Section 12.2
Broadcasting and Gossiping

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INTRODUCTION

A communication network can be modeled by a connected graph with vertices representing nodes of the network and edges representing communication links between them. A low level operation in a communication network is the transmission of a message from one node to an adjacent node. A collection of such transmissions can be combined to achieve higher level goals such as broadcasting, in which information originating at one node in the network must be distributed to all other nodes of the network. A protocol specifies the transmissions and the order in which they are made to achieve a higher level goal and can be described by a labeled subgraph. We consider two broad categories of problems — finding the best network for a given high level goal and finding the best protocol for a given goal and network.

We assume that the nodes of the network are synchronized and that transmitting a message from a node to its neighbor takes one unit of time. A protocol can be specified by a sequence of sets, each containing the transmissions made during a particular time unit. We represent this in a graph by assigning labels to each edge indicating the time(s) during which that edge transmits a message. In this model, a message originating at vertex u is received by a vertex v if there is a path from u to v with increasing edge labels.

We assume that every vertex can send a message to at most one neighbor during a given time unit. In the labeled graph, this means that no two edges incident on a single vertex have the same label. We will discuss variants of this model at the end of this section.

12.2.1 Broadcasting

In this subsection, we consider broadcasting. We begin by looking in more detail at the notion of broadcast time in this model, giving some basic definitions and simple facts.

DEFINITIONS

D1: ***Broadcasting*** is the goal of sending a single message from a particular vertex (called the ***originator***) to all of the other vertices in the graph.

D2: A ***broadcast protocol for originator (source)*** s is represented by a graph $G = (V, E)$ on n vertices such that every edge e is labeled with at most one positive integer i and the labels satisfy the following constraints:

- the set of labels on the edges incident on any vertex are disjoint,
- there is exactly one path with increasing labels from s to each of the other vertices in the graph.

D3: A vertex v is ***informed at time*** t by a broadcast protocol for originator s if the last edge in the path from s to v is labeled with t. The ***completion time*** of a broadcast protocol for originator s is the least integer t such that every vertex v in the graph is informed by time t.

D4: A ***broadcast protocol for a graph*** G is a collection of broadcast protocols for each originator $s \in V(G)$.

D5: The ***broadcast time of a broadcast protocol for a graph*** G is the maximum completion time for any originator $s \in V(G)$.

D6: The ***broadcast time*** $b(G)$ ***of a graph*** G is the minimum broadcast time for any broadcast protocol for G.

D7: Let $b(n)$ be the minimum $b(G)$ over all graphs G with n vertices.

EXAMPLE

E1: Figure 12.2.1 shows examples of two broadcast protocols for a given originator s.

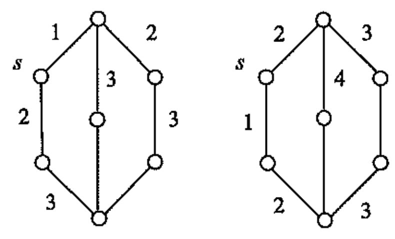

Figure 12.2.1: Two broadcast protocols for originator s.

REMARK

R1: To broadcast quickly, the labeled edges incident on s should have labels $1, 2, \ldots, i$ and for each other vertex v, one of its incident edges is labeled with the time t at which v is informed, and the other labeled edges incident on v should be labeled $t+1, t+2, \ldots, t+j$.

FACTS

F1: Due to the restrictions on the labels, at most 2^t vertices can be informed at time $t > 0$ by any broadcast protocol for any originator.

F2: $b(n) = \lceil \log n \rceil$. The previous fact implies $b(n) \geq \lceil \log n \rceil$. Protocols showing $b(n) \leq \lceil \log n \rceil$ are discussed next.

Minimum Broadcast Trees

Here we discuss those trees which correspond to minimum time broadcast protocols.

DEFINITIONS

D8: A ***minimum broadcast tree*** is a rooted tree on n vertices with root s for which there exists a broadcast protocol for originator s with completion time $\lceil \log n \rceil$.

D9: The ***binomial tree*** on 2^k vertices is a rooted tree defined recursively as follows. A single vertex is a binomial tree on 2^0 vertices. A binomial tree on 2^k vertices is formed from two binomial trees on 2^{k-1} vertices by joining their roots with an edge and making one of them the root of the binomial tree on 2^k vertices.

EXAMPLE

E2: Figure 12.2.2 shows a binomial tree and two of its subtrees that are minimum broadcast trees.

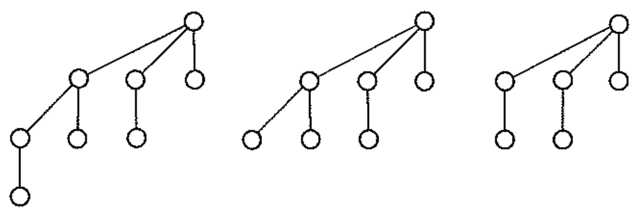

Figure 12.2.2: Binomial tree (left) and two subtrees.

FACTS

F3: The binomial tree on 2^k vertices is a minimum broadcast tree [Prosk].

F4: Every n vertex connected subtree, $2^{k-1} < n \leq 2^k$, of the binomial tree on 2^k vertices that includes the root is a minimum broadcast tree and these subtrees are the only minimum broadcast trees on n vertices [Prosk].

$B(n)$ and Minimum Broadcast Graphs

One research goal has been to determine the graphs on n vertices with the fewest edges that allow minimum time broadcasting from each vertex.

DEFINITIONS

D10: A graph $G = (V, E)$ on n vertices is a ***broadcast graph*** if there is a broadcast protocol with completion time $\lceil \log n \rceil$ for each originator $v \in V$. In other words, for every $v \in V$ there is a spanning subgraph of G that is a minimum broadcast tree rooted at v. A broadcast graph on n vertices is a ***minimum broadcast graph*** (MBG) if the number of its edges is the minimum over all broadcast graphs on n vertices. The number of edges in an MBG on n vertices is denoted $\boldsymbol{B(n)}$.

