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Comparison of Mixed Integer Programming Formulations for the Shared Multicast Tree Problem

Tightening the LP bounds

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Abstract In this paper we focus on the Shared Multicast Tree problem (SMT), which is a task in wireless network design aiming to establish a wireless communication network minimizing necessary energy consumption. SMT is a generalization of the Shared Broadcast Tree problem (SBT), and can be regarded as a Steiner tree problem with a nonlinear objective functinon that reflects the use in wireless communication. In particular, we consider two integer linear programming formulations and investigate how they relate to each other. Both models are subsequently extended by additional variables and corresponding constraints. We also present several valid inequalities. Our goal is to achieve a stronger LP bound than models studied in previous works, and also to devise a method which allows computing these lower bounds for instances as large as possible. Numerical experiments suggest that both models are much stronger than previous formulations, however, the number of constraints makes them impractical for solving instances of even fairly small size as the computation takes prohibitively long time. Applying a constraint generation scheme on one of the studied models substantially increases the size of the instances for which it is possible to obtain a strong LP bound.

Keywords Wireless communication, broadcast tree, multicast, Steiner tree, LP bound, valid inequalities

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1 Introduction

The purpose of a multicast communication in a wireless ad-hoc network is to route information from a sending device to a set of receiving devices. Given a set of wireless devices and distances between them, the task is to assign power to each device, so that the demands of the communication are met and the energy consumption is as low as possible, assuming their locations are fixed. Power efficiency is an important measure in designing ad-hoc wireless networks since the devices typically use batteries as power supply and are therefore heavily energy-constrained. Individual devices work as transceivers, which means that they have the ability to both transmit and receive a signal. Moreover, the power level of a device can be dynamically adjusted during a multicast session.

Unlike wired networks, where signal passing takes place along pre-defined links, nodes in ad-hoc wireless networks use omnidirectional antennas, and hence a message reaches all nodes within the communication range of its sender. This range is determined by the power assigned to the sender, which is the maximum rather than the sum of the powers necessary to reach all intended receivers. This feature is often referred to as *wireless advantage* [1].

A well known and extensively studied task in wireless network design is the Minimum Energy Broadcast (MEB) problem. Given a set of wireless devices with one designated source node among them, the goal is to assign powers to individual nodes which determines their communication ranges, inducing a broadcast tree such that a signal initiated by the source reaches all the remaining nodes, and the energy consumption for this communication is minimized. Typically, not only one node can act as a source. Every node may initiate a message intended for the remaining nodes. In general, two different sources have two different optimal broadcast trees, which means that the optimal broadcast trees must be calculated separately for every possible source node. Furthermore, in order to route signals correctly, the nodes must be able to recognize which node initiated currently received signal and therefore which broadcast tree is used, or from the relaying device's perspective, which power level should be set. It is obvious that such overhead calculations require additional energy and certain abilities of used devices.

The idea of the SBT problem is to maintain a single broadcast tree regardless the source of a signal. Such a tree would not be optimal for individual sources, but routing at each node would be considerably simplified. Provided that a single broadcast tree is used, the nodes are no longer required to identify the source of the message in order to set a correct power level. Instead, only the immediate neighbour from which the signal was received must be recognized. The objective function in SBT captures not only the power levels of the nodes, but depends also on how often a node actually transmits using certain power level. A natural extension of this concept and a forefront of this paper is the Shared Multicast Tree (SMT) problem, in which some of the nodes never initiate any transmission and do not have to receive any signals. They are called *non-destinations*, and can be used as intermediate forwarding nodes whenever it reduces the resulting power, and thus play the role of Steiner nodes. Devices that can initiate a transmission and also have to receive every message are referred to as *destinations*.

1.1 Related work

1.2 Assumptions and notation

An ad-hoc wireless network is modeled by a complete graph $G = (V, E)$, where the set V of nodes represents the set of wireless devices and the set of edges $E = \{\{i, j\} : i, j \in V, i \neq j\}$ corresponds to the potential links between them. Often we use the set $A = \{(i, j) : i, j \in V, \{i, j\} \in E\}$ that contains all arcs derived from E. The set $D \subseteq V$ of *destinations* denotes selected devices that initiate a communication and also are required to receive every message initiated by some other destination. The remaining devices represented by $V \setminus D$ do not have to receive the messages, but can be used as intermediate nodes relaying a transmission. For an arbitrary $i \in V$, sets $V \setminus \{i\}$ and $D \setminus \{i\}$ are abbreviated as V_i and D_i, respectively.

Next, $d : V \times V \to \mathbb{R}$ is a function that determines a distance between every two nodes. The constant α represents an environmentally dependent parameter typically valued between 2 and 4. Power requirement p_{ij} for sending a message from node i to node j is then calculated as $p_{ij} = d_{ij}^{\alpha}$, implying the symmetry $p_{ij} = p_{ji}$. The task is to find a Steiner tree minimizing the objective function clarified in the next section.

If $\{i, j\}$ is an edge in a tree $T = (V_T, E_T)$ in G, we use $T_{i/j}$ to denote the subtree of T consisting of all vertices k such that the path from k to j visits i, as introduced in [2]. Additionally, we define a function $\text{nod}(T_{i/j})$ that returns the number of destinations in $T_{i/j}$. Neighbours of i in T are denoted i_1^T, i_2^T, $i_3^T, \ldots$ in non-increasing order of distance from i. If there is no risk of confusion, we omit the superscript T. The highest and second highest power levels of i are defined by its neighbours i_1 and i_2, respectively. For a leaf i of T, we define $p_{ii_2} = 0$.

Let $z \in \{0, 1\}^E$ be a binary vector with components corresponding to edges in E. Then undirected graph induced by z is defined as $G_z = (V, E_z)$, where $\{i, j\} \in E_z \Leftrightarrow z_{ij} = 1$. Directed graph induced by $x \in \{0, 1\}^A$ is defined analogously. In both cases, the induced (directed) graph is not necessarily connected. Vector $f^s = (f^s_{ij})_{(i,j) \in A}$ for some $s \in D$ is often used in discussions of IP models. A continuous relaxation of an IP model M is denoted as LP(M).

The remainder of this paper is organized as follows: Section 2 describes the SMT problem and gives detailed explanation of its objective function. Integer linear programming formulations, valid inequalities and their analysis are presented in Section 3, followed by Section 4 that compares the studied models. Section 5 describes a constraint generation procedure used for experimental evaluation with results reported in Section 6. Future work and concluding remarks are summarized in Section 7.

2 Shared Broadcast and Multicast Tree problem

A feasible solution to an SMT instance is a Steiner tree spanning a set D of destinations in G. Assume the tree $T = (V_T, E_T)$ depicted in Fig. 1 to be one such solution. Any node $s \in D$ can initiate a transmission, and all the remaining destinations must receive it. Consider the node i with three neighbours i_1, i_2 and i_3 ordered by decreasing distance from i. If the transmitting node is a, b or i_1, then the signal reaches i via arc (i_1, i) and all nodes in the subtree $T_{i_1/i}$ highlighted by the grey area have already received the signal, and so i does not have to send it back to i_1. It suffices that i forwards the signal to its most distant neighbour except from i_1, which is i_2. By using the power level p_{ii_2} and due to the wireless advantage, the message reaches all the neighbours that have not received it yet. On the other hand, if the transmission is initiated by a destination from $T \setminus T_{i_1/i}$ (outside the grey area), then i has to forward it to its most distant neighbour i_1, from where it will be relayed to all nodes that have not received the signal.

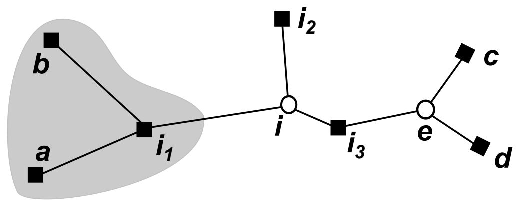


Fig. 1: A simple feasible solution illustrating the calculation of a contribution of node i to the objective function. Destinations and Steiner nodes are denoted by solid squares and empty circles, respectively.

