u \in V\}$. The requirement for partition into binomial trees has to be adjusted accordingly. The subtrees contained in G remain unchanged, every arc $(u, v) \in A_i, i = 1, \ldots, |S|$ in G'' with $u \in V$ and $v \in V_\alpha$ becomes (u, v_0) in G', and every $(u, v) \in A_\alpha$ becomes (v_0, v_0). So, v_0 acts as a universal node that can substitute several nodes in each binomial tree.

The values of the variables must fulfill constraints

$$\sum_{i\in I}\sum_{j\in J} y_{ij}^v = 1 \qquad v \in V, \tag{4a}$$

$$\sum_{v\in V'} y_{ij}^v = 1 \qquad i \in I, j \in J, \tag{4b}$$

$$y_{1s}^s = 1 \qquad s \in S, \tag{4c}$$

$$y_{ij}^u + y_{lj}^v \le 1 \qquad i \in I, l \in C(i), j \in S, u, v \in V', (u,v) \notin A', \tag{4d}$$

$$y \in \{0,1\}^{I\times S\times V'}. \tag{4e}$$

As the model solves the decision problem, it suffices to find any feasible solution, and no objective function is needed. The interpretation of constraints (4a) is that every node in the extended graph G' belongs to exactly one binomial tree. By (4b) is ensured that there is always exactly one i-th node of each binomial tree. Next, (4c) enforce that source nodes are the first nodes in corresponding binomial trees, in accordance with definition of the mapping β. The remaining constraints (4d) guarantee that the arcs of binomial trees follow arcs in A'. The definition of A' also prevents arcs of the binomial trees to be oriented from V_α to V. In other words, once the signal leaves the original graph G and enters K_α, it cannot return back to G. Without this requirement, it could be possible to find a partition of G' into binomial trees, even though no partition of G into truncated binomial trees exists.

References

1. Chu, X., Chen, Y., Time division intersatellite link topology generation problem: Modeling and solution, International Journal of Satellite Communications and Networking, 194 – 206, 36 (2017)
2. Jansen, K., Müller, H., The minimum broadcast time problem for several processor networks, Theoretical Computer Science, 69 – 85, 147 (1995)
3. Middendorf, M., Minimum broadcast time is NP-complete for 3-regular planar graphs and deadline 2, Information Processing Letters, 281 – 287, 46 (1993)