D11: A ***k-dimensional binary hypercube*** $\mathcal{H}_k$ is a k-regular vertex-transitive bipartite graph with $n = 2^k$ vertices and $k \cdot 2^{k-1}$ edges. Each vertex is labeled with a different binary string $x_1 x_2 \cdots x_k$, and there is an edge between two vertices if and only if their labels differ in exactly one bit position $1 \leq \ell \leq k$. Such vertices are called dimension ℓ neighbors and the edge connecting them is a dimension ℓ edge. The set of 2^{k-1} dimension ℓ edges is a perfect matching in $\mathcal{H}_k$. Thus, $\mathcal{H}_k$ can be defined recursively as two copies of $\mathcal{H}_{k-1}$ joined by a perfect matching corresponding to dimension k.

D12: The ***Cayley graph*** $\mathcal{D}_k$ is a $(k-1)$-regular vertex-transitive bipartite graph with $n = 2^k - 2$ vertices and $(k-1) \cdot (2^{k-1} - 1)$ edges. The vertices are labeled with the integers mod $2^k - 2$. There is an edge between two vertices with labels i and j if and only if $(i + j) \pmod{2^k - 2} = 2^\ell - 1$ for some $1 \leq \ell \leq k - 1$. Such vertices are called dimension ℓ neighbors and the edge connecting them is a dimension ℓ edge. The set of $2^{k-1} - 1$ dimension ℓ edges is a perfect matching in $\mathcal{D}_k$.

This definition is specifically for Cayley graphs based on dihedral groups.

D13: The ***recursive circulant graph*** $\mathcal{G}_{n,d}$, $d \geq 2$ has n vertices labeled with the integers $\bmod\, n$ and an edge between two vertices with labels i and j if and only if $i + d^\ell = j \pmod{n}$ for some $0 \leq \ell \leq \lceil \log_d n \rceil - 1$.

D14: The ***Knödel graph*** $\mathcal{W}_{n,\Delta}$ is a Δ-regular bipartite graph, $2 \leq \Delta \leq \lfloor \log n \rfloor$, with n vertices, n even, and $\frac{n\Delta}{2}$ edges. The vertices are labeled (i, j), $i = 1, 2$, $0 \leq j \leq \frac{n}{2} - 1$. There is a dimension ℓ edge, $0 \leq \ell \leq \Delta - 1$, between every pair of vertices $(1, j)$ and $(2, j + 2^\ell - 1 \pmod{\frac{n}{2}})$, $0 \leq j \leq \Delta - 1$. The set of $\frac{n}{2}$ dimension ℓ edges is a perfect matching in $\mathcal{W}_{n,\Delta}$.

EXAMPLES

E3: Figure 12.2.3 shows three non-isomorphic MBGs on 16 vertices.

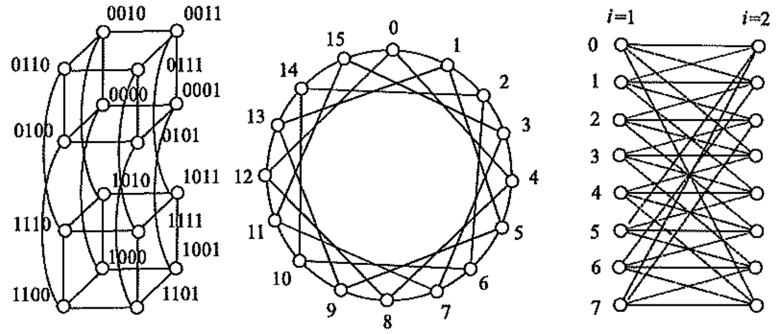

Figure 12.2.3: $\mathcal{H}_4$ (left), $\mathcal{G}_{16,4}$ (center), $\mathcal{W}_{16,4}$ (right).

E4: Figure 12.2.4 shows two isomorphic MBGs on 14 vertices.

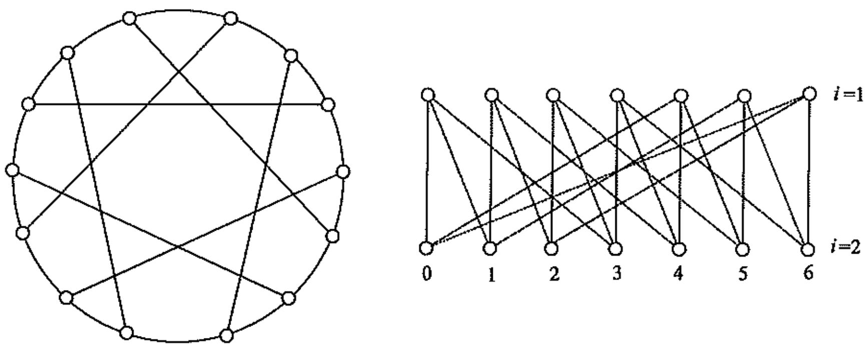

Figure 12.2.4: Heywood graph $\mathcal{D}_4$ (left), $\mathcal{W}_{14,3}$ (right).

FACTS

F5: The set $\{\mathcal{H}_k \mid k = 1,2,3,\ldots\}$ of binary hypercubes is an infinite family of MBGs [FHMP]. $\mathcal{H}_k$ is a broadcast graph because there is a spanning subgraph isomorphic to a binomial tree on 2^k vertices rooted at each vertex. One broadcast protocol is to use ***dimension order***: each vertex that is informed at time t informs its dimension $t+1, t+2, \ldots, k$ neighbors during time steps $t+1, t+2, \ldots, k$, respectively. $\mathcal{H}_k$ is an MBG because the number of informed vertices must double during each of k time units to inform all 2^k vertices and this requires that the originator has at least k neighbors. Any vertex can be the originator, so the minimum number of edges is $k \cdot 2^{k-1}$.

F6: The set $\{\mathcal{G}_{2^k,4} \mid k = 2,3,4,\ldots\}$ of recursive circulant graphs is an infinite family of MBGs. $\mathcal{G}_{2^k,4}$ is isomorphic to $\mathcal{H}_k$ for $k = 2$ and non-isomorphic to $\mathcal{H}_k$ for $k \geq 3$ [FR98].

F7: The set $\{\mathcal{W}_{2^k,k} \mid k = 2, 3, \ldots\}$ of Knödel graphs is an infinite family of MBGs. $\mathcal{W}_{2^k,k}$ is isomorphic to $\mathcal{H}_k$ for $k = 2, 3$ and non-isomorphic for $k \geq 4$ [FR98, FP01]. It is isomorphic to $\mathcal{G}_{2^k,4}$ for $k = 2$ and non-isomorphic for $k \geq 3$ [FR98].