The objective function captures the entire network structure, and takes account of the frequency of usage of certain power levels. In the example above, node i uses power level p_{ii_2} every time source of the relayed signal lies in the subtree $T_{i_1/i}$ which contains three potential sources. The power level p_{ii_1} is used whenever the source lies outside of $T_{i_1/i}$, which applies to four sources. The contribution of node i to the objective function is thus $3p_{ii_2} + 4p_{ii_1}$. The total cost of T is the sum over all nodes' contributions. In general, the total power consumption, or cost, is

$$c(T) = \sum_{i \in V_T} \left[\text{nod}(T_{i_1/i}) p_{ii_2} + \text{nod}(T \setminus T_{i_1/i}) p_{ii_1} \right].$$

Problem 1 (SMT): Find a Steiner tree T of (G, D) minimizing $c(T)$.

Like most of the wireless network design problems presented in the literature, SMT is NP-hard. This follows from the NP-hardness of SBT[6], which is the special case of Problem 1 where $D = V$.

3 MILP Formulations Based on Broadcast Trees

In this section, we state and explain MILP formulations of the SMT problem. A basic element of every MIP formulation for SMT is a set of constraints modelling a Steiner tree. We investigate two such Steiner tree models with variables of up to 3 node indices and compare SMT models based on them. Both models are subsequently strengthened by valid inequalities. Variables with 4 node indices and associated constraints further extend the models. Valid inequalities added to the extended models result in the strongest known SMT formulations.

3.1 Original SMT Model [SMT-X1]

The first model extends the SBT formulation [2] by the Steiner nodes in order to formulate the multicast version of the problem.

3.1.1 Formulation

Define the binary variables

$$z_{ij} = \begin{cases} 1 & \text{if edge } \{i,j\} \in E \text{ is in the solution,} \\ 0 & \text{otherwise,} \end{cases}$$

$$x^s_{ij} = \begin{cases} 1 & \text{if arc } (i,j) \in A \text{ is used to transmit a message from } s \in D, \\ 0 & \text{otherwise.} \end{cases}$$

$$y^s_{ij} = \begin{cases} 1 & \text{if node } i \in V \text{ uses power } p_{ij} \text{ to transmit a message from } s \in D, \\ 0 & \text{otherwise.} \end{cases}$$

The model SMT-X1 is formulated as:

$$\min \sum_{(i,j)\in A} \sum_{s\in D} p_{ij} y^s_{ij} \tag{1a}$$

s.t.

$$\sum_{j\in V_i} x^s_{ji} = 1 \qquad i,s\in D, i\neq s, \tag{1b}$$

$$\sum_{j\in V_i} x^s_{ji} \leq 1 \qquad i\in V\setminus D, s\in D, \tag{1c}$$

$$x^s_{ij} \leq \sum_{k\in V_i\setminus\{j\}} x^s_{ki} \qquad i\in V\setminus D, (i,j)\in A, s\in D, \tag{1d}$$

$$x^s_{ij} + x^s_{ji} = z_{ij} \qquad \{i,j\}\in E, s\in D, \tag{1e}$$

$$x^s_{js} = 0 \qquad s\in D, (j,s)\in A, \tag{1f}$$

$$x^s_{ij} \leq \sum_{k\in V_i : p_{ik}\geq p_{ij}} y^s_{ik} \qquad s\in D, (i,j)\in A, \tag{1g}$$

$$z\in\{0,1\}^E, x,y\in\{0,1\}^{A\times D}. \tag{1h}$$

This model is a slightly modified version of the SMT model introduced in [4], which contains a constraint disallowing non-destination leaves and a weaker version of constraint (1d). Let (x,y,z) be an optimal solution to SMT-X1. Then, $x^s \in \{0,1\}^A$ induces a broadcast Steiner arborescence H rooted at source $s\in D$. From $z\in\{0,1\}^E$ we obtain the corresponding (undirected) broadcast Steiner tree. Finally, $y^s\in\{0,1\}^A$ describes links determining the power levels used by nodes when relaying a message originated in s. The graph induced by y is a subgraph of the tree induced by x, and is not necessarily connected.

Constraints (1b)-(1f) model a Steiner tree. Constraint (1b) ensures that a message from source s reaches a destination i from exactly one neighbour $j\in V_i$. Analogously, (1c) covers the case when $i\in V\setminus D$: for every source s, there is at most one inbound arc to a non-destination i.

If a non-destination i forwards a message from s towards j, it receives the same message from a neighbour k different from j. This is enforced by (1d).

Expression (1e) enforces that an edge $\{i,j\}$ is part of a solution if and only if for every $s\in D$, either (i,j) or (i,j) is an arc used for sending a message from s. The next constraint (1f) expresses that a transmission initiated by $s\in D$ cannot reach s again, which implies non-existence of a directed cycle containing s.

Finally, by (1g), we define a relation between x-variables and y-variables used in the objective function. Whenever the arc (i,j) is used for transmission of a message from $s\in D$, the power assigned to node i must be at least p_{ij}.

3.1.2 Valid inequalities [SMT-X1-VI]

It is possible to strengthen SMT-X1 by adding the valid inequalities

$$\sum_{j \in V_i} x_{ji}^s \leq \sum_{j \in V_i} x_{ij}^s \qquad i \in V \setminus D, s \in D, \tag{1i}$$

$$\sum_{j \in V_s} y_{sj}^s = 1 \qquad s \in D, \tag{1j}$$

$$\sum_{j \in V_i \setminus \{s\}} y_{ij}^s \geq \sum_{j \in V_i} x_{ji}^s \qquad i \in V \setminus D, s \in D. \tag{1k}$$

Constraint (1i) ensures that G_{x^s}, and also any feasible solution, does not contain Steiner nodes as leaves. Even though the presence of such leaves does not increase the objective value of an integral solution, it is desirable to eliminate them, because by definition, a Steiner tree does not contain non-destination leaves. Inequality (1j) says that there has to be exactly one neighbour $j \in V$ of $s \in D$, such that s uses the power p_{sj} in order to transmit its own signal. A signal never disappears in a non-destination. As (1k) states, if a non-destination i receives a signal from s, then there is a node $j \in V_i \setminus \{s\}$ to which the signal is forwarded requiring power p_{ij} assigned to node i. Index s can be excluded from the first sum's limits, because (1f), (1g) and optimality imply $y_{is}^s = 0$.

3.2 Multi-flow Extension [SMT-X2]

This section shows how to use multi commodity network flow in order to strengthen the model further.

3.2.1 Formulation

Consider a network flow problem where one unit of commodity (s, t) must be sent from $s \in D$ to $t \in D$. For this purpose, let $S = \{(s, t) \in D \times D, s \neq t\}$ be the set of ordered pairs of distinct destinations. In order to model the connectivity requirements, we introduce a variable f_{ij}^{st} as follows:

$$f_{ij}^{st} = \begin{cases} 1 & \text{if arc } (i, j) \in A \text{ carries flow from } s \text{ to } t,\ (s, t) \in S, \\ 0 & \text{otherwise.} \end{cases}$$

The relation between the x-variables in SMT-X1 and the f-variables is easy to see. If an arc (i, j) carries flow from s to t, then (i, j) is used for transmitting a signal initiated by s. SMT-X1 can be strengthened by flow conservation constraints for each (s, t)-pair, which gives

$$\min \sum_{(i,j)\in A} \sum_{s\in D} p_{ij} y_{ij}^s \tag{2a}$$

s.t.

$(1b) - (1g), (1i) - (1k)$

$$\sum_{j\in V_i} f_{ij}^{st} - \sum_{j\in V_i} f_{ji}^{st} = 0 \qquad (s,t) \in S, i \in V \setminus \{s,t\}, \tag{2b}$$

$$\sum_{j\in V_t} f_{tj}^{st} - \sum_{j\in V_t} f_{jt}^{st} = -1 \qquad (s,t) \in S, \tag{2c}$$

$$f_{ij}^{st} \leq x_{ij}^s, \qquad (i,j) \in A, (s,t) \in S, \tag{2d}$$

$$f_{ij}^{st} = f_{ji}^{ts}, \qquad (i,j) \in A, (s,t) \in S, \tag{2e}$$

$$\mathbf{z} \in \{0,1\}^E, \mathbf{x}, \mathbf{y} \in \{0,1\}^{A\times D}, \mathbf{f} \in \{0,1\}^{A\times S}. \tag{2f}$$

The flow conservation constraints (2b)-(2c) guarantee that for each $(s,t) \in S$, one unit of commodity (s,t) flows from s to t. Next, constraint (2d) expresses that if an arc (i,j) carries an s,t-flow, then this arc is used for sending a message initiated in s. The flow symmetry (2e) states that arc (i,j) carries flow from s to t if and only if arc (j,i) carries flow from t to s.

3.2.2 Valid inequalities [SMT-X2-VI]

The flow variables introduced in Section 3.2.1 suggest strengthening SMT-X2 by more valid inequalities involving these variables:

$$f_{ij}^{st_1} - f_{ij}^{st_2} + f_{ij}^{t_1t_2} \geq 0 \qquad (i,j) \in A, \quad (s,t_1),(s,t_2),(t_1,t_2) \in S, \tag{2g}$$

$$x_{ij}^s \leq \sum_{t\in D_s} f_{ij}^{st} \qquad (i,j) \in A, s \in D, \tag{2h}$$

$$\sum_{i\in V_j, p_{ji}\geq p_{jk}} f_{ji}^{st} \leq \sum_{i\in V_j, p_{ji}\geq p_{jk}} y_{ji}^s \quad j,k \in V, (s,t) \in S. \tag{2i}$$

Assume $s, t_1, t_2 \in D$. If there is a flow via (i,j) from s to t_2, then t_1 lies either in $T_{i/j}$ or in $T_{j/i}$. In the former case, (i,j) also carries a flow from t_1 to t_2. In the latter case, (i,j) carries flow from s to t_1. This is accomplished by (2g). By (2h) we state that whenever an arc (i,j) carries a signal from s, there is at least one destination other than s receiving it. That means that (i,j) carries an (s,t)-flow from s to t. Consider nodes $j,k \in V$ and a pair of destinations (s,t). If an (s,t)-flow is sent through (j,i) such that $p_{ji} \geq p_{jk}$, then a message from s must be relayed by j using power level at least p_{jk}. This is expressed by (2i).

3.3 SMT based on F1 [SMT -F1]

There are many formulations for the Steiner minimum tree problem, that can serve as a basis for modelling SMT. We consider the formulation F1, a multi-commodity network flow based model studied in [3], where the authors use abbreviation P_F. The model assumes a given $s_0 \in D$ that plays a role of a unique source. To simplify the notation, let $D_0 = D_{s_0}$.

3.3.1 Formulation

Model F1 for the Steiner minimum arborescence problem contains variables

$$f_{ij}^t = \begin{cases} 1 & \text{if arc } (i,j) \in A \text{ carries flow from } s_0 \text{ to } t \in D_0, \\ 0 & \text{otherwise,} \end{cases}$$
$$x_{ij} = \begin{cases} 1 & \text{if arc } (i,j) \in A \text{ is a part of the solution,} \\ 0 & \text{otherwise.} \end{cases}$$

The x-variables inducing the resulting tree correspond to arcs, and analogous z-variables in the SMT-X1 model correspond to edges. Hence, an optimal solution to SMT-X1 is an undirected tree, whereas optimal solutions to F1 are arborescences rooted at s_0. The vector f^t defines a directed path from s_0 to $t \in D$ in the arborescence.

We aim to create the model SMT-F1 based on F1. For this purpose, it is necessary to find a way to represent the constraint (1g) in the F1 space. The y-variables from SMT-X1 have to be used in the extended F1, because they appear in the objective function which remains unchanged. By considering the role of individual sets of variables in both models, the x-variables used in SMT-X1 are expressed by the variables used in SMT-F1 as

$$\begin{aligned} x_{ij}^s &= x_{ij} - f_{ij}^s + f_{ji}^s && (i,j) \in A, s \in D_0, \\ x_{ij}^0 &= x_{ij} && (i,j) \in A. \end{aligned} \tag{3}$$

Having this transformation in hand, it is easy to construct a model for Problem 1 based on the minimum Steiner tree model F1. We denote the model SMT-F1, and formulate it as follows:

$$\min \sum_{(i,j)\in A} \sum_{s\in D} p_{ij} y_{ij}^s \tag{4a}$$

s.t.

$$f_{ij}^t \leq x_{ij} \qquad t \in D_0, (i,j) \in A, \tag{4b}$$

$$\sum_{j\in V_i} f_{ji}^t - \sum_{j\in V_i} f_{ij}^t = \begin{cases} 1 & t \in D_0, t = i, \\ 0 & t \in D_0, i \in V \setminus \{s_0, t\}, \end{cases} \tag{4c}$$

$$x_{ij} - f_{ij}^t + f_{ji}^t \leq \sum_{\substack{k\in V: \\ p_{ik}\geq p_{ij}}} y_{ik}^t \qquad t \in D_0, (i,j) \in A, \tag{4d}$$

$$x_{ij} \leq \sum_{\substack{k\in V: \\ p_{ik}\geq p_{ij}}} y_{ik}^0 \qquad (i,j) \in A, \tag{4e}$$

$$\sum_{j\in V_i} x_{ji} \leq 1 \qquad i \in V \setminus D, \tag{4f}$$

$$f_{ti}^t = 0 \qquad t \in D_0, i \in V_t, \tag{4g}$$

$$f_{it}^t = x_{it} \qquad t \in D_0, i \in V_t, \tag{4h}$$

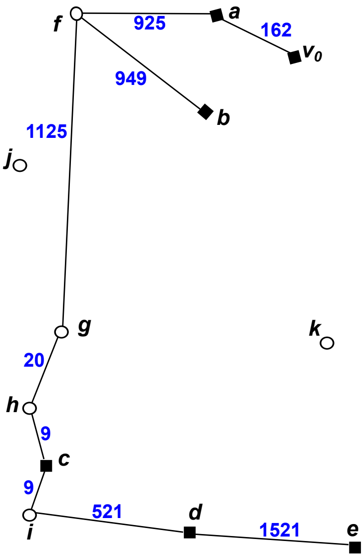
$$x_{i0} = 0 \qquad i \in V_0, \tag{4i}$$

$$x \in \{0,1\}^A, f \in \{0,1\}^{A\times D_0}, \tag{4j}$$

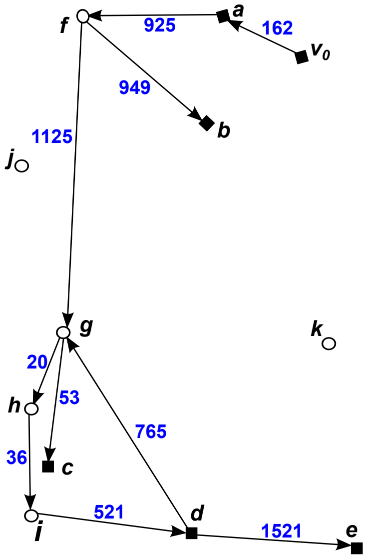
$$y \in \{0,1\}^{A\times D}. \tag{4k}$$

Constraints (4b)-(4c) together with (4j) imply that $\mathbf{x}$ induces an arborescence spanning D with node s_0 as the root. Constraint (4d) and (4e) have the same purpose as (1g), and are expressed in SMT-F1 space using transformations (3). Note that the y_{ij}^s-variables determining power levels are defined for all destinations $s \in D$, while in the model F1 [3] of the minimum Steiner tree problem, the f_{ij}^s variables are defined only for $s \in D_0$.

By (4f) we prevent a non-destination from having multiple entering arcs. This is not necessary in the minimum Steiner tree problem formulation F1, because the objective function causes that such solutions are filtered out by optimality. The necessity of this constraint in SMT is demonstrated in Fig. 4. The optimal solution with objective value 25156 to the depicted instance obtained by solving SMT-X1 is shown in Fig. 2a. The solution in Fig 2b yielded by solving SMT-F1 without the constraint (4f) has objective value 25148, but is not a feasible solution to Problem 1, because of the cycle (g, h, i, d, g). The non-existence of such a cycle in a solution given by model SMT-X1 is ensured by constraints (1b), (1c) and (1e). A detailed proof of this claim can be found in [4]. A transmission commenced in node c is sent via arc (g, f).