F8: If G_1 and G_2 are MBGs on 2^k vertices, then the graph that results from adding any perfect matching between the vertices of G_1 and G_2 is an MBG on 2^{k+1} vertices. This construction can be applied recursively with different matchings at each stage to construct many non-isomorphic MBGs [FR98].

F9: The set $\{\mathcal{D}_k \mid k = 2, 3, 4, \ldots\}$ of Cayley graphs is an infinite family of MBGs with $2^k - 2$ vertices and $(k-1) \cdot (2^{k-1} - 1)$ edges [DFF91, KH90]. One broadcast protocol is a dimension order protocol similar to the hypercube protocol except that there are $k - 1$ dimensions instead of k. The originator and its dimension 1 neighbor are idle during the k^{th} time step because they have no uninformed neighbors after $k - 1$ time steps; every other informed vertex informs its dimension 1 neighbor during the k^{th} time step. $\mathcal{D}_k$ is an MBG because any originator with degree less than $k - 1$ cannot inform $2^k - 3$ other vertices in k time units, so every vertex must have degree at least $k - 1$.

F10: The Cayley graph $\mathcal{D}_k$ is isomorphic to the Knödel graph $\mathcal{W}_{2^k-2,k-1}$ for $k \geq 3$ [FR98, HMP97], so $\{\mathcal{W}_{2^k-2,k-1} \mid k = 3, 4, \ldots\}$ is an infinite family of MBGs.

F11: Table 12.1 shows the known values of $B(n)$ for small n, indicated by asterisks, and the best upper bounds currently known for other small n. Some of the graphs that verify these values are the results of ad hoc constructions, while others are the results of construction methods that have produced several MBGs. The contents of this table are based on [BFP92, BHLP92b, DFF91, FHMP, KH90, Labahn, MS94, MitHed, Sac96, VenWen, WenVen95, WenVen, XiaWan, ZZ01].

n	$B(n)$	n	$B(n)$	n	$B(n)$	n	$B(n)$	n	$B(n)$	n	$B(n)$	n	$B(n)$
1	0*	10	12*	19	25*	28	48*	37	57	46	82	55	111
2	1*	11	13*	20	26*	29	52*	38	57	47	83	56	111
3	2*	12	15*	21	28*	30	60*	39	60	48	83	57	126
4	4*	13	18*	22	31*	31	65*	40	60	49	94	58	121*
5	5*	14	21*	23	34	32	80*	41	65	50	95	59	124*
6	6*	15	24*	24	36	33	48	42	66	51	100	60	130*
7	8*	16	32*	25	40	34	49	43	71	52	99	61	136*
8	12*	17	22*	26	42*	35	51	44	72	53	107	62	155*
9	10*	18	23*	27	44*	36	52	45	81	54	108	63	162*

Table 12.1: Upper bounds on $B(n)$ for small n. Values that are known to be optimal are indicated by asterisks.

F12: $B(n) \in \Theta(L(n)n)$, where $L(n)$ is the number of consecutive leading 1's in the binary representation of $n - 1$ [GP91].

F13: $B(n) \leq (n \lceil \log n \rceil)/2$ [Far].

F14: $B(n) \leq (n \lfloor \log n \rfloor)/2$ for even values of n [FR98].

F15: $B(n)$ is monotonically non-decreasing for any n in the first quarter of the range between any two consecutive powers of 2. More precisely, $B(n) \leq B(n+1)$ for $2^{m-1}+1 \leq n \leq 2^{m-1} + 2^{m-3} - 1$ [HL03].

RESEARCH PROBLEMS

RP1: Is $B(n)$ monotonically non-decreasing between two consecutive powers of 2? This question has only been solved for the first quarter of each range (see **F15**).

RP2: In a broadcast graph on $n = 2^k - 1$ vertices, every vertex must have degree at least $k-1$ and if the originator has degree $k-1$ then it must have at least one neighbor of degree k. This gives a lower bound $B(2^k - 1) \geq \left\lceil \frac{1}{2}(n(k-1) + \left\lceil \frac{n}{k+1} \right\rceil) \right\rceil$. MBGs matching this lower bound are known for $k \leq 7$ [BHLP92b, FHMP, H08, Labahn]. The question is open for $k > 7$. A good starting point to explore this problem would be the values of k that are one less than a prime because the lower bound formula is exact for these values by Fermat's little theorem (i.e., the ceilings disappear).

RP3: $B(n)$ is only known for a few infinite families of graphs and some small values of n. The general problems of determining $B(n)$ and of finding more infinite families are open. MBGs on $n = 2^k$ and $n = 2^k - 2$ vertices are k-regular and $(k-1)$-regular graphs, respectively. For most other values of n the MBGs will not be regular and the difficulty of proving upper and lower bounds seems to increase the farther that n is from a power of 2. MBGs for $n = 2^k - 1$ vertices are known for small k (see **RP2**). General lower bounds for $n = 2^k - l$, $l = 3, 4, 5, 6$, $k \geq 4$ were proved in [Sac96] and MBGs matchings these lower bounds are known for $k = 4, 5, 6$. It is unknown if there are MBGs matching these bounds for $k \geq 7$. The degree of any originator in an MBG on $n = 2^k - 2^p - r$ vertices is at least $k - p$ [GarVac] and this gives a general lower bound on $B(n)$.

Construction of Sparse Broadcast Graphs

As MBGs have proven to be difficult to construct, numerous methods have been proposed to construct sparse broadcast graphs that have a small number of edges. Almost all of these methods are combinations and variations of a few techniques.

DEFINITIONS

D15: Given a broadcast protocol in a broadcast graph G on n vertices, a vertex u is ***idle at time*** $t \leq \lceil \log n \rceil$ if and only if u is aware of the message at (the beginning of) time step t and u does not communicate with any of its neighbors during time step t.

D16: A subset of vertices C in a broadcast graph G is a ***solid 1-cover*** if and only if C is a vertex cover of G, and for each $u \notin C$, there is a broadcast protocol for u such that at least one neighbor of u is idle at some time during the broadcast.

D17: Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. The ***compound of*** G ***into*** H ***relative to a set*** $S \subseteq V(G)$, denoted $G_S[H]$, is obtained by replacing each vertex x of H by a graph G_x isomorphic to G and adding a matching between two sets S_x and S_y if x and y are adjacent in H. More precisely, the matching between S_x and S_y connects each vertex in S_x with its copy in S_y. For any vertex $u \in S$, we use H_u to denote the graph isomorphic to H that interconnects the copies of $u \in V(G)$.