(a) Optimal solution obtained by SMT-X1. The z-variables inducing the resulting tree in this model represent undirected edges. The objective value of this tree is 25156.



(b) Solution obtained by SMT-F1 without constraint 4f with objective value 25148. The nodes are connected by arrows because the tree yielded by this model is rooted in a predefined source s_0.

Fig. 2: An exemplary instance showing why constraint (4f) is necessary in SMT-F1. Blue numbers denote power requuirements of connection between nodes. For better legibility, the distances of the links are not proportional.

As a consequence of the link between g and d in Fig. 2b, the node d also receives the message. This link is absent in Fig. 2a, and so i has to relay the signal using arc (i, d), causing the higher total objective value. Similarly, the obviously valid inequalities (4g)-(4i) are not necessary in the minimum Steiner tree formulation, but have to be included in the formulation of SMT, because they disallow nodes in D_0 to have multiple entering arcs. The same restriction has to be imposed on s_0 by adding (4i).

Proposition 1 *If (f, x) satisfies (4b) - (4c) and (4f)-(4j) then G_x is an arborescence spanning D rooted at s_0.*

Proof The connectivity of G_x as well as coverage of all nodes from D is ensured by flow constraints (4c) and relation (4b). The absence of both directed and undirected cycles is enforced by (4f)-(4i). These constraints together imply that no node has more than one entering arc. □

3.3.2 Valid inequalities [SMT-F1-VI]

The same valid inequalities as in SMT-X1-VI can be added to SMT-F1, leading to the SMT-F1-VI model. Inequality (1j) can be added without any change. The x-variable in (1k) has to be replaced by the equivalent expression defined by (3). Subsequent application of flow conservation yields

$$\sum_{j \in V_i} y_{ij}^s \geq \sum_{j \in V_i} x_{ji} \qquad i \in V \setminus D, s \in D. \tag{4l}$$

Further strengthening can be achieved by including

$$\sum_{j \in V_i} x_{ji} - \sum_{j \in V_i} x_{ij} \leq 0 \qquad i \in V \setminus D \tag{4m}$$

introduced in [3]. This constraint is analogous to ($1i$), and can also by obtained by applying (3) together with (4c).

3.4 F2 Extension [SMT-F2]

Similarly to the extension SMT-X2 of SMT-X1 by s,t-flow variables, the SMT-F1 model can also be extended by variables with four node indices. Analogously to S, let $\check{S} = \{\{s,t\} \subseteq D : s \neq t\}$ be the set of unordered pairs of destinations, and let $\check{S}_0 = \{\{s,t\} \in S : s \neq s_0 \neq t\}$.

3.4.1 Formulation

The authors of [3] use variables

$$\check{f}_{ij}^{st} = \begin{cases} 1 & \text{if arc } (i,j) \in A \text{ carries flow from } s_0 \text{ to both } s \text{ and } t, \ \{s,t\} \in \check{S}_0, \\ 0 & \text{otherwise,} \end{cases}$$

describing a common flow from s_0 to s and t. This allows formulation of an extended model denoted as SMT-F2:

$$\min \sum_{(i,j)\in A} \sum_{s\in D} p_{ij} y_{ij}^s \tag{5a}$$

s.t.

$(4b) - (4i), (1j), (4l), (4m)$,

$$\sum_{j\in V_i} \check{f}_{ji}^{st} - \sum_{j\in V_i} \check{f}_{ij}^{st} \geq \begin{cases} -1 & \{s,t\} \in \check{S}_0, i = 0, \\ 0 & \{s,t\} \in \check{S}_0, i \in V \setminus \{s_0\}, \end{cases} \tag{5b}$$

$$\check{f}_{ij}^{st} \leq f_{ij}^s \qquad \{s,t\} \in \check{S}_0, (i,j) \in A, \tag{5c}$$

$$\check{f}_{ij}^{st} \leq f_{ij}^t \qquad \{s,t\} \in \check{S}_0, (i,j) \in A, \tag{5d}$$

$$f_{ij}^s + f_{ij}^t - \check{f}_{ij}^{st} \leq x_{ij} \qquad \{s,t\} \in \check{S}_0, (i,j) \in A, \tag{5e}$$

$$x \in \{0,1\}^A, f \in \{0,1\}^{A\times D}, \check{f} \in \{0,1\}^{A\times \check{S}}, \tag{5f}$$

$$y \in \{0,1\}^{A\times D}. \tag{5g}$$

By (5b) is ensured that the common flow is non-increasing. The inequalities (5e) replace a weaker (4b). It follows from the domain of $\check{f}$, that

$$\check{f}_{ij}^{st} = \check{f}_{ij}^{ts}, \tag{6}$$

because S_0 consists of unordered pairs. By the implicit assumption of (6) in SMT-F2, it is possible to infer additional valid inequalities for SMT. We can also write

$$\begin{aligned} \check{f}_{ij}^{st} + \check{f}_{ji}^{st} = \check{f}_{ij}^{ts} + \check{f}_{ji}^{ts} \Rightarrow f_{ij}^t + f_{ji}^s - f_{ij}^{st} = f_{ij}^s + f_{ji}^t - f_{ij}^{ts} \Rightarrow \\ \Rightarrow f_{ij}^{0t} + f_{ji}^{0s} - f_{ij}^{st} = f_{ij}^{0s} + f_{ji}^{0t} - f_{ij}^{ts}. \end{aligned}$$

The first and second implication follow from the transformation (7a) and (8b), respectively. The last equality consists of only variables from SMT-X2 space, and so the valid inequality

$$f_{ij}^{ut} + f_{ji}^{us} + f_{ij}^{ts} = f_{ij}^{us} + f_{ji}^{ut} + f_{ij}^{st} \qquad (u,t),(u,s),(s,t),(t,s) \in S_0, i,j \in V$$

can be added to SMT-X2. All the occurances of s_0 were replaced by a general destination $u \in D$, because s_0 does not have any special role in SMT-X2.

3.4.2 Valid inequalities [SMT-F2-VI]

To complete the listing of models, we state the SMT-F2-VI model created by adding transformed valid inequalities (2g)-(2i) to SMT-F2.

4 Relations Between the Models

In order to create the SMT-F1 model, it is necessary to express variables x^s_{ij} in F1 space using relation (3). The aim of this section is to show how the entire SMT-X2 model can be converted into an equivalent model that uses only variables of SMT-F2.

The following equations express all variables from SMT-X2 in SMT-F2 space:

$$\begin{aligned} f^{st}_{ij} &= f^t_{ij}(1-\check{f}^{st}_{ij}) + f^s_{ji}(1-\check{f}^{st}_{ji}) = \\ &= f^t_{ij} + f^s_{ji} - \check{f}^{st}_{ij} - \check{f}^{st}_{ji} && (i,j)\in A, \{s,t\}\in S_0 \qquad (7a) \\ x^s_{ij} &= x_{ij}(1-f^s_{ij})(1-f^s_{ji}) + x_{ji}f^s_{ji} = \\ &= x_{ij} - f^s_{ij} + f^s_{ji} && (i,j)\in A, s\in D_0 \qquad (7b) \\ z_{ij} &= x_{ij} + x_{ji} && \{i,j\}\in E \qquad (7c) \end{aligned}$$

The transformations can be explained as follows: Let $T = (V_T, E_T)$ be a tree covering D, and consider an edge $\{i,j\} \in E_T$ dividing T into two subtrees T_i and T_j rooted in i and j, respectively. If the arc (i,j) carries flow from $s \in D$ to $t \in D$, then s and t must lie in different subtrees. Node s_0 lies either in T_i or T_j. These two cases are captured by the first equality in (7a). If both s_0 and s lie in T_i, then $f^t_{ij} = 1$. Similarly, if s_0 and t lie in T_j, then $f^s_{ji} = 1$. The expressions in parentheses prevent s and t belonging to the same subtree. Using the implications $\check{f}^{st}_{ij} = 1 \Rightarrow f^t_{ij} = 1$ and $\check{f}^{st}_{ji} = 1 \Rightarrow f^s_{ji} = 1$ that follow from the interpretation of variables, we justify the second equality expressing this relation linearly. In the transformation (7b) of x^s_{ij}, we distinguish the situation when s_0 and s are in the same subtree, in which case none of the arcs (i,j) and (j,i) carries a flow to s, and when s and s_0 belong to different subtrees, and there is a flow via (j,i) towards s. Again, the last equality is justified since $f^s_{ij} = 1 \Rightarrow x_{ij} = 1$. The relation (7c) is obvious.

By a similar approach, we achieve the transformation from SMT-X2 space to SMT-F2 space.

$$\begin{aligned} x_{ij} &= x^0_{ij} && (i,j)\in A \qquad (8a) \\ f^t_{ij} &= x^t_{ji}x^0_{ij} = f^{0t}_{ij} && (i,j)\in A, t\in D_0 \qquad (8b) \\ \check{f}^{st}_{ij} &= x^s_{ji}x^t_{ji}x^0_{ij} && (i,j)\in A, \{s,t\}\in \check{S}_0 \qquad (8c) \end{aligned}$$

We aim to compare the models presented in Section 3 in terms of strength. The results obtained by numerical experiments presented in the next section suggest that SMT-F1-VI model is at least as strong as SMT-X2. This section proves this conjecture. First, we express the SMT-X2 model in SMT-F2 space using transformations (7a)-(7c).

$$\min \sum_{(i,j)\in A} \sum_{s\in D} p_{ij} y^s_{ij} \tag{9a}$$

s.t.

$$\sum_{j\in V_i} (x_{ji} - f^s_{ji} + f^s_{ij}) = 1 \qquad i \in D, s \in D, i \neq s, \tag{9b}$$

$$\sum_{j\in V_i} (x_{ji} - f^s_{ji} + f^s_{ij}) \leq 1 \qquad i \in V \setminus D, s \in D, \tag{9c}$$

$$x_{ij} - f^s_{ij} + f^s_{ji} \leq \sum_{k\in V_i\setminus\{j\}} (x_{ki} - f^s_{ki} + f^s_{ik}) \qquad i \in V \setminus D, j \in V_i, s \in D, \tag{9d}$$

$$x_{js} - f^s_{js} + f^s_{sj} = 0 \qquad s \in D, j \in V_s, \tag{9e}$$

$$x_{ij} - f^s_{ij} + f^s_{ji} \leq \sum_{k\in V: p_{ik}\geq p_{ij}} y^s_{ik} \qquad s \in D, (i,j) \in A, \tag{9f}$$

$$\sum_{j\in V_i} \big(x_{ji} - f^s_{ji} + f^s_{ij}\big) \leq \sum_{j\in V_i} \big(x_{ij} - f^s_{ji} + f^s_{ij}\big) \qquad i \in V \setminus D, s \in D, \tag{9g}$$

$(1j)$

$$\sum_{j\in V_i\setminus\{s\}} y^s_{ij} \geq \sum_{j\in V_i} \big(x_{ji} - f^s_{ji} + f^s_{ij}\big) \qquad i \in V \setminus D, s \in D, \tag{9h}$$

$$\sum_{j\in V_i} (f^t_{ij} + f^s_{ji}) = \sum_{j\in V_i} (f^t_{ji} + f^s_{ij}) \qquad i \in V, t \in D, i \neq t, \tag{9i}$$

$$\sum_{j\in V_t} (f^t_{tj} + f^s_{jt}) = \sum_{j\in V_t} (f^t_{jt} + f^s_{tj}) = -1 \qquad t \in D, \tag{9j}$$

$$x_{ij} + x_{ji} \leq 1 \qquad \{i,j\} \in E, \tag{9k}$$

$$0 \leq x_{ij} - f^s_{ij} + f^s_{ji} \leq 1 \qquad (i,j) \in A, s \in D, \tag{9l}$$

$$f^t_{ij} - \check{f}^{st}_{ij} - \check{f}^{st}_{ji} \leq x_{ij} - f^s_{ij} \qquad (i,j) \in A, \{s,t\} \in \check{S}, \tag{9m}$$

$$0 \leq f^t_{ij} + f^s_{ji} - \check{f}^{st}_{ij} - \check{f}^{st}_{ji} \leq 1 \qquad \{i,j\} \in E, \{s,t\} \in \check{S}, \tag{9n}$$

$$f^0_{ij} = 0 \qquad (i,j) \in A, \tag{9o}$$

$$\mathbf{x} \in \{0,1\}^A, \mathbf{f} \in \{0,1\}^{A\times D}, \check{\mathbf{f}} \in \{0,1\}^{A\times \check{S}}, \tag{9p}$$

$$\mathbf{y} \in \{0,1\}^{A\times D}. \tag{9q}$$

The following lemmata are useful for analysis of the relations between the models.

Lemma 1 *All feasible solutions to LP(SMT-F1-VI) satisfy*

$$\sum_{j\in V_i} x_{ji} = 1, \qquad i \in D_0. \tag{10}$$

Proof Utilizing first (4h), next (4c) for $t = i$, and finally (4g), we get

$$\sum_{j\in V_i} x_{ji} = \sum_{j\in V_i} f^i_{ji} = 1 + \sum_{j\in V_i} f^i_{ij} = 1.$$

□

Lemma 2 *LP(SMT-F1-VI) has an optimal solution such that* $\forall (i, j) \in A, t \in D_0 : \min\{f^t_{ij}, f^t_{ji}\} = 0$.

Proof Assume that $\exists t \in D_0, (i, j) \in A : \min\{f^t_{ij}, f^t_{ji}\} = \epsilon > 0$ in an optimal solution. This assumption applies only for $j \neq t$ because (4g) defines $f^t_{ti} = 0$. It is then possible to reduce the flow towards t along the cycle (i, j, i) by ϵ. Flow conservation remains satisfied because for both i and j, entering and leaving flow towards t is reduced by the same amount. All remaining constraints are satisfied and the left-hand side of (4d) does not change because

$$x_{ij} - (f^t_{ij} - \epsilon) + (f^t_{ji} - \epsilon) = x_{ij} - f^t_{ij} + f^t_{ji},$$

and thereby the objective value is not altered. Such a solution is an alternative optimal solution satisfying the property stated by this lemma. □

In the following, let $f^*_{ij} = \max_{t\in D_0}\{f^t_{ij}\}$.

Lemma 3 *LP(SMT-F1-VI) has an optimal solution such that* $\forall (i, j) \in A : \min\{x_{ij} - f^*_{ij}, x_{ji} - f^*_{ji}\} = 0$.

Proof If $\exists (i, j) \in A : \min\{x_{ij} - f^*_{ij}, x_{ji} - f^*_{ji}\} = \epsilon > 0$, it is possible to decrease both x_{ij} and x_{ji} by ϵ which does not increase the objective value, and does not violate any constraint. In particular, the left-hand side in (4m) remains unchanged after this operation. □

Lemma 4 *LP(SMT-F1-VI) has an optimal solution where for each arc* $(i, j) \in A$*, at least one of the following properties holds:*

- $x_{ij} = f^*_{ij}$, *(L.4a)*
- *(4m) is satisfied with equality at* i*, i.e.,* $\sum_{k\in V_i} x_{ki} = \sum_{k\in V_i} x_{ik}$ *(L.4b)*

Proof If $f^*_{ij} = f^t_{ij}$ and $j = t$, the option (L.4a) already holds as stated by (4h). Let $j \neq t$. By decreasing x_{ij}, the objective value can not increase because x_{ij} appears only in the left-hand side of (4d) and (4e) with a positive sign. This variable can be reduced as long as it preserves feasibility of the solution. The only constraints that could be affected by reducing x_{ij} are (4b) and (4m), and so assigning

$$x_{ij} := \max\left\{f^*_{ij}, \sum_{k\in V_i} x_{ki} - \sum_{k\in V_i\setminus\{j\}} x_{ik}\right\}$$

either does not change the value x_{ij}, or yields an alternative optimum of LP(SMT-F1-VI). □

The only constraint that could prevent $x_{ij} = f^*_{ij}$ is (4m). If flows destined for two different destinations enter $i \in V \setminus D$ along two different entering arcs, and then continue via some shared arc (i,j), then $x_{ij} > f^*_{ij}$ can occur. An example of such situation is demonstrated in Fig. 3.