EXAMPLES

E5: Figure 12.2.5 shows a broadcast protocol for C_6 (left) in which the vertex to the left of the originator is idle during time step 3, and two copies of C_6 joined by a partial matching between their solid covers of size 3 (right).

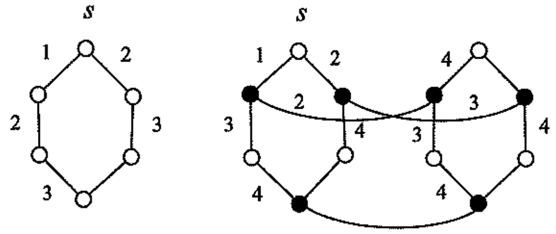

Figure 12.2.5: Partial matching of two copies of C_6.

E6: Figure 12.2.6 shows a solid 1-cover of the Heywood graph.

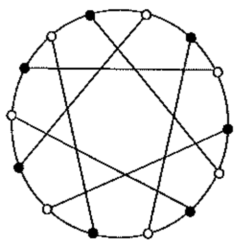

Figure 12.2.6: Solid 1-cover of the Heywood graph $\mathcal{D}_4$.

FACTS

F16: If $G = (V, E)$ is a broadcast graph on n vertices and C is a solid 1-cover of G, then $B(2n) \leq 2|E| + |C|$ [BFP92, KH90]. Construct G' by joining two isomorphic copies of G, G_1 and G_2, with a perfect matching between their solid covers C_1 and C_2. Let $u_1 \in V(G_1)$ be the originator of the broadcast. If $u_1 \in C_1$, then u_1 sends the message to its copy u_2, and then u_1 and u_2 broadcast in G_1 and G_2, respectively. If $u_1 \notin C_1$, then u_1 initiates a broadcast in G_1 and whenever a vertex $v_1 \in C_1$ is informed, it first informs its copy v_2 and then both copies of v continue to broadcast in their copies of G. C_2 is a solid cover so u_2 can be informed by an idle vertex. See Figure 12.2.5 for an example.

F17: Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be broadcast graphs on n_1 and n_2 vertices, respectively, and let C be a solid 1-cover of G. If $\lceil \log n_1 n_2 \rceil = \lceil \log n_1 \rceil + \lceil \log n_2 \rceil$, then $G_C[H]$ is a broadcast graph and therefore $B(n_1 n_2) \leq n_2|E(G)| + |C||E(H)|$ [BFP92]. A specific case is seen in **F16**, where H is $\mathcal{H}_2$.

F18: $B(2^k - 2^p) \leq \frac{2^k - 2^p}{4}(2k - p - 1)$ for any $1 \leq p \leq k - 2$ and $k \geq 4$ [KH90]. This also follows from **F17**.

F19: Vertex deletion was introduced in [XiaWan]. If G is a broadcast graph with n vertices, $2^{k-1} + 1 < n \leq 2^k$, e edges, and a vertex v of degree d, then $B(n - 1) \leq e + \frac{1}{2}d(d - 3)$. Construct G' on $n - 1$ vertices by deleting v and adding enough edges to form a clique among the former neighbors of v. Each broadcast protocol for G can be modified so that the vertex u that informed v in G informs the former neighbors of v in the same order that v informed them in G.

REMARKS

R2: Vertex addition was introduced in [BHLP92b]. In this construction method, a vertex is added to a broadcast graph on $n-1$ vertices and is connected to some of the original vertices, yielding a broadcast graph on n vertices. This method is used in [BHLP92b, H08, HL12].

R3: Solid 1-covers are vertex covers for which the broadcast protocols have enough idle vertices to "cover" all paths of length 1 (i.e., edges). They can be generalized to ***solid h-covers*** which cover all paths of length h. Solid h-covers with $h > 1$ are used in [BFP92, KH90, MS94].

R4: m-way splits were introduced in [ChaLie] (for $m = 5, 6, 7$) and generalized in [BFP92]. The construction is similar to graph compounding and uses a graph G that has a perfect matching with certain broadcast properties, and graphs that have ***even adjacency splits.*** (A graph on ℓ vertices has an even adjacency split if it has two dominating sets of size $\lceil \frac{\ell}{2} \rceil$ and $\lfloor \frac{\ell}{2} \rfloor$.) The method produces sparse broadcast graphs on $n = mi + j$ vertices for which i is not a power of 2, $0 \leq j < m$, and $\lceil \log(mi + j) \rceil = \lceil \log m \rceil + \lceil \log i \rceil$. This method produces broadcast graphs for more values of n than solid h-covers, which require that $n = mi$ for $m, i \in \mathbb{N}$.

R5: Several other construction methods have been developed. Most are combinations of matchings, partial matchings, compounding, m-way splits, vertex deletion, vertex addition, and operations on hypercubes. The methods in [Di99, VenWen, WenVen95] are similar to the solid covers method with vertex deletion and give similar results. The methods in [Che, GarVac] use solid covers, hypercubes, and vertex deletion. Matchings were used in the first construction methods [Far] and early uses of solid covers, vertex addition, and vertex deletion appear in [BHLP92a]. The methods in [HL99] include compounding, merging vertices, and deleting edges based on a broadcast graph construction using binomial trees.

R6: The constructions in [GP91] use hypercubes and generalized Fibonacci numbers to produce asymptotic bounds on $B(n)$.

Bounded Degree Broadcast Graphs

If we restrict our attention to graphs of given maximum degree, the time required to broadcast increases. Researchers have developed efficient broadcast protocols for some bounded degree graphs that are of interest to network designers.

DEFINITIONS

D18: The ***k-dimensional cube-connected cycles graph*** $\mathcal{CCC}_k$ is derived from $\mathcal{H}_k$ by replacing each vertex $x = x_1x_2 \ldots x_k$ of $\mathcal{H}_k$ by a cycle of length k with vertices labeled (i, x), $0 \leq i \leq k-1$. Each of the k vertices on a cycle that replaces x inherits one of the k edges that were incident to x in $\mathcal{H}_n$. In particular, vertex (i, x) is connected to its neighbors $(i+1, x)$ and $(i-1, x)$ on its cycle and to vertex $(i, x_1x_2 \cdots \overline{x_{i+1}} \cdots x_k)$ where $\overline{x_{i+1}}$ is the complement of the binary value x_{i+1}.

D19: The ***k-dimensional (wrapped) butterfly graph*** $\mathcal{BF}_k$ is derived in a similar way to $\mathcal{CCC}_k$. $\mathcal{BF}_k$ has the same vertex set as $\mathcal{CCC}_k$. The difference is that $\mathcal{BF}_k$ has two edges corresponding to each former hypercube edge instead of one: (i, x) is connected to $(i+1, x_1x_2 \cdots \overline{x_{i+1}} \cdots x_k)$ and $(i-1, x_1x_2 \cdots \overline{x_i} \cdots x_k)$.

D20: The ***k*-dimensional binary shuffle-exchange graph** $\mathcal{SE}_k$ has 2^k vertices with the same labels as the vertices of $\mathcal{H}_k$. Each vertex $x_1x_2\cdots x_k$ is connected to its *shuffle neighbor* $x_2x_3\cdots x_kx_1$ and its *unshuffle neighbor* $x_kx_1\cdots x_{k-1}$, and to its *exchange neighbor* $x_1x_2\cdots\overline{x_k}$. Parallel edges and loops are removed.

D21: The ***k*-dimensional binary de Bruijn graph** $\mathcal{UB}_k$ has the same vertex set as $\mathcal{H}_k$ and $\mathcal{SE}_k$. As in $\mathcal{SE}_k$, each vertex $x_1x_2\cdots x_k$ is connected to its shuffle neighbor $x_2x_3\cdots x_kx_1$ and its unshuffle neighbor $x_kx_1\cdots x_{k-1}$. It is also connected to its *shuffle-exchange neighbor* $x_2x_3\cdots x_k\overline{x_1}$ and its *unshuffle-exchange neighbor* $x_kx_1\cdots\overline{x_{k-1}}$. Parallel edges and loops are removed.

D22: The ***de Bruijn graph*** $\mathcal{UB}_{d,k}$ ***with degree*** $2d$ ***and diameter*** k has d^k vertices with labels that are strings of length k over the alphabet $\{0,1,\ldots,d-1\}$. There is an edge between vertex $x_1,x_2,\ldots,x_k$ and each vertex $x_2,x_3,\ldots,x_k,\lambda$ and $\lambda,x_1,x_2,\ldots,x_{k-1}$ with $\lambda\in\{0,1,\ldots,d-1\}$. $\mathcal{UB}_{2,k}$ is the special case $\mathcal{UB}_k$ in **D21**.

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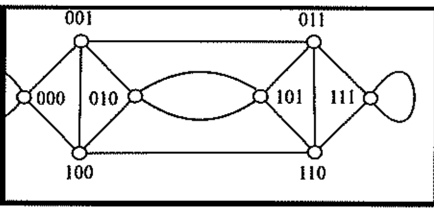

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Table 12.2 lists some parameters of hypercubes and bounded-degree approximations of hypercubes.

	vertices	edges	degree	diameter
$\mathcal{H}_k$	2^k	$k \cdot 2^{k-1}$	regular k	k
$\mathcal{CCC}_k$	$k2^k$	$\frac{3}{2} \cdot k2^k$	regular 3	$2k - 1 + \max(1, \lfloor \frac{k-2}{2} \rfloor)$
$\mathcal{BF}_k$	$k2^k$	$2 \cdot k2^k$	regular 4	$\lfloor \frac{3k}{2} \rfloor$
$\mathcal{SE}_k$	2^k	$\frac{3}{2} \cdot 2^k - 3 + (k \pmod 2)$	maximum 3	$2k - 1$
$\mathcal{UB}_k$	2^k	$2 \cdot 2^k - 3$	maximum 4	k

Table 12.2: Some parameters of hypercube-derived graphs.

FACTS

F20: Let a_t^Δ be the maximum number of vertices that can be informed after t time units in any graph with maximum degree Δ. There is one informed vertex at time $t = 0$ (the originator), and $a_t^\Delta \leq 2 \cdot a_{t-1}^\Delta$ for $t \geq 1$ (see **F1**). Furthermore, at time $t + \Delta - 1$, $t \geq 1$, all vertices that were informed by time t have no more uninformed neighbors, so $a_{t+\Delta}^\Delta \leq a_{t+\Delta-1}^\Delta + (a_{t+\Delta-1}^\Delta - a_t^\Delta) = 2a_{t+\Delta-1} - a_t^\Delta$. Based on this recurrence, $1.440 \log_2 n - 1.769$ is a lower bound on $b(n, 3)$ and $1.137 \log_2 n - 0.637$ is a lower bound on $b(n, 4)$ [LP88]. Closed form lower bounds are not known for $\Delta > 4$, but asymptotically $b(n, \Delta) > p(n, \Delta)$ where $p(n, \Delta) \approx (1 + \frac{\log_2 e}{2\Delta})$ [BHLP92a].

F21: $b(\mathcal{CCC}_k) = \lceil \frac{5k}{2} \rceil - 1$ [LP88].

F22: $1.7417k \leq b(\mathcal{BF}_k) \leq 2k - 1$. The upper bound may be improved to $2k - \frac{1}{2} \log \log k + c$ for some constant c and a sufficiently large k [KMPS94].

F23: $b(\mathcal{SE}_k) = 2k - 1$ [HJM93].

F24: $1.3171k \leq b(\mathcal{UB}_k) \leq \frac{3}{2}(k + 1)$. The upper bound is from [BP88] and the lower bound is from [KMPS94].

F25: $b(n, \Delta) < p(n, \Delta)$ where $p(n, \Delta) \approx (1 + \frac{0.415}{\Delta}) \log_2 n$ [BHLP92a]. This bound is obtained using graph compounds $\mathcal{H}[\mathcal{UB}_{d,k}]$ where $\mathcal{H}$ is a hypercube and $\mathcal{UB}_{d,k}$ is a de Bruijn graph. Better upper bounds for particular values of Δ can be obtained by compounding different graphs in de Bruijn graphs [BHLP92a]. See **F20** for a lower bound on $p(n, \Delta)$.

12.2.2 Gossiping

In this subsection, we turn our attention to gossiping — a related problem that was first investigated before broadcasting. Results on broadcasting provide lower bounds on gossiping and graphs constructed for broadcasting are also useful for gossiping.