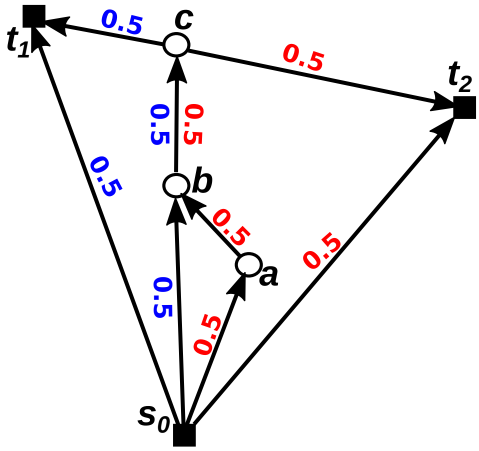


Fig. 3: A feasible solution to LP(SMT-F1-VI) of an instance on six vertices with destinations $D = \{s_0, t_1, t_2\}$. Blue and red labels along the edges represent the amount of flow through a given edge towards t_1 and t_2, respectively. In this solution, $x_{ij} > f^*_{ij}$, because if $x_{s_0 i} = x_{ki} = 0.5$, then x_{ij} must be equal to 1 in order to fulfill (4m).

Consider an instance with optimal solution (f, x, y) to LP(SMT-F1-VI) satisfying conditions in Lemma 2. If (f, x, y) violates conditions in Lemma 3, it is possible to alter x and obtain another optimum (f, x', y) satisfying conditions in both lemmata 2 and 3. Similarly, if (f, x', y) still violates Lemma 4 at some arc (i, j), the solution is altered by decreasing x'_{ij} while preserving feasibility. Once conditions in Lemma 2 and Lemma 3 are satisfied, such a decrease cannot violate them. These remarks are summarized as

Observation 1 *Instances of LP(SMT-F1-VI) have optimal solutions satisfying the conditions in all of Lemma 2, Lemma 3 and Lemma 4.*

Proposition 2 *LP(SMT-F1-VI) is at least as strong as LP(SMT-X2).*

Proof Let (x,f,y) be an optimal solution to LP(SMT-F1-VI) satisfying the conditions in Lemmata 2, 2 and 4. We show that each inequality in LP(SMT-X2) is implied by inequalities in LP(SMT-F1-VI).

(9b): Assume $t \in D_0$ and $i \in D_0 \setminus \{t\}$. Flow conservation (4c) implies

$$\sum_{j \in V_i} x_{ji} - \sum_{j \in V_i} f^t_{ji} + \sum_{j \in V_i} f^t_{ij} = \sum_{j \in V_i} x_{ji}.$$

Then (9b) follows from Lemma 1. Assume $i = s_0$. Due to (4i) and (4b), the following first two sums equal zero, which gives

$$\sum_{j\in V_i} x_{ji} - \sum_{j\in V_i} f^t_{ji} + \sum_{j\in V_i} f^t_{ij} = \sum_{j\in V_i} f^t_{ij} = 1,$$

where the latter equality follows by summing (4c) over all $i \in V \setminus \{s_0\}$. If $t = s_0$, then after applying definition (9o), i. e., $f^0_{ij} = 0$, (9b) also follows from Lemma 1.

(9c): The proof is analogous to (9b), with (4f) replacing (10).

(9d): The inequality can be rewritten as

$$\begin{aligned} x_{ij} \leq \sum_{k\in V_i\setminus\{j\}} x_{ki} - \sum_{k\in V_i\setminus\{j\}} f^s_{ki} + \sum_{k\in V_i\setminus\{j\}} f^s_{ik} + f^s_{ij} - f^s_{ji} = \\ = \sum_{k\in V_i\setminus\{j\}} x_{ki} - \sum_{k\in V_i} f^s_{ki} + \sum_{k\in V_i} f^s_{ik} = \sum_{k\in V_i\setminus\{j\}} x_{ki}, \end{aligned}$$

where the last equality follows from the flow conservation (4c). Now assume the contrary that for some $(i, j) \in A$, where $i \in V \setminus D$,

$$x_{ij} > \sum_{k\in V_i\setminus\{j\}} x_{ki}. \tag{11}$$

The proof is divided into two parts that capture the two cases stated by Lemma 4. Assume first that (L.4a) holds. We have that $\exists t \in D_0$ s. t. $f^t_{ij} = x_{ij}$. Besides the strict inequality, assumption (11) also implies $x_{ij} > 0$ which together with Lemma 2 gives $f^t_{ji} = 0$. By applying (4b) to arcs entering i,

$$\sum_{k\in V_i} f^t_{ik} \geq f^t_{ij} = x_{ij} > \sum_{k\in V_i\setminus\{j\}} x_{ki} \geq \sum_{k\in V_i\setminus\{j\}} f^t_{ki} = \sum_{k\in V_i} f^t_{ki},$$

contradicting flow conservation constraints (4c). Assume next that (L.4a) does not hold, i.e., $f^*_{ij} < x_{ij}$. We know from Lemma 3 that $x_{ji} = f^*_{ji}$, and so $\exists t \in D_0$ s. t. $f^t_{ji} = x_{ji}$. Moreover, Lemma 2 says that if for some $s \in D_0 : f^s_{ji} > 0$, then $f^s_{ij} = 0$, i.e. any flow that enters i via (j, i) must leave it through an arc different from (i, j). Together with the flow conservation and (4b),

$$x_{ji} = f^t_{ji} \leq \sum_{k\in V_i} f^t_{ik} = \sum_{k\in V_i\setminus\{j\}} f^t_{ik} \leq \sum_{k\in V_i\setminus\{j\}} x_{ik}.$$

Note that for $x_{ji} = f^t_{ji} = 0$, $f^t_{ij} \geq 0$ in which case the second equality above would not hold, but we could directly write $x_{ji} \leq \sum_{k\in V_i\setminus\{j\}} x_{ik}$. Combined with the assumption (11) we obtain

$$\sum_{k\in V_i} x_{ki} = x_{ji} + \sum_{k\in V_i\setminus\{j\}} x_{ki} < x_{ij} + \sum_{k\in V_i\setminus\{j\}} x_{ik} = \sum_{k\in V_i} x_{ik},$$

contradicting (L.4b), and thereby Lemma 4. The proof applies with minor simplifications also for $s = s_0$.

(9e): Follows from (4g) and (4h) if $s \in D_0$, or (4i) and (9o) if $s = s_0$.

(9f): Identical to (4d) if $s \in D_0$, or follows from (4e) and (9o) if $s = s_0$.

(9g): Follows immediately from (4m) by utilizing (4c) at node i if $s \in D_0$, or by utilizing (9o) if $s = s_0$.

(9h) Follows from (4l) by applying flow conservation at i if $s \in D_0$, and by applying (9o) if $s = s_0$.

(9i): All four-index variables cancel out. Thus, (9i) follows from flow conservation (4c) at $i \in D_0$. and from (9o) if $i = s_0$.

(9j): Is implied by flow conservation at t if $t \in D_0$. For $t = s_0$, by combining (9o) and (4i) together with (4b), we arrive at $\sum_{j \in V_0} f^s_{0j} = 1$, which obviously (?) holds.

(9k): Adding x_{ji} to (9d) gives the desired relation

$$x_{ij} + x_{ji} \leq \sum_{k \in V_i \setminus \{j\}} x_{ki} + x_{ji} = \sum_{k \in V_i} x_{ki} \leq 1,$$

where the last inequality follows from (4f) if $i \in V \setminus D$, and from Lemma 1 if $i \in D_0$. Finally, if $i = s_0$, (9k) is implied by combining (4i) together with relaxed integrality constraints.