DEFINITIONS

D24: ***Gossiping*** is the goal of sending a unique message from each vertex to all of the other vertices in the graph.

D25: A ***gossip protocol*** is represented by a graph $G = (V, E)$ on n vertices such that every edge e is labeled with a set of positive integers c_e and the labels satisfy the following constraints:

- the sets of labels on any two edges incident on a vertex are disjoint,
- there is a ***transmission path*** between any two vertices u and v; in particular a path of edges $(e_1, e_2, \ldots, e_k)$ such that there exist elements of the corresponding label sets $t_i \in c_{e_i}$ such $t_1 < t_2 < \cdots < t_k$. Such a transmission path "ends" at time t_k.

D26: A vertex v is ***informed at time t by a gossip protocol*** if there exists a transmission path from every other vertex to v that ends at or before time t.

D27: The ***completion time*** of a gossip protocol is the least integer t such that every vertex v in the graph is informed at time t.

D28: The ***gossip time*** $g(G)$ of a graph G is the minimum completion time of any gossip protocol for G.

EXAMPLE

E8: Figure 12.2.8 shows an example of a gossip protocol for a unicyclic graph.

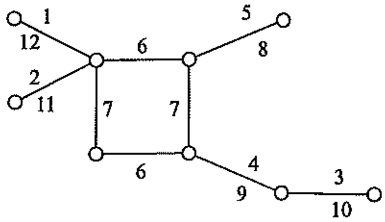

Figure 12.2.8: A gossip protocol.

REMARK

R7: Implicit in the above definitions is a basic model of communication. This model allows "calls" between two vertices such that all messages currently known to both vertices are exchanged. The model allows this unbounded exchange of information to occur in one time unit. Further, the model allows calls on disjoint edges, e_1 and e_2, to occur during the same time unit t, when $t \in c_{e_1}$ and $t \in c_{e_2}$; each time unit is a "round." When these assumptions are found to be untenable, researchers have augmented the basic model in a variety of ways that are too numerous to consider within the bounds of this paper.

FACTS

F26: Clearly $g(G) \geq b(G)$ since from a gossip protocol a broadcast protocol for each vertex of G can be inferred that has a completion time which is at most the completion time of the gossip protocol.

F27: Given a gossip protocol, the time-reversed protocol derived from it is also a valid gossip protocol. In particular, if for each edge e we replace each $t \in c_e$ with $g(G) - t + 1$, transmission paths still exist for all pairs of vertices [T71].

Optimal Gossip Graphs

Here, we discuss the structure of gossiping in terms of the number of calls made without regard to the time needed to make the calls.

DEFINITIONS

D29: The ***size of a gossip protocol*** for a graph G is the total number of calls used. In particular, the size is $\sum_{e \in E(G)} |c_e|$. Let $f(G)$ be the minimum size of any gossip protocol for G. A protocol of size $f(G)$ is an ***optimal gossip protocol*** for G.

D30: The ***optimal gossip size*** $f(n)$ is the minimum $f(G)$ over all graphs with n vertices.

REMARKS

R8: As remarked above, the basic model of communication allows calls to occur "at the same time." For the purpose of understanding $f(G)$ this is de-emphasized; it is as if parallel calls have been linearized. Bandwidth consumed by a protocol is measured by this quantity, not time.

FACTS

F28: $f(n+1) \leq f(n)+2$ for $n \geq 1$ [T71]. This follows from the following observation: Given a graph G with $f(G) = f(n)$, add an edge from any vertex v of G to a new vertex v' to form a new graph G'. A valid gossip protocol for G' can be obtained by adding a call between v and v' both before and after a valid gossip protocol for G. This observation can be used to recursively construct gossip protocols from protocols for smaller graphs.

F29: $f(G) \leq 2n-3$ for any (connected) graph G. This follows from the previous fact, building a protocol recursively using a trivial protocol for the subgraph of two adjacent vertices as the basis. For some graphs G with n vertices we have $f(G) > f(n)$; for example, the graph which is a path of four vertices requires, by exhaustive analysis, 5 calls, while $f(n) = 4$.

F30: $f(n) = 2n - 4$ for $n \geq 4$, while $f(1) = 0$, $f(2) = 1$, and $f(3) = 2$. Various researchers independently proved this fact using very different proof techniques [BS72, HMS72, T71]. Other proofs appeared later as corollaries to stronger theorems. The difficulty was the lower bound; the upper bound follows from the next fact.

F31: $f(G) = 2n - 4$ for any graph G which contains C_4 as a subgraph; C_4 is a cycle of four edges. Note that $f(C_4) = 4$. This fact follows from the above facts, building a protocol recursively using a protocol for the C_4 subgraph as the basis. An example of such a protocol was seen in Figure 12.2.8.

F32: $f(G) = 2n-4$ if and only if the graph G contains C_4 as a subgraph. This resisted proof for many years even though it was widely conjectured. (The difficulty is that a C_4 subgraph is necessary even though the protocol in the previous fact can be replaced by a fundamentally different protocol.) Eventually two independent proofs appeared [B81, KS80]. Later proofs used deeper results about "information flow" [K96, L95].

Minimum Gossip Graphs

In this subsection, we return our focus to minimum time gossiping and consider the graphs on n vertices with the fewest edges that allow minimum time gossiping.

DEFINITIONS

D31: Recall that $g(G)$ is the gossip time for G, the number of rounds necessary to gossip in G. The ***minimum gossip time*** $g(n)$ is the minimum $g(G)$ over all graphs with n vertices.

D32: Let $M_n = \{G \mid G \text{ has } n \text{ vertices and } g(G) = g(n)\}$. A ***minimum gossip graph*** G (MGG) is a member of M_n, for some n, and the number of its edges is the minimum over all graphs in M_n. The number of edges in an MGG on n vertices is denoted $\boldsymbol{G(n)}$.

FACTS

F33: As noted above $g(G) \geq b(G)$ for every G, so it follows that $g(n) \geq b(n)$.

F34: $g(n) = \lceil \log_2 n \rceil$ if n is even and $g(n) = \lceil \log_2 n \rceil + 1$ if n is odd. The lower bound follows from the bound on $b(n)$ and the fact that with odd n some node does not participate in the first round. The upper bounds come from constructions [K75]. This result was anticipated in earlier literature where the problem was posed in a different context, e.g. [B50].

F35: It follows from the above that $G(n) \geq B(n)$ for even n [F00].

F36: For any tree T with n vertices $g(T) \geq 2\lceil \log_2 n \rceil - 1$ [L86].

F37: For all $n = 2^k$, $k \geq 1$, $G(n) = \frac{n}{2} \log_2 n$. In particular, the set $\{\mathcal{H}_k \mid k = 1, 2, 3, \ldots\}$ of binary hypercubes is an infinite family of MGGs for such n [L93]. $\mathcal{H}_k$ has a gossip protocol that makes all calls across each dimension at the same time and proceeds in dimension order; it will broadcast from each originator in the same manner as was discussed for broadcasting. For $n = 16$, all MGGs have been characterized; they are all formed from $\mathcal{H}_4$ by perturbing C_4 subgraphs [LP91]. Further, when $k \geq 2$ the families $\mathcal{W}_{2^k,k}$ and $\mathcal{G}_{2^k,4}$ are also MGGs [FR98].

F38: For all $n = 2^k - 2$, $k \geq 3$, $G(n) = \frac{n}{2}(\lceil \log_2 n \rceil - 1)$. In particular, the set $\{\mathcal{D}_k \mid k = 3, 4, 5, \ldots\}$ of Cayley graphs is an infinite family of MGGs for such n [DFF91, L93]. One gossip protocol is a dimension order protocol similar to the hypercube protocol except there are $k - 1$ dimensions instead of k; after calling across all dimensions the first dimension is repeated.

F39: For all $n = 2^k - 4$, $k \geq 6$, $G(n) = \frac{n}{2}(\lceil \log_2 n \rceil - 1)$. In particular, the infinite family $\mathcal{W}_{n,k-1}$ are MGGs for such n [FR98, L93].

F40: $G(n) \leq (n \lfloor \log n \rfloor)/2$ for even values of n [BHLP97].

F41: Table 12.3 shows the known exact values of $G(n)$ for small n, indicated by asterisks, and the best upper bounds currently known for other small n. These are apparently the only known values, beyond the families in the above facts. Some of the graphs that verify these values are the results of ad hoc constructions, while others are the results of construction methods that have produced several MBGs. The contents of this table are based on [F00, FL00, FR98, L93].

n	$G(n)$	n	$G(n)$	n	$G(n)$	n	$G(n)$	n	$G(n)$	n	$G(n)$	n	$G(n)$	n	$G(n)$
1	0*	5	5*	9	9*	13	17	17	20	21	27	25	32	29	45
2	1*	6	6*	10	13*	14	21*	18	25	22	36	26	52	30	60*
3	2*	7	7*	11	11*	15	19*	19	22	23	29	27	34	31	47
4	4*	8	12*	12	18*	16	32*	20	28	24	36*	28	56*	32	80*

Table 12.3: Upper bounds on $B(n)$ for small n. Values that are known to be optimal are indicated by asterisks.

12.2.3 Other Variations of Broadcasting and Gossiping

Many variations of broadcasting and gossiping have been investigated. Most of these involve changes to the model that we have been using, considering digraphs, hypergraphs, multiple originators, multiple messages from a single source, random transmissions, or restricted protocols. The model described here is somewhat unrealistic for gossiping, as it assumes that combined messages can be sent in a single transmission in constant time. Many of the other papers on gossiping use different models that account for the size of the combined messages sent.

For more information on these and other variations, see [FL94, HHL88, HKMP96, HKPRU05].

References

[BS72] B. Baker , R. Shostak, Gossips and telephones, *Discrete Math.* 2 (1972), 191–193.

[B50] A. Bavelas, Communication patterns in task-oriented groups, *J. Acous. Soc. Am.* 22(6) (1950), 725–730.

[BFP92] J.-C. Bermond, P. Fraigniaud, and J.G. Peters, Antepenultimate broadcasting, *Networks* 26 (1995), 125–137.

[BHLP97] J.-C. Bermond, H.A. Harutyunyan, A.L. Liestman, and S. Pérennes, A note on the dimensionality of modified Knödel graphs, *Int. J. Found. Comput. Sci.* 8(2) (1997), 109–116.

[BHLP92a] J.-C. Bermond, P. Hell, A.L. Liestman, and J.G. Peters, Broadcasting in bounded degree graphs, *SIAM J. Discrete Math.* (5) (1992), 10–24.

[BHLP92b] J.-C. Bermond, P. Hell, A. L. Liestman, and J.G. Peters, Sparse broadcast graphs, *Discrete Appl. Math.* 36 (1992), 97–130.

[BP88] J.-C. Bermond and C. Peyrat, Broadcasting in de Bruijn networks. In: *Proc. 19th SE Conference on Combinatorics, Graph Theory and Computing*, Congr. Numer. (1988), 283–292.

[B81] R. Bumby, A Problem with telephones, *SIAM J. Alg. Discrete Math.* 2 (1981), 13–18.

[ChaLie] S. C. Chau and A. L. Liestman, Constructing minimal broadcast networks, *J. Comb. Inf. Syst. Sci.* 10 (1985), 110–122.

[Che] X. Chen, An upper bound for the broadcast function $B(n)$, *Chinese J. Comput.* 13 (1990), 605-611.

[R94] J. de Rumeur, *Communications dans les réseaux de processeurs*, Collection Etudes et Recherches en Informatique, Masson, Paris, 1994.

[DFF91] M. J. Dinneen, M. R. Fellows, and V. Faber, Algebraic constructions of efficient broadcast networks, In *Applied Algebra, Algebraic Algorithms and Error Correcting Codes 9.* Lecture Notes in Computer Science 539 (1991), 152–158.

[Di99] M. J. Dinneen, J. A. Ventura, M. C. Wilson, and G. Zakeri, Compound constructions of broadcast networks, *Discrete Math.* 93 (1999), 205–232.

[Far] A. M. Farley, Minimal broadcast networks, *Networks* 9 (1979), 313–332.

[FHMP] A. M. Farley, S. Hedetniemi, S. Mitchell, and A. Proskurowski, Minimum broadcast graphs, *Discrete Math.* 25 (1979), 189-193.

[F00] G. Fertin, A study of minimum gossip graphs, *Discrete Math.* 215 (2000), 33–57.