(9l): The lower bound follows from (4b) for $s \in D_0$, and from (9o) and non-negativity of x_{ij} for $s = s_0$. The upper bound follows from (9b) for $i \in D$ and from (9c) for $i \in V \setminus D$. To see this, observe that each term in the sums in (9b)-(9c) is non-negative because of (4b).

Due to the symmetry $\check{f}^{st}_{ij} = \check{f}^{ts}_{ij}$, constraints (9m) and (9n) represent for each $(i,j) \in A$ and for each $\{s,t\} \in \check{S}_0$ relations

$$\check{f}^{st}_{ij} + \check{f}^{st}_{ji} \geq f^t_{ij} + f^s_{ij} - x_{ij}, \tag{12a}$$

$$\check{f}^{st}_{ij} + \check{f}^{st}_{ji} \geq f^t_{ji} + f^s_{ji} - x_{ij}, \tag{12b}$$

$$f^t_{ij} + f^s_{ji} \geq \check{f}^{st}_{ij} + \check{f}^{st}_{ji} \geq f^t_{ij} + f^s_{ij} - 1, \tag{12c}$$

$$f^s_{ij} + f^t_{ji} \geq \check{f}^{st}_{ij} + \check{f}^{st}_{ji} \geq f^s_{ij} + f^t_{ij} - 1. \tag{12d}$$

From these inequalities,

$$\check{f}^{st}_{ij} + \check{f}^{st}_{ji} \in \left[\max\{f^s_{ij} + f^t_{ji}, f^t_{ij} + f^s_{ji}\} - 1, \min\{f^s_{ij} + f^t_{ji}, f^t_{ij} + f^s_{ji}\}\right].$$

Assume without loss of generality that $f^s_{ij} + f^t_{ji} \leq f^t_{ij} + f^s_{ji}$. Assigning $\check{f}^{st}_{ij} := f^s_{ij}$ and $\check{f}^{st}_{ji} := f^t_{ji}$ does not violate any other constraint as the $\check{f}$-variables appear only in (9m) and (9n).

(9m): Using $\check{f}^{st}_{ij} + \check{f}^{st}_{ji} = f^s_{ij} + f^t_{ji}$ and (9l) we obtain inequalities

$$f^s_{ij} + f^t_{ji} - f^t_{ij} - f^s_{ij} + x_{ij} \geq 0,$$
$$f^s_{ij} + f^t_{ji} - f^t_{ji} - f^s_{ji} + x_{ij} \geq 0,$$

which shows that (12a) and (12b) hold under the selected values for $\check{f}^{st}_{ij}$ and $\check{f}^{st}_{ji}$.

(9n): Again, from $\check{f}^{st}_{ij} + \check{f}^{st}_{ji} = f^s_{ij} + f^t_{ji}$ we get for (12c)

$$f^s_{ij} + f^t_{ji} + 1 \geq f^s_{ij} + f^t_{ji},$$

which obviously holds. Finally, utilizing (9l) we obtain for (12d)

$$\begin{aligned} f^s_{ij} + f^t_{ji} + 1 - f^t_{ij} - f^s_{ji} = f^s_{ij} - f^s_{ji} + f^t_{ji} - f^s_{ij} + 1 \geq \\ \geq x_{ij} - 1 - x_{ij} + 1 = 0. \end{aligned}$$

□

Note that in parts (9d) and (9l) of this proof it is necessary to assume SMT-F1 instead of SMT-F2. The arguments work with (4b), but could not be used with stronger (5e). Proposition 2 suggests that additional 4-index variables in SMT-X2 model are not very beneficial, because the formulation is implied by the smaller SMT-F1-VI. Nonetheless, introducing SMT-X2 is justified because of valid inequalities (2g)-(2i) that significantly strengthen the model and can also be converted into SMT-F2 space and also increase the LP bound.

Proposition 3 *LP(SMT-X2) is at least as strong as LP(SMT-F1-VI).*

Proof The approach is analogous the the proof of Proposition 2. We express SMT-F1-VI in the space of SMT-X2 using transformations (8a) and (8b). After applying these transformations on (4b), (4c), (4e), (4f), (4i), (4j) and (4m), we immediately obtain constraints already present in SMT-X1-VI. Similarly, applying transformation (7b) on constraints (4d) and (4l) results in (1g) and (1k). For (4g) we get $f^t_{ti} = f^{0t}_{ti} = f^{t0}_{it} \leq x^t_{it} = 0$ by utilizing (2e) and (9m). Finally, we have

$$1 = \sum_{i \in V_t} f^{0t}_{it} \leq \sum_{i \in V_t} x^0_{it} = 1,$$

where the first equality follows from (2c) and already proved $f^{0t}_{ti} = 0, t \in D_0, i \in V_t$, and the second equality is a part of (1b). The inequality is a consequence of (2d), but clearly it must be satisfied with equality, and thus individual corresponding summands must be equal too. □

The combination of Propositions 2 and 3 implies

Corollary 1 *Models SMT-X2 and SMT-F1-VI are equally strong.*

5 Constraint Generation Scheme

The stronger models SMT-X2-VI and SMT-F2-VI are too large and are therefore not very practical for solving even fairly small instances. The main idea of how to tackle larger instances and thereby make these model more useful in practice is to solve a relaxation of the model where some of the constraints are omitted. It is assumed that the omitted constraints can often be satisfied in solution of the relaxed problem, without being explicitly included in the model. Relaxed constraints that are violated in the obtained solutions can be dynamically added to the model and the whole process is repeated, until some termination criteria are fulfilled. This approach is known as a *constraint generation scheme.*

5.1 Implementation

Experimental evaluation from the next section reveals that the valid inequality (2i) makes this model very strong, in fact it often gives an integral solution. To the contrary, the other valid inequalities rarely improve the LP bounds, and moreover increase the runtime. For these practical reasons, we consider the model SMT-X2+(2i) for constraint generation. Also note that the shortest runtime is given by solving LP(SMT-X1), and this runtime does not increase much by adding valid inequalities and solving LP(SMT-X1-VI).

For each $(s,t) \in S$, constraints (2b)-(2d)form a classical maximum $s-t$ flow formulation which can be solved very quickly. Observe that according to (2d), x^s plays the role of a capacitiy vector in the network $G_{x^s} = (V, E, s, t, x^s)$ in which we want to find the size of maximum flow, and thereby verify whether the flow conservation constraints are already satisfied even though they are not explicitly included in the model. The formulation can also be extended by (2i). Unfortunately, including (2e) is slightly problematic here, because it contains variables of different commodity, namely (t,s). It is not possible to simply extend the model by these constraints, because a potential violation of flow conservation for the commodity (t,s) would not be discovered, as there are no corresponding capacity constraints. Nevertheless, this can be easily resolved by combining the two commodities in one maximum flow formulation. Let us consider two distinct destinations $s,t \in D$ and define variables

$\acute{f}_{ij} \in [0,1]$ representing the amount of flow of commodity (s,t) carried by arc $(i,j) \in A$, and

$\grave{f}_{ij} \in [0,1]$ representing the amount of flow of commodity (t,s) carried by arc $(i,j) \in A$.