[FL00] G. Fertin and R. Labahn, Compounding of gossip graphs, *Networks* 36(2) (2000), 126–137.

[FR98] G. Fertin and A. Raspaud, Families of graphs having broadcasting and gossiping properties. In: *Proc. 24th Int. Workshop on Graph-Theoretic Concepts in Computer Science (WG '98)*, Smolenice, Lecture Notes in Computer Science 1517, Springer-Verlag, 1998, 63–77.

[FL94] P. Fraigniaud and E. Lazard, Methods and problems of communication in usual networks, *Discrete Appl. Math.* 53 (1994), 79–133.

[FP01] P. Fraigniaud and J. G. Peters, Minimum linear gossip graphs and maximal linear (Δ, k)-gossip graphs, *Networks* 38 (2001), 150–162.

[GarVac] L. Gargano and U. Vaccaro, On the construction of minimal broadcast networks, *Networks* 19 (1989), 673-689.

[GP91] M. Grigni and D. Peleg, Tight bounds on minimum broadcast networks, *SIAM J. Discrete Math.* 4(2) (1991), 207–222.

[HMS72] A. Hajnal, E. C. Milner, and E. Szemeredi, A cure for the telephone disease, *Canad. Math. Bull.* 15(3) (1972), 447–450.

[H08] H. A. Harutyunyan, An efficient vertex addition method for broadcast networks, *Internet Math.* 5(3) (2008), 211–225.

[HL99] H. A. Harutyunyan and A. L. Liestman, More broadcast graphs, *Discrete Appl. Math.* 98 (1999), 81–102.

[HL03] H. A. Harutyunyan and A. L. Liestman, On the monotonicity of the broadcast function, *Discrete Math.* 262(1-3) (2003), 149–157.

[HL12] H. A. Harutyunyan and A. L. Liestman, Upper bounds on the broadcast function using minimum dominating sets, *Discrete Math.* 312(20) (2012), 2992–2996.

[HHL88] S. M. Hedetniemi, S. T. Hedetniemi, and A. L. Liestman, A survey of gossiping and broadcasting in communication networks, *Networks* 18 (1988), 319–349.

[HMP97] M.-C. Heydemann, N. Marlin and S. Pérennes, Complete rotations in Cayley graphs, *Eur. J. Comb.* 22(2) (2001), 179–196.

[HJM93] J. Hromkovič, C. D. Jeschke, and B. Monien, Optimal algorithms for dissemination of information in some interconnection networks, *Algorithmica* 10 (1993), 24–40.

[HKMP96] J. Hromkovič, R. Klasing, B. Monien, and R. Peine, Dissemination of information in interconnection networks (broadcasting & gossiping). In F. Hsu and D.-Z. Du, editors, *Combinatorial Network Theory.* Kluwer Academic Publishers, 1996, 125–212.

[HKPRU05] J. Hromkovič, R. Klasing, A. Pelc, P. Ružička, and W. Unger, Dissemination of information in communication networks: part 1. broadcasting, gossiping, leader election, and fault-tolerance, Springer Monograph, Springer-Verlag, 2005.

[KH90] L. H. Khachatrian and O. S. Haroutunian, Construction of new classes of minimal broadcast networks. In *Proc. 3rd International Colloquium on Coding Theory, Dilijan, Armenia,* 1990, 69–77.

[KMPS94] R. Klasing, B. Monien, R. Peine, and E. A. Stöhr, Broadcasting in butterfly and de Bruijn networks, *Discrete Appl. Math.* 53(1-3) (1994), 183–197.

[KS80] D. J. Kleitman and J. B. Shearer, Further gossip problems, *Discrete Math.* 30 (1980), 151–156.

[K75] W. Knödel, New gossips and telephones, *Discrete Math.* 13 (1975), 95.

[K96] D. W. Krumme, Reordered gossip schemes, *Discrete Math.* 156 (1996), 113–140.

[Labahn] R. Labahn, A minimum broadcast graph on 63 vertices, *Discrete Appl. Math.* 53 (1994), 247–250.

[L95] R. Labahn, Kernels of minimum size gossip schemes, *Discrete Math* 143 (1995), 99–139.

[L93] R. Labahn, Some minimum gossip graphs, *Networks* 23 (1993), 333–341.

[L86] R. Labahn, The telephone problem for trees, *Elektron. Info. u Kybernet.* 22 (1986), 475–485.

[LP91] R. Labahn and C. Pietsch, Characterizing minimum gossip graphs on 16 vertices, *Technical Report,* Univ. Bonn (1991).

[LP88] A. L. Liestman and J.G. Peters, Broadcast networks of bounded degree, *SIAM J. Discrete Math.* 1(4) (1988), 531–540.

[MS94] M. Mahéo and J.-F. Saclé, Some minimum broadcast graphs, Technical Report 685, LRI, Université de Paris-Sud, 1991. (Some of the results of this report have since been published with the same title in *Discrete Appl. Math.* 53 (1994), 275–285.)

[MitHed] S. Mitchell and S. Hedetniemi, A census of minimum broadcast graphs, *J. Comb. Inf. Syst. Sci.* 5 (1980), 141–151.

[Prosk] A. Proskurowski, Minimum broadcast trees, *IEEE Trans. Comput.* 30 (1981), 363–366.

[Sac96] J.-F. Saclé, Lower bounds for the size in four families of minimum broadcast graphs, *Discrete Math.* 150 (1996), 359–369.

[T71] R. Tijdeman, On a telephone problem, *Nieuw Arch. Wiskd.* 19(3) (1971), 188–192.

[VenWen] J. A. Ventura and X. Weng, A new method for constructing minimal broadcast networks, *Networks* 23 (1993), 481–497.

[WenVen95] X. Weng and J. A. Ventura, A doubling procedure for constructing minimal broadcast networks, *Telecommunication Systems* 3 (1995), 259–293.

[WenVen] X. Weng and J. A. Ventura, Constructing optimal broadcast networks, Working Paper 91-132, Dept. of Industrial and Management Systems Engineering, Pennsylvania State Univ., 1991.

[XiaWan] J. Xiao and X. Wang, A research on minimum broadcast graphs, *Chinese J. Comput.* 11 (1988), 99–105.

[ZZ01] J.-G. Zhou and K.-M. Zhang, A minimum broadcast graph on 26 vertices, *Appl. Math. Lett.* 14 (2001), 1023–1026.