The extended maximum flow formulation denoted as 2MF is constructed as follows:

$$\min \sum_{i \in V_t} \acute{f}_{ti} - \sum_{i \in V_t} \acute{f}_{it} + \sum_{i \in V_s} \grave{f}_{si} - \sum_{i \in V_s} \grave{f}_{is} \tag{13a}$$

s.t.

$$\sum_{j \in V_i} \acute{f}_{ij} - \sum_{j \in V_i} \acute{f}_{ji} = 0 \qquad i \in V \setminus \{s,t\}, \tag{13b}$$

$$\acute{f}_{ij} \le x^s_{ij} \qquad (i,j) \in A, \tag{13c}$$

$$\grave{f}_{ij} \le x^t_{ij} \qquad (i,j) \in A, \tag{13d}$$

$$\acute{f}_{ij} = \grave{f}_{ji}, \qquad (i,j) \in A, \tag{13e}$$

$$\sum_{i \in V_j, p_{ji} \ge p_{jk}} \acute{f}_{ji} \le \sum_{i \in V_j, p_{ji} \ge p_{jk}} y^s_{ji} \qquad j,k \in V, \tag{13f}$$

$$\sum_{i \in V_j, p_{ji} \ge p_{jk}} \grave{f}_{ji} \le \sum_{i \in V_j, p_{ji} \ge p_{jk}} y^t_{ji} \qquad j,k \in V. \tag{13g}$$

$$\acute{f}, \grave{f} \in [0,1]^A. \tag{13h}$$

Note that the right-hand sides in (13c)-(13g) are not variables. These values are input data obtained by solving a weaker model. The problem modelled by this formulation can be understood as a maximum 2-commodity network flow, where the source of the first commodity is the target of the second commodity and vice versa. Furthermore, the arcs have different capacities for the two commodities, even though it is required that the amount of one commodity passing through an arc (i,j) equals the amount of the other commodity passing through (j,i).

Based on these observations we build the constraint generation procedure. First, we solve LP(SMT-X1-VI) and obtain vectors x and y. For each $(s,t) \in S$ we then check whether the constraints (2b)-(2d) together with (2i) can already be satisfied by solving the extended maximum $s-t$ flow model. If $\mathrm{mf}^{st}(G_{x^s})$=1, flow conservation constraints for (s,t) are satisfied, otherwise we remember (s,t), and after processing all pairs, new flow constraints for violated paris selected according to a certain strategy are added to the original model. This augmented model is again solved and the $s-t$ pairs are again checked for satisfying maximum flow constraints. The whole process is repeated until there are no violated flow constraints for any $s-t$ pair. The algorithm **??** describes the process formally.

There are various strategies how to determine which of the violated flow constraints will be added to the model. Here we describe those that we considered in the experimental part of this work.

5.1.1 Add all

One of the simplest and most basic strategies adds max-flow constraints of all the violated $s-t$ pairs. This ensures that in the next iteration is the last one, because all the constraints will be satisfied. This is turns out to be less practical, because after the first iteration, most of the $s - t$ pairs violate the flow constraints and we end up adding too many of them which usually leads to a very long runtime of the next iteration.

5.1.2 Add first k

Another simple strategy adds first k found $s - t$ pairs violating the flow constraints. An advantage of this strategy is that it is not necessary to run solve the maximum flow problem for all the $s - t$ pairs. Nevertheless, this does not have very noticeable effect, because solving the maximum-flow problem is very fast. Like in the previous case, this strategy does not exploit the structure of the graph induced by the violated $s - t$pairs.

5.1.3 Add best k

This strategy is based on the assumption that adding $s - t$ pairs for which the maximum flow value is lower, and are thus 'more violated', are more likely to cause satisfaction of a bigger number of $s - t$ pairs that are violated but not yet included.

5.1.4 Add maximum matching

6 Experimental Evaluation

The practical part of this work focuses on comparison of the models presented in the previous section. As the main focus of this study is to determine tighter bounds, the conducted experiments are designed for this purpose. Instances of intended number of vertices are generated with random coordinates uniformly distributed between [0, 0] and [100, 100]. All computations were made on an Intel Core 2 Quad CPU at 2.83 GHz and 8 GB RAM.

6.1 Comparison of the models

In the following experiments, two different scenarios are considered. First, we create instances with constant number of destinations, and the number of non-destinations gradually increases. Conversely, in the second scenario, the number of non-destinations is fixed, while the number of destinations increases. The models are compared with respect to the objective value of their solutions and CPU time.

7 Conclusion and Future Work

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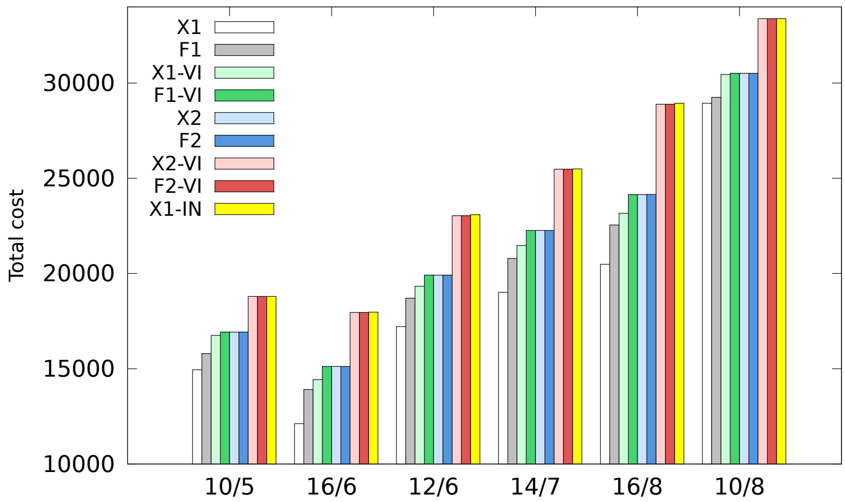
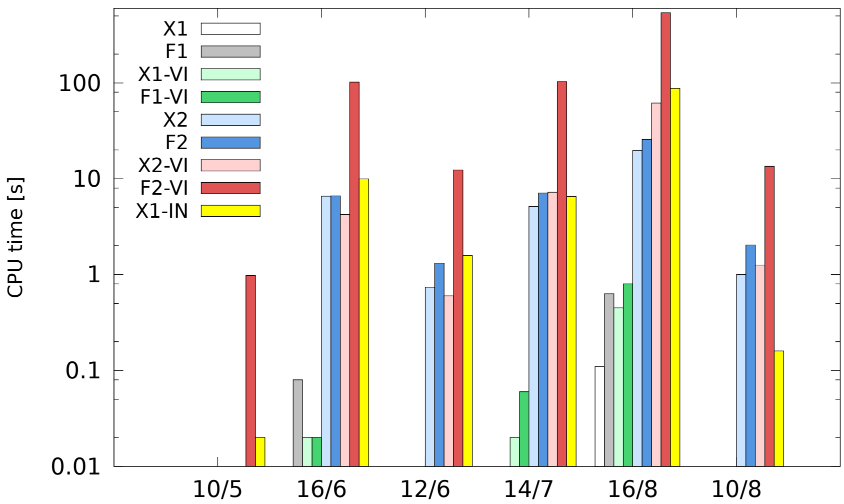


(a) Average objective values of solutions obtained for all the formulations.



(b) Average CPU times required for solving considered models.

Fig. 4

Table 1: AVG costs for 50 instances

Model	X1	F1	X1-VI	F1-VI	X2	F2	X2-VI	F2-VI	X1-INT
10/5	14944.73	15795.17	16754.53	16924.56	16924.56	16924.73	18794.82	18794.82	18794.82
16/6	12115.42	13909.75	14427.49	15124.43	15124.43	15125.53	17954.54	17954.54	17973.14
12/6	17215.43	18701.96	19332.55	19912.02	19912.02	19913.00	23033.58	23033.58	23084.48
14/7	19015.17	20793.59	21468.73	22261.69	22261.69	22262.70	25476.38	25476.38	25492.02
16/8	20485.22	22548.66	23162.98	24145.73	24145.73	24148.42	28887.35	28887.35	28943.36
10/8	28938.88	29248.02	30447.23	30513.38	30513.38	30514.54	33378	33378	33382.9

Table 2: AVG CPU time for 50 instances

Model	X1	F1	X1-VI	F1-VI	X2	F2	X2-VI	F2-VI	X1-INT
10/5	0	0	0	0	0	0	0	0.98	0.02
16/6	0	0.08	0.02	0.02	6.58	6.62	4.24	102.04	9.98
12/6	0	0	0	0	0.74	1.32	0.6	12.34	1.58
14/7	0	0	0.02	0.06	5.14	7.1	7.24	103.18	6.54
16/8	0.11	0.63	0.45	0.8	19.74	25.78	61.66	539.77	87.57
10/8	0	0	0	0	1	2.04	1.26	13.48	0